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SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

SMEA1503-FINITE ELEMENT ANALYSIS

UNIT 1

ENGINEERING ANALYSIS

Objectives of Analysis:

Engineering analysis is adopted for machineries and building structures before and after assembling their parts in order to determine

i. The type and quality of load
ii. Location of loading
iii. Developed stress
iv. Permissible deflection
v. Vibration properties
vi. Pressure and temperature variation

Methods of Engineering Analysis:

The three methods adopted for analyzing engineering products to evaluate their mechanical and other properties are:

1) Experimental methods
2) Analytical methods
3) Numerical methods or approximation methods

Experimental methods:

1) In these methods, the actual product or their prototype model are really tested by using testing equipment.
2) If there is a need to change the dimensions of the prototype, the entire prototype is to be disassembled and to be reassembled and then testing should be carried out.
3) It needs man power and materials.

Analytical Methods:

1) These methods are theoretically analyzing methods.

2) Only simple and regular shaped products like beams, columns, shafts, plates can be analyzed by these methods
3) The products and their loadings specified by mathematical expressions and they are analyzed.

Numerical Methods:

1) For the products of complicated sizes, shapes with complicated material properties and boundary conditions, getting solutions using analytical method is highly difficult. In such situation engineer prefers numerical methods that gives approximate but acceptable solutions.
2) By this method, the approximate but acceptable solutions will be obtained.

Three methods in Numerical methods

i. Functional Approximating Method
ii. Finite Difference Method (FDM).
iii. Finite Element Method (FEM).

Functional Approximation:

1. In this method, the physical problems are first written in terms of differential equation or any possible mathematical expressions.
2. Then the approximate solution can be obtained by integration and by applying boundary condition.
3. The variation method specifically known as Rayleigh-Ritz methods and weighted residual methods are some of the functional approximating methods.

Finite Difference Method (FDM):

1. The finite difference method approximates the derivatives in the governing differential equation using difference equations.
2. Finite difference method is useful for solving heat transfer and fluid mechanics problems.
3. This method cannot be effectively used for regions having or irregular boundaries.

Finite Element Method (FEM):

1. In this method, the complex region defining the domain is divided into smaller elements called finite elements.
2. The physical properties like shape, dimensions and other boundary conditions are imposed on the elements.
3. Then these elements are assembled in a proper way and the solution for the entire system can be obtained.

Steps in FEA

1. **Discretization of structure-** Dividing the whole complex structure into finite elements by lines or surfaces.
2. **Numbering of nodes and elements-** In FEM, physical problems are solved using matrices and the size of the matrix depends on the number of nodes of the element.
3. **Selection of displacement function-** Linear, quadratic and cubic polynomials are used to evaluate the value of the field variable at any part of the element.
4. **Formation of element stiffness matrix and load vector-** Based on equilibrium conditions or variational principles stiffness matrix is formulated.
5. **Formation of global stiffness matrix and load vector-** The element stiffness matrices are assembled using the following formulae to get the global stiffness matrix

   $$[K][\delta] = [F]$$

   where $[K]$ – Global stiffness matrix, $[\delta]$ – Nodal displacement vector and $[F]$ – Nodal force vector
6. **Incorporation of boundary conditions**
7. **Compute element stresses and strain**
8. **Analysis and interpretation of results**

Classification of Functional Approximation Methods:

1. Variational Methods
2. Weighted residual methods

Variational Method:

Rayleigh-Ritz method:

1. It is a typical variational method in which the principle of integral approach is adopted for solving mostly the complex structural problems.
2. In this method, the potential energy π is considered as the function of Ritz parameters which are one to infinity.
3. In practice, the displacement function y(x) can be expressed in terms of polynomial series or trigonometric series such as,

$$y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \ldots. \quad (1.1)$$

or

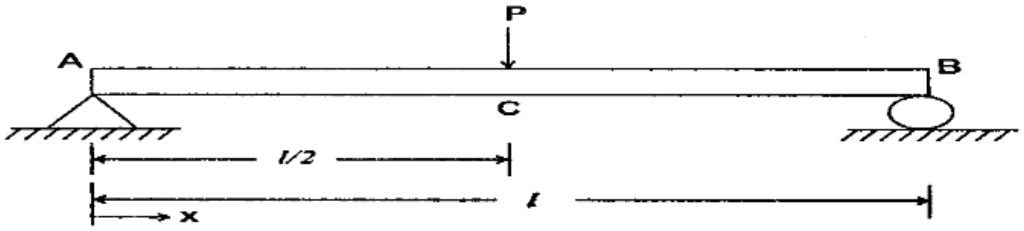
$$y(x) = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} + a_3 \sin\frac{5\pi x}{l} + \ldots \quad (1.2)$$

where a_1 , a_2 , a_3 … are known as Ritz parameters or Ritz coefficients.

4. Selecting anyone of the above two functions appropriately as a trial function, the total potential energy can be formulated.
5. The total potential energy is the algebraic sum of "Integral strain energy and external work done".
   Mathematically, Total potential energy, $\pi = U - W$
   Where U – Internal strain energy and W – Work done by external force
6. By making the total potential energy to reach minimum value (i.e., stationary condition), the approximate solution can be determined.
7. The accuracy of the solution depends on the number of Ritz coefficients.

Problem 1

Find the deflection at the center of a simply supported beam of span length l, subjected to a concentrated load P at its mid-point.



The total potential energy for a beam is given by, $\pi = U - W$

Strain energy for a beam, $U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$

Where E is the modulus of elasticity, I is the area moment of inertia of the beam section and y is the deflection which can be expressed as,

$$y = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \ldots. \quad (1)$$

to simplify the problem, consider the first three terms such as,

$$y = a_1 + a_2 x + a_3 x^2 \quad (2)$$

The boundary conditions are $y = 0$ at $x = 0$ and $x = l$

Hence equations (2) becomes $0 = a_1$ and $0 = a_2 l + a_3 l^2$ which gives $a_2 = - a_3 l$

Then y can be expressed as,

$$y = - a_3 lx + a_3 x^2 = a_3 (x^2 - lx) \quad (3)$$

Differentiating two times we get,

$$\frac{dy}{dx} = a_3(2x - l) and \frac{d^2y}{dx^2} = 2a_3$$

Then strain energy is given by,

$$U = \frac{EI}{2}\int_0^l (2a_3)^2 dx = \frac{EI}{2} 4a_3{}^2 l = 2EIa_3{}^2 l$$

Work done, $W = P * y_{\text{at } x=l/2}$

$= P\ a_3(x^2 - lx)_{\text{at } x=l/2}$ (from equation (3))

$$= Pa_3\left(\frac{l^2}{4} - \frac{l*l}{2}\right) \quad = -Pa_3\frac{l^2}{4}$$

The total potential energy is given by, $\pi = U - W$

$$= 2EIa_3{}^2 l - \left(-Pa_3\frac{l^2}{4}\right) = 2EIa_3{}^2 l + Pa_3\frac{l^2}{4}$$

For minimum potential energy condition,

$$\frac{\partial \pi}{\partial a_3} = 0$$

That is,

$$4EIa_3 l = -P\frac{l^2}{4}$$

Therefore,

$$a_3 = -P\frac{l^2}{4} * \frac{1}{4EIl}$$

hence

$$a_3 = -\frac{Pl}{16EI}$$

Substituting the value of a_3 in Equation (3) we get,

$$y = a_3(x^2 - lx) \quad = -\frac{Pl}{16EI}(x^2 - lx)$$

Maximum deflection occurs at x = l / 2

Hence,

$$y_{max} = -\frac{Pl}{16EI}\left(\frac{l}{4}^2 - l\frac{l}{2}\right) \; = -\frac{Pl}{16EI}\left(-\frac{l}{4}^2\right)$$

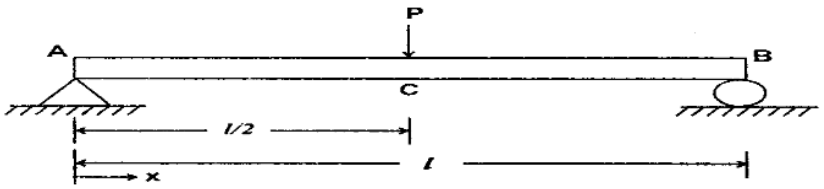
Therefore

$y_{max} = -\frac{Pl^3}{64EI}$ is the approximate solution.

To get more accurate solution, the displacement function should contain more number of Ritz parameters.

Problem 2

Find the deflection at the center of a simply supported beam of span length l subjected to a concentrated load P at its mid-point using trail function from trigonometric series.



$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} + a_3 \sin\frac{5\pi x}{l} + \ldots$

To simply the problem, select one term function as,

$y = a_1 \sin\frac{\pi x}{l} = a \sin\frac{\pi x}{l}$ (1)

Now consider the potential energy as $\pi = U - W$

Strain energy for a beam, $U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$

Differentiating the displacement function two times we get,

$$\frac{dy}{dx} = a \cos\frac{\pi x}{l} * \frac{\pi}{l} = \frac{a\pi}{l} \cos\frac{\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

Then strain energy, $U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$

$$= \frac{EI}{2}\int_0^l \left(-a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}\right)^2 dx \qquad (2)$$

$$= \frac{EI}{2}\left(-a\frac{\pi^2}{l^2}\right)^2 \int_0^l \sin^2\frac{\pi x}{l}\,dx$$

Now, $\int_0^l \sin^2\frac{\pi x}{l}\,dx = \int_0^l \frac{1}{2}\left(1-\cos\frac{2\pi x}{l}\right)dx$ (Since $\sin^2 A = \frac{1-\cos 2A}{2}$)

$$= \frac{1}{2}\left[x - \left\{\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right\}\right]_0^l = \frac{1}{2}\left[(l-0) - \frac{l}{2\pi}\{\sin 2\pi - \sin 0\}\right]$$

Now the equation (2) implies

Strain energy, $U = \frac{EI}{2}\int_0^l \left(-a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}\right)^2 dx = \frac{EI}{2}\left(-a\frac{\pi^2}{l^2}\right)^2\frac{l}{2} = \frac{a^2\pi^4 EI}{4l^3}$

Work Done, W = P * y_{max}

= P * $y_{\text{at } x=l/2}$ $= P\left(a\sin\frac{\pi x}{l}\right)_{at\ x=l/2}$ From Equation (1)

$$= Pa\sin\frac{\pi}{2} = Pa \quad (\text{Since } \sin\frac{\pi}{2} = 1)$$

The total potential energy, π = U – W

$$\pi = \frac{a^2\pi^4 EI}{4l^3} - Pa$$

$$\frac{\partial \pi}{\partial a} = 0 \rightarrow \frac{2a\pi^4 EI}{4l^3} = P \quad \text{Therefore, } a = \frac{2pl^3}{\pi^4 EI}$$

Maximum deflection occurs at x=l/2

$$\text{Hence } y_{max} = \left(a \sin \frac{\pi x}{l}\right)_{at\ x=l/2} \quad y_{max} = \left(\frac{2pl^3}{\pi^4 EI} \sin \frac{\pi x}{l}\right)_{at\ x=l/2} \quad = \frac{2pl^3}{\pi^4 EI} \sin \frac{\pi}{2}$$

$$\text{Therefore, } y_{max} = \frac{pl^3}{48.7EI}$$

Weighted Residual Method

The weighted residual method is employed to obtain approximate solutions to linear and non linear non structural problems whose characteristics are expressed in terms of differential equations. The required simultaneous equations to find the solution can be derived from the governing differential equation, without knowing the functional. The methods are

1. Point Collocation method
2. Sub domain collocation method
3. Least square method
4. Galerkin's method.

Steps

1. Let $y_e(x)$ is the exact solution of the differential equation
2. An approximate function called the trial function is considered $y(x)= f(x,a_i)$, i=1,2…. And is substituted in the differential equation to find the residual R. The trial function should satisfy the boundary conditions.
3. This residual is further treated to evaluate the required solution. It is essential that the residual multiplied by a weighing function and the domain integral of the product should be zero.
4. The number of weighing functions is equal to the number of unknown coefficients in the approximate function.

Point Collocation method:

1. In the collocation method, also called point collocation, the residual $R(x, a_i)$ is set equal to zero at n specific points $x_1, x_2, x_3, \ldots x_n$.
2. The weighting function Wi can be expressed as
   $$W_i = \int_D \delta(x - x_i) R(x, a_i) dx = 0$$
3. At point $x=x_i$, $W_i = 1$ and hence $R(x, a_i) = 0.$
4. At other points in the domain, $W_i = 0$.

Sub-domain Collocation method:

1. In this method, the domain is subdivided into n subdomains and the integral of the residual over each sub-domain is then required to be zero.
2. That is the weighting function selected is unity ($W_i = 1$) over the domain
3. Here $\int_D R(x, a_i) dx = 0$

Least squares method:

1. In this method, the integral of the weighted square of the residual over the domain is required to be minimum.
2. That is $I = \int_D [R(x, a_i)]^2 dx = minimum$
3. The minimization of the integral is with respect to the unknown coefficients in the approximate solution.
4. That is $\frac{\partial I}{\partial a} = 0$

Galerkin's method:

1. In this method, the trial function y(x) itself considered as the weighting function.

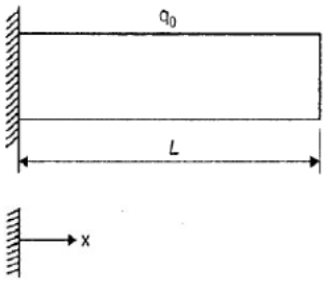
2. The domain integral of the product of trial function with the residual is then set equal to zero.

3. That is $\int_D W_i R(x, a_i)\, dx = \int_D y(x) R(x, a_i)\, dx = 0$

Problem

Consider a uniform rod subjected to a uniform axial load as illustrated in Figure. It can be readily shown that the deformation of bar is governed by the differential equation

$$AE\frac{d^2u}{dx^2} + q_0 = 0 \qquad (1)$$



$$u(0) = 0, \left|\frac{du}{dx}\right|_{x=L} = 0$$

Find an approximate solution of the above differential equation by using (i) Point Collocation method (ii) Sub domain Collocation method (iii) Least Square method and (iv) Galerkin's method

Solution:

The given differential equation is $AE\frac{d^2u}{dx^2} + q_0 = 0$

The boundary conditions are, $u(0) = 0, \left|\frac{du}{dx}\right|_{x=L} = 0$

Let us assume an approximate solution, $u(x) = a_1 + a_2 x + a_3 x^2$ (2)

(Order of approximate solution is equal to order of differential equation)

From Boundary equation, $u(0) = 0$

Hence, $u(0) = a_1 + a_2(0) + a_3(0)^2$ Therefore $a_1 = 0$

The approximate solution becomes, $u(x) = a_2 x + a_3 x^2$

From $\left|\frac{du}{dx}\right|_{x=L} = 0$ $\left|\frac{du}{dx}\right|_{x=L} = a_2 + 2a_3 L = 0$ Hence, $a_2 = -2a_3 L$

Now the approximate solution becomes, $u(x) = -2a_3 Lx + a_3 x^2$

$$u(x) = a_3(x^2 - 2Lx) \quad (3)$$

To get residual equation, by substituting (3) in (1)

We get, $AE\frac{d^2}{dx^2}[a_3(x^2 - 2Lx)] + q_0 = R_d$

$$AE\frac{d}{dx}[a_3(2x - 2L)] + q_0 = R_d$$

$$AE2a_3 + q_0 = R_d \quad (4)$$

Point Collocation method:

In Collocation method, residual are set to zero. $(R_d = 0)$

Therefore, $AE2a_3 + q_0 = 0$

$$AE2a_3 = -q_0$$

Therefore, $a_3 = \frac{-q_0}{2AE}$ (5)

By substituting (5) in (3), the final solution is, $u(x) = \frac{-q_0}{2AE}(x^2 - 2Lx)$

Sub domain collocation method:

This method requires $\int_0^l R_d \, dx = 0$ (6)

Substituting (4) in (6) we get,

$$\int_0^l (AE2a_3 + q_0) \, dx = 0$$

Upon integration,

$$[AE2a_3 x + q_0 x]_0^l = 0$$

By substituting the limits we get,

$$[AE2a_3 l + q_0 l] = 0$$

Hence $a_3 = \frac{-q_0}{2AE}$ (7)

