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SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

SMEA1306 SOLID AND FLUID MECHANICS

UNIT 1 STRESS STRAIN AND DEFORMATION OF SOLIDS, STATES OF STRESS

STRESS STRAIN AND DEFORMATION OF SOLIDS, STATES OF STRESS

When an external force acts on a body, the body tends to undergo some deformation. Due to cohesion between the molecules, the body resists deformation. This resistance by which material of the body opposes the deformation is known as strength of material, within a certain limit (i.e., in the elastic stage). Whenever a load is attached to a thin hanging wire, it elongates and the load moves downwards (sometimes through a negligible distance). The amount, by which the wire elongates, depends upon the amount of load and the nature as well as cross-sectional area of the wire material.

Elasticity

Whenever a force acts on a body, it undergoes some deformation and the molecules offer some resistance to the deformation. It will be interesting to know that when the external force is removed, the force of resistance also vanishes; and the body springs back to its original position. But it is only possible, if the deformation, caused by the external force, is within a certain limit. Such a limit is called elastic limit.

The property of certain materials of returning back to their original position, after removing the external force, is known as elasticity.

Stress

The force of resistance per unit area, offered by a body against deformation is known as stress. The external force acting on the body is called the load or force. The load is applied on the body while the stress is induced in the material of the body. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are equal.

$$Stress = \sigma = \frac{Force}{Area} = \frac{F}{A}$$

where F = Load or force acting on the body, and
A = Cross-sectional area of the body.

The unit of stress depends upon the unit of load (or force) and unit of area. In M.K.S. units, the force is expressed in kgf and area in metre square (i.e., m^2). Hence unit of stress becomes as kgf/m^2. In the S.L units, the force is expressed in newtons (written as N) and area is expressed as m^2. Hence unit of stress becomes as N/m^2.

Strain

Whenever a single force (or a system of forces) acts on a body, it undergoes some deformation. This deformation per unit length is known as strain. Mathematically strain may be defined as the deformation per unit length. *i.e.*, strain

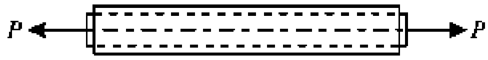
$$Strain = \varepsilon = \frac{x}{L}$$

Types of Stresses

Though there are many types of stresses, yet the following two types of stresses are important from the subject point of view: **1.** Tensile stress, **2.** Compressive stress.

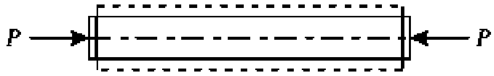
1. Tensile Stress

When a section is subjected to two equal and opposite pulls and the body tends to increase its Length. The stress induced is called tensile stress. The corresponding strain is called tensile strain. As a result of the tensile stress, the *cross-sectional area of the body gets reduced.



2. Compressive Stress

When a section is subjected to two equal and opposite pushes and the body tends to shorten its Length. The stress induced is called compressive stress. The corresponding strain is called compressive strain. As a result of the compressive stress, the cross-sectional area of the body gets increased.



Hooke's Law

It states, "*When a material is loaded, within its elastic limit, the stress is proportional to the strain.*"

$$\frac{\text{Stress}}{\text{Strain}} = E = \text{Constant}$$

Modulus of Elasticity or Young's Modulus (E)

Whenever a material is loaded, within its elastic limit, the stress is proportional to strain

$$\sigma \propto \varepsilon$$
$$= E \times \varepsilon$$
$$E = \frac{\sigma}{\varepsilon}$$

Where, σ = Stress,
ε = Strain, and
E = A constant of proportionality known as modulus of elasticity or Young's modulus.

Numerically, it is that value of tensile stress, which when applied to a uniform bar will increase its length to double the original length if the material of the bar could remain perfectly elastic throughout such an excessive strain.

S. No.	Material	Modulus of elasticity (E) in GPa i.e. GN/m^2 or kN/mm^2		
1.	Steel	200	to	220
2.	Wrought iron	190	to	200
3.	Cast iron	100	to	160
4.	Copper	90	to	110
5.	Brass	80	to	90
6.	Aluminium	60	to	80
7.	Timber	10		

Deformation of a Body Due to Force Acting on it

Consider a body subjected to a tensile stress.

Let P = Load or force acting on the body,
l = Length of the body,
A = Cross-sectional area of the body,
σ = Stress induced in the body,

E = Modulus of elasticity for the material of the body,
ε = Strain, and
δl = Deformation of the body.

$$\sigma = \frac{P}{A} \qquad \text{Strain,} \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

$$\delta l = \varepsilon . l = \frac{\sigma . l}{E} = \frac{Pl}{AE}$$

Example: *A steel rod 1 m long and 20 mm × 20 mm in cross-section is subjected to a tensile force of 40 kN. Determine the elongation of the rod, if modulus of elasticity for the rod material is 200 GPa.*

Given:

Length (l) = 1 m = 1 × 10^3 mm
Cross-sectional area (A) = 20 × 20 = 400 mm^2
Tensile force (P) = 40 kN = 40 × 10^3 N
Modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2

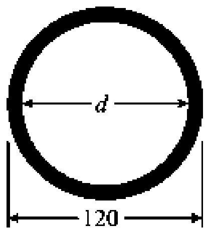
elongation of the road,

$$\delta l = \frac{P.l}{A.E} = \frac{(40 \times 10^3) \times (1 \times 10^3)}{400 \times (20 \times 10^3)} = 0.5 \text{ mm}$$

Example *A hollow steel tube 3.5 m long has external diameter of 120 mm. In order to determine the internal diameter, the tube was subjected to a tensile load of 400 kN and extension was measured to be 2 mm. If the modulus of elasticity for the tube material is 200 GPa, determine the internal diameter of the tube.*

Given:

Length (l) = 3.5 m = 3.5 × 10^3 mm
External diameter (D) = 120 mm
Load (P) = 400 kN = 400 × 10^3 N
Extension (δl) = 2 mm
Modulus of elasticity E = 200 GPa = 200 × 10^3 N/mm^2



area of the tube,

$$A = \frac{\pi}{4}[(120)^2 - d^2] = 0.7854\,[(120)^2 - d^2]$$

extension of the tube (δl),

$$2 = \frac{P.l}{A.E} = \frac{(400 \times 10^3) \times (3.5 \times 10^3)}{0.7854\,[(120)^3 - d^2\ (200 \times 10^3)} = \frac{8913}{14400 - d^2}$$

$$\therefore \qquad 28800 - 2d^2 = 8913 \qquad \text{or} \qquad 2d^2 = 28800 - 8913 = 19887$$

$$\text{or} \qquad d^2 = \frac{19887}{2} = 9943.5 \qquad \text{or} \qquad d = 99.71 \text{ mm} \qquad \textbf{Ans.}$$

Example: Two wires, *one of steel and the other of copper, are of the same length and are subjected to the same tension. If the diameter of the copper wire is 2 mm, find the diameter of the steel wire, if they are elongated by the same amount. Take E for steel as 200 GPa and that for copper as 100 GPa.*

Given:

Diameter of copper wire (d_C) = 2 mm

Modulus of elasticity for steel (E_S) = 200 GPa = 200 × 10^3 N/mm^2
Modulus of elasticity for Copper (E_C) = 100 GPa = 100 × 10^3 N/mm^2

Let d_S = Diameter of the steel wire,
l = Lengths of both the wires and
P = Tension applied on both the wires.

$$A_C = \frac{\pi}{4} \times (d_C)^2 = \frac{\pi}{4} \times (2)^2 = 3.142 \text{ mm}^2$$

and area of steel wire,
$$A_S = \frac{\pi}{4} \times (d_S)^2 = 0.7854\, d_S^2 \text{ mm}^2$$

We also know that increase in the length of the copper wire

$$\delta l_C = \frac{Pl}{A_C E_C} = \frac{Pl}{3.142 \times (100 \times 10^3)} = \frac{Pl}{314.2 \times 10^3} \quad \text{...}(i)$$

and increase in the length of the steel wire,

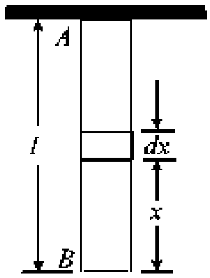
$$\delta l_S = \frac{Pl}{A_S E_S} = \frac{Pl}{0.7854\, d_S^2 \times (200 \times 10^3)} = \frac{Pl}{157.1 \times 10^3 \times d_S^2} \quad \text{...}(ii)$$

Since both the wires are elongated by the same amount, therefore equating equations (*i*) and (*ii*).

$$\frac{Pl}{314.2 \times 10^3} = \frac{Pl}{157.1 \times 10^3 \times d_S^2} \quad \text{or} \quad d_S^2 = \frac{314.2}{157.1} = 2$$

$$\therefore \quad d_S = \sqrt{2} = 1.41 \text{ mm} \quad \textbf{Ans.}$$

Deformation of a Body Due to Self Weight



Consider a bar AB hanging freely under its own weight as shown.

Let l = Length of the bar.
A = Cross-sectional area of the bar.
E = Young's modulus for the bar material,
and w = Specific weight of the bar material.

