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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL-ANCILLARY MATHEMATICS – II (SMT1213)

ANCILLARY MATHEMATICS II

COURSE CODE SMT 1213

COMMON TO BSC., (PHYSICS & CHEMISTRY)

COURSE MATERIAL

SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

UNIT I TRIGONOMETRY

Section : II

Expansion of sin $(n\theta)$ and cos $(n\theta)$ in powers of sin θ and cos θ where n is a positive integer.

By De-Moivre's theorem

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

$$= (\cos\theta)^n + n c_1 (\cos\theta)^{n-1}(i\sin\theta)$$

$$+ n c_2 (\cos\theta)^{n-2}(i\sin\theta)^2$$

$$+ n c_3 (\cos\theta)^{n-3}(i\sin\theta)^3$$

$$+ n c_4 (\cos\theta)^{n-4}(i\sin\theta)^4 + \ldots\ldots \text{ by Binomial result}$$

$$\left[(x+y)^n = x^n + nc_1 x^{n-1} y + nc_2 x^{n-2} y^2 + \ldots \right]$$

$$= \cos^n\theta + i\, n c_1 \cos^{n-1}\theta \sin\theta$$

$$- nc_2 \cos^{n-2}\theta \sin^2\theta - nc_3\, i \cos^{n-3} \sin^3\theta$$

$$+ n c_4 \cos^{n-4}\theta \sin^4\theta + \ldots\ldots$$

$$\cos n\theta = \textbf{Real part of R.H.S}$$

$$= \cos^n\theta - n c_2 \cos^{n-2}\theta \sin^2\theta$$

$$+ n c_4 \cos^{n-4}\theta \cdot \sin^4\theta - \ldots$$

$$\sin(n\theta) = Im \textbf{ part of R.H.S}$$

$$= n c_1 \cdot \cos^{n-1}\theta \cdot \sin\theta - n c_3 \cos^{n-3}\theta \cdot \sin^3\theta$$

$$+ n c_5 \cos^{n-5}\theta \sin^5\theta - \ldots\ldots$$

Expansion of tan $(n\theta)$ in powers of tan θ

$$\tan(n\theta) = \frac{\sin(n\theta)}{\cos(n\theta)}$$

$$= \frac{n c_1 \cos^{n-1}\theta \sin\theta - n c_3 \cos^{n-3}\theta \sin^3\theta + n c_5 \cos^{n-5}\theta \cdot \sin^5\theta - \ldots}{\cos^n\theta - n c_2 \cos^{n-2}\theta \cdot \sin^2\theta + n c_4 \cos^{n-4}\theta \cdot \sin^4\theta - \ldots}$$

$$= \frac{n c_1 \cos^{n-1}\theta \sin\theta/\cos^n\theta - n c_3 \cos^{n-3}\sin^3\theta/\cos^n\theta + n c_5 \cos^{n-5}\theta \cdot \sin^5\theta/\cos^n\theta - \ldots}{1 - nc_2\cos^{n-2}\theta \cdot \sin^2\theta/\cos^n\theta + n c_4 \cos^{n-4}\theta \sin^4\theta/\cos^n\theta}$$

$$\tan n\theta = \frac{n c_1 \tan\theta - n c_3 \tan^3\theta + n c_5 \tan^5\theta - \ldots\ldots}{1 - n c_2 \tan^2\theta + n c_4 \tan^4\theta - \ldots}$$

Note : In working our problems on expansion, we often use the following results

(i) $\sin^2\theta = 1 - \cos^2\theta$

(ii) $\sin^4\theta = (1 - \cos^2\theta)^2$

$$= 1 - 2\cos^2\theta + \cos^4\theta$$

(iii) $\sin^6\theta = (1 - \cos^2\theta)^3$

$$= 1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta$$

(iv) $\sin^8\theta = (1 - \cos^2\theta)^4$

$$= 1 - 4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta$$

(a) $\cos^2\theta = 1 - \sin^2\theta$

(b) $\cos^4\theta = (1 - \sin^2\theta)^2$

$$= 1 - 2\sin^2\theta + \sin^4\theta$$

(c) $\cos^6\theta = (1 - \sin^2\theta)^3$

$$= 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta$$

(d) $\cos^8\theta = (1 - \sin^2\theta)^4$

$$= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta$$

Eg: Show that $\dfrac{\cos 3\theta}{\cos\theta} = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$

Solution : $\dfrac{\cos 3\theta}{\cos\theta} = \dfrac{\cos^3\theta - 3c_2\cos\theta\cdot\sin^2\theta}{\cos\theta}$

$$= \cos^2\theta - 3\sin^2\theta = (1 - \sin^2\theta) - 3\sin^2\theta$$

$$= 1 - 4\sin^2\theta = \cos^2\theta - 3(1 - \cos^2\theta)$$

$$= 4\cos^2\theta - 3$$

Eg: Express $\dfrac{\sin 4\theta}{\sin\theta}$ **in terms of** $\cos\theta$.

Solution : $\dfrac{\sin 4\theta}{\sin\theta} = \dfrac{4c_1\cos^3\theta\cdot\sin\theta - 4c_3\cos\theta\cdot\sin^3\theta}{\sin\theta}$

$$= 4\cos^3\theta - 4\sin^2\theta\cos\theta$$

$$= 4[\cos^3\theta - \cos\theta(1 - \cos^2\theta)]$$

$$= 4[\cos^3\theta - \cos\theta + \cos^3\theta]$$

$$= 8\cos^3\theta - 4\cos\theta$$

Eg: Express $\cos 4\theta$ **interms of** $\sin\theta$ **and** $\cos\theta$.

Solution :

$$\cos 4\theta = \cos^4\theta - 4c_2\cos^2\theta\cdot\sin^2\theta + 4c_4\sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 + \cos^4\theta - 2\cos^2\theta$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= (1 - \sin^2\theta)^2 - 6(1 - \sin^2\theta)\sin^2\theta + \sin^4\theta$$

$$= 1 - 2\sin^2\theta + \sin^4\theta - 6\sin^2\theta + 6\sin^4\theta + \sin^4\theta$$

$$= 8\sin^4\theta - 8\sin^2\theta + 1.$$

Eg: Show that $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

$$= 16\sin^4\theta - 20\sin^2\theta + 5$$

Solution :

$$\frac{\sin 5\theta}{\sin\theta} = \frac{5c_1\cdot\cos^4\theta\sin\theta - 5c_3\cos^2\theta\sin^3\theta + 5c_5\sin^5\theta}{\sin\theta}$$

$$= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= 5\cos^4\theta - 10\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$$

$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 16\cos^4\theta - 12\cos^2\theta + 1$

Similarly $\dfrac{\sin 5\theta}{\sin\theta} = 5(1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + \sin^4\theta$

$= 5(1 - 2\sin^2\theta + \sin^4\theta)$

$-(10)\sin^2\theta + 10\sin^4\theta + \sin^4\theta$

$= 16\sin^4\theta - 20\sin^2\theta + 5$

Eg: Express cos 6 θ interms of cos θ.

Solution : $\cos 6\theta = \cos^6\theta - 6c_2\cos^4\theta\cdot\sin^2\theta$

$+ 6c_4\cos^2\theta\sin^4\theta - 6c_6\sin^6\theta$

$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\cdot\sin^4\theta - \sin^6\theta$

$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2$

$-(1-\cos^2\theta)^3$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$-(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta$

$- 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

Eg: Express cos (6θ) interms of sin θ.

Solution :

$\cos(6\theta) = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

by the previous eg

$= (1-\sin^2\theta)^3 - 15(1-\sin^2\theta)^2\sin^2\theta$

$+ 15(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$

$= 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta - 15\sin^2\theta(1 - 2\sin^2\theta + \sin^4\theta)$

$+ 15\sin^4\theta - 15\sin^6\theta - \sin^6\theta$

$= 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta - 15\sin^2\theta + 30\sin^4\theta - 15\sin^6\theta$

$+ 15\sin^4\theta - 15\sin^6\theta - \sin^6\theta$

$= 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta$

Eg: S.T. $\dfrac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

Solution :

L.H.S. $= \dfrac{\sin 6\theta}{\sin\theta} = \dfrac{6c_1\cos^5\theta\sin\theta - 6c_3\cos^3\theta\sin^3\theta + 6c_5\cos\theta\cdot\sin^5\theta}{\sin\theta}$

$= 6\cos^5\theta - 20\cos^3\theta\ \sin^2\theta + 6\cos\theta\cdot\sin^4\theta$

$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$

$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta$

$+ 6\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= 6\cos^5\theta - 20\cdot\cos^3\theta + 20\cdot\cos^5\theta + 6\cos\theta$

$- 12\cos^3\theta + 6\cos^5\theta$

$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

Eg: $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$

Solution : Given expression

$= \dfrac{7c_1\cos^6\theta\sin\theta - 7c_3\cos^4\theta\cdot\sin^3\theta + 7c_5\cdot\cos^2\theta\cdot\sin^5\theta - 7c_7\sin^7\theta}{\sin\theta}$

$= 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 2\cos^2\theta\sin^4\theta - \sin^6\theta$

$$= 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta) - \sin^6\theta$$

$$= 7(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta) - 35(1 - 2\sin^2\theta + \sin^4\theta)\sin^2\theta$$

$$+ 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$$

$$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta + 70\sin^4\theta - 35\sin^6\theta$$

$$+ 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

Eg. Find $\dfrac{\cos 7\theta}{\cos\theta}$ in powers of $\cos\theta$.

Solution : we now

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

Substituting $\frac{\pi}{2} - \theta$ instead of θ,

$$\frac{\sin 7\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = 7 - 56\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]^2 + 112\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]^4$$

$$-64\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]^6$$

$$\frac{\sin\left(\frac{3\pi}{2}-7\theta\right)}{\cos\theta} = 7 - 56\cos^2\theta + 112\cos^4\theta - 64\cos^6\theta$$

$$-\frac{\cos 7\theta}{\cos\theta} = 7 - 56\cos^2\theta + 112\cos^4\theta - 64\cos^6\theta$$

$$\frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$$

Eg: Express $\cos 7\theta$ in powers of $\cos\theta$.

Solution :

$$\cos 7\theta = \cos^7\theta - 7c_2\cos^5\theta\sin^2\theta + 7c_4\cos^3\theta\cdot\sin^4\theta$$

$$-7c_6\cos\theta\cdot\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta)$$

$$+ 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta)$$

$$-7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

$$= \cos^7\theta + 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta$$

$$-7\cos\theta + 21\cdot\cos^3\theta - 21\cdot\cos^5\theta + 7\cos^7\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

Eg: Express $\cos(8\theta)$ interms of $\sin\theta$.

Solution :

$$\cos 8\theta = \cos^8\theta - 8c_2\cdot\cos^6\theta\cdot\sin^2\theta + 8c_4\cos^4\theta\cdot\sin^4\theta$$

$$-8c_6\cos^2\theta\cdot\sin^6\theta + 8c_8\sin^8\theta$$

$$= (1-\sin^2\theta)^4 - 28(1-\sin^2\theta)^3\sin^2\theta$$

$$+ 70(1-\sin^2\theta)^2\sin^4\theta - 28(1-\sin^2\theta)\sin^6\theta + \sin^8\theta$$

$$= (1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta)$$

$$- 28\sin^2\theta\,(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)$$

$$+ 70\sin^4\theta\,(1 - 2\sin^2\theta + \sin^4\theta) - 28\sin^6\theta + 28\sin^8\theta + \sin^8\theta$$

$$= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta$$

$$- 28\sin^2\theta + 84\sin^4\theta - 84\sin^6\theta$$

$$+ 28\sin^8\theta + 70\sin^4\theta - 140\sin^6\theta + 70\cdot\sin^8\theta - 28\sin^6\theta + 28\sin^8\theta$$

$$+ \sin^8\theta$$

$$= 128\sin^8\theta - 256\sin^6\theta + 160\,\sin^4\theta - 32\cdot\sin^2\theta + 1$$

Eg: Express tan (6θ) interms of tan θ.

Solution :

$$\tan 6\theta = \frac{\sin(6\theta)}{\cos(6\theta)} = \frac{6c_1\tan\theta - 6c_3\tan^3\theta + 6c_5\cdot\tan^5\theta}{1 - 6c_2\tan^2\theta + 6c_4\tan^4\theta - 6c_6\tan^6\theta}$$

$$= \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

Eg. If $x = 2\cos\theta$. Prove that $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$

Solution : $2(1 + \cos 8\theta) = 2\cdot 2\cdot\cos^2(4\theta)$

$$= (2\cos 4\theta)^2$$

$$= \left[2(2\cos^2(2\theta) - 1)\right]^2$$

$$= (4\cos^2 2\theta - 2)^2$$

$$= \left[4(2\cos^2\theta - 1)^2 - 2\right]^2$$

$$= \left[4(4\cos^4\theta - 4\cos^2\theta + 1) - 2\right]^2$$

$$= (16\cos^4\theta - 16\cos^2\theta + 2)^2$$

$$= \left[16\left(\frac{x}{2}\right)^4 - (16)\left(\frac{x}{2}\right)^2 + 2\right]^2 \quad \left(\text{since } \cos\theta = \frac{x}{2}\right)$$

$$= (x^4 - 4x^2 + 2)^2$$

Eg. If $x = 2\cos\theta$, Prove that $\dfrac{1 + \cos 7\theta}{1 + \cos\theta} = (x^3 - x - 2x + 1)^2$

Solution : L.H.S. $= \dfrac{1 + \cos 7\theta}{1 + \cos\theta} = \dfrac{2\cos^2\frac{7\theta}{2}}{2\cos^2\frac{\theta}{2}} = \left(\dfrac{\cos\frac{7\theta}{2}}{\cos\frac{\theta}{2}}\right)^2$

$$= \left(\frac{\cos 7\phi}{\cos\phi}\right)^2 \quad \text{where } \phi = \frac{\theta}{2}$$

$$= \left(64\cos^6\phi - 112\cos^4\phi + 56\cos^2\phi - 7\right)^2$$

Given that $2\cos\theta = x \Leftrightarrow \cos\theta = \dfrac{x}{2}$

$$1 + \cos\theta = 1 + \frac{x}{2}$$

$$2\cos^2\frac{\theta}{2} = \frac{2 + x}{2}$$

$$\cos^2\phi = \frac{2 + x}{4}$$

Substituting this in (1),

$$\text{L.H.S.} = \left[64\left(\frac{2+x}{4}\right)^3 - 112\left(\frac{2+x}{4}\right)^2 + 56\left(\frac{2+x}{4}\right) - 7\right]^2$$

Hence (1) has root $\pm \sin\frac{2\pi}{7}, \pm \sin\frac{4\pi}{7}, \pm \sin\frac{6\pi}{7}$

Put $x^2 = y$ in (1).

Then $7 - 56y + 112y^2 - 64y^3 = 0$

(6) $64y^3 - 112y^2 + 56y - 7 = 0$

has the roots $\sin^2\frac{2\pi}{7}, \sin^2\frac{4\pi}{7}, \sin^2\frac{6\pi}{7}$.

EXERCISES

If $c = \cos\theta$, $s = \sin\theta$, Prove that

(i) $\cos 6\theta = 1 - 18s^2 + 48s^4 - 32s^6$

(2) $\sin 5\theta = 5s - 20s^3 + 16s^5$

(3) $\frac{\sin 7\theta}{\sin\theta} = 7 - 56s^2 + 112s^4 - 64s^6$

(4) $\cos 5\theta = 16c^5 - 20c^3 + 5c$

(5) $\cos 4\theta = 8c^4 - 8c^2 + 1$

(6) If $x = 2\cos\theta$, then $\frac{1+\cos 9\theta}{1+\cos\theta} = (x^4 - x^3 - 3x^2 + 2x + 1)^2$

Section : III EXPANSION IN SERIES

Expansion of $\sin^n\theta$ and $\cos^n\theta$ interms of sines and cosines of multiples of θ, n being a positive integer.

Let $x = \cos\theta + i\sin\theta = e^{i\theta}$

$$\frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad \text{....(1)}$$

$$= e^{i\theta} + e^{-i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$x - \frac{1}{x} = 2i\sin\theta \quad \text{....(2)}$$

$$= e^{i\theta} - e^{-i\theta}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Also $x^n = (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{i(n\theta)}$

$$= \cos(n\theta) + i\sin(n\theta)$$

$$\frac{1}{x^n} = \cos(n\theta) - i\sin(n\theta) = e^{-in\theta}$$

$$x^n + \frac{1}{x^n} = 2\cos(n\theta)$$
$$x^n - \frac{1}{x^n} = 2i\cdot\sin(n\theta)$$

From (1), $(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$

$$= x^n + nc_1 x^{n-1}\frac{1}{x} + nc_2 x^{n-2}\frac{1}{x^{2}} + nc_3 x^{n-3}\frac{1}{x^3} + \ldots\ldots + nc_{n-3}x^3\frac{1}{x^{n-3}} + nc_{n-2}x^2\frac{1}{x^{n-2}} + nc_{n-1}x\frac{1}{x^{n-1}} + nc_n\frac{1}{x^n}$$

$$(2\cos\theta)^n = \left(x^n + \frac{1}{x^n}\right) + {}^nc_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nc_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + {}^nc_3\left(x^{n-6} + \frac{1}{x^{n-6}}\right) + \ldots\ldots \quad \text{..... (A)}$$

From (2), $(2i\sin\theta)^n = x^n + nc_1 x^{n-1}\left(-\frac{1}{x}\right) + nc_2 x^{n-2}\left(\frac{-1}{x}\right)^2$

$$+ n c_3 x^{n-3}\left(\frac{-1}{x}\right)^3 + \ldots\ldots + n c_{n-3} x^3\left(\frac{-1}{x}\right)^{n-3}$$

$$+ n c_{n-2} x^2\left(-\frac{1}{x}\right)^{n-2} + nc_{n-1} x\left(-\frac{1}{x}\right)^{n-1} + n c_n\left(-\frac{1}{x}\right)^n$$

$$(2i \sin \theta)^n = \left(x^n + \frac{1}{x^n}(-1)^n\right) - {}^n c_1\left(x^{n-2} + \frac{1}{x^{n-2}}(-1)^n\right) + {}^n c_2\left(x^{n-4} + \frac{1}{x^{n-4}}(-1)^n\right) - {}^n c_3\left(x^{n-6} + \frac{1}{x^{n-6}}(-1)^n\right) + \ldots \quad \ldots (B)$$

Remark : The expression for a power of a sine will be in a series of sizes (or) cosines according as the power is odd (or) even.

for all egs, $x = \cos\theta + i \sin\theta$

Eg. Expand $\cos^4\theta$ in a series of cosines of multiples of θ.

Solution : Using result (A) $(n = 4)$,

$$(2\cos\theta)^4 = \left(x + \frac{1}{x}\right)^4 = \left(x^4 + \frac{1}{x^4}\right) + (4 c_1)\left(x^2 + \frac{1}{x^2}\right) + 4 c_2.$$

$$= 2\cos 4\theta + 4\,(2\cos 2\theta) + 6$$

$$2^4 \cos^4\theta = (2\cos 4\theta + 8\cos 2\theta + 6)$$

$$\cos^4\theta = \frac{1}{2^3}(\cos 4\theta + 4\cos 2\theta + 3)$$

Eg. Expand $\sin^4\theta$ in a series of cosines of multiples of θ.

Solution : $(-1)^4 = 1$.

Using result (B) $(n = 4)$.

$$(2\, i \sin\theta)^4 = \left(x - \frac{1}{x}\right)^4 = \left(x^4 + \frac{1}{x^4}\right) - 4 c_1\left(x^2 + \frac{1}{x^2}\right) + 4 c_2$$

$$\therefore 2^4 i^4 \sin^4\theta = 2\cos 4\theta - 4\,(2\cos 2\theta) + 6$$

$$2^3 (1) \sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3.$$

$$\sin^4\theta = \frac{1}{2^3}[\cos 4\theta - 4\cos 2\theta + 3]$$

Eg. Expand $\cos^5\theta$ in a series cosines of multiples of θ.

Solution : Using the result (A) (n = 5),

$$(2\cos\theta)^5 = \left(x + \frac{1}{x}\right)^5 = \left(x^5 + \frac{1}{x^5}\right) + (5 c_1)\left(x^3 + \frac{1}{x^3}\right) + (5 c_2)\left(x + \frac{1}{x}\right)$$

$$2^5 \cos^5\theta = 2\cos 5\theta + 5\,(2\cos 3\theta) + (10)\,(2\cos\theta)$$

$$\therefore 2^4 \cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$$

$$\cos^5\theta = \frac{1}{16}[\cos 5\theta + 5\cos 3\theta + 10\cos\theta]$$

Eg. Express $\sin^5\theta$ in a series of sines of multiples of θ.

Solution : Using the result (B) (n = 5), and $(-1)^5 = -1$,

$$(2\, i \sin\theta)^5 = \left(x - \frac{1}{x}\right)^5 = \left(x^5 - \frac{1}{x^5}\right) - 5 c_1\left(x^3 - \frac{1}{x^3}\right) + 5 c_2\left(x - \frac{1}{x}\right)$$

$$2^5 i^5 \sin^5\theta = 2\, i \cdot \sin 5\theta - 5\,(2\, i \sin 3\theta) + 10\,(2\, i \sin\theta)$$

$$2^4 \sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta \ (\text{since } i^5 = i)$$

$$\sin^5\theta = \frac{1}{2^4}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$$

Eg. Express $\sin^6\theta$ in a series of cosines of multiples of θ.