By substituting (7) in (3) we get,

$$u(x) = \frac{-q_0}{2AE}(x^2 - 2Lx)$$

(or) $u(x) = \frac{q_0}{2AE}\left(2Lx - x^2\right)$

Least Squares Method:

The functional $I = \int_0^l R_d^2 dx = minimum$ (8)

It can also be written as, $\frac{\partial I}{\partial a_3} = \int_0^l R_d \frac{\partial R_d}{\partial a_3} dx$

$\frac{\partial R_d}{\partial a_3} = 2AE$ (From (4))

$$\frac{\partial I}{\partial a_3} = \int_0^l (AE2a_3 + q_0)(2AE) \, dx \quad \frac{\partial I}{\partial a_3} = \int_0^l (A^2E^2 4a_3 + 2AEq_0) dx$$

Upon integration,

$$\frac{\partial I}{\partial a_3} = [A^2E^2 4a_3 x + 2AEq_0 x]_0^l$$

By substituting the limits we get,

$$\frac{\partial I}{\partial a_3} = [A^2E^2 4a_3 l + 2AEq_0 l]$$

The requirement is $\frac{\partial I}{\partial a_3} = 0$ Therefore, $[A^2E^2 4a_3 l + 2AEq_0 l] = 0$

$$A^2E^2 4a_3 l = -2AEq_0 l$$

Hence $a_3 = \frac{-q_0}{2AE}$ (10)

By substituting (10) in (3) we get,

$$u(x) = \frac{-q_0}{2AE}(x^2 - 2Lx) \qquad \text{(or) } u(x) = \frac{q_0}{2AE}(2Lx - x^2)$$

(iv) Galerkin's Method:

In this method, the trial function itself is considered as the weighting function and this method requires,

$$\int_0^l W_i R_d dx = 0 \text{ where i = 1 to n} \quad (11)$$

Hence $u(x) = W_i = a_3(x^2 - 2Lx)$ (12)

Substituting (4) and (12) in (11) we get,

$$\int_0^l a_3(x^2 - 2Lx)(AE2a_3 + q_0)dx = 0$$

$$\int_0^l [2AE{a_3}^2x^2 + a_3x^2q_0 - 4AELx{a_3}^2 - 2Lxa_3q_0]dx = 0$$

Upon integration,

$$\left[\frac{2AE{a_3}^2x^3}{3} + \frac{a_3x^3q_0}{3} - \frac{4AELx^2{a_3}^2}{2} - \frac{2Lx^2a_3q_0}{2}\right]_0^l = 0$$

$$\left[\frac{2AE{a_3}^2x^3}{3} + \frac{a_3x^3q_0}{3} - 2AELx^2{a_3}^2 - Lx^2a_3q_0\right]_0^l = 0$$

By substituting the limits we get,

$$\left[\frac{2AE{a_3}^2L^3}{3} + \frac{a_3L^3q_0}{3} - 2AELL^2{a_3}^2 - LL^2a_3q_0\right] - [0] = 0$$

$$\left[\frac{2AE{a_3}^2L^3}{3} + \frac{a_3L^3q_0}{3} - 2AEL^3{a_3}^2 - L^3a_3q_0\right] = 0$$

Dividing by a_3 and L^3, we get,

$$\left[\frac{2AEa_3}{3} + \frac{q_0}{3} - 2AEa_3 - q_0\right] = 0$$

$$\frac{2AEa_3}{3} - 2AEa_3 = q_0 - \frac{q_0}{3} \qquad -\frac{4AEa_3}{3} = \frac{2q_0}{3}$$

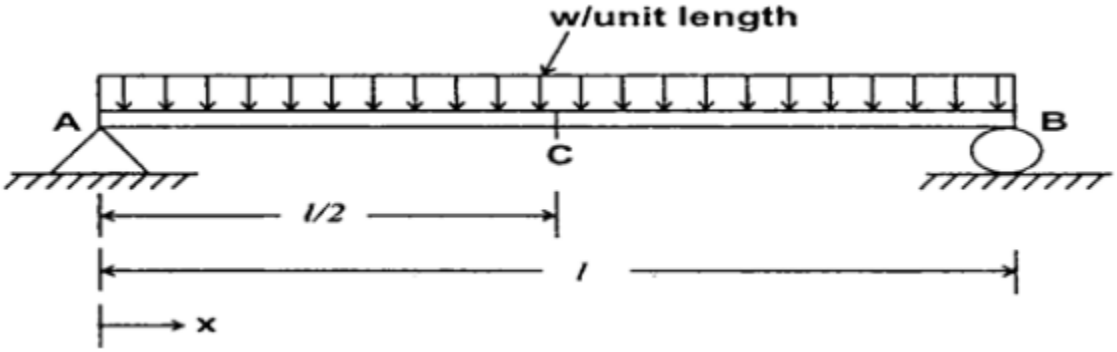
Hence $a_3 = \frac{-q_0}{2AE}$ (13)

By substituting (13) in (3) we get,

$$u(x) = \frac{-q_0}{2AE}(x^2 - 2Lx) \qquad \text{(or) } u(x) = \frac{q_0}{2AE}\left(2Lx - x^2\right)$$

Problem

Find the deflection at the centre of a simply supported beam of span length l subjected to uniformly distributed load throughout its length as shown in figure using (a) Point Collocation method, (b) Sub-domain collocation method (c) Least squares method, (d) Galerkin's method.



The differential equation governing the deflection of beam subjected to uniformly distributed load is given by

$$EI\frac{d^4y}{dx^4} - w = 0, \qquad 0 \le x \le l$$

Now, let us select the trial function for deflection as

$$y = a \sin\frac{\pi x}{l} \qquad \ldots. (1)$$

The boundary conditions to be satisfied are $y = 0$ at $x = 0$ and $x = l$ where y is the deflection and $EI\frac{d^2y}{dx^2} = 0$ at $x = 0$ and $x = l$ where $EI\frac{d^2y}{dx^2} = M$ (Bending moment) and E = Young's modulus, I = Moment of inertia of the beam.

$$\frac{dy}{dx} = a\frac{\pi}{l}\cos\frac{\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} \qquad \ldots. (2)$$

Eqn. (1) satisfies the boundary conditions as $y = 0$ at $x = 0$ and $x = l$

Similarly the eqn. (2) satisfies the boundary conditions as $EI\frac{d^2y}{dx^2} = 0$ at $x = 0$ and $x = l$

$$\frac{d^3y}{dx^3} = -a\frac{\pi^3}{l^3}\cos\frac{\pi x}{l} \qquad \text{and} \qquad \frac{d^4y}{dx^4} = a\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} \qquad \ldots. (3)$$

Substituting the eqn. (3) in the governing differential equation, we get the residual as

$$R = EI\ a \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w \qquad \ldots. (4)$$

(a) Point Collocation Method:

In this method the residual is set to zero.

i.e., $EI\ a \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w = 0$

To get maximum deflection, take $x = \frac{l}{2}$ (i.e., at the centre of beam)

Then, $EI\ a \frac{\pi^4}{l^4} \sin \frac{\pi}{l}\left(\frac{l}{2}\right) = w$ $\qquad a = \frac{w\,l^4}{\pi^4\,EI}$

The trial function $y = \frac{w\,l^4}{\pi^4\,EI} \sin \frac{\pi x}{l}$ $\qquad$ At $x = \frac{l}{2}$, $y_{max} = \frac{w\,l^4}{\pi^4\,EI} \sin \frac{\pi}{2}$

(b) Sub-domain Collocation Method:

In this method, the integral of the residual over the sub-domain is set to zero.

i.e., $\int_0^l R\,dx = 0$ $\qquad$ i.e., $\int_0^l \left(a\,EI \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w \right) dx = 0$

i.e., $\left[a\,EI \frac{\pi^4}{l^4} \left(-\cos \frac{\pi x}{l} \right) \left(\frac{l}{\pi} \right) - w\,x \right]_0^l = 0$

$\therefore a = \frac{w\,l^4}{2\,\pi^3\,EI}$ $\qquad$ At $x = \frac{l}{2}$, $y_{max} = \frac{wl^4}{62EI} \sin \frac{\pi}{2}$

(c) Least Squares Method:

In this method the functional

$I = \int_0^l R^2 \, dx$ is minimum

$$\text{Now } I = \int_0^l \left(a\,EI \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w \right)^2 dx$$

$$= \int_0^l \left(a^2 E^2 I^2 \frac{\pi^8}{l^8} \sin^2 \frac{\pi x}{l} + w^2 - 2a\,EI\,w \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} \right) dx$$

$$= \int_0^l \left(a^2 E^2 I^2 \frac{\pi^8}{l^8} \left\{ \frac{1 - \cos\left(\frac{2\pi x}{l} \right)}{2} \right\} + w^2 - 2a\,EI\,w \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} \right) dx$$

$$= \frac{a^2 E^2 I^2 \pi^8}{2 l^7} + w^2 l - 4\,a\,EI\,w \frac{\pi^3}{l^3}$$

Now,

$$\frac{\partial I}{\partial a} = 0 \Rightarrow a E^2 I^2 \frac{\pi^8}{l^7} = 4\,EI\,w \frac{\pi^3}{l^3} \qquad \therefore a = \frac{4\,w\,l^4}{\pi^5\,EI}$$

At $x = \frac{l}{2}$, Maximum deflection, $y_{max} = \frac{4\,w\,l^4}{\pi^5\,EI} \cdot \sin \frac{\pi}{2}$

Galerkin's Method:

In this method, $\int_0^l (y\,R)\,dx = 0$

$$\text{i.e., } \int_0^l \left\{ \left(a \sin \frac{\pi x}{l} \right) \left(a\,EI \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w \right) \right\} dx = 0$$

Solving we get

$$a = \frac{2\,w\,l}{\pi} \frac{2 l^3}{EI\,\pi^4} = \frac{4\,w\,l^4}{\pi^5\,EI} \qquad \text{At } x = \frac{l}{2},\ y_{max} = \frac{4w\,l^4}{\pi^5\,EI}$$

Problems for practice

1. Derive the expression for deflection and bending moment in a simply supported beam of span of length l, subjected to UDL over entire span using two term trigonometric trial function using Rayleigh Ritz method.

   Consider the differential equation for a problem such as $\frac{d^2y}{dx^2} + 300x^2 = 0; 0 \le x \le 1$ with the boundary conditions y(0) = y(1) = 0. The functional corresponding to this problem to be extremized is given by, $I = \int_0^1 \left\{-\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300x^2y\right\} dx$ Find the solution of the problem using Rayleigh-Ritz method using a one term solution as $y = ax(1 - x^3)$

2. **Consider the differential equation for a problem as**

   $$\frac{d^2y}{dx^2} + 300\,x^2 = 0, \qquad 0 \le x \le 1.$$

   with the boundary conditions y (0) = 0, y (1) = 0. Find the solution of the problem using a one coefficient trial function as $y = a_1 x (1 - x^3)$. Use (i) Point collocation method, (ii) Sub-domain collocation method, (iii) Least square method and (iv) Galerkin's method.

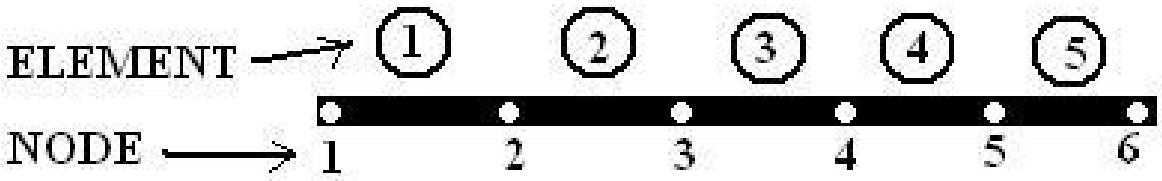
3. **Solve the following equation using a two parameter trial solution by (a) Point Collocation method and (b) Galerkin's method**

   $$\frac{dy}{dx} + y = 0, 0 \le x \le 1, y(0) = 1$$

UNIT-II

ONE DIMENSIONAL ELEMENT

Bar and beam elements are considered as One Dimensional elements. These elements are often used to model trusses and frame structures. Bar is a member which resists only axial loads. A beam can resist axial, lateral and twisting loads. A truss is an assemblage of bars with pin joints and a frame is an assemblage of beam elements.



As shown in the figure, a one dimensional structure is divided into several elements and the each element has 2 nodes.

Shape function

$N_1N_2N_3$ are usually denoted as shape function. In one dimensional problem, the displacement $u = \sum N_i u_i = N_1 u_1$

For two noded bar element, the displacement at any point within the element is given by, $u = \sum N_i u_i = N_1 u_1 + N_2 u_2$

For three noded triangular element, the displacement at any point within the element is given by,

$u = N_1 u_1 + N_2 u_2 + N_3 u_3$

$v = N_1 v_1 + N_2 v_2 + N_3 v_3$

Shape function need to satisfy the following

(a) First derivatives should be finite within an element;

(b) Displacement should be continuous across the element boundary

Properties of Stiffness Matrix

1. It is a symmetric matrix,
2. The sum of elements in any column must be equal to zero,
3. It is an unstable element. So the determinant is equal to zero.

Assumptions

Nodal Forces and Moments

Forces and moments can only be applied at the nodes of the beam element, not between the nodes. The nodal forces and moments, f_n, are related to the nodal displacements and rotations, $\underline{E}$through the ele ment stiffness matrix, $\underline{K}_e$.

Constant Load

The loads that are applied to the beam element are assumed to be static and not to vary over the time period being considered, this assumption is only valid if the r ate of change of the force is much less than the applied force ($F >> dF/dt$). If the loads vary significantly, (if the variation in load is not much less than the applied force) then the pro blem must be considered as dynamic.

Weightless Member

The weight (W) of the beam is neglected, if it is much less than the total resultant forces (F) acting on the beam. If the weight of the beam is not neglected, then its effects must be represented as vertical forces acting at the nodes, by dividing up the weight and lumping it at the nodes, proportionally according to it's placement along the beam.

Prismatic Member

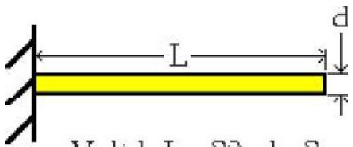
The beam element is assumed to have a constant cross-section, which means that the cross-sectional area and the moment of inertia will both be constant (i.e., the be am element is a prismatic member). If a beam is stepped, then it must be divided up into sections of constant cross-section, in order to obtain an exact solution. If a beam is tape red, then the beam can be approximated by using many small beam elements, each having the same cross-section as the

middle of the tapered length it is approximating. The more sections that are used to approximate a tapered be am, the more accurate the solution will be.

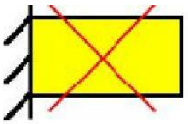
The moment of inertia is a geometric property of a beam element, which describes the beams resistance to bending and is assumed to be constant through the length of the element. The moment of inertia can be different along different axes if the beam element is not symmetric, we use the moment of inertia (I) of the axis about which the bendin g of the beam occurs Where (I_z) refers to the moment of inertia, resisting bending about the "*z*" axis and (I_y) about the "*y*" axis.

The Beam Element is a Slende r Member

A beam is assumed to be a slender member, when it's length (L) is moree than 5 times as long as either of it's cross-sec tional dimensions (d) resulting in (d/L<.2). A beam must be slender, in order for the beam equations to apply, that were used to derive our FEM equations.



Valid L=20, d=2
d/L=.1



Invalid L 10, d 5
d/L .5

The Beam Bends without Twisting.

It is assumed that the c ross-section of the beam is symmetric about the plane of bending (*x*-*y* plane in this case) and will undergo symmetric bending (where no twisting of the beam occurs during the bending process). If the beam is not symmetric about this plane, then the beam will twist during bending and the situation will no longer be one-dimensional and must be approached as an unsymmetric bending problem (where the beam twists while bending) in order to obtain a correct solution.

Cross Section Remains Plane

When a beam element b ends, it is assumed that it will deflect uniformly, thus the cross section will move uniformly and remain plane to the beam centerline. In other words, plane sections remain plane and normal to the *x* axis before and after bending.

Axially Rigid

The one-dimensional bea m element is assumed to be axially rigid, meaning that there will be no axial displacement (u) along the beams centroidal axis. This implies that forces will only be applied perpendicular to the beams centroidal axis. The one-dimensional beam element can be used only when the degrees of freedom are limited to vertical displacements (perpendicular to the beams centriodal axis) and rotations in one plane. If axial displacements are present then a one-dimensional bar element must be superimposed with the one-dimensional beam element in order to obtain a valid solution.