Now consider a small section dx of the bar at a distance x from B. We know that weight of the bar for a length of x,

$$P = wAx$$

Elongation of the small section of the bar, due to weight of the bar for a small section of length x,

$$= \frac{Pl}{AE} = \frac{(wAx).dx}{AE} = \frac{wx.dx}{E}$$

Total elongation of the bar may be found out by integrating the above equation between zero and l. Therefore total elongation,

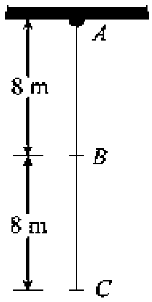
$$\delta l = \int_0^l \frac{wx.dx}{E}$$

$$= \frac{w}{E}\int_0^l x.dx$$

$$= \frac{w}{E}\left[\frac{x^2}{2}\right]_0^l$$

$$\delta l = \frac{wl^2}{2E} = \frac{Wl}{2AE} \quad \text{...}(\because W = wAl = \text{Total weight})$$

Example *A steel wire ABC 16 m long having cross-sectional area of 4 mm² weighs 20N as shown in Fig. If the modulus of elasticity for the wire material is 200 GPa, find the deflections at C and B.*



Given:

Length (l) = 16 m = 16×10^3 mm

Cross-sectional area (A) = 4 mm²

Weight of the wire ABC (W) = 20 N

Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm²

Deflection of wire at C due to self-weight of the wire AC,

$$dl_C = \frac{Wl}{2AE} = \frac{20 \times (16 \times 10^3)}{2 \times 4 \times (200 \times 10^3)} = 0.2 \text{ mm} \quad \textbf{Ans.}$$

Deflection at B consists of deflection of wire AB due to self-weight plus deflection due to weight of the wire BC. We also know that deflection of the wire at B due to self-weight of wire AB

$$\delta l_1 = \frac{(W/2) \times (l/2)}{2AE} = \frac{10 \times (8 \times 10^3)}{2 \times 4 \times (200 \times 10^3)} = 0.05 \text{ mm} \qquad ...(i)$$

and deflection of the wire at B due to weight of the wire BC.

$$\delta l_2 = \frac{(W/2) \times (l/2)}{AE} = \frac{10 \times (8 \times 10^3)}{4 \times (200 \times 10^3)} = 0.1 \text{ mm} \qquad ...(ii)$$

∴ Total deflection of the wire at B.

$$\delta l_B = \delta l_1 + \delta l_2 = 0.05 + 0.1 = 0.15 \text{ mm} \qquad \textbf{Ans.}$$

Principle of Superposition

A body is subjected to a number of forces acting on its outer edges as well as at some other sections, along the length of the body. In such a case, the forces are split up and their effects are considered on individual sections. The resulting deformation, of the body, is equal to the algebraic sum of the deformations of the individual sections. Such a principle, of finding out the resultant deformation, is called the principle of superposition. The relation for the resulting deformation may be modified as:

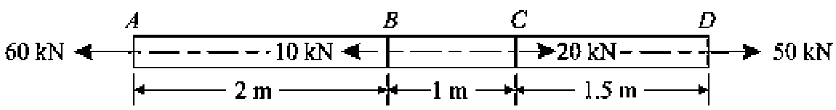
$$\delta l = \frac{Pl}{AE} = \frac{1}{AE}(P_1 l_1 + P_2 l_2 + P_3 l_3 + ...)$$

P_1 = Force acting on section 1,

l_1 = Length of section 1,

P_2, l_2 = Corresponding values of section 2, and so on.

Example *A steel rod ABCD 4.5 m long and 25 mm in diameter is subjected to the forces as shown in Fig. If the value of Young's modulus for the steel is 200 GPa, determine its deformation.*



Given:

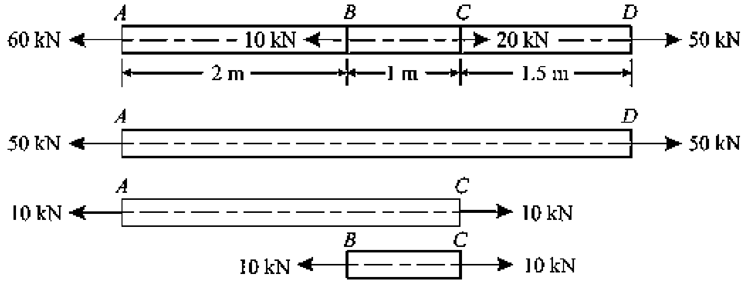
Diameter (D) = 25 mm and

Young's modulus (E) = 200 GPa = 200 kN/mm²

We know that cross-sectional area of the steel rod.

$$A = \frac{\pi}{4}(D)^2 = \frac{\pi}{4} \times (25)^2 = 491 \text{ mm}^2$$

For the sake of simplification, the force of 60 kN acting at *A* may be split up into two forces of 50 kN and 10 kN respectively. Similarly the force of 20 kN acting at *C* may also be split up into two forces of 10 kN and 10 kN respectively.



Now it will be seen that the bar AD is subjected a tensile force of 50 kN, part AC is subjected to a tensile force of 10 kN and the part BC is subjected to a tensile force of 10 kN as shown in Fig. We know that deformation of the bar.

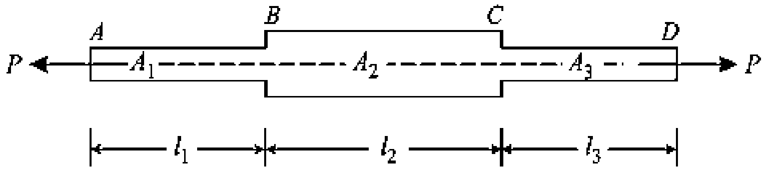
$$\delta l = \frac{1}{AE}[P_1 l_1 + P_2 l_2 + P_3 l_3]$$

$$= \frac{1}{491 \times 200}\left[[50 \times (4.5 \times 10^3)] + [10 \times (3 \times 10^3)] + [10 \times (1 \times 10^3)\right] \text{mm}$$

$$= \frac{1}{491 \times 200} \times (265 \times 10^3) = 2.70 \text{ mm} \qquad \textbf{Ans.}$$

Stresses in the Bars of Different Sections

A bar is made up of different lengths having different cross-sectional areas



In such cases, the stresses, strains and hence changes in lengths for each section is worked out separately as usual. The total change in length is equal to the sum of the changes of all the individual lengths. It may be noted that each section is subjected to the same external axial pull or push.

Let

P = Force acting on the body,

E = Modulus of elasticity for the body,

l_1 = Length of section 1,

A_1 = Cross-sectional area of section 1,

l_2, A_2 = Corresponding values for section 2 and so on.

We know that the change in length of section 1.

$$\delta l_1 = \frac{Pl_1}{A_1 E} \qquad \text{Similarly} \qquad \delta l_2 = \frac{Pl_2}{A_2 E} \qquad \text{and so on}$$

$\therefore$ Total deformation of the bar,

$$\delta l = \delta l_1 + \delta l_2 + \delta l_3 + \ldots\ldots\ldots$$

$$= \frac{Pl_1}{A_1 E} + \frac{Pl_2}{A_2 E} + \frac{Pl_3}{A_3 E} + \ldots\ldots\ldots$$

$$= \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} + \ldots\ldots\ldots\right)$$

NOTE. Sometimes, the modulus of elasticity is different for different sections. In such cases, the total deformation,

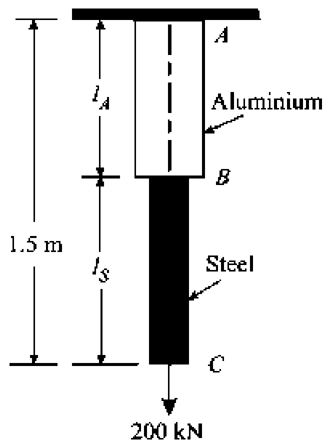
$$\delta l = P\left(\frac{l_1}{A_1 E_1} + \frac{l_2}{A_2 E_2} + \frac{l_3}{A_3 E_3} + \dots\dots\right)$$

Example *A compound bar ABC 1.5 m long is made up of two parts of aluminium and steel and that cross-sectional area of aluminium bar is twice that of the steel bar. The rod is subjected to an axial tensile load of 200 kN. If the elongations of aluminium and steel parts are equal, find the lengths of the two parts of the compound bar. Take E for steel as 200 GPa and E for aluminium as one-third of E for steel.*

Given:

Total length (L) = 1.5 m = 1.5 × 103 mm
Cross-sectional area of aluminium bar (A_A) = 2 A_S
Axial tensile load (P) = 200 kN = 200 × 103 N
Modulus of elasticity of steel (ES) = 200 GPa = 200 × 103 N/mm2
Modulus of elasticity of aluminium (EA) = $\frac{E_S}{3} = \frac{200 \, x \, 10^3}{3} N/mm^2$

Let, l_A = Length of the aluminium part,
and l_S = Length of the steel part.



We know that elongation of the aluminium part AB,

$$\delta l_A = \frac{P.l_A}{A_A.E_A} = \frac{(200\times10^3)\times l_A}{2A_S\times\left(\frac{200\times10^3}{3}\right)}$$

$$= \frac{1.5\, l_A}{A_S} \qquad ...(i)$$

and elongation of the steel part BC,

$$\delta l_S = \frac{P.l_S}{A_S.E_S} = \frac{(200\times10^3)\times l_S}{A_S\times(200\times10^3)} = \frac{l_S}{A_S}$$

Since elongations of aluminium and steel parts are equal, therefore equating equations (i) and (ii),

$$\frac{1.5 l_A}{A_S} = \frac{l_S}{A_S} \quad \text{or} \quad l_S = 1.5\, l_A$$

We also know that total length of the bar ABC (L)

$$1.5 \times 10^3 = l_A + l_S = l_A + 1.5\, l_A = 2.5\, l_A$$

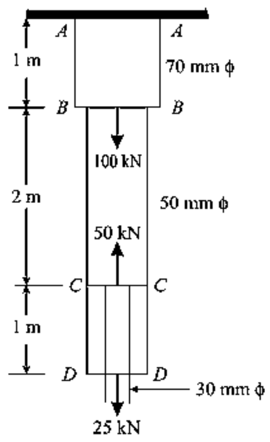
$\therefore \qquad l_A = \frac{1.5\times10^3}{2.5} = 600 \text{ mm}$ **Ans.**

and $\qquad l_S = (1.5 \times 10^3) - 600 = 900 \text{ mm}$ **Ans.**

Example *A circular steel rod ABCD of different cross-sections is loaded as shown in Fig. Find the maximum stress induced in the rod and its deformation. Take E = 200 GPa.*

Given:

Length of first part AB (l_1) = 1 m = 1 × 10^3 mm
Diameter of first part AB (D_1) = 70 mm
Length of second part BC (l_2) = 2 m = 2 × 10^3 mm
Diameter of second part BC (D_2) = 50 mm
Length of third part CD (l_3) = 1 m = 1 × 10^3 mm
Diameter of third part CD (D_3) = 50 mm
Internal diameter of hole (d_3) = 30 mm.



Maximum stress induced in the rod

We know that area of the first part (*AB*) of the rod,

$$A_1 = \frac{\pi}{4}(D_1)^2 = \frac{\pi}{4}(70)^2 \text{ mm}^2$$
$$= 3848.5 \text{ mm}^2$$

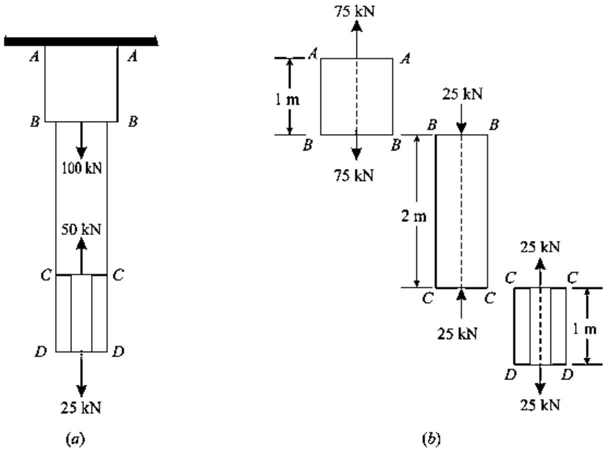
Similarly area of the second part (*BC*) of the rod,

$$A_2 = \frac{\pi}{4}(D_2)^2 = \frac{\pi}{4}(50)^2 = 1963.5 \text{ mm}^2$$

and area of the third part *CD* of the rod,

$$A_3 = \frac{\pi}{4}[D_3)^2 - d_3^2]$$

For simplification, the force of 100 kN acting at *B*-*B* may be split up into two forces of 75 kN and 25 kN. Similarly the force of 50 kN acting at *C*-*C* may be split up into two forces of 25 kN and 25 kN respectively as shown in Fig.



