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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT - I - MATRICES - SMT1113

1. Introduction

1.1 CHARACTERISTIC EQUATION:

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

   If X_r be the non-zero vector satisfying AX = λX, when $\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

1.2 CHARACTERISTIC POLYNOMIAL:

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A

1.3 Working rule to find characteristic equation:

1.3.1 For a 3 x 3 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements$,

$S_3 = Determinant of A = \ |A|$

1.3.2 For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements$, $S_2 = Determinant\ of A = \ |A|$

Problems:

1. **Find the characteristic equation of the matrix** $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}$

Solution: Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$where $S_1 =$

$sumofthemaindiagonalelements = 1 + 2 = 3,$

$S_2 = Determinant of A = \ |A|$ = 1(2) – 2(0) = 2

Therefore, the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

2. **Find the characteristic equation of** $\begin{pmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{pmatrix}$

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 = sumofthemaindiagonalelements$ = 8 + 7 + 3 = 18,

$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 +$

$20 + 20 = 45$, $S_3 = Determinant of A = \ |A|$ = 8(5)+6(-10)+2(10) = 40 -60 + 20 = 0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

3. **Find the characteristic polynomial of** $\begin{pmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$

Solution: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic polynomial of A is $\lambda^2 - S_1\lambda + S_2$ where $S_1 = sumofthemaindiagonalelements$ = 3 + 2 = 5 and $S_2 = DeterminantofA = |A|$ = 3(2) – 1(-1) = 7

Therefore, the characteristic polynomial is $\lambda^2 - 5\lambda + 7$

2.1 CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation

2.2 Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A
(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ satisfies its own characteristic equation**

Solution:Let A = $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =\ 1 + 1 = 2$

$S_2 = \ |A| = 1 - (-4) = 5$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Therefore, the given matrix satisfies its own characteristic equation

2. **If A = $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} \end{bmatrix}$ $\boldsymbol{write\ A^2\ interms\ of\ A\ and\ I, using\ Cayley - Hamilton\ theorem}$**

Solution:Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 6$

$S_2 = \ |A| = 5$

Therefore, the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$

By Cayley-Hamilton theorem, $A^2 - 6A + 5I = 0$

i.e., $A^2 = 6A - 5I$

3. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = \left|A\right| = 2(4-1) + 1(-2+1) + 2(1-2) = 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

$(1) \Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ------------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by $A^{-1}, A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

4. **Verify that A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4**

Solution:Given A =$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = \text{Sum of the main diagonal elements} = 0$

$S_2 = |A| = -1 - 4 = -5$

Therefore, the characteristic equation is $\lambda^2 - 0\lambda - 5 = 0$ i.e., $\lambda^2 - 5 = 0$

To prove: $A^2 - 5I = 0$---------- (1)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

To find A^4:

From (1), we get, $A^2 - 5I = 0 \Rightarrow A^2 = 5I$

Multiplying by A^2 on both sides, we get, $A^4 = A^2(5I) = 5A^2 = 5\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

5. **Find A^{-1} if A = $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{4} \\ \mathbf{3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$, using Cayley-Hamilton theorem**

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 - 1 = 2$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (-2+1) + (-1-8) + (2+3)$
$= -1 - 9 + 5 = -5$

$S_3 = |A| = 1(-2+1) + 1(-3+2) + 4(3-4) = -1 - 1 - 4 = -6$

The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley- Hamilton theorem, $A^3 - 2A^2 - 5A + 6I = 0$ ------------ (1)

To find A^{-1}:

Multiplying (1) by A^{-1}, we get, $A^2 - 2A - 5A^{-1}A + 6A^{-1}I = 0 \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = -A^2 + 2A + 5I \Rightarrow A^{-1} = \frac{1}{6}(-A^2 + 2A + 5I)$ ------------- (2)

$$A^2 = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2), $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

6. **If A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$, $\boldsymbol{find\ A^n\ in\ terms\ of\ A}$**

Solution:The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 = 3$

$S_2 = |A| = 2 - 0 = 2$

The characteristic equation of A is $\lambda^2 - 3\lambda + 2 = 0$ i.e., $\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm 1}{2} = 2,1$

To find A^n:

When λ^n is divided by $\lambda^2 - 3\lambda + 2$, let the quotient be $Q(\lambda)$ and the remainder be $a\lambda + b$

$\lambda^n = (\lambda^2 - 3\lambda + 2)Q(\lambda) + a\lambda + b$ ---------- (1)

When $\lambda = 1, 1^n = a + b$

When $\lambda = 2, 2^n = 2a + b$

$2a + b = 2^n$ ------------ (2)

$a + b = 1^n$ ------------ (3)

Solving (2) and (3), we get, (2) - (3)⇒ $a = 2^n - 1^n$

$$(2) - 2 \times (3) \Rightarrow b = -2^n + 2(1)^n$$

$$\text{i.e., } a = 2^n - 1^n$$

$$b = 2(1)^n - 2^n$$

Since $A^2 - 3A + 2I = 0$ by Cayley-Hamilton theorem, (1) ⇒ $A^n = aA + bI$

$$A^n = (2^n - 1^n)\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} + [2(1)^n - 2^n]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Use Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ to express as a linear polynomial in A (i) $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ (ii) $A^4 - 4A^3 - 5A^2 + A + 2I$

Solution: Given A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

By Cayley-Hamilton theorem, we get, $A^2 - 4A - 5I = 0$ ------------ (1)

$\lambda^3 - 2\lambda + 3$

$\lambda^2 - 4\lambda - 5 \,\big)\, \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$

$\lambda^5 - 4\lambda^4 - 5\lambda^3$

$-2\lambda^3 + 11\lambda^2 - \lambda$

(-)$-2\lambda^3 + 8\lambda^2 + 10\lambda$

$3\lambda^2 - 11\lambda - 10$

$(-)\ 3\lambda^2 - 12\lambda - 15$

$\lambda + 5$

$$\mathrm{A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I = 0 + A + 5I}$$

$= A + 5I$ (by (1)) which is a linear polynomial in A

(i)

λ^2

$\lambda^2 - 4\lambda - 5 \,\big)\, \lambda^4 - 4\lambda^3 - 5\lambda^2 + \lambda + 2$

$\lambda^4 - 4\lambda^3 - 5\lambda^2$

(-) $\lambda + 2$

$A^4 - 4A^3 - 5A^2 + A + 2I$ **=** $A^2(A^2 - 4A - 5I) + A + 2I$ = 0 + A + 2I = A +2I (by (1)) which is a linear polynomial in A

8. **Using Cayley-Hamilton theorem, find A^{-1} when A =** $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 + 1 = 3$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (1-1) + (1-3) + (1-0)$
$= 0 - 2 + 1 = -1$

$S_3 = |A| = 1(1-1) + 0(2+1) + 3(-2-1) = 1(0) + 0 - 9 = -9$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

Pre-multiplying by A^{-1}, we get, $A^2 - 3A - I + 9A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{9}(-A^2 + 3A + I)$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-A^2 = \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix}; 3A = \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

9. **Verify Cayley-Hamilton theorem for the matrix** $A = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

Solution: Given A = $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The Characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = Sum of the main diagonal elements = 1+2+1 = 4

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (2-6) + (1-7) + (2-12)$
$= \quad -4 - 6 - 10 = -20$

$S_3 = |A| = 1(2-6) - 3(4-3) + 7(8-2) = -4 - 3 + 42 = 35$

The characteristic equation is $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

To prove that: $A^3 - 4A^2 - 20A - 35I = 0$

$$A^2 = \begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix} = \begin{bmatrix}1+12+7 & 3+6+14 & 7+9+7\\4+8+3 & 12+4+6 & 28+6+3\\1+8+1 & 3+4+2 & 7+6+1\end{bmatrix} = \begin{bmatrix}20 & 23 & 23\\15 & 22 & 37\\10 & 9 & 14\end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix}20 & 23 & 23\\15 & 22 & 37\\10 & 9 & 14\end{bmatrix}\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix} = \begin{bmatrix}20+92+23 & 60+46+46 & 140+69+23\\15+88+37 & 45+44+74 & 105+66+37\\10+36+14 & 30+18+28 & 70+27+14\end{bmatrix}$$

$$= \begin{bmatrix}135 & 152 & 232\\140 & 163 & 208\\60 & 76 & 111\end{bmatrix}$$

$$A^3 - 4A^2 - 20A - 35I = \begin{bmatrix}135 & 152 & 232\\140 & 163 & 208\\60 & 76 & 111\end{bmatrix} - 4\begin{bmatrix}20 & 23 & 23\\15 & 22 & 37\\10 & 9 & 14\end{bmatrix} - 20\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix} - 35\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}135 & 152 & 232\\140 & 163 & 208\\60 & 76 & 111\end{bmatrix} - \begin{bmatrix}80 & 92 & 92\\60 & 88 & 148\\40 & 36 & 56\end{bmatrix} - \begin{bmatrix}20 & 60 & 140\\80 & 40 & 60\\20 & 40 & 20\end{bmatrix} - \begin{bmatrix}35 & 0 & 0\\0 & 35 & 0\\0 & 0 & 35\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = 0$$

Therefore, Cayley-Hamilton theorem is verified.

10. Verify Cayley-Hamilton theorem for the matrix (i) A = $\begin{bmatrix}\mathbf{3} & \mathbf{-1}\\\mathbf{-1} & \mathbf{5}\end{bmatrix}$ $(\boldsymbol{ii})\boldsymbol{A} = \begin{bmatrix}\mathbf{1} & \mathbf{4}\\\mathbf{2} & \mathbf{3}\end{bmatrix}$

Solution:(i) Given A = $\begin{bmatrix}3 & -1\\-1 & 5\end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 3 + 5 = 8$

$S_2 = |A| = 15 - 1 = 14$

The characteristic equation is $\lambda^2 - 8\lambda + 14 = 0$

To prove that: $A^2 - 8A + 14I = 0$

$$A^2 = \begin{bmatrix}3 & -1\\-1 & 5\end{bmatrix}\begin{bmatrix}3 & -1\\-1 & 5\end{bmatrix} = \begin{bmatrix}9+1 & -3-5\\-3-5 & 1+25\end{bmatrix} = \begin{bmatrix}10 & -8\\-8 & 26\end{bmatrix}$$

$8A = 8\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix}$

$14I = 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

$A^2 - 8A + 14I = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix} - \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence Cayley-Hamilton theorem is verified.

(ii) Given A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

To prove that: $\boldsymbol{A^2 - 4A - 5I = 0}$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$4A = 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix}; 5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence Cayley-Hamilton theorem is verified.

3.EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX:

3.1 Working rule to find eigen values and eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called eigen values
3. To find the eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct eigen values, we get n independent eigen vectors
2. If 2 or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values

3. If X_i is a solution for an eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique but may be any one of the vectors cX_i
4. Algebraic multiplicity of an eigen value λ is the order of the eigen value as a root of the characteristic polynomial (i.e., if λ is a double root, then algebraic multiplicity is 2)
5. Geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ

3.2 Non-symmetric matrix:

If a square matrix A is non-symmetric, then A ≠ A^T

Note:

1. In a non-symmetric matrix, if the eigen values are non-repeated then we get a linearly independent set of eigen vectors
2. In a non-symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent eigen vectors.
   If we form a linearly independent set of eigen vectors, then diagonalization is possible through similarity transformation

3.3 Symmetric matrix:

If a square matrix A is symmetric, then A = A^T

Note:

1. In a symmetric matrix, if the eigen values are non-repeated, then we get a linearly independent and pair wise orthogonal set of eigen vectors
2. In a symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent and pair wise orthogonal set of eigen vectors
   If we form a linearly independent and pair wise orthogonal set of eigen vectors, then diagonalization is possible through orthogonal transformation

Problems:

1. **Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = sumofthemaindiagonalelements = 1 - 1 = 0,$

$S_2 = DeterminantofA = |A|$ = 1(-1) – 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

A is a non-symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$**--------------- (1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [**From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$

$$3x_1 + x_2 = 0$$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [**From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$$

i.e., we get only one equation $x_1 - x_2 = 0$

$$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. **Find the eigen values and eigen vectors of** $\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sumofthemaindiagonalelements = 2 + 3 + 2 = 7,$

$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 =$ $11,$

$S_3 = DeterminantofA = |A|$ = 2(4)-2(1)+1(-1) = 5

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & 0 \end{array}$$

$(\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda = 1,$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \frac{6+4}{2}, \frac{6-4}{2} = 5, 1$$

Therefore, the eigen values are 1, 1, and 5

A is a non-symmetric matrix with repeated eigen values

<u>To find the eigen vectors:</u>

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 5$, $\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

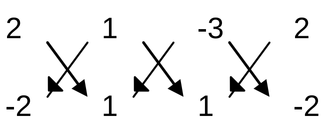
$\Rightarrow -3x_1 + 2x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + 2x_2 - 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$



$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 1$, $\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

All the three equations are one and the same. Therefore, $x_1 + 2x_2 + x_3 = 0$

Put $x_1 = 0 \Rightarrow 2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3 . Taking x_3 = 2 , x_2 = -1$

Therefore, $X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Put $x_2 = 0 \Rightarrow x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 . Taking x_1 = 1, x_3 = -1$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

3. Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 1 = 2,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} =$

$-4 - 4 + 4 = -4,$

$S_3 = Determinant of A = |A|$ = 2(-4)+2(-2)+2(2) = - 8 – 4 + 4 = - 8

Therefore, the characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

2	1	-2	-4	8

0 2 0 -8

1 0 -4 0

$$(\lambda-2)(\lambda^2-4)=0 \Rightarrow \lambda=2, \quad \lambda=\ 2,-2$$

Therefore, the eigen values are 2, 2, and -2

A is a non-symmetric matrix with repeated eigen values

To find the eigen vectors:

$[A-\lambda I]X=0$

$$\begin{bmatrix}2-\lambda & -2 & 2\\ 1 & 1-\lambda & 1\\ 1 & 3 & -1-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Case 1: If $\lambda=-2$, $\begin{bmatrix}2-(-2) & -2 & 2\\ 1 & 1-(-2) & 1\\ 1 & 3 & -1-(-2)\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

i.e., $\begin{bmatrix}4 & -2 & 2\\ 1 & 3 & 1\\ 1 & 3 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow 4x_1-2x_2+2x_3=0$ --------- (1)

$x_1+3x_2+x_3=0$ ------------- (2)

$x_1+3x_2+x_3=0$ ------------ (3) . Equations (2) and (3) are one and the same.

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$x_1x_2x_3$

-1 1 2 -1

3 1 1 3

$$\Rightarrow \frac{x_1}{-4}=\frac{x_2}{-1}=\frac{x_3}{7} \Rightarrow \frac{x_1}{4}=\frac{x_2}{1}=\frac{x_3}{-7}$$

Therefore, $X_1=\begin{bmatrix}4\\1\\-7\end{bmatrix}$

Case 2: If $\boldsymbol{\lambda = 2}$**,** $\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0x_1 - 2x_2 + 2x_3 = 0$---------- (1)

$x_1 - x_2 + x_3 = 0$---------------- (2)

$x_1 + 3x_2 - 3x_3 = 0$------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$x_1 x_2 x_3$

-2	2	0	-2
-1	1	1	-1

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We get one eigen vector corresponding to the repeated root $\lambda_2 = \lambda_3 = 2$

4. **Find the eigen values and eigen vectors of** $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 1 + 5 + 1 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0,$

$S_3 = Determinant of A =$ $|A|$ = 1(4)-1(-2)+3(-14) = - 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

A is a symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 7x_2 + x_3 = 0$ ------------- (2)

$3x_1 + x_2 + 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1 3 3 1

7 1 1 7

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{0} = \frac{x_3}{4} = \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case 2: If$\lambda = 3$, $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 + 2x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1 3 -2 1

2 1 1 2

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1} = \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If$\lambda = 6$, $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 - x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

1 3 -5 1

-1 1 1 -1

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

5. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Determine the algebraic and geometric multiplicity

Solution: Let A $= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$

$-1 - 1 - 1 = -3,$

$S_3 = Determinant\ of\ A = |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

-1	1	0	-3	-2
	0	-1	1	2
	1	-1	-2	0

$(\lambda-(-1))(\lambda^2-\lambda-2)=0 \Rightarrow \lambda=-1,$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2-4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

A is a symmetric matrix with repeated eigen values. The algebraic multiplicity of $\lambda = -1$ is 2

To find the eigen vectors:

$[A-\lambda I]X = 0$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\boldsymbol{\lambda = 2}$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$x_1 \quad x_2 \quad x_3$

1	1	-2	1
-2	1	1	-2

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\boldsymbol{\lambda = -1}$**,** $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ---------------- (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 is orthogonal to X_1 *and* X_2 .

$[1 \quad 1 \quad 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$[0 \quad 1 \quad -1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$-------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

l m n

1 1 1 1

1 -1 0 1

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1$, there corresponds two linearly independent eigen vectors X_2 and X_3. So, the geometric multiplicity of eigen value $\lambda = -1$ is 2

Problems under properties of eigen values and eigen vectors.

1. **Find the sum and product of the eigen values of the matrix $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$**

Solution: Sum of the eigen values = Sum of the main diagonal elements = -3

Product of the eigen values = |A| = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Product of two eigen values of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigen value**

Solution: Let the eigen values of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2 = 16$

We know that $\lambda_1\lambda_2\lambda_3 = |A|$ (Since product of the eigen values is equal to the determinant of the matrix)

$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ = 6(9-1)+2(-6+2) +2(2-6) = 48-8-8 = 32

$$There\ fore, \lambda_1\lambda_2\lambda_3 = 32 \Rightarrow 16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

3. **Find the sum and product of the eigen values of the matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ without finding the roots of the characteristic equation**

Solution:We know that the sum of the eigen values = Trace of A = a + d

Product of the eigen values = |A| = ad – bc

4. If 3 and 15 are the two eigen values of A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find | A | , without expanding the determinant

Solution:Given $\lambda_1 = 3 \, and \, \lambda_2 = 15, \lambda_3 = ?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$$

$We \, know \, that \, the \, product \, of \, the \, eigen \, values = |A|$

$\Rightarrow (3)(15)(0) = |A|$

$\Rightarrow |A| = 0$

5. If 2, 2, 3 are the eigen values of A = $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the eigen values of A^T

Solution:By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of A^T are 2,2,3

6. Find the eigen values of A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ .Clearly, A is a lower triangular matrix. Hence, by the property "the characteristic roots of a triangular matrix are just the diagonal elements of the matrix", the eigen values of A are 2, 3, 4

7. Two of the eigen values of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1}

Solution:Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, 3 + 6 + k = 11 ⇒ $k = 2$

Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

8. **Find the eigen values of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-5} & \mathbf{4} \end{bmatrix}$. Hence, form the matrix whose eigen values are $\frac{1}{6}$ $and - 1$**

Solution: Let A = $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. The characteristic equation of the given matrix is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =$ 5 and $S_2 = |A| = -6$

Therefore, the characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm 7}{2} = 6, -1$

Therefore, the eigen values of A are 6, -1

Hence, the matrix whose eigen values are $\frac{1}{6}$ $and - 1$ is A^{-1}

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

|A| = 4 -10 = - 6; adj A = $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

9. **Find the eigen values of the inverse of the matrix A = $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$**

Solution:We know that A is an upper triangular matrix. Therefore, the eigen values of A are 2, 3, 4. Hence, by using the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

10. **Find the eigen values of A^3 given A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} & \mathbf{-7} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$**

Solution:Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 1, 2, 3

Therefore, the eigen values of A^3 are $1^3, 2^3, 3^3$ i.e., 1,8,27

11. If 1 and 2 are the eigen values of a 2 x 2 matrix A, what are the eigen values of A^2 and A^{-1}?

Solution:Given 1 and 2 are the eigen values of A.