Homogenous Material

A beam element has the same material composition throughout and therefore the same mechanical properties at every position in the material. Therefore, the modulus of elasticity E is constant throughout the beam element. A member in which the material properties varies from one point to the next in the member is called inhomogenous (non-homogenous). If a beam is composed of different ty pes of materials, then it must be divide up into elements that are each of a single homogeneous material, otherwise the solution will not be exact.

Isotropic Material

A beam element has the same mechanical and physical properties in all directions, i.e., they are independent of direction. For instance, cutting out three tensile test specimens, one in the x-direction, one in the y-direction and the other oriented 45 degrees in the x-y plane, a tension test on each specimen, will result in the same value for the modulus of elasticity (E), yield strength (σ_Y) and ultimate strength (σ_U) . Most metals are considered isotropic. In contrast fibrous materials, such as wood, typically have properties that are directionaly dependant and are generally considered anisotropic (not isotropic).

Rigid body motion occurs when forces and/or moments are applied to an unrestrained mesh (body), resulting in motion that occurs without any deformations in the entire mesh (body). Since no strains (deformations) occur during rigid body motion, there can be no stresses developed in the mesh. In order to obtain a unique FEM solution, rigid body motion must be constrained. If rigid body motion is not constrained, then a singular system of equations will result, since the determinate of the mesh stiffness matrix is equal to zero (i.e., $|K| = 0$).

There are two rigid body modes for the one-dimensional beam element , a translation (displacement) only and a rotation only. These two rigid body modes can occur at the same time resulting in a displacement and a rotation simultaneously. In order to eliminate rigid body motion in a 1-D beam elem ent (body), one must prescribe at least two no dal degrees of freedom (DOF), either two displacements or a displacement and a rotation. A DOF can be equal to zero or a non-zero know n value, as long as the element is restrained from rigid body motion (deformation can take place when forces and moments are applied) .

For simplicity we will introduce the rigid body modes using a mesh composed of a single element. If only *translatio nal rigid body motion* occurs, then the displacement at local node **I** will be equal to the displacement at local node **J**. Since the displacements are equal there is no strain developed in the element and the applied nodal forces cause the element to move in a rigid (non-deflected) vertical motion (which can be either up as shown below or it can be in the downward direction depending on the direction of the applied forces).

Derivation of shape function and stiffness matrix for a 1 dimensional bar element

Consider a bar element with nodes 1 and 2 as shown with displacements of u_1 and u_2 at the respective nodes

The displacement u can be given as $u=a_0+a_1x$ ------(1)

where a_0 and a1 are generalised coordinates.

$u = [1\ x] \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ ----- (2)

at node 1, $u=u_1$, $x=0$

at node 2, $u=u_2$, $x=l$

Substituting in (1) we get, $u_1=a_0$ and $u_2=a_0+a_1l$

In matrix form, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Substituting in eqn 2,

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \frac{1}{l} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

u= $[N_1\ N_2] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where N_1= l-x/l and N_2= x/l

N_1 and $N_2$are the shape functions

We know that Stiffness matrix [K]= $\int_v [B]^T [D][B] dv$ ------(3)

Where [B]- Strain displacement relationship matrix

[D]- Elasticity matrix

[B]= $\begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$ and $[B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix}$

In one dimensional problems, [D] = Youngs Modulus and dv= Adx

Substituting the values in Eqn 3, we get

[K]= $\frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ which is the stiffness matrix for a one dimensional bar element.

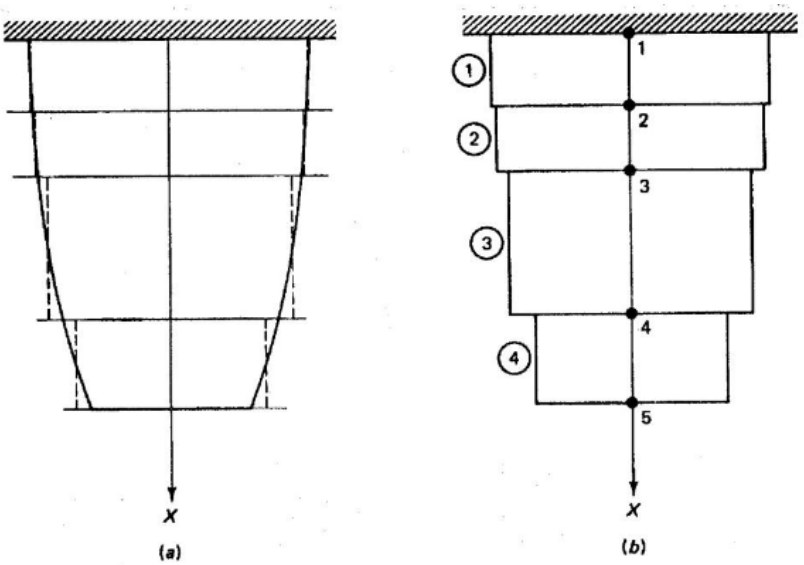
Types of loading

a) Body force (f)- Distributed force per unit volume (N/m3). Eg. Self weight due to gravity
b) Traction force (T)- Force per unit area (N/m2). Eg. Frictional forces, viscous drag and surface shear

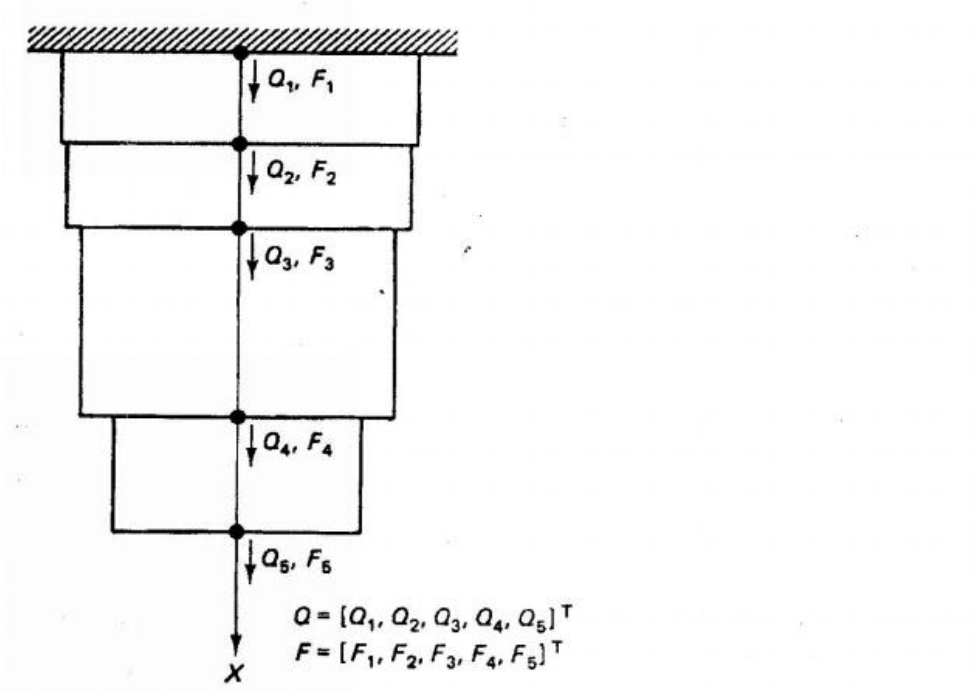
c) Point load- Concentrated load at a point

Steps in Finite Element modelling of a one dimensional bar element

(1) The first step is to subdivide the bar called discretization. A non uniform bar is transformed to a uniform stepped bar.

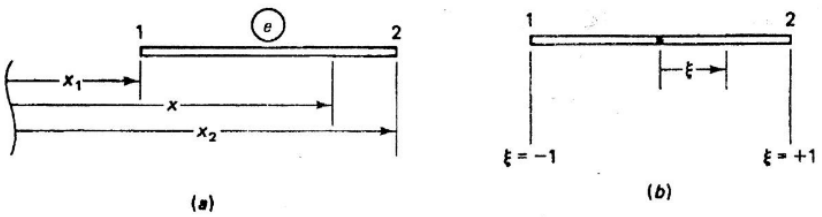


(2) Numbering of nodes

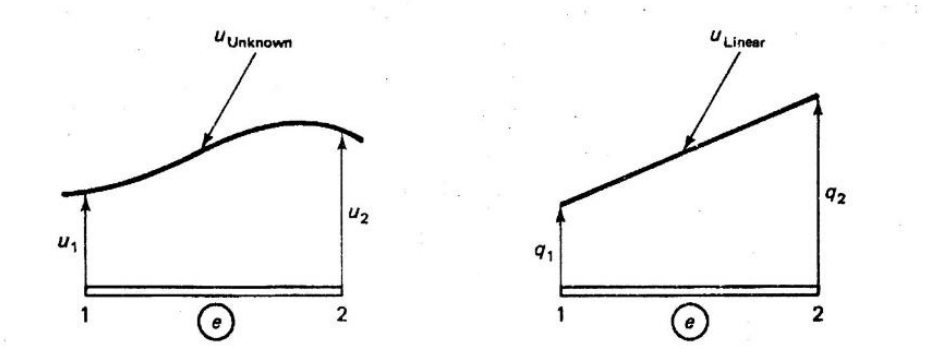


(3) Natural coordinate

Consider a single element. Local node 1 is at a distance of x_1 from datum and node 2 is x_2 measured from same datum point.



(4) Shape functions

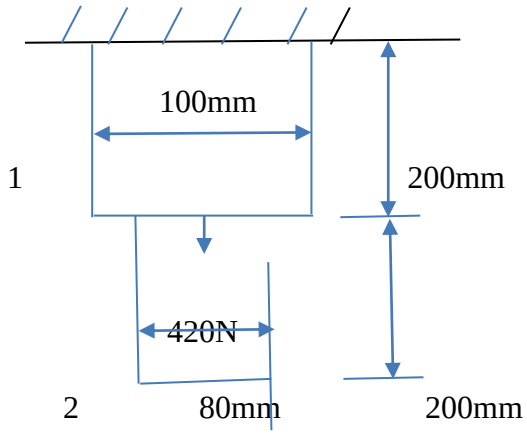


Establish a linear interpolation function to represent the linear displacement field within the element. Linear shape functions are given by

$$N_1(\xi)=\frac{1-\xi}{2} \quad \text{and} \quad N_2(\xi)=\frac{1+\xi}{2}$$

Example

A thin steel plate of uniform thickness 25mm is subjected to a point load of 420N at mid depth as shown. The plate is also subjected to self weight. If Young's modulus, E=2 x 10^5N/mm^2 and unit weight density 0.8 x 10^{-4}N/mm^2. Calculate (i) Displacement at each nodal point (ii) Stresses in each element.



Thickness t=25mm, A_1= 100 x 25=2500mm^2, A_2= 80x 25=2000mm^2

Point load p= 420N

Young's modulus E=2 x 10^5 N/mm^2, Unit weight density= 0.8 x 10^{-4} N/mm^3

Body Force vector $\{F\} = \frac{\rho A l}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$$\begin{pmatrix}F_1\\F_2\end{pmatrix} = \frac{\rho_1 A_1 l_1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix} = \begin{Bmatrix}20\\36\\16\end{Bmatrix}$$

A point load of 420N is acting at middepth at nodal point 2. Hence

$$\begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix} = \begin{Bmatrix}20\\456\\16\end{Bmatrix}$$

Finite element equation for 1st element is

$$\frac{2500 \times 2 \times 10^5}{200}\begin{Bmatrix}1 & -1\\-1 & 1\end{Bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \begin{Bmatrix}F_1\\F_2\end{Bmatrix}$$

Finite element equation for 2nd element is

$$\frac{2000 \times 2 \times 10^5}{200}\begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Assembling the equations

$$2 \times 10^5 \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 12.5+10 & -10 \\ 0 & -10 & 10 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Apply boundary conditions at node 1 displacement is 0 and substituting the values of forces we get,

u_2= 1.888 x 10^{-4}mm , u_3= 1.9688 x 10^{-4}mm

Stress $\sigma = E\frac{du}{dx}$

$$\sigma_1 = \frac{2 \times 10^5 \times 1.888 \times 10^{-4}}{200} = 0.188N/mm^2$$

$$\sigma_2 = \frac{2 \times 10^5 \times 1.968 \times 10^{-4} - 1.888 \times 10^{-4}}{200} = 0.008N/mm^2$$

Temperature effects

When the free expansion is prevented in the member, the change in temperature causes stress in the member. Let δT be the rise in temperature and α be the coefficient of thermal expansion. The nodal vector can be given by

$$\{F\} = EA \propto \partial T\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Where E is the Young's modulus, A is the area of the element.

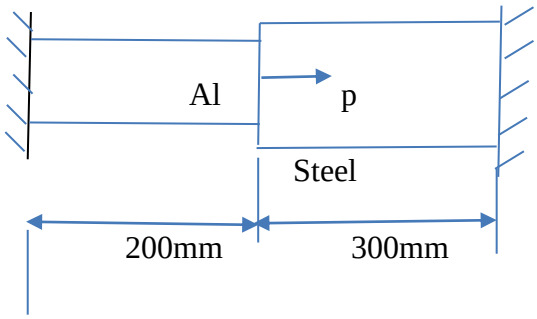
Thermal stress is given by

$$\{\sigma\} = E\frac{du}{dx} - E\alpha\delta T$$

Problems for practice

1. An axial load of 4 x 10^5N is applied at 30 degree centigrade to the rod as shown. The temperature is then raised to 60 degree centigrade. Calculate the following
   (i) Nodal displacements
   (ii) Stresses in each material

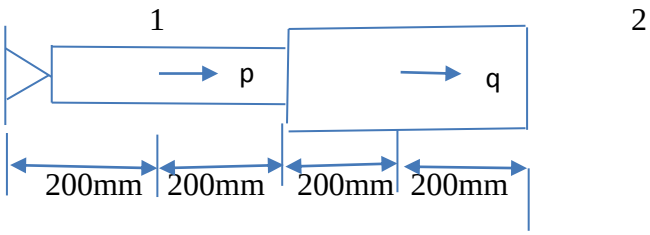
(iii) Reactions at each nodal point



For Aluminium- Area = 1000mm^2, E= 0.7 x10^{5}N/mm^2, α=23 x 10^{-6}/C

For steel- Area- 1500mm^2, E= 2 x 10^{5}N/mm^2, α=12 x10^{-6}/C

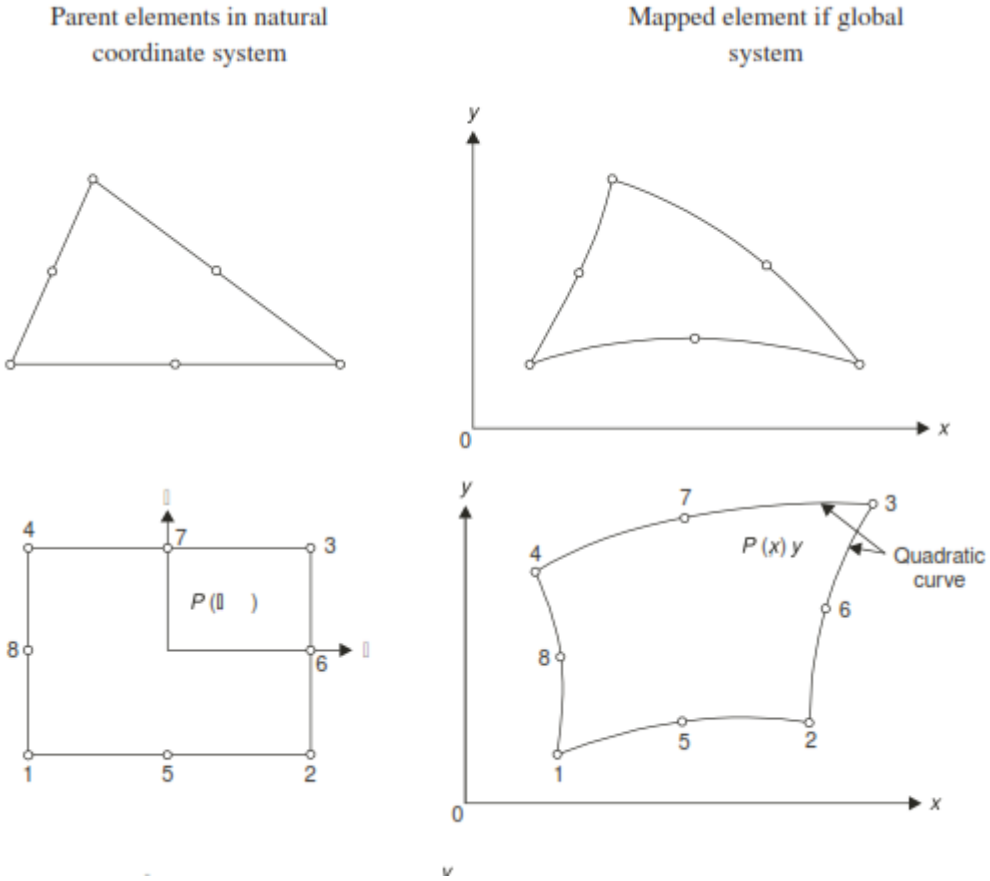
2. Calculate the nodal displacements, element stresses and support reactions. A_1=300mm^2, A_2=500mm^2, E=2 x10^{5}N/mm^2



UNIT –III

ISOPARAMETRIC FORMULATIONS

The isoparametric concept brought out by Taig and latter on generalized by B.M. Irons revolutionized the finite elements analysis and it also helped in properly mapping the curved boundaries. They brought out the concept of mapping regular triangular and rectangular elements in natural coordinate system, to arbitrary shapes in global system as shown in Fig. The coordinate transformation of natural coordinates to global coordinate system is presented. The terms isoparametric, super parametric and subparametrics are defined.The basic theorems on which isoparametric concept is based are explained and need for satisfying uniqueness theorem of mapping is presented. Assembling of stiffness matrix is illustrated. For assembling stiffness matrix integration is to be carried out numerically.