Solution : Using the result (B) (n = 6) and $(-1)^6 = 1$,

$$(2i \sin\theta)^6 = \left(x^6 + \frac{1}{x^6}\right) - 6 c_1\left(x^4 + \frac{1}{x^4}\right) + 6 c_2\left(x^2 + \frac{1}{x^2}\right) - 6 c_3$$

$$2^6 i^6 \sin^6\theta = 2\cos 6\theta - 6\,(2\cos 4\theta) + 15\,(2\cos 2\theta) - 20$$

$$-2^5 \sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10$$

$$\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

Eg. Express $\cos^6\theta$ in a series of cosines of multiples of θ

Solution : Using the result (A) ($n = 6$),

$$(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6 = \left(x^6 + \frac{1}{x^6}\right) + (6c_1)\left(x^4 - \frac{1}{x^4}\right) + 6c_2\left(x^2 + \frac{1}{x^2}\right) + 6c$$

$$2^6\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

$$\cos^6\theta = \frac{1}{2^5}\,[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$$

Eg. Express $\sin^7\theta$ in a series of sines of multiples of θ

Solution : Using the result (B) ($n = 7$), and $(-1)^7 = -1$,

$$(2i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7 = \left(x^7 - \frac{1}{x^7}\right) - 7c_1\left(x^5 - \frac{1}{x^5}\right) + 7c_2\left(x^3 - \frac{1}{x^3}\right) - 7c_3\left(x - \frac{1}{x}\right)$$

$$-i\,2^7\sin^7\theta = 2^7 i^7\sin^7\theta = (2i\sin 7\theta)$$

$$- 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)$$

$$-2^6\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$$

$$\sin^7\theta = -\frac{1}{64}\,[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta]$$

Eg. Express $\cos^7\theta$ in a series of cosines of multiples of θ

Solution : Using the result (A) ($n = 7$),

$$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7 = \left(x^7 + 1/x^7\right) + 7c_1(x^5 + 1/x^5)$$

$$+ 7c_2(x^3 + 1/x^3) + 7c_3(x + 1/x)$$

$$2^7\cos^7\theta = 2\cos 7\theta + 7(2\cos 5\theta)$$

$$+ (21)(2\cos 3\theta) + 35(2\cos\theta)$$

$$2^6\cos^7\theta = \cos(7\theta) + 7\cos(5\theta) + 21\cos(3\theta) + 35\cos\theta$$

$$\therefore \cos^7\theta = \frac{1}{64}\,[\cos(7\theta) + 7\cos(5\theta) + 21\cos(3\theta) + 35\cos\theta]$$

Eg. Express $\sin^8\theta$ in a series of cosines of multiples of θ.

Solution : Using the result (B) ($n = 8$), and $(-1)^8 = 1$,

$$(2i\sin\theta)^8 = \left(x^8 + \frac{1}{x^8}\right) - 8c_1\left(x^6 + \frac{1}{x^6}\right) + 8c_2\left(x^4 + \frac{1}{x^4}\right)$$

$$- 8c_3\left(x^2 + \frac{1}{x^2}\right) + 8c_4$$

$$2^8 i^8\sin^8\theta = 2^8(1)\sin^8\theta$$

$$= 2\cos(8\theta) - 8(2\cos 6\theta) + 28(2\cos 4\theta)$$

$$- 56(2\cos 2\theta) + 70$$

$$2^7\sin^8\theta = \cos(8\theta) - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35$$

$$\sin^8\theta = \frac{1}{2^7}\,[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35]$$

Eg. Express $\cos^8\theta$ in a series of cosines of multiples of θ.

Solution : Using the result (A) ($n = 8$),

$$(2\cos\theta)^8 = \left(x^8 + \frac{1}{x^8}\right) + 8c_1\left(x^6 + \frac{1}{x^6}\right) + 8c_2\left(x^4 + \frac{1}{x^4}\right)$$

$$+ 8c_3\left(x^2 + \frac{1}{x^2}\right) + 8c_4$$

$$2^8\cos^8\theta = 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70$$

$$2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$$

$$\therefore \cos^8\theta = \frac{1}{2^7}\,[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$$

Eg. Expand $\sin^3\theta \cdot \cos^4\theta$ in a series of sines of multiples of θ.

Solution : Let $x = \cos\theta + i\sin\theta \Rightarrow \frac{1}{x} = \cos\theta - i\sin\theta$

Then $x + \frac{1}{x} = 2\cos\theta,\ x - \frac{1}{x} = 2i\sin\theta$

$$\therefore (2i\sin\theta)^3(2\cos\theta)^4 = \left(x - \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)^4$$

$$= \left(x - \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x + \frac{1}{x}\right)$$

$$2^3 i^3 \sin^3 2^4 \cos^4\theta = \left[\left(x^6 - \frac{1}{x^6}\right) - 3c_1\left(x^2 - \frac{1}{x^2}\right)\right]\left[x + \frac{1}{x}\right]$$

(using the result (B) ($n = 3$) and replace x^2 instead of

$$-2^7 i\sin^3\theta \cdot \cos^4\theta = \left(x^7 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x}\right) + \left(x^5 - \frac{1}{x^7}\right) - 3\left(x - \frac{1}{x^3}\right)$$

$$= \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 7\theta + 2i\sin 5\theta - 3(2i\sin 3\theta) - 3(2i\sin\theta)$$

$$-2^6\sin^3\theta\cos^4\theta = \sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta$$

$$\therefore \sin^3\theta\cos^4\theta = -\frac{1}{64}[\sin 7\theta + \sin(5\theta) - 3\sin 3\theta - 3\sin\theta]$$

g. Expand $\sin^4\theta\ \cos^3\theta$ in a series of cosines of multiples θ.

lution : Let $x = \cos\theta + i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta,\ x - \frac{1}{x} = 2i\sin\theta.$

$$\sin\theta)^4(2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)\left(x - \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x - \frac{1}{x}\right)\left[\left(x^6 - \frac{1}{x^6}\right) - 3c_1\left(x^2 - \frac{1}{x^2}\right)\right]$$

$$= \left(x^7 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x}\right) - x^5 + \frac{1}{x^7} + 3(x) - 3\ \frac{1}{x^3}.$$

$$i^4\sin^4\theta\ 2^3\cos^3\theta = \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) + 3\left(x + \frac{1}{x}\right) - 3\left(x^3 + \frac{1}{x^3}\right)$$

$$2^7(1)\sin^4\theta\cos^3\theta = 2\cos 7\theta - 2\cos 5\theta - 3(2\cos 3\theta) + 3(2\cos\theta)$$

$$2^6\sin^4\theta\cos^3\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$$

$$\therefore \sin^4\theta\cos^3\theta = \frac{1}{64}[\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta]$$

Eg. Expand $\sin^5\theta \cdot \cos^4\theta$ in a series of sines of multi[ples] of θ.

Solution : Let $x = \cos\theta + i\sin\theta$; $x + \dfrac{1}{x} = 2\cos\theta$, $x - \dfrac{1}{x} = 2i\sin\theta$

$$\therefore (2i\sin\theta)^5(2\cos\theta)^4 = \left(x - \frac{1}{x}\right)^5\left(x + \frac{1}{x}\right)^4 = \left(x - \frac{1}{x}\right)\left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^4$$

$$= \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)^4$$

$$2^5 i^5\sin^5\theta \cdot 2^4\cos^4\theta = \left(x - \frac{1}{x}\right)\left[\left(x^8 + \frac{1}{x^8}\right) - 4c_1\left(x^4 + \frac{1}{x^4}\right) + 6\right]$$

(using the result (B) ($n = 4$) substituting x^2 instead [of x])

$$2^9 i\sin^5\theta \cdot \cos^4\theta = x^9 + \frac{1}{x^7} - 4\left(x^5 + \frac{1}{x^3}\right) + 6x$$

$$- x^7 - \frac{1}{x^9} + 4\left(x^3 + \frac{1}{x^5}\right) - 6\frac{1}{x}$$

$$= \left(x^9 - \frac{1}{x^9}\right) - \left(x^7 - \frac{1}{x^7}\right) + 4\left(x^3 - \frac{1}{x^3}\right)$$

$$- 4\left(x^5 - \frac{1}{x^5}\right) + 6\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 9\theta - 2i\sin 7\theta - 4(2i\sin 5\theta)$$

$$+ 4(2i\sin 3\theta) + 6(2i\sin\theta)$$

$$2^8\sin^5\theta\cos^4\theta = \sin 9\theta - \sin 7\theta - 4\sin 5\theta + 4\sin 3\theta + 6\sin\theta$$

Eg. Expand $\sin^7\theta\cos^5\theta$ in a series of sines of multi[ples] of θ.

Solution : Let $x = \cos\theta + i\sin\theta$.

$$\therefore x + \frac{1}{x} = 2\cos\theta,\ x - \frac{1}{x} = 2i\sin\theta,\ x^n - \frac{1}{x^n} = 2i\sin(n\theta)$$

$$(2i\sin\theta)^7(2\cos\theta)^5 = \left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^5 = \left(x - \frac{1}{x}\right)^2\left(x - \frac{1}{x}\right)^5\left(x + \frac{1}{x}\right)^5$$

$$= \left(x - \frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^5$$

$$i^7\sin^7\theta\ 2^5\cos^5\theta = \left(x^2 + \frac{1}{x^2} - 2\right)\left[(x^{10} - 1/x^{10})\right.$$

$$\left. - 5c_1(x^6 - 1/x^6) + 5c_2(x^2 - 1/x^2)\right]$$

$$-2^{12} i\sin^7\theta \cdot \cos^5\theta = \left(x^{12} - \frac{1}{x^8}\right) - 5\left(x^8 - \frac{1}{x^4}\right) + 10(x^4 - 1)$$

$$+ \left(x^8 - \frac{1}{x^{12}}\right) - 5\left(x^4 - \frac{1}{x^8}\right) + 10\left(1 - \frac{1}{x^4}\right)$$

$$- 2\left(x^{10} - \frac{1}{x^{10}}\right) + (10)\left(x^6 - \frac{1}{x^6}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - \left(x^{10} - \frac{1}{x^{10}}\right)$$

$$- 4\left(x^8 - \frac{1}{x^8}\right) + (10)\left(x^6 - \frac{1}{x^6}\right)$$

$$+ 5\left(x^4 - \frac{1}{x^4}\right) - (20)\left(x^2 - \frac{1}{x^2}\right)$$

$$= 2i\sin(12\theta) - 2(2i\sin(10\theta))$$

$$- 4(2i\sin 8\theta) + 10(2i\sin 6\theta)$$

$$+ 5(2i\sin 4\theta) - 20(2i\sin 2\theta)$$

$$-2^{11}\sin^7\theta \cdot \cos^5\theta = \sin(12\theta) - 2\sin(10\theta) - 4\sin 8\theta$$

$$+ 10\sin(6\theta) + 5\sin(4\theta) - 20\sin(2\theta).$$

[Eg. Expand] $\sin^6\theta\cos^5\theta$ in a series of cosines of multiples of θ.

[So]lution : Let

$$x = \cos\theta + i\sin\theta \Leftrightarrow x + \frac{1}{x} = 2\cos\theta,\ x - \frac{1}{x} = 2i\sin\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$(2i\sin\theta)^6(2\cos\theta)^5 = \left(x - \frac{1}{x}\right)^6\left(x + \frac{1}{x}\right)^5 = \left(x - \frac{1}{x}\right)\left(x - \frac{1}{x}\right)^5 ...$$

$$2^6 i^6 \sin^6\theta \cdot 2^5\cos^5\theta = \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)^5$$

$$-2^{11}\sin^6\theta\cos^5\theta = \left(x - \frac{1}{x}\right)\left[\left(x^{10} - \frac{1}{x^{10}}\right) - 5c_1\left(x^6 ...\right)\right.$$

$$+ 5c_2\left(x^2 ...\right)$$

$$= \left(x^{11} - \frac{1}{x^9}\right) - 5\left(x^7 - \frac{1}{x^5}\right) + (10)\left(x^3 ...\right)$$

$$-\left(x^9 - \frac{1}{x^{11}}\right) + 5\left(x^5 - \frac{1}{x^7}\right) - (10)\left(x ...\right)$$

$$= \left(x^{11} + \frac{1}{x^{11}}\right) - \left(x^9 - \frac{1}{x^9}\right) - 5\left(x^7 + \frac{1}{...}\right)$$

$$+ 5\left(x^5 + \frac{1}{x^5}\right) + (10)\left(x^3 + \frac{1}{x^3}\right) - (10)\left(x ...\right)$$

$$= 2\cos(11\theta) - 2\cos(9\theta) - 5(2\cos 7\theta)$$

$$+ 5(2\cos 5\theta) + 10(2\cos 3\theta) - 10(2\cos ...)$$

$$-2^{10}\cdot\sin^6\theta\cdot\cos^5\theta = \cos(11\theta) - \cos(9\theta) - 5\cos(7\theta)$$

$$+ 5(\cos(5\theta) + 10\cos(3\theta) - 10\cos ...$$

EXERCISES

Prove that the following :

(1) $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

...) $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

...) $\cos^7\theta = \frac{1}{2^6}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta]$

...) $\sin^4\theta = \frac{1}{8}[\cos 4\theta - 4\cos + 3]$

...) $\sin^7\theta = -\frac{1}{64}[\sin 7\theta - 7\sin(5\theta) + 21\sin(3\theta) - 3\sin\theta]$

...) $256\cdot\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin(3\theta) + 126\sin\theta.$

...) $2^5\cos^2\theta\cdot\sin^4\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

...) $\sin^3\theta\cdot\cos\theta = -\frac{1}{8}(\sin 4\theta - 2\sin 2\theta)$

...) $2^6\sin^5\theta\cdot\cos^2\theta = \sin 7\theta - 3\sin 5\theta + \sin(3\theta) + 5\sin\theta$

...0) If $\cos^5\theta = A\cos\theta + B\cos 3\theta + C\cos 5\theta$, Prove that $\sin^5\theta = A\sin\theta - B\sin 3\theta + C\sin(5\theta)$.

...1) $\cos^3\theta\cdot\sin^4\theta = A_1\cos\theta + B_3\cos 3\theta + A_5\cos(5\theta) + A_7\cos(7\theta)$, then P.T. $A_1 - \frac{1}{3}A_3 + \frac{1}{5}A_5 - \frac{1}{7}A_i = \frac{2}{35}$

...2) Express $\sin^7\theta\cdot\cos^3\theta$ as the sum of sines if multiples of θ.

...3) Express $\sin\theta\cdot\cos^5\theta$ interms of sines of multiples of θ.

...4) Express $\sin^2\theta\cdot\cos\theta$ interms of cosines of multiples of θ.

...5) Express $\sin^6\theta\cdot\cos^3\theta$ interms of cosines of multiples of θ.

...ction : IV Expansion of sin θ, cos θ and tan θ in powers θ.

...sult. When θ is expressed in radians, Prove that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \text{ to } \infty$$

(13) Find $\underset{\theta \to 0}{Lt} \dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

(14) Evaluate $\underset{x \to 0}{Lt} \dfrac{\cos 1 - \cos(\cos x)}{x^2}$

(15) Evaluate $\underset{x \to 0}{Lt} \dfrac{\sin 2x - 2\sin x}{x^3}$

Section : V HYPERBOLIC FUNCTIONS

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\cos\theta + i\sin\theta = e^{i\theta}, \quad \cos\theta - i\sin\theta = e^{-i\theta}$$

$$\therefore 2\cos\theta = e^{i\theta} + e^{-i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

The expression $\frac{1}{2}(e^{\theta} + e^{-\theta})$ and $\frac{1}{2}(e^{\theta} - e^{-\theta})$ are defined hyperbolic cosine and sine respectively of the angle θ.

$$\cos h\,\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh x = \frac{e^{\theta} - e^{-\theta}}{2}$$

Thus $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$; $\coth x = \dfrac{1}{\tanh x}$

$$\text{cosech}\, x = \frac{1}{\sinh x}; \quad \text{sech}\, x = \frac{1}{\cosh x}$$

Further $\cos(i\theta) = \dfrac{e^{i(i\theta)} + e^{-i(i\theta)}}{2} = \dfrac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta$

$$\sin(i\theta) = \frac{e^{i(i\theta)} - e^{-i(i\theta)}}{2i} = \frac{e^{-\theta} - e^{\theta}}{2i} = -\frac{1}{i}\sinh\theta$$

$$= -(-i)\sinh\theta = i\sinh\theta$$

$$e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots \quad e^{-\theta} = 1 - \frac{\theta}{1!} + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \dots$$

$$\cos(i\theta) = \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots$$

$$\sin(i\theta) = i\sinh\theta, \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$$

$$\cosh(i\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$\sinh(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} = i\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) = i\sin\theta$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\tan(i\theta) = \frac{\sin(i\theta)}{\cos(i\theta)} = i\tanh\theta$$

$$\cot(i\theta) = \frac{\cos(i\theta)}{\sin(i\theta)} = \frac{\cosh\theta}{i\sinh\theta} = (-i)\coth\theta$$

$$\text{cosec}(i\theta) = \frac{1}{\sin(i\theta)} = \frac{1}{i\sinh\theta} = (-i)\,\text{cosech}\,\theta$$

$$\sec(i\theta) = \frac{1}{\cos(i\theta)} = \frac{1}{\cosh\theta} = \text{sech}\,\theta$$

Using these relations, we can derive relations between hyperbolic functions corresponding to relations between circular functions. Put $\theta = ix$.

(i) $\sin^2\theta + \cos^2\theta = 1$

$$[\sin(ix)]^2 + [\cos(ix)]^2 = 1 \Leftrightarrow (i\sinh x)^2 + (\cosh x)^2 = 1$$

$$\Leftrightarrow i^2\sinh^2 x + \cosh^2 x = 1 \Leftrightarrow \boxed{\cosh^2 x - \sinh^2 x = 1}$$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta.$

Then $\cos(2ix) = [\cos(ix)]^2 - [\sin(ix)]^2 = (\cosh x)^2 - (i \sinh x)^2$

$= \cosh^2 x - i^2 \sinh^2 x$

$$\boxed{\cosh 2x = \cosh^2 x + \sinh^2 x}$$

(iii) $\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$\therefore \sin(2ix) = 2\sin(ix)\cos(ix)$

$\Rightarrow i\sinh(2x) = 2(i \sinh x)(\cosh x)$

$$\boxed{\sinh(2x) = 2\sinh x \cdot \cosh x}$$

(iv) $1 + \tan^2\theta = \sec^2\theta$

$\therefore 1 + [\tan(ix)]^2 = [\sec(ix)]^2$

$1 + (i\tanh x)^2 = (\text{sech}\, x)^2 \Leftrightarrow 1 + i^2\tanh^2 = \text{sech}^2 x$

$$\boxed{\therefore 1 - \tanh^2 x = \text{sech}^2 x}$$

(v) $1 + \cot^2\theta = \text{cosec}^2\theta$

$\therefore 1 + [\cot(ix)]^2 = [\text{cosec}(ix)]^2$

$\Leftrightarrow 1 + (-i\coth x)^2 = (-i\, \text{cosech}\, x)^2$

$\Leftrightarrow 1 + i^2\coth^2 x = i^2 \text{cosech}^2 x$

$1 - \coth^2 x = -\text{cosech}^2 x$

$$\boxed{\coth^2 x - 1 = \text{cosech}^2 x}$$

(vi) $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$

Put $A = ix, B = iy$

$\therefore \sin(ix \pm iy) = \sin(ix)\cos(iy) \pm \cos(ix)\sin(iy)$

$\sin[(i)(x \pm y)] = (i\sinh x)(\cosh y) \pm (\cosh x)(i\sinh y)$

$i\sinh(x \pm y) = i\sinh x \cdot \cosh y \pm i\cosh x \cdot \sinh y$

$$\boxed{\therefore \sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \sinh y}$$

(vii) $\cos(A \pm B) = \cos A \cdot \cos B \mp \sin A \sin B$

Put $A = ix, B = iy$

$\cos(ix \pm iy) = \cos(ix)\cos(iy) \mp \sin(ix)\sin(iy)$

$\cos[i(x \pm y)] = \cosh x \cdot \cosh y \mp (i\sinh x)(i\sinh y)$

$\cosh(x \pm y) = \cosh x \cdot \cosh y \mp (i^2)(\sinh x \cdot \sinh y)$

$$\boxed{\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y}$$

We can express $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$ interms of logarithmic functions.