Now it will be seen that the bar *AB* is subjected to a tensile load of 75 kN, part *BC* is subjected to a compressive load of 25 kN and the part *CD* is subjected to a tensile load of 25 kN as shown in Fig. We know that tensile stress in part 1,

$$\sigma_1 = \frac{P_{AB}}{A_1} = \frac{75 \times 10^3}{3848.5} = 19.49 \text{ N/mm}^2 = 19.49 \text{ MPa}$$

Similarly, $$\sigma_2 = \frac{P_{BC}}{A_2} = \frac{25 \times 10^3}{1963.5} = 12.73 \text{ N/mm}^2 = 12.73 \text{ MPa}$$

and $$\sigma_3 = \frac{P_{CD}}{A_3} = \frac{25 \times 10^3}{1256.6} = 19.89 \text{ N/mm}^2 = 19.89 \text{ MPa}$$

From the above three values of the stresses, we find that maximum stress induced in the rod is in *CD* and is equal to 19.89 MPa. **Ans.**

We also know that elongation of the part AB, due to tensile load of 75 kN,

$$\delta l_1 = \frac{P_1 l_1}{A_1 E} = \frac{(75\times10^3)\times(1\times10^3)}{3848.5\times(200\times10^3)} = 0.097 \text{ mm}$$

Similarly shortening of the part BC due to compressive load of 25 kN.

$$\delta l_2 = \frac{P_2 l_2}{A_2 E} = \frac{(25\times10^3)\times(2\times10^3)}{1963.5\times(200\times10^3)} = 0.127 \text{ mm}$$

and elongation of the part CD due to tensile load of 25 kN.

$$\delta l_3 = \frac{P_3 l_3}{A_3 E} = \frac{(25\times10^3)\times(1\times10^3)}{1256.6\times(200\times10^3)} = 0.099 \text{ mm}$$

∴ Deformation of the rod,

$$\delta l = \delta l_1 - \delta l_2 + \delta l_3 = 0.097 - 0.127 + 0.099 = 0.069 \text{ mm}$$

Stresses in the Bars of Uniformly Tapering Circular Sections

Consider a circular bar AB of uniformly tapering circular section as shown in Fig.

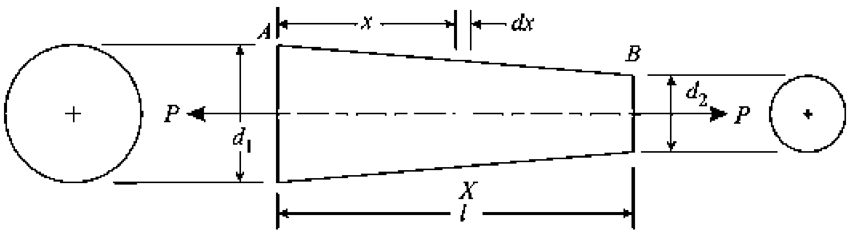
Let P = Pull on the bar.

l = Length of the bar,

$d1$ = Diameter of the bigger end of the bar, and

$d2$ = Diameter of the smaller end of the bar.

Now consider a small element of length dx of the bar, at a distance x from the bigger end as shown in Fig. We know that diameter of the bar at a distance x, from the left end A,



$$dx = d_1 - (d_1 - d_2)\frac{x}{l} = d_1 - kx, \qquad \text{...(where } k = \frac{d_1 - d_2}{l}\text{)}$$

and cross-sectional area of the bar at this section,

$$A_X = \frac{\pi}{4}(d_1 - kx)^2$$

∴ Stress,

$$\sigma_X = \frac{P}{\frac{\pi}{4}(d_1 - kx)^2} = \frac{4P}{\pi(d_1 - kx)^2}$$

and strain,

$$\varepsilon_X = \frac{\text{Stress}}{E} = \frac{\frac{4P}{\pi(d_1 - kx)^2}}{E} = \frac{4P}{\pi(d_1 - kx)^2 E}$$

∴ Elongation of the elementary length

$$= \varepsilon_X \cdot dx = \frac{4P \cdot dx}{\pi(d_1 - kx)^2 E}$$

Total extension of the bar may be found out by integrating the above equation between the limit 0 and l. Therefore total elongation,

$$\delta l = \int_0^l \frac{4P \cdot dx}{\pi(d_1 - kx)^2 E}$$

$$= \frac{4P}{\pi E}\int_0^l \frac{dx}{(d_1 - kx)^2}$$

$$= \frac{4P}{\pi E}\left[\frac{(d_1 - kx)^{-1}}{-1 \times -k}\right]_0^l$$

$$= \frac{4P}{\pi E k}\left[\frac{1}{d_1 - kx}\right]_0^l$$

$$= \frac{4P}{\pi E k}\left[\frac{1}{d_1 - kl} - \frac{1}{d_1}\right]$$

Substituting the value of $k = \frac{d_1 - d_2}{l}$ in the above equation,

$$\delta l = \frac{4P}{\pi E \frac{(d_1 - d_2)}{l}}\left[\frac{1}{d_1 - \frac{(d_1 - d_2)\, l}{l}} - \frac{1}{d_1}\right]$$

$$= \frac{4Pl}{\pi E (d_1 - d_2)}\left[\frac{1}{d_2} - \frac{1}{d_1}\right] = \frac{4Pl}{\pi E (d_1 - d_2)}\left[\frac{d_1 - d_2}{d_2 d_1}\right]$$

$$\delta l = \frac{4Pl}{\pi E d_2 d_1}$$

Example *If the tension test bar is found to taper from (D + a) diameter to (D – a) diameter, prove that the error involved in using the mean diameter to calculate Young's modulus is* $(\frac{10a}{D})^2$ *per cent.*

Given:

Larger diameter ($d1$) = (D + a)

Smaller diameter ($d2$) = (D – a).

Let P = Pull on the bar,

l = Length of the bar,

$E1$ = Young's modulus by the tapering formula,

$E2$ = Young's modulus by the mean diameter formula and

δl = Extension of the bar.

First of all, let us find out the values of Young's modulus for the test bar by the tapering formula and then by the mean diameter formula. We know that extension of the bar by uniformly varying formula

$$\delta l = \frac{4\,Pl}{\pi E_1 d_1 d_2} = \frac{4\,Pl}{\pi E_1 (D + a)(D - a)} = \frac{4\,Pl}{\pi E_1 (D^2 - a^2)}$$

or $$E_1 = \frac{4\,Pl}{\pi (D^2 - a^2)\,.\,\delta l} \qquad ...(i)$$

and extension of the bar by mean diameter (D) formula,

$$\delta l = \frac{P\,l}{AE_2} = \frac{P\,l}{\frac{\pi}{4}(D)^2 \times E_2} = \frac{4\,Pl}{\pi\, D^2\, E_2}$$

or $$E_2 = \frac{4\,Pl}{\pi D^2\,.\,\delta l} \qquad ...(ii)$$

$\therefore$ Percentage error involved (in using the mean diameter to calculate the Young's modulus)

$$= \left(\frac{E_1 - E_2}{E_1}\right) \times 100 = \frac{\left(\dfrac{4\,Pl}{\pi(D^2 - a^2)\,\delta l}\right) - \left(\dfrac{4\,Pl}{\pi\,D^2 \,.\, \delta l}\right)}{\dfrac{4\,Pl}{\pi\,(D^2 - a^2)\,\delta l}} \times 100$$

$$= \frac{\dfrac{1}{(D^2 - a^2)} - \dfrac{1}{D^2}}{\dfrac{1}{(D^2 - a^2)}} \times 100 = \frac{\dfrac{D^2 - (D^2 - a^2)}{(D^2 - a^2)\,(D^2)}}{\dfrac{1}{(D^2 - a^2)}} \times 100$$

$$= \frac{a^2}{D^2} \times 100 = \left(\frac{10\,a}{D}\right)^2 \qquad \textbf{Ans.}$$

Example *A steel plate of 20 mm thickness tapers uniformly from 100 mm to 50 mm in a length of 400 mm. What is the elongation of the plate, if an axial force of 80 kN acts on it? Take E = 200 Gpa.*

Given :

Plate thickness = 20 mm ;
Width at A = 100 mm ; Width at B = 50 mm;
Length (l) = 400 mm ;
Axial force (P) = 80 kN = 80×10^3 N
Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2

Now consider a small element of length dx, of the bar, at a distance x from A as shown in Fig. From the geometry of the figure, we find that the width of the plate at a distance x from A.

$$= 100 - (100 - 50) \times \frac{x}{400} = 100 - 0.125\,x \text{ mm}$$

$\therefore$ Cross-sectional area of the plate at this section.

$$A_x = 20 \times (100 - 0.125\,x)$$

and stress,
$$\sigma_x = \frac{P}{A_x} = \frac{80 \times 10^3}{20 \times (100 - 0.125\,x)} = \frac{4 \times 10^3}{100 - 0.125\,x}$$

$\therefore$ Strain,
$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{\dfrac{4 \times 10^3}{100 - 0.125\,x}}{200 \times 10^3} = \frac{1}{50\,(100 - 0.125\,x)}$$

and increase in the length of the small element

$$= \varepsilon_x \,.\, dx = \frac{dx}{50(100 - 0.125\,x)}$$

Now total elongation of the plate may be found out by integrating the above equation between 0 and 400.

$$\therefore \qquad \delta l = \int_0^{400} \frac{dx}{50(100 - 0.125\,x)}$$

$$= \frac{1}{50} \int_0^{400} \frac{dx}{(100 - 0.125\,x)}$$

$$= \frac{1}{50(-0.125)} [\log_e (100 - 0.125\,x)]_0^{400}$$

$$= -\frac{1}{6.25} [\log_e (50 - \log_e 100)]$$

$$= 0.16\,[\log_e 100 - \log_e 50] \qquad \text{...(Taking minus sign outside)}$$

$$= 0.16 \times \log_e\left(\frac{100}{50}\right) = 0.16 \times \log_e 2 = 0.16 \times 2.3 \log 2$$

$$\text{...}(\because \log_e = 2.3 \log_{10})$$

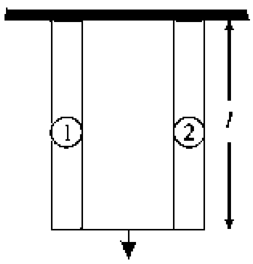
$$= 0.16 \times 2.3 \times 0.3010 = 0.11 \text{ mm} \qquad \textbf{Ans.}$$

Stresses in the Bars of Composite Structures

A bar made up of two or more different materials, joined together is called a composite bar. The bars are joined in such a manner, that the system extends or contracts as one unit, equally, when subjected to tension or compression. Following two points should always be kept in view, while solving example on composite bars:

1. Extension or contraction of the bar is equal.
2. The total external load, on the bar, is equal to the sum of the loads carried by the different materials.