Therefore, 1^2 *and* 2^2 i.e., 1 and 4 are the eigen values of A^2 and 1 and $\frac{1}{2}$ are the eigen values of A^{-1}

12. If 1,1,5 are the eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigen values of 5A

Solution:By the property "If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of kA, the eigen values of 5A are 5(1), 5(1), 5(5) ie., 5,5,25

13. Find the eigen values of A, $A^2, A^3, A^4, 3A, A^{-1}, A - I, 3A^3 + 5A^2 - 6A + 2I$ if A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 2, 5

The eigen values of A^2 are $2^2, 5^2$ i.e., 4, 25

The eigen values of A^3 are $2^3, 5^3$ i.e., 8, 125

The eigen values of A^4 are $2^4, 5^4$ i.e., 16, 625

The eigen values of 3A are 3(2), 3(5) i.e., 6, 15

The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{5}$

A – I = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

Since A - I is an upper triangular matrix, the eigen values of A- I are its main diagonal elements i.e., 1,4

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$ and $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$ where $\lambda_1 = 2$ and $\lambda_2 = 5$

First eigen value = $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$

= $3(2)^3$+ $5(2)^2$ - 6(2) + 2 = 24 + 20 -12 + 2=34

Second eigen value = $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$

= $3(5)^3$+ $5(5)^2$- 6(5) + 2

= 375+ 125-30 + 2 = 472

14. Find the eigen values of adj A if A = $\begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{4} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

<u>**Solution**</u>:Given A =$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 3, 4, 1

We know that $A^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|}\ adj\ A$

Adj A = |A| A^{-1}

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

|A| =Product of the eigen values = 12

Therefore, the eigen values of adj A is equal to the eigen values of 12 A^{-1} i.e., $\frac{12}{3}, \frac{12}{4}, 12$ i.e., 4, 3, 12

<u>**Note:**</u> A =$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Here, A is an upper triangular matrix, B is a lower triangular matrix and C is a diagonal matrix. In all the cases, the elements in the main diagonal are the eigen values. Hence, the eigen values of A, B and C are 1, 4, 6

15. Two eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and they are $\frac{1}{5}$ times the third. Find them

Solution:Let the third eigen value be λ_3

We know that $\lambda_1 + \lambda_2 + \lambda_3$ =2+3+2 = 7

Given $\lambda_1 = \lambda_2 = \frac{\lambda_3}{5}$

$$\frac{\lambda_3}{5} + \frac{\lambda_3}{5} + \lambda_3 = 7$$

$$\left[\frac{1}{5} + \frac{1}{5} + 1\right]\lambda_3 = 7 \Rightarrow \frac{7}{5}\lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

Therefore, $\lambda_1 = \lambda_2 = 1$ and hence the eigen values of A are 1,1, 5

16. If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, *find the value of a*

Solution:Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$. Let the eigen values of A be 2, 3, k

We know that the sum of the eigen values = sum of the main diagonal elements

Therefore, 2 +3 +k = 2+ 2+2 = 6 $\Rightarrow k = 1$

We know that product of the eigen values = |A|

⇒2(3)(k) = |A|

$$\Rightarrow 6 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Rightarrow 6 = 2(4) - 0 + 1(-2a) \Rightarrow 6 = 8 - 2a \Rightarrow 2a = 2 \Rightarrow a = 1$$

17. Prove that the eigen vectors of the real symmetric matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are orthogonal in pairs

Solution:The characteristic equation of A is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$where $S_1 = sum\ of\ the\ main\ diagonal\ elements = 7$;

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements\ = 4 + (-8) + 4 = 0$

$$S_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(4) - 1(-2) + 3(-14) = -36$$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 36 = 0$

3	1-7	0	36	
	0	3	-12	-36
	1	-4	-12	0

Therefore, $\lambda = 3, \lambda^2 - 4\lambda - 12 = 0 \Rightarrow \lambda = 3, \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-12)}}{2(1)} = \frac{4 \pm 8}{2} = 6, -2$

Therefore, the eigen values of A are -2, 3, 6

To find the eigen vectors:

$$(A - \lambda I)X = 0$$

Case 1: When $\lambda = -2, \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 + 7x_2 + x_3 = 0$ ---------- (2)

$3x_1 + x_2 + 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$$x_1 x_2 x_3$$

1 3 3 1

7 1 1 7

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2: When $\lambda = 3$, $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 2x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 2x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

1 3 -2 1

2 1 1 2

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case 3: When $\lambda = 6$, $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 - x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 5x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

1 3 -5 1

-1 1 1 -1

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

To prove that: $X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0$

$X_1^T X_2 = [-1 \quad 0 \quad 1] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$

$$X_2^T X_3 = [1 - 1 \quad 1]\begin{bmatrix}1\\2\\1\end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = [1 \quad 2 \quad 1]\begin{bmatrix}-1\\0\\1\end{bmatrix} = -1 + 0 + 1 = 0$$

Hence, the eigen vectors are orthogonal in pairs

18. Find the sum and product of all the eigen values of the matrix A = $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{2} & \mathbf{2} & \mathbf{4}\\ \mathbf{1} & \mathbf{2} & \mathbf{7}\end{bmatrix}$**.Is the matrix singular?**

Solution:Sum of the eigen values = Sum of the main diagonal elements =Trace of the matrix

Therefore, the sum of the eigen values = 1+2+7=10

Product of the eigen values = │A│ = 1(14 - 8) -2(14 - 4) + 3(4 - 2) = 6-20+ 6= - 8

│A│≠0. Hence the matrix is non-singular.

19. Find the product of the eigen values of A = $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-2}\\ \mathbf{1} & \mathbf{0} & \mathbf{3}\\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3}\end{bmatrix}$

Solution:Product of the eigen values of A = │A│= $\begin{vmatrix}1 & 2 & -2\\ 1 & 0 & 3\\ -2 & -1 & -3\end{vmatrix} = 1(3) - 2(3) - 2(-1) =$

$3 - 6 + 2 = -1$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT - II - DIFFERENTIATION AND ITS APPLICATIONS - SMT1113

Definition 1. Differentiation

The rate at which a function changes with respect to the independent variable is called the derivative of the function.

(i.e) If y= f(x) be a function, where x and y are real variables which are independent and dependent variables respectively, then the derivative of y with respect to x is $\frac{dy}{dx}$.

Definition 2. Derivative of addition or subtraction of functions

If f(x) and g(x) are two functions of x, then $\frac{d[f(x) \pm g(x)]}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$

Definition 3. Product rule

If $y = uv$, where u and v are functions of x, then $\frac{d[uv]}{dx} = v\frac{d[u]}{dx} + u\frac{d[v]}{dx}$

Definition 4. Quotient rule

If $y = \frac{u}{v}$, where u and v are functions of x, then $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Important Derivatives Formulae

1. $\frac{d}{dx}(c) = 0$ where 'c' is any constant.
2. $\frac{d}{dx}(x^n) = nx^{n-1}$.
3. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.
4. $\frac{d}{dx}(a^x) = a^x \log a$
5. $\frac{d}{dx}(e^x) = e^x$.
6. $\frac{d}{dx}(\sin x) = \cos x$.
7. $\frac{d}{dx}(\cos x) = -\sin x$.
8. $\frac{d}{dx}(\tan x) = \sec^2 x$.

9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

10. $\frac{d}{dx}(\sec x) = \sec x \tan x$.

11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

13. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

15. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

16. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{1-x^2}}$.

17. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$

1.1. Ordinary Differentiation Problems

1. **Differentiate** $\mathbf{x + \frac{1}{x}}$

   Solution Let $y = x + \frac{1}{x}$

   Then $\frac{dy}{dx} = \frac{d\left(x + \frac{1}{x}\right)}{dx} \quad = \frac{d(x)}{dx} + \frac{d(x^{-1})}{dx} = 1 - \frac{1}{x^2}$

2. **Differentiate** $\mathbf{3\tan x + 2\cos x - e^x + 5}$

   Solution:

   Let $y = 3\tan x + 2\cos x - e^x + 5$

   Then $\frac{dy}{dx} = \frac{d(3\tan x + 2\cos x - e^x + 5)}{dx} \quad = 3\frac{d(\tan x)}{dx} + 2\frac{d(\cos x)}{dx} - \frac{d(e^x)}{dx} + \frac{d(5)}{dx}$

   $= 3sec^2 x - 2\sin x - e^x$

3. **Differentiate** $\mathbf{y = e^{2x}\cos 3x}$

   Solution: $\frac{dy}{dx} = \frac{d(e^{2x}\cos 3x)}{dx} \quad = \cos 3x \, \frac{d(e^{2x})}{dx} + e^{2x} \, \frac{d(\cos 3x)}{dx}$

   $= 2\cos 3x \, e^{2x} - 3e^{2x}\sin 3x$

4. **Differentiate** $\mathbf{y = e^{\sin x}x^2}$

   Solution: $\frac{dy}{dx} = \frac{d(e^{\sin x}x^2)}{dx}$

$= x^2 \frac{d(e^{sinx})}{dx} + e^{sinx}\frac{d(x^2)}{dx}$
$= x^2 e^{sinx}(cosx) + 2xe^{sinx}$

5. **Differentiate y = $\mathbf{x^3e^{-x}tanx}$**
Solution: $\frac{dy}{dx} = \frac{d(x^3e^{-x}tanx)}{dx}$
$= e^{-x}tanx\frac{d(x^3)}{dx} + x^3\, tanx\frac{d(e^{-x})}{dx} + x^3e^{-x}\frac{d(tanx)}{dx}$
$= 3x^2e^{-x}tanx - x^3e^{-x}tanx + x^3e^{-x}sec^2x$

6. **Differentiate y** $= \frac{\mathbf{e^x}}{\mathbf{cosx}}$
Solution: $\frac{dy}{dx} = \frac{d\left(\frac{e^x}{cosx}\right)}{dx} = \frac{cosx\, e^x - e^x\,(-sinx)}{cos^2x}$
$= \frac{cosx\, e^x + e^x\,(sinx)}{cos^2x}$

7. **Differentiate y** $= \frac{\mathbf{ax+b}}{\mathbf{cx+d}}$
Solution:$\frac{dy}{dx} = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$ (by quotient rule)

8. **Differentiate** $\frac{\mathbf{x^2+2x+3}}{\sqrt{\mathbf{x}}}$
Solution: $\frac{dy}{dx} = \frac{\sqrt{x}(2x+2)-(x^2+2x+3)\frac{1}{2}x^{-1/2}}{(\sqrt{x})^2} = \frac{2\sqrt{x}(x+1)-(x^2+2x+3)\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$
$= \frac{2\sqrt{x}\times 2\sqrt{x}(x+1)-(x^2+2x+3)}{2\sqrt{x}(\sqrt{x})^2} = \frac{4x(x+1)-(x^2+2x+3)}{2x^{3/2}}$
$= \frac{4x^2+4x-x^2-2x-3}{2x^{3/2}} = \frac{3x^2+2x-3}{2x^{3/2}}$

9. **Differentiate y** $= \mathbf{(3x^2-1)^3}$
Solution: Given $y = (3x^2-1)^3$
Differentiating w.r.to x, we get
$\Rightarrow \frac{dy}{dx} = 3(3x^2-1)^2 6x$
$= 3(9x^4 - 6x^2 + 1) = 27x^4 - 18x^2 + 3$

10. **Differentiate: $\mathbf{log\left(\frac{1+sinx}{1-sinx}\right)}$**
Solution: Let $y = log\left(\frac{1+sinx}{1-sinx}\right)$
$\Rightarrow y = log(1+sinx) - log(1-sinx)$
Differentiate y w.r.to x, we get
$\frac{dy}{dx} = \frac{1}{1+sinx}cosx - \frac{1}{1-sinx}(-cosx)$

$= \frac{(1-sinx)cosx+cosx(1+sinx)}{(1+sinx)(1-sinx)}$

$= \frac{cosx - sinx\, cosx + cosx + cosx\, sinx}{1-sin^2x}$
$= \frac{2\, cosx}{cos^2x} = 2\frac{1}{cosx} = 2\, secx$

2.1. Differentiation Problems on Logarithmic Functions

1. **Differentiate $\mathbf{x^{sinx}}$**
Solution: Let $y = x^{sinx}$
Taking log on both sides, we get $logy = sinx\, logx$

Now differentiating with respect to x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x(\cos x) + \sin x\frac{1}{x}$ (Using product rule)

$\Rightarrow \frac{dy}{dx} = y\left(\log x(\cos x) + \sin x\frac{1}{x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{y(x\cos x \log x+\sin x)}{x}$

$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{x\cos x \log x+\sin x}{x}\right)$

2. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution: Given $x^y = e^{x-y}$

Taking log on both sides, we get $\log x^y = \log e^{x-y}$

$\Rightarrow y\log x = (x-y)\log_e e$

$\Rightarrow y\log x = (x-y)\ldots\ldots\ldots(1)$

$\Rightarrow \frac{1}{x}y + \log x\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \log x\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx}(\log x + 1) = \frac{x-y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x(1+\log x)}$

$\Rightarrow \frac{dy}{dx} = \frac{y\log x}{x(1+\log x)}$....(2)

Again from (1) $y + y\log x = x$

$\Rightarrow y(1+\log x) = x, \frac{y}{x} = \frac{1}{1+\log x}$

$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

3. If $y = x^{x^{x^{\cdots\infty}}}$, then find $\frac{dy}{dx}$

Solution:

Given $y = x^{x^{x^{\cdots\infty}}} = x^y$

Taking log on both sides

$\log y = y\log x$

Differentiating w. r. to x we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \left(\frac{1-y\log x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{y}{1-y\log x}\right) = \frac{y^2}{x(1-y\log x)}$

4. Differentiate $y = \log\left(\frac{x^2+1}{x^2-1}\right)$

Solution:

$y = \log(x^2+1) - \log(x^2-1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2+1}2x - \frac{1}{x^2-1}2x$

$\Rightarrow \frac{dy}{dx} = 2x\left(\frac{1}{x^2+1} - \frac{1}{x^2-1}\right)$

$$\Rightarrow \frac{dy}{dx} = 2x\left(\frac{x^2-1-(x^2+1)}{(x^2+1)(x^2-1)}\right) = 2x\left(\frac{x^2-1-x^2-1)}{x^4-1}\right) = 2x\left(\frac{-2)}{x^4-1}\right) = \frac{-4x}{x^4-1}$$

5. **Differentiate $y = e^{3x^2+2x+3}$**

   Solution: $\frac{dy}{dx} = e^{3x^2+2x+3}(6x+2)$

3.1. Differentiation of Implicit functions

If two variables x and y are connected by the relation f(x, y) = 0 and none of the variable is directly expressed in terms of the other, then the relation is called an implicit function.

Problems

1. **Find $\frac{dy}{dx}$, if $x^3+y^3 = 3axy$**

   Solution:

   Differentiating w.r.to x, we get

   $\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$

   $\Rightarrow 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$

   $\Rightarrow \frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$

   $\Rightarrow \frac{dy}{dx} = \frac{(3ay-3x^2)}{3y^2-3ax} = \frac{3(ay-x^2)}{3(y^2-ax)} = \frac{(ay-x^2)}{(y^2-ax)}$

2. **Find $\frac{dy}{dx}$, if $x^2 + y^2 = 16$**

   Solution:

   Given $x^2 + y^2 = 16$

   $\Rightarrow y^2 = 16 - x^2$

   $\Rightarrow y = \sqrt{16 - x^2}$

   $\Rightarrow \frac{dy}{dx} = \frac{1}{2}(16 - x^2)^{-1/2} \times (-2x)$

   $\Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{16-x^2}} = -\frac{x}{y}$

3. **Find $\frac{dy}{dx}$, if $x = at^2, y = 2at$**

   Solution: Given $x = at^2, y = 2at$

   $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

   Now $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$

4. **Find $\frac{dy}{dx}$, if $y^2 + x^3 - xy + \cos y = 0$**

   Solution:

   Given $y^2 + x^3 - xy + \cos y = 0$

   $\Rightarrow 2y\frac{dy}{dx} + 3x^2 - \frac{d}{dx}(xy) - \sin y\frac{dy}{dx} = 0$

   $\Rightarrow (2y - \sin y)\frac{dy}{dx} + 3x^2 - \left(x\frac{dy}{dx} + y \times 1\right) = 0$

   $\Rightarrow (2y - \sin y - x)\frac{dy}{dx} + 3x^2 - y = 0$

   $\Rightarrow (2y - \sin y - x)\frac{dy}{dx} = y - 3x^2$

   $\Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{2y-\sin y-x}$

4.1 Equations of Tangent and Normal

According to Leibniz, tangent is the line through a pair of very close points on the curve.

Definition

The tangent line (or simply tangent) to a plane curve at a given point is the straight line that just touches the curve at that point.

Definition

The normal at a point on the curve is the straight line which is perpendicular to the tangent at that point.

The tangent and the normal of a curve at a point are illustrated in Fig. 1

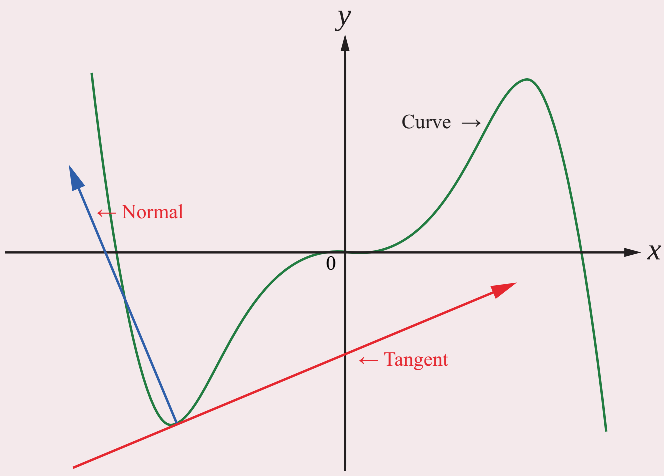


Fig.1

Consider the given curve $y = f(x)$.

The equation of the tangent to the curve at the point, say (a,b), is given by

$$y - b = (x-a)\times\left(\frac{dy}{dx}\right)_{(a,b)} \text{ or } y - b = f'(a)\cdot(x-a).$$

In order to get the equation of the normal to the same curve at the same point, we observe that normal is perpendicular to the tangent at the point. Therefore, the slope of the normal at (a,b) is the negative of the reciprocal of the slope of the tangent which is $-\left(\frac{1}{\frac{dy}{dx}}\right)_{(a,b)}$.

Hence, the equation of the normal is ,

$$(y-b) = -\left(\frac{1}{\frac{dy}{dx}}\right)_{(a,b)}\times(x-a) \text{ or } (y-b)\times\left(\frac{dy}{dx}\right)_{(a,b)} = -(x-a).$$

Remark

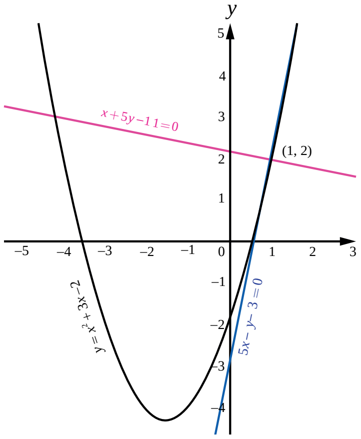
(i) If the tangent to a curve is horizontal at a point, then the derivative at that point is 0. Hence, at that point (x_1, y_1) the equation of the tangent is $y = y_1$ and equation of the normal is $x = x_1$.

(ii) If the tangent to a curve is vertical at a point, then the derivative exists and infinite (∞) at the point. Hence, at that point (x_1, y_1) the equation of the tangent is $x = x_1$ and the equation of the normal is $y = y_1$.

Example

Find the equations of tangent and normal to the curve $y = x^2 + 3x - 2$ at the point $(1, 2)$.

Solution



We have, $\frac{dy}{dx} = 2x + 3$. Hence at $(1, 2), \frac{dy}{dx} = 5$.

Therefore, the required equation of tangent is

$(y - 2) = 5(x - 1) \Rightarrow 5x - y - 3 = 0$.

The slope of the normal at the point $(1, 2)$ is $-\frac{1}{5}$.

Therefore, the required equation of normal is

$(y - 2) = -\frac{1}{5}(x - 1) \Rightarrow x + 5y - 11 = 0 \cdot$

Example

Find the points on the curve $y = x^3 - 3x^2 + x - 2$ at which the tangent is parallel to the line $y = x$.

Solution

The slope of the line $y = x$ is 1. The tangent to the given curve will be parallel to the line, if the slope of the tangent to the curve at a point is also 1. Hence,

$$\frac{dy}{dx} = 3x^2 - 6x + 1 = 1$$

which gives $3x^2 - 6x = 0$.

Hence, $x = 0$ and $x = 2$.

Therefore, at (0, –2) and (2, –4) the tangent is parallel to the line $y = x$.

Example

Find the equation of the tangent and normal at any point to the Lissajous curve given by $x = 2\cos 3t$ and $y = 3\sin 2t, t \in \mathbb{R}$.