(viii) Let $y = \sinh^{-1} x$

$$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{1}{2}\left[e^y - \frac{1}{e^y}\right]$$

$$= \frac{1}{2}\frac{[e^y \cdot e^y - 1]}{e^y} = \frac{e^{2y} - 1}{2e^y}$$

$2xe^y = e^{2y} - 1 \Leftrightarrow e^{2y} - 2xe^y - 1 = 0$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2 - 4(1)(-1)}}{2(1)} = \frac{2x + 2\sqrt{x^2 + 1}}{2}$$

$e^y = x + \sqrt{x^2 + 1}$

$$\boxed{\therefore y = \log\left(x + \sqrt{x^2 + 1}\right) = \sinh^{-1} x}$$

(ix) $y = \cosh^{-1} x \Rightarrow x = \cosh y \Rightarrow x = \dfrac{e^y + e^{-y}}{2} = \dfrac{1}{2}\dfrac{[e^{2y} + 1]}{e^y}$

$\Leftrightarrow 2xe^y = e^{2y} + 1$

Thus $e^{2y} - 2xe^y + 1 = 0$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4(1)(1)}}{2} = x + \sqrt{x^2 - 1}$$

$$\boxed{y = \log\left(x + \sqrt{x^2 - 1}\right) = \cosh^{-1} x}$$

(x) Let $y = \tan h^{-1} x \Leftrightarrow x = \tan h\, y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}} = \dfrac{e^{2y} - 1}{e^{2y} + 1}$

$$\therefore \frac{x}{1} = \frac{e^{2y} - 1}{e^{2y} + 1} \Leftrightarrow \frac{1+x}{1-x} = \frac{e^{2y} + 1 + e^{2y} - 1}{(e^{2y} + 1) - (e^{2y} - 1)}$$

$$\therefore \frac{1+x}{1-x} = \frac{2\,e^{2y}}{2} = e^{2y}$$

$$\therefore \log \frac{1+x}{1-x} = 2y \Leftrightarrow \quad \boxed{y = \frac{1}{2} \log_e \left(\frac{1+x}{1-x} \right) = \tanh^{-1} x}$$

Eg: Separate into real and imaginary parts of (i) sin $(x + iy)$ (ii) cos $(x + iy)$ (iii) tan $(x + iy)$ (iv) cot $(x + iy)$ (v) cosec $(x + iy)$ (vi) sec $(x + iy)$ (vii) sinh $(x + iy)$ (viii) cosh $(x + iy)$ (ix) tanh $(x + iy)$ (x) coth $(x + iy)$ (xi) cosech $(x + iy)$ (xii) sech $(x + iy)$

Solution :

(i) $\sin (x + i\,y) = \sin x \cos (i\,y) + \cos x \cdot \sin (i\,y)$

$= \sin x \cosh y + \cos x\, (i \sinh y)$

$= \sin x \cosh y + i \cos x \cdot \sinh y$

Real part $= \sin \cosh y$, Im. Part $= \cos x \cdot \sinh y$

(ii) $\cos (x + iy) = \cos x \cdot \cos (iy) - \sin x \cdot \sin (iy)$

$= \cos x \cdot \cosh y - \sin x \cdot (i \sinh y)$

Real part $= \cos x \cdot \cosh y$, Im. Part $= -\sin x \cdot \sinh y$

(iii) $\tan (x + i\,y) = \dfrac{\sin (x + i\,y)}{\cos (x + i\,y)} = \dfrac{2 \sin (x + i\,y) \cos (x - i\,y)}{2 \cdot \cos (x + i\,y) \cos (x - i\,y)}$

$= \dfrac{\sin (2x) + \sin (i2y)}{\cos (2x) + \cos (2iy)} = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$

Real part $= \dfrac{\sin 2x}{\cos 2x + \cosh 2y}$, Im. Part $= \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$

(iv) $\cot (x + iy) = \dfrac{\cos (x + iy)}{\sin (x + iy)} = \dfrac{2 \cos (x + iy) \sin (x - iy)}{2 \sin (x + iy) \sin (x - iy)}$

$= \dfrac{\sin 2x - \sin (2iy)}{\cos (i2y) - \cos (2x)} = \dfrac{\sin (2x) - i \sinh (2y)}{\cosh (2y) - \cos (2x)}$

Real part $= \dfrac{\sin 2x}{\cosh 2y - \cos (2x)}$, Im. Part $= -\dfrac{\sinh (2y)}{\cosh (2y) - \cos (2x)}$

(v) $\text{cosec}(x + iy) = \dfrac{1}{\sin (x + iy)} = \dfrac{2 \sin (x - iy)}{2 \sin (x + iy) \sin (x - iy)}$

$= \dfrac{2\,[\sin x \cos (iy) - \cos x \cdot \sin (iy)]}{\cos (i\,2y) - \cos (2x)}$

$= \dfrac{2\,[\sin x \cos (iy) - \cos x \cdot \sin (iy)]}{\cos (i\,2y) - \cos (2x)}$

$= \dfrac{2 \sin x \cosh y - 2 \cos x\, (i \sinh y)}{\cosh 2y - \cos 2x}$

Real part $= \dfrac{2 \sin x \cdot \cosh y}{\cosh 2y - \cos 2x}$; Im. Part $= \dfrac{-2 \cos x \sinh y}{\cosh 2y - \cos 2x}$

(vi) $\sec (x + iy) = \dfrac{1}{\cos (x + iy)} = \dfrac{2 \cos (x - iy)}{2 \cos (x + iy) \cos (x - iy)}$

$= \dfrac{2\,(\cos x \cos (iy) + \sin x \cdot \sin (iy))}{\cos 2x + \cos (i2y)}$

$= \dfrac{2 \cos x \cdot \cosh y + 2 (\sin x)(i \sinh y)}{\cos (2x) + \cosh (2y)}$

Real part $= \dfrac{2 \cos x \cdot \cosh y}{\cos 2x + \cosh 2y}$; Im Part $= \dfrac{2 \sin x \cdot \sinh y}{\cos 2x + \cosh 2y}$

(vii) we have $\sin (x + iy) = \sin x \cosh y + i \cos x \cdot \sinh y$

$\therefore \sin (i)(x + iy) = \sin (i\,x) \cosh (iy) + i \cos (i\,x) \sinh (iy)$

$i \sinh (x + iy) = i \sinh x \cos y + i \cosh x\, (i \sin y)$

$\therefore \sinh (x + iy) = \sinh x \cdot \cos y + i \cosh x \cdot \sin y$

Real Part $= \sinh x \cdot \cos y$; Im. Part $= \cosh x \cdot \sin y$

(viii) $\cos (x + iy) = \cos x \cosh y - i \sin x \cdot \sinh y$

$\therefore \cos (i)(x + iy) = \cos (i\,x) \cosh (iy) - i \sin (ix) \sinh (iy)$

$\cosh (x + iy) = \cosh x \cos y - i\,(i \sinh x)(i \sin y)$

$= \cosh x \cos y + i \sinh x \cdot \sin y$

In $\cosh (x + iy)$, **Real part is** $\cos x \cdot \cos y$,

Im. Part $= \sinh x \cdot \sin y$

(ix) $\tan(x+iy) = \dfrac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh(2y)}$ by (iii)

$\therefore \tan i(x+iy) = \dfrac{\sin(i2x) + i\sinh(2iy)}{\cos(i2x) + \cosh(i2y)}$

$i\tan h(x+iy) = \dfrac{i\sinh 2x + i\cdot i\sin(2y)}{\cosh 2x + \cos 2y}$

$\therefore \tan h(x+iy) = \dfrac{\sinh 2x + i\sin 2y}{\cosh 2x + \cos 2y}$

In $\tanh(x+iy)$, Real part is $\dfrac{\sinh 2x}{\cosh 2x + \cos 2y}$, Im

Part $= \dfrac{\sin 2y}{\cosh 2x + \cos 2y}$

(x) $\cot(x+iy) = \dfrac{\sin 2x - i\sinh(2y)}{\cosh 2y - \cos 2x}$ by (iv)

$\therefore \cot(i)(x+iy) = \dfrac{\sin(i2x) - i\sinh(i2y)}{\cosh(i2y) - \cos(i2x)}$

$(-i)\coth(x+iy) = \dfrac{i\sinh(2x) - i\cdot i\sin(2y)}{\cos(2y) - \cosh(2x)}$

$\coth(x+iy) = \dfrac{-\sin 2y + i\sinh 2x}{\cos(2y) - \cosh(2x)}$

Im $\cot h(x+iy)$, Real part is $\dfrac{-\sin(2y)}{\cos(2y) - \cosh(2x)}$,

Im. Part $= \dfrac{\sinh(2x)}{\cos 2y - \cosh 2x}$

(xi) $\operatorname{cosec}(x+iy) = \dfrac{2\sin x\cdot\cosh y - i\cdot 2\cos x\cdot\sinh y}{\cosh 2y - \cos 2x}$

$\therefore \operatorname{cosec} i(x+iy) = \dfrac{2\sin(ix)\cosh(iy) - i\,2\cos(ix)\sinh(iy)}{\cosh(i2y) - \cos(i2x)}$

$(-i)\operatorname{cosech}(x+iy) = \dfrac{2i\sinh x\cos y - i\cdot 2\cosh x\,(i\sin y)}{\cos 2y - \cosh(2x)}$

$\operatorname{cosech}(x+iy) = \dfrac{(-2)\,i\sinh x\cdot\cos y + 2\cosh x\cdot\sin y}{\cos(2y) - \cosh(2x)}$

In $\operatorname{cosec}(x+iy)$, Real part $= \dfrac{-2\sinh x\cdot\cos y}{\cos 2y - \cosh 2x}$,

Im. Part $= \dfrac{2\cosh x\cdot\sin y}{\cos(2y) - \cosh(2x)}$

(xii) $\sec(x+iy) = \dfrac{2\cos x\cosh y + i\,2\sin x\sinh y}{\cos 2x + \cosh 2y}$

$\therefore \sec(i)(x+iy) = \dfrac{2\cos(ix)\cosh(iy) + i\cdot 2\cdot\sin(ix)\sinh(iy)}{\cos(i2x) + \cosh(i2y)}$

$\operatorname{sech}(x+iy) = \dfrac{2\cosh x\cos y + i2\,(i\sinh x)(i\sin y)}{\cosh(2x) + \cos 2y}$

$\therefore$ In $\operatorname{sech}(x+iy)$, Real part is $\dfrac{2\cosh x\cdot\cos x}{\cosh(2x) + \cos(2y)}$,

Im Part $= \dfrac{-2\sinh x\ \sin y}{\cosh(2x) + \cos(2y)}$

Eg: If $\tanh\frac{y}{2} = \tan\frac{x}{2}$

Show that

(i) $\cos x\cdot\cosh y = 1$

(ii) $\tan x = \sinh y$

(iii) $y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

Proof :

(i) $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{1-\tanh^2\frac{y}{2}}{1+\tanh^2\frac{y}{2}} = \dfrac{1}{\cosh y} \Leftrightarrow \cos x\cdot\cosh y = 1$

(ii) $\tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{2\tanh\frac{y}{2}}{1-\tanh^2\frac{y}{2}} = \dfrac{2\tanh\frac{y}{2}}{\operatorname{sech}^2\frac{y}{2}}$

$= 2\sinh\frac{y}{2}\cosh\frac{y}{2} = \sinh y$

(iii) $\tanh\frac{y}{2} = \tan\frac{x}{2} \Rightarrow \frac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}} = \frac{\tan\frac{x}{2}}{1}$

$$\therefore \frac{\left(e^{y/2}+e^{-y/2}\right)+\left(e^{y/2}-e^{-y/2}\right)}{\left(e^{y/2}+e^{-y/2}\right)-\left(e^{y/2}-e^{-y/2}\right)} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$\frac{2e^{y/2}}{2e^{-y/2}} = \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\left(\tan\frac{\pi}{4}\right)\left(\tan\frac{\pi}{2}\right)}$$

$$\therefore e^{y} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \Leftrightarrow y = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

Eg: If $\sin\theta = \tanh x$, Prove that $\tan\theta = \sinh x$.

Solution :

$$\sinh x = \frac{\sinh x}{\cosh x}\cosh x = \frac{(\tanh x)}{\text{sec}\,h\,(x)} = (\sin\theta)\frac{1}{\sqrt{1-\tanh^2 x}}$$

$$= (\sin\theta)\frac{1}{\sqrt{1-\sin^2\theta}} = \sin\theta\cdot\frac{1}{\cos\theta} = \tan\theta$$

Eg: If $y = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, show that $\tanh\frac{y}{2} = \tan\frac{x}{2}$.

Solution :

$$y = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \Leftrightarrow e^{y} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$\Leftrightarrow \frac{e^{y}}{1} = \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$\frac{e^{y}-1}{e^{y}+1} = \frac{\left(1+\tan\frac{x}{1}\right)-\left(1-\tan\frac{x}{2}\right)}{\left(1+\tan\frac{x}{2}\right)+\left(1-\tan\frac{x}{2}\right)} = \frac{2\tan\frac{x}{2}}{2} = \tan\frac{x}{2}$$

$$\therefore \frac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}} = \tan\frac{x}{2} \Leftrightarrow \tanh\frac{y}{2} = \tan\frac{x}{2}.$$

Eg: If $\sin(A+iB) = x+iy$, Prove that

(i) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

(ii) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

Proof : $x+iy = \sin(A+iB) = \sin A\cosh B + i\cos A\sinh B$

$\therefore x = \sin A\cosh B$

$y = \cos A\sinh B$

$\frac{x}{\sin A} = \cosh B;\ \frac{y}{\cos A} = \sinh B$

$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1$

$\frac{x}{\cosh B} = \sin A;\ \frac{y}{\sinh B} = \cos A$

$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 B = i$

Eg: If $\sin(A+iB) = \cos\theta + i\sin\theta$, show that

(i) $\cos^2 A = \sinh^2 B$, (ii) $\cos^2 A = \pm\sin\theta$.

Proof :

$\cos\theta + i\sin\theta = \sin(A+iB) = \sin A\cosh B + i\cos A\sinh B$

$\cos\theta = \sin A\cosh B;\ \sin\theta = \cos A\sinh B.$ .....(1)

$1 = \cos^2\theta + \sin^2\theta$

$= \sin^2 A\cosh^2 B + \cos^2 A\sinh^2 B$

$= (1-\cos^2 A)\cdot(1+\sinh^2 B) + \cos^2 A\sinh^2 B$

$1 = 1 + \sinh^2 B - \cos^2 A - \cos^2 A\sinh^2 B + \cos^2 A\sinh^2 B$

$0 = \sinh^2 B - \cos^2 A$

$\therefore \cos^2 A = \sinh^2 B$ This proves (i),

$\cosh B = \dfrac{\cos\theta}{\sin A}$; $\sinh B = \dfrac{\sin\theta}{\cos A}$ from (1)

$1 = \cosh^2 B - \sinh^2 B = \dfrac{\cos^2\theta}{\sin^2 A} - \dfrac{\sin^2\theta}{\cos^2 A}$

$\therefore \sin^2 A \cdot \cos^2 A = \cos^2 A \cdot \cos^2\theta - \sin^2 A \sin^2\theta$

$(1-\cos^2 A)\cos^2 A = \cos^2 (1-\sin^2\theta) - (1-\cos^2 A)\sin^2\theta$

$\cos^2 A - \cos^4 A = \cos^2 A - \cos^2 A \sin^2\theta - \sin^2\theta + \cos^2 A \sin^2\theta$

$\therefore \cos^4 A = \sin^2\theta$

$\cos^2 A = \pm \sin\theta$

Eg: If $\sin(A + iB) = r(\cos\theta + i\sin\theta)$, show that

(i) $r^2 = \frac{1}{2}(\cosh 2B - \cos 2A)$

(ii) $\tan\theta = \tanh B \cdot \cot A$

Proof :

$r\cos\theta + i\, r\sin\theta = \sin(A + iB) = \sin A \cosh B + i\cos A \sinh B$

$\therefore r\cos\theta = \sin A \cdot \cosh B$; $r\sin\theta = \cos A \cdot \sinh B$.

$r^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= \sin^2 A \cosh^2 B + \cos^2 A + \sinh^2 B$

$= \dfrac{(1-\cos 2A)}{2}\dfrac{(1+\cosh 2B)}{2} + \dfrac{(1+\cos 2A)}{2}\dfrac{(-1+\cosh 2B)}{2}$

$(-1 + \cosh^2 B - \cos 2A + \cosh 2B) +$

$= \dfrac{-(1+\cosh 2B - \cos^2 A - \cos 2A - \cosh 2B)}{4}$

$= \dfrac{2(\cosh 2B - \cos 2A)}{4}$

$= \dfrac{\cosh 2B - \cos 2A}{2}$

$\tan\theta = \dfrac{r\sin\theta}{r\cos\theta} = \dfrac{\cos A \cdot \sinh B}{\sin A \cdot \cosh B}$

$= \tanh B \cdot \cot A$

Eg: If $\cos(x + iy) = \cos\theta + i\sin\theta$, show that

$\cos 2x + \cosh 2y = 2$.

Proof :

$\cos\theta + i\sin\theta = \cos x \cosh y - i\sin x \sinh y$

$\therefore \cos\theta = \cos x \cdot \cosh y$; $\sin\theta = -\sin x \cdot \sinh y$

$1 = \cos^2\theta + \sin^2\theta = \cos^2 x \cosh^2 y + \sin^2 x \; \sinh^2 y$.

$= \left(\dfrac{1+\cos 2x}{2}\right)\left(\dfrac{1+\cosh 2y}{2}\right) + \left(\dfrac{1-\cos 2x}{2}\right)\left(\dfrac{\cosh 2y - 1}{2}\right)$

$= \dfrac{1 + \cosh 2y + \cos 2x + \cos 2x \cdot \cosh 2y) + (\cosh 2y - 1 - \cos 2x \cosh 2y - \cos 2x)}{4}$

$= \dfrac{2(\cosh 2y + \cos 2x)}{4}$

$\therefore 2 = \cosh 2y + \cos 2x$

Eg: If $\cos(x + iy) = r(\cos\theta + i\sin\theta)$, Prove that $y = \frac{1}{2}\log\dfrac{\sin(x-\theta)}{\sin(x+\theta)}$

Proof : $r\cos\theta + i\, r\sin\theta = \cos x \cdot \cosh y - i\sin x \cdot \sinh y$

$\therefore r\cos\theta = \cos x \cosh y$; $r\sin\theta = -\sin x \cdot \sinh y$

$\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{-\sin x \cdot \sinh y}{\cos x \cdot \cosh y} \Leftrightarrow \tan\theta = -\tan x \cdot \tanh y$.

$\dfrac{e^y - e^{-y}}{e^y + e^{-y}} = \tanh y = -\dfrac{\tan\theta}{\tan x} = -\tan\theta \cdot \cot x = -\dfrac{\sin\theta}{\cos\theta}\dfrac{\cos x}{\sin x}$

$\dfrac{(e^y + e^{-y}) + (e^y - e^{-y})}{(e^y + e^{-y}) - (e^y - e^{-y})} = \dfrac{\sin x\cos\theta - \sin\theta\cos x}{\sin x\cos x + \sin\theta\cos x}$

$$e^{2y} = \frac{2e^{y}}{2e^{-y}} = \frac{\sin(x-\theta)}{\sin(x+\theta)}$$

$$\therefore 2y = \log\frac{\sin(x-\theta)}{\sin(x+\theta)} \Leftrightarrow y = \frac{1}{2}\log\frac{\sin(x-\theta)}{\sin(x+\theta)}$$

Eg: If $\cos(x+iy) = \cos\theta + i\sin\theta$, prove that $\sin^2 x = \pm\sin\theta$

Proof : $\cos\theta = \cos x\cdot\cosh y;\ \sin\theta = -\sin x\cdot\sinh y$

$$\cosh y = \frac{\cos\theta}{\cos x};\ \sin 2y = -\frac{\sin\theta}{\sin x}$$

$$1 = \cosh^2 y - \sinh^2 y = \frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x}$$

$$\therefore \cos^2 x\cdot\sin^2 x = \cos^2\theta\cdot\sin^2 x - \sin^2\theta\cdot\cos^2\theta$$

$$(1-\sin^2 x)\sin^2 x = (1-\sin^2\theta)\sin^2 x - (\sin^2\theta)(1-\sin^2 x)$$

$$\sin^2 x - \sin^4 x = \sin^2 x - \sin^2\theta\cdot\sin^2 x - \sin^2\theta + \sin^2\theta\cdot\sin^2 x$$

$$\sin^4 x = \sin^2\theta \Leftrightarrow \sin^2 x = \pm\sin\theta$$

Eg: If $\cos(u+iv) = x+iy$ where u, v, x, y as real, Prove that

(i) $(1+x)^2 + y^2 = (\cosh v + \cos u)^2$

(ii) $(1-x)^2 + y^2 = (\cosh v - \cos u)^2$

Proof : $x + iy = \cos(u+iv) = \cos u\cdot\cosh v - i\sin u\cdot\sinh v$

$x = \cos u\cdot\cosh v;\ y = -\sin u\cdot\sinh v$

$$(1+x)^2 + y^2 = (1+\cos u\cosh v)^2 + \sin^2 u\cdot\sinh^2 v$$

$$= 1 + 2\cos u\cdot\cosh v + \cos^2 u\cdot\cosh^2 v + \sin^2 u\cdot\sinh^2 v$$

$$= 1 + 2\cos u\cdot\cosh v + \cos^2 u\cdot\cosh^2 v + (1-\cos^2 u)(\cosh^2 v - 1)$$

$$= 1 + 2\cos u\cdot\cosh v + \cos^2 u\cdot\cosh^2 v + \cosh^2 v - 1 - \cos^2 u\cdot\cosh^2 v + \cos^2 u$$

$$= \cosh^2 v + 2\cos u\cdot\cosh v + \cos^2 u$$

$$= (\cosh v + \cos u)^2$$

$$(1-x)^2 + y^2 = (1-\cos u\cdot\cosh v)^2 + \sin^2 u\cdot\sinh^2 v$$

$$= 1 - 2\cos u\cdot\cosh v + \cos^2 u\cdot\cosh^2 v + (1-\cos^2 u)(\cosh^2 v - 1)$$

$$= \cosh^2 v - 2\cos u\cdot\cosh v + \cos^2 u$$

$$= (\cosh v - \cos u)^2$$

Eg: If $x + iy = \tan(A + iB)$, prove that

(i) $x^2 + y^2 + 2x\cot 2A = 1$

(ii) $x^2 + y^2 - 2y\coth 2B + 1 = 0$

Proof : .