Consider a composite bar made up of two different materials as shown in Fig.



Let P = Total load on the bar,

l_1 = Length of the bar 1

l_2 = Length of the bar 2

A_1 = Area of bar 1,

E_1 = Modulus of elasticity of bar 1.

P_1 = Load shared by bar 1, and

A_2, E_2, P_2= Corresponding values for bar 2,

Total load on the bar,

$$P = P_1 + P_2 \qquad \text{...}(i)$$

$\therefore$ Stress in bar 1,
$$\sigma_1 = \frac{P_1}{A_1}$$

and strain in bar 1,
$$\varepsilon_1 = \frac{\sigma_1}{E_1} = \frac{P_1}{A_1 E_1}$$

$\therefore$ Elongation,
$$\delta l_1 = \varepsilon_1 . l_1 = \frac{\sigma_1 l_1}{E_1} = \frac{P_1 l_1}{A_1 E_1} \qquad \text{...}(ii)$$

Similarly, elongation of bar 2,

$$\delta l_2 = \varepsilon_2 l_2 = \frac{\sigma_2 l_2}{E_1} = \frac{P_2 l_2}{A_2 E_2} \qquad \text{...}(iii)$$

Since both the elongations are equal, therefore equating (*ii*) and (*iii*), we get $\delta l_1 = \delta l_2$

$$\frac{P_1 l}{A_1 E_1} = \frac{P_2 l}{A_2 E_2} \quad \text{or} \quad \frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \qquad \text{...}(iv)$$

or
$$P_2 = P_1 \times \frac{A_2 E_2}{A_1 E_1}$$

But
$$P = P_1 + P_2 = P_1 + P_1 \times \frac{A_2 E_2}{A_1 E_1}$$

$$= P_1\left(1 + \frac{A_2 E_2}{A_1 E_1}\right) = P_1\left(\frac{A_1 E_1 + A_2 E_2}{A_1 E_1}\right)$$

or
$$P_1 = P \times \frac{A_1 E_1}{A_1 E_1 + A_2 E_2} \qquad \text{...}(v)$$

Similarly,
$$P_2 = P \times \frac{A_2 E_2}{A_1 E_1 + A_2 E_2} \qquad \text{...}(vi)$$

From these equations we can find out the loads shared by the different materials. We have also seen in equation (*iv*) that

$$\frac{Pl_1}{A_1 E_1} = \frac{Pl_2}{A_2 E_2}$$

or $$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \qquad \ldots\left(\because \frac{P}{A} = \sigma = \text{Stress}\right)$$

$\therefore$ $$\sigma_1 = \frac{E_1}{E_2} \times \sigma_2 \qquad \ldots(vii)$$

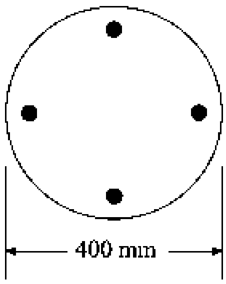
Similarly, $$\sigma_2 = \frac{E_2}{E_1} \times \sigma_1 \qquad \ldots(viii)$$

From the above equations, we can find out the stresses in the different materials. We also know that the total load,

$$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$$

Example *A reinforced concrete circular column of 400 mm diameter has 4 steel bars of 20 mm diameter embedded in it. Find the maximum load which the column can carry, if the stresses in steel and concrete are not to exceed 120 MPa and 5 MPa respectively. Take modulus of elasticity of steel as 18 times that of concrete.*

Given:



Diameter of column (D) = 400 mm
No. of reinforcing bars = 4
Diameter of bars (d) = 20 mm
Maximum stress in steel ($\sigma_{S(max)}$) = 120 MPa = 120 N/mm^2
Maximum stress in concrete ($\sigma_{C(max)}$) = 5 MPa = 5 N/mm^2
Modulus of elasticity of steel (E_S) = 18 E_C

Total area of the circular column.

$$= \frac{\pi}{4} \times (D)^2 = \frac{\pi}{4} \times (400)^2 = 125\,660 \text{ mm}^2$$

and area of reinforcement (*i.e.*, steel),

$$A_S = 4 \times \frac{\pi}{4} \times (d)^2 = 4 \times \frac{\pi}{4} \times (20)^2 \text{ mm}^2$$
$$= 1257 \text{ mm}^2$$

$\therefore$ Area of concrete,

$$A_C = 125\,660 - 1257 = 124\,403 \text{ mm}^2$$

First of all let us find out the maximum stresses developed in the steel and concrete. We know that if the stress in steel is 120 N/mm^2, then stress in the concrete.

$$\sigma_C = \frac{E_C}{E_S} \times \sigma_S = \frac{1}{18} \times 120 = 6.67 \text{ N/mm}^2 \qquad \ldots(i)$$

It is more than the stress in the concrete (*i.e.*, 5 N/mm^2). Thus these stresses are not accepted. Now if the stress in concrete is 5 N/mm^2, then stress in steel,

$$\sigma_S = \frac{E_S}{E_C} \times \sigma_C = 18 \times 5 = 90 \text{ N/mm}^2 \qquad \ldots(ii)$$

It is less than the stress is steel (*i.e.*, 120 N/mm^2). It is thus obvious that stresses in concrete and steel will be taken as 5 N/mm^2 and 90 N/mm^2 respectively. Therefore maximum load, which the column can carry.

$$P = (\sigma_C \cdot A_C) + (\sigma_S \cdot A_S) = (5 \times 124\,403) + (90 \times 1257) \text{ N}$$
$$= 735\,150 \text{ N} = 735.15 \text{ kN} \qquad \textbf{Ans.}$$

Stresses and Strains in Statically Indeterminate Structures

Simple equations of statics were sufficient to solve the examples. But, sometimes, the simple equations are not sufficient to solve such problems. Such problems are called statically indeterminate problems and the structures are called statically indeterminate structures. For solving statically indeterminate problems, the deformation characteristics of the structure are also taken into account along with the statical equilibrium equations. Such equations, which contain the deformation characteristics, are called compatibility equations.

Types of Statically Indeterminate Structures

1. Simple statically indeterminate structures.
2. Indeterminate structures supporting a load.
3. Composite structures of equal lengths.
4. Composite structures of unequal lengths.

Stresses in Simple Statically Indeterminate Structures

Example *A square bar of 20 mm side is held between two rigid plates and loaded by an axial force P equal to 450 kN as shown. Find the reactions at the ends A and C and the extension of the portion AB. Take E = 200 Gpa*

Given:

Area of bar (A) = $20 \times 20 = 400\ mm^2$

Axial force (P) = $450\ kN = 450 \times 10^3$ N

Modulus of elasticity (E) = $200\ GPa$

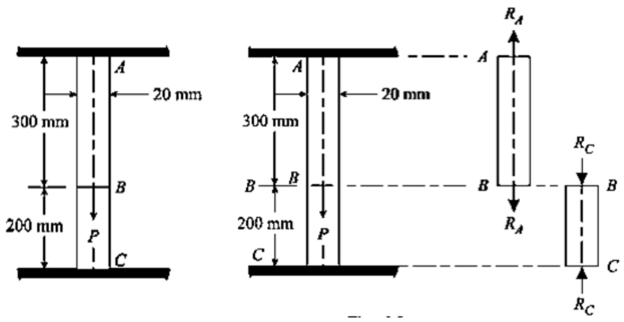
$= 200 \times 10^3\ \text{N/mm}^2$

Length of AB (l_{AB}) = 300 mm and

length of BC (l_{BC}) = 200 mm.

R_A = Reaction at A, *and*

R_C = Reaction at C.



Since the bar is held between the two rigid plates *A and C, therefore, the upper portion will be C* subjected to tension, while the lower portion will be subjected to compression as shown

Moreover, the increase of portion AB will be equal to the decrease of the portion BC.

We know that sum of both the reaction is equal to the axial force, *i.e.*,

$$R_A + R_C = 450 \times 10^3 \quad ...(i)$$

Increase in the portion AB,

$$\delta l_{AB} = \frac{R_A\, l_{AB}}{A\,E} = \frac{R_A \times 300}{A\,E}$$

and decrease in the portion BC,

$$\delta l_{BC} = \frac{R_C\, l_{BC}}{A\,E} = \frac{R_C \times 200}{A\,E} \quad ...(ii)$$

Since the value δl_{AB} is equal to that of δl_{BC}, therefore equating the equations (ii) and (iii),

$$\frac{R_A \times 300}{A\,E} = \frac{R_C \times 200}{A\,E}$$

$$R_C = \frac{R_A \times 300}{200} = 1.5\,R_A$$

Now substituting the value of R_C in equation (ii),

$$R_A + 1.5\,R_A = 450 \qquad \text{or} \qquad 2.5\,R_A = 450$$

$$\therefore \qquad R_A = \frac{450}{2.5} = 180\ \text{kN} \qquad \textbf{Ans.}$$

and $R_C = 1.5\, R_A = 1.5 \times 180 = 270$ kN **Ans.**

Extension of the portion AB

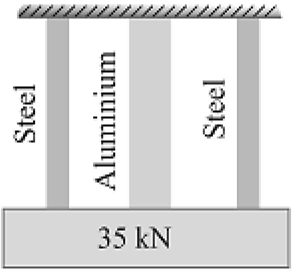
Substituting the value of R_A in equation (*ii*)

$$\delta_{AB} = \frac{R_A \times 300}{A\,E} = \frac{(180 \times 10^3) \times 300}{400 \times (200 \times 10^3)} = 0.675 \text{ mm} \quad \textbf{Ans.}$$

Stresses in Indeterminate Structures Supporting a Load

Example *A block weighing 35 kN is supported by three wires. The outer two wires are of steel and have an area of 100 mm² each, whereas the middle wire of aluminium and has an area of 200 mm². If the elastic modulii of steel and aluminium are 200 GPa and 80 GPa respectively, then calculate the stresses in the aluminium and steel wires.*

Given:



Total load (P) = 35 kN
= 35×10^3 N

Total area of steel rods (A) = 2×100
= 200 mm²

Area of aluminium rod (A_A) = 200 mm²

Modulus of elasticity of steel (E) = 200 Gpa
= 200×10^3 N/mm²

Modulus of elasticity of aluminium (E_A) = 80 GPa
= 80×10^3 N/mm²

Load supported by wires (P) = 35 kN = 35×10^3 N

Let σ_S = Stress in steel wires,
σ_A = Stress in aluminium wire and
l = Length of the wires.