COORDINATE TRANSFORMATION

So far we have used the shape functions for defining deflection at any point interms of the nodal displacement. Taig suggested use of shape function for coordinate transformation form natural local coordinate system to global Cartesian system and successfully achieved in mapping parent element to required shape in global system. Thus the Cartesian

coordinate of a point in an element may be expressed as or in matrix form where N are shape functions and (x)

$$x = N_1 x_1 + N_2 x_2 + \ldots + N_n x_n$$
$$y = N_1 y_1 + N_2 y_2 + \ldots + N_n y_n$$
$$z = N_1 z_1 + N_2 z_2 + \ldots + N_n z_n$$

or in matrix form

$$\{x\} = [N]\{x\}_e$$

where N are shape functions and (x)e are the coordinates of nodal points of the element. The shape functions are to be expressed in natural coordinate system. The shape function of this element is given as

$$N_1 = \frac{(1-\xi)(1-\eta)}{4},\ N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

P is a point with coordinate $(\xi\eta)$. In global system the coordinates of the nodal points are

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \text{ and } (x_4, y_4)$$

To get this mapping we define the coordinate of point P as

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

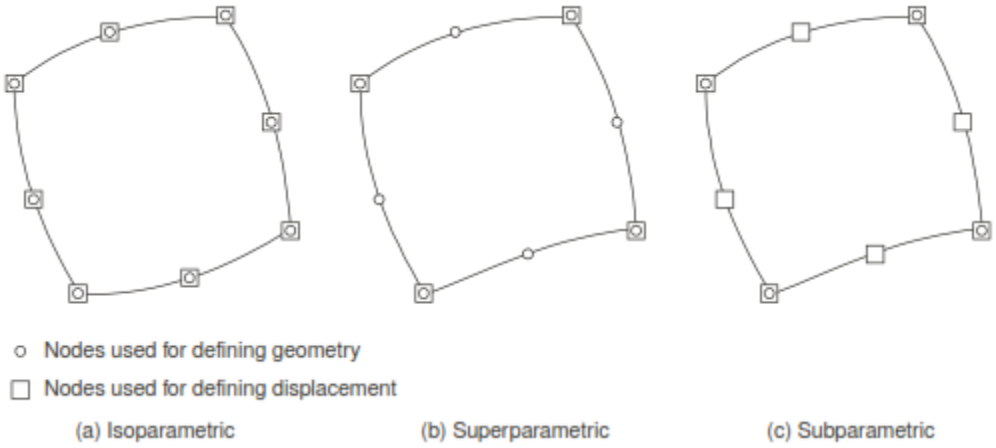
and

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Noting that shape functions are such that at node i, Ni = 1 and all others are zero, it satisfy the coordinate value at all the nodes.

ISOPARAMETRIC,SUBPARAMETRIC, AND SUPERPARAMETRIC

The finite element analysis with isoparametric elements, shape functions are used for defining the geometry as well as displacements. If the shape functions defining the boundary and displacements are the same, the element is called as isoparametric element. For example, in Fig. all the eight nodes are used in defining the geometry and displacement.



.

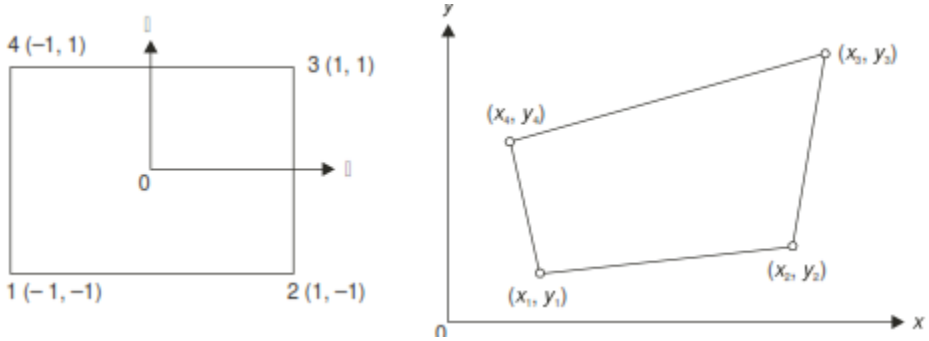
Thus, in this case $u = [N]\{\delta\}_e$, $x = [N]\{x\}_e$ and $y = [N]\{y\}_e$

where [N] is quadratic shape function of serendipity family.

The element in which more number of nodes are used to define geometry compared to the number of nodes used to define displacement are known as superparametric element. One such element is shown in Fig. in which 8 nodes are used to define the geometry and displacement is defined using only 4 nodes. In the stress analysis where boundary is highly curved but stress gradient is not high, one can use these elements advantageously. Figure shows a subparametric element in which less number of nodes are used to define geometry compared to the number of nodes used for defining the displacements. Such elements can be used advantageously in case of geometry being simple but stress gradient high.

STIFFNESS MATRIX ASSEMBLY

Assembling element stiffness matrix is a major part in finite element analysis. Since it involves coordinate transformation from natural local coordinate system to Cartesian global system, isoparametric elements need special treatment. Here the assembling of element stiffness matrix for 4 noded quadrilateral element is explained in detail. The procedure can be easily extended to higher order elements by using suitable functions and noting the increased number of nodes.



The shape functions are given as

$$N_i = \frac{(1+\xi_i)(1+\eta_i)}{4}$$

$$N_1 = \frac{(1-\xi)(1-\eta)}{4},\quad N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

The displacement at any point is given as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix}$$

the relationship between the coordinates can be done by partial differentiation

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}$$

i.e.,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

where

$$[J] = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix}$$

[J]= Jacobian matrix

Jacobian matrix relates the local coordinates to global coordinate system. In case of three dimensional the jacobian matrix is given as

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \varsigma} & \frac{\partial y}{\partial \varsigma} & \frac{\partial z}{\partial \varsigma} \end{bmatrix}$$

Now going back to isoparametric quadrilateral element,
Let

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Where

$$J_{11} = \frac{\partial x}{\partial \xi} \quad J_{12} = \frac{\partial y}{\partial \xi}$$

$$J_{21} = \frac{\partial x}{\partial \eta} \quad J_{22} = \frac{\partial y}{\partial \eta}$$

we know,

$$x = \sum_{i=1}^{4} N_i x_i = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$\therefore J_{11} \quad \frac{\partial x}{\partial \xi} \quad \frac{\partial N_1}{\partial \xi} x_1 \quad \frac{\partial N_2}{\partial \xi} x_2 \quad \frac{\partial N_3}{\partial \xi} x_3 \quad \frac{\partial N_4}{\partial \xi} x_4$$

Similarly J_{12}, J_{21} and J_{22} can be assembled.

$$J = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

For any specified point the above matrix can be assembled. Now,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$= \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

J_{11}, J_{12}, J_{13}, J_{14} are the elements of Jacobian inverse matrix. Since for a given point Jacobian matrix is known its inverse can be calculated and Jacobian inverse matrix is assembled. With this transformation relation known, we can expresses derivatives of the displacements as shown below

$$= \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* \\ 0 & 0 & J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

The strain displacement relation is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix}$$

$$= \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

But $u = \sum_{i=1}^{4} N_i u_i$ and $v = \sum_{i=1}^{4} N_i v_i$

$$\therefore \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\{B\} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

Then element stiffness matrix is given by

$$[k] = \oint [B]^T [D][B]\, dV$$

In this case,

$$[k] = t \iint [B]^T [D][B]\, \partial x\, \partial y$$

where t is the thickness.

Where $|J|$ is the determinant of the Jacobian.

$$\therefore [k] \quad t \iint [B]^T [D][B] |J|\, \partial\xi\partial\eta$$

NUMERICAL INTEGRATION

In Gauss quadrature formula sampling points are cleverly placed. In this, both n sampling points and n weights are treated as variables to make exact 2n – 1 degree polynomial. This is an open quadrature formula, the function values need not be known at end points but they must be known at predetermined sampling points. The location of sampling points ε_ι and weight function w are determined using Legendre polynomials. Hence this method is some times called as Gauss Legendre quadrature formula. Table shows gauss sampling points

Location of sampling points and weight functions

in Gauss Integration $\int_{-1}^{1} f(\xi)\,d\xi = \sum_{i=1}^{n} W_i f(\xi_i)$

n	ξ	W_i
1	$\xi_1 = 0.00000000$	$W_1 = 2.00000000$
2	$-\xi_1 = \xi_2 = 0.57735027$	$W_1 = W_2 = 1.00000000$
3	$-\xi_1 = \xi_3 = 0.77459667$ $\xi_2 = 0.00000000$	$W_1 = W_3 = 0.55555556$ $W_2 = 0.88888889$
4	$-\xi_1 = \xi_4 = 0.86113631$ $-\xi_2 = \xi_3 = 0.33998104$	$W_1 = W_4 = 0.34785485$ $W_2 = W_3 = 0.65214515$

For two dimensional problem n = 2 means 2 × 2 = 4 Gaussian points and for three dimensional problems it works out to be 2 × 2 × 2 = 8. Thus,

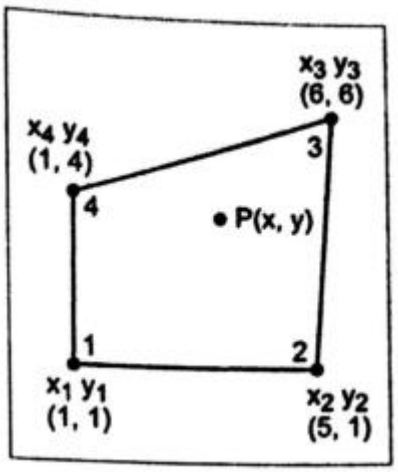
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1}\sum_{i=1}^{n} W_i f(\xi,\eta)\,d\eta$$

$$= \sum_{j=1}^{n} W_j \left\{\sum_{i=1}^{n} W_i f(\xi,\eta)\right\}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} W_j W_i f(\xi,\eta)$$

Problems

1. For the isoparametric four noded quadrilateral element show in fig determine the cartisian co-ordinates of point P which has local co-ordinates ε= 0.5 and η=0.5.



Solution:

Cartesian co-ordinates of points 1, 2, 3 and 4,

$$x_1 = 1; \qquad y_1 = 1$$
$$x_2 = 5; \qquad y_2 = 1$$
$$x_3 = 6; \qquad y_3 = 6$$
$$x_4 = 1; \qquad y_4 = 4$$

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

$$N_1 = \frac{1}{4}(1-0.5)(1-0.5) = 0.0625$$

$$N_2 = \frac{1}{4}(1+0.5)(1-0.5) = 0.1875$$

$$\Rightarrow N_3 = \frac{1}{4}(1+0.5)(1+0.5) = 0.5625$$

$$\Rightarrow N_4 = \frac{1}{4}(1-0.5)(1+0.5) = 0.1875$$

We know that,

$$\text{Co-ordinate, } x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$= 0.0625 \times 1 + 0.1875 \times 5 + 0.5625 \times 6 + 0.1875 \times 1$$
$$\boxed{x = 4.5625}$$

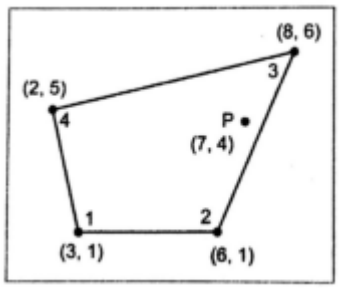
Similarly,

$$\text{Co-ordinate, } y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$
$$= 0.0625 \times 1 + 0.1875 \times 1 + 0.5625 \times 6 + 0.1875 \times 4$$
$$\boxed{y = 4.375}$$

The Cartesian co-ordinates of the point P are (4.5625,4.375}

Problem 2

For the isoparametric quadrilateral element shown in fig. determine the local co-ordinates of the point P which has Cartesian co-ordinates (7,4).



Solution:

Cartesian co-ordinates of point 1, 2, 3 and 4

$x_1 = 3; \quad y_1 = 1$

$x_2 = 6; \quad y_2 = 1$

$x_3 = 8; \quad y_3 = 6$

$x_4 = 2; \quad y_4 = 5$

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

Cartesian co-ordinates of point P (x, y),

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Substitute N_1, N_2, N_3, N_4, x, x_1, x_3 and x_4 values in equation (1),

$$7 = \frac{1}{4}[(1-\varepsilon)(1-\eta)\times 3 + (1+\varepsilon)(1-\eta)\times 6 + (1+\varepsilon)(1+\eta)\times 8 + (1-\varepsilon)(1+\eta)\times 2]$$

$$28 = [(1-\eta-\varepsilon+\varepsilon\eta)3 + (1-\eta+\varepsilon-\varepsilon\eta)6 + (1+\eta+\varepsilon+\varepsilon\eta)8 + (1+\eta-\varepsilon-\varepsilon\eta)2]$$

$$28 = 3 - 3\eta - 3\varepsilon + 3\varepsilon\eta + 6 - 6\eta + 6\varepsilon - 6\varepsilon\eta + 8 + 8\eta + 8\varepsilon + 8\varepsilon\eta + 2 + 2\eta - 2\varepsilon - 2\varepsilon\eta$$

$$28 = 19 + \eta + 9\varepsilon + 3\varepsilon\eta$$

$$\boxed{\eta + 9\varepsilon + 3\varepsilon\eta = 9}$$

Substitute N_1, N_2, N_3, N_4, y, y_1, y_2, y_3 and y_4 values in equation (2),

$$4 = \frac{1}{4}[(1-\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1+\eta)\times 6 + (1-\varepsilon)(1+\eta)\times 5]$$

$$= \frac{1}{4}[1-\eta-\varepsilon+\varepsilon\eta+1-\eta+\varepsilon-\varepsilon\eta+6+6\eta+6\varepsilon+6\varepsilon\eta+5+5\eta-5\varepsilon-5\varepsilon\eta]$$

$$4 = \frac{1}{4}[13+9\eta+\varepsilon+\varepsilon\eta]$$

$$\Rightarrow 16 = 13+9\eta+\varepsilon+\varepsilon\eta$$

$$\Rightarrow \boxed{9\eta+\varepsilon+\varepsilon\eta = 3}$$

Equation (4) multiplied by (–3),

$$-27\eta-3\varepsilon-3\varepsilon\eta = -9$$

Solving equation (3) and (5),

$$\eta+9\varepsilon+3\varepsilon\eta = 9$$
$$-27\eta-3\varepsilon-3\varepsilon\eta = -9$$

Solving,

$$-26\eta+6\varepsilon = 0$$
$$-26\eta = -6\varepsilon$$
$$\Rightarrow \boxed{\varepsilon = 4.3333\,\eta}$$

Substitute ε value in equation (3),

$$(3) \Rightarrow \eta+9(4.3333\,\eta)+3(4.3333\,\eta)\times\eta = 9$$
$$\eta+39\eta+13\eta^2 = 9$$
$$\Rightarrow 13\eta^2+40\eta = 9$$
$$\Rightarrow 13\eta^2+40\eta-9 = 0$$
$$\eta = \frac{-40\pm\sqrt{(40)^2-4(13)(-9)}}{2(13)}$$

$$[a x^2+b x+c=0;\ \text{Roots: } \frac{-b\pm\sqrt{b^2-4ac}}{2a}]$$

$$= \frac{-40+45.475}{26}$$

$$\boxed{\eta = 0.210587}$$

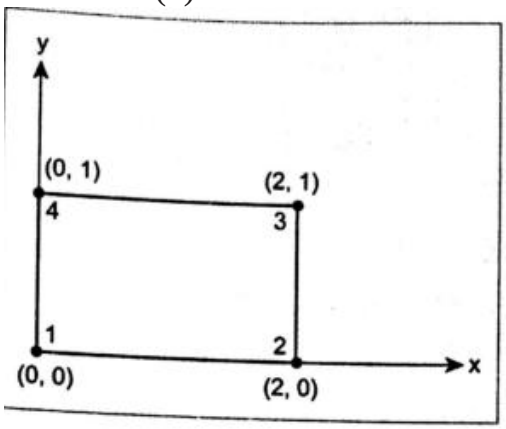
Substitute η value in equation (6),

$$\Rightarrow \varepsilon = 4.33333 \times 0.210587$$

$$\boxed{\varepsilon = 0.912545}$$

Local co-ordinates of the point P η=0.200557
ε=0.912545

problem 3
a four noded rectangular element is shown in fig. Determine the following (1) Jacobian matrix (2) strain displacement matrix (3) element stresses



Solution:

Take $E = 2 \times 10^5$ ***N/mm²;*** $\nu = 0.25$; $u = [0, 0, 0.003, 0.004, 0.006, 0.004, 0, 0]^T$
$\varepsilon = 0$; $\eta = 0$
Assume plane stress condition.