$x + iy = \tan(A+iB)$, $i2B = (A+iB) - (A-iB)$

$2A = (A-iB) + (A+iB)$

$\therefore x - iy = \tan(A-iB) \Leftrightarrow x^2 + y^2 = \tan(A+iB)\cdot\tan(A-iB)$

$\tan(2A) = \tan[(A+iB) + (A-iB)]$

$$= \frac{\tan(A+iB) + \tan(A-iB)}{1 - \tan(A+iB)\tan(A-iB)} = \frac{(x+iy)+(x-iy)}{1-(x^2+y^2)} = \frac{2x}{1-(x^2+y^2)}$$

$$\therefore \cot 2A = \frac{1-(x^2+y^2)}{2x} \Leftrightarrow 2x\cot 2A = 1-(x^2+y^2)$$

$$\therefore x^2 + y^2 + 2x\cot 2A = 1$$

we have $\tan(i2B) = \tan(\,(A+iB) - (A-iB)\,)$

$$i\tanh 2B = \frac{\tan(A+iB) - \tan(A-iB)}{1 + \tan(A+iB)\tan(A-iB)}$$

$$= \frac{(x+iy)-(x-iy)}{1+(x^2+y^2)} = \frac{i2y}{1+x^2+y^2}$$

$$\therefore \tanh 2B = \frac{2y}{1+x^2+y^2} \Leftrightarrow \coth 2B = \frac{1+x^2+y^2}{2y}$$

$$2y\coth(2B) = 1 + x^2 + y^2 \Leftrightarrow x^2 + y^2 - 2y\coth(2B) + 1 = 0$$

Eg: If $\tan(\theta + i\phi) = \sin(x + iy)$, prove that

$$\coth y \cdot \sinh 2\phi = \cot x \cdot \sin 2\theta$$

Proof : $\tan(\theta + i\phi) = \sin(x + iy)$

$$\Rightarrow \frac{\sin 2\theta}{\cos(2\theta) + \cosh(2\phi)} + i\frac{\sinh(2\phi)}{\cos(2\theta) + \cosh(2\phi)}$$

$$= \sin x \cosh y + i \cos u \cdot \sinh y$$

(by earlier result)

$$\therefore \frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y$$

$$\text{and} \frac{\sinh(2\phi)}{\cos 2\theta + \cosh(2\phi)} = \cos x \sinh y$$

$$\therefore \frac{\sin x \cosh y}{\cos x \sinh y} = \frac{(\sin 2\theta) \mid (\cos 2\theta + \cosh 2\phi)}{(\sinh 2\phi) \mid (\cos 2\theta + \cosh 2\phi)} = \frac{\sin(2\theta)}{\sinh(2\phi)}$$

$$\tan x \cdot \coth y = \frac{\sin 2\theta}{\sinh(2\phi)}$$

$$\sinh(2\phi) \coth y = \frac{\sin 2\theta}{\tan x} = (\sin 2\theta) \cot x.$$

Eg: Separate into real and imaginary part of $\tan^{-1}(\alpha + i\beta)$.

Solution : Let $\tan^{-1}(\alpha + i\beta) = x + iy$

$$\therefore \alpha + i\beta = \tan(x + iy)$$

$$\alpha - i\beta = \tan(x - iy)$$

To find x & y, we use first

$$2x = (x + iy) + (x - iy)$$

$$i2y = (x + iy) - (x - iy)$$

$$\tan 2x = \tan[(x + iy) + (x + iy)]$$

$$= \frac{\tan(x + iy) + \tan(x - iy)}{1 - \tan(x + iy)\tan(x - iy)}$$

$$= \frac{(\alpha + i\beta) + (\alpha - i\beta)}{1 - (\alpha - i\beta)(\alpha - i\beta)}$$

$$= \frac{2\alpha}{1 - (\alpha^2 + \beta^2)}$$

$$2x = \tan^{-1}\left(\frac{2\alpha}{1 - \alpha^2 - \beta^2}\right)$$

$$\text{Real part } = x = \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1 - \alpha^2 - \beta^2}\right)$$

$$\tan(i2y) = \tan[(x + iy) - (x - iy)]$$

$$i \tanh 2y = \frac{\tan(x + iy) - \tan(x - iy)}{1 + \tan(x + iy)\tan(x - iy)}$$

$$= \frac{(\alpha + i\beta) - (\alpha - i\beta)}{1 + \alpha^2 + \beta^2}$$

$$= \frac{i2\beta}{1 + \alpha^2 + \beta^2}$$

$$\therefore \tanh(2y) = \frac{2\beta}{1 + \alpha^2 + \beta^2}$$

$$2y = \tanh^{-1}\left(\frac{2\alpha}{1 + \alpha^2 + \beta^2}\right)$$

$$\text{Im part } = y = \frac{1}{2}\tanh^{-1}\left(\frac{2\alpha}{1 + \alpha^2 + \beta^2}\right)$$

EXERCISES

(1) If $(x + iy) = c\cos(A - iB)$, show that

(i) $\dfrac{x^2}{c^2\cosh^2 B} + \dfrac{y^2}{c^2\sinh^2 B} = 1$

(ii) $\dfrac{x^2}{c^2\cos^2 A} - \dfrac{y^2}{c^2\sin^2 A} = 1$

(2) If $\cosh u = \sec\theta$, prove that $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$

(3) If $\tan y = \tan\alpha \tanh\beta$, $\tan z = \cot\alpha \tanh\beta$

Prove that $\tan(y + z) = \sinh(2\beta)\,\text{cosec}(2x)$

(4) If $\cos^{-1}(\alpha + i\beta) = \theta + i\phi$, show that

$$\alpha^2 \text{sech}^2\phi + \beta^2 \text{cosech}^2\phi = 1$$

UNIT I TRIGONOMETRY

PART - A

1. Prove that Cotix = –i.cothx

2. If $\tanh\frac{y}{2} = \tan\frac{x}{2}$ then evaluate Cosx . Coshy = ----------

3. If Sinθ = tanhx, then Prove that Sinhx = tanθ

4. Prove that $\text{Sinh}^{-1}x = \log\left(x+\sqrt{x^2+1}\right)$

5. $\tanh^{-1}x = \log\sqrt{\frac{1+x}{1-x}}$

6. Find the Real Part of Cosech(x+iy)

7. Evaluate $\frac{Cos3\theta}{Cos\theta}$

8. Find the expansion of Cos2θ in terms of Cosθ is -----------

9. Find the expansion of Sin2θ in terms of Sinθ and Cosθ

10. Evaluate Cos3θ in terms of Sinθ and Cosθ

PART- B

11. Express $\text{Sin}^6\theta$ in a series of Cosines of multiples of θ.

12. Express $\text{Cos}^8\theta$ in a series of cosines of multiples of θ.

13. Expand $\text{Sin}^6\theta.Cos^5\theta$ in a series of cosines of multiples of θ.

14. If Cos(x+iy) = r(Cosθ+iSinθ), Prove that y = $\frac{1}{2}\log\frac{Sin(x-\theta)}{Sin(x+\theta)}$.

15. If x+iy = tan(A+iB), prove that
   (i) $x^2+y^2+2xCot2A=1$
   (ii) $x^2+y^2-2y\text{Coth}2B+1=0$

16. Separate into real and imaginary part of $\tan^{-1}(\alpha+i\beta)$

17. If $\text{Cos}(x+iy) = \cos\theta + i\sin\theta$, Prove that $\sin^2 x = \pm\sin\theta$.

18. If $y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, Show that $\tanh\frac{y}{2} = \tan\frac{x}{2}$.

19. If $\text{Sin}(A+iB) = \text{Cos}\,\theta + i\text{Sin}\,\theta$, Show that
(i) $\text{Cos}^2 A = \text{Sinh}^2 B$, (ii) $\text{Cos}^2 A = \pm Sin\theta$.

20. If $\text{Cos}(x+iy) = \text{Cos}\,\theta + i\text{Sin}\,\theta$, Show that $\text{Cos}2x + \text{Cosh}2y = 2$

UNIT II FOURIER SERIES

COURSE MATERIAL

UNIT II –FOURIER SERIES

TOPICS IN FOURIER SERIES

Eulers Formula -Dirichlets Conditions – Statement only – change of interval – odd and even functions – half range series – RMS value – Parseval's formula – complex form of Fourier series- Harmonic Analysis.

INTRODUCTION

The Fourier series is named in honour of Jean-Baptiste Joseph Fourier (1768–1830), who made important contributions to the study of trigonometric series, after preliminary investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli.Fourier introduced the series for the purpose of solving the heat equation in a metal plate. Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems, and especially those involving linear differential equations with constant coefficients, for which the eigen solutions are sinusoids.Fourier series has many such applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics.

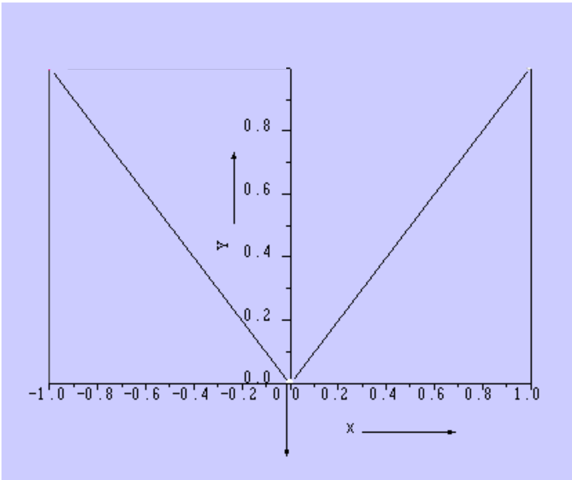
PRELIMINARIES

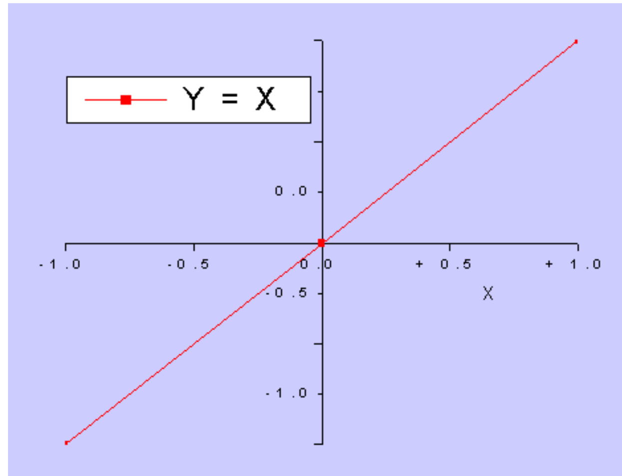
Definitions :

A function y = f(x) is said to be <u>even</u>, if f(-x) = f(x). The graph of the even function is always symmetrical about the y-axis.

A function y=f(x) is said to be <u>odd</u>, if f(-x) = - f(x). The graph of the odd function is always symmetrical about the origin.

For example, the function f(x) = $|x|$ in [-1,1] is even as f(-x) = $|-x| = |x|$ = f(x) and the function f(x) = x in [-1,1] is odd as f(-x) = -x = -f(x). The graphs of these functions are shown below :





Graph of f(x) = $|x|$

Graph of f(x) = x

Note that the graph of f(x) = $|x|$ is symmetrical about the y-axis and the graph of f(x) = x is symmetrical about the origin.

1. If f(x) is even and g(x) is odd, then

   - h(x) = f(x) . g(x) is odd
   - h(x) = f(x) . f(x) is even
   - h(x) = g(x) .g(x) is even

   For example,

   1. h(x) = x^2 cosx is even, since both x^2 and cosx are even functions

   2. h(x) = xsinx is even, since x and sinx are odd functions

   3. h(x) = x^2 sinx is odd, since x^2 is even and sinx is odd.

2. If f(x) is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

3. If f(x) is odd, then $\int_{-a}^{a} f(x)dx = 0$

   For example,

$$\int_{-a}^{a} \cos x dx = 2\int_{0}^{a} \cos x dx, \quad \text{as cosx is even}$$

and $$\int_{-a}^{a} \sin x dx = 0, \quad \text{as sinx is odd}$$

PERIODIC FUNCTIONS

A periodic function has a basic shape which is repeated over and over again. The fundamental range is the time (or sometimes distance) over which the basic shape is defined. The length of the fundamental range is called the period.

A general periodic function f(x) of period T satisfies the condition f(x+T) = f(x)

Here f(x) is a real-valued function and T is a positive real number.

As a consequence, it follows that

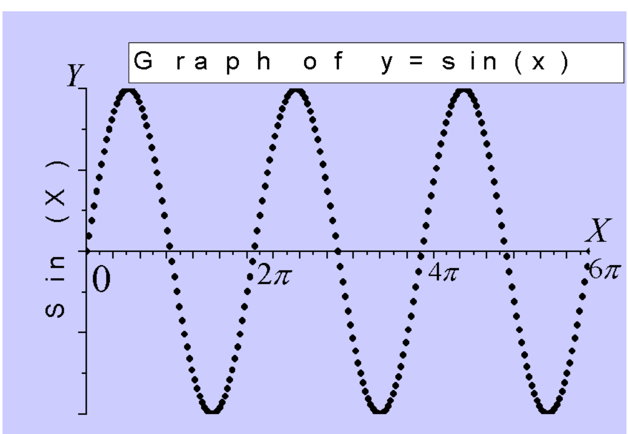
f(x) = f(x+T) = f(x+2T) = f(x+3T) = ….. = f(x+nT)

Thus, f(x) = f(x+nT), n=1,2,3,…..

The function f(x) = sinx is periodic of period 2π since

Sin(x+2nπ) = sinx, n=1,2,3,……..

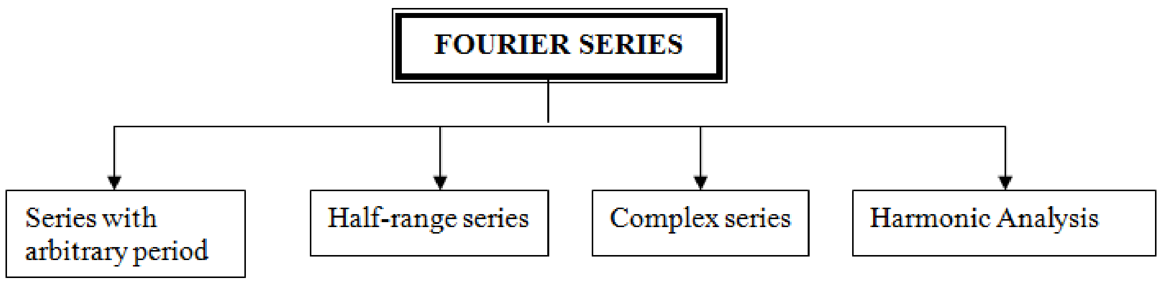
The graph of the function is shown below :



FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



FORMULA FOR FOURIER SERIES

Dirichlet conditions

Dirichlet conditions are sufficient conditions for a real-valued, periodic function *f*(*x*) to be equal to the sum of its Fourier series at each point where *f* is continuous. Moreover, the behavior of the Fourier series at points of discontinuity is determined as well (it is the midpoint of the values of the discontinuity). These conditions are named after Peter Gustav Lejeune Dirichlet.

The conditions are:

- *f*(*x*) must be absolutely integrable over a period.
- *f*(*x*) must have a finite number of extrema in any given bounded interval, i.e. there must be a finite number of maxima and minima in the interval.
- *f*(*x*) must have a finite number of discontinuities in any given bounded interval, however the discontinuity cannot be infinite.

Let

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx \qquad (1)$$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,3,..... \quad (2)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,3,...... \quad (3)$$

Then, the infinite series f(x)= $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x$ (4)

is called the Fourier series of f(x) in the interval (a,a+2*l*). Also, the real numbers a_0, a_1, a_2, ….a_n, and b_1, b_2 , ….b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \,\ldots\ldots \qquad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where $f(x^+)$ and $f(x^-)$ are the values of f(x) immediately to the right and to the left of f(x) respectively.

Some useful results :

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$$\int uvdx = uv_1 - u'v_2 + u''v_3 + ...$$

Here u', u'' ,….. are the successive derivatives of u and

$$v_1 = \int vdx, v_2 = \int v_1 dx, ...$$

We illustrate the rule, through the following examples :

$$\int x^2 \sin nxdx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x} dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful :

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

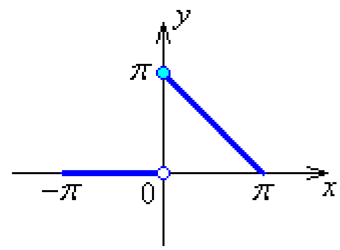
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

3. If ‘n’ is integer, then

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $\sin 2n\pi = 0$, $\cos 2n\pi = 1$

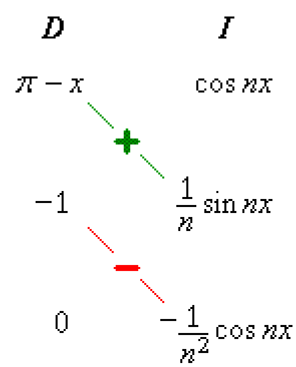
Examples for the interval $(0,2\pi)$ and $(-\pi, \pi)$

1.Expand $f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$ in a Fourier series.



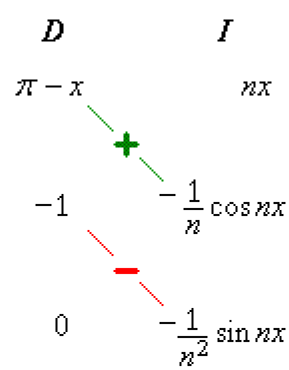
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{(\pi-x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi-x)\sin nx - \cos nx}{n^2}\right]_0^{\pi} = \frac{1-(-1)^n}{n^2\pi}$$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi-x)\cos nx + \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$

Therefore the Fourier series for $f(x)$ is

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)$$

2. Obtain the Fourier expansion of f(x) = $\frac{1}{2}(\pi - x)$ in $-\pi < x < \pi$

We have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we get, $$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

3. Obtain the Fourier expansion of $f(x)=e^{-ax}$ in the interval $(-\pi, \pi)$. Deduce that

$$\operatorname{cosech}\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi}-e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx + n\sin nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

Thus,

$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

For $x=0$, $a=1$, the series reduces to

$$f(0)=1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$ or

$$1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

or $$1 = \frac{2\sinh\pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Thus,

$$\pi\cos ech\pi = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

This is the desired deduction.

4. Obtain the Fourier expansion of f(x) = x^2 over the interval ($-\pi$, π). Deduce that

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+......+\infty$$

Sol:

The function f(x) is even. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

or $$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx,$$ since f(x)cosnx is even

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$ since f(x).sin*nx* is odd.

Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

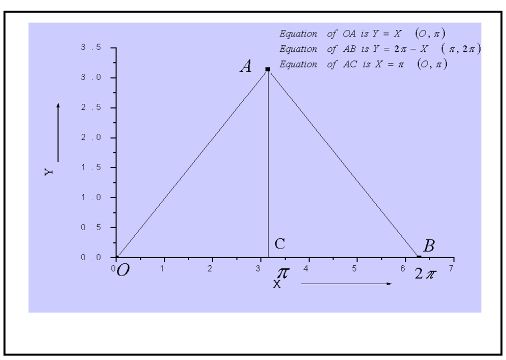
$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$

5. Obtain the Fourier expansion of $f(x) = \begin{cases} x, 0 \le x \le \pi \\ 2\pi - x, \pi \le x \le 2\pi \end{cases}$

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$

The graph of f(x) is shown below.



Here OA represents the line f(x)=x, AB represents the line f(x)=(2π-x) and AC represents the line x=π. Note that the graph is symmetrical about the line AC, which in turn is parallel to y-axis. Hence the function f(x) is an even function.

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_{0}^{\pi} x dx = \pi$$

since f(x)cosnx is even.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0,$$ since f(x)sinnx is odd

Thus the Fourier series of f(x) is

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$$

For x=π , we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos n\pi$$

or
$$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus, $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

or
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

This is the series as required.

6. Obtain the Fourier expansion of

f(x) = $\begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Here,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right]$$
$$= \frac{1}{n^2 \pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$
$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier series is

$$f(x) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{\left[1-2(-1)^n\right]}{n}\sin nx$$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+) = 0$, $f(x^-) = -\pi$ at x=0. Hence

$$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0 - \pi) = \frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$
$$or\ \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

Simplifying we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Change of interval

7.. Obtain the Fourier series of $f(x) = 1-x^2$ over the interval (-1,1).

Sol:

The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$

$$= 2\int_0^1 (1-x^2)dx = 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_0^1 f(x)\cos(n\pi x)dx$$ as f(x) cos(nπx) is even

$$= 2\int_0^1 (1-x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n = 2\left[\left(1-x^2\right)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx$$ =0, since f(x)sin(nπx) is odd.

The Fourier series of f(x) is $f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$

8.Find the Fourier series expansion for the standard square wave, $f\left(x\right) = \begin{cases} -1 & \left(-1 < x < 0\right) \\ +1 & \left(0 \le x < +1\right) \end{cases}$

Sol: $\ell = 1$.

The function is odd ($f(-x) = -f(x)$ for all x).

Therefore $a_n = 0$ for all n. We will have a Fourier sine series only.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x \, dx = \int_{-1}^{0} -\sin n\pi x \, dx + \int_{0}^{1} \sin n\pi x \, dx$$

$$= \left[\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[\frac{-\cos n\pi x}{n\pi}\right]_{0}^{1} = \frac{2\left(1-(-1)^n\right)}{n\pi}$$

(can use symmetry)

9. Obtain the Fourier expansion of

$$f(x) = \begin{cases} 1+\dfrac{4x}{3} in -\dfrac{3}{2} < x \le 0 \\ 1-\dfrac{4x}{3} in 0 \le x < \dfrac{3}{2} \end{cases}$$

And hence deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$

Sol:

The period of f(x) is $\frac{3}{2} - \left(\frac{-3}{2}\right) = 3$

Also f(-x) = f(x). Hence f(x) is even. Thus b_n =0.

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_{0}^{3/2} f(x)dx$$

$$= \frac{4}{3}\int_{0}^{3/2}\left(1-\frac{4x}{3}\right)dx = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3/2}\int_{0}^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - \left(\frac{-4}{3}\right)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\Bigg|_{0}^{3/2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Also, $$b_n = \frac{1}{3}\int_{-3/2}^{3/2} f(x)\sin\left(\frac{n\pi x}{3/2}\right)dx = 0$$

Thus $$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{2n\pi x}{3}\right)$$

putting x=0, we get

$$f(0) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$$

or $$1 = \frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+......\right]$$

Thus, $$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+......$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$

10. Find a Fourier series for $f(x) = x$, $0 < x < 4$, $f(x + 4) = f(x)$.

Sol:

$$a_0 = \frac{1}{2}\int_0^4 x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^4 = \frac{1}{2}(8-0) = 4$$

For $n = 1, 2, 3, \ldots$:

$$a_n = \frac{1}{2}\int_0^4 x\cos\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left(\left(0+\frac{4}{n^2\pi^2}\cos(2n\pi)\right) - \left(0+\frac{4}{n^2\pi^2}\cos(0)\right)\right) = 0$$

$$b_n = \frac{1}{2}\int_0^4 x \sin\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left(\left(\frac{-8}{n\pi}\cos(2n\pi) - 0\right) - (0-0)\right) = \frac{-4}{n\pi}$$

Consequently,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) = 2 + \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}.$$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period 2*l* may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval (0,*l*) which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Half Range Sine series :

Suppose f(x) is given in the interval (0,*l*). Then Half range sine series of f(x) over (0,*l*) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The half-range sine series of f(x) over (0,π) given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Half Range cosine series :

The half-range cosine series of f(x) over (0,*l*) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The half-range cosine series over (0, π) I given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \quad \text{n} = 1,2,3,\ldots..$$

The Fourier series of $f(x) = |x|$ in $[-1, 1]$.