Solution

Observe that the given curve is neither a circle nor an ellipse.

$$\text{Now, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\frac{6\cos 2t}{6\sin 3t} = -\frac{\cos 2t}{\sin 3t}.$$

Therefore, the tangent at any point is

$$y - 3\sin 2t = -\frac{\cos 2t}{\sin 3t}(x - 2\cos 3t)$$

That is, $x\cos 2t + y\sin 3t = 3\sin 2t\sin 3t + 2\cos 2t\cos 3t$.

The slope of the normal is the negative of the reciprocal of the tangent which in this case is $\frac{\sin 3t}{\cos 2t}$. Hence, the equation of the normal is

$$y - 3\sin 2t = \frac{\sin 3t}{\cos 2t}(x - 2\cos 3t).$$

That is, $x\sin 3t - y\cos 2t = 2\sin 3t\cos 3t - 3\sin 2t\cos 2t = \sin 6t - \frac{3}{2}\sin 4t$.

4.2 Angle between two curves

Angle between two curves, if they intersect, is defined as the acute angle between the tangent lines to those two curves at the point of intersection.

For the given curves, at the point of intersection using the slopes of the tangents, we can measure the acute angle between the two curves. Suppose $y = m_1x + c_1$ and $y = m_2x + c_2$ are two lines, then the acute angle θ between these lines is given by,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \quad \text{................ (1)}$$

where m_1 and m_2 are finite.

Remark

(i) If the two curves are parallel at (x_1, y_1), then $m_1 = m_2$.

(ii) If the two curves are perpendicular at (x_1, y_1) and if m_1 and m_2 exists and finite then $m_1m_2 = -1$.

Example

Find the angle between $y = x^2$ and $y = (x-3)^2$.

Solution

Let us now find the point of intersection of the two given curves. Equating $x^2 = (x-3)^2$ we get, $x = \frac{3}{2}$. Therefore, the point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$. Let θ be the angle between the curves. The slopes of the curves are as follows :

For the curve $y = x^2$,

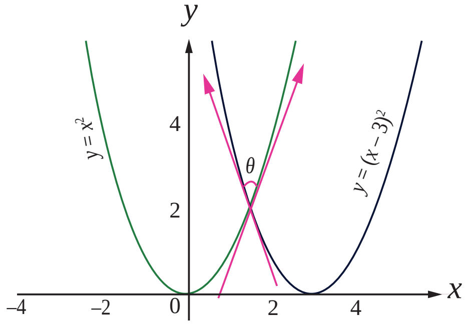
$$\frac{dy}{dx} = 2x.$$

$$\text{Let } m_1 = \frac{dy}{dx} \text{ at } \left(\frac{3}{2}, \frac{9}{4}\right) = 3.$$

For the curve $y = (x-3)^2$,

$$\frac{dy}{dx} = 2(x-3).$$

$$\text{Let } m_2 = \frac{dy}{dx} \text{ at } \left(\frac{3}{2}, \frac{9}{4}\right) = -3.$$



Using (1), we get

$$\tan\theta = \left|\frac{3-(-3)}{1-9}\right| = \frac{3}{4}$$

$$\text{Hence, } \theta = \tan^{-1}\left(\frac{3}{4}\right).$$

Example

Find the angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection (0,0) and (1,1).

Solution

Let us now find the slopes of the curves.

Let m_1 be the slope of the curve $y = x^2$,

$$\text{then } m_1 = \frac{dy}{dx} = 2x.$$

Let m_2 be the slope of the curve $x = y^2$,

$$\text{then } m_2 = \frac{dy}{dx} = \frac{1}{2y}.$$

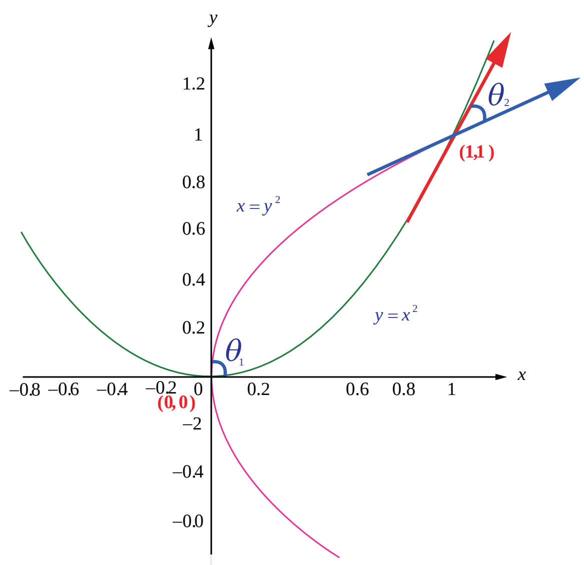
Let θ_1 and θ_2 be the angles at (0,0) and (1,1) respectively.

At $(0,0)$, we come across the indeterminate form of $0\times\infty$ in the denominator of $\tan\theta_1 = \left|\dfrac{2x-\frac{1}{2y}}{1+(2x)\left(\frac{1}{2y}\right)}\right|$ and so we follow the limiting process.

$$\tan\theta_1 = \lim_{(x,y)\to(0,0)}\left|\frac{2x-\frac{1}{2y}}{1+(2x)\left(\frac{1}{2y}\right)}\right|$$

$$= \lim_{(x,y)\to(0,0)}\left|\frac{4xy-1}{2(y+x)}\right|$$

$$= \infty$$



$$\text{which gives } \theta_1 = \tan^{-1}(\infty) = \frac{\pi}{2}.$$

At $(1,1)$, $m_1 = 2$, $m_2 = \dfrac{1}{2}$

$$\tan\theta_2 = \left|\frac{2-\frac{1}{2}}{1+(2)\left(\frac{1}{2}\right)}\right|$$

$$= \frac{3}{4}$$

$$\text{which gives } \theta_2 = \tan^{-1}\left(\frac{3}{4}\right).$$

5. MAXIMA AND MINIMA

5.1. Introduction

Here, we show how differentiation can be used to find the maximum and minimum values of a function. Because the derivative provides information about the gradient or slope of the graph of a function we can use it to locate points on a graph where the gradient is zero. We shall see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality. In many applications, a scientist, engineer, or economist for example, will be interested in such points for obvious reasons such as maximising power, or profit, or minimising losses or costs.

5.2 . Stationary points

When using mathematics to model the physical world in which we live, we frequently express physical quantities in terms of **variables**. Then, **functions** are used to describe the ways in which these variables change. A scientist or engineer will be interested in the ups and downs of a function, its maximum and minimum values, its turning points. Drawing a graph of a function using a graphical calculator or computer graph plotting package will reveal this behaviour, but if we want to know the precise location of such points we need to turn to algebra and differential calculus. In this section we look at how we can find maximum and minimum points in this way.

Consider the graph of the function, $y(x)$, shown in Figure 1. If, at the points marked A, B and C, we draw tangents to the graph, note that these are parallel to the x axis. They are horizontal. This means that at each of the points A, B and C the gradient of the graph is zero.

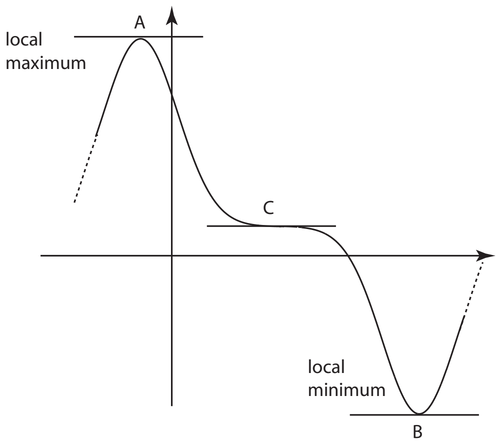


Figure 1. The gradient of this graph is zero at each of the points A, B and C.

We know that the gradient of a graph is given by $\frac{\mathrm{d}y}{\mathrm{d}x}$ Consequently, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at points A, B and C. All of these points are known as **stationary points**.

Key Point

Any point at which the tangent to the graph is horizontal is called a **stationary point**.

We can locate stationary points by looking for points at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

5.3. Turning points

Refer again to Figure 1. Notice that at points A and B the curve actually turns. These two stationary points are referred to as **turning points**. Point C is not a turning point because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

So, all turning points are stationary points.

But not all stationary points are turning points (e.g. point C).

In other words, there are points for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ which are not turning points.

Key Point

At a **turning point** $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Not all points where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ are turning points, i.e. not all stationary points are turning points.

Point A in Figure 1 is called a **local maximum** because in its immediate area it is the highest point, and so represents the greatest or maximum value of the function. Point B in Figure 1 is called a **local minimum** because in its immediate area it is the lowest point, and so represents the least, or minimum, value of the function. Loosely speaking, we refer to a local maximum as simply a **maximum**. Similarly, a local minimum is often just called a **minimum**.

5. 4. Distinguishing maximum points from minimum points

Think about what happens to the gradient of the graph as we travel through the minimum turning point, from left to right, that is as x increases. Study Figure 2 to help you do this.

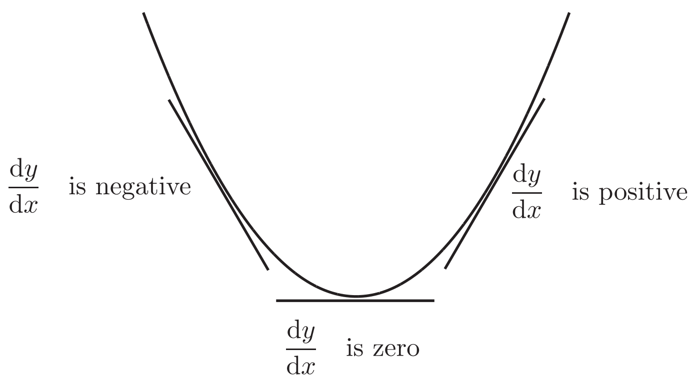


Figure 2. $\frac{\mathrm{d}y}{\mathrm{d}x}$ goes from negative through zero to positive as x increases.

Notice that to the left of the minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative because the tangent has negative gradient. At the minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. To the right of the minimum point $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive, because here the tangent has a positive gradient. So, $\frac{\mathrm{d}y}{\mathrm{d}x}$ goes from negative, to zero, to positive as x increases. In other words, $\frac{\mathrm{d}y}{\mathrm{d}x}$ must be increasing as x increases.

In fact, we can use this observation, once we have found a stationary point, to check if the point is a minimum. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing near the stationary point then that point must be minimum. Now, if the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive then we will know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing; so we will know that the stationary point is a minimum. Now the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$, called the **second derivative**, is written $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. We conclude that if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is positive at a stationary point, then that point must be a minimum turning point.

Key Point

if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at a point, and if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ there, then that point must be a minimum.

It is important to realise that this test for a minimum is not conclusive. It is possible for a stationary point to be a minimum even if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ *equals* 0, although we cannot be certain: other types of behaviour are possible. (However, we cannot have a minimum if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative.)

To see this consider the example of the function $y = x^4$. A graph of this function is shown in Figure 3. There is clearly a minimum point when $x = 0$. But $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3$ and this is clearly zero when $x = 0$. Differentiating again $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2$ which is also zero when $x = 0$.

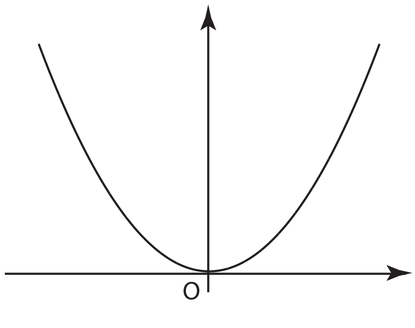


Figure 3. The function $y = x^4$ has a minimum at the origin where $x = 0$, but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ and so is not greater than 0.

Now think about what happens to the gradient of the graph as we travel through the maximum turning point, from left to right, that is as x increases. Study Figure 4 to help you do this.

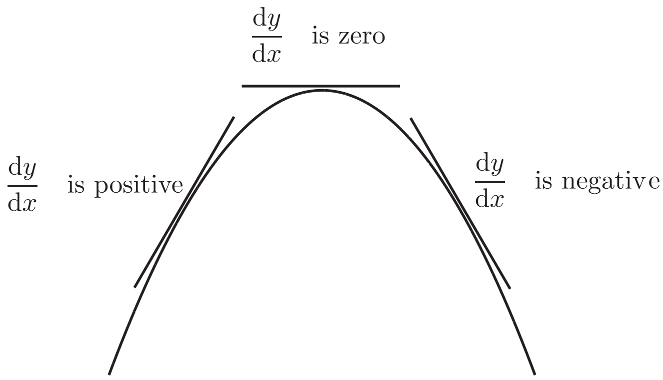


Figure 4. $\frac{dy}{dx}$ goes from positive through zero to negative as x increases.

Notice that to the left of the maximum point, $\frac{dy}{dx}$ is positive because the tangent has positive gradient. At the maximum point, $\frac{dy}{dx} = 0$. To the right of the maximum point $\frac{dy}{dx}$ is negative, because here the tangent has a negative gradient. So, $\frac{dy}{dx}$ goes from positive, to zero, to negative as x increases.

In fact, we can use this observation to check if a stationary point is a maximum. If $\frac{dy}{dx}$ is decreasing near a stationary point then that point must be maximum. Now, if the derivative of $\frac{dy}{dx}$ is negative then we will know that $\frac{dy}{dx}$ is decreasing; so we will know that the stationary point is a maximum. As before, the derivative of $\frac{dy}{dx}$, the **second derivative** is $\frac{d^2y}{dx^2}$. We conclude that if $\frac{d^2y}{dx^2}$ is negative at a stationary point, then that point must be a maximum turning point.

Key Point

if $\frac{dy}{dx} = 0$ at a point, and if $\frac{d^2y}{dx^2} < 0$ there, then that point must be a maximum.

It is important to realise that this test for a maximum is not conclusive. It is possible for a stationary point to be a maximum even if $\frac{d^2y}{dx^2} = 0$, although we cannot be certain: other types of behaviour are possible. But we cannot have a maximum if $\frac{d^2y}{dx^2} > 0$, because, as we have already seen the point would be a minimum.

Key Point

The second derivative test: summary

We can locate the position of stationary points by looking for points where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

As we have seen, it is possible that some such points will not be turning points.

We can calculate $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at each point we find.

If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is positive then the stationary point is a minimum turning point.

If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative, then the point is a maximum turning point.

If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ it is possible that we have a maximum, or a minimum, or indeed other sorts of behaviour. So if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ this second derivative test does not give us useful information and we must seek an alternative method (see Section 5).

Example

Suppose we wish to find the turning points of the function $y = x^3 - 3x + 2$ and distinguish between them.

We need to find where the turning points are, and whether we have maximum or minimum points. First of all we carry out the differentiation and set $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ equal to zero. This will enable us to look for any stationary points, including any turning points.

$$\begin{aligned} y &= x^3 - 3x + 2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - 3 \end{aligned}$$

At stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and so

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \qquad \text{(factorising)} \\ 3(x-1)(x+1) &= 0 \qquad \text{(factorising the difference of two squares)} \end{aligned}$$

It follows that either $x - 1 = 0$ or $x + 1 = 0$ and so either $x = 1$ or $x = -1$.

We have found the x coordinates of the points on the graph where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, that is the stationary points. We need the y coordinates which are found by substituting the x values in the original function $y = x^3 - 3x + 2$.

<u>when $x = 1$</u>: $\quad y = 1^3 - 3(1) + 2 = 0.$

<u>when $x = -1$</u>: $\quad y = (-1)^3 - 3(-1) + 2 = 4.$

To summarise, we have located two stationary points and these occur at $(1,0)$ and $(-1,4)$.

Next we need to determine whether we have maximum or minimum points, or possibly points such as C in Figure 1 which are neither maxima nor minima.

We have seen that the first derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3$. Differentiating this we can find the second derivative:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$$

We now take each point in turn and use our test.

when $x = 1$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x = 6(1) = 6$. We are not really interested in this value. What is important is its sign. Because it is positive we know we are dealing with a minimum point.

when $x = -1$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x = 6(-1) = -6$. Again, what is important is its sign. Because it is negative we have a maximum point.

Finally, to finish this off we produce a quick sketch of the function now that we know the precise locations of its two turning points (See Figure 5).

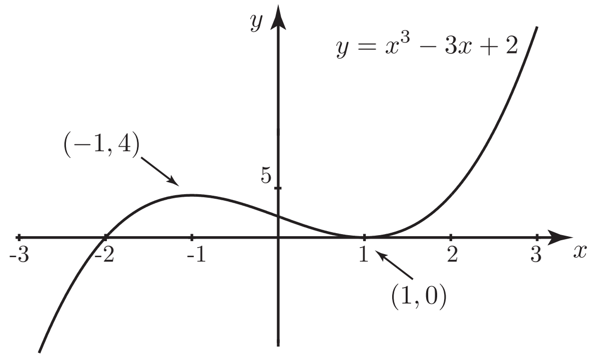


Figure 5. Graph of $y = x^3 - 3x + 2$ showing the turning points

5.5 An example which uses the first derivative to distinguish maxima and minima

Example

Suppose we wish to find the turning points of the function $y = \frac{(x-1)^2}{x}$ and distinguish between them.

First of all we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

In this case we need to apply the quotient rule for differentiation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x \cdot 2(x-1) - (x-1)^2 \cdot 1}{x^2}$$

This does look complicated. Don't rush to multiply it all out if you can avoid it. Instead, look for common factors, and tidy up the expression.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x\cdot 2(x-1)-(x-1)^2\cdot 1}{x^2}\\ &= \frac{(x-1)(2x-(x-1))}{x^2}\\ &= \frac{(x-1)(x+1)}{x^2}\end{aligned}$$

We now set $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to zero in order to locate the stationary points including any turning points.

$$\frac{(x-1)(x+1)}{x^2} = 0$$

When equating a fraction to zero, it is the top line, the numerator, which must equal zero. Therefore

$$(x-1)(x+1) = 0$$

from which $x-1=0$ or $x+1=0$, and from these equations we find that $x=1$ or $x=-1$.

The y co-ordinates of the stationary points are found from $y=\frac{(x-1)^2}{x}$.

when $x=1$: $y=0$.

when $x=-1$: $y=\frac{(-2)^2}{-1}=-4$.

We conclude that stationary points occur at $(1,0)$ and $(-1,-4)$.

We now have to decide whether these are maximum points or minimum points. We could calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and use the second derivative test as in the previous example. This would involve differentiating $\frac{(x-1)(x+1)}{x^2}$ which is possible but perhaps rather fearsome! Is there an alternative way ? The answer is yes. We can look at how $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes as we move through the stationary point. In essence, we can find out what happens to $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ without actually calculating it.

First consider the point at $x=-1$. We look at what is happening a little bit before the point where $x=-1$, and a little bit afterwards. Often we express the idea of 'a little bit before' and 'a little bit afterwards' in the following way. We can write $-1-\epsilon$ to represent a little bit less than -1, and $-1+\epsilon$ to represent a little bit more. The symbol ϵ is the Greek letter epsilon. It represents a small positive quantity, say 0.1. Then $-1-\epsilon$ would be -1.1, just a little less than -1. Similarly $-1+\epsilon$ would be -0.9, just a little more than -1.

We now have a look at $\frac{\mathrm{d}y}{\mathrm{d}x}$; not its value, but its sign.

When $x=-1-\epsilon$, say -1.1, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive.

When $x=-1$ we already know that $\frac{\mathrm{d}y}{\mathrm{d}x}$=0.

When $x = -1 + \epsilon$, say -0.9, $\dfrac{dy}{dx}$ is negative.

We can summarise this information as shown in Figure 6.

	$x = -1 - \epsilon$	$x = -1$	$x = -1 + \epsilon$
sign of $\dfrac{dy}{dx}$	$+$	0	$-$
shape of graph	$\nearrow$	$\longrightarrow$	$\searrow$

Figure 6. Behaviour of the graph near the point $(-1, -4)$

Figure 6 shows us that the stationary point at $(-1, -4)$ is a maximum turning point. Then we turn to the point $(1, 0)$. We carry out a similar analysis, looking at the sign of $\dfrac{dy}{dx}$ at $x = 1 - \epsilon$, $x = 1$, and $x = 1 + \epsilon$. The results are summarised in Figure 7.