Cartesian co-ordinates of point 1, 2, 3 and 4

$x_1 = 0$; $y_1 = 0$
$x_2 = 2$; $y_2 = 0$
$x_3 = 2$; $y_3 = 1$
$x_4 = 0$; $y_4 = 1$

Young's modulus, $E = 2 \times 10^5$ N/m²
Poisson's ratio, $\nu = 0.25$

$$\text{Displacements, } u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

Natural co-ordinates, $\varepsilon = 0, \ \eta = 0$

Jacobian matrix for quadrilateral element is

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where,
$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4]$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4]$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4]$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4]$$

Substitute $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, and ε , η

$$J_{11} = \frac{1}{4}[0+2+2-0]$$

$$\boxed{J_{11} = 1}$$

$$J_{12} = \frac{1}{4}[0+0+1-1]$$

$$\boxed{J_{12} = 0}$$

$$J_{21} = \frac{1}{4}[0-2+2+0]$$

$$\boxed{J_{21} = 0}$$

$$J_{22} = \frac{1}{4}[-0-0+1+1]$$

$$\boxed{J_{22} = 0.5}$$

$$\Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$\text{Jacobian matrix, } [J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\Rightarrow |J| = 1 \times 0.5 - 0$$

$$\boxed{|J| = 0.5}$$

The strain displacement matrix is

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

$$[B] = \frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{0.5 \times 4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 & -1 & 0.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$= \frac{0.5}{0.5 \times 4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$[B] = 0.25\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

Element stress, $\sigma = [D][B]\{u\}$

For plane stress condition,

Stress-strain relationship matrix, $[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

$$= \frac{2 \times 10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= 213.33 \times 10^3\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= 213.33 \times 10^3 \times 0.25 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\sigma\} = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 53.333 \times 10^3 \times 0.25 \begin{bmatrix} -4 & -2 & 4 & -2 & 4 & 2 & -4 & 2 \\ -1 & -8 & 1 & -8 & 1 & 8 & -1 & 8 \\ -3 & -1.5 & -3 & 1.5 & 3 & 1.5 & 3 & -1.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 13.333 \times 10^3 \begin{Bmatrix} 0 + 0 + (4 \times 0.003) + (-2 \times 0.004) + (4 \times 0.006) + (2 \times 0.004) + 0 + 0 \\ 0 + 0 + (1 \times 0.003) + (-8 \times 0.004) + (1 \times 0.006) + (8 \times 0.004) + 0 + 0 \\ 0 + 0 + (-3 \times 0.003) + (1.5 \times 0.004) + (3 \times 0.006) + (1.5 \times 0.004) + 0 + 0 \end{Bmatrix}$$

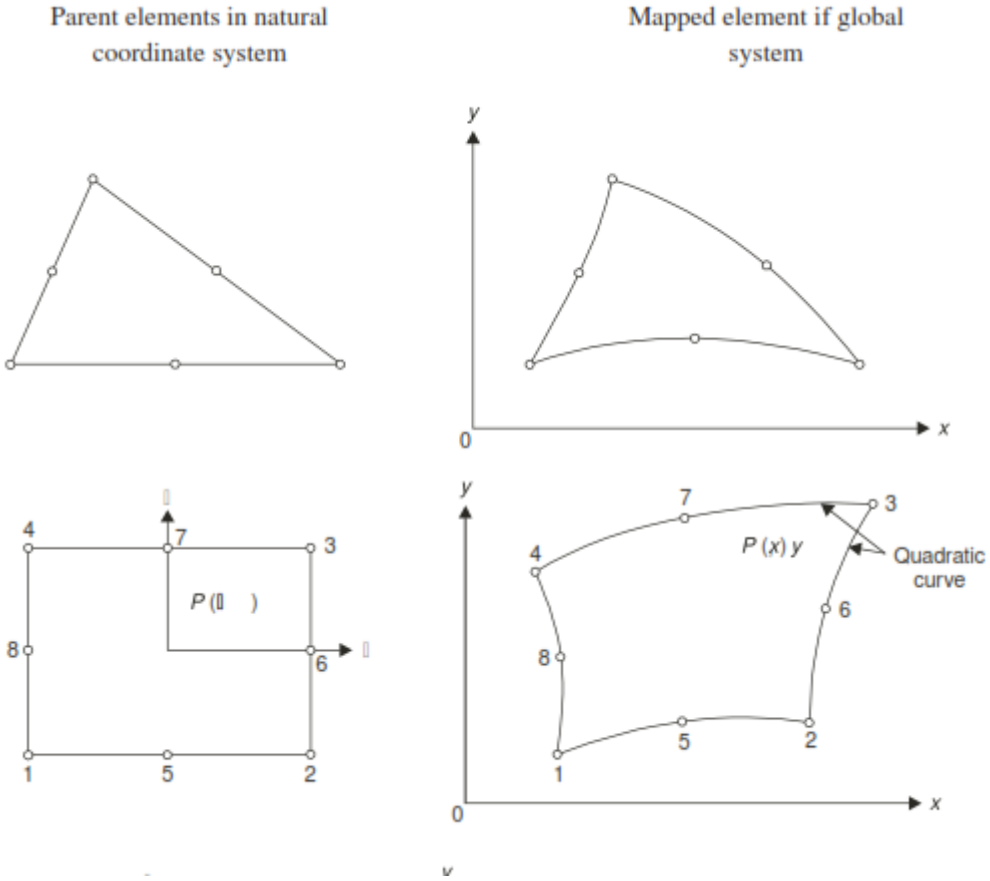
$$\{\sigma\} = 13.333 \times 10^3 \begin{Bmatrix} 0.036 \\ 0.009 \\ 0.021 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 480 \\ 120 \\ 280 \end{Bmatrix} \text{ N/m}^2$$

UNIT –III

ISOPARAMETRIC FORMULATIONS

The isoparametric concept brought out by Taig and latter on generalized by B.M. Irons revolutionized the finite elements analysis and it also helped in properly mapping the curved boundaries. They brought out the concept of mapping regular triangular and rectangular elements in natural coordinate system, to arbitrary shapes in global system as shown in Fig. The coordinate transformation of natural coordinates to global coordinate system is presented. The terms isoparametric, super parametric and subparametrics are defined.The basic theorems on which isoparametric concept is based are explained and need for satisfying uniqueness theorem of mapping is presented. Assembling of stiffness matrix is illustrated. For assembling stiffness matrix integration is to be carried out numerically.



COORDINATE TRANSFORMATION

So far we have used the shape functions for defining deflection at any point interms of the nodal displacement. Taig suggested use of shape function for coordinate transformation form natural local coordinate system to global Cartesian system and successfully achieved in mapping parent element to required shape in global system. Thus the Cartesian

coordinate of a point in an element may be expressed as or in matrix form where N are shape functions and (x)

$$x = N_1 x_1 + N_2 x_2 + \ldots + N_n x_n$$
$$y = N_1 y_1 + N_2 y_2 + \ldots + N_n y_n$$
$$z = N_1 z_1 + N_2 z_2 + \ldots + N_n z_n$$

or in matrix form

$$\{x\} = [N]\{x\}_e$$

where N are shape functions and (x)e are the coordinates of nodal points of the element. The shape functions are to be expressed in natural coordinate system. The shape function of this element is given as

$$N_1 = \frac{(1-\xi)(1-\eta)}{4},\ N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

P is a point with coordinate $(\xi\eta)$. In global system the coordinates of the nodal points are

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \text{ and } (x_4, y_4)$$

To get this mapping we define the coordinate of point P as

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

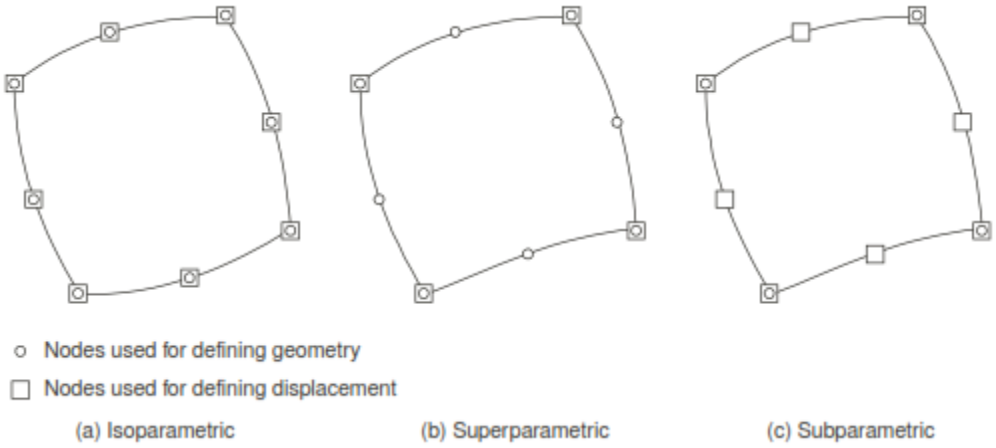
and

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Noting that shape functions are such that at node i, Ni = 1 and all others are zero, it satisfy the coordinate value at all the nodes.

ISOPARAMETRIC,SUBPARAMETRIC, AND SUPERPARAMETRIC

The finite element analysis with isoparametric elements, shape functions are used for defining the geometry as well as displacements. If the shape functions defining the boundary and displacements are the same, the element is called as isoparametric element. For example, in Fig. all the eight nodes are used in defining the geometry and displacement.



.

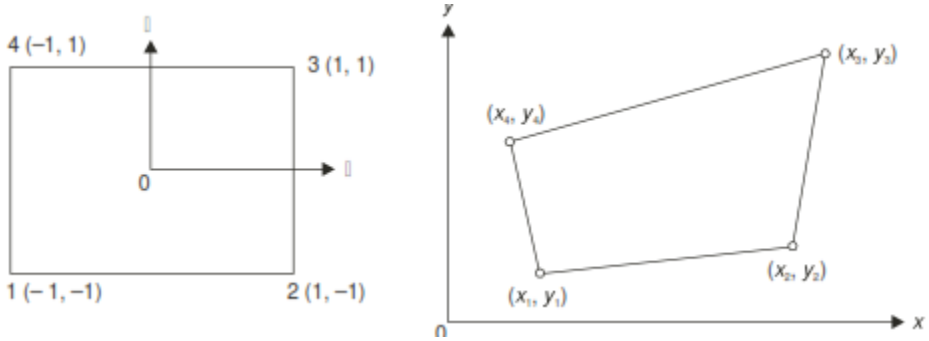
Thus, in this case $u = [N]\{\delta\}_e$, $x = [N]\{x\}_e$ and $y = [N]\{y\}_e$

where [N] is quadratic shape function of serendipity family.

The element in which more number of nodes are used to define geometry compared to the number of nodes used to define displacement are known as superparametric element. One such element is shown in Fig. in which 8 nodes are used to define the geometry and displacement is defined using only 4 nodes. In the stress analysis where boundary is highly curved but stress gradient is not high, one can use these elements advantageously. Figure shows a subparametric element in which less number of nodes are used to define geometry compared to the number of nodes used for defining the displacements. Such elements can be used advantageously in case of geometry being simple but stress gradient high.

STIFFNESS MATRIX ASSEMBLY

Assembling element stiffness matrix is a major part in finite element analysis. Since it involves coordinate transformation from natural local coordinate system to Cartesian global system, isoparametric elements need special treatment. Here the assembling of element stiffness matrix for 4 noded quadrilateral element is explained in detail. The procedure can be easily extended to higher order elements by using suitable functions and noting the increased number of nodes.



The shape functions are given as

$$N_i = \frac{(1+\xi_i)(1+\eta_i)}{4}$$

$$N_1 = \frac{(1-\xi)(1-\eta)}{4},\quad N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

The displacement at any point is given as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix}$$

the relationship between the coordinates can be done by partial differentiation

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}$$

i.e.,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

where

$$[J] = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix}$$

[J]= Jacobian matrix

Jacobian matrix relates the local coordinates to global coordinate system. In case of three dimensional the jacobian matrix is given as

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \varsigma} & \frac{\partial y}{\partial \varsigma} & \frac{\partial z}{\partial \varsigma} \end{bmatrix}$$

Now going back to isoparametric quadrilateral element,
Let

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Where

$$J_{11} = \frac{\partial x}{\partial \xi} \quad J_{12} = \frac{\partial y}{\partial \xi}$$

$$J_{21} = \frac{\partial x}{\partial \eta} \quad J_{22} = \frac{\partial y}{\partial \eta}$$

we know,

$$x = \sum_{i=1}^{4} N_i x_i = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$\therefore J_{11} \quad \frac{\partial x}{\partial \xi} \quad \frac{\partial N_1}{\partial \xi} x_1 \quad \frac{\partial N_2}{\partial \xi} x_2 \quad \frac{\partial N_3}{\partial \xi} x_3 \quad \frac{\partial N_4}{\partial \xi} x_4$$

Similarly J_{12}, J_{21} and J_{22} can be assembled.

$$J = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

For any specified point the above matrix can be assembled. Now,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$= \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

J_{11}, J_{12}, J_{13}, J_{14} are the elements of Jacobian inverse matrix. Since for a given point Jacobian matrix is known its inverse can be calculated and Jacobian inverse matrix is assembled. With this transformation relation known, we can expresses derivatives of the displacements as shown below

$$= \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* \\ 0 & 0 & J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

The strain displacement relation is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix}$$

$$= \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

But $u = \sum_{i=1}^{4} N_i u_i$ and $v = \sum_{i=1}^{4} N_i v_i$

$$\therefore \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\{B\} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

Then element stiffness matrix is given by

$$[k] = \oint [B]^T [D][B]\, dV$$

In this case,

$$[k] = t \iint [B]^T [D][B]\, \partial x\, \partial y$$

where t is the thickness.