Sol:

Evaluating the Fourier cosine coefficients,

D		*I*
x		$\cos n\pi x$
	+	
1		$\frac{1}{n\pi}\sin n\pi x$
	−	
0		$\frac{-1}{(n\pi)^2}\cos n\pi x$

$$a_n = \frac{2}{1}\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

$$\Rightarrow \quad a_n = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

and $a_0 = \frac{2}{1}\int_0^1 x\,dx = \left[x^2\right]_0^1 = 1$

Evaluating the first few terms,

$$a_0 = 1, \quad a_1 = \frac{-4}{\pi^2}, \quad a_2 = 0, \quad a_3 = \frac{-4}{9\pi^2}, \quad a_4 = 0, \quad a_5 = \frac{-4}{25\pi^2}, \quad a_6 = 0, \ldots$$

or $$a_n = \begin{cases} 1 & (n=0) \\ \frac{-4}{(n\pi)^2} & (n=1,3,5,\ldots) \\ 0 & (n=2,4,6,\ldots) \end{cases}$$

Therefore the Fourier cosine series for *f* (*x*) = *x* on [0, 1] (which is also the Fourier series for *f* (*x*) = | *x* | on [–1, 1]) is

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\left((2k-1)\pi x\right)}{(2k-1)^2}$$

or

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \frac{\cos 7\pi x}{49} + \ldots\right)$$

Problems

1. Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).

Sol:

We have,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[\left(\pi x - x^2\right)\left(\frac{-\cos nx}{n}\right) - \left(\pi - 2x\right)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1 - (-1)^n\right]\sin nx$$

2. Obtain the cosine series of $f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases} \qquad over(0,\pi)$

Sol:

Here

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2\pi}\left[1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2,6,10,.....$$

Thus, the Fourier cosine series is

$$\text{f(x)} = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$$

3. Obtain the half-range cosine series of f(x) = c-x in 0<x<c

Sol:

Here $$a_0 = \frac{2}{c}\int_0^c (c-x)dx = c$$

$$a_n = \frac{2}{c}\int_0^c (c-x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Integrating by parts and simplifying we get,

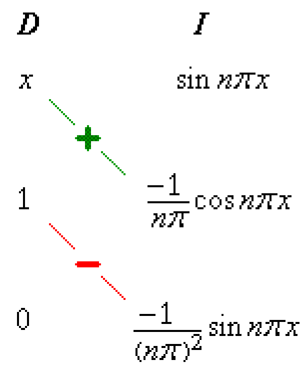
$$a_n = \frac{2c}{n^2\pi^2}\left[1-(-1)^n\right]$$

The cosine series is given by f(x) = $\frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$

4.Find the Fourier sine series and the Fourier cosine series for $f(x) = x$ on [0, 1].

Sol:

$f(x) = x$ happens to be an odd function of x for any domain centred on $x = 0$. The odd extension of $f(x)$ to the interval [–1, 1] is $f(x)$ itself.



Evaluating the Fourier sine coefficients,

$$b_n = \frac{2}{1}\int_0^1 x\sin\left(\frac{n\pi x}{1}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

$$\Rightarrow \quad b_n = 2\left[-\frac{x}{n\pi}\cos\left(\frac{n\pi x}{1}\right) + \frac{1}{(n\pi)^2}\sin\left(\frac{n\pi x}{1}\right)\right]_0^1$$

$$= \frac{2}{n\pi}\times(-1)^{n+1}$$

Therefore the Fourier sine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x) = x$ on

[–1, 1]) is

$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi x)}{n\pi}$$

or

$$f(x) = \frac{2}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \frac{\sin 4\pi x}{4} + \dots\right)$$

4.Find Fourier sine and cosine series of f(t) = t; 0 <t<π.

Sol:

Fourier sine series of f(x) is given by

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} t \sin nt \; dt = \frac{-2t\cos nt}{n\pi} \Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi} \cos nt \; dt = \frac{2(-1)^{n+1}}{n}.$$

Therefore

$$t = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nt, \qquad 0 < t < \pi.$$

Fourier Cosine series.

$$a_n = \frac{2}{\pi}\int_0^{\pi} t \cos nt \; dt = \frac{2t\sin nt}{n\pi} \Big|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin nt \; dt = \frac{2[(-1)^n - 1}{n^2\pi},$$

for $n \geq 1$ *and* $a_0 = \frac{2}{\pi}\int_0^{\pi} t \; dt = \pi$, *so*

$$t = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{\cos(2j-1)t}{(2j-1)^2}, \qquad 0 < t < \pi.$$

5. Expand *f(x) = cos x*, *0 < x < π* in a Fourier sine series.

Sol. Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\cos x\ \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin nx\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x]\,dx\ , \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos(n+1)x}{n+1}\right) + \left(\frac{-\cos(n-1)x}{n-1}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\left\{\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\} - \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$ 2SinACosB = Sin(A+B) + Sin(A–B)

$$= -\frac{1}{\pi}\left[(-1)^n\left\{\frac{-1}{n+1} + \frac{-1}{n-1}\right\} - \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$\cos(n+1)\pi = (-1)^{n+1}$$

$$\cos(n-1)\pi = (-1)^{n-1}$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} + \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{2n}{n^2-1}\right\} + \left\{\frac{2n}{n^2-1}\right\}\right]$$

$$b_n = \frac{2n}{\pi(n^2-1)}\left[(-1)^n + 1\right], \quad n \neq 1$$

When n = 1, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = \frac{2}{\pi}\int_0^{\pi}\cos x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}(1-1) = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1\sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= 0 + \sum_{n=2}^{\infty}\frac{2n[(-1)^n + 1]}{\pi(n^2-1)}\sin nx$$

$$= \frac{2}{\pi}\left[\frac{4\sin 2x}{3} + 0 + \frac{8\sin 4x}{15} + 0 + \frac{12\sin 6x}{35} + 0 + \dots\dots\dots\dots\right]$$

$$= \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{2\sin 4x}{15} + \frac{3\sin 6x}{35} + \dots\dots\dots\dots\right]$$

Interval	Fourier series of f(x)=	a_0	a_n	b_n
(0,2*l*)	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x$	$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$	$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx,$	$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$
(-*l* , *l*)		$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$	$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx$	$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$
(0,2π)	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$	$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$	$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx$	$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx$
(-π, π)		$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$	$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$	$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$
	Half Range sine series			
(0,*l*)	$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$	-	-	$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$
(0,π)	$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$	-	-	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$
	Half Range cosine series			
(0,*l*)	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$	$a_0 = \frac{2}{l}\int_0^{l} f(x)dx$	$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	-
(0,π)	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$	$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$	

Root Mean Square Value(RMS value)

The RMS value of a function *f(x)* in (a,b) is defined by

$$\bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2\,dx$$

Parseval's Identity For Fourier Series

The Parseval's identity for Fourier series in the interval (c, c + 2*l*) is

$$\frac{1}{l}\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

The Parseval's identity for Fourier series in the interval (c, c + 2π) is

$$\frac{1}{\pi}\int_c^{c+2\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Where a_o, a_n, b_n are Fourier coefficients.

1. Expand $f(x) = x - x^2$ as a Fourier series in $-l < x < l$ and using this series find the root square mean value of *f(x)* in the interval.

Sol. Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\,dx$$

$$= \frac{1}{l}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{\frac{l^2}{2} - \frac{l^3}{3}\right\} - \left\{\frac{l^2}{2} + \frac{l^3}{3}\right\}\right]$$

$$= \frac{1}{l}\left(\frac{-2l^3}{3}\right) = \frac{-2l^2}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \;= \frac{1}{l}\int_{-l}^{l}(x - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(x - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1 - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{0 + (1 - 2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\} - \left\{0 + (1 + 2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\}\right]$$

$$= \frac{(-1)^n l^2}{l\, n^2\pi^2}\left[1 - 2l - 1 - 2l\right]$$

$$= \frac{(-1)^n l}{n^2\pi^2}\left[-4l\right] = \frac{4\, l^2\,(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \;= \frac{1}{l}\int_{-l}^{l}(x - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(x - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1 - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{-(l-l^2)\left(\frac{(-1)^n l}{n\pi}\right)+0-\frac{2(-1)^n l^3}{n^3\pi^3}\right\}-\left\{-(-l-l^2)\left(\frac{(-1)^n l}{n\pi}\right)+0-\frac{2(-1)^n l^3}{n^3\pi^3}\right\}\right]$$

$$= \frac{-(-1)^n l}{l\, n\pi}\left[l-l^2+l+l^2\right]$$

$$= \frac{(-1)^{n+1}}{n\pi}[2l] = \frac{2l\,(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2}\left(\frac{-2l^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{4l^2(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{l} + \frac{2l(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$(i.e.)\ f(x) = \frac{-l^2}{3} + \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} - \frac{1}{4^2}\cos\frac{4\pi x}{l} + \dots\dots\right]$$

$$+ \frac{2l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \frac{1}{4}\sin\frac{4\pi x}{l} + \dots\dots\right]$$

RMS value of $f(x)$ in $(-l, l)$ is

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$= \frac{1}{4}\left(\frac{-2l^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{16l^4(-1)^{2n+2}}{n^4\pi^4} + \frac{4l^2(-1)^{2n+2}}{n^2\pi^2}\right]$$

$$(i.e.)\ \bar{y}^2 = \frac{l^4}{9} + \sum_{n=1}^{\infty}\left[\frac{8l^4}{n^4\pi^4} + \frac{2l^2}{n^2\pi^2}\right]$$

2. Find the half range cosine series for $f(x) = x(\pi - x)$ in $0 < x < \pi$.

Deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\dots = \frac{\pi^4}{90}$

Sol. Half range fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

7.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0-0)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right]$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 + \frac{(-\pi)(-1)^n}{n^2} + 0\right\} - \left\{0 + \frac{(\pi)(1)}{n^2} + 0\right\}\right]$$

$$= \frac{2\pi}{\pi n^2}\left[-(-1)^n - 1\right]$$

$$= -\frac{2}{n^2}\left[(-1)^n + 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{1}{2}\left(\frac{\pi^2}{3}\right) + \sum_{n=1}^{\infty} -\frac{2}{n^2}\left[(-1)^n + 1\right]\cos nx$$

$$= \frac{\pi^2}{6} - 2\left[0 + \frac{2\cos 2x}{2^2} + 0 + \frac{2\cos 4x}{4^2} + 0 + \frac{2\cos 6x}{6^2} + 0 + \ldots\ldots\ldots\ldots\right]$$

$$= \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \ldots\ldots\ldots\ldots\right]$$

Parseval's identity for half range fourier cosine series is

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty} a_n^{\,2}$$

$$\frac{2}{\pi}\int_0^{\pi}[\pi x - x^2]^2\,dx = \frac{1}{2}\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\frac{4}{n^4}[(-1)^n+1]^2$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi^2x^2 + x^4 - 2\pi x^3)dx \;= \frac{\pi^4}{18} + 4\left[0 + \frac{4}{2^4} + 0 + \frac{4}{4^4} + 0 + \frac{4}{6^4} + 0 + \ldots\ldots\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^2x^3}{3} + \frac{x^5}{5} - \frac{2\pi x^4}{4}\right]_0^{\pi} = \frac{\pi^4}{18} + \frac{16}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\left(\frac{\pi^5}{3} + \frac{\pi^5}{5} - \frac{\pi^5}{2}\right) - 0\right] = \frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{30}\right] - \frac{\pi^4}{18} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\pi^4}{15} - \frac{\pi^4}{18} \;= \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\ldots\ldots$$

$$(i.e.)\;\; \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\ldots\ldots$$

3. Find the Fourier series expansion of $f(x) = \begin{cases} l - x, & 0 < x < l \\ 0, & l < x < 2l \end{cases}$

Hence deduce the value of the series (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$ (ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots$

Sol. Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx \;= \frac{1}{l}\int_0^{l}(l-x)\,dx + \frac{1}{l}\int_l^{2l}(0)\,dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^2}{-2}\right]_0^{l}$$

$$= \frac{1}{-2l}\left[0 - l^2\right]$$

$$= \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\cos\frac{n\pi x}{l}\,dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\left\{0 - \frac{(-1)^n l^2}{n^2\pi^2}\right\} - \left\{0 - \frac{l^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{1}{l}\frac{l^2}{n^2\pi^2}\left[(-1)^{n+1} + 1\right]$$

$$= \frac{l}{n^2\pi^2}\left[(-1)^{n+1} + 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\sin\frac{n\pi x}{l}\,dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\{0-0\} - \left\{-\frac{l^2}{n\pi} - 0\right\}\right]$$

$$= \frac{l}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

$$= \frac{l}{4} + \sum_{n=1}^{\infty}\left(\frac{l[(-1)^{n+1}+1]}{n^2\pi^2}\cos\frac{n\pi x}{l} + \frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$= \frac{l}{4} + \frac{l}{\pi^2}\left[\frac{2}{1^2}\cos\frac{\pi x}{l} + 0 + \frac{2}{3^2}\cos\frac{3\pi x}{l} + 0 + \frac{2}{5^2}\cos\frac{5\pi x}{l} + 0 + \ldots\ldots\ldots\ldots\right]$$

$$+ \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\ldots\ldots\ldots\right]$$

$$(i.e.)\; f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\ldots\ldots\ldots\right]$$

$$+ \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\ldots\ldots\ldots\right] \;---\;(1)$$

Put $x = \frac{l}{2}$ (which is point of continuity) in equation (1), we get

$$l - \frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}(0) + \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin 4\pi + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\ldots\ldots\right]$$

$$\frac{l}{2} = \frac{l}{4} + \frac{l}{\pi}\left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + \ldots\ldots\ldots\right]$$

$$\frac{l}{2} - \frac{l}{4} = \frac{l}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\right]$$

$$\frac{l}{4} = \frac{l}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$$

Put *x* = *l* in equation (1) we get

$$f(l) = \frac{l}{4} + \frac{2l}{\pi^2}\left[-\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \ldots\ldots\ldots\right] \quad \text{--------------- (2)}$$

But *x* = *l* is the point of discontinuity. So we have

$$f(l) = \frac{f(l-) + f(l+)}{2} = \frac{(0)+(0)}{2} = 0$$

f(x) = l – x
f(l–) = l – l =0

f(x) = 0
f(l) = 0

Hence equation (2) becomes

$$0 = \frac{l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\right]$$

$$-\frac{l}{4} = -\frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots$$

4.Find the half range cosine series for the function *f(x)* = *x* in 0 < x < *l*.

Hence deduce the value of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

Sol. Half range Fourier cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{2}{l}\left[\frac{l^2}{2} - 0\right] = l$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 + \frac{(-1)^n l^2}{n^2\pi^2}\right\} - \left\{0 + \frac{l^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l[(-1)^n - 1]}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l}{2} + \frac{2l}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{l} + 0 - \frac{2}{3^2}\cos\frac{3\pi x}{l} + 0 - \frac{2}{5^2}\cos\frac{5\pi x}{l} + 0 - \dots\dots\dots\right]$$

$$(i.e.)\ f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\dots\dots\right]$$

Using Parseval's identity for half range Fourier cosine series we have

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty} a_n^{\,2}$$

$$\frac{2}{l}\int_0^l (x)^2\,dx = \frac{l^2}{2} + \sum_{n=1}^{\infty}\left[\frac{4l^2\{(-1)^n - 1\}^2}{n^4\pi^4}\right]$$

$$\frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{2} + \frac{4l^2}{\pi^4}\left[\frac{4}{1^4} + 0 + \frac{4}{3^4} + 0 + \frac{4}{5^4} + 0 + \dots\dots\dots\right]$$

$$\frac{2}{l}\left[\frac{l^3}{3} - 0\right] = \frac{l^2}{2} + \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots\right]$$

$$\frac{2l^2}{3} - \frac{l^2}{2} = \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots\right]$$

$$\frac{l^2}{6} = \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots \quad (i.e.)\ \frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

Question Bank
FOURIER SERIES

UNIT II

PART A

1. What is the value of b_n when the function f(x)=cosax is expanded as a Fourier series in (-π, π)?
2. State Dirichlet conditions for a function to be expanded as a Fourier series.
3. If $f(x) = x^2$ in $-\pi \le x \le \pi$, find b_n
4. Write the complex form of Fourier series of f(x) in $0<x<2\pi$.
5. Find the Fourier sine series for the function f(x)=k, $0<x<\pi$
6. Find a_0 in the Fourier series corresponding to f(x) =x in $0 \le x \le 2\pi$
7. If f(x) = $\sum_{-\infty}^{\infty} c_n e^{inx}$ is the complex form of the Fourier series corresponding to f(x) in (0,2π), write the formula for c_n.
8. Write the formulas for finding the Euler's constants in the Fourier series expansion of f(x) in (-π, π).
9. Find a_0 in the Fourier series corresponding to $f(x) = x^2$ in $0 \le x \le 2\pi$
10. find the Fourier series for the function f(x) =k in $0 \le x \le 2\pi$ What is the value of b_n when the function $f(x)=x^2$ expanded as a Fourier series in (-π, π)?
11. State Dirichlet conditions for a function to be expanded as a Fourier series.
12. If f(x) = sinx in $-\pi \le x \le \pi$, find a_n
13. Write the complex form of Fourier series of f(x) in $0<x<l$.
14. Find the Fourier cosine series for the function f(x)=k, $0<x<\pi$
15. State parseval's identity on Fourier series
16. If f(x) = $\sum_{-\infty}^{\infty} c_n e^{inx}$ is the complex form of the Fourier series corresponding to f(x) in (0,2π), Write the formula for C_n.
17. Write the formulas for finding the Euler's constants in the Fourier series expansion of f(x) in (-π, π).
18. Find the root mean square value of the function f(x) = x in the interval (0,1).
19. Define Harmonic analysis.

PART B

1. Obtain the Fourier Series for $f(x) = x + x^2$ in (-π, π). Deduce that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\infty = \frac{\pi^2}{6}.$$

2. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $0 \le x \le 2\pi$.

3. Express $f(x) = x$ in half range cosine series in the range $0 < x < 1$ and deduce the value of $\left(\frac{1}{1^4}\right)+\left(\frac{1}{3^4}\right)+\left(\frac{1}{5^4}\right)+.....to\ \infty$

4. Obtain the Fourier series of $f(x)=|\cos x|$ in $-\pi \leq x \leq \pi$

5. (a) Find the complex form of Fourier series for $f(x) = e^{-ax}$ in the interval $-1 \leq x \leq 1$.

   (b) Expand $f(x) = 2x - x^2$, $0 < x < 3$ as a half range sine series.

6. Find the Fourier series expansion of the periodic function f(x) of period 2, defined by

   $f(x)=\begin{cases}1+x, & -1<x\leq 0\\ 1-x, & 0\leq x\leq 1\end{cases}$ Deduce that $\sum_{1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

7. Obtain Fourier series for $f(x)=\begin{cases}l-x, & 0<x\leq l\\ 0, & l\leq x\leq 2l\end{cases}$. Hence deduce that

   $(i)\ 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.......=\frac{\pi}{8}$ $\qquad (ii)\ \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+..........\infty=\frac{\pi^2}{8}$

8. Compute the first two harmonics of the Fourier series of f(x) given by the following table.

X	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	π
f(x)	10.	1.4	1.9	1.7	1.5	1.2	1.0

9. Find the Fourier series of $f(x) = x^2$ in the interval $(-\pi \leq X \leq \pi)$ and deduce $\sum_{1}^{\infty}\frac{1}{n^2}=\pi^2/6$

10. Obtain the first two harmonics in the Fourier series expansion in (0,6) for the function $y = f(x)$ defined by the table given below.

X	0	1	2	3	4	5
Y	9	18	24	28	26	20

UNIT III PARTIAL DIFFERENTIAL EQUATIONS

COURSE MATERIAL

PARTIAL DIFFERENTIAL EQUATIONS

Formation of equations by elimination of arbitrary constants and arbitrary functions - Solutions of PDE - general, particular and complete integrals - Solutions of First order Linear PDE (Lagrange's linear equation) - Solution of Linear Homogeneous PDE of higher order with constant coefficients.

INTRODUCTION

A partial differential equation is an equation involving a function of two or more variables and some of its partial derivatives. Therefore a partial differential equation contains one dependent variable and more than one independent variable

Notations in PDE

$p = \partial z/\partial x$ $q = \partial z/\partial y$ $r = \partial^2 z/\partial x^2$ $s = \partial^2 z/\partial x\partial y$ $t = \partial^2 z/\partial y^2$

Formation of partial differential equations:

There are two methods to form a partial differential equation.

(i) By elimination of arbitrary constants.

(ii) By elimination of arbitrary functions.

Formation of partial differential equations by elimination of arbitrary constants:

1. Form a p.d.e by eliminating the arbitrary constants a and b from $Z=(x+a)^2+(y-b)^2$

Solution:

Given $Z= (x+a)^2+(y-b)^2$

$P = \frac{\partial z}{\partial x} = 2(x+a)$, ie) $x+a = \frac{p}{2}$

$q = \frac{\partial z}{\partial y} = 2(y-b)$, ie) $y-b = \frac{q}{2}$

$$\therefore (1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$4z = p^2+q^2$$

which is the required p.d.e.

2. Find the p.d.e of all planes having equal intercepts on the X and Y axis.

Solution:

Intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Given : a=b. [Equal intercepts on the x and y axis]

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad .. \qquad (1)$$

Here a and c are the two arbitrary constants.