	$x = 1 - \epsilon$	$x = 1$	$x = 1 + \epsilon$
sign of $\dfrac{dy}{dx}$	$-$	0	$+$
shape of graph	$\searrow$	$\longrightarrow$	$\nearrow$

Figure 7. Behaviour of the graph near the point $(1, 0)$

We see that the point is a minimum.

This, so-called **first derivative test**, is also the way to do it if $\dfrac{d^2y}{dx^2}$ is zero in which case the second derivative test does not work. Finally, for completeness a graph of $y = \dfrac{(x-1)^2}{x}$ is shown in Figure 8 where you can see the maximum and minimum points.

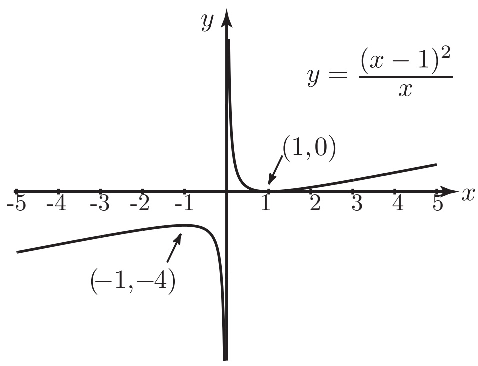


Figure 8. A graph of $y = \dfrac{(x-1)^2}{x}$ showing the turning points

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT - III - INTEGRATION - SMT1113

1.1 Standard results

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

2. $\int \frac{1}{x}\, dx = \log x + c$

3. $\int e^x dx = e^x + c$

4. $\int \sin x\, dx = -\cos x + c$

5. $\int \cos x\, dx = \sin x + c$

6. $\int \tan x\, dx = \log \sec x + c$

7. $\int \cot x\, dx = \log \sin x + c$

8. $\int \sec x \tan x\, dx = \sec x + c$

9. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

10. $\int \sec^2 x\, dx = \tan x + c$

11. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

12. $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$

13. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$

14. $\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x + c$

1.2 INTEGRATION BY PARTS:

$$\int u\, dv = u\, v - \int v\, du$$

1. Find $\int x \cos x\, dx$

Solution:

Let u =x, dv =cosx dx

Then integration by parts gives,

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c$$

2. Find $\int \log x\,dx$

Solution:

Let u = logx , dv = dx

Then, $\int \log x\,dx = (\log x)x - \int \frac{1}{x} x\,dx$

$= x(\log x) - x + c$

3. Find $\int x e^x\,dx$

Solution:

Let u = x , dv = e^xdx

$$\int x e^x\,dx = xe^x - \int 1e^x\,dx = xe^x - e^x + c$$

4. Find $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Solution:

Let u =$\sin^{-1}$x , dv = x/√(1-x^2)dx

For finding v,

Put t =1 – x^2 then dt = -2x dx

Then v = $\int \frac{-dt}{2\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$

$$\therefore \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \sin^{-1}x\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)dx$$

$= -\sqrt{1-x^2}\,\sin^{-1}x + x + c$

2.1 BERNOULLI'S FORMULA

$$\int u\,dv = uv - u' v_1 + u'' v_2 + \cdots$$

Problems

1. Solve $\int x^2 e^x \, dx$

Solution:

$$\int x^2 e^x \, dx = x^2 e^x - 2x(e^x) + 2e^x + C$$

2. Solve $\int x \sin ax \, dx$

Solution:

$$\int x \sin ax \, dx = x\left(\frac{-\cos ax}{a}\right) - \left(\frac{-\sin ax}{a^2}\right) + C$$

3. Solve $\int (ax^2 + bx + c) \cos x \, dx$

Solution:

$$\int (ax^2 + bx + c) \cos x \, dx = (ax^2 + bx + c)(\sin x) + (2ax + b)(-\cos x) + 2a(-\sin x) + c$$

3 Definite Integrals

Definite Integrals is defined as

$\int_a^b f(x)\text{dx} = [F(x)]_a^b = F(b) - F(a)$

3.1 Properties of Definite

Integrals Property:1

$$\int_a^b f(x)\text{dx} = \int_a^b f(y)\text{dy} = \int_a^b f(t)\text{dt} .$$

Property: 2

$$\int_a^b f(x)\text{dx} = -\int_b^a f(x)\text{dx}$$

Property: 3

$\int_a^b f(x)\text{dx} = \int_a^c f(x)\text{dx} + \int_c^b f(x)\text{dx}$,if $a \le c \le b$

Property : 4

$$\int_0^a f(x)\text{dx} = \int_0^a f(a-x)\text{dx}$$. a : any real constant.

Proof : $\int_0^a f(a-x)\text{dx}$

Let

$$a - x = t$$
$$dx = -dt$$
$$when x = 0; t = a$$
$$when x = a; t = 0$$

$= \int_a^0 f(t)\frac{dt}{-1} = -\int_a^0 f(t)\text{dt}$

$= \int_0^a f(t)\text{dt}$ (by prop 1)

$= \int_0^a f(x)\text{dx}$

Property : 5

$$\int_0^{2a} f(x)\text{dx} = \int_0^a \left[f(x) + f(2a-x) \right] \text{dx}$$

Property : 6

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Property: 7

(i)If $f(-x) = f(x)$ **(Even Function)** **then** $\int_{-a}^{a} f(x)\,\text{dx} = 2\int_0^a f(x)\,\text{dx}$

(ii)If $f(-x) = -f(x)$ **(Odd Function)** **then** $\int_{-a}^{a} f(x)\,\text{dx} = 0$

Solved Problems

1. $Evaluate \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,\text{dx}$

Solution Let $I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,\text{dx}$ ………..(1)

Applying property 4 in (1),we have

$$I = \int_0^{\frac{\pi}{2}} \frac{a\sin(\frac{\pi}{2} - x) + b\cos(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}\,\text{dx}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{a\cos x + b\sin x}{\sin x + \cos x}\,\text{dx}$$ ……………….(2)

Adding (1)and (2),we have

$$2I = \int_0^{\frac{\pi}{2}} \frac{(a+b)\sin x + (a+b)\cos x}{\sin x + \cos x}\,\text{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)(a+b)}{\sin x + \cos x}\,\mathrm{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} (\mathrm{a}+\mathrm{b})\,\mathrm{dx}$$

$$2I = \frac{\pi}{2}(a+b)$$

$$I = \frac{\pi}{4}(a+b)$$

2. $Show that \int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{dx} = -\frac{\pi}{2}\log 2$

$$.Let.I = \int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{dx} \ldots\ldots\ldots\ldots(1)$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin(\frac{\pi}{2} - x)\mathrm{dx}$$

$$I = \int_0^{\frac{\pi}{2}} \log\cos(x)\mathrm{dx} \ldots\ldots\ldots\ldots(2)$$

(1)+(2) implies $2I = \int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{dx} + \int_0^{\frac{\pi}{2}} \log(\cos x)\mathrm{dx}$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x\cos x\mathrm{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} \log\frac{\sin 2x}{2}\mathrm{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin 2x\mathrm{dx} - \int_0^{\frac{\pi}{2}} \log 2\mathrm{dx} \ldots\ldots\ldots\ldots\ldots(.I)$$

$Consider \int_0^{\frac{\pi}{2}} \log\sin 2x\mathrm{dx}$

Let θ=2x then dx =dθ/2

$x = 0$ *then* $\theta = 0$ *and* $x = \frac{\pi}{2}$ *then* $\theta = \pi$

$$= \int_0^{\pi} \log(\sin\theta)\frac{d\theta}{2}$$

(since $\int_0^a f(x)\,dx = 2\int_0^{\frac{a}{2}} f(x)\,dx$)

$$= 2\int_0^{\frac{\pi}{2}} \log(\sin\theta)\frac{d\theta}{2}$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin x)dx = I \quad \text{(by prop 1)}$$

Subsuiting in I,we have

$$2I = I - \int_0^{\frac{\pi}{2}} \log 2dx$$

$$I = -\int_0^{\frac{\pi}{2}} \log 2dx$$

$$I = -\log 2\int_0^{\frac{\pi}{2}} dx$$

$$I = -\frac{\pi}{2}\log 2 .$$

3.*Show that* $\int_0^{\frac{\pi}{4}} \log.(1+\tan x)dx = \frac{\pi}{8}\log 2$

$$Let\ I = \int_0^{\frac{\pi}{4}} \log.(1+\tan x)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[.1+\tan(\frac{\pi}{4}-x)\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log.\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log.\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log.\left[\frac{2}{1+\tan x}\right] dx$$

$$I = \int_0^{\frac{\pi}{4}} [(\log 2 - \log(1+\tan x)] dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2 dx - \int_0^{\frac{\pi}{4}} \log.(1+\tan x) dx$$

$$I = \int_0^{\frac{\pi}{4}} \log 2 dx - I$$

$$2I = \log 2 \int_0^{\frac{\pi}{4}} dx$$

$$2I = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

4. $Evaluate \int_0^{\pi} \log(1+\cos x) dx$

$$Let I = \int_0^{\pi} \log(1+\cos x) dx \quad \ldots\ldots\ldots\ldots(1)$$

$$I = \int_0^{\pi} \log(1+\cos(\pi - x) dx$$

$$I = \int_0^{\pi} \log(1-\cos x) dx \quad \ldots\ldots\ldots\ldots(2)$$

(1)+(2) implies $2I = \int_0^{\pi} \log(1+\cos x) dx + \int_0^{\pi} \log(1+\cos x) dx$

$$2I = \int_0^{\pi} \log(1-\cos x)(1+\cos x) dx$$

$$2I = \int_0^{\pi} \log(1-\cos^2 x) dx = \int_0^{\pi} \log \sin^2 x \, dx = 2\int_0^{\pi} \log \sin x \, dx$$

$$I = \int_0^{\pi} \log \sin x \, dx$$

By the property $f(2a-x) = f(x) then \int_0^{2a} f(2a-x) \, dx = 2\int_0^{a} f(x) \, dx$

$$= 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx$$

$$=2\left[-\frac{\pi}{2}\log 2\right]$$

$$I=\pi\log\frac{1}{2}$$

4. Gamma Function

Definition :

Gamma function is defined as follows

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx \quad ; \ n>0$$

1. $\Gamma(n+1)=n\Gamma(n)$

2. $\Gamma(1)=1$

3. $\Gamma(n+1)=n\,!$,n>0

4. $\Gamma(n).\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}.0<n<1$

5. $\Gamma(\frac{1}{2})=\sqrt{\pi}$

Proof :

$$WKT\ \ \Gamma(n)=\int_0^\infty t^{n-1}e^{-t}dx$$

$$\Gamma(\frac{1}{2})=\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt$$

$put\ \ t=x^2\ \ \ dt=2x\ dx$

$No\ change\ in\ \lim its$

$$\Gamma(\frac{1}{2})=\int_0^\infty (x^2)^{-\frac{1}{2}}e^{-x^2}\,2x\ dx$$

$$\Gamma(\frac{1}{2})=2\int_0^\infty (\frac{1}{x})e^{-x^2}x\ dx$$

$$\Gamma(\frac{1}{2})=2\int_0^\infty e^{-x^2}\ \ dx$$

$Similarly\ ,we\ have\ \ \Gamma(\frac{1}{2})=2\int_0^\infty e^{-y^2}\ \ dx$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \left[2\int_0^\infty e^{-x^2}\ dx\right]\left[2\int_0^\infty e^{-y^2}\ dy\right]$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \left[4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\ dy\right]$$

Transforming into polar coordinates

$$\left\{\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\} \Rightarrow x^2 + y^2 = r^2$$

$dxdy = rdrd\theta$

$r : 0 \to \infty$

$\theta : 0 \to \frac{\pi}{2}$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \left[4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\ dr\ d\theta\right]$$

$$= \left[4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\ d(\frac{r^2}{2})\ d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\ d(r^2)\ d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2}}{-1}\right]_0^\infty d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} \left[\frac{e^{-\infty} - e^0}{-1}\right] d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} d\theta\right] = 2[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2}\right] = \pi$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

$6.\ \Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\, dx$

Proof :

$Sub\ x = t^2\ \ in\ the\ \ formula\ \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$then\ dx = 2\ t\ dt$

$$\Gamma(n) = \int_0^\infty t^{2(n-1)} e^{-t^2}\, (2t\ d\,t)$$

$$\Gamma(n) = 2\int_0^\infty t^{2n-1} e^{-t^2}\, dt$$

$$\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\, dx \qquad (by\ \ property\ 1)$$

5. **Beta Function**

Definition

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx, m > 0\ \&\ n > 0$$

Results:

1. $\beta(m,n) = \beta(n,m)$

Proof :

By definition of beta function, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Using property 4 of definite integral

$$\beta(m,n) = \int_0^1 (1-x)^{m-1}\left[(1-(1-x)\right]^{n-1} dx$$

$$\beta(m,n) = \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$\beta(m,n) = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$$

$$\beta(m,n) = \beta(n,m)$$

2. $\beta(m,n) = 2\int_0^\infty \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

Proof:

$$WKT\ \beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$Take\quad x=\sin^2\theta\quad dx=2\sin\theta\cos\theta\ d\theta$$

$$if\ \ x=0\ then\ \theta=0$$

$$if\ \ x=1\ then\ \theta=\frac{\pi}{2}$$

$$\beta(m,n)=\int_0^{\frac{\pi}{2}}(\sin^2\theta)^{m-1}(\cos^2\theta\}^{n-1}\ 2\sin\theta\cos\theta\ d\theta$$

$$\beta(m,n)=2\int_0^{\infty}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

3. $\beta(m,n)=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx$

6. Relation between Beta and Gamma function

$$\beta(m,n)=\frac{\Gamma\ m\ \Gamma\ n}{\Gamma\ m+n}$$

Proof :WKT $\Gamma(n)=2\int_0^{\infty}x^{2n-1}e^{-x^2}dx$

$$\Gamma(m)\Gamma(n)=\left[2\int_0^{\infty}x^{2m-1}e^{-x^2}dx\right]\left[2\int_0^{\infty}y^{2n-1}e^{-y^2}dx\right]$$

$$\Gamma(m)\Gamma(n)=4\int_0^{\infty}\int_0^{\infty}e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}dxdy$$

Transforming into polar coordinates

$$\left.\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\right\}\Rightarrow x^2+y^2=r^2$$

$$dxdy=rdrd\theta$$

$$r:0\to\infty$$

$$\theta:0\to\frac{\pi}{2}$$

$$\Gamma(m)\Gamma(n)=4\int_0^{\frac{\pi}{2}}\int_0^{\infty}e^{-r^2}r^{2m+2n-2}\cos^{2m-1}\theta\sin^{2n-1}\theta(rdrd\theta)$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r^{2m+2n-1}\cos^{2m-1}\theta \sin^{2n-1}\theta dr d\theta$$

$$\Gamma(m)\Gamma(n) = \left[2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta d\theta)\right]\left[2\int_0^{\infty} r^{2(m+n)-1} e^{-r^2} dr\right]$$

$$\Gamma\ m\ \Gamma\ n = \beta(m,n)\Gamma\ m+n$$

$$\beta(m,n) = \frac{\Gamma\ m\ \Gamma\ n}{\Gamma\ m+n}$$

We can also prove Γ1/2 using the beta gamma relation

Put m=n=1/2

$$\beta(m,n) = \frac{\Gamma\ m\ \Gamma\ n}{\Gamma\ m+n}$$

$$\beta(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma\ \frac{1}{2}\ \Gamma \frac{1}{2}}{\Gamma\ 1}$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \beta(\frac{1}{2},\frac{1}{2}) = \int_0^{\frac{\pi}{2}} \sin^{2(\frac{1}{2})-1}\theta \cos^{2(\frac{1}{2})-1}\theta\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta = 2\ [\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2}\right]$$

$$\left[\Gamma\left[\frac{1}{2}\right]\right]^2 = \pi$$

$$\Rightarrow \Gamma\left[\frac{1}{2}\right] = \sqrt{\pi}$$

Problems :

1.*Evaluate* $\int_0^1 x^6(1-x)^5 dx$

$WKT\,\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m>0\ \&\ n>0$

Taking m-1=6 and n-1=5 we get m=7 and n=6

$$\beta(7,6)=\frac{\Gamma 7\Gamma 6}{\Gamma 13}=\frac{6!x5!}{13!}=\frac{(6x5x4x3x2x1)(5x4x3x2x1)}{13x12x11x10x9x8x7x6x5x4x3x2x1}=\frac{1}{72072}$$

2. $Evaluate\beta(\frac{5}{2},\frac{7}{2})=\int_0^1 x^{m-1}(1-x)^{n-1}dx, m>0 \;\&\; n>0$

$$=\frac{\Gamma\frac{5}{2}\Gamma\frac{7}{2}}{\Gamma 6}=\frac{\frac{3}{2}.\frac{1}{2}.\Gamma\frac{1}{2}.\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}{5!}=\frac{3\pi}{256}$$

3. $Evaluate\int_0^{\frac{\pi}{2}}\sin^6\theta d\theta$

$$\int_0^{\frac{\pi}{2}}\sin^6\theta d\theta=\beta(\frac{7}{2},\frac{1}{2})=\frac{\Gamma\frac{7}{2}\Gamma\frac{1}{2}}{\Gamma 4}=\frac{1}{2}\frac{.\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{3!}=\frac{15\pi}{96}$$

4. $Evaluate\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^5\theta d\theta$

$Take 2m-1=6 and 2n-1=5$

$then m=\frac{7}{2} and n=3$

$$=\frac{1}{2}\beta(\frac{7}{2},3)$$

$$=\frac{1}{2}\frac{\Gamma\frac{7}{2}\Gamma 3}{\Gamma\frac{13}{2}}=\frac{1}{2}\frac{\Gamma\frac{7}{2}\Gamma 3}{\frac{11}{2}\frac{9}{2}\frac{7}{2}\Gamma\frac{7}{2}}=\frac{1}{2}\frac{2!}{\left[\frac{693}{8}\right]}=\frac{8}{693}$$