Where $|J|$ is the determinant of the Jacobian.

$$\therefore [k] \quad t \iint [B]^T [D][B] |J|\, \partial\xi\partial\eta$$

NUMERICAL INTEGRATION

In Gauss quadrature formula sampling points are cleverly placed. In this, both n sampling points and n weights are treated as variables to make exact 2n – 1 degree polynomial. This is an open quadrature formula, the function values need not be known at end points but they must be known at predetermined sampling points. The location of sampling points ε_ι and weight function w are determined using Legendre polynomials. Hence this method is some times called as Gauss Legendre quadrature formula. Table shows gauss sampling points

Location of sampling points and weight functions

in Gauss Integration $\int_{-1}^{1} f(\xi)\,d\xi = \sum_{i=1}^{n} W_i f(\xi_i)$

n	ξ	W_i
1	$\xi_1 = 0.00000000$	$W_1 = 2.00000000$
2	$-\xi_1 = \xi_2 = 0.57735027$	$W_1 = W_2 = 1.00000000$
3	$-\xi_1 = \xi_3 = 0.77459667$ $\xi_2 = 0.00000000$	$W_1 = W_3 = 0.55555556$ $W_2 = 0.88888889$
4	$-\xi_1 = \xi_4 = 0.86113631$ $-\xi_2 = \xi_3 = 0.33998104$	$W_1 = W_4 = 0.34785485$ $W_2 = W_3 = 0.65214515$

For two dimensional problem n = 2 means 2 × 2 = 4 Gaussian points and for three dimensional problems it works out to be 2 × 2 × 2 = 8. Thus,

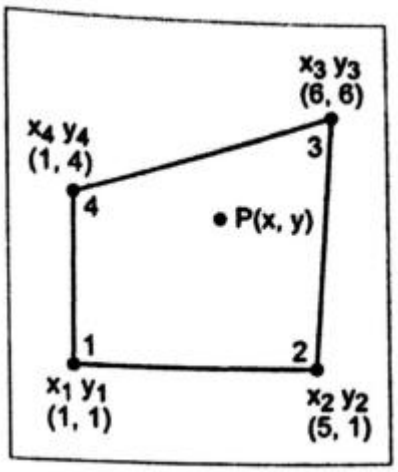
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1}\sum_{i=1}^{n} W_i f(\xi,\eta)\,d\eta$$

$$= \sum_{j=1}^{n} W_j \left\{\sum_{i=1}^{n} W_i f(\xi,\eta)\right\}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} W_j W_i f(\xi,\eta)$$

Problems

1. For the isoparametric four noded quadrilateral element show in fig determine the cartisian co-ordinates of point P which has local co-ordinates ε= 0.5 and η=0.5.



Solution:

Cartesian co-ordinates of points 1, 2, 3 and 4,

$$x_1 = 1; \qquad y_1 = 1$$
$$x_2 = 5; \qquad y_2 = 1$$
$$x_3 = 6; \qquad y_3 = 6$$
$$x_4 = 1; \qquad y_4 = 4$$

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

$$N_1 = \frac{1}{4}(1-0.5)(1-0.5) = 0.0625$$

$$N_2 = \frac{1}{4}(1+0.5)(1-0.5) = 0.1875$$

$$\Rightarrow N_3 = \frac{1}{4}(1+0.5)(1+0.5) = 0.5625$$

$$\Rightarrow N_4 = \frac{1}{4}(1-0.5)(1+0.5) = 0.1875$$

We know that,

$$\text{Co-ordinate, } x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$= 0.0625 \times 1 + 0.1875 \times 5 + 0.5625 \times 6 + 0.1875 \times 1$$
$$\boxed{x = 4.5625}$$

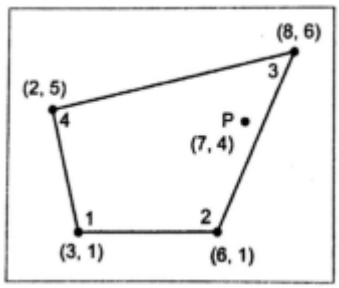
Similarly,

$$\text{Co-ordinate, } y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$
$$= 0.0625 \times 1 + 0.1875 \times 1 + 0.5625 \times 6 + 0.1875 \times 4$$
$$\boxed{y = 4.375}$$

The Cartesian co-ordinates of the point P are (4.5625,4.375}

Problem 2

For the isoparametric quadrilateral element shown in fig. determine the local co-ordinates of the point P which has Cartesian co-ordinates (7,4).



Solution:

Cartesian co-ordinates of point 1, 2, 3 and 4

$x_1 = 3; \quad y_1 = 1$

$x_2 = 6; \quad y_2 = 1$

$x_3 = 8; \quad y_3 = 6$

$x_4 = 2; \quad y_4 = 5$

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

Cartesian co-ordinates of point P (x, y),

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Substitute N_1, N_2, N_3, N_4, x, x_1, x_3 and x_4 values in equation (1),

$$7 = \frac{1}{4}[(1-\varepsilon)(1-\eta)\times 3 + (1+\varepsilon)(1-\eta)\times 6 + (1+\varepsilon)(1+\eta)\times 8 + (1-\varepsilon)(1+\eta)\times 2]$$

$$28 = [(1-\eta-\varepsilon+\varepsilon\eta)3 + (1-\eta+\varepsilon-\varepsilon\eta)6 + (1+\eta+\varepsilon+\varepsilon\eta)8 + (1+\eta-\varepsilon-\varepsilon\eta)2]$$

$$28 = 3 - 3\eta - 3\varepsilon + 3\varepsilon\eta + 6 - 6\eta + 6\varepsilon - 6\varepsilon\eta + 8 + 8\eta + 8\varepsilon + 8\varepsilon\eta + 2 + 2\eta - 2\varepsilon - 2\varepsilon\eta$$

$$28 = 19 + \eta + 9\varepsilon + 3\varepsilon\eta$$

$$\boxed{\eta + 9\varepsilon + 3\varepsilon\eta = 9}$$

Substitute N_1, N_2, N_3, N_4, y, y_1, y_2, y_3 and y_4 values in equation (2),

$$4 = \frac{1}{4}[(1-\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1+\eta)\times 6 + (1-\varepsilon)(1+\eta)\times 5]$$

$$= \frac{1}{4}[1-\eta-\varepsilon+\varepsilon\eta+1-\eta+\varepsilon-\varepsilon\eta+6+6\eta+6\varepsilon+6\varepsilon\eta+5+5\eta-5\varepsilon-5\varepsilon\eta]$$

$$4 = \frac{1}{4}[13+9\eta+\varepsilon+\varepsilon\eta]$$

$$\Rightarrow 16 = 13+9\eta+\varepsilon+\varepsilon\eta$$

$$\Rightarrow \boxed{9\eta+\varepsilon+\varepsilon\eta = 3}$$

Equation (4) multiplied by (–3),

$$-27\eta-3\varepsilon-3\varepsilon\eta = -9$$

Solving equation (3) and (5),

$$\eta+9\varepsilon+3\varepsilon\eta = 9$$
$$-27\eta-3\varepsilon-3\varepsilon\eta = -9$$

Solving,

$$-26\eta+6\varepsilon = 0$$
$$-26\eta = -6\varepsilon$$
$$\Rightarrow \boxed{\varepsilon = 4.3333\,\eta}$$

Substitute ε value in equation (3),

$$(3) \Rightarrow \eta+9(4.3333\,\eta)+3(4.3333\,\eta)\times\eta = 9$$
$$\eta+39\eta+13\eta^2 = 9$$
$$\Rightarrow 13\eta^2+40\eta = 9$$
$$\Rightarrow 13\eta^2+40\eta-9 = 0$$
$$\eta = \frac{-40\pm\sqrt{(40)^2-4(13)(-9)}}{2(13)}$$

$$[a x^2+b x+c=0;\ \text{Roots: } \frac{-b\pm\sqrt{b^2-4ac}}{2a}]$$

$$= \frac{-40+45.475}{26}$$

$$\boxed{\eta = 0.210587}$$

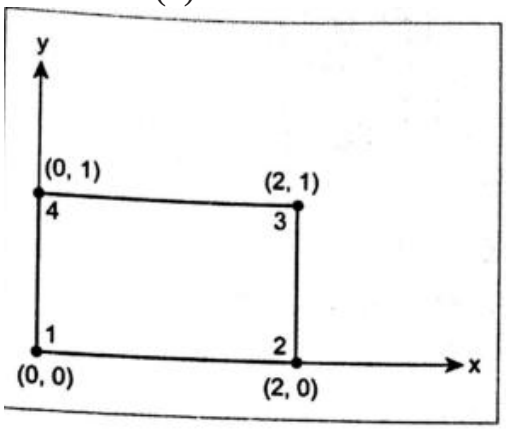
Substitute η value in equation (6),

$$\Rightarrow \varepsilon = 4.33333 \times 0.210587$$

$$\boxed{\varepsilon = 0.912545}$$

Local co-ordinates of the point P η=0.200557
ε=0.912545

problem 3
a four noded rectangular element is shown in fig. Determine the following (1) Jacobian matrix (2) strain displacement matrix (3) element stresses



Solution:

Take $E = 2 \times 10^5$ ***N/mm²;*** $\nu = 0.25$; $u = [0, 0, 0.003, 0.004, 0.006, 0.004, 0, 0]^T$
$\varepsilon = 0$; $\eta = 0$
Assume plane stress condition.

Cartesian co-ordinates of point 1, 2, 3 and 4

$x_1 = 0$; $y_1 = 0$
$x_2 = 2$; $y_2 = 0$
$x_3 = 2$; $y_3 = 1$
$x_4 = 0$; $y_4 = 1$

Young's modulus, $E = 2 \times 10^5$ N/m²
Poisson's ratio, $\nu = 0.25$

$$\text{Displacements, } u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

Natural co-ordinates, $\varepsilon = 0, \ \eta = 0$

Jacobian matrix for quadrilateral element is

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where,
$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4]$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4]$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4]$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4]$$

Substitute $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, and ε , η

$$J_{11} = \frac{1}{4}[0+2+2-0]$$

$$\boxed{J_{11} = 1}$$

$$J_{12} = \frac{1}{4}[0+0+1-1]$$

$$\boxed{J_{12} = 0}$$

$$J_{21} = \frac{1}{4}[0-2+2+0]$$

$$\boxed{J_{21} = 0}$$

$$J_{22} = \frac{1}{4}[-0-0+1+1]$$

$$\boxed{J_{22} = 0.5}$$

$$\Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$\text{Jacobian matrix, } [J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\Rightarrow |J| = 1 \times 0.5 - 0$$

$$\boxed{|J| = 0.5}$$

The strain displacement matrix is

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

$$[B] = \frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{0.5 \times 4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 & -1 & 0.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$= \frac{0.5}{0.5 \times 4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$[B] = 0.25\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

Element stress, $\sigma = [D][B]\{u\}$

For plane stress condition,

Stress-strain relationship matrix, $[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

$$= \frac{2 \times 10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= 213.33 \times 10^3\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= 213.33 \times 10^3 \times 0.25 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\sigma\} = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 53.333 \times 10^3 \times 0.25 \begin{bmatrix} -4 & -2 & 4 & -2 & 4 & 2 & -4 & 2 \\ -1 & -8 & 1 & -8 & 1 & 8 & -1 & 8 \\ -3 & -1.5 & -3 & 1.5 & 3 & 1.5 & 3 & -1.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 13.333 \times 10^3 \begin{Bmatrix} 0 + 0 + (4 \times 0.003) + (-2 \times 0.004) + (4 \times 0.006) + (2 \times 0.004) + 0 + 0 \\ 0 + 0 + (1 \times 0.003) + (-8 \times 0.004) + (1 \times 0.006) + (8 \times 0.004) + 0 + 0 \\ 0 + 0 + (-3 \times 0.003) + (1.5 \times 0.004) + (3 \times 0.006) + (1.5 \times 0.004) + 0 + 0 \end{Bmatrix}$$

$$\{\sigma\} = 13.333 \times 10^3 \begin{Bmatrix} 0.036 \\ 0.009 \\ 0.021 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 480 \\ 120 \\ 280 \end{Bmatrix} \text{ N/m}^2$$

UNIT 5

DYNAMIC ANALYSIS

Nonlinear problems

Various non linear problems in finite element analysis may be group into the following three categories.

1. Material non linear problems
2. Geometric non linear problems
3. Non linear boundary or initial conditions

Nonlinear Material Behavior

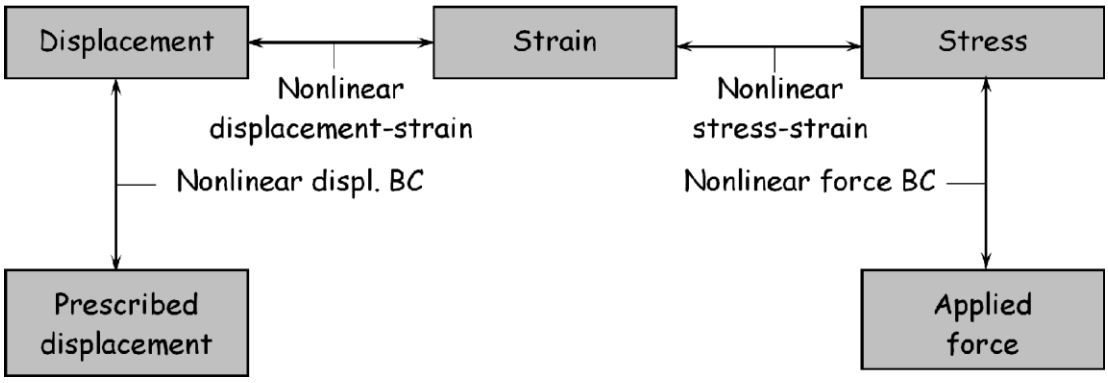
This is one of the most common forms of nonlinearity, and would include nonlinear elastic, plastic, and visco elastic behavior. For thermal problems, a temperature dependent thermal conductivity will produce nonlinear equations.

Large Deformation Theory (Geometric Nonlinearity)

If a continuum body under study undergoes large finite deformations, the strain-displacement relations will become nonlinear. Also for structural mechanics problems under large deformations, the stiffness will change with deformation thus making the problem nonlinear. Buckling problems are also nonlinear.

Nonlinear Boundary or Initial Conditions

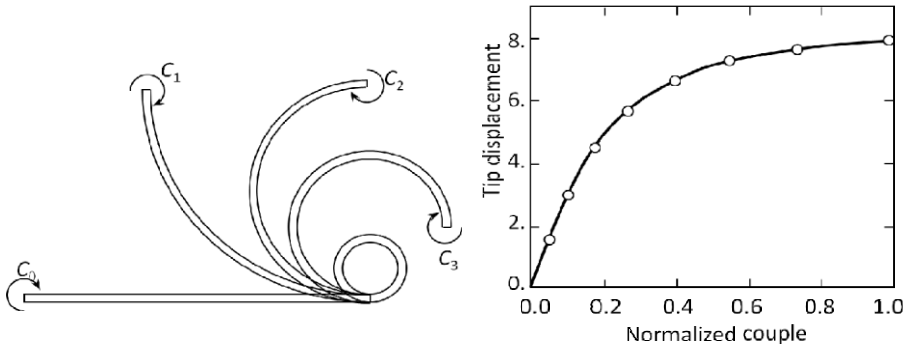
Problems involving contact mechanics normally include a boundary condition that depends on the deformation thereby producing a nonlinear formulation. Thermal problems involving melting or freezing (phase change) also include such nonlinear boundary conditions.

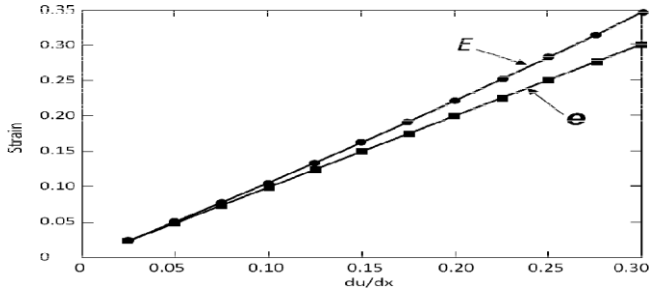


Non linearity in structural problem

GEOMETRIC NONLINEARITY

Relations among kinematic quantities (i.e., displacement, rotation and strains) are nonlinear





Displacement-strain relation

i. E has a higher-order term

ii. (du/dx) << 1 → e(x) ~ E(x).