Differentiating (1) p.w.r.to 'x' we get

$$\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0$$

$$\frac{1}{a} + \frac{1}{c}p = 0.$$

$$\frac{1}{a} = -\frac{1}{c}p. \qquad (2)$$

Diff (1) p.w.r.to. 'y' we get

$$0+\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0.$$

$$\frac{1}{a}+\frac{1}{c}q=0$$

$$\frac{1}{a}=-\frac{1}{c}q \qquad (3)$$

From (2) and (3) $\Rightarrow -\frac{1}{c}p=-\frac{1}{c}q$

p = q ,which is the required p.d e.

3. Form the p.d.e by eliminating the constants a and b from $z = ax^n+by^n$.

Solution:

Given: $z = ax^n+by^n$. (1)

$P = \frac{\partial z}{\partial x} = anx^{n-1}$

$\frac{p}{n} = ax^{n-1}$

Multiply ' x' we get, $\frac{px}{n} = ax^n$ (2)

$q = \frac{\partial z}{\partial y} = bny^{n-1}$

$\frac{q}{n} = by^{n-1}$

Multiply ' y' we get , $\frac{qy}{n} = by^n$ (3)

Substitute (2) and (3) in (1) we get the required p.d.e $z = \frac{px}{n}+\frac{qy}{n}$

zn = px+qy.

Formation of partial differential equations by elimination of arbitrary functions:

1. Eliminate the arbitrary function f from z= $f\left(\frac{y}{x}\right)$ and form a partial differential equation.

   Solution:

   Given z = $f\left(\frac{y}{x}\right)$ (1)

   Differentiating (1) p.w.r.to 'x' we get

   $P = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$ (2)

   Differentiating (1) p.w.r.to y we get

   $q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$ (3)

   $\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{-y}{x}$

   $\therefore$ px = -qy

   ie) px+qy = 0 is the required p.d.e.

2. Eliminate the arbitrary functions f and g from z = f(x+iy)+g(x-iy) to obtain a partial differential equation involving z,x,y.

   Solution:

   Given : z = f(x+iy)+g(x-iy) (1)

   $P = \frac{\partial z}{\partial x} = f'(x+iy)+g'(x-iy)$ (2)

   $q = \frac{\partial z}{\partial y} = i f'(x+iy)-ig'(x-iy)$ (3)

$r = \frac{\partial^2 z}{\partial x^2}$ = f″(x+iy)+g″(x-iy) (4)

$t = \frac{\partial^2 z}{\partial y^2}$ = -f″(x+iy)-g″(x-iy) (5)

r + t = 0 is the required p.d.e.

3. Form the p.d.e by eliminating arbitrary function ϕ from the relation
   $\phi(xyz, x^2 + y^2 + z^2) = 0$

   Solution:

   The pde is obtained from $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

   $\begin{vmatrix} yz + xyp & 2x + 2zp \\ xz + xyq & 2y + 2zq \end{vmatrix} = 0$

   (yz+xyp)(2y+2zq)-(xz+xyq)(2x+2zp)=0

SOLUTION OF PDE

Complete solution: A solution which contains as many arbitrary constants as there are independent variables is called a complete integral (or)complete solution.(number of arbitrary constants=number of independent variables)

Particular solution: A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral (or) particular solution.

General solution: A solution of a p.d.e which contains the maximum possible number of arbitrary functions is called a general integral (or) general solution.

1. Find the general solution of $\frac{\partial^2 z}{\partial y^2} = 0$

   Solution:

   Given $\frac{\partial^2 z}{\partial y^2} = 0$

ie) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=0$

Integrating w.r.to 'y' on both sides

$\frac{\partial z}{\partial y}$ = a (constants)

ie) $\frac{\partial z}{\partial y}$ =f(x)

Again integrating w.r.to 'y' on both sides.

z = f(x) y + b which is the required solution.

Lagrange's linear equations:

The equation of the form Pp + Qq = R is known as Lagrange's equation, where P, Q and R are functions of x, y and z. To solve this equation it is enough to solve the subsidiary equations.

dx/P = dy/Q = dz/R

If the solution of the subsidiary equation is of the form u(x, y, z) = c_1 and v(x, y, z) = c_2 then the solution of the given Lagrange's equation is Φ(u, v) = 0.

To solve the subsidiary equations we have two methods:

1 Method of Grouping:

Consider the subsidiary equation dx/P = dy/Q = dz/R..Take any two members say first two or last two or first and last members. Now consider the first two members dx/P = dy/Q. If P and Q contain z (other than x and y) try to eliminate it. Now direct integration gives u(x, y) = c_1. Similarly take another two members dy/Q = dz/R. If Q and R contain x(other than y and z) try to eliminate it. Now direct integration gives v(y, z) = c_2 . Therefore solution of the given Lagrange's equation is Φ(u, v) = 0.

1. Solve px + qy = z

Solution:

The Lagrange's eqn is Pp + Qq = R

and the auxilliary eqn. is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

ie $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ (1)

Taking the first two ratios,

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, logx = logy + loga

$$\frac{x}{y} = a \qquad (2)$$

Similarly, taking last two ratios of eqn (1),

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, logy = logz + logb

$$\frac{y}{z} = b \qquad (3)$$

Eqns (2) and (3) are independent solns of (1).

Hence the complete soln of the given eqn. is φ(u,v)=0

ie; $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

Method of multiplier's

Choose any three multipliers l, m, n may be constants or function of x, y and z such that in $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$

the expression lP + mQ + nR = 0. Hence ldx + mdy + ndz = 0

[since each of the above ratios equal to a constant $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{ldx + mdy + ndz}{lP + mQ + nR} = k(say)$

$$ldx + mdy + ndz = k(lP + mQ + nR)$$

If $lP + mQ + nR = 0$ then $ldx + mdy + ndz = 0$]

Now direct integration gives u(x, y, z) = c_1.

similarly choose another set of multipliers l′, m′, n′

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

the expression $l'P + m'Q + n'R = 0$

therefore $l'dx + m'dy + n'dz = 0$ (as explained earlier)

Now direct integration gives v(x, y, z) = c_2.

Therefore solution of the given Lagrange's equation is Φ(u, v) = 0.

1. Solve $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$

Solution:

The Lagrange's eqn is Pp + Qq = R

and the auxilliary eqn. is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$$

Taking multpliers as x,y,z;

$$\frac{dx}{x\left(y^2 - z^2\right)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = k(say)$$

$$xdx + ydy + zdz = k(x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2))$$

$$xdx + ydy + zdz = 0$$

Integrating , $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c}{2}$

ie; $x^2 + y^2 + z^2 = c$

$u = x^2 + y^2 + z^2$ (1)

Again taking the multipliers as 1/x,-1/y,-1/z,

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = k(say)$$

$$\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz = k(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)$$

$$\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz = 0$$

Integrating, log x- log y- log z= log C'

$$\frac{x}{yz} = c'$$

$$v = \frac{x}{yz} \qquad (2)$$

solution is $\phi(x^2 + y^2 + z^2, \frac{x}{yz}) = 0$

Homogeneous Linear partial differential equations:

Equation of the form $a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^{n-1} z}{\partial x^{n-1}\partial y} + ...a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$

$F(x, y) = [a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n]z$

where $D = \partial/\partial x$ and $D' = \partial/\partial y$

Solution of Homogeneous Linear partial differential equations:

The Complete solution consists of two parts namely complementary function and particular integral.

i.e) Z = C.F + P.I

To find the Complementary function (C.F.):

The complementary function is the solution of the equation

$$a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n = 0.$$

In this equation, put D = m and D′ = 1 then we get an equation, which is called auxiliary equation.Hence the auxiliary equation is

$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n = 0.$

Let the root of this equation be $m_1, m_2, m_3, \ldots\ldots\ldots m_n.$

Case 1: If the roots are real or imaginary and different say $m_1 \neq m_2 \neq m_3 \neq \ldots\ldots \neq m_n$. then the

C.F. is $Z = f_1(y + m_1x) + f_2(y + m_2x) + \ldots\ldots + f_n(y + m_nx)$

Case 2: If any two roots are equal, say $m_1 = m_2 = m$, and others are different then the C.F. is

$$Z = f_1(y + mx) + xf_2(y + mx) + f_3(y + m_3x) + \ldots\ldots + f_n(y + m_nx)$$

Case 3: If three roots are equal, say $m_1 = m_2 = m_3 = m$, then the C.F. is

$$Z = f_1(y + mx) + xf_2(y + mx) + x^2f_3(y + mx) + \ldots\ldots + f_n(y + m_nx) .$$

To find the Particular Integral:

Rule1: If $F(x, y) = e^{ax+by}$ then

$$\text{P.I.} = \frac{1}{\phi(D, D')} e^{ax+by}$$

$= 1 / \Phi(a, b).\ e^{ax+by}$ provided $\Phi(a, b) \neq 0$ [Replace D by a and D′ by b]

If $\Phi(a, b) = 0$ refer rule 4.

Rule2: If F(x, y) = sin (mx + ny) or cos (mx + ny) then

$$\text{P.I.} = \frac{1}{\phi(D, D')} \sin(mx + ny) \quad or \quad \cos(mx + ny)$$

Replace D^2 by $-m^2$, D'^2 by $-n^2$ and DD' by $-mn$ in provided the denominator is not equal to zero. If the denominator is zero refer rule 4.

Rule3: If $F(x, y) = x^m y^n$

$$\text{P.I.} = \frac{1}{\phi(D, D')} x^m y^n$$

$= [\Phi (D, D')]^{-1} x^m y^n$

Expand $[\Phi (D, D')]^{-1}$ by using binomial theorem and then operate on $x^m y^n$

Note: 1/ D denotes integration w.r.t x, 1/ D′ denotes integration w.r.t y.

Rule4: If F(x, y) is any other function, resolve Φ (D, D′) in to linear factor say $(D - m_1 D')$

$(D - m_2 D')$ etc. then the P.I. = $\frac{1}{(D - m_1 D')(D - m_2 D')} F(x, y)$

Note:1

$$\frac{1}{(D - mD)} F(x, y) = \int F(x, c-mx)\, dx, \text{ where } y = c-mx.$$

Note:2

If the denominator is zero in rule (1) and (2) then apply Rule (4)

1. Solve $(D^2-2DD'+ D'^2)z = 0$

   Solution:

   Given $(D^2-2DD'+ D'^2) z = 0$

   The auxiliary eqn is $m^2-2m+1=0$

   ie) $(m-1)^2 =0$

   m =1,1

   The roots are equal.

   ∴ $C.F = f_1(y+x)+xf_2(y+x)$

   Hence z = C.F

   $z = f_1(y+x)+xf_2(y+x)$.

2. Solve $(D^4-D'^4)z = 0$

Solution:

Given $(D^4-D'^4)\, z = 0$

The auxiliary equation is $m^4-1= 0$

[Replace D by m and D′ by 1]

Solving $(m^2-1)\,(m^2+1) = 0$

$m^2-1=0$, $m^2+1 =0$

$m^2 =1$, $m^2 = -1$

$m =\pm 1$, $m = \pm\sqrt{-1} = \pm\, i$

ie) $m =1,-1,i,-i$

The solution is $z = f_1(y+x)+ f_2(y-x)+f_3(y+ix)+f_4(y-ix)$.

3. Find the P.I of $[D^2 + 4DD']y = e^x$

Solution:

$$\text{P.I} = \frac{1}{D^2 + 4DD'} e^x$$

$$= \frac{1}{D^2 + 4DD'} e^{x+0y}$$

$$= e^x \left[\frac{1}{1+4(1)(0)} \right] \quad \text{Replace D by 1 and } D' \text{ by 0}$$

$$= e^x .$$

Solution is $y = e^x$.

4. Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

Solution:

The symbolic form is $(D^3 - 3D^2 D' + 4D'^3)z = e^{x+2y}$

A.E is $m^3 - 3m^2 + 4 = 0$

$m = -1, 2, 2$

C.F is $z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

$$P.I = \frac{1}{D^3 - 3D^2 D^1 + 4D'^3} e^{x+2y}$$

$$= \frac{1}{1-(3)(1)(2)+(4)(8)} e^{x+2y}$$

$$= \frac{1}{27} e^{x+2y}$$

The complete solution is

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

5. Solve $[D^2 - 2DD' + D'^2]$ z = cos(x-3y).

Solution:

Given $[D^2 - 2DD' + D'^2]z = \cos(x-3y)$.

The auxiliary equation is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

C.F = $f_1(y+x) + x f_2(y+x)$.

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y)$$

$$= \frac{\cos(x-3y)}{-1-2(3)-9}$$

$$= \frac{-1}{16} \cos(x-3y)$$

$\therefore$ The complete solution is $Z = f_1(y+x) + xf_2(y+x) - \frac{1}{16}\cos(x-3y)$.

6. Solve $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

Solution:

The symbolic form is $[D^2 + 3DD' + 2D'^2]z = x + y$

A.E is $m^2 + 3m + 2 = 0$

$m = -1 , -2$

C.F is $z = f_1(y-x) + f_2(y-2x)$

$$\text{P.I} = \frac{1}{D^2 + 3DD' + 2D'^2} x + y$$

$$= \frac{1}{D^2\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]} x + y$$

$$= \frac{1}{D^2}\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]^{-1} x + y$$

$$= \frac{1}{D^2}\left[1 - (\frac{3D'}{D} + \frac{2D'^2}{D^2}) + ...\right] x + y$$

$$= \frac{1}{D^2}\left[1 - \frac{3D'}{D}\right] x + y$$

$$= \frac{1}{D^2}\left[(x + y) - \frac{3D'}{D}(x + y)\right]$$

$$= \frac{1}{D^2}\left[(x+y)-3x\right]$$

$$= \frac{1}{D^2}\left[y-2x\right]$$

$$= \frac{1}{D^2}\left[(x+y)-3x\right]$$

$$= \frac{1}{D^2}\left[y-2x\right]$$

$$= \frac{yx^2}{2} - \frac{x^3}{3}$$

The complete solution is

$$z = f_1(y-x) + f_2(y-2x) + \frac{yx^2}{2} - \frac{x^3}{3}$$

7. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

Solution:

The symbolic form is $[D^2 + DD' - 6D'^2]z = y\cos x$

A.E is $m^2 + m - 6 = 0$

$m = -3 , 2$

C.F is $z = f_1(y-3x) + f_2(y+2x)$

$$\text{P.I} = \frac{1}{D^2 + DD' - 6D'^2} y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')} y\cos x$$

$$= \frac{1}{(D+3D')} \int (c-2x)\cos x \; dx$$

$$= \frac{1}{(D+3D')}\int [\left(c-2x\right)\sin x - \int -2\sin x]\ dx$$

$$= \frac{1}{(D+3D')}[(y+2x-2x)\sin x - 2\cos x]$$

$$= \frac{1}{(D+3D')}[y\sin x - 2\cos x]$$

$$= \int [(c+3x)\sin x - 2\cos x]\ dx$$

$$= (y - 3x + 3x)\cos x + 3\sin x - 2\sin x$$

$$= -y\cos x + \sin x$$

The complete solution is

$$z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x$$

UNIT III
PARTIAL DIFFERENTIAL EQUATIONS
PART A

1. Form the partial differential equation from $z = (x-a)^2 + (y-b)^2 + 1$ by eliminating a and b.
2. Find the partial differential equation by eliminating arbitrary constants a & b . from $z = (x+a)(y+b)$
3. Write the complete integral for the partial differential equation $z = px + qy + pq$.
4. Find the complete solution of $z = px + qy + p^2q^2$
5. Form the p.d.e by eliminating the arbitrary function from $z = f(x^2 + y^2)$.
6. Find the particular integral of $(D^2 - 6DD' + 9D'^2)z = \cos(3x+y)$.
7. Solve $(D^4 - D'^4)z = 0$.
8. Find the particular integral of $(D^2 - DD')z = \sin x \cos 2y$.
9. Find the complete integral of $p + q = pq$.
10. Solve $\left(D^3 - 3DD'^2 + 2D'^3\right)z = 0$.

PART B

1. Find the singular integral of the PDE $z = px + qy + p^2 - q^2$.
2. Form the partial differential equation by eliminating the arbitrary functions 'f ' and 'g' in $z = f(2x + y) + g(3x-y)$
3. Solve $(D^2 + DD' - 2D'^2)z = e^{2x+y} + \sin(x+y)$.
4. Solve : $(x^2-yz)p + (y^2 - zx)q = z^2 - xy$.
5. Solve : $x(y^2 + z^2)p + y(z^2 + x^2)q = z(y^2 - x^2)$.
6. Solve $(D^2 - 6DD' + 5D'^2)z = e^x \sinh y$.
7. Solve $(mz - ny)p + (nx - lz)q = ly - mx$.
8. Solve $(D^2 + 4DD' - 5D'^2)z = x + y^2 + \pi$.
9. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x.$
10. Find the general solution of $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

UNIT IV NUMERICAL INTERPOLATION

SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name :Ancillary Mathematics II

Difference Operators

Newton's Forward Interpolation Formula

Lagrange's Interpolation

Numerical Integration

Subject Name :Ancillary Mathematics II

DIFFERENCE OPERATORS

FORWARD DIFFERENCE OPERATOR (Δ):

$$\Delta f(x) = f(x+h) - f(x)$$

SHIFTING OPERATOR(E):

$$Ef(x)=f(x+h)$$

RELATIONS BETWEEN DIFFERENCE OPERATORS

We note that

$$Ef(x) = f(x+h) \quad = [f(x+h) - f(x)] + f(x) \quad = \Delta f(x) + f(x) = (\Delta + 1)f(x).$$

Thus,

$$\boxed{E \equiv 1 + \Delta} \quad \text{or} \quad \Delta \equiv E - 1.$$

5.2 NEWTON FORWARD INTERPOLATION FORMULA

$$P_N(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x - x_0)(x - x_1) + \cdots + \frac{\Delta^k y_0}{k!\,h^k}(x - x_0)\cdots(x - x_{k-1})$$

$$+\frac{\Delta^N y_0}{N!\,h^N}(x - x_0)\ldots(x - x_{N-1}).$$

Let $u = \frac{x - x_0}{h}$, then

$$x - x_1 = hu + x_0 - (x_0 + h) = h(u-1), x - x_2 = h(u-2), \ldots, x - x_k = h(u-k), \quad \text{etc..}$$

With this transformation the above forward interpolation formula is simplified to the following form:

$$P_N(u) = y_0 + \frac{\Delta y_0}{h}(hu) + \frac{\Delta^2 y_0}{2!\,h^2}\{(hu)(h(u-1))\} + \cdots + \frac{\Delta^k y_0 h^k}{k!\,h^k}[u(u-1)\cdots(u-k+1)]$$

$$+ \cdots + \frac{\Delta^N y_0}{N!\,h^N}\Big[(hu)\big(h(u-1)\big)\cdots\big(h(u-N+1)\big)\Big].$$

$$= y_0 + \Delta y_0(u) + \frac{\Delta^2 y_0}{2!}(u(u-1)) + \cdots + \frac{\Delta^k y_0}{k!}\Big[u(u-1)\cdots(u-k+1)\Big]$$

$$+ \cdots + \frac{\Delta^N y_0}{N!}\Big[u(u-1)\ldots(u-N+1)\Big].$$

If $N = 1$, we have a linear interpolation given by

$$f(u) \approx y_0 + \Delta y_0(u).$$

For $N = 2$, we get a quadratic interpolating polynomial:

$$f(u) \approx y_0 + \Delta y_0(u) + \frac{\Delta^2 y_0}{2!}[u(u-1)]$$

PROBLEM 1

Obtain the Newton's forward interpolating polynomial, $P_5(x)$ for the following tabular data and interpolate the value of the function at $x = 0.0045$.

x	0	0.001	0.002	0.003	0.004	0.005
y	1.121	1.123	1.1255	1.127	1.128	1.1285

Solution: For this data, we have the Forward difference difference table

x_i	y_i	Δy_i	$\Delta^2 y_3$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$	
0	1.121	0.002	0.0005	-0.0015	0.002	-.0025	
.001	1.123	0.0025	-0.0010	0.0005	-0.0005		
.002	1.1255	0.0015	-0.0005	0.0			
.003	1.127	0.001	-0.0005				
.004	1.128	0.0005					
.005	1.1285						

Thus, for $x = x_0 + hu$, where $x_0 = 0$, $h = 0.001$ and $u = \dfrac{x - x_0}{h}$, we get

$$P_5(x) = 1.121 + u \times .002 + \frac{u(u-1)}{2}(.0005) + \frac{u(u-1)(u-2)}{3!} \times (-.0015)$$

$$+\frac{u(u-1)(u-2)(u-3)}{4!}(.002) + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!} \times (-.0025).$$

$$\begin{aligned} P_5(0.0045) &= P_5(0 + 0.001 \times 4.5) \\ &= 1.121 + 0.002 \times 4.5 + \frac{0.0005}{2} \times 4.5 \times 3.5 - \frac{0.0015}{6} \times 4.5 \times 3.5 \times 2.5 \\ &\quad + \frac{0.002}{24} \times 4.5 \times 3.5 \times 2.5 \times 1.5 - \frac{0.0025}{120} \times 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \\ &= 1.12840045. \end{aligned}$$

PROBLEM 2

Using the following table for $\tan x$, approximate its value at 0.71. Also, find an error estimate (Note $\tan(0.71) = 0.85953$).

x_i	0.70	72	0.74	0.76	0.78
$\tan x_i$	0.84229	0.87707	0.91309	0.95045	0.98926

Solution: As the point $x = 0.71$ lies towards the initial tabular values, we shall use Newton's Forward formula. The forward difference table is:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$		
0.70	0.84229	0.03478	0.00124	0.0001	0.00001		
0.72	0.87707	0.03602	0.00134	0.00011			
0.74	0.91309	0.03736	0.00145				
0.76	0.95045	0.03881					
0.78	0.98926						

In the above table, we note that $\Delta^3 y$ is almost constant, so we shall attempt 3^{rd} degree polynomial interpolation.