5. $Evaluate\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}d\theta$

$$Given I=\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}\theta\cos^{\frac{-1}{2}}\theta d\theta$$

$$=\frac{1}{2}\beta(\frac{3}{4},\frac{1}{4})=\frac{1}{2}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma 1}$$

$$=\frac{1}{2}\Gamma\frac{3}{4}\Gamma\frac{1}{4}=\frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{\sqrt{2}}\left[\Theta\;\Gamma(n).\Gamma(1-n)=\frac{\pi}{\sin n\pi}.\right]$$

$$6..Evaluate\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \Gamma n, n > 0$$

$$Put\ \ \log\frac{1}{y} = t$$

$$\frac{1}{y} = e^t$$

$$y = e^{-t}$$

$$dy = -e^{-t} dt$$

$$Limits : y = 0 \Rightarrow t = \infty and y = 1 \Rightarrow t = 0$$

$$\therefore \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \int_\infty^0 (t)^{n-1} (-e^{-t}) dt (by prop 2)$$

$$\int_0^\infty (t)^{n-1} e^{-t} dt = \Gamma n$$

$$7.Evaluate\int_0^\infty e^{-(hx)^2} dx$$

$$put (hx)^2 = t$$

$$hx = t^{\frac{1}{2}}$$

$$dx = \frac{1}{2h} t^{-\frac{1}{2}} dt$$

$$\int_0^\infty e^{-(hx)^2} dx$$

$$when x = 0 \Rightarrow t = 0$$

$$when x = \infty \Rightarrow t = \infty$$

$$then\int_0^\infty e^{-(hx)^2} dx = \int_0^\infty e^{-t} \frac{t^{\frac{-1}{2}}}{2h} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} \frac{t^{\frac{1}{2}-1}}{h} dt$$

$$= \frac{\Gamma\frac{1}{2}}{2h} = \frac{\sqrt{\pi}}{2h}$$

8. Prove that $\int_0^\infty \frac{t^2}{1+t^4}\,dt = \frac{\pi}{2\sqrt{2}}$

$Put \;\; t = \sqrt{\tan\theta}$

$$dt = \frac{1}{2}\tan^{\frac{-1}{2}}\theta(\sec^2\theta)d\theta$$

$$dt = \frac{1+\tan^2\theta}{2\sqrt{\tan\theta}}\,d\theta$$

$$\therefore \int_0^\infty \frac{t^2}{1+t^4}\,dt = \int_0^{\frac{\pi}{2}} \frac{\tan\theta(1+\tan^2\theta)}{(1+\tan^2\theta)2\sqrt{\tan\theta}}d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\cos^{\frac{-1}{2}}\theta d\theta$$

$$= \frac{1}{4}\beta(\frac{3}{4},\frac{1}{4})$$

$$= \frac{1}{4}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma 1}$$

$$= \frac{1}{4}\Gamma\frac{3}{4}\Gamma\frac{1}{4}$$

$$= \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{2\sqrt{2}}$$

9. Evaluate $\int_0^\infty \frac{x^a}{a^x}\,dx \quad where \quad a > 1$

$Let \;\; a^x = e^t$

$t = x\log a \quad , by \; definition \; of \;\; \log arithm$

$\therefore x = \dfrac{t}{\log a} \Rightarrow dx = \dfrac{dt}{\log a}$

$when \; x = 0 \Rightarrow t = 0$

$when \;\; x = \infty \Rightarrow t = \infty$

$$\int_0^\infty \frac{x^a}{a^x}\,dx \;\; = \int_0^\infty \frac{\left(\dfrac{t}{\log\; a}\right)^a}{e^t}\,\frac{dt}{\log\; a}$$

$$= \frac{1}{(\log\; a)^{a+1}} \int_0^\infty e^{-t} t^a\, dt$$

$$= \frac{1}{(\log\; a)^{a+1}} \Gamma(a+1).$$

$$10.\text{Prove that } \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}$$

$$WKT \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

$$Then\ \ \beta(m+1,n) = \frac{\Gamma m+1\ \Gamma n}{\Gamma(m+n+1)} = \frac{m\Gamma m \Gamma n}{\Gamma(m+n+1)}$$

$$\therefore \frac{\beta(m+1,n)}{m} = \frac{\Gamma m \Gamma n}{\Gamma(m+n+1)} \text{..........}1$$

$$similarly\ \ \beta(m,n+1) = \frac{\Gamma m \Gamma n+1}{\Gamma(m+n+1)} = \frac{\Gamma m\ n \Gamma n}{\Gamma(m+n+1)}$$

$$\frac{\beta(m,n+1)}{n} = \frac{\Gamma m \Gamma n}{\Gamma(m+n+1)} \text{.....................}2$$

$$also\ multiply by \frac{1}{m+n} on\ both\ sides\ in \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

$$\frac{\beta(m,n)}{m+n} = \frac{\Gamma m \Gamma n}{(m+n)\Gamma(m+n)}$$

$$\frac{\beta(m,n)}{m+n} = \frac{\Gamma m \Gamma n}{\Gamma(m+n+1)} \text{.................}3$$

from equation 1,2,3 *we have*

$$\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}$$

$$11.\text{Prove that } \frac{\beta(n,\frac{1}{2})}{2^{2n-1}} = \beta(n.n) \text{ and hence deduce the duplication formula}$$

$$WKT\ \ \beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\ \ d\theta$$

$$\beta(n,\frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\ d\theta$$

Also we have $\beta(n,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2n-1}\theta\ \ d\theta$

$$\beta(n,n) = 2\int_0^{\frac{\pi}{2}} (\sin\ \ \theta \cos\theta)^{2n-1}\ d\theta$$

$$\beta(n,n) = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2n-1} d\theta$$

$$\beta(n,n) = \frac{2}{2^{2n-1}} \int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta \ d\theta$$

$$let\ \phi = 2\theta \quad \therefore d\theta = \frac{1}{2} d\phi$$

$$\beta(n,n) = \frac{2}{2^{2n-1}} \int_0^{\frac{\pi}{2}} \sin^{2n-1} \phi \ \frac{d\phi}{2}$$

$$\beta(n,n) = \frac{1}{2^{2n-1}} \int_0^{\frac{\pi}{2}} \sin^{2n-1} \phi \ d\phi$$

$$\beta(n,n) = \frac{1}{2^{2n-1}} \beta(n, \frac{1}{2})$$

$$\beta(n\ , \frac{1}{2}) = 2^{2n-1} \beta(n,n)$$

$$\frac{\Gamma n \Gamma \frac{1}{2}}{\Gamma\left(n + \frac{1}{2}\right)} = 2^{2n-1} \beta(n,n)$$

$$\frac{\Gamma n \Gamma \frac{1}{2}}{\Gamma\left(n + \frac{1}{2}\right)} = 2^{2n-1} \frac{\Gamma n \Gamma n}{\Gamma 2n}$$

$$\Gamma 2n = 2^{2n-1} \frac{\Gamma n \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}}$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT - IV - ORDINARY DIFFERENTIAL EQUATIONS - SMT1113

1 LINEAR EQUATIONS OF HIGHER ORDER

A linear equation of n[th] order with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots.a_n y = X \qquad (1)$$

where a_1, a_2,…….a_n are constants and X is a function of x. This equation can also be written in the form

$$\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots.. + a_n\right)y = X \text{ where } D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, \ldots\ldots.. D^n = \frac{d^n}{dx^n}$$

Consider $\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots.. + a_n\right)y = 0$ (2)

The general solution of equation (2) is given by $Y = c_1 y_1 + c_2 y_2 + \ldots\ldots c_n y_n$

where $y_1, y_2, \ldots\ldots y_n$ are n independent solutions and $c_1, c_2, \ldots\ldots c_n$ are arbitrary constants.

Y is called the complementary function (C.F) of equation (1).

Suppose u is a particular solution (particular integral) of equation (1)

Then the general solution of equation (1) is of the form y=Y+u where Y is the complementary function

and u is a particular integral (P.I).

Thus y=C.F + P.I

To find Complementary functions

Case (1)

Roots of the A.E are real and distinct say m_1 and m_2

y=$c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case (2)

Roots of the A.E are imaginary then

y=$e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$

Case (3)

Roots of the A.E are real and equal say $m_1 = m_2$ then

$y = e^{m_1 x}\left(c_1 x + c_2\right)$

1.Solve $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$

Put $\frac{d}{dx} = D$

$\left(D^2 y - 2Dy + 3y\right) = 0$

$\left(D^2 - 2D + 3\right)y = 0$

The auxiliary equation is $m^2 - 2m + 3 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - (4)(1)(3)}}{(2)(1)}$$

$$m = 2 \pm \frac{\sqrt{-8}}{2}$$

$$m = \frac{2 \pm i2\sqrt{2}}{2}$$

$$m = 1 \pm i\sqrt{2}$$

C.F = $e^{x}\left[c_1 \cos\left(\sqrt{2}x\right)+c_2 \sin\left(\sqrt{2}x\right)\right]$

The general solution is y = C.F+P.I

$$y=e^{x}\left[c_1 \cos\left(\sqrt{2}x\right)+c_2 \sin\left(\sqrt{2}x\right)\right]+0$$

To find Particular integral

When the R.H.S of the given differential equation is a function of x , we have to find particular Integral.

Case (i)

If $f(x)=e^{ax}$, then $P.I=\frac{1}{F(D)}e^{ax}$. Replace D by a in F(D), provided F(D)$\neq$ 0.

If F(a) = 0 then $P.I=\frac{x}{F'(D)}e^{ax}$ provided $F'(a)\neq 0$

If F'(a) = 0 then $P.I=\frac{x}{F''(D)}e^{ax}$ provided $F''(a)\neq 0$ and so on

Case (ii)

If f(x) = sinax or cosax then $P.I=\frac{1}{F(D)}\sin ax or \cos ax$

Replace $D^2 by -a^2$ in F(D), provided F(D)$\neq$ 0.

If F(D) = 0, when we replace D^2 by $-a^2$ then proceed as case (i)

Case (iii)

If f(x) = a^n then $P.I=\frac{1}{F(D)}x^n$

$P.I=\left[F(D)\right]^{-1}x^n$, Expand $\left[F(D)\right]^{-1}$ by using binomial theorem and then operate on x^n.

Case (iv)

If f(x)=e^{ax}x, where X is sinax (or) cosax (or) x then

$$P.I=\frac{1}{F(D)}e^{ax}X=e^{ax}\frac{1}{F(D+a)}X$$

Here $\frac{1}{F(D+a)}X$ can be evaluated by using anyone of the first three types.

Problems

1.Solve $(D^2+6D+9)y=5e^{3x}$

$m^2+6m+9=0$

$(m+3)^2=0$

m=-3,-3

$C.F=(c_1x+c_2)e^{-3x}$

P.I $=\left(\frac{1}{(D^2+6D+9)}\right)5e^{3x}$

$=\left(\frac{1}{(3)^2+6(3)+9}\right)5e^{3x}$

$=\frac{5}{36}e^{3x}$

The general solution is y = C.F + P.I

y= $(c_1x+c_2)e^{-3x}+\frac{5}{36}e^{3x}$

2. Solve $(D^2+6D+5)y=e^{-x}$

$m^2+6m+5=0$

$(m+5)(m+1)=0$

m=-1,-5

C.F = $c_1e^{-x}+c_2e^{-5x}$

P.I $=\left(\frac{1}{(D^2+6D+5)}\right)e^{-x}$

$=\left(\frac{1}{(-1)^2+6(-1)+5}\right)e^{-x}$

$= \frac{x}{2D+6} e^{-x} = \frac{x}{2(-1)+6} e^{-x}$

$= \frac{x}{4} e^{-x}$

The general solution is y = C.F + P.I

y= $c_1 e^{-x} + c_2 e^{-5x} + \frac{x}{4} e^{-x}$

2.Solve $\left(D^2 + D + 1\right) y = \sin 2x$

Solution:

The auxiliary equation is $m^2 + m + 1 = 0$

$m = \frac{-1 \pm i\sqrt{3}}{2}$

C.F= $e^{\frac{-x}{2}} \left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right) \right]$

P.I $= \left(\frac{1}{\left(D^2 + D + 1\right)}\right) \sin 2x$

$= \left(\frac{1}{\left(-4 + D + 1\right)}\right) \sin 2x$

$= \left(\frac{1}{D-3}\right) \sin 2x$

$= \left(\frac{D+3}{D^2 - 9}\right) \sin 2x$

$= \left(\frac{D+3}{-13}\right) \sin 2x$

$=- \frac{2\cos 2x}{13} - \frac{3\sin 2x}{13}$

The general solution is y = C.F + P.I

$$y = e^{\frac{-x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] - \frac{2\cos 2x}{13} - \frac{3\sin 2x}{13}$$

3.Solve $(D^2 + 3D + 2)y = x^2$

Solution:

The auxiliary equation is $m^2 + 3m + 2 = 0$

(m + 2) (m + 1) = 0

Hence m = -2, -1

C.F = $c_1 e^{-2x} + c_2 e^{-x}$

$$\text{P.I} \quad = \left(\frac{1}{(D^2 + 3D + 2)}\right) x^2$$

$$= \frac{1}{2}\left(1 + \frac{3D + D^2}{2}\right)^{-1} x^2$$

$$= \frac{1}{2}\left(1 - \left(\frac{3D + D^2}{2}\right) + \left(\frac{3D + D^2}{2}\right)^2\right) x^2$$

$$= \frac{1}{2}\left(1 - \frac{3D}{2} + \frac{7D^2}{4}\right) x^2$$

$$= \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

The general solution is y = C.F + P.I

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

4. Solve $(D^2 - 4D + 3)y = e^x \cos 2x$

Solution:

The auxiliary equation is $m^2 - 4m + 3 = 0$

(m -1) (m -3) = 0

Hence m = 1, 3

C.F = $c_1e^x + c_2e^{3x}$

P.I = $\left(\frac{1}{(D^2-4D+3)}\right)e^x\cos 2x$

$= \left(\frac{e^x}{(D+1)^2-4(D+1)+3}\right)\cos 2x$

$= \left(\frac{e^x}{D^2-2D}\right)\cos 2x$

$= \left(\frac{e^x}{-4-2D}\right)\cos 2x$

$= -\frac{1}{2}\left(\frac{e^x}{D+2}\right)\cos 2x$

$= -\frac{e^x}{2}\left(\frac{D-2}{D^2-4}\right)\cos 2x$

$= -\frac{e^x}{2}\left[\frac{(D-2)\cos 2x}{-8}\right]$

$= \frac{e^x}{16}(-2\sin 2x - 2\cos 2x)$

$= -\frac{e^x}{8}(\sin 2x + \cos 2x)$

The general solution is y =C.F+ P.I

y = $c_1e^x + c_2e^{3x} - \frac{e^x}{8}(\sin 2x + \cos 2x)$

5. Solve $(D^2 - 2D + 2)y = e^x \sin x$

The auxiliary equation is $m^2 - 2m + 2 = 0$

$m = 1 \pm i$

C.F = $e^x[c_1\cos x + c_2\sin x]$

P.I $= \left(\frac{1}{(D^2-2D+2)}\right)e^x \sin x$

$= \left[\frac{e^x}{(D+1)^2-2(D+1)+2}\right]\sin x$

$= \left[\frac{e^x}{D^2+1}\right]\sin x$

$= \left[\frac{e^x}{(D+i)(D-i)}\right]\sin x$

$= e^x$ Imaginary part of $\left[\frac{1}{(D+i)(D-i)}\right]e^{ix}$

$= e^x$ Imaginary part of $\left[\frac{1}{2i}xe^{ix}\right]$

$= e^x$ Imaginary part of $\left[-\frac{1}{2}ix(\cos x + i\sin x)\right]$

$= -\frac{1}{2}xe^x \cos x$

The general solution is y = C.F + P.I

y = $e^x[c_1\cos x + c_2 \sin x] - \frac{1}{2}xe^x\cos x$

6. Solve $(D^3-3D^2+3D-1)y = x^2e^x$

The auxiliary equation is $m^3-3m^2+3m-1=0$

$(m-1)^3 = 0$

m=1 (thrice)

C.F $= e^x(c_1+c_2x+c_3x^2)$

P.I $= \frac{1}{D^3-3D^2+3D-1}x^2e^x$

$= \left[\frac{e^x}{(D+1)^3-3(D+1)^2+3(D+1)-1}\right]x^2$

$= \qquad e^x\left(\frac{1}{D^3}\right)x^2$

$= \qquad \frac{e^x x^5}{60}$ (By integrating x^2 thrice with respect to x)

The general solution is y = C.F + P.I

$$y = e^x\left(c_1 + c_2 x + c_3 x^2\right) + \frac{e^x x^5}{60}$$

2. Linear Differential Equations with variable coefficients

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + ...a_n y = X$$

Where a_0 , a_1 ,…….a_n are constants and X is a function of x is called Euler's homogeneous linear differential equation.

Equation can be reduced to constant coefficient by means of transformation z = log x.Then

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$, $x^3D^3 = \theta(\theta-1)(\theta-2)$ where $\theta = \frac{d}{dz}$.

1.Solve $x^2y'' - xy' + 4y = \cos(\log x) + x\sin(\log x)$

Solution:

Put z = log x and $\theta = \frac{d}{dz}$

The given equation reduces to

$$[\theta(\theta-1) - \theta + 4]y = \cos z + e^z \sin z$$

$$[\theta^2 - 2\theta + 4]y = \cos z + e^z \sin z$$

The auxiliary equation is m^2 -2m + 4=0

$$m = 1 \pm i\sqrt{3}$$

Hence $C.F = e^z\left(c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z\right)$

$$= x\left[c_1 \cos\left(\sqrt{3}\log x\right) + c_2 \sin\left(\sqrt{3}\log x\right)\right]$$

P.I = $\left[\frac{1}{\theta^2 - 2\theta + 4}\right]\cos z + \left[\frac{1}{\theta^2 - 2\theta + 4}\right]\left(e^z \sin z\right)$

$$= \left[\frac{1}{3 - 2\theta}\right]\cos z + e^z\left[\frac{1}{(\theta+1)^2 - 2(\theta+1) + 4}\right](\sin z)$$

$$= \left[\frac{1}{3 - 2\theta}\right]\cos z + e^z\left[\frac{1}{\theta^2 + 3}\right](\sin z)$$

$$= \left[\frac{3 + 2\theta}{9 - 4\theta^2}\right]\cos z + \frac{e^z \sin z}{(-1+3)}$$

$$= \left[\frac{3 + 2\theta}{13}\right]\cos z + \frac{e^z \sin z}{2} \quad = \frac{1}{13}\left(3\cos z - 2\sin z\right) + \frac{1}{2}e^z \sin z$$

$$= \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{1}{2}x\sin(\log x)$$

The solution is y = C.F + P.I

$$y = x\left[c_1 \cos\left(\sqrt{3}\log x\right) + c_2 \sin\left(\sqrt{3}\log x\right)\right] + \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{1}{2}x\sin(\log x)$$

2.Solve $(x^2D^2 + 2xD + 4)y = x^2 + 2\log x$

Solution:

Put z = log x and $\theta = \frac{d}{dz}$

The given equation reduces to

$$\left[\theta(\theta - 1) + 2\theta + 4\right]y = e^{2z} + 2z$$

$$\left[\theta^2 + \theta + 4\right]y = e^{2z} + 2z$$

The auxiliary equation is $m^2+m+4=0$

$$m = \frac{-1 \pm \sqrt{1-16}}{2} = \frac{-1 \pm i\sqrt{15}}{2}$$

$$\text{C.F} = e^{-\frac{z}{2}}\left[c_1 \cos\left(\frac{\sqrt{15}}{2}\right)z + c_2 \sin\left(\frac{\sqrt{15}}{2}\right)z\right]$$

$$= x^{-\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{15}}{2}\right)\log x + c_2 \sin\left(\frac{\sqrt{15}}{2}\right)\log z\right]$$

$$\text{P.I} = \left[\frac{1}{\theta^2 + \theta + 4}\right]\left(e^{2z} + 2z\right) = P.I_1 + P.I_2$$

$$P.I_1 = \left[\frac{1}{\theta^2 + \theta + 4}\right]\left(e^{2z}\right)$$

$$= = \frac{e^{2z}}{10} = \frac{x^2}{10}$$

$$P.I_2 = \left[\frac{1}{\theta^2 + \theta + 4}\right](2z)$$

$$= \frac{1}{2}\left[\frac{1}{1 + \left(\frac{\theta + \theta^2}{4}\right)}\right](z)$$

$$= \frac{1}{2}\left[1 + \frac{\theta + \theta^2}{4}\right]^{-1} z = \frac{1}{2}\left[1 - \frac{\theta}{4}\right]z$$

$$= \frac{1}{2}\left[z - \frac{1}{4}\right]$$

$$= \frac{1}{2}\log x - \frac{1}{8}$$

The general solution is y= C.F+ $P.I_1 + P.I_2$

$$\text{Y} = x^{-\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{15}}{2}\right)\log x + c_2 \sin\left(\frac{\sqrt{15}}{2}\right)\log z\right] + \frac{x^2}{10} + \frac{1}{2}\log x - \frac{1}{8}$$

3. METHOD OF VARIATION OF PARAMETERS

Example:1

Use the method of variation of parameter to solve $(D^2+4)y = \cot 2x$.

Solution:

A.E is $m^2+4=0$; $m=\pm 2i$

The C. F = $e^{ox}[A\cos 2x+B\sin 2x]$

Now,

$$f_1=\cos 2x \qquad f_2=\sin 2x$$

$$f_1^{'}=-2\sin 2x \qquad f_2^{'}=2\cos 2x$$

$$f_1f_2^{'}-f_1^{'}f_2=2(\cos^2 2x+\sin^2 2x)=2$$

P.I = $P\, f_1+Q\, f_2$

$$P = -\int \frac{f_2 X}{f_1f_2^{'}-f_1^{'}f_2}dx$$

$$=-\int \frac{\sin 2x\ \cot 2x}{2}dx$$

$$P = -\frac{1}{2}\int \cos 2x dx$$

$$=-\frac{1}{4}\sin 2x$$

$$Q=\int \frac{f_1 X}{f_1f_2^{'}-f_1^{'}f_2}dx$$

$$=\int \frac{\cos 2x\ \cot 2x}{2}dx$$

$$=\frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}dx$$

$$=\frac{1}{2}\int \frac{1-\sin^2 2x}{\sin 2x}dx$$

$$=\frac{1}{2}\int (\operatorname{cosec}2x-\sin 2x)dx$$

$$=\frac{1}{2}\left\{-\frac{1}{2}\log(\operatorname{cosec}2x+\cot 2x)+\frac{1}{2}\cos 2x\right\}$$

$\therefore$ P.I=Pf_1+Qf_2

$$=\frac{1}{4}\sin 2x[\cos 2x-\log(\cos ec2x+\cot 2x)]-\frac{1}{4}\cos 2x\sin 2x$$

$$=-\frac{1}{4}\sin 2x\ \log(\cos ec2x+\cot 2x)$$

$\therefore$ The complete solution is

$$y=(A\cos 2x+B\sin 2x)-\frac{1}{4}\sin 2x\ \log(\cos ec2x+\cot 2x)$$

Examples :2

Solve $(D^2+a^2)y$ = secax by the method of variation of parameters.