Domain of integration

i. Undeformed domain W_0

ii. Deformed domain W_x

$$a(\mathbf{u},\bar{\mathbf{u}}) = \iint_{\Omega} \varepsilon(\bar{\mathbf{u}}) : \sigma(\mathbf{u})\, d\Omega$$

MATERIAL NONLINEARITY

Linear (elastic) material

$$\{\sigma\} = [D]\{\varepsilon\}$$

Only for infinitesimal deformation

Nonlinear (elastic) material

i. [C] is not a constant but depends on deformation

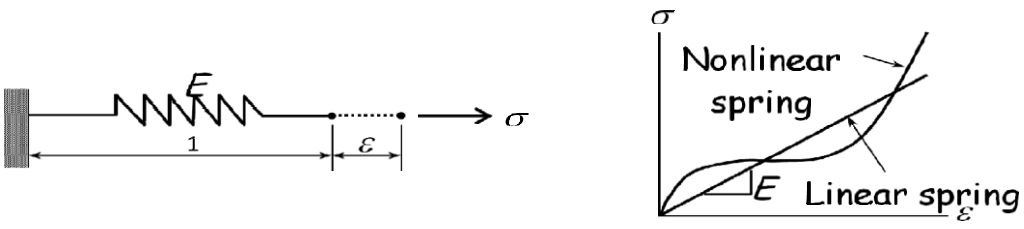
ii. Stress by differentiating strain energy density U

iii. Linear material:

$$U = \frac{1}{2} E\varepsilon^2$$

$$\sigma = \frac{dU}{d\varepsilon} = E\varepsilon$$

Stress is a function of strain (deformation): potential, path independent



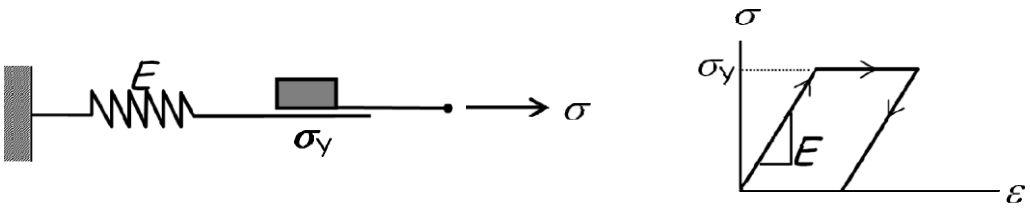
Linear and nonlinear elastic spring models

Elasto-plastic material (energy dissipation occurs)

i. Friction plate only support stress up to s_y

ii. Stress cannot be determined from stress alone

iii. History of loading path is required: path-dependent

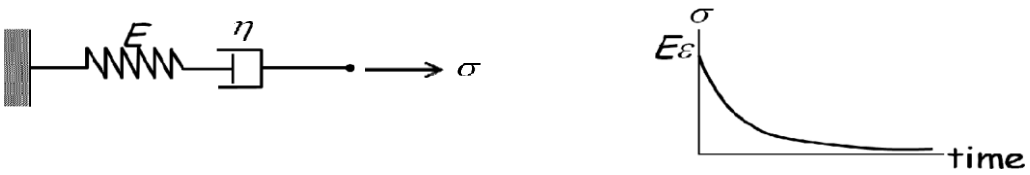


Elasto-plastic spring model

Visco-elastic material

i. Time-dependent behavior

ii. Creep, relaxation

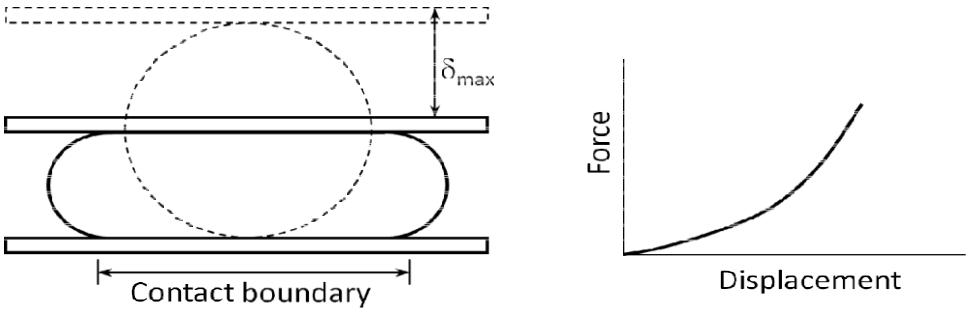


Visco-elastic spring model

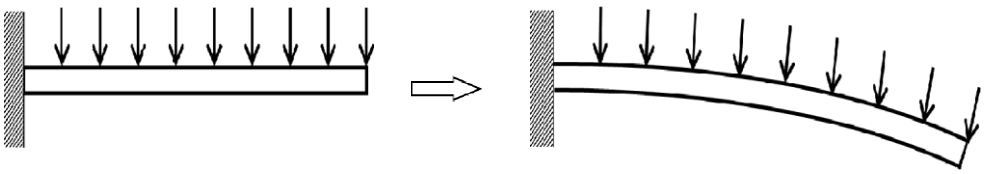
Boundary and Force Nonlinearities

Nonlinear displacement BC (kinematic nonlinearity)

Contact problems, displacement dependent conditions

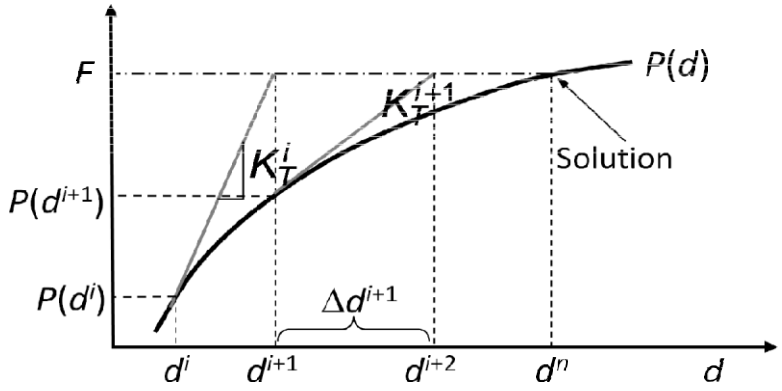


Nonlinear force BC (Kinetic nonlinearity)



NON LINEAR ANALYSIS

Newton-Raphson Method



i. Most popular method

ii. Assume $\mathbf{d}^i$ at i-th iteration is known

iii. Looking for $\mathbf{d}^{i+1}$ from first-order Taylor series expansion

$$\mathbf{P}(\mathbf{d}^{i+1}) \approx \mathbf{P}(\mathbf{d}^i) + \mathbf{K}_T^i(\mathbf{d}^i) \cdot \Delta\mathbf{d}^i = \mathbf{F}$$

$$\mathbf{K}_T^i(\mathbf{d}^i) \equiv \left(\frac{\partial \mathbf{P}}{\partial \mathbf{d}} \right)^i$$

: **Jacobian matrix or Tangent stiffness matrix**

iv. **Solve for incremental solution**

$$\mathbf{K}_T^i \Delta\mathbf{d}^i = \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$$

v. Update solution

$$\mathbf{d}^{i+1} = \mathbf{d}^i + \Delta\mathbf{d}^i$$

Consistent System Matrices

To do dynamic and vibration finite element analysis, you need at least a mass matrix to pair with the stiffness matrix. As a general rule, the construction of the master mass matrix M largely parallels of the master stiffness matrix K. Mass matrices for individual elements are formed in local coordinates, transformed to global, and merged into the master mass matrix following exactly the same techniques used for K. In practical terms, the assemblers for K and M can be made identical. This procedural uniformity is one of the great assets of the Direct Stiffness Method. A notable difference with the stiffness matrix is the possibility of using a *diagonal* mass matrix based on direct lumping. A master diagonal mass matrix can be stored simply as a vector. If all entries are nonnegative, it is easily inverted, since the inverse of a diagonal matrix is also diagonal. Obviously a lumped mass matrix entails significant computational advantages for calculations that involve M−1. This is balanced by some negative aspects that are examined in some detail later.

Mass Matrix Construction

The master mass matrix is built up from element contributions, and we start at that level. The construction of the mass matrix of individual elements can be carried out through several methods. can be categorized into three groups: direct mass lumping, variational mass lumping, and template mass lumping. The last group is more general in that includes all others. Variants of the first two techniques are by now standard in the FEM literature, and implemented in all general purpose codes.

Direct Mass Lumping

The total mass of element *e* is directly apportioned to nodal freedoms, ignoring any cross coupling. The goal is to build a *diagonally lumped mass matrix* or DLMM, denoted here by M*eL* .As the simplest example, consider a 2-node prismatic bar element with length *L*, cross section area *A*, and mass density ρ, which can only move in the axial direction *x*,. The total mass of the element is *Me* = ρ *AL*. This is divided into two equal parts and assigned to each end

Dynamic condensation

The accuracy of the resulting reduced model is generally very low for dynamic problems. To achieve reasonably accurate results, the masters must be chosen with great care and number of masters should be greater than the number of modes interested. To alleviate the limitations, the inertia effects could be partially or fully included in the condensation. The corresponding condensation approaches are generally called dynamic condensation

The equation of motion is cast as a shifted Eigen problem. A shift value, f, is introduced into the set of equations describing the dynamic system, thus

$$[[K_n]-(\lambda-f)[M_n]]\{x_n\}=\{0\}$$

The terms are rearranged to group the constant term f times the mass matrix with the stiffness matrix to yield

$$[[[K_n]+f[M_n]]-\lambda[M_n]]\{x_n\}=\{0\}$$

Then let a new system matrix [D] be used to describe the 'effective' stiffness matrix as

$$[D_n]=[[K_n]+f[M_n]]$$

This 'effective' stiffness equation

$$[D_n]\{x_n\}=\{F_n\}$$

can be partitioned into the 'a' active DOF and the 'd' deleted or omitted DOF to form two equations given as

$$\begin{bmatrix}[D_{aa}] & [D_{ad}]\\ [D_{da}] & [D_{dd}]\end{bmatrix}\begin{Bmatrix}x_a\\ x_d\end{Bmatrix}=\begin{Bmatrix}F_a\\ F_d\end{Bmatrix}$$

Assuming that the forces on the deleted DOF are zero, then the second equation can be written as

$$[D_{da}]\{x_a\}+[D_{dd}]\{x_d\}=\{0\}$$

which can be solved for the displacement at the deleted DOF as

$$\{x_d\}=-[D_{dd}]^{-1}[D_{da}]\{x_a\}$$

The first equation can be written as

$$[D_{aa}]\{x_a\}+[D_{ad}]\{x_d\}=\{F_a\}$$

and upon substituting for the 'd' deleted DOF this equation becomes

$$[D_{aa}]\{x_a\}+[D_{ad}][D_{dd}]^{-1}[D_{da}]\{x_a\}=\{F_a\}$$

This can be manipulated to yield the desired transformation to be

$$[T_f]=\begin{bmatrix}[I]\\ [t_f]\end{bmatrix}=\begin{bmatrix}[I]\\ -[D_{dd}]^{-1}[D_{da}]\end{bmatrix}$$

Using this transformation, the reduced stiffness can be written as

$$\left[K_a^f\right]=[T_f]^T[K_n][T_f]$$

This same transformation can be applied to the mass matrix given by

$$\left[M_a^f\right]=[T_f]^T[M_n][T_f]$$

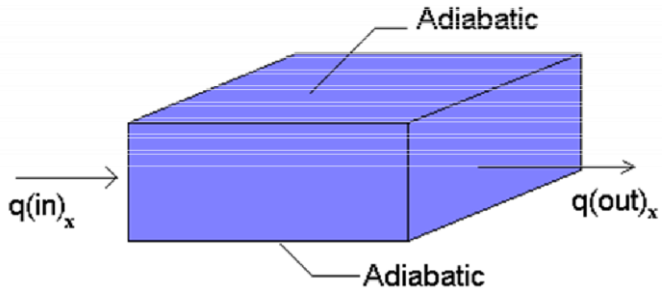
THERMAL ANALYSIS

One dimensional conduction

$$q_x=-K_{xx}\cdot\frac{dT}{dx}$$

K = thermal conductivity

dT = temperature change



$$E_{in}+E_{gen}=\Delta U+E_{out}$$

$$q_{in}\cdot A\cdot dt+Q\cdot A\cdot dx\cdot dt=\Delta U+q_{out}\cdot A\cdot dt$$

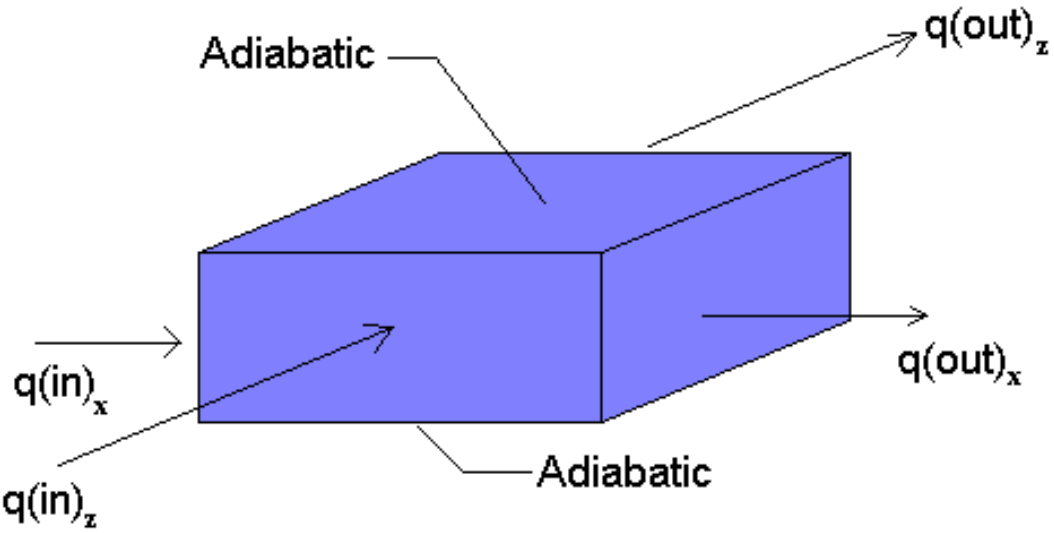
q = heat conducted E = kinetic energy

Q = internal heat source U = stored energy

A = cross-sectional area t = time

Two-Dimensional Conduction

$$q_x = -K_{xx} \cdot \frac{dT}{dx} + -K_{zz} \cdot \frac{dT}{dz}$$

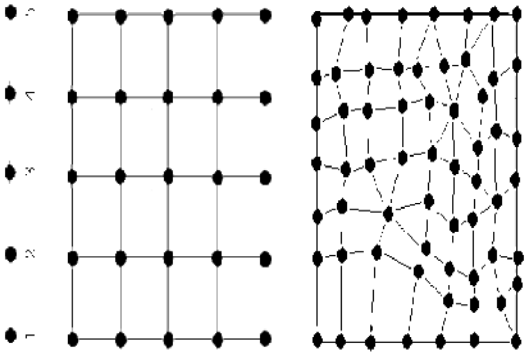


$$E_{in} + E_{gen} = \Delta U + E_{out}$$

$$q_{inX} \cdot A \cdot dt + q_{inZ} \cdot A \cdot dt + Q \cdot A \cdot dx \cdot dt = \Delta U + q_{outX} \cdot A \cdot dt + q_{outZ} \cdot A \cdot dt$$

Finite Element 2-D Conduction

Select Element Type



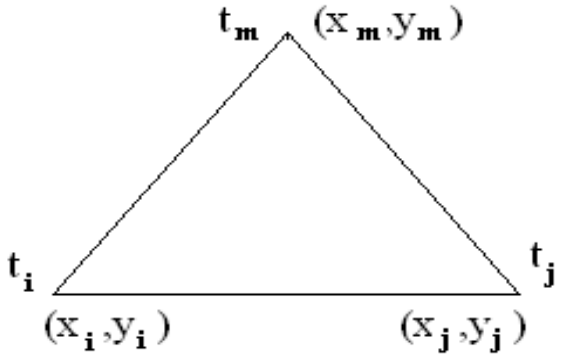
i. 1-d elements are lines

ii. 2-d elements are either **triangles**, quadrilaterals, or a mixture as shown

iii. Label the nodes so that the difference between two nodes on any element is minimized.

Finite Element 2-D Conduction

1. Assume (Choose) a Temperature Function



Assume a linear temperature function for each element as:

$$t(x, y) = a_1 + a_2 x + a_3 y$$

$$\{\Psi\} = \{a_1 + a_2 x + a_3 y\} = [1 \text{ x y}] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Where u and v describe temperature gradients at (x_i, y_i).