We know that increase in the length of steel wires,

$$\delta l_S = \frac{\sigma_S \times l_S}{E_S} = \frac{\sigma_S \times l}{200 \times 10^3}$$

Similarly,

$$\delta l_A = \frac{\sigma_A \times l_A}{E_A} = \frac{\sigma_A \times l}{80 \times 10^3}$$

Since increase in the lengths of steel and aluminium wires is equal, therefore equating equations (*i*) and (*ii*), we get

$$\frac{\sigma_S \times l}{200 \times 10^3} = \frac{\sigma_A \times l}{80 \times 10^3} \quad \text{or} \quad \sigma_S = \frac{200}{80} \times \sigma_A = 2.5\, \sigma_A$$

We also know that load supported by the three wires (P),

$$35 \times 10^3 = (\sigma_S \,.\, A_S) + (\sigma_A \,.\, A_A) = (2.5\, \sigma_A \times 200) + (\sigma_A \times 200) = 700\, \sigma_A$$

$$\therefore \quad \sigma_A = \frac{35 \times 10^3}{700} = 50 \text{ N/mm}^2 = 50 \text{ MPa} \quad \textbf{Ans.}$$

and $\sigma_S = 2.5\, \sigma_A = 2.5 \times 50 = 125$ MPa **Ans.**

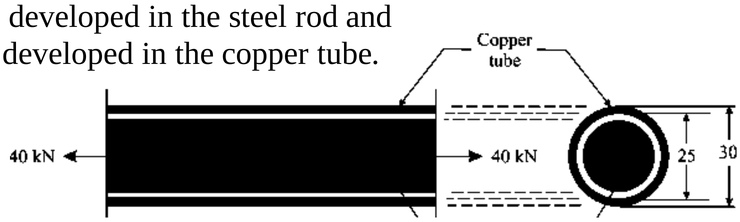
Stresses in Composite Structures of Equal Lengths

Example *A mild steel rod of 20 mm diameter and 300 mm long is enclosed centrally inside a hollow copper tube of external diameter 30 mm and internal diameter 25 mm. The ends of the rod and tube are brazed together, and the composite bar is subjected to an axial pull of 40 kN as shown. If E for steel and copper is 200 GPa and 100 GPa respectively, find the stresses developed in the rod and the tube.*

Given :

Diameter of steel rod = 20 mm;
External diameter of copper tube = 30 mm;
Internal diameter of copper tube = 25 mm;
Total load (P) = $40\ kN = 40 \times 103$ N;
Modulus of elasticity of steel (E_S) = 200 GPa and
Modulus of elasticity of copper (E_C) = 100 GPa

Let σ_s = Stress developed in the steel rod and
σ_c= Stress developed in the copper tube.



We know that area of steel rod,

$$A_S = \frac{\pi}{4} \times (20)^2 = 314.2 \text{ mm}^2$$

and area of copper tube,

$$A_C = \frac{\pi}{4} [(30)^2 - (25)^2] = 216 \text{ mm}^2$$

We also know that stress in steel,

$$\sigma_S = \frac{E_S}{E_C} \times \sigma_C = \frac{200}{100} \times \sigma_C = 2\,\sigma_C$$

and total load (P), $$40 \times 10^3 = (\sigma_S.A_S) + (\sigma_C.A_C)$$
$$= (2\sigma_C \times 314.2) + (\sigma_C \times 216) = 844.4\,\sigma_C$$

$\therefore$ $$\sigma_C = \frac{40\times10^3}{844.4} = 47.4 \text{ N/mm}^2 = 47.4 \text{ MPa} \qquad \textbf{Ans.}$$

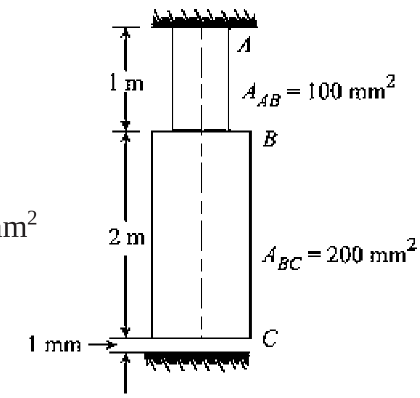
and $$\sigma_S = 2\,\sigma_C = 2 \times 47.4 = 94.8 \text{ MPa} \qquad \textbf{Ans.}$$

Stresses in Composite Structures of Unequal Lengths

Example *A composite bar ABC, rigidly fixed at A and 1 mm above the lower support, is subjected to an axial load of 50 kN at B as shown. If the cross-sectional area of the section AB is 100 mm² and that of section BC is 200 mm², find the reactions at both the ends of the bar. Also find the stresses in both the section. Take E = 200 GPa.*

Given:

Length of AB (l_{AB}) = 1 m = 1×10^3 mm
Area of AB (A_{AB}) = 100mm^2
Length of BC (l_{BC}) = 2 m = 2×10^3 mm
Area of BC (A_{BC}) = 200 mm^2
Axial load (P) = 50 kN = 50×10^3 N
Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2



Reactions at both the ends of the bar

The bar is rigidly fixed at A and loaded at B, therefore, upper portion AB is subjected to tensions. We also know that increase in length of the portion AB due to the load at B

$$\delta l = \frac{P.l_{AB}}{A_{AB}.E} = \frac{(50\times10^3)\times(1\times10^3)}{100\times(200\times10^3)} = 2.5 \text{ mm}$$

We find that of increase in the length of the portion *AB* would have been less than 1 mm (*i.e.*, gap between *C* and lower support), then the lower portion of the bar *BC* should not have been subjected to any stress. Now it will be interesting to know that as the increase in length *AB* is 2.5 mm, therefore, first action of the 50 kN load will be to increase the length AB by 1 mm, till the end C touches the lower support. And a part of the load will be required for this increase. Then the remaining load will be shared by both the portions of the bar *AB* and *BC* of the bar.

Let P = Load required to increase 1 mm length of the bar *AB*,

We know that increase in length

$$1 = \frac{P_1 \times l_{AB}}{A_{AB} \cdot E} = \frac{P_1 \times (1 \times 10^3)}{100 \times (200 \times 10^3)} = 0.05 \times 10^{-3} P_1$$

$$\therefore \quad P_1 = \frac{1}{0.05 \times 10^{-3}} = 20 \times 10^3 \text{ N} = 20 \text{ kN}$$

and the remaining loas, which will be shared by the portion *AB* and *CD*

$$= 50 - 20 = 30 \text{ kN}$$

Let R_A = Reaction at A due to 30 kN load, and

R_C = Reaction at C due to 30 kN load.

Thus, $$R_A + R_C = 30 \text{ kN} = 30 \times 10^3 \text{ N} \quad ...(i)$$

We know that increase in length *AB* due to reaction R_A (beyond 1 mm),

$$\delta l_1 = \frac{R_A \cdot l_{AB}}{A_{AB} \cdot E} = \frac{R_A \times (1 \times 10^3)}{100 \times (200 \times 10^3)} = 0.05 \times 10^{-3} R_A \quad ...(ii)$$

and decrease in length *BC* due to reaction R_C,

$$\delta l_2 = \frac{R_C \cdot l_{BC}}{A_{BC} \cdot E} = \frac{R_C \times (2 \times 10^3)}{200 \times (200 \times 10^3)} = 0.05 \times 10^{-3} R_C \quad ...(iii)$$

Since δl_1 is equal to δl_2, therefore equating equations (*i*) and (*ii*),

$$0.05 \times 10^{-3} R_A = 0.05 \times 10^{-3} R_C \quad \text{or} \quad R_A = R_C$$

Now substituting the value of R_C in equation (*i*)

$$R_A + R_A = 30 \quad \text{or} \quad R_A = R_C = \frac{30}{2} = 15 \text{ kN}$$

$\therefore$ Total reaction at $A = (20 + 15) = 35$ kN **Ans.**

and total reaction at $C = 15$ kN **Ans.**

Stresses in both the sections

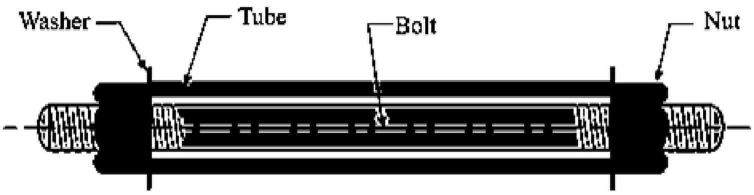
We know that stress in the bar *AB*,

$$\sigma_{AB} = \frac{35 \times 10^3}{100} = 350 \text{ N/mm}^2 = 350 \text{ MPa} \quad \textbf{Ans.}$$

and $$\sigma_{BC} = \frac{15 \times 10^3}{200} = 75 \text{ N/mm}^2 = 75 \text{ MPa} \quad \textbf{Ans.}$$

Stresses in Nuts and Bolts

Nuts and bolts to tighten the components of a machine or structure. It is generally done by placing washers below the nuts as shown. A nut can be easily tightened, till the space between the two washers becomes exactly equal to the body placed between them. It will be interesting to know that if we further tighten the nut, it will induce some load in the assembly. As a result of this, bolt will be subjected to some tension, whereas the washers and body between them will be subjected to some compression. And the induced load will be equally shared between the bolt and the body. Now consider an assembly consisting of two nuts and a bolt along with a tube as shown



Let P = Tensile load induced in the bolt as a result of tightening the nut,
l = Length of the bolt,
A_1= Area of the bolt,
σ_1 = Stress in the bolt due to induced load,
E_1 = Modulus of elasticity for the bolt material.
A_2, σ_2, E_2 = Corresponding values for the tube