Solution:

Given $(D^2+a^2)y$ = secax

A.E is $m^2+a^2=0$

$m=\pm ai$

$\therefore$ $C.F=A\cos ax+B\sin ax$

$$f_1=\cos ax \qquad f_2=\sin ax$$
$$f_1'=-a\sin ax \qquad f_2'=a\cos ax$$
$$f_1f_2'-f_1'f_2=a\cos^2 ax+a\sin^2 ax=a$$

$$P=-\int\frac{f_2X}{f_1f_2'-f_1'f_2}dx$$

$$=-\int\frac{\sin ax\ \sec ax}{a}dx$$

$$=-\frac{1}{a}\int\sin ax\frac{1}{\cos ax}dx$$

$$=-\frac{1}{a}\int\frac{\sin ax}{\cos ax}dx = \frac{1}{a^2}\log[\cos ax]$$

$$Q=\int\frac{f_1X}{f_1f_2'-f_1'f_2}dx$$

$$=\int\frac{\cos ax\ \sec ax}{a}dx$$

$$=\frac{1}{a}\int\cos ax\frac{1}{\cos ax}dx=\frac{1}{a}x$$

$$\therefore P.I = P\ f_1 + Qf_2 = \frac{1}{a^2}\log(\cos ax)\cos ax + \frac{1}{a}x\sin ax$$

$\therefore$ Complete solution y=C.F+P.I.

Example :3

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x\ \log x$ by using method of variation of parameter.

Solution:

A.E is m^2-2m+1=0

C.F is $(Ax+B)e^x$

Where $f_1 = xe^x$ $\quad\quad$ $f_2 = e^x$

$$f_1' = xe^x + e^x \quad\quad f_2' = e^x$$

$$f_1 f_2' - f_1' f_2 = xe^{2x} - (xe^x + e^x)e^x = -e^{2x}$$

P.I=Pf_1+Qf_2 Where

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{e^x e^x \log\ x}{-e^{2x}} dx$$

$$= \int \log\ x\ \ dx$$

$$= x\ \log x - x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}$$

$$= \int \frac{xe^x.e^x \log x}{-e^{2x}} dx = -\int x\ \log x\ dx$$

$$= -\int \log\ x d\left(\frac{x^2}{2}\right)$$

$$= -\frac{x^2}{2}\log\ x + \frac{x^2}{4}$$

$\therefore$ P.I $= Pf_1 + Qf_2$

$$= (x \log x - x)xe^x + \left(\frac{-x^2 \log x}{2} + \frac{x^2}{4} \right) e^x$$

s

$$= x^2 e^x \log x - x^2 e^x - \frac{x^2 e^x \log x}{2} + \frac{x^2 e^x}{4}$$

$$= \frac{x^2 e^x \log x}{2} - \frac{3x^2 e^x}{4} = \frac{1}{4} x^2 e^x (2 \log x - 3)$$

The complete solution is

$$y = (Ax + B)e^x + \frac{x^2 e^x}{4}(2 \log x - 3)$$

Example:4

Use the method of variation of parameter to solve $\left(D^2 + a^2\right) y = \cot ax$.

Solution:

A.E is $m^2 + a^2 = 0 \quad m = \pm ai$

Then $C.F = e^{ox} \left[A\cos ax + B\sin ax \right]$

Now,

$$f_1 = \cos ax \qquad\qquad f_2 = \sin ax$$

$$f_1' = -a \sin ax \qquad\qquad f_2' = a \cos ax$$

$$f_1 f_2' - f_1' f_2 = a(\cos^2 ax + \sin^2 ax) = a$$

P.I $= P\, f_1 + Q\, f_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin ax \cot ax}{a} dx$$

$$P = -\frac{1}{a} \int \cos ax dx$$

$$= -\frac{1}{a^2} \sin ax$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos ax \cot ax}{a} dx$$

$$= \frac{1}{a}\int \frac{\cos^2 ax}{\sin ax}dx$$

$$= \frac{1}{a}\int \frac{1-\sin^2 ax}{\sin ax}dx$$

$$= \frac{1}{a}\int (\cos ecax - \sin ax)dx$$

$$= \frac{1}{a}\left\{-\frac{1}{a}\log(\cos ecax + \cot ax) + \frac{1}{a}\cos ax \right\}$$

$\therefore$ P.I=Pf_1+Qf_2

$$= \frac{1}{a^2}\sin ax[\cos ax - \log(\cos ecax + \cot ax)] - \frac{1}{a^2}\cos ax \sin ax$$

$$=- -\frac{1}{a^2}\sin ax \log(\cos ecax + \cot ax)$$

$\therefore$ The complete solution is

$$y = (A\cos ax + B\sin ax) - \frac{1}{a^2}\sin ax \log(\cos ecax + \cot ax)$$

Example:5

Solve $(D^2-1)y = \frac{1}{1+e^x}$ by using method of variation of parameter.

Solution:

A.E is $m^2-1=0$

C.F is $Ae^x + Be^{-x}$

Where $f_1 = e^x$ $\quad f_2 = e^{-x}$

$f_1^{'} = e^x$ $\quad f_2^{'} = -e^{-x}$

$f_1 f_2^{'} - f_1^{'} f_2 = -e^x e^{-x} - e^{-x}e^x = -2$

P.I=Pf_1+Qf_2 Where

$$P = -\int \frac{f_2 X}{f_1 f_2^{'} - f_1^{'} f_2}dx$$

$$= -\int \frac{e^{-x}}{-2(1+e^x)}dx$$

$put\ e^x = t \Rightarrow e^x dx = dt$

$$= \frac{1}{2}\int \frac{1}{t^2(1+t)}dt$$

$$= \frac{1}{2}\int (\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{1+t})dt$$

$$= \frac{1}{2}\left[-\log t - \frac{1}{t} + \log(1+t) \right]$$

$$= \frac{1}{2}\left[-x - e^{-x} + \log(1+e^x) \right]$$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2}$$

$$= \int \frac{e^x}{-2(1+e^x)} dx$$

$$put\ 1+e^x = t \Rightarrow e^x dx = dt$$

$$= -\frac{1}{2}\int \frac{1}{t} dt$$

$$= -\frac{1}{2}\log(1+e^x)$$

P.I = $Pf_1 + Qf_2$ $= \frac{e^x}{2}\left[-x - e^{-x} + \log(1+e^x) \right] - \frac{e^{-x}}{2}\log(1+e^x)$

The complete solution is $y(x) = Ae^x + Be^{-x} + \frac{e^x}{2}\left[-x - e^{-x} + \log(1+e^x) \right] - \frac{e^{-x}}{2}\log(1+e^x)$

MORE PROBLEMS

1. **Solve** $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

Solution: Given $\left(D^2-5D+6\right)y=0$

The Auxiliary equation (A.E) is $m^2-5m+6=0$

$$(m-2)(m-3)=0$$

$m_1=2,\ m_2=3$ The roots are real and distinct.

Complementary function is (C.F) $= Ae^{m_1x}+Be^{m_2x}=Ae^{2x}+Be^{3x}$, Since $R.H.S=0$ $\therefore P.I.=0$

$\therefore$ The general solution is $y=Ae^{2x}+Be^{3x}$

2. **Solve** $\left(D^3+D^2-D-1\right)y=0$

Solution: The A.E. is $m^3+m^2-m-1=0$

$m^2(m+1)-1(m+1)=0$

$\left(m^2-1\right)(m+1)=0$

$m^2=1,\ m=-1$ $\quad m=\pm 1,\ m=-1$ $\quad m_1=1,\ m_2=m_3=-1$

Roots are real, distinct and equal

$\therefore C.F.=Ae^{m_1x}+(Bx+C)e^{m_2x}=Ae^{x}+(Bx+C)e^{-x}$

$\because R.H.S.=0, \therefore P.I.=0$ $\quad \therefore \quad y=Ae^{x}+(Bx+C)e^{-x}$

3. **Solve** $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$

Solution: Given $\left(D^2-6D+13\right)y=0$

The Auxiliary equation (A.E) is $m^2-6m+13=0$

$m=3\pm 2i$ $\quad (\alpha\pm i\beta)$ $\therefore$ The roots are complex $(\alpha=3,\beta=2)$

C.F. $=e^{\alpha x}\left(A\cos\beta x+B\sin\beta x\right)=e^{3x}\left(A\cos 2x+B\sin 2x\right)$, $\quad \because \quad R.H.S=0$ $\quad \therefore P.I.=0$

$\therefore \quad y=e^{3x}\left(A\cos 2x+B\sin 2x\right)$

4. **Find the solution of x from** $\frac{dy}{dt}=x,\ \frac{dx}{dt}=y$

Solution: Given $Dy=x,\ Dx=y$

$Dy-x=0$ -------- (1) $\quad -y+Dx=0$ ------------ (2)

Eliminate y from (1) and (2), we get

$$\left(D^2 - 1\right)x = 0$$

A.E. is $m^2 - 1 = 0$, $\quad m = \pm 1$

$C.F. = Ae^t + Be^{-t}$ $\quad$ Since $R.H.S. = 0 \Rightarrow P..I. = 0 \;\therefore\; x(t) = Ae^t + Be^{-t}$

5. Solve by the method of variation of parameters $\dfrac{d^2y}{dx^2} + 4y = \sec 2x$

SOLUTION:

Given $\left(D^2 + 4\right)y = \sec 2x$

The A.E. is

$m^2 + 4 = 0 \qquad m = \pm 2i$

$C.F. = c_1 \cos 2x + c_2 \sin 2x$

$f_1 = \cos 2x, \quad f_2 = \sin 2x$

$f_1' = -2\sin 2x, \quad f_2' = 2\cos 2x$

$$f_1 f_2' - f_1' f_2 = 2\left(\cos^2 2x + \sin^2 2x\right) = 2$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}dx = -\int \frac{\sin 2x \sec 2x}{2}dx = -\frac{1}{2}\int \tan 2x\,dx = -\frac{1}{4}\log\left[\sec 2x\right]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}dx = \int \frac{\cos 2x \sec 2x}{2}dx = \frac{1}{2}\int dx = \frac{x}{2}$$

$$P.I. = f_1 P + f_2 Q = -\frac{1}{4}\cos 2x \log\left[\sec 2x\right] + \sin 2x\left(\frac{x}{2}\right)$$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log\left[\sec 2x\right] + \sin 2x\left(\frac{x}{2}\right)$$

6. Solve by the method of variation of parameters $\dfrac{d^2y}{dx^2} + y = \tan x$

SOLUTION:

Given $\left(D^2 + 1\right)y = \tan x$

The A.E. is $m^2 + 1 = 0$

$m = \pm i$

$C.F. = c_1 \cos x + c_2 \sin x$

$f_1 = \cos x, \quad f_2 = \sin x$

$f_1' = -\sin x, \quad f_2' = \cos x$

$$f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}dx = -\int \frac{\sin x \tan x}{1}dx = -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx$$

$$= -\int\left(\sec x - \cos x\right)dx = -\log\left(\sec x + \tan x\right) + \sin x$$

$$Q=\int \frac{f_1 X}{f_1\, f_2\, '-f_1\, '\, f_2}dx=\int \frac{\cos x\ \tan x}{1}dx=\int \sin x\, dx=-\cos x$$

$$P.I.=f_1 P+f_2 Q=\cos x\left[-\log\left(\sec x+\tan x\right)+\sin x\right]-\sin x.\cos x$$

$$y=c_1\cos x+c_2\sin x+\cos x\left[-\log\left(\sec x+\tan x\right)+\sin x\right]-\sin x.\cos x$$

7. Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+y=x\sin x$

SOLUTION:

Given $\left(D^2+1\right)y=x\sin x$

The A.E. is

$m^2+1=0$

$m=\pm i$

$C.F.=c_1\cos x+c_2\sin x$

$f_1=\cos x,\quad f_2=\sin x$

$f_1\,'=-\sin x,\quad f_2\,'=\cos x$

$f_1\, f_2\,'-f_1\,'\, f_2=\cos^2 x+\sin^2 x=1$

$$P=-\int \frac{f_2 X}{f_1\, f_2\, '-f_1\, '\, f_2}dx=-\int \frac{\sin x\ x\sin x}{1}dx=-\int x\sin^2 x\, dx=-\int x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=-\frac{1}{2}\int\left(x-x\cos 2x\right)dx=-\frac{1}{2}\left[\frac{x^2}{2}\right]+\frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right)-(1)\left(\frac{-\cos 2x}{4}\right)\right]$$

$$=-\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x$$

$$Q=\int \frac{f_1 X}{f_1\, f_2\, '-f_1\, '\, f_2}dx=\int \frac{\cos x\ x\sin x}{1}dx=\frac{1}{2}\int x\sin 2x\, dx$$

$$=\frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]=-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x$$

$$P.I.=f_1 P+f_2 Q=\cos x\left[-\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x\right]+\sin x\left[-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x\right]$$

$$y=c_1\cos x+c_2\sin x+\cos x\left[-\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x\right]+\sin x\left[-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x\right]$$

8. Solve $\left(D^2+4D+3\right)y=e^{-x}\sin x$

SOLUTION:

Given $\left(D^2+4D+3\right)y=e^{-x}\sin x$

The A.E. is $m^2+4m+3=0$

$(m+1)(m+3)=0$

$$m=-1,-3$$

$$C.F.=Ae^{-x}+Be^{-3x}$$

$$P.I.=\frac{1}{D^2+4D+3}\left(e^{-x}\sin x\right)=e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$$

$$=e^{-x}\frac{1}{D^2-2D+1+4D-4+3}\sin x=e^{-x}\frac{1}{D^2+2D}\sin x=e^{-x}\frac{1}{-1+2D}\sin x=e^{-x}\frac{2D+1}{(2D)^2-1}\sin x$$

$$=e^{-x}\frac{2D+1}{4D^2-1}\sin x=e^{-x}\frac{2D+1}{-4-1}\sin x=e^{-x}\frac{2D+1}{-5}\sin x$$

$$P.I.=-\frac{e^{-x}}{5}\left[2\cos x+\sin x\right]$$

$$y=C.F.+P.I.=Ae^{-x}+Be^{-3x}-\frac{e^{-x}}{5}\left[2\cos x+\sin x\right]$$

9. **Solve $\left(D^2-4D+4\right)y=e^{2x}+x^2$**

Solution:

The A.E. is $m^2-4m+4=0$

$$(m-2)^2=0$$

$$m=2,2$$

$$C.F.=(Ax+B)e^{2x}$$

$$P.I_1=\frac{1}{D^2-4D+4}e^{2x}$$

$$=\frac{1}{4-8+4}e^{2x}=x\frac{1}{2D-4}e^{2x}=x\frac{1}{4-4}e^{2x}=\frac{x^2}{2}e^{2x}$$

$$P.I_2=\frac{1}{D^2-4D+4}x^2=\frac{1}{4\left(1+\frac{D^2-4D}{4}\right)}x^2=\frac{1}{4}\left(1+\frac{D^2-4D}{4}\right)^{-1}x^2$$

$$=\frac{1}{4}\left(1-\left(\frac{D^2-4D}{4}\right)+\left(\frac{D^2-4D}{4}\right)^2-\cdots\right)x^2=\frac{1}{4}\left(1-\frac{D^2}{4}+D+\frac{D^4+16D^2-8D^3}{16}-\cdots\right)x^2$$

$$=\frac{1}{4}\left(1+D-\frac{D^2}{4}+D^2\right)x^2=\frac{1}{4}\left(1+D+\frac{3D^2}{4}\right)x^2=\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$$

$$P.I=P.I_1+P.I_2$$

$$\therefore\ y=(Ax+B)e^{2x}+\frac{x^2}{2}e^{2x}+\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$$

10. **Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=\sin 3x\cos 2x$**

SOLUTION:

Given $\left(D^2-4D+3\right)y=\sin 3x\cos 2x$

The A.E. is $m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$$m = 1, 3$$

$C.F. = Ae^{x} + Be^{3x}$

$$P.I = \frac{1}{D^2 - 4D + 3}(\sin 3x \cos 2x) = \frac{1}{D^2 - 4D + 3}\frac{1}{2}(\sin 5x + \sin x)$$

$$= \frac{1}{D^2 - 4D + 3}\frac{1}{2}\sin 5x + \frac{1}{D^2 - 4D + 3}\frac{1}{2}\sin x$$

$$= P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{2}\frac{1}{D^2 - 4D + 3}\sin 5x = \frac{1}{2}\frac{1}{-25 - 4D + 3}\sin 5x = \frac{1}{2}\frac{1}{-4D - 22}\sin 5x = -\frac{1}{4}\frac{1}{2D + 11}\sin 5x$$

$$= -\frac{1}{4}\frac{1}{(2D+11)}\frac{(2D-11)}{(2D-11)}\sin 5x = -\frac{1}{4}\frac{2D - 11}{4D^2 - 121}\sin 5x = -\frac{1}{4}\frac{2D-11}{4(-25) - 121}\sin 5x$$

$$= -\frac{1}{4}\frac{2D - 11}{-221}\sin 5x = \frac{1}{884}[10\cos 5x - 11\sin 5x]$$

$$P.I_2 = \frac{1}{2}\frac{1}{D^2 - 4D + 3}\sin x = \frac{1}{2}\frac{1}{-1 - 4D + 3}\sin x = \frac{1}{2}\frac{1}{2 - 4D}\sin x$$

$$= \frac{1}{2}\frac{2 + 4D}{4 - 16D^2}\sin x = \frac{1}{2}\frac{2\sin x + 4\cos x}{20} = \frac{\sin x + 2\cos x}{20}$$

$$y = C.F + P.I_1 + P.I_2 = Ae^{x} + Be^{3x} + \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{\sin x + 2\cos x}{20}$$

11. **Solve $\frac{dx}{dt} + 2y = -\sin t, \frac{dy}{dt} - 2x = \cos t$**

SOLUTIO N:

$$\frac{dx}{dt} + 2y = -\sin t, \frac{dy}{dt} - 2x = \cos t$$

$(i.e.)\ Dx + 2y = -\sin t$........(1); $Dy - 2x = \cos t$.......(2)

$(1)\times 2 \Rightarrow 2Dx + 4y = -2\sin t$...........(3)

$(2)\times D \Rightarrow -2Dx + D^2 y = -\sin t$............(4)

$(3)+(4) \Rightarrow (D^2 + 4)y = -3\sin t$

The A.E. is $m^2 + 4 = 0$

$$m = \pm 2i$$

$C.F. = A\cos 2t + B\sin 2t$

$$P.I. = \frac{1}{D^2 + 4}(-3\sin t) = -3\frac{\sin t}{3} = -\sin t$$

$y = A\cos 2t + B\sin 2t - \sin t$

$Dy = -2A\sin 2t + 2B\cos 2t - \cos t$

$\therefore (2) \Rightarrow \quad 2x = Dy - \cos t$

$$= -2A\sin 2t + 2B\cos 2t - \cos t - \cos t$$

$$x = -A\sin 2t + B\cos 2t - \cos t$$

12. **Solve $\frac{dx}{dt}-7x+y=0,\ \frac{dy}{dt}-2x-5y=0$**

SOLUTION:

$(D-7)x+y=0$.............(1)

$-2x+(D-5)y=0$..........(2)

$(1)\times 2 \Rightarrow \quad 2(D-7)x+2y=0$...........(3)

$(2)\times(D-7)\Rightarrow \ -2(D-7)x+(D-5)(D-7)y=0$.............(4)

$(3)+(4)\Rightarrow \left(D^2-12D+37\right)y=0$

A.E. is $\quad m^2-12m+37=0$

$m^2-12m+36+1=0$

$(m-6)^2+1=0$

$(m-6)^2=-1$

$m=6\pm i$

$\therefore \quad y=e^{6t}\left(A\cos t+B\sin t\right)$

$(2)\Rightarrow -2x=-(D-5)y$

$x=\frac{1}{2}Dy-\frac{5}{2}y$

$=\frac{1}{2}\left[e^{6t}\left(-A\sin t+B\cos t\right)+6e^{6t}\left(A\cos t+B\sin t\right)\right]-\frac{5}{2}e^{6t}\left(A\cos t+B\sin t\right)$