Note that $x_0 = 0.70,\ h = 0.02$ gives $u = \dfrac{0.71 - 0.70}{0.02} = 0.5.$ Thus, using forward interpolating polynomial of degree 3, we get

$$P_3(u) = 0.84229 + 0.03478u + \frac{0.00124}{2!}u(u-1) + \frac{0.0001}{3!}u(u-1)(u-2).$$

$$\text{Thus,} \qquad \tan(0.71) \approx 0.84229 + 0.03478(0.5) + \frac{0.00124}{2!} \times 0.5 \times (-0.5)$$

$$+\frac{0.0001}{3!} \times 0.5 \times (-0.5) \times (-1.5)$$

$$= 0.859535.$$

An error estimate for the approximate value is

$$\frac{\Delta^4 y_0}{4!} u(u-1)(u-2)(u-3)\bigg|_{u=0.5} = 0.00000039.$$

Note that exact value of $\tan(0.71)$ (upto 5 decimal place) is 0.85953. and the approximate value, obtained using the Newton's interpolating polynomial is very close to this value. This is also reflected by the error estimate given above.

PROBLEM 3

Apply third degree polynomial for the set of values given by to estimate the value of f(10.3) by taking

$$(i)\ \ x_0 = 9.0, \qquad\qquad (ii)\ \ x_0 = 10.0.$$

Also, find approximate value of $f(13.5)$.

Solution: Note that $x = 10.3$ is closer to the values lying in the beginning of tabular values, while $x = 13.5$ is towards the end of tabular values. Therefore, we shall use forward difference formula for $x = 10.3$ and the backward difference formula for $x = 13.5$. Recall that the interpolating polynomial of degree 3 is given by

$$f(x_0 + hu) = y_0 + \Delta y_0 u + \frac{\Delta^2 y_0}{2!} u(u-1) + \frac{\Delta^3 y_0}{3!} u(u-1)(u-2).$$

Therefore,

1. for $x_0 = 9.0$, $h = 1.0$ and $x = 10.3$, we have $u = \frac{10.3 - 9.0}{1} = 1.3$. This gives,

$$f(10.3) \approx 5 + .4 \times 1.3 + \frac{.2}{2!}(1.3) \times .3 + \frac{.0}{3!}(1.3) \times .3 \times (-0.7)$$

$$= 5.559.$$

2. for $x_0 = 10.0,\ h = 1.0$ and $x = 10.3$, we have $u = \frac{10.3 - 10.0}{1} = .3$. This gives,

$$f(10.3) \approx 5.4 + .6 \times .3 + \frac{.2}{2!}(.3) \times (-0.7) + \frac{-0.3}{3!}(.3) \times (-0.7) \times (-1.7)$$
$$= 5.54115.$$

Note: as $x = 10.3$ is closer to $x = 10.0$, we may expect estimate calculated using $x_0 = 10.0$ to be a better approximation.

for $x_0 = 13.5$, we use the backward interpolating polynomial, which gives,

$$f(x_N + hu) \approx y_0 + \nabla y_N u + \frac{\nabla^2 y_N}{2!}u(u+1) + \frac{\Delta^3 y_N}{3!}u(u+1)(u+2).$$

Therefore, taking $x_N = 14,\ h = 1.0$ and $x = 13.5$, we have $u = \frac{13.5 - 14}{1} = -0.5$. This gives,

$$f(13.5) \approx 8.1 + .6 \times (-0.5) + \frac{-0.1}{2!}(-0.5) \times 0.5 + \frac{0.0}{3!}(-0.5) \times 0.5 \times (1.5)$$
$$= 7.8125.$$

5.3 LAGRANGES INTERPOLATION FORMULA

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)\cdots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1)$$
$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n)$$

PROBLEM 1

Using the following data, find by Lagrange's formula, the value of $f(x)$ *at* $x = 10$:

i	*0*	*1*	*2*	*3*	*4*
x_i	*9.3*	*9.6*	*10.2*	*10.4*	*10.8*
$y_i = f(x_i)$	*11.40*	*12.80*	*14.70*	*17.00*	*19.80*

Also find the value of x *where* $f(x) = 16.00$.

SOLUTION:

$$f(10) \approx -0.01792 \times \left[\frac{11.40}{0.7 \times 0.4455} + \frac{12.80}{0.4 \times (-0.1728)} + \frac{14.70}{(-0.2) \times 0.0648} + \frac{17.00}{(-0.4) \times (-0.0704)} + \frac{19.80}{(-0.8) \times 0.4320}\right]$$

$$= 13.197845.$$

Now to find the value of x *such that* $f(x) = 16$, *we interchange the roles of* x *and* y *and calculate the following products:*

5.4 NUMERICAL INTEGRATION

TRAPEZOIDAL RULE FOR INTEGRATION:

UNIT IV
NUMERICAL INTERPOLATION
PART A

1. Using Newton-Raphson's formula find an iterative formula To find $\sqrt{n}$, where n is a positive number.
2. Show that a root of $x^3 - 6x - 13 = 0$ lies between 3 and 4 and give the first approximation by N-R method.
3. Write Newton's Forward difference formula to find dy/dx, d^2y/dx^2, for y = f(x).
4. Write Simpson's 1/3rd rule and Simpson's 3/8th rule to integrate $\int_a^b y(x)dx$

5. Write the order of converge for the Newton Raphson Method.
6. What is the condition for convergence in solving systems of equations using iterative methods?
7. Form the difference table for the sequence 2, 9, 28, 65, 126, 217.
8. State Newton – Raphson formula for iteration.
9. State the condition for convergence of Gauss – Seidel method.
10. State Newton's backward difference interpolation formula.
11. State Trapezoidal rule.
12. State the Criteria for the convergence conditions for Newton's method.
13. Write the Newton's forward Interpolation formula.
14. Define Simpson's 3/8th rule in Numerical Integration.
15. Write an iterative formula for finding a root of an equation by Newton Raphson's method.
16. State Newton's forward formula for first and second derivatives at $x = x_0$.
17. Why is Trapezoidal rule so called?

PART B

1.Solve by Gauss elimination method x – 3y – z = -30, 2x – y – 3z = 5, 5x – y – 2z = 142.

2.. Solve by Gauss Seidal method 3x + y – z - w = 0, x + 3y - z + 2w +3= 0, -2x + 2y + 3z - 2w = 4,
x + 2y + z -5w + 1 = 0.

3.. Find the first and second derivative of the function at x = 15 from the table given below

X:	15	17	19	21	23	25
Y:	3.873	4.123	4.359	4.583	4.796	5.000

4. The velocity of a train which starts from rest is given by the following table time being reckoned in minutes from the start and speed in miles per hour.

Minutes :	2	4	6	8	10	12	14	16	18	20
Miles / hr :	10	18	25	29	32	20	11	5	2	0

5. Find the negative root of $x^3 - 4x + 9 = 0$ by Regula falsi method.
6. Solve 8x - 3y + 2z = 20, 4x + 11y – z = 33, 6x + 3y + 12z =35 by Gauss Seidel Method.
7. From the following table estimate y value at x = 46.

X :	45	50	55	60	65
y :	114.84	96.16	83.32	74.48	68.48

8. Evaluate $\int_0^{\pi} \sin x \, dx$ using Simpsons 1/3rd rule by taking ten equal intervals.

9. Evaluate $\int_4^{5.2} \log_e x \, dx$ by

(a) Trapezoidal rule

(b) Simpson's rule taking h=0.2

10. Find the positive root of $x^3 - x = 1$ correct to four decimal places by N-R method.

11. Solve the following system by Gauss – Jacobi method:

$10x - 5y - 2z = 3; 4x - 10y + 3z = -3; x + 6y + 10z = -3$

12.Using Newton's interpolation formulae find the values of y at $x = 21$ *and* $x = 28$ from the following

X	20	23	26	29
Y	0.3420	0.3907	0.4384	0.4848

13. Evaluate $\int_0^6 \frac{dx}{1+x}$

a) Trapezoidal rule (b) Simpson's rules by dividing the range in to six equal parts.

14. Find a real root of $2x^3 + 3x - 10 = 0$ correct to four decimal place using Newton's method

15. Solve 2x + 3y + 3z = 10 3x – y + 2z = 13, x + 2y + z = 3 by Gauss Jordan method.

16. Find a positive root of $xe^x = \cos x$ correct to 3 decimal places using Newton's method.

17. Solve the equations : $4x - 2y + z = 3$ $3x + 9y - 2z = 10$ $4x + 2y + 13z = 19$ by using Gauss Seidel method.

18. Construct a polynomial for the data given below. Also find the value of y(x = 5)

X	4	6	8	10
Y	1	3	8	16

19. Find $\int_0^1 \frac{dx}{1+x^2}$ by Trapezoidal rule by taking h = 0.25 and hence find the approximate value of π.

20. Find the rate of change of pressure corresponding volume when V = 2 from the following table

V	2	4	6	8	10
P	105	42.7	25.3	16.7	13

21. Find a real root of $2x^3 + 3x - 10 = 0$ correct to four decimal place using Newton's method

22. Find a positive root of $xe^x = \cos x$ correct to 3 decimal places using Newton's method.

23.Solve the equations : $4x - 2y + z = 3;$ $3x + 9y - 2z = 10;$ $4x + 2y + 13z = 19$ by using Gauss Seidel method.

24. Construct a polynomial for the data given below. Also find the value of y(x = 5)

X	4	6	8	10
Y	1	3	8	16

25.Find a root of the equation $x^3 - 4x - 9 = 0$ using the Regula falsi method in four stages. (4)

26. Solve the equation x + 2y + z = 3; 2x + 3y + 3z = 10 and 3x – y + 2z = 13 by using Gauss – Jacobi's method.

27. Find the root of $\cos x - xe^x = 0$ by N-R method correct to three decimal places.

28. Using Gauss Seidel method, solve the system of equations

$$2x - 6y + 8z = 24,\ 5x + 4y - 3z = 2,\ 3x + y + 2z = 16$$

29. Using Newton's forward and backward interpolation formula, find y(43) and y(84) from the following data:

x	40	50	60	70	80	90
y	184	204	226	250	276	304

UNIT V NUMERICAL DIFFERENTIATION AND INTEGRATION

$$\int_a^b f(x)dx = \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \cdots + \frac{h}{2}[y_k + y_{k+1}] + \cdots + \frac{h}{2}[y_{n-2} + y_{n-1}] + \frac{h}{2}[y_{n-1} + y_n]$$

i.e.

$$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_k + \cdots + 2y_{n-1} + y_n]$$

(h/2) [(sum of the first and last ordinates) +
(Sum of the remaining ordinates)]

This is called TRAPEZOIDAL RULE. It is a simple quadrature formula, but is not very accurate.

Remark *An estimate for the error E_1 in numerical integration using the Trapezoidal rule is given by*

$$E_1 = -\frac{b-a}{12}\overline{\Delta^2 y},$$

where $\overline{\Delta^2 y}$ is the average value of the second forward differences.

Recall that in the case of linear function, the second forward differences is zero, hence, the Trapezoidal rule gives exact value of the integral if the integrand is a linear function.

Example 1 Using Trapezoidal rule compute the integral $\int_0^1 e^{x^2}dx$, where the table for the values of $y = e^{x^2}$ is given below:

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.00000	1.01005	1.04081	1.09417	1.17351	1.28402	1.43332	1.63231	1.89648	2.2479	2.71828

Solution: Here, $h = 0.1,\ n = 10,$

$$\frac{y_0 + y_{10}}{2} = \frac{1.0 + 2.71828}{2} = 1.85914,$$

and

$$\sum_{i=1}^{9} y_i = 12.81257.$$

Thus,

$$\int_0^1 e^{x^2}dx = 0.1 \times [1.85914 + 12.81257] = 1.467171$$

Simpson's Rule

If we are given odd number of tabular points, *i.e.* n is even, then we can divide the given integral of integration in even number of sub-intervals $[x_{2k}, x_{2k+2}]$. Note that for each of these sub-intervals, we have the three tabular points $x_{2k}, x_{2k+1}, x_{2k+2}$ and so the integrand is replaced with a quadratic interpolating polynomial.

$$\begin{aligned}\int_a^b f(x)dx &= \frac{h}{3}[(y_0 + y_n) + 4 \times (y_1 + y_3 + \cdots + y_{2k+1} + \cdots + y_{n-1}) \\ &+ 2 \times (y_2 + y_4 + \cdots + y_{2k} + \cdots + y_{n-2})] \\ &= \frac{h}{3}\left[(y_0 + y_n) + 4 \times \left(\sum_{i=1,\ i-odd}^{n-1} y_i\right) + 2 \times \left(\sum_{i=2,\ i-even}^{n-2} y_i\right)\right].\end{aligned}$$

An estimate for the error E_2 in numerical integration using the Simpson's rule

$$E_2 = -\frac{b-a}{180}\overline{\Delta^4 y},$$

Simpson's one third Rule

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

From the above values, we want to find the integration of *$y = f(x)$ with the range x_0 and $x_0 + n^h$*

$\int_{x_0}^{x_n} f(x)\,dx = (h/3)\,[\,(y_0 + y_n) + 2\,(y_2 + y_4 + \quad) + 4\,(y_1 + y_3 + \ \ldots\ )\,]$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

The above equation is called *Simpson's one third rule and* it is applicable only when **number of ordinates must be odd** (no. of pairs).

CALCULATE $\int_0^1 e^{x^2} dx$, by Simpson's rule.

Solution: Here, $h = 0.1,\ n = 10$, thus we have odd number of nodal points. Further,

$$y_0 + y_{10} = 1.0 + 2.71828 = 3.71828, \qquad \sum_{i=1,\ i-odd}^{9} y_i = y_1 + y_3 + y_5 + y_7 + y_9 = 7.26845,$$

and

$$\sum_{i=2,\ i-even}^{8} y_i = y_2 + y_4 + y_6 + y_8 = 5.54412.$$

Thus,

$$\int_0^1 e^{x^2} dx = \frac{0.1}{3} \times [3.71828 + 4 \times 7.268361 + 2 \times 5.54412] = 1.46267733$$

1. Evaluate $_{-3}{}^{3}\, x^4\, dx$ by using Trapezoidal rule. Verify result by actual integration.

Step 1. We are given that $f(x) = x^4$. Interval length $(b - a) = (3 - (-3)) = 6$. So we divide 6 equal intervals with $h = 6/6 = 1.0$ And tabulate the values as below

x	:	-3	-2	-1	0	1	2	3
y	:	81	16	1	0	1	16	81

Step2. Write down the trapezoidal rule and put the respective values in that rule

$_{-3}{}^{3} f(x)\, dx = (h/2) [(y_0 + y_n) + 2 (y_1 + y_2 + y_3 + \ldots + y_{n-1})]$

$= (h/2)$ [(sum of the first and last ordinates) + (Sum of the remaining ordinates)]

$= (1/2) [(81+81) + 2 (16+1+0+1+16)]$

$= 115$

By actual integration $_{-3}{}^{3} f(x)\, dx = {}_{-3}{}^{3} x^4\, dx$

$= [(3^5/5) - (-3^5/5)]$

$= [(243/5) + (243/5)]$

$= 97.5$

Evaluate $_0{}^{1}\, 1/(1+x^2)\, dx$ *by using Trapezoidal rule with* $h = 0.2$

$_{-3}{}^{3} f(x)\, dx = (h/2) [(y_0 + y_n) + 2 (y_1 + y_2 + y_3 + \ldots + y_{n-1})]$

$= (h/2)$ [(sum of the first and last ordinates) + (Sum of the remaining ordinates)]

$= (0.2/2) [(1+0.5) + 2 (0.96154+0.86207+0.73529+0.60976)]$

$= (0.1) [(1.05) + 6.33732]$

$= 0.783732$

Evaluate $_0{}^6$ 1/ (1+x) dx by using Trapezoidal rule .

Step 1. We are given that f(x) = 1/(1+x). Interval length (b –a) = (6 – 0) = 6. So we divide 6 equal intervals with h= 1. And tabulate the values as below

x :	0	1	2	3	4	5	6
y1/(1+x^2):	1	0.5	1/3	1/4	1/5	1/6	1/7

Step2. Write down the trapezoidal rule and put the respective values of y in that rule

$_{-3}{}^3 f(x)\,dx = (h/2)\,[\,(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \ldots + y_{n-1})\,]$

= (h/2) [(sum of the first and last ordinates) + (Sum of the remaining ordinates)]

= (1/2) [(1+1/7) + 2 (0.5+1/3 + 1/4 + 1/5 +1/6)]

= (0.5) [(1.05) + 6.33732]

Evaluate $_4{}^{5.2}$ $\log_e x$ dx by using Trapezoidal rule .

Step 1. We are given that f(x) = $\log_e x$ Interval length (b –a) = (5.2 – 4) = 1.2. So we divide 6 equal intervals with h= 0.2. And tabulate the values as below

x :	4	4.2	4.4	4.6	4.8	5.0	5.2
y :	1.39	1.44	1.48	1.53	1.57	1.61	1.65

Step2. Write down the trapezoidal rule and put the respective values of y in that rule

$_{-3}{}^3 f(x)\,dx = (h/2)\,[\,(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \ldots + y_{n-1})\,]$

= (h/2) [(sum of the first and last ordinates) + (Sum of the remaining ordinates)]

= (0.2/2) [(1.39+1.65) + 2 (1.44 +1.48 + 1.53+ 1.57 +1.61)]

= (0.1) [3.04 + 2(7.63)]

= 1.83

Evaluate $\int_0^\pi \sin x\,dx$ by using Trapezoidal rule, by dividing the range into ten equal parts .

Solution :

Step 1. *We are given that $f(x) = \sin x$ Interval length $(b - a) = (\pi - 0) = \pi$.*
So we divide 10 equal intervals with $h = \pi/10$ (specified in the question itself), and tabulate the values as below

x:	0	π/10	2π/10	3π/10	4π/10	
Y:	0.0	0.3090	0.5878	0.8090	0.9511	
x:	5π/10	6π/10	7π/10	8π/10	9π/10	π
Y:	1.0	0.9511	0.8090	0.5878	0.3090	0

Step2. Write down the trapezoidal rule and put the respective values of y in that rule

$$\int_{-3}^{3} f(x)\,dx = (h/2)\,[\,(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \ldots + y_{n-1})\,]$$

$$= (h/2)\,[\,(\text{sum of the first and last ordinates}) + (\text{Sum of the remaining ordinates})\,]$$

$$=(\pi/20)[(0+0)+2(0.3090+0.5878+0.8090+0.9511+1.0+0.9511+0.8090+0.5878+0.309)]$$

$$= 1.9843$$

2. *Evaluate $_0{}^{1.2}$ $1/(1+x^2)$ dx by using Simpson's one third rule with h = 0.2*

Solution:

Step 1. We are given that f(x) = $1/(1+x^2)$. Interval length (b –a) = (1.2 – 0) = 1.2 . So we divide 6 equal intervals with h= 0.2 And tabulate the values as below

x :	0	0.2	0.4	0.6	0.8	1.0	1.2
$y1/(1+x^2)$:	1	0.9615	0.8621	0.7353	0.6098	0.5000	0.4098

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\,dx = (h/3)\,[\,(y_0 + y_6) + 2\,(y_2 + y_4\,) + 4\,(y_1 + y_3 + y_5\,)\,]$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

= (0.2 /3) [(1+0.4098) + 2 (0.8621 +0.6098) + 4 (0.9615+0.7353+0.5)]

= (0.0667) [(1.4098) +2(1.4719) + 4 (2.1503)]

= (0.0667) [1.4098 + 2.9438 + 8.6012]

=0.8641

Evaluate $\int_{-3}^{3} x^4\,dx$ by using Simpson's one third rule. Verify result by actual integration.