The tensile load on the bolt is equal to the compressive load on the tube, therefore

$$\sigma_1 \cdot A_1 = \sigma_2 \cdot A_2$$

$$\therefore \quad \sigma_1 = \frac{A_2}{A_1} \times \sigma_2 \qquad \text{Similarly,} \qquad \sigma_2 \frac{A_1}{A_2} \times \sigma_1$$

and the total load $\quad (P) = \sigma_1 A_1 + \sigma_2 A_2$

We also know that increase in the length of the bolt due to tensile stress in it,

$$\delta l_1 = \frac{\sigma_1 . l}{E_1} \qquad ...(i)$$

and decrease in the length of the tube due to compressive stress in it,

$$\delta l_2 = \frac{\sigma_2 . l}{E_2} \qquad ...(ii)$$

$\therefore$ Axial advancement (*i.e.*, movement) of the nut

$$= \delta l_1 + \delta l_2$$

Example *A solid copper rod 300 mm long and 40 mm diameter passes axially inside a steel tube of 50 mm internal diameter and 60 mm external diameter. The composite bar is tightened by using rigid washers of negligible thickness. Determine the stresses in copper rod and steel tube, when the nut is tightened so as to produce a tensile load of 100 kN in the copper rod.*

Given:

Length of copper rod (l) = 300 mm
Diameter of copper rod (DC) = 40 mm
Internal diameter of steel tube (dS) = 50 mm
External diameter of steel tube (DS) = 60 mm
Tensile load in copper rod (P) = 100 kN = 100×10^3 N

Let σ_C = Stress in the copper rod and
σ_S = Stress in the steel rod.

We know that area of the copper rod,

$$A_C = \frac{\pi}{4} \times (D_C)^2 = \frac{\pi}{4} \times (40)^2 = 400\,\pi \text{ mm}^2$$

and area of the steel tube,

$$A_S = \frac{\pi}{4} \times [D_S^2 - d_C^2] = \frac{\pi}{4} \times [(60)^2 - (50)^2 = 275\,\pi \text{ mm}^2$$

We also know that tensile load on the copper rod is equal to the compressive load on the steel tube. Therefore stress in steel rod,

$$\sigma_S = \frac{A_C}{A_S} \times \sigma_C = \frac{400\pi}{275\pi} \times \sigma_C = \frac{16\sigma_C}{11} = 1.455\,\sigma_C$$

and load (P) $$100 \times 10^3 = (\sigma_C \cdot A_C) + (\sigma_S \cdot A_S) = (\sigma_C \times 400\,\pi) + (1.455\,\sigma_C \times 275\,\pi)$$
$$= 800\,\pi\,\sigma_C$$

$$\therefore \quad \sigma_C = \frac{100 \times 10^3}{800\pi} = 39.8 \text{ N/mm}^2 = 39.8 \text{ MPa (tension)} \qquad \textbf{Ans.}$$

and $$\sigma_S = 1.455\,\sigma_C = 1.455 \times 39.8 \text{ N/mm}^2 = 57.9 \text{ N/mm}^2 \qquad \textbf{Ans.}$$
$$= 57.9 \text{ MPa (compression)} \qquad \textbf{Ans.}$$

Thermal Stresses and Strains

Whenever there is some increase or decrease in the temperature of a body, it causes the body to expand or contract. A little consideration will show that if the body is allowed to expand or contract freely, with the rise or fall of the temperature, no stresses are induced in the body. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called thermal stresses or temperature stresses. The corresponding strains are called thermal strains or temperature strains.

Thermal Stresses in Simple Bars

The thermal stresses or strains, in a simple bar, may be found out as discussed below:

1. Calculate the amount of deformation due to change of temperature with the assumption that bar is free to expand or contract.

2. Calculate the load (or force) required to bring the deformed bar to the original length.

3. Calculate the stress and strain in the bar caused by this load.

The thermal stresses or strains may also be found out first by finding out amount of deformation due to change in temperature, and then by finding out the thermal strain due to the deformation. The thermal stress may now be found out from the thermal strain as usual. Now consider a body subjected to an increase in temperature.

Let l = Original length of the body,
t = Increase of temperature and
α = Coefficient of linear expansion.

We know that the increase in length due to increase of temperature

$$\delta l = l.\alpha.t$$

If the ends of the bar are fixed to rigid supports, so that its expansion is prevented, then compressive strain induced in the bar.

$$\varepsilon = \frac{\delta l}{l} = \frac{l.\alpha.t}{l} = \alpha.t$$

$\therefore$ Stress $$\sigma = \varepsilon.E = \alpha.t.E.$$

Cor. If the supports yield by an amount equal to Δ, then the actual expansion that has taken place,

$$\delta l = l\alpha t - \Delta$$

and strain, $$\varepsilon = \frac{\delta l}{l} = \frac{l\alpha t - \Delta}{l} = \left(\alpha t - \frac{\Delta}{l}\right)$$

$\therefore$ Stress, $$\sigma = \varepsilon.E = \left(\alpha t - \frac{\Delta}{l}\right)E$$

Example *Two parallel walls 6 m apart are stayed together by a steel rod 25 mm diameter passing through metal plates and nuts at each end. The nuts are tightened home, when the rod is at a temperature of 100°C. Determine the stress in the rod, when the temperature falls down to 60°C, if (a) the ends do not yield, and (b) the ends yield by 1 mm. Take E = 200 GPa and* $\alpha = 12 \times 10^{-6}$ */°C*

Given:

Length (l) = 6 m = 6×10^3 mm

Diameter (d) = 25 mm
Decrease in temperature (t) = 100° – 60° = 40°C
Amount of yield in ends (Δ) = 1 mm
Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2
Coefficient of linear expansion (α) = 12×10^{-6}/°C.

(*a*) ***Stress in the rod when the ends do not yield***

We know that stress in the rod when the ends do not yield,

$$\sigma_1 = \alpha.t.E = (12 \times 10^{-6}) \times 40 \times (200 \times 10^3) \text{ N/mm}^2$$
$$= 96 \text{ N/mm}^2 = 96 \text{ MPa} \quad \textbf{Ans.}$$

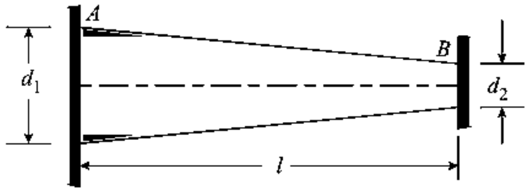
(*b*) ***Stress in the rod when the ends yield by 1 mm***

We also know that stress in the rod when the ends yield,

$$\sigma_2 = \left[\alpha t - \frac{\Delta}{l}\right] E = \left[(12 \times 10^{-6})\, 40 - \frac{1}{6 \times 10^3}\right] 200 \times 10^3 \text{ N/mm}^2$$
$$= 62.6 \text{ N/mm}^2 = 62.6 \text{ MPa} \quad \textbf{Ans.}$$

Thermal Stresses in Bars of Circular Tapering Section

Consider a circular bar of uniformly tapering section fixed at its ends A and B and subjected to an increase of temperature as shown



Let l = Length of the bar.
d_1= Diameter at the bigger end of the bar,
d_2= Diameter at the smaller end of the bar,
t = Increase in temperature and
a = Coefficient of linear expansion.

The increase in temperature, the bar AB will tend to expand. But since it is fixed at both of its ends, therefore it will cause some compressive stress. We also know that the increase in length due to increase in temperature,

$$\delta l = l.\alpha.t \quad \text{...}(i)$$

Now let P = Load (or force) required to bring the deformed bar to the original length.

We know that decrease in the length of the circular bar due to load P

$$\delta l = l.\alpha.t \quad \text{...}(i)$$

Now let P = Load (or force) required to bring the deformed bar to the original length.

We know that decrease in the length of the circular bar due to load P

$$\delta l = \frac{4Pl}{\pi E d_1 d_2} \quad \text{...}(ii)$$

Equating equations (i) and (ii),

$$l.\alpha.t = \frac{4Pl}{\pi E d_1 d_2} \quad \text{or} \quad P = \frac{\pi E d_1 d_2 .\alpha t}{4}$$

$\therefore$ *Max. stress,

$$\sigma_{max} = \frac{P}{\frac{\pi}{4} \times d_2^2} = \frac{\pi E d_1 d_2 .\alpha t}{4 \times \frac{\pi}{4} \times d_2^2} = \frac{\alpha t E d_1}{d_2}$$

NOTE. If we substitute $d_1 = d_2$, the above relation is reduced to

$$\sigma = \alpha.t.E \qquad \text{...(Same as for simple bars)}$$

Example *A circular bar rigidly fixed at its both ends uniformly tapers from 75 mm at one end to 50 mm at the other end. If its temperature is raised through 26 K, what will be the maximum stress developed in the bar. Take E as 200 GPa and α as 12 × 10–6 /K for the bar material.*

Given:

Diameter at end 1 (d_1) = 75 mm
Diameter at end 2 (d_2) = 50 mm
Rise in temperature (t) = 26 K
E = 200 GPa = 200 × 10^3 N/mm^2
$\alpha = 12 \times 10^{-6}$ /K
maximum stress developed in the bar,

$$\alpha_{max} = \frac{\alpha t.E.d_1}{d_2} = \frac{(12\times10^{-6})\times26\times(200\times10^3)\times75}{50} \text{ N/mm}^2$$

$$= 93.6 \text{ N/mm}^2 = 93.6 \text{ MPa} \qquad \textbf{Ans.}$$

Thermal Stresses in Bars of Varying Section

Consider a bar ABC fixed at its ends A and C and subjected to an increase of temperature as shown

Let

l_1 = Length of portion *AB*,
σ_1 = Stress in portion *AB*,
A_1 = Cross-sectional area of portion *AB*,
l_2, σ_2, A_2 = Corresponding values for the portion *BC*,
α= Coefficient of linear expansion and
t = Increase in temperature

We know that as a result of the increase in temperature, the bar *ABC* will tend to expand. But since it is fixed at its ends *A* and *C*, therefore it will cause some compressive stress in the body. Moreover, as the thermal stress is shared equally by both the portions, therefore

$$\sigma_1 A_1 = \sigma_2 A_2$$

Moreover, the total deformation of the bar (assuming it to be free to expand),

$$\delta l = \delta l_1 + \delta l_2 = \frac{\sigma_1 l_1}{E} + \frac{\sigma_2 l_2}{E} = \frac{l}{E}(\sigma_1 l_1 + \sigma_2 l_2)$$