2.Assume (Choose) a Temperature Function

$$T = N_i t_i + N_j t_j + N_m t_m$$

$$[T] = [N_i + N_j + N_m] \begin{Bmatrix} t_i \\ t_j \\ t_m \end{Bmatrix}$$

T = temperature function

N = shape function

t = nodal temperature

3.Define Temperature Gradient Relationships

$$\{g\} = \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_m}{\partial y} \end{bmatrix} \begin{Bmatrix} t_i \\ t_j \\ t_m \end{Bmatrix}$$

$$[B] = \frac{\partial}{\partial x}[N] = \frac{1}{|x|} \begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma & \gamma & \gamma \end{bmatrix}$$

Analogous to strain matrix: {g}=[B]{t}

[B] is derivative of [N]

Heat flux/Temperature Gradient :

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = - \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix} \{g\} = -[D]\{g\}$$

4. Derive Element Conduction Matrix and Equations

$$[k] = \underbrace{\iiint_V [B]^T [D][B] dV}_{\text{Conduction}} + \underbrace{\iint_S h[N]^T [N] dS}_{\text{Convection}}$$

$$= tA[B]^T [D][B] + hP\int_0^L \begin{Bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} dx$$

$$= \frac{AK_{xx}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

5. Derive Element Conduction Matrix and Equations

$$\{f_Q\} = Q\iiint_V [V]^T dV = \frac{QV}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \text{ for constant heat source}$$

$$\{f\} = [k]\{t\}$$

for each element

Stiffness matrix is general term for a matrix of known coefficients being multiplied by unknown degrees of freedom, i.e., displacement OR temperature, etc. Thus, the element conduction matrix is often referred to as the stiffness matrix.

6. Assemble Element Equations, Apply BC's

$$\{F\} = [K]\{t\}$$

From here on virtually the same as structural approach. Heat flux boundary conditions already accounted for in derivation. Just substitute into above equation and solve for the following:

7. Solve for Nodal Temperatures

8. Solve for Element Temperature Gradient & Heat Flux

Dynamic equation of motion

In dynamic problems the displacements, velocities, strains, stresses and loads are all time dependent. The procedure involved in deriving the FE equations of a dynamic problem can be stated by the following steps:

1. Idealize the body into E finite elements
2. Assume the displacement model of element e as

$$\vec{U}(x,y,z,t) = \begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix} = [N(x,y,z)]\vec{Q}^{(e)}(t)$$

3. Derive the element characteristic (stiffness and mass) matrices and characteristic (load) vector.

$$\vec{\varepsilon} = [B]\vec{q}^{(e)}$$

$$\vec{\sigma} = [D]\vec{\varepsilon} = [D][B]\vec{Q}^{(e)}$$

4. Assemble the element matrices and vectors and derive the overall system equations of motion.
5. Solve the equation of motion by applying the boundary conditions.

Consistent and lumped mass matrices

The above mass matrix is called as **"consistent"** mass matrix of the element.It is called consistent because the same displacement model that is used for deriving the element stiffness matrix is used to for the derivation of mass matrix.

It is interest to note that several dynamic problems have been and are being solved with simpler forms of mass matrices.the simplest form of mass matrix

that can be used is that obtained by placing point (concentrated) mass mi at node point I in the directions of assumed displacement degrees of freedom.

The concentrated masses refer to translational and rotational inertia of the element and are calculated by assuming that the material within the mean locations on either side of the particular displacement behaves like a rigid body while the remainder of the element does not participate in the motion.

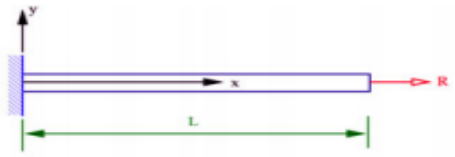
Thus, this assumption excludes the dynamic coupling that exists between the element displacements and hence the resulting element mass matrix is purely diagonal and is called the **"lumped"** mass matrix.

Natural frequencies and mode shapes

The oscillatory motion occurs at certain frequencies known as natural frequencies or characteristic values, and it follows well defined deformation patterns known as mode shapes and characteristic modes.

AXIALLY LOADED BAR

A. Constant End Load



Given: Length L, Section Area A, Young's modulus E
Find: stresses and deformations.

Assumptions:
The cross-section of the bar does not change after loading.
The material is linear elastic, isotropic, and homogeneous.
The load is centric.
End-effects are not of interest to us.

Strength of Materials Approach

From the **equilibrium equation**, the axial force at a random point x along the bar is:

$$\mathbf{f}(x) = \mathbf{R}(= const) \Rightarrow \boldsymbol{\sigma}(x) = \frac{\mathbf{R}}{A}$$

From the **constitutive equation (Hooke's Law)**:

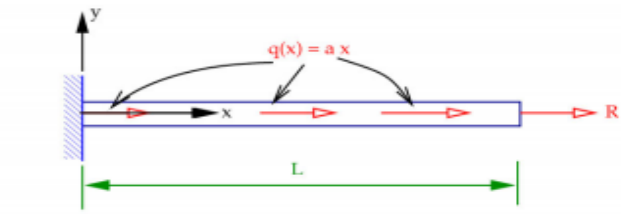
$$\boldsymbol{\epsilon}(x) = \frac{\boldsymbol{\sigma}(x)}{E} = \frac{\mathbf{R}}{AE}$$

Hence, the deformation $\delta(x)$ is obtained from kinematics as:

$$\boldsymbol{\epsilon} = \frac{\delta(x)}{x} \Rightarrow \delta(x) = \frac{\mathbf{R}x}{AE}$$

Note: The stress & strain is independent of x for this case of loading.

B. Linearly Distributed Axial + Constant End Load



From the **equilibrium equation**, the axial force at random point x along the bar is:

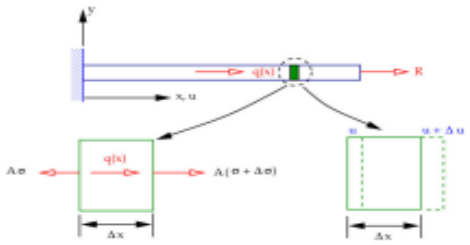
$$\mathbf{f}(x) = \mathbf{R} + \frac{aL + ax}{2}(L - x) = \mathbf{R} + \frac{a(L^2 - x^2)}{2} (\text{ depends on x})$$

In order to now find stresses & deformations (which depend on x) we have to repeat the process for every point in the bar. This is computationally inefficient.

From the **equilibrium equation**, for an infinitesimal element:

$$A\sigma = \mathbf{q}(x)\Delta x + A(\sigma + \Delta\sigma) \Rightarrow A \underbrace{lim}_{\Delta x \to 0} \frac{\Delta\sigma}{\Delta x} + \mathbf{q}(x) = 0 \Rightarrow A\frac{d\sigma}{dx} + \mathbf{q}(x) = 0$$

$$\text{Also, } \epsilon = \frac{d\mathbf{u}}{dx}, \sigma = E\epsilon, \mathbf{q}(x) = ax \Rightarrow AE\frac{d^2\mathbf{u}}{dx^2} + ax = 0$$



Strong Form

$$AE\frac{d^2\mathbf{u}}{dx^2} + ax = 0$$

$\mathbf{u}(0) = 0$ essential BC

$$\mathbf{f}(L) = \mathbf{R} \Rightarrow AE\left.\frac{d\mathbf{u}}{dx}\right|_{x=L} = \mathbf{R} \text{ natural BC}$$

Analytical Solution

$$\mathbf{u}(x) = \mathbf{u}_{hom} + \mathbf{u}_p \Rightarrow \mathbf{u}(x) = C_1 x + C_2 - \frac{ax^3}{6AE}$$

C_1, C_2 are determined from the BC

An analytical solution cannot always be found

Approximate Solution - The Galerkin Approach (#3): Multiply by the weight function **w** and integrate over the domain

$$\int_0^L AE\frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx + \int_0^L ax\mathbf{w}dx = 0$$

Apply integration by parts

$$\int_0^L AE\frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx = \left[AE\frac{d\mathbf{u}}{dx}\mathbf{w}\right]_0^l - \int_0^L AE\frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx \Rightarrow$$

$$\int_0^L AE\frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx = \left[AE\frac{d\mathbf{u}}{dx}(L)\mathbf{w}(L) - AE\frac{d\mathbf{u}}{dx}(0)\mathbf{w}(0)\right] - \int_0^L AE\frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx$$

But from BC we have $u(0) = 0$, $AE\frac{du}{dx}(L)\mathbf{w}(L) = \mathbf{R}\mathbf{w}(L)$, therefore the approximate weak form can be written as

$$\int_0^L AE\frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx = \mathbf{R}\mathbf{w}(L) + \int_0^L ax\mathbf{w}dx$$

In Galerkin's method we assume that the approximate solution, $\mathbf{u}$ can be expressed as

$$\mathbf{u}(x) = \sum_{j=1}^{n} u_j N_j(x)$$

$\mathbf{w}$ is chosen to be of the same form as the approximate solution (but with arbitrary coefficients w_i),

$$\mathbf{w}(x) = \sum_{i=1}^{n} w_i N_i(x)$$

Plug $\mathbf{u}(x)$,$\mathbf{w}(x)$ into the approximate weak form:

$$\int_0^L AE \sum_{j=1}^{n} u_j \frac{dN_j(x)}{dx} \sum_{i=1}^{n} w_i \frac{dN_i(x)}{dx} dx = \mathbf{R} \sum_{i=1}^{n} w_i N_i(L) + \int_0^L ax \sum_{i=1}^{n} w_i N_i(x) dx$$

w_i is arbitrary, so the above has to hold $\forall\ w_i$:

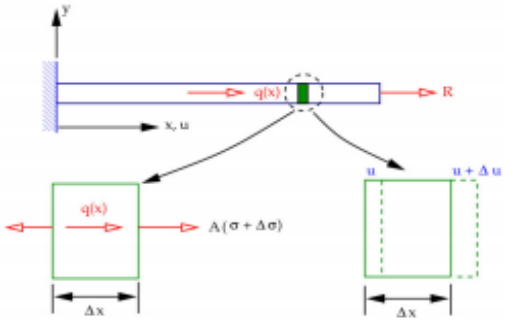
$$\sum_{j=1}^{n} \left[\int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx \right] u_j = \mathbf{R} N_i(L) + \int_0^L ax N_i(x) dx \quad i = 1 \dots n$$

which is a system of n equations that can be solved for the unknown coefficients u_j.

The matrix form of the previous system can be expressed as

$$\mathbf{K}_{ij} u_j = f_i \text{ where } K_{ij} = \int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx$$
$$\text{and } f_i = \mathbf{R} N_i(L) + \int_0^L ax N_i(x) dx$$

Finite Element Solution - using 2 discrete elements, of length h (3 nodes)
From the iso-parametric formulation we know the element stiffness matrix $\mathbf{K}^e = \frac{AE}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Assembling the element stiffness matrices we get:



$$\mathbf{K}^{tot} = \begin{bmatrix} K_{11}^e & K_{12}^1 & 0 \\ K_{12}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ 0 & K_{12}^2 & K_{22}^2 \end{bmatrix} =$$

$$\mathbf{K}^{tot} = \frac{AE}{h} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

ANSYS

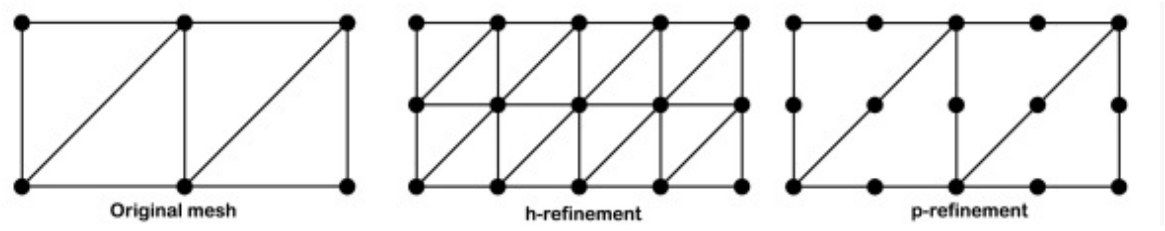
h AND p METHOD

Two types method are used to demonstrate the numerical convergence of the solution :

1). h – method

2). p – method

The h- and p- versions of the finite element method are different ways of adding degrees of freedom (dof) to the model



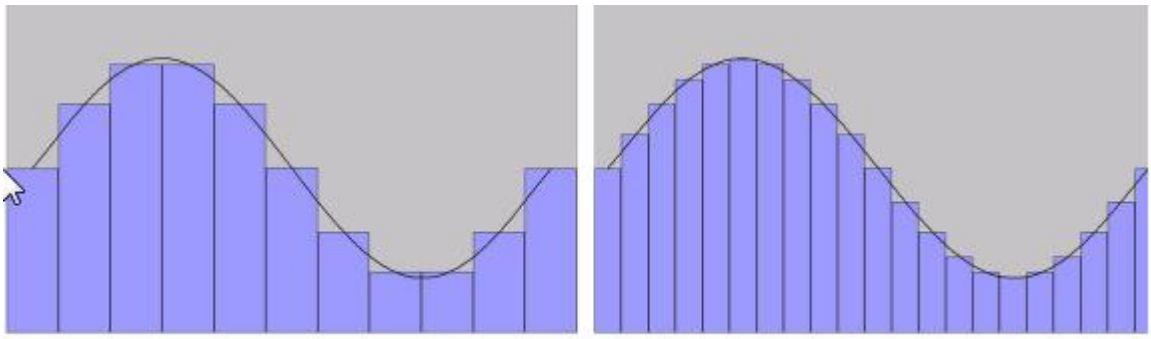
h-method –> The h-method improves results by using a finer mesh of the same type of element. This method refers to decreasing the characteristic length (h) of elements, dividing each existing element into two or more elements without changing the type of elements used.

p-method –> The p-method improves results by using the same mesh but increasing the displacement field accuracy in each element. This method refers to increasing the degree of the highest complete polynomial (p) within an element without changing the number of elements used.

The difference between the two methods lies in how these elements are treated. The h-method uses many simple elements, whereas the p-method uses few complex elements.

H-Method

More accurate information is obtained by increasing the number of elements.The name for the h-method is borrowed from mathematics. The variable h is used to specify the step size in numeric integration. If a part is modeled with a very course mesh, then the stress distribution across the part will be very inaccurate. In order to increase the accuracy of the solution, more elements must be added. This means creating a finer mesh. As an initial run, a course mesh is used to model the problem. A solution is obtained. To check this solution, a finer mesh is created. The mesh must always be changed if a more accurate solution is desired. The problem is run again to obtain a second solution. If there is a large difference between the two solutions, then the mesh must be made even finer and then solve the solution again. This process is repeated until the solution is not changing much from run to run.

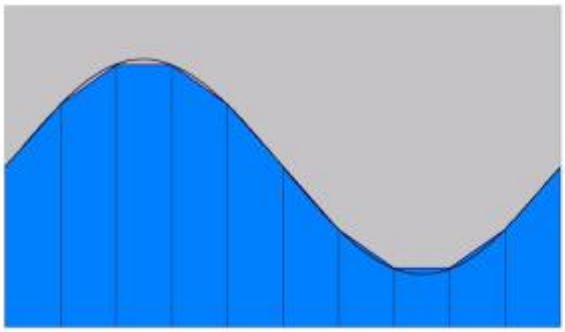


P-Method

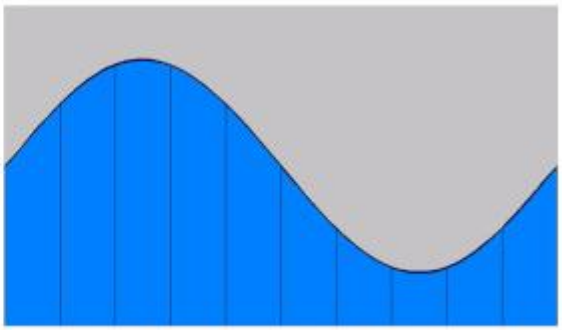
The p in p-method stands for polynomial. Large elements and complex shape functions are used in p-method problems. In order to increase the accuracy of the solution, the complexity of the shape function must be increased. Increasing the polynomial order increases the complexity of the shape function.The mesh does not need to be changed when using the p-method.

As an initial run, the solution might be solved using a first order polynomial shape function. A solution is obtained. To check the solution the problem will be solved again using a more complicated shape function. For the second run, the solution may be solved using a third order polynomial shape function. A second solution is obtained. The output from the two runs is compared.

If there is a large difference between the two solutions, then the solution should be run using a third order polynomial shape function. This process is repeated until the solution is not changing much from run to run.



P-Method with 2nd Order Polynomial



P-Method with 3rd Order Polynomial