Step 1. We are given that $f(x) = x^4$. Interval length $(b - a) = (3 - (-3)) = 6$. So we divide 6 equal intervals with $h = 6/6 = 1.0$ And tabulate the values as below

x	:	-3	-2	-1	0	1	2	3
y	:	81	16	1	0	1	16	81

Step2. Write down the Simpson's one third rule and put the respective values in that rule

$$\int_{-3}^{3} f(x)\,dx = (h/3)\,[\,(y_0 + y_6) + 2\,(y_2 + y_4) + 4\,(y_1 + y_3 + y_5)\,]$$

$= (h/3)$ [(sum of the first and last ordinates) + + 2 (Sum of remaining even ordinates) +4 (sum of remaining odd ordinates)]

$$= (1/3)\,[\,(81+81) + 2\,(1+1) + 4(16+1+16)\,]$$

$$= 98$$

By actual integration $\int_{-3}^{3} f(x)\,dx = \int_{-3}^{3} x^4\,dx$

$$= [\,(3^5/5) - (-3^5/5)\,]$$

$$= [\,(243/5) + (243/5)\,]$$

$$= 97.5$$

3. *Evaluate $_0{}^6$ 1/ (1+x) dx by using Simpson's one third rule .*

Solution:

Step 1. We are given that f(x) = 1/(1+x). Interval length (b –a) = (6 – 0) = 6. So we divide 6 equal intervals with h= 1. And tabulate the values as below

x :	*0*	*1*	*2*	*3*	*4*	*5*	*6*
y1/(1+x^2):	*1*	*0.5*	*1/3*	*1/4*	*1/5*	*1/6*	*1/7*

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$_{-3}{}^3 f(x)\,dx = (h/3)\,[\,(y_0 + y_6) + 2\,(y_2 + y_4\,) + 4\,(y_1 + y_3 + y_5\,)\,]$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

= (1/3) [(1+1/7) + 2 (1/3 + 1/5) + 4(0.5 +1/4 +1/6)]

= 1.9587

4. *Evaluate* $\int_4^{5.2} \log_e x \, dx$ *by using Simpson's one third rule .*

Solution:

Step 1. *We are given that* $f(x) = \log_e x$ *Interval length* $(b - a) = (5.2 - 4) = 1.2$*. So we divide 6 equal intervals with* $h = 0.2$*. And tabulate the values as below*

x	: 4	4.2	4.4	4.6	4.8	5.0	5.2
y	: 1.39	1.44	1.48	1.53	1.57	1.61	1.65

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$\int_{-3}^{3} f(x)\,dx = (h/3)\,[\,(y_0 + y_6) + 2\,(y_2 + y_4) + 4\,(y_1 + y_3 + y_5)\,]$

$= (h/3)$ *[(sum of the first and last ordinates) +*
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

$= (0.2/3)\,[\,(1.39+1.65) + 2\,(1.48+ 1.57) + 4\,(1.44+ 1.53++1.61)\,]$

$= (0.0667)\,[\,3.04 + 2(3.05)+ 4\,(4.58)\,]$

$= 1.83$

5. *Evaluate* $\int_0^{\pi} \sin x \, dx$ *by using Simpson's one third rule, by dividing the range into ten equal parts .*

Solution :

Step 1. We are given that f(x) = sin x Interval length (b –a) = (π – 0) =π.
So we divide 10 equal intervals with h= π/10 (specified in the question itself), and tabulate the values as below

x:	0	π/10	2π/10	3π/10	4π/10	
Y:	0.0	0.3090	0.5878	0.8090	0.9511	
x:	5π/10	6π/10	7π/10	8π/10	9π/10	π
Y:	1.0	0.9511	0.8090	0.5878	0.3090	0

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$\int_{-3}^{3} f(x)\, dx = (h/3)\, [\, (y_0 + y_{10}) + 2\, (y_2 + y_4 + y_6 + y_8\,) + 4\, (y_1 + y_3 + y_5 + y_7 + y_9\,)\,]$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates

$=(\pi/20)[(0+0)+2(0.5878+0.9511+0.9511+0.5878) + 4(0.3090+0.8090+1+0.8090+0.3090)]$

= 2.0009

Simpson's three-eighth Rule

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……… x_n
y:	y_0	y_1	y_2	y_3 ……… y_n

From the above values, we want to find the integration of $y = f(x)$ *with the range* x_0 *and* $x_0 + h$

$$\int_{x_0}^{x_n} f(x)\,dx = (3h/8)\,[\,(y_0 + y_n) + 2\,(y_3 + y_6 + y_9 + \ldots) + 3\,(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1})\,]$$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three ordinates)
+3 (sum of remaining ordinates)]

The above equation is called Simpson's three-eighths rule which **is applicable** *only when* **n is multiple of** *3 .*

1. *Evaluate* $\int_{-3}^{3} x^4\,dx$ *by using Simpson's three-eighth rule. Verify result by actual integration.*

Step 1. We are given that $f(x) = x^4$. *Interval length* $(b - a) = (3 - (-3)) = 6$. *So we divide 6 equal intervals with* $h = 6/6 = 1.0$ *And tabulate the values as below*

x	:	-3	-2	-1	0	1	2	3
y	:	81	16	1	0	1	16	81

Step2. Write down the Simpson's three-eighth rule and put the respective values in that rule

$$\int_{-3}^{3} f(x)\,dx = (3h/8)\,[\,(y_0 + y_6) + 2\,(y_3) + 3\,(y_1 + y_2 + y_4 + y_5)\,]$$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

$$= (3/8)\,[\,(81+81) + 2\,(0)) + 3(16+1+1+16)\,]$$

$$= 99$$

By actual integration $\int_{-3}^{3} f(x)\,dx = \int_{-3}^{3} x^4\,dx$

$$= [\,(3^5/5) - (-3^5/5)\,]$$

$$=[\,(243/5) + (243/5)]$$

$$= 97.5$$

Evaluate $\int_{0}^{1.2} 1/(1+x^2)\,dx$ *by using Simpson's three-eighth rule with* $h = 0.2$

Step 1. We are given that $f(x) = 1/(1+x^2)$. Interval length $(b - a) = (1 - 0) = 1$. So we divide 6 equal intervals with h= 0.2 and tabulate the values as below

x :	0	0.2	0.4	0.6	0.8	1.0	1.2
$y1/(1+x^2)$:	1	0.9615	0.8621	0.7353	0.6098	0.5000	0.4098

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$\int_{-3}^{3} f(x)\,dx = (3h/8) [(y_0 + y_6) + 2 (y_3) + 3 (y_1 + y_2 + y_4 + y_5)]$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

$= (3 \times 0.2 /3) [(1+0.4098) + 2 (0.7353) + 3 (0.9615+0.8621+0.6098 + 0.5)]$

$= (0.075) [1.4098 + 1.4706 + 3 (2.9334)]$

$= (0.075) [1.4098 + 1.4706 + 8.8002]$

=0.8760

7. *Evaluate $\int_0^6 1/(1+x)\,dx$ by using Simpson's three-eighth rule .*

Solution:

Step 1. We are given that $f(x) = 1/(1+x)$. Interval length $(b - a) = (6 - 0) = 6$. So we divide 6 equal intervals with h= 1. And tabulate the values as below

X :	0	1	2	3	4	5	6
$y = 1/(1+x^2)x$:	1	0.5	1/3	1/4	1/5	1/6	1/7

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\, dx = (3h/8) [(y_0 + y_6) + 2 (y_3) + 3 (y_1 + y_2 + y_4 + y_5)]$

= *(3h/8) [(sum of the first and last ordinates) +*
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

= (3/8) [(1+1/7) + 2 (1/4) + 3 (0.5 + 1/3 + 1/5 +1/6)]

= **1.9661**

8. *Evaluate* $_{4}{}^{5.2} \log_e x\, dx$ *by using Simpson's three-eighth rule .*

Solution:

Step 1. We are given that $f(x) = \log_e x$ *Interval length (b –a) = (5.2 – 4) = 1.2. So we divide 6 equal intervals with h= 0.2. And tabulate the values as below*

x	:	4	4.2	4.4	4.6	4.8	5.0	5.2
y	:	1.39	1.44	1.48	1.53	1.57	1.61	1.65

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\, dx = (3h/8) [(y_0 + y_6) + 2 (y_3) + 3 (y_1 + y_2 + y_4 + y_5)]$

= *(3h/8) [(sum of the first and last ordinates) +*
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

= (3x 0.2 /8) [(1.39+1.65) + 2 (1.53) + 3 (1.44+ 1.48 +1.57++1.61)]

= (0.075) [3.04 + 3.06 + 3 (6.1)]

= 1.83

9. *Evaluate $\int_0^9 x^2\,dx$ by using Simpson's three-eighth rule, by dividing the range into nine equal parts and verify your answer with actual integration.*

Solution :

Step 1. We are given that $f(x) = x^2$ Interval length $(b - a) = (9 - 0) = 9$.

So, we divide 9 equal intervals with h=9 /9 = 1 (specified in the question itself), and tabulate the values as below

X :	0	1	2	3	4
$Y = x^2$:	0	1	4	9	16
x:	5	6	7	8	9
Y:	25	36	49	64	81

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\,dx = (3h/8)\,[\,(y_0 + y_9) + 2\,(y_3 + y_6\,) + 3\,(y_1 + y_2 + y_4 + y_5 + y_7 + y_8\,)\,]$

$= (3h/8)$ [*(sum of the first and last ordinates)* +
+ 2 *(Sum of multiples of three, other than last ordinates)*
+3 *(sum of remaining ordinates)]*

$= (3/8)\,[\,(0 + 81) + 2\,(9 + 36) + 3\,(1 + 4 + 16 + 25 + 49 + 64)\,]$

$= (.375)\,[\,81 + 90 + 477\,]$

$= \mathbf{243}$

By actual integration $_0{}^9 f(x)\,dx = {}_0{}^9 x^2\,dx$

$= [\,(9^3/3) - (0^3/3)\,]$

$= [\,(729 / 3) + 0\,]$

$= \mathbf{243}$

Thus,the required value of x is obtained as:

$$x \approx 217.3248 \times \left[\frac{9.3}{4.6 \times 217.3248} + \frac{9.6}{3.2 \times (-78.204)} + \frac{10.2}{1.3 \times 73.5471} + \frac{10.40}{(-1.0) \times (-151.4688)} + \frac{10.80}{(-3.8) \times 839.664}\right]$$

$$\approx 10.39123.$$

Difference equations	Order	Degree
$\Delta^2 u_x - 5\Delta u_x + 7u_x = 0$	2	1
$y_{x+3} - 7y_{x+1} + 8y_x = \cos x$	3	1
$y(x+3) - y(x+2) + 7y(x+1) + 10y(x) = 0$	3	1
$(E^2 - 5E + 16)\, y_x = e^x$	2	1
$y_x\, y_{x+1}^2 - y_{x+2}\, y_x + 5y_x = x^2 + 7$	2	2

Note: $y_{x+3} - 5y_{x+2} + 7y_{x+1} = x^2$ is of order 2 only since it can be written as $u_{x+2} - 5u_{x+1} + 7u_x = x^2$ where $y_{x+1} = u_x$.

After getting the value of u_x, we can get y_x using $y_x = u_{x-1}$.

Note: In most of the physical situations, the interval of differencing h is unity. Hence, we take $h = 1$ and proceed unless otherwise specifically mentioned.

A *solution* of a difference equation is a function which satisfies the difference equation.

A *general solution* of a difference equation of order n is a solution which contains n arbitrary constants or n arbitrary functions which are periodic of period equal to the interval of differencing.

A *particular solution* of a difference equation is a solution got from the general solution by giving particular values to the arbitrary constants. For example, $y_x = A \cdot 3^x + B(-3)^x$ is the general solution of

$$y_{x+2} - 9y_x = 0 \quad \text{...(5)}$$

while $y_x = 3^x$ or $y_x = (-3)^x$ or $y_x = 2 \cdot 3^x + 5 \cdot (-3)^x$ are particular solutions of (5).

Linear Difference Equation:

An equation of the form

$$a_0 y_{x+n} + a_1 y_{x+n-1} + a_2 y_{x+n-2} + \cdots + a_{n-1} y_{x+1} + a_n y_x = \phi(x) \quad \text{...(1)}$$

i.e., $$(a_0 E^n + a_1 E^{n-1} + a_2 E^{n-2} + \cdots + a_n) y_x = \phi(x) \quad \text{...(2)}$$

where $a_0, a_1, a_2, \cdots a_n$ and $\phi(x)$ are known functions of x is called a linear difference equation in y_x. In a linear difference equation.

Let $f(E)\, y_x = \phi(x)$ be the linear equation. ...(1)

Write down the auxiliary equation $f(a) = 0$ and get the roots $a_1, a_2, \cdots a_n$.

Case 1. If the roots $a_1, a_2, \cdots a_n$ are all real and distinct the corresponding complementary function of (1) or complete solution of $f(E)\, y_x = 0$ is

$$y_x = c_1 a_1^x + c_2 a_2^x + \cdots + c_n a_n^x.$$

Case 2. If $a_1 = a_2$, the corresponding C.F. is

$$(c_1 + c_2 x) a_1^x + c_3 a_3^x + \cdots + c_n a_n^x.$$

Case 3. If $a_1 = \alpha + i\beta$, $a_2 = \alpha - i\beta$, and $r = |\alpha + i\beta| = \sqrt{\alpha^2 + \beta^2}$

$$\theta = \text{amp.}\,(\alpha + i\beta) = \tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

Complementary function is

$$r^x (c_1 \cos\theta x + c_2 \sin\theta x) + c_3 a_3^x + \cdots + c_n a_n^x.$$

Example 1:

Form the difference equation given $y_n=(An+B)3^n$

$$y_{n+1} = (An + A + B)\,3^{n+1}$$

$$y_{n+2} = (An + 2A + B)\,3^{n+2}$$

$$y_n = (An + B)\,3^n$$

$$\frac{1}{3}y_{n+1} = (An + A + B)\,3^n$$

$$\frac{1}{9}y_{n+2} = (An + 2A + B)\,3^n$$

(4) + (6) − 2 (·5) gives,

$$y_n + \frac{1}{9}y_{n+2} - \frac{2}{3}y_{n+1} = 0$$

$$y_{n+2} - 6\,y_{n+1} + 9y_n = 0.$$

Example 2:

Solve $y_{x+3} - 2y_{x+2} - y_{x+1} + 2y_x = 0$

Solution. Writing in the shift operator form,

$$(E^3 - 2E^2 - E + 2)\, y_x = 0$$

The auxiliary equation is $a^3 - 2a^2 - a + 2 = 0$

$$(a-1)(a+1)(a-2) = 0$$

$$\therefore \quad a = 1, -1, 2$$

Since the equation is homogeneous equation, the complete solution is

$$y_x = A \cdot 1^x + B(-1)^x + C(2)^x$$
$$= A + B(-1)^x + C \cdot 2^x.$$

Example 3:

Solve $y_{x+2} - y_{x+1} + y_x = 0$

given $y_0 = 1, y_1 = \frac{\sqrt{3}+1}{2}$

Solution. The given equation becomes,

$$(E^2 - E + 1)\, y_x = 0$$

The auxiliary equation is $a^2 - a + 1 = 0$

$$a = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2} i = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \therefore\ r = 1, \theta = \pi/3$$

$$\left[\text{or, } r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1,\ \ \theta = \tan^{-1}\sqrt{3} = \pi/3\right]$$

$$\therefore \qquad y_x = 1^x\left(A\cos\frac{\pi}{3}x + B\sin\frac{\pi}{3}x\right)$$

$$y_x = A\cos\frac{\pi}{3}x + B\sin\frac{\pi}{3}x.$$

Since $y_0 = 1,\ A = 1$

$$y_1 = 1\cos\frac{\pi}{3} + B\sin\frac{\pi}{3} = \frac{\sqrt{3}+1}{2}$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}B = \frac{\sqrt{3}+1}{2}$$

$$\therefore \qquad B = 1$$

Hence, the solution is

$$y_x = \cos\frac{\pi x}{3} + \sin\frac{\pi x}{3}$$

Example 4:

Solve $y_{n+2} - 4y_{n+1} + 4y_n = 2^n + 3^n + \pi.$

Solution. The equation reduces to

$$(E^2 - 4E + 4)\, y_n = 2^n + 3^n + \pi$$

Auxiliary equation is $a^2 - 4a + 4 = 0$

$\therefore \qquad a = 2, 2$

$\therefore$ C.P. is $(A + Bn) \cdot 2^n$

$$(PI)_1 = \frac{2^n}{(E-2)^2} = \frac{n^{(2)}}{2!}\, 2^{n-2} = n\,(n-1)\, 2^{n-3}$$

$$(PI)_2 = \frac{3^n}{(E-2)^2} = \frac{3^n}{1}, \text{ replacing } E \text{ by } 3$$

$$(PI)_3 = \frac{\pi \cdot 1^n}{(E-2)^2} = \frac{\pi}{(1-2)^2} = \pi$$

Hence, the complete solution is

$$y_n = (A + Bn)\, 2^n + n\,(n-1) \cdot 2^{n-3} + 3^n + \pi.$$

Example 5:

Solve $y_{x+2} - 4y_x = 9x^2$

Solution. This equation becomes,

$$(E^2 - 4)\, y_x = 9x^2$$

Auxiliary equation is $a^2 - 4 = 0$

$\therefore \quad a = \pm 2$

C.F. is $A2^x + B(-2)^x$

$$\text{P.I.} = \frac{9x^2}{E^2 - 4}$$

$$= \frac{9x^2}{(1+\Delta)^2 - 4} = \frac{9\,(x^2)}{\Delta^2 + 2\Delta - 3}$$

$$= -\frac{1}{3}\frac{9x^2}{1-\frac{(\Delta^2+2\Delta)}{3}}$$

$$= -3\left[1-\frac{\Delta^2+2\Delta}{3}\right]^{-1} x^2$$

$$= -3\left[1+\frac{\Delta^2+2\Delta}{3}+\left(\frac{\Delta^2+2\Delta}{3}\right)^2+\cdots\right]x^2$$

$$= -3\left[1+\frac{2\Delta}{3}+\frac{7}{9}\Delta^2+\cdots\right](x^{(2)}+x^{(1)})$$

$$= -3\left[x^{(2)}+x^{(1)}+\frac{2}{3}(2x^{(1)}+1)+\frac{7}{9}(2)\right]$$

$$= -3\left[x(x-1)+x+\frac{2}{3}(2x+1)+\frac{14}{9}\right]$$

$$= -3\left[x(x-1) + x + \frac{2}{3}(2x+1) + \frac{14}{9}\right]$$

$$= -3\left[x^2 + \frac{4x}{3} + \frac{20}{9}\right]$$

Hence the complete solution is

$$y_x = A \cdot 2^x + B(-2)^x - 3\left(x^2 + \frac{4x}{3} + \frac{20}{9}\right).$$

Aliter: To find the P.I., assume the particular integral

$$y_x = ax^2 + bx + c$$

Substituting in (1),

$$a(x+2)^2 + b(x+2) + c - 4(ax^2 + bx + c) = 9x^2$$

$$-3ax^2 + x(4a - 3b) + 4a + 2b - 3c = 9x^2$$

Equating the coefficients,

$$-3a = 9;\ 4a - 3b = 0,\ 4a + 2b - 3c = 0$$

$$\therefore\ a = -3;\ b = -4,\ c = \frac{20}{-3}$$

$$\therefore\ \text{P.I. is } -3x^2 - 4x - \frac{20}{3}.$$

Example 6:

Solve: $y_{x+2} - 7y_{x+1} - 8y_x = x(x-1)\,2^x$

Solution. The given equation can be written as

$$(E^2 - 7E - 8)\, y_x = x(x-1) \cdot 2^x$$

Auxiliary equation is $a^2 - 7a - 8 = 0$

$$(a-8)(a+1) = 0$$

$$\therefore \qquad a = 8, -1$$

C.F. is $A \cdot 8^x + B(-1)^x$

$$\text{P.I.} = \frac{x(x-1) \cdot 2^x}{E^2 - 7E - 8}$$

$$= 2^x \cdot \frac{x(x-1)}{(2E)^2 - 7(2E) - 8}$$

$$= 2^x \cdot \frac{x(x-1)}{4E^2 - 14E - 8}$$

$$= 2^x \frac{x(x-1)}{4(1+\Delta)^2 - 14(1+\Delta) - 8}$$

$$= 2^x \frac{x(x-1)}{4\Delta^2 - 6\Delta - 18}$$

$$= 2^{x-1} \frac{x(x-1)}{2\Delta^2 - 3\Delta - 9}$$

$$= \frac{2^{x-1}}{-9}\left(1 + \frac{2\Delta^2 - 3\Delta}{-9}\right)^{-1} x(x-1)$$

$$= -\frac{1}{9} \cdot 2^{x-1}\left[1 + \left(\frac{2\Delta^2 - 3\Delta}{9}\right) + \left(\frac{2\Delta^2 - 3\Delta}{9}\right)^2 + \cdots\right] x^{(2)}$$

$$= -\frac{1}{9} \cdot 2^{x-1}\left[1 - \frac{\Delta}{3} + \frac{1}{3}\Delta^2 + \cdots\right] x^{(2)}$$

$$= -\frac{1}{9} \cdot 2^{x-1}\left[x^{(2)} - \frac{2}{3}x^{(1)} + \frac{1}{3}(2)\right]$$

$$= -\frac{1}{9} \cdot 2^{x-1}\left[x(x-1) - \frac{2x}{3} + \frac{2}{3}\right]$$

Hence, the solution is,

$$y_x = A \cdot 8^x + B \cdot (-1)^x - \frac{1}{9} \cdot 2^{x-1}\left(x^2 - \frac{5x}{3} + \frac{2}{3}\right).$$

UNIT-V
NUMERICAL DIFFERENTIATION AND INTEGRATION

PART-A

1.Write down the Crank-Nicholson difference formula.

2.Write down the Explicit scheme to solve one dimensional wave Equation.

3.Define a difference quotient.

4.Write down standard five point formula in solving Laplace equation over a region.

5.Classify the following partial differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$.

6. Write down the difference scheme for poisson equation.

7.Express u_{xx} and u_{yy} in terms of difference quotient.

8.Classify the partial differential equation $xf_{xx} + yf_{yy} = 0$,x>0 ,y>0.

9.Write down diagonal five point formula in solving laplace equation over a region.

10.Write Bender-Schmidt recurrence equation.

PART-B

11.Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units satisfying the following boundary conditions.

(i) u(0,y)=0 for 0≤y≤4
(ii) u(4,y)=12+y for 0≤y≤4
(iii) u(x,0)=3x for 0≤x≤4
(iv) u(x,4)= x^2 for 0≤x≤4

12a) Solve $25u_{xx} - u_{tt} = 0$ for u at the pivotal points given u(0,t)=u(5,t)=0

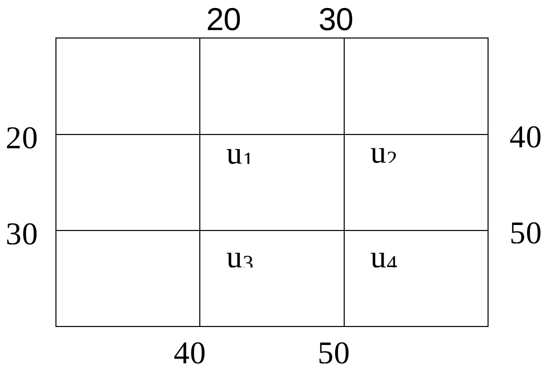
$u_t(x,0) = 0$ and u(x,0)= $\begin{cases} 2x, & 0 \le x \le 2.5 \\ 10 - 2x, & 2.5 \le x \le 5 \end{cases}$

for one half period of vibration.

b) Using Bender-Schmidt method ,solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given u(0,t)=u(4,t)=0,u(x,0)=x(4-x).Assume h=1.Find the values of ‘u’ upto t=5.

3. Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides x=0, y=0, x=3, y=3 with u=0 on the boundary and mesh length 1 unit.

4.Solve the laplace equation $\nabla^2 u = 0$ at the interior points of the square mesh given below.



5.Apply Crank Nicholson method with h=0.2 and λ=1 and find u(x,t) in the rod by considering two time steps of the heat equation $u_{xx} = u_t$ given u(x,0)=sinπx, u(0,t)=0, u(1,t)=0.

6.Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the nodal points for the following square region given the boundary conditions.