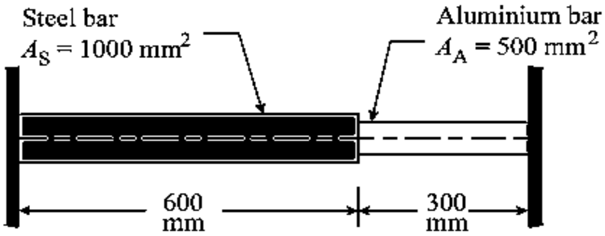
NOTE. Sometimes, the modulus of elasticity is different for different sections. In such cases, the total deformation.

$$\delta l = \left(\frac{\sigma_1 l_1}{E_1} + \frac{\sigma_2 l_2}{E_2}\right)$$

Example *A composite bar made up of aluminium and steel, is held between two supports as shown. The bars are stress-free at a temperature of 38°C. What will be the stresses in the two bars, when the temperature is 21°C, if (a) the supports are unyielding, (b) the supports come nearer to each other by 0.1 mm? It can be assumed that the change of temperature is uniform all along the length of the bar. Take E for steel as 200 GPa; E for aluminium as 75 GPa and coefficient of expansion for steel as 11.7 × 10–6 per °C and coefficient of expansion for aluminium as 23.4 × 10-6 per °C.*

Given:

Length of steel bar (l_S) = 600 mm
Area of steel bar (A_S) = *1000 mm*^{2}
Length of aluminium bar (l_A) = 300 mm
Area of aluminium bar (A_A) = *500 mm*^{2}
Decrease in temperature (t) = 38 – 21 = 17°C
Modulus of elasticity of steel (E_S) = 200 GPa = 200 × 10^3 N/mm^2
Modulus of elasticity of aluminium (E_A) = 75 GPa = 75 x 10^3 N/mm^2
Coefficient of expansion for steel (α_S) = 11.7 × 10^{-6}/°C
Coefficient of expansion for aluminium (α_A) = 23.4 × 10^{-6}/°C.



Let σ_S = Stress in the steel bar, and
σ_A = Stress in the aluminium bar.

(*a*) *Stresses when the supports are unyielding*

$$\sigma_S \cdot A_S = \sigma_A \cdot A_A \qquad \text{or} \qquad \sigma_S \times 1000 = \sigma_A \times 500$$

$$\therefore \quad \sigma_S = \sigma_A \times 500/1000 = 0.5\,\sigma_A$$

We know that free expansion of steel bar due to increase in temperature,

$$\delta l_S = l_S . \alpha_S . t = 600 \times (11.7 \times 10^{-6}) \times 17 = 0.119 \text{ mm}$$

and

$$\delta l_A = l_A . \alpha_A . t = 300 \times (23.4 \times 10^{-6}) \times 17 = 0.119 \text{ mm}$$

$\therefore$ Total contraction of the bar,

$$\delta l = \delta l_S + \delta l_A = 0.119 + 0.119 = 0.238 \text{ mm}$$

Now let us assume a tensile force to be applied at A and C, which will cause an expansion of 0.238 mm of the rod (*i.e.*, equal to the total contraction). Therefore

$$0.238 = \frac{\sigma_S . l_S}{E_S} + \frac{\sigma_A . l_A}{E_A} = \frac{(0.5\,\sigma_A) \times 600}{200 \times 10^3} + \frac{\sigma_A \times 300}{75 \times 10^3} = 5.5 \times 10^{-3}\,\sigma_A$$

$$\therefore \quad \sigma_A = \frac{0.238}{5.5 \times 10^{-3}} = 43.3 \text{ N/mm}^2 = 43.3 \text{ MPa} \qquad \textbf{Ans.}$$

and

$$\sigma_S = 0.5\,\sigma_A = 0.5 \times 43.3 = 21.65 \text{ MPa} \qquad \textbf{Ans.}$$

(*b*) *Stresses when the supports come nearer to each other by 0.1 mm*

In this case, there is an expansion of composite bar equal to 0.238 – 0.1 = 0.138 mm. Now let us assume a tensile force, which will cause an expansion of 0.138 mm. Therefore

$$0.138 = \frac{\sigma_S . l_S}{E_S} + \frac{\sigma_A . l_A}{E_A} = \frac{(0.5\,\sigma_A) \times 600}{200 \times 10^3} + \frac{\sigma_A \times 300}{75 \times 10^3} = 5.5 \times 10^{-3}\,\sigma_A$$

$$\therefore \quad \sigma_A = \frac{0.138}{5.5 \times 10^{-3}} = 25.1 \text{ N/mm}^2 = 25.1 \text{ MPa} \qquad \textbf{Ans.}$$

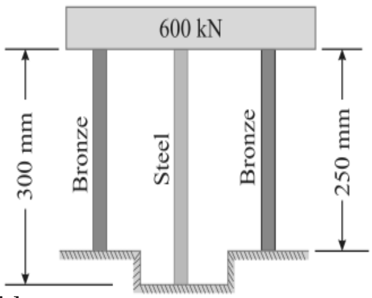
and

$$\sigma_S = 0.5\,\sigma_A = 0.5 \times 25.1 = 12.55 \text{ MPa} \qquad \textbf{Ans.}$$

Superposition of Thermal Stresses

Example *A rigid slab weighing 600 kN is placed upon two bronze rods and one steel rod each of 6000 mm² area at a temperature of 15°C as shown in Fig. Find the temperature, at which the stress in steel rod will be zero. Take: Coefficient of expansion for steel = 12 × 10^{-6} /°C, Coefficient of expansion for bronze = 18 × 10^{-6}/°C*
Young's modulus for steel = 200 Gpa, Young's modulus for bronze = 80 GPa.

Given:



Weight = 600 kN = 600 × 10^3 N
Area of bronze rod (A_B) = A_S = 6000mm²
Coefficient of expansion for steel (α_s) = 12 × 10^{-6} /°C
Coefficient of expansion for bronze (α_B) = 18 × 10^{-6} /°C
Modulus of elasticity of steel (E_S) = 200 GPa = 200 × 10^3 N/mm²
Modulus of elasticity of bronze (E_B) = 80 GPa = 80 × 10^3 N/mm²

Let t = Rise in temperature, when the stress in the steel rod will be zero.

Due to increase in temperature all the three rods will expand. The expansion of bronze rods will be more than the steel rod (because α_B is greater than α_S). If the stress in the steel rod is to be zero, then the entire load should be shared by the two bronze rods. Or in other words, the decrease in the length of two bronze rods should be equal to the difference of the expansion of the bronze rods and steel rod. We know that free expansion of the steel rod

$$= l_S.\alpha_S.t = 300 \times 12 \times 10^{-6} \times t = 3.6 \times 10^{-3}\, t$$

Similarly, free expansion of the bronze rods,

$$= l_B.\alpha_B.t = 250 \times 18 \times 10^{-6} \times t = 4.5 \times 10^{-3}\, t$$

∴ Difference in the expansion of the two rods

$$= (4.5 \times 10^{-3})\, t - (3.6 \times 10^{-3})\, t = 0.9 \times 10^{-3}\, t \qquad ...(i)$$

We also know that the contraction of the bronze rods due to load of 600 kN

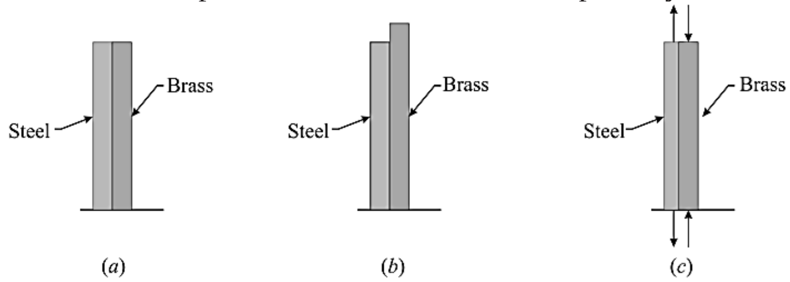
$$= \frac{Pl}{AE} = \frac{(600 \times 10^3) \times 250}{(2 \times 6000) \times (80 \times 10^3)} = 0.156 \text{ mm} \qquad ...(ii)$$

Now equating equations (i) and (ii),

$$0.9 \times 10^{-3} \times t = 0.156 \qquad \text{or} \qquad t = \frac{0.156}{9 \times 10^{-4}} = 173.3°\text{C} \qquad \textbf{Ans.}$$

Thermal Stresses in Composite Bars

Whenever there is some increase or decrease in the temperature of a bar, consisting of two or more different materials, it causes the bar to expand or contract. The different coefficients of linear expansions the two materials do not expand or contract by the same amount, but expand or contract by different amounts. The steel and brass could have been free to expand, and then no internal stresses would have induced. The two members are rigidly fixed, therefore the composite bar, as a whole, will expand by the same amount.



We know that the brass expands more than the steel (because the coefficient of linear expansion of the brass is greater than that of the steel). Therefore the free expansion of the brass will be more than that of the steel. But since both the members are not free to expand, therefore the expansion of the composite bar, as a whole, will be less than that of the brass; but more than that of the steel as shown. *It is thus obvious that the brass will be subjected to compressive force*, *whereas* the steel will be subjected to tensile force as shown.

σ_1 = Stress in brass

ε_1 = Strain in brass,

α_1 = Coefficient of linear expansion for brass,

A_1 = Cross-sectional area of brass bar,

$\sigma_2, \varepsilon_2, \alpha_2\, A_2$ = Corresponding values for steel, and

ε = Actual strain of the composite bar per unit length.

As the compressive load on the brass is equal to the tensile load on the steel, therefore

$$\sigma_1.A_1 = \sigma_2.A_2$$

Now strain in brass,

$$\varepsilon_1 = \alpha_1.t - \varepsilon \quad ...(i)$$

and strain in steel,

$$\varepsilon_2 = \alpha_2.t - \varepsilon \quad ...(ii)$$

Adding equation (*i*) and (*ii*), we get

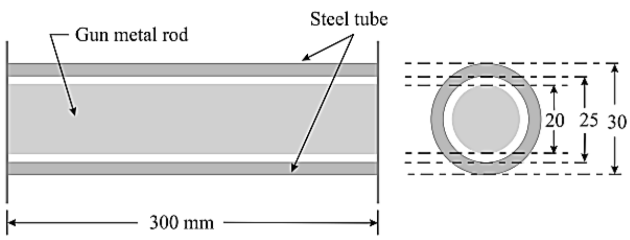
$$\varepsilon_1 + \varepsilon_2 = -t\,(\alpha_1 + \alpha_2)$$

Notes : 1. In the above equation the value of α_1 is taken as greater of the two values of α_1 and α_2.