$x=\frac{1}{2}\left[(A+B)e^{6t}\cos t+(B-A)\mathrm{e}^{6t}\sin t\right]$

13. **Solve $\frac{dx}{dt}+\frac{dy}{dt}+x+y=10e^t,\ \frac{dx}{dt}-\frac{dy}{dt}+x-y=0$ given $x(0)=2,\ y(0)=3$**

SOLUTION:

Given $\frac{dx}{dt}+\frac{dy}{dt}+x+y=10e^t,\ \frac{dx}{dt}-\frac{dy}{dt}+x-y=0$

$(i.e.)\ Dx+Dy+x+y=10e^t$.........(1); $\quad Dx-Dy+x-y=0$.........(2)

$(1)+(2)\Rightarrow 2Dx+2x=10e^t$

$Dx+x=5e^t$

$(D+1)x=5e^t$......(3)

A.E. is $m+1=0;\quad m=-1$

$\therefore \quad C.F.=Ae^{-t}$

$P.I.=\frac{1}{D+1}5e^t=5\frac{e^t}{2}$

$\therefore\ x=C.F+P.I=Ae^{-t}+\frac{5}{2}e^t$.......(4)

$(1)-(2)\Rightarrow 2Dy+2y=10e^t$

$$Dy + y = 5e^t$$
$$(D+1)y = 5e^t$$

A.E. is $m+1=0;\ m=-1$

$$C.F. = Be^{-t}$$
$$P.I. = \frac{1}{D+1}5e^t = 5\frac{e^t}{2}$$
$$\therefore \quad y = Be^{-t} + \frac{5}{2}e^t \text{...........(5)}$$

Given $x(0)=2,\ y(0)=3$

$$(4) \Rightarrow 2 = A + \frac{5}{2}$$
$$A = 2 - \frac{5}{2} = \frac{-1}{2}$$
$$(5) \Rightarrow 3 = B + \frac{5}{2}$$
$$B = 3 - \frac{5}{2} = \frac{1}{2}$$
$$\therefore \quad (4) \Rightarrow x = \frac{-1}{2}e^{-t} + \frac{5}{2}e^t$$
$$(5) \Rightarrow y = \frac{1}{2}e^{-t} + \frac{5}{2}e^t$$

14. **Solve $\left(x^2D^2 - 3xD + 4\right)y = x^2\cos(\log x)$**

SOLUTION:

Put $x = e^z \Rightarrow \log x = z$

$$xD = D'$$
$$x^2D^2 = D'(D'-1)$$
$$(1) \Rightarrow \left[D'(D'-1) - 3D' + 4\right]y = e^{2z}\cos z$$
$$\left[D'^2 - 4D' + 4\right]y = e^{2z}\cos z$$
$$\left[D' - 2\right]^2 y = e^{2z}\cos z$$

The A.E. is

$$(m-2)^2 = 0 \qquad m = 2, 2$$
$$C.F. = (Az+B)e^{2z}$$
$$P.I. = \frac{1}{(D'-2)^2}e^{2z}\cos z = e^{2z}\frac{1}{D'^2}\cos z = e^{2z}\frac{1}{D'}(-\sin z) = -e^{2z}\cos z$$
$$\therefore y = C.F + P.I = (Az+B)e^{2z} - e^{2z}\cos z = (A\log x + B)x^2 - x^2\cos(\log x)$$

15. **Solve $\left(x^2D^2 - 2xD - 4\right)y = x^2 + 2\log x$**

SOLUTION:

Given $\left(x^2D^2-2xD-4\right)y=x^2+2\log x$.............(1)

Put $x=e^z \Rightarrow \log x=z$

$$xD=D'$$

$$x^2D^2=D'\left(D'-1\right)$$

$(1)\Rightarrow \left[D'\left(D'-1\right)-2D'-4\right]y=e^{2z}+2z$

$$\left[D'^2-3D'-4\right]y=e^{2z}+2z$$

The A.E. is $m^2-3m-4=0$

$$m^2-4m+m-4=0$$

$$m\left(m-4\right)+\left(m-4\right)=0$$

$$\left(m-4\right)\left(m+1\right)=0$$

$$m=4,\ -1$$

$C.F.=Ae^{-z}+Be^{4z}$

$$P.I_1=\frac{1}{D'^2-3D'-4}e^{2z}=\frac{1}{4-6-4}e^{2z}=\frac{-1}{6}e^{2z}$$

$$P.I_2=\frac{1}{D'^2-3D'-4}\left(2z\right)=\frac{-2}{4}\left[\frac{1}{1-\frac{D'^2-3D'}{4}}\right]z=-\frac{1}{2}\left[1-\frac{D'^2-3D'}{4}+\ldots\right]z$$

$$=-\frac{1}{2}\left[z-\frac{3}{4}(1)\right]=-\frac{1}{2}z+\frac{3}{8}$$

$$\therefore\ y=Ae^{-z}+Be^{4z}-\frac{1}{6}e^{2z}-\frac{1}{2}z+\frac{3}{8}$$

$$=Ae^{-\log x}+Be^{4\log x}-\frac{1}{6}e^{2\log x}-\frac{1}{2}\log x+\frac{3}{8}$$

$$y=\frac{A}{x}+Bx^4-\frac{1}{6}x^2-\frac{1}{2}\log x+\frac{3}{8}$$

16. **Solve** $\boldsymbol{\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}}$

SOLUTION:

Given $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$ (i.e.) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=12\log x$

$\left[x^2D^2+xD\right]y=12\log x$..........(1)

Put $x=e^z \Rightarrow \log x=z$

$xD=D'$

$x^2D^2=D'\left(D'-1\right)$

$(1) \Rightarrow \left[D'\left(D'-1\right)+D'\right]y=12z$

$\left[D'^2\right]y=12z$

The A.E. is $m^2=0$

$C.F.=\left(Az+B\right)e^{0z}=Az+B$

$P.I.=\frac{1}{D'^2}\left(12z\right)=12\frac{1}{D'}\left(\frac{z^2}{2}\right)=12\frac{z^3}{6}=2z^3$

$\therefore\ y=\left(Az+B\right)+2z^3=A\log x+B+2\left(\log x\right)^3$

17. **Solve** $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=\left(1+x\right)^2$

SOLUTION:

Given $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=\left(1+x\right)^2$ (*i.e.*) $\left(x^2D^2-3xD+4\right)y=\left(1+x\right)^2$ ....(1)

Put $x=e^z \Rightarrow \log x=z$

$xD=D'$

$x^2D^2=D'\left(D'-1\right)$

$(1) \Rightarrow \left[D'\left(D'-1\right)-3D'+4\right]y=\left(1+e^z\right)^2$

$\left[D'^2-D'-3D'+4\right]y=1+e^{2z}+2e^z$

$\left[D'-2\right]^2 y=e^{0z}+e^{2z}+2e^z$

The A.E. is $\left(m-2\right)^2=0$

$m=2,2$

$C.F.=\left(Az+B\right)e^{2z}=\left(A\log x+B\right)x^2$

$P.I_1=\frac{1}{\left(D'-2\right)^2}e^{0z}=\frac{1}{4}e^{0z}=\frac{1}{4}$

$P.I_2=\frac{1}{\left(D'-2\right)^2}e^{2z}$

$=\frac{1}{\left(2-2\right)^2}e^{2z}$

$=z\frac{1}{2\left(D'-2\right)}e^{2z}$

$=\frac{z}{2}\frac{1}{2-2}e^{2z}$

$=\frac{z^2}{2}e^{2z}=\frac{\left(\log x\right)^2x^2}{2}$

$$P.I_3 = \frac{1}{(D'-2)^2} 2e^z = 2e^z \frac{1}{(1-2)^2} = 2e^z = 2x$$

$$\therefore \quad y = C.F + P.I_1 + P.I_2 + P.I_3 = (A\log x + B)x^2 + \frac{1}{4} + \frac{(\log x)^2 x^2}{2} + 2x$$

18. **Solve $(3x+2)^2 y'' + 3(3x+2)y' - 36y = 3x^2 + 4x + 1$**

SOLUTION:

Given $(3x+2)^2 y'' + 3(3x+2)y' - 36y = 3x^2 + 4x + 1$

Put $3x + 2 = e^z \Rightarrow \log(3x+2) = z$

$$x = \frac{e^z}{3} - \frac{2}{3}$$

Let $(3x+2)D = 3D'$

$$(3x+2)^2 D^2 = 9D'(D'-1)$$

$$\left[9D'(D'-1) + 3(3D') - 36\right]y = 3\left[\frac{e^z}{3} - \frac{2}{3}\right]^2 + 4\left[\frac{e^z}{3} - \frac{2}{3}\right] + 1$$

$$\left[9D'^2 - 9D' + 9D' - 36\right]y = 3\left[\frac{e^{2z}}{9} + \frac{4}{9} - \frac{4}{9}e^z\right] + \frac{4}{3}e^z - \frac{8}{3} + 1$$

$$\left[9D'^2 - 36\right]y = \frac{e^{2z}}{3} - \frac{1}{3}$$

$$\left[D'^2 - 4\right]y = \frac{1}{27}e^{2z} - \frac{1}{27}$$

The A.E. is $m^2 - 4 = 0$

$$m = \pm 2$$

$$C.F. = Ae^{2z} + Be^{-2z} = A(3x+2)^2 + B(3x+2)^{-2}$$

$$P.I_1 = \frac{1}{D'^2-4}\left(\frac{e^{2z}}{27}\right) = \frac{1}{27}\frac{1}{4-4}e^{2z} = \frac{1}{27}z\frac{1}{2D'}e^{2z} = \frac{z}{54}\frac{e^{2z}}{2} = \frac{ze^{2z}}{108} = \frac{\log(3x+2)}{108}(3x+2)^2$$

$$P.I_2 = \frac{1}{D'^2-4}\left(\frac{e^{0z}}{27}\right) = -\frac{1}{108}$$

$$y = C.F + P.I_1 - P.I_2 = A(3x+2)^2 + B(3x+2)^{-2} + \frac{\log(3x+2)}{108}(3x+2)^2 + \frac{1}{108}$$

19. **Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\left[\log(1+x)\right]$**

SOLUTION:

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\left[\log(1+x)\right]$$

Put $1 + x = e^z$

$$z = \log(1+x)$$

Then $(x+1)D = D'$

$$(x+1)^2 D^2 = D'(D'-1)$$

$$\therefore \quad \left[D'(D'-1) + D' + 1\right] y = 2\sin z$$

$$\left[D'^2 + 1\right] y = 2\sin z$$

The A.E. is $m^2 + 1 = 0$

$$m = \pm i$$

$$C.F. = A\cos z + B\sin z = A\cos\left[\log(1+x)\right] + B\sin\left[\log(1+x)\right]$$

$$P.I. = \frac{1}{D'^2+1} 2\sin z = 2\frac{1}{D'^2+1}\sin z \; = 2\frac{1}{-1+1}\sin z$$

$$= 2z\frac{1}{2D'}\sin z = z\frac{1}{D'}\sin z = z(-\cos z) = -\log(1+x)\cos\left[\log(1+x)\right]$$

$$\therefore \quad y = C.F. + P.I$$

$$= A\cos\left[\log(1+x)\right] + B\sin\left[\log(1+x)\right] - \log(1+x)\cos\left[\log(1+x)\right]$$

20. **Solve** $\boldsymbol{(2+x)^2 \frac{d^2y}{dx^2} - (2+x)\frac{dy}{dx} + y = 2+x}$

SOLUTION:

Put $2 + x = e^z \Rightarrow \log(2+x) = z$

$$x = e^z - 2$$

Let $(2+x)D = D'$

$$(2+x)^2 D^2 = D'(D'-1)$$

Then

$$\left[D'(D'-1) - D' + 1\right] y = e^z$$

$$\left[D'^2 - 2D' + 1\right] y = e^z$$

$$\left[D'-1\right]^2 y = e^z$$

The A.E. is

$$(m-1)^2 = 0$$

$$m = 1,1$$

$$\text{C.F.} = (\text{Az}+\text{B})e^z = \left(A\log(2+x) + B\right)(2+x)$$

$$P.I. = \frac{1}{\left[D'-1\right]^2} e^z = z\frac{1}{2D'} e^z = \frac{ze^z}{2} = \frac{1}{2}\left[\log(2+x)\right](2+x)$$

$$y = \left(A\log(2+x) + B\right)(2+x) + \frac{1}{2}\left[\log(2+x)\right](2+x)$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT - V - PROBABIBILITY AND DISTRIBUTIONS - SMT1113

1. Introduction to Probability

Probabilities are associated with experiments where the outcome is not known in advance or cannot be predicted. For example, if you toss a coin, will you obtain a head or tail? If you roll a die will obtain 1, 2, 3, 4, 5 or 6? Probability measures and quantifies "how likely" an event, related to these types of experiment, will happen. The value of a probability is a number between 0 and 1 inclusive. An event that cannot occur has a probability (of happening) equal to 0 and the probability of an event that is certain to occur has a probability equal to 1.(see probability scale below).

In order to quantify probabilities, we need to define the sample space of an experiment and the events that may be associated with that experiment.

2. Sample space and Events

The sample space is the set of all possible outcomes in an experiment. We define an event as some specific outcome of an experiment. An event is a subset of the sample space.

Examples :

(i) If a die is rolled, the sample space S is given by S = {1,2,3,4,5,6}.

(ii) If two coins are tossed, the sample space S is given by S = {HH,HT,TH,TT} , where H = head and T = tail.

(iii) If two dice are rolled, the sample space S is given by

S = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) }

Probability theory is based on some axioms that act as the foundation for the theory, so let us state and explain these axioms.

3. Axioms of Probability:

Axiom 1: For any event A, $P(A) \geq 0$.

Axiom 2: Probability of the sample space S is $P(S) = 1$.

Axiom 3: If A_1, A_2, A_3, ⋯ are disjoint events, then $P(A_1 \cup A_2 \cup A_3 \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$

4. Mutually Exclusive Events

Two events are mutually exclusive if they cannot occur at the same time. Example, A die is rolled. The event of getting an even number on the face of the die and the event of getting an odd number are mutually exclusive events.

Independent events

Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

Example 1:

Two coins are tossed, find the probability that two heads are obtained.

Solution:

The sample space S is given by S = {(H,T),(H,H),(T,H),(T,T)}
Let E be the event "two heads are obtained" E = {(H,H)}
$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$.

A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamond.

Example 2:

A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamond.

Solution:

Let E be the event "getting the 3 of diamond". An examination of the sample space shows that there is one "3 of diamond" so that n(E) = 1 and n(S) = 52. Hence the probability of event E occurring is given by $P(E) = \frac{n(E)}{n(S)} = \frac{1}{52}$.

Example 3:

A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

Solution:

Total number of marbles in the jar is n(S) = 20
Let E be the event "getting a white marble" is n(E) = 10.
$P(E) = \frac{n(E)}{n(S)} = \frac{10}{20} = \frac{1}{2}$.

Example 4:

Two dice are rolled, find the probability that the sum is equal to 5.
Solution:

Two dice are rolled, the sample space S is given by

S = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) }

Let E be the event "getting a sum equal to 5". Then n(E) = 4 and Then n(S) = 36.
P(E) = $\frac{n(E)}{n(S)} = \frac{4}{36} = \frac{1}{9}$.

Example 5:
A committee of 5 people is to be formed randomly from a group of 10 women and 6 men. Find the probability that the committee has
a) 3 women and 2 men.
b) 4 women and 1men.
c) 5 women.
d) at least 3 women

Solution:

There are $16C_5$ ways to select 5 people (committee members) out of a total of 16 people (men and women). There are $10C_3$ ways to select 3 women out of 10. There are $6C_2$ ways to select 2 men out of 6. There are $10C_3 \times 6C_2$ ways to select 3 women out of 10 and 2 men out of 6.

(a) P(3 women AND 2 men) = $\frac{10C_3 \times 6C_2}{6C_5}$ = 0.412087

(b) P(4 women AND 1 men) = $\frac{10C_4 \times 6C_1}{16C_5}$ = 0.288461

(c) P(5 women) = $\frac{10C_5 \times 6C_0}{16C_5}$ = 0.0576923

(d) P(at least 3 women) = P(3 women or 4 women or 5 women).
Since the events "3 women", "4 women" and "5 women" are all mutually exclusive, then P(at least 3 women) = P(3 women or 4 women or 5 women) = P(3 women) + P(4 women) + P(5 women) = 0.412087 + 0.288461 + 0.0576923 = 0.758240

Example 6:

In a presidential election, there are four candidates. Call them A, B, C, and D. Based on our polling analysis, we estimate that A has a 2020 percent chance of winning the election, while B has a 4040 percent chance of winning. What is the probability that A or B win the election?

Solution:

The events that {A wins}, {B wins}, {C wins}, and {D wins} are disjoint since more than one of them cannot occur at the same time. For example, if A wins, then B cannot win. From the third axiom of probability, the probability of the union of two disjoint events is the summation of individual probabilities. Therefore,

P(A wins or B wins) = P({A wins}∪{B wins})
= P({A wins})+P({B wins})=P({A wins})+P({B wins})
= 0.2+0.4=0.2+0.4
= 0.6

5. Conditional Probability

In this section, we discuss one of the most fundamental concepts in probability theory. Here is the question: as you obtain additional information, how should you update probabilities of events? For example, suppose that in a certain city, 23 percent of the days are rainy. Thus, if you pick a random day, the probability that it rains that day is 23 percent: P(R) = 0.23, where R is the event that it rains on the randomly chosen day.

Now suppose that we pick a random day, but we also tell that it is cloudy on the chosen day. Now that we have this extra piece of information, how do we update the chance that it rains on that day? In other words, what is the probability that it rains given that it is cloudy? If C is the event that it is cloudy, then we write this as P(R|C), the conditional probability of R given that C has occurred. It is reasonable to assume that in this example, P(R|C) should be larger than the original P(R), which is called the prior probability of R. For calculating P(R|C) we have a general formula which is given below.

If A and B are two events in a sample space S, then the conditional probability of A given B is defined as $P(A|B) = \frac{P(A\cap B)}{P(B)}$, when P(B) > 0.

Example 7:

A fair die is rolled. Let A be the event that the outcome is an odd number and let B be the event that the outcome is less than or equal to 3. What is P(A) and P(A|B)?

Solution

Given that S = { 1,2,3,4,5,6}, A = {1,3,5} and B ={1,2,3}.

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

$P(A|B) = \frac{n(A\cap B)}{n(B)} = \frac{2}{3}$.

Example 8:

In a factory there are 100 units of a certain product, 5 of which are defective. We pick three units from the 100 units at random. What is the probability that none of them are defective?

Solution

Let us define A_i as the event that the ith chosen unit is not defective, for i=1,2,3. We are interested in $P(A_1 \cap A_2 \cap A_3)$. Note that $P(A_1) = \frac{95}{100}$

Given that the first chosen item was good, the second item will be chosen from 94 good units and 5 defective units, thus $P(A_2|A_1) = \frac{94}{99}$.

Given that the first and second chosen items were okay, the third item will be chosen from 93 good units and 5 defective units, thus $P(A_3|A_2, A_1) = \frac{93}{98}$.

Thus, we have $P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_2,A_1) = \frac{95}{100} \times \frac{94}{99} \times \frac{93}{98} = 0.8560$

6. BAYE'S THEOREM

Let S be a sample space.

Let A1, A2, ... An be disjoint events in S and B be any arbitrary event in S with

$P(B) \neq 0$. Then Baye's theorem says

$$P(A_r/B) = \frac{P(A_r)\,P(B/A_r)}{\sum_{r=1}^{n} P(A_r)\,P(B/A_r)}$$

EXAMPLE :1

There are two identical boxes containing respectively 4 white and 3 red balls, 3 white and 7 red balls. A box is chosen at random and a ball is drawn from it. Find the probability that the ball is white. If the ball is white, what is the probability that it is from first box?

Solution:

Let A_1, A_2 be the boxes containing 4 white and 3 red balls, 3 white and 7 red balls.

i.e

A_1	A_2
4 White 3 Red Total 7 Balls	3 White 7 Red Total 10 balls

One box is chosen at random out of two boxes.

$\therefore P(A_1) = P(A_2) = \frac{1}{2}$

One ball is drawn from the chosen box. Let B be the event that the drawn ball is white.

$\therefore P(B/A_1)$ = P(that the drawn ball is white from the Ist Box)

$$P(B/A_1) = \frac{4}{7}$$

$\therefore P(B/A_2)$ = P (that the white ball drawn from the IInd Box)

$$\Rightarrow P(B/A_2) = \frac{3}{10}$$

$$\begin{aligned} P(B) &= P \text{ (that the drawn ball is white)} \\ &= P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) \\ &= \frac{1}{2}\,\frac{4}{7} + \frac{1}{2}\,\frac{3}{10} \\ &= \frac{61}{140} \end{aligned}$$

Now by Baye's Theorem, probability that the white ball comes from the Ist Box is,

$$P(B_1/A) = \frac{P(A_1)P(B/A_1)}{P(A_1)P(B/A_1)+P(A_2)P(B/A_2)}$$

$$= \frac{\frac{1}{2}\frac{4}{7}}{\frac{1}{2}\frac{4}{7}+\frac{1}{2}\frac{3}{10}} = \frac{\frac{4}{7}}{\frac{4}{7}+\frac{3}{10}} = \frac{40}{61}$$

EXAMPLE: 2

A factory has 3 machines A_1, A_2, A_3 producing 1000, 2000, 3000 bolts per day respectively. A_1 produces 1% defectives, A_2 produces 1.5% and A_3 produces 2% defectives. A bolt is chosen at random at the end of a day and found defective. What is the probability that it comes from machine A_1?

Solution:

$P(A_1)$ = P(that the machine A_1 produces bolts)

$= \frac{1000}{6000} = \frac{1}{6}$

$P(A_2)$ = P(that the machine A_2 produces bolts)

$= \frac{2000}{6000} = \frac{1}{3}$

$P(A_3)$ = P(that the machine A_3 produces bolts)

$= \frac{3000}{6000} = \frac{1}{2}$

Let B be the event that the chosen bolt is defective

$\therefore P(B/A_1)$= P(that defective bolt from the machine A_1)

$= .01$

Similarly $P(B/A_2)$ = P(that the defective bolt from the machine A_2)
= .015 and

$P(B/A_3)$ = P(that the defective bolt from the machine A_3)
= .02

We haev to find $P(A_1/B)$

Hence by Baye's theorem, we get

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)P(B/A_3)}$$

$$= \frac{\frac{1}{6}\text{x } (.01)}{\frac{1}{6}\text{x } (.01) + \frac{1}{3}\text{x } (.015) + \frac{1}{2}\text{x}(.02)}$$

$$= \frac{.01}{.01 + .03 + .06} = \frac{.01}{.1} = \frac{1}{10}$$

EXAMPLE:3

In a bolt factory machines A_1, A_2, A_3 **manufacture respectively 25%, 35% and 40% of the total output. Of these 5, 4, 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine A_2?**

Solution:

$P(A_1)$ = P(that the machine A_1 manufacture the bolts)

$$= \frac{25}{100} = .25$$

Similarly $P(A_2)$ $= \frac{35}{100} = .35$ and

$P(A_3)$ $= = \frac{40}{100} = .4$

Let B be the event that the drawn bolt is defective.

$\therefore$ $P(B/A_1)$= P(that the defective bolt from the machine A_1)

$$= \frac{5}{100} = .05$$

Similarly $P(B/A_2) = \frac{4}{100} = .04$ and $P(B/A_3) = \frac{2}{100} = .02$

we have to find $P(A_2/B)$

Hence by Baye's theorem, we get

$$P(A_2/B) = \frac{P(A_2)P(B/A_2)}{P(A_1)P(B/A_1) + P(A)P(B/A_2) + P(A_3)P(B/A_3)}$$

$$= \frac{(.35)(.04)}{(.25)(.05) + (.35)(.04) + (.4)(.02)}$$

$$= \frac{28}{69}$$

EXERCISE PROBLEMS

1. There are 3 boxes containing respectively 1 white, 2 red, 3 black balls; 2white, 3 red, 1 black ball : 2 white, 1 red, 2 black balls. A box is chosen at random and from it two balls are drawn at random. The two balls are 1 red and 1 white. What is the probability that they come from the second box?

2. In a company there are three machines A_1, A_2 and A_3. They produce 20%, 35% and 45% of the total output respectively. Previous experience shows that 2% of the products produced by machines A_1 are defective. Similarly defective percentage for machine A_2 and A_3 are 3% and 5% respectively. A product is chosen at random and is found to be defective. Find the probability that it would have been produced by machine A_3?

3. Let U_1, U_2, U_3 be 3 urns with 2 red and 1 black, 3 red and 2 black, 1 red and 1 black ball respectively. One of the urns is chosen at random and a ball is drawn from it. The colour of the ball is found to be black. What is the probability that it has been chosen from U_3?

7. Random Variables

Introduction: Consider an experiment of throwing a coin twice. The outcomes {HH, HT, TT} constitute the sample space. Each of these outcomes can be associated with a number by specifying a rule of association (e g. the number of heads). Such a rule of association is called a random variable. We denote random variable by a capital letter (X,Y) and any particular value of the random variable by x or y.

Discrete Random Variable:

A discrete random variable is a random variable X whose possible values constitute finite set of values count ably infinite set of values.

Continuous Random Variable:

A random variable X which takes all possible values in a given interval is called continuous random variable.

Probability Mass Function (or) Probability Function:

The numbers $p_i = p(x_i)$ satisfies the following conditions

(i) $p(x_i) \geq 0$

(ii) $\sum_{i=1}^{\infty} p(x_i) = 0$

The function p satisfying the above two conditions is called the probability mass function (or) probability function.

Probability Density Function

The p .d .f f(x) of a random variable X has the following properties

(i) $f(x) \geq 0, \quad -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) If E is any event, then P(E) = $\int_E f(x)dx$

Moment Generating Function

The moment generating function (m.g.f.) of a random variable X (about origin) whose probability function f(x) is given by

$M_X(t) = E(e^{tX})$

$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$, for continuous probability distribution

$= \sum_{x}^{\infty} e^{tx} p(x)$,   for discrete probability distribution

Example 1: A random variable X has the following probability function

Values of X	0	1	2	3	4	5	6	7	8
Probability P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Determine the value of a
(ii) Find P (X<3), P(X≥3), P(0<X<5)

Solution: We know that if P(x) is the probability mass function, then

$$\sum_{i=1}^{\infty} P(x_i) = 1$$

a + 3a+ 5a + 7a + 9a + 11a + 13a + 15a + 17a =1

$81a = 1 \Rightarrow a = \frac{1}{81}$

$P(X < 3) = P(0) + P(1) + P(2) = a + 3a + 5a = \frac{1}{81} + \frac{3}{81} + \frac{5}{81} = \frac{9}{81}$

$P(X \geq 3) = 1 - P(X < 3) = 1 - \frac{9}{81} = \frac{72}{81}$

P (0 < X < 5) = P (1) + P (2) + P (3) + P (4) = 3a + 5a + 7a + 9a

$= \frac{3}{81} + \frac{5}{81} + \frac{7}{81} + \frac{9}{81} = \frac{24}{81}$

Example 2: If the random variable X takes the value 1, 2, 3 and 4 such that 2P(X =1) = 3P(X=2) = P(X=3) = 5P(X=4). Find the probability distribution.

Solution: Let 2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)

$P(X = 1) = \frac{k}{2}$ $P(X = 2) = \frac{k}{3}$ $P(X = 3) = k$ $P(X= 4) = \frac{k}{5}$

We know that $\sum p_i = 1$

i.e., $\frac{k}{2}+\frac{k}{3}+\frac{k}{5}+k = 1 \Rightarrow \frac{15k+10k+30k+6k}{30} = 1 \Rightarrow 61k = 30$

i.e., $k = \frac{30}{61}$

∴ The probability distribution of X is given by the following table.

x_i	1	2	3	4
$P(x_i)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Example 3: The diameter of an electric cable is assumed to be a continuous random variable with p.d.f $f(x) = 6x(1 - x), 0 \leq x \leq 1$

(i) Check that above is a p.d.f.
(ii) Determine a number "b" such that P(X< b) = P(X > b)

Solution: We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

(i) $\int_0^1 f(x)dx = \int_0^1 6x(1-x)dx = 6[x^2/2 - x^3/3] = 6[\frac{1}{2} - 1/3] = 1$ ∴ f(x) is a p.d.f.

(ii)Given P(X< b) = P(X > b)

$$\int_0^b 6x(1-x)dx = \int_b^1 6x(1-x)dx$$

$$[6x(1-x)]_0^b = [6x(1-x)]_0^b$$

$$6\left[\frac{b^2}{2}-\frac{b^3}{3}\right] = \left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{b^2}{2}-\frac{b^3}{3}\right)$$

$4b^3 - 6b^2 + 1 = 0$

Solving , b = 1/2, (1±i) / 2 where b = ½ lies in (0 ,1)

Example 4: If a random variable X has the probability density function

f (x) = $\frac{1}{2}(x+1)$, if -1 < x <1

0, otherwise Find the mean and variance of X

Solution: Mean = $\int_{-1}^{1} xf(x)dx = \frac{1}{2}\int_{-1}^{1} x(x+1)dx = \frac{1}{2}\int_{-1}^{1}(x^2+x)dx = \frac{1}{2}[\frac{x^3}{3}+\frac{x^2}{2}]_{-1}^{1}$

$$\frac{1}{2}[\frac{1}{3}+\frac{1}{2}+\frac{1}{3}-\frac{1}{2}] = \frac{1}{3}$$

$\therefore$ Mean = $\frac{1}{3}$

Variance = $\int_{-1}^{1}[x-\frac{1}{3}]^2[\frac{x+1}{2}]dx = \frac{1}{18}\int_{-1}^{1}(9x^2+1-6x)(x+1)dx$

$= \frac{1}{18}\int_{-1}^{1}(9x^3+3x^2-5x+1)dx = \frac{1}{18}[\frac{9x^4}{4}+x^3-\frac{5x^2}{2}+x]_{-1}^{1} = \frac{2}{9}$

$\therefore$ Variance = $\frac{2}{9}$

Example 5: A continuous random variable X that can assume any value between x=2 and x=5 has a density function given by f(x) = k (1+x). Find P (x<4)

Solution: We know that $\int_2^5 f(x)dx = 1$

i.e. $\int_2^5 k(1+x)dx = 1$

k $[\frac{(1+x)^2}{2}]_2^5 = 1 \Rightarrow$ k = $\frac{2}{27}$

$\therefore$ P (X < 4) = P (2<X<4) = $\int_2^4 k(1+x)dx = \frac{16}{27}$

Example 6: A random variable X has the density function :

$f(x) = K \cdot \frac{1}{1+x^2}$ in $-\infty < x < \infty$

$= 0$ otherwise. Find K and the distribution function F (x)

Solution: Since f(x) is a p.d.f. ,we have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} k \frac{1}{1+x^2} dx = 1$$

$$k\left(\tan^{-1} x\right)_{-\infty}^{\infty} = 1$$

$$k(\pi/2 + \pi/2) = 1, \therefore k = 1/\pi$$

To find F(x) : $F(x) = \int_{-\infty}^{x} f(x)dx = 1/\pi \int_{-\infty}^{x} \frac{1}{1+x^2} dx = 1/\pi\left(\tan^{-1} x\right)_{-\infty}^{x}$

$= 1/\pi\ [\tan^{-1} x + \pi/2]$

Example 7: Find the moment generating function of a random variable X having the p.d.f $f(x) = \frac{1}{3}$, $-1 < x < 2$

$= 0$, otherwise

Solution: We know that the m.g.f. for a continuous random variable X is

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-1}^{2} e^{tx} \frac{1}{3} dx = \frac{1}{3}\left[\frac{e^{tx}}{t}\right]_{-1}^{2} = \frac{1}{3}\left[\frac{e^{2t} - e^{-t}}{t}\right]$$

$$\therefore M_X(t) = \frac{1}{3}\left[\frac{e^{2t} - e^{-t}}{t}\right]$$

8. Some special Distributions:

1. **Binomial Distribution**
2. **Poisson Distribution**
3. **Normal Distribution**

Binomial Distribution:

A random variable X is said to follow a discrete binomial distribution if its probability mass function is given by $P(X=x) = nc_x p^x q^{n-x}$ where $p+q=1$

Mean of Binomial Distribution:

The mean of Binomial distribution is np

Variance of Binomial Distribution:

The variance of Binomial distribution is npq

Poisson distribution:

A random variable X is said to follow a discrete poisson distribution if its probability mass function is given by P(X=x) = $\frac{e^{-\lambda}\lambda^{x}}{x!}$, x= 0,1,2,……,∞

= 0, otherwise

Mean of Poisson distribution:

The mean of Poisson distribution is λ

Variance of Poisson distribution:

The variance of Poisson distribution is λ

Normal Distribution:

A random variable X is said to follow a continuous normal distribution with mean μ and variance σ^2 if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty,\ \sigma > 0,\ -\infty < \mu < \infty$$

The total area bounded by the curve is 1.

Mean, Median and Mode of the normal distribution coincide

Example 1: Find the binomial distribution for which the mean is 4 and variance is 3.

Solution: We know that, for binomial distribution

Mean = np, Variance = npq

Given mean = 4 i.e. np = 4 variance = 3 i.e. npq = 3

$$\frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}$$

$$\therefore\ p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

Substituting p = $\frac{1}{4}$ in mean we get n = 16

$$\therefore p(x) = 16C_x (\frac{1}{4})^x (\frac{3}{4})^{n-x}$$

Example 2: 6 dice are thrown 729 times. How times do you expect at least three dice to show a five or a six?

Solution: p = probability of getting 5 or 6 with one die = $\frac{2}{6} = \frac{1}{3}$

$$\therefore q = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

P (at least three dice showing five or six) = p (x≥3)

$= p(3) + p(4) + p(5) + p(6)$

$= 6C_3\,(\frac{1}{3})^3(\frac{2}{3})^3 + 6C_4(\frac{1}{3})^4(\frac{2}{3})^2 + 6C_5(\frac{1}{3})^5(\frac{2}{3}) + 6C_6(\frac{1}{3})^6$

$= \frac{160+60+12+1}{3^6} \quad = \frac{233}{3^6}$

For 729 times, the expected number of times at least 3 dice showing 5 or 6

$= N \times \frac{233}{3^6}$

$= 729 \times \frac{233}{3^6}$

= 233 times

Example 3: Ten coins are thrown simultaneously. Find the chance of obtaining at least 7 heads

Solution: Given $p=\frac{1}{2}$, $q=\frac{1}{2}$, $n=10$

The probability of getting x successes $= p(x) = nc_x p^x q^{n-x}$

(1)Probability of getting at least 7 heads = $p(x \geq 7)$

$= p(7) + p(8) + p(9) + p(10)$

$=10C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 + 10C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + 10C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 + 10C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0$

$=\left(\frac{1}{2}\right)^{10}[10C_7 + 10C_8 + 10C_9 + 10C_{10}]$

=0.171875

Example 4: If X is a Poisson variate P(X= 2) = 9 P(X=4) + 90 P(X=6), find mean and variance of X.

Solution: $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2…

Given P(X=2) = 9 P(X=4) + 90 P(X=6)

i.e. $\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$

$= e^{-\lambda}\lambda^2(\frac{9\lambda^2}{4!} + \frac{90\lambda^2}{6!})$

$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$

$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$ i.e. $\lambda^4 + 3\lambda^2 - 4 = 0$

$\lambda^2 = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2}$

$\lambda^2 = 1 or \lambda^2 = -1 \quad \Rightarrow \lambda = \pm 1 or \lambda = \pm i$

$\therefore$ Mean = $\lambda = 1$

Variance = $\lambda = 1$ Standard deviation = 1

Example 5: Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experiences show that 2% of such fuses are defective.

Solution: Given n = 200, p = 2% = $\frac{2}{100}$ = 0.02

$\therefore$ Mean $\lambda =$ n x p = 200 x 0.02 = 4 $\Rightarrow \lambda$ = 4

The Poisson distribution is p(x) = $\frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-4}4^x}{x!}$

P (at most 5 defective fuses) = p(x ≤ 5)

= p(0) + p(1) + p(2) + p(3) + P(4) + p(5)

$$= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!} + \frac{e^{-4}4^5}{5!}$$

$$= e^{-4}[1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}] = e^{-4}[42.866] = 0.785$$

Example 6: A manufacturer of cotterpins knows that 5% of his product is defective.If he sells cotterpins in boxes of 100 and guarantees that not more than 10 pins will be defective,what is the approximate probability that a box will fail to meet the guaranteed quality?

Solution: n=100, p =5% =0.05, $\lambda =$ nxp =100x0.05 =5

The Poisson distribution = P(X= x) = $\frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}5^x}{x!}$

P(a box will fail to meet the guaranteed quality) = p(x > 10)

=1 - p(x≤10)

=1 – [p(0)+p(1)+…p(10)]

$$= \frac{e^{-5}5^0}{0!} + \frac{e^{-5}5^1}{1!} + \frac{e^{-5}5^2}{2!} + \frac{e^{-5}5^3}{3!} + \frac{e^{-5}5^4}{4!} + \frac{e^{-5}5^5}{5!} +$$

$$\frac{e^{-5}5^6}{6!} + \frac{e^{-5}5^7}{7!} + \frac{e^{-5}5^8}{8!} + \frac{e^{-9}5^9}{9!} + \frac{e^{-10}5^{10}}{10!}$$

$$=1- e^{-5}[\ 1+ 5+ \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \ldots \frac{5^{10}}{10!}\ ]$$

= 1- e^{-5}[146.36]

=0.014

Example 7: If X is normally distributed and the mean of X is 12 and the S.D. is 4. Find out the probability of the following. (i) x ≥ 20, (ii) x ≤ 20, (iii) 0 ≤ x ≤ 12

Solution: Given μ = 12, σ = 4

(i) To find p (x ≥ 20)

When x = 20, z = $\frac{x-\mu}{\sigma} = \frac{20-12}{4} = 2$

i.e., When x = 20, z = 2 $\therefore$ p (x ≥ 20) = p (z ≥ 2)

= 0.5 – p (0≤ z≤2)

= 0.5 – 0.4772 = 0.0228----- (1)

(ii) To find p (x ≤ 20)

When x = 20, z = $\frac{x-\mu}{\sigma} = \frac{20-12}{4} = 2$

$\therefore$ p (x ≤ 20) = p (z ≤ 2) = 1 – p (z ≥ 2) = 1 – 0.0228 (from(1))

= 0.9772

(iii) To find p (0 ≤ x ≤ 12)

When x = 0, z = $\frac{x-\mu}{\sigma} = \frac{0-12}{4} = -3$

When x = 12, z = $\frac{x-\mu}{\sigma} = \frac{12-12}{4} = 0$

$\therefore$ p (0 ≤ x ≤12) = p (-3 ≤ z ≤ 0) = p (0 ≤ z ≤ 3) = 0.4987 (from table)

Example 8: In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution?

Solution: Let the mean and standard deviation of the given normal distribution be μ and σ. The area lying to the left of the ordinate at x = 35 is 0.07. The corresponding value of z is negative.

The area lying to the right of the ordinates at x = 63 up to the mean is 0.5 – 0.07 = 0.43.

The value of z corresponding to the area 0.43 is 1.4757

i.e. $\frac{35-\mu}{\sigma} = -1.4757$

$\frac{\mu-35}{\sigma} = 1.4757$

μ - 35 1.4757 σ ……………. (1)

Similarly the area lying to the left of the ordinate at x = 63 upto the mean is 0.39 (39%)

The value of z corresponding to the area 0.39 is 1.2263

i.e. $\frac{63-\mu}{\sigma} = 1.2263$

63 - μ = 1.2263 σ ……………… (2)

Solving (1) and (2) we get mean μ = 50.288

S.D σ = 10.36

Example 9: Assume that mean height of soldiers to be 68.22 inches with a variance of 10.8 inches. How many soldiers in a regiment of 1000 would you expect to e over 6 feet tall?

Solution: Given μ =68.22, σ^2=10.8, σ =3.286

p (x > 6feet) = p (x > 72 inches)

When x =72, z = $\frac{x-\mu}{\sigma} = \frac{72-68.22}{3.286} = 1.1503$

p (x > 72) = p (z >1.1503)

=0.5 – p (0 < z < 1.1503)

=0.5 - 0.3749

=0 .1251

For 1000 soldiers, the number of soldiers greater than 6 feet = 1000x 0.1251

= 125 soldiers