Example *A gun metal rod 20 mm diameter, screwed at the ends, passes through a steel tube 25 mm and 30 mm internal and external diameters respectively. The nuts on the rod are screwed tightly home on the ends of the tube. Find the intensity of stress in each metal, when the common temperature rises by 200°F. Take. Coefficient of expansion for steel = 6 × 10–6/°F Coefficient of expansion for gun metal = 10 × 10 –6 /°F Modulus of elasticity for steel = 200 Gpa, Modulus of elasticity for gun metal = 100 GPa.*

Given:

Diameter of gun metal rod = 20 mm
Internal diameter of steel tube = 25 mm
External diameter of steel tube = 30 mm
Rise in temperature (t) = 200°F
Coeff of expansion for steel (α_S) = 6 × 10^{-6} /°F
Coeff of expansion for gun metals (α_G) = 10 × 10^{-6} /°F
(E_S) = 200 GPa = 200 × 10^3 N/mm^2
(E_G) = 100 GPa = 100 × 10^3 N/mm^2



The temperature of the gun metal rod and steel tube will increase; the free expansion of gun metal rod will be more than that of steel tube. Thus the gun metal rod will be subjected to compressive stress and the steel tube will be subjected to tensile stress.

$$A_G = \frac{\pi}{4} \times (20)^2 = 100\,\pi \text{ mm}^2$$

$$A_S = \frac{\pi}{4}\,[(30)^2 - (25)^2] = 68.75\,\pi \text{ mm}^2$$

$$\sigma_S = \frac{A_G}{A_S} \times \sigma_S = \frac{100\pi}{68.75\pi} \times \sigma_G = 1.45\ \sigma_G$$

We know that strain in steel tube,

$$\varepsilon_S = \frac{\sigma_S}{E_S} = \frac{\sigma_S}{200 \times 10^3}$$

and

$$\varepsilon_G = \frac{\sigma_G}{E_G} = \frac{\sigma_G}{100 \times 10^3}$$

We also know that total strain,

$$\varepsilon_S + \varepsilon_G = t\,(\alpha_G - \alpha_S)$$

$$\frac{\sigma_S}{200 \times 10^3} + \frac{\sigma_G}{100 \times 10^3} = 200\ [(10 \times 10^{-6}) - (6 \times 10^{-6})]$$

$$\frac{1.45\sigma_G}{200 \times 10^3} + \frac{\sigma_G}{100 \times 10^3} = 200 \times (4 \times 10^{-6})$$

$$\frac{3.45\sigma_G}{200 \times 10^3} = 800 \times 10^{-6}$$

$$3.45\ \sigma G = (800 \times 10^{-6}) \times (200 \times 10^3) = 160$$

$\therefore$

$$\sigma_G = \frac{160}{3.45} = 46.4\ \text{N/mm}^2 = 46.4\ \text{MPa} \qquad \textbf{Ans.}$$

and

$$\sigma_S = 1.45\ \sigma_G = 1.45 \times 46.4 = 67.3\ \text{MPa} \qquad \textbf{Ans.}$$

Elastic constant

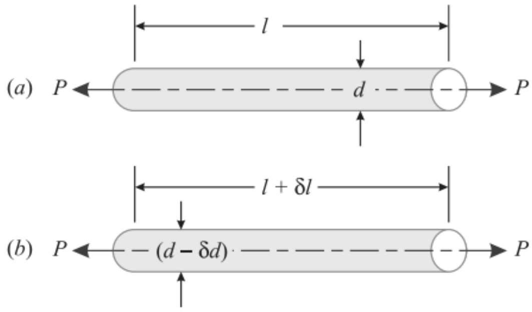
The axial deformation of a body, when it is subjected to a direct tensile or compressive stress. But we have not discussed the lateral or side effects of the pulls or pushes. It has been experimentally found, that the axial strain of a body is always followed by an opposite kind of strain in all directions at right angle to it. Thus, in general, there is always a set of the following two types of strains in a body, when it is subjected to a direct stress.

- Primary or linear strain, and
- Secondary or lateral strain

Whenever some external force acts on a body, it undergoes some deformation. Now consider a circular bar subjected to a tensile force as shown. Let

l = Length of the bar,
d = Diameter of the bar,
P = Tensile force acting on the bar, and
dl = Increase in the length of the bar

The deformation of the bar per unit length in the direction of the force is known as **linear strain**. The linear deformation of a circular bar of length l and diameter d subjected to a tensile force P. The deformation of the bar, we will find that bar has extended through a length dl, which will be followed by the decrease of diameter from d to (d – δd) as shown. Similarly, if the bar is subjected to a compressive force, the length of the bar will decrease by dl which will be followed by the increase of Diameter from d to (d + δd). It is thus obvious that every direct stress is always accompanied by a strain in its own direction and an opposite kind of strain in every direction at right angles to it. Such a strain is known as **secondary or lateral strain.**



Poisson's ratio

If a body is stressed within its elastic limit, the lateral strain bears a constant ratio to the linear strain.

$$\frac{\text{Lateral strain}}{\text{Linear strain}} = \text{(constant)}$$

This constant is known as Poisson's ratio and is denoted by $\frac{1}{m}$ or μ. Mathematically,

$$\text{Lateral strain} = \frac{1}{m} \times \varepsilon = \mu\,\varepsilon$$

Example *A steel bar 2 m long, 40 mm wide and 20 mm thick is subjected to an axial pull of 160 kN in the direction of its length. Find the changes in length, width and thickness of the bar. Take E = 200 GPa and Poisson's ratio = 0.3.*

Given: Length (l) = 2 m = 2 × 103 mm

Width (b) = 40 mm ;

Thickness (t) = 20 mm;

Axial pull (P) = 160 kN = 160 × 103 N ;

Modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2

poisson's ratio (1/m) = 0.3

Change in length

We know that change in length,

$$\delta l = \frac{Pl}{AE} = \frac{(160 \times 10^3) \times (2 \times 10^3)}{(40 \times 20) \times (200 \times 10^3)} = 2 \text{ mm} \qquad \textbf{Ans.}$$

Change in width

We know that linear strain,

$$\varepsilon = \frac{\delta l}{l} = \frac{2}{2 \times 10^3} = 0.001$$

and lateral strain

$$= \frac{1}{m} \times \varepsilon = 0.3 \times 0.01 = 0.0003$$

∴ Change in width,

$$\delta b = b \times \text{Lateral strain} = 40 \times 0.0003 = 0.012 \text{ mm} \qquad \textbf{Ans.}$$

Change in thickness

We also know that change in thickness,

$$\delta t = t \times \text{Lateral strain} = 20 \times 0.0003 = 0.006 \text{ mm} \qquad \textbf{Ans.}$$

Volumetric strain

Whenever a body is subjected to a single force (or a system of forces), it undergoes some changes in its dimensions. The change in dimensions of a body will cause some changes in its volume. The ratio of change in volume, to the original volume, is known as **volumetric strain**

The following are important from the subject point of view:

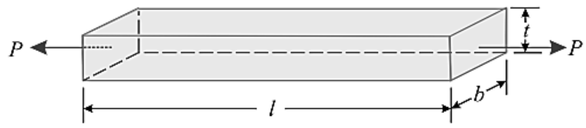
1. A rectangular body subjected to an axial force.
2. A rectangular body subjected to three mutually perpendicular force

$$\varepsilon_V = \frac{\delta V}{V}$$

δV = Change in volume, and

V = Original volume.

Volumetric Strain of a Rectangular Body Subjected to an Axial Force



Consider a bar, rectangular in section, subjected to an axial tensile force as shown in Fig. 6.2.

Let l = Length of the bar,

b = Breadth of the bar,

t = Thickness of the bar,

P = Tensile force acting on the bar,

E = Modulus of elasticity and

$\frac{1}{m}$ = Poisson's ratio.

We know that change in length,

$$\delta l = \frac{Pl}{AE} = \frac{Pl}{bt\,E} \qquad \text{...}(i)$$

and linear stress,

$$\sigma = \frac{\text{Force}}{\text{Area}} = \frac{P}{b\,t}$$

$\therefore$ Linear strain $= \frac{\text{Stress}}{E} = \frac{P}{bt\,E}$

and lateral strain $= \frac{1}{m} \times \text{Linear strain} = \frac{1}{m} \times \frac{P}{bt\,E}$

$\therefore$ Change in thickness,

$$\delta t = t \times \frac{1}{m} \times \frac{P}{bt\,E} = \frac{P}{mb\,E} \qquad \text{...}(ii)$$

and change in breadth,

$$\delta b = b \times \frac{1}{m} \times \frac{P}{bt\,E} = \frac{P}{mt\,E} \qquad \text{...}(iii)$$

As a result of this tensile force, let the final length

$= l + \delta l$

Final breadth $= b - \delta b$ ...(Minus sign due to compression)

and final thickness $= t - \delta t$ ...(Minus sign due to compression)

We know that original volume of the body,

$V = l.b.t.$

and final volume $= (l + \delta l)(b - \delta b)(t - \delta t)$

$$= lbt\left(1 + \frac{\delta l}{l}\right)\left(1 - \frac{\delta b}{b}\right)\left(1 - \frac{\delta t}{t}\right)$$

$$= lbt\left[1+\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right] \qquad \text{...(Ignoring other negligible values)}$$

$\therefore$ Change in volume,

$$\delta V = \text{Final volume} - \text{Original volume}$$

$$= lbt\left(1+\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)-lbt = lbt\left(\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)$$

$$= V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)$$

and volumetric strain,

$$\frac{\delta V}{V} = \frac{V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)}{V}=\frac{P}{btE}\left(1-\frac{2}{m}\right)$$

$$= \varepsilon\left(1-\frac{2}{m}\right) \qquad \ldots\left(\because \frac{P}{btE}=\varepsilon=\text{Strain}\right)$$

Example *A steel bar 2 m long, 20 mm wide and 15 mm thick is subjected to a tensile load of 30 kN. Find the increase in volume, if Poisson's ratio is 0.25 and Young's modulus is 200 GPa.*

Given: Length (l) = 2 m = 2 × 10 3 mm ; Width (b) = 20 mm ; Thickness (t) = 15 mm

Tensile load (P) = 30 kN = 30×10^3 N ; Poisson's ratio $\left(\frac{1}{m}\right)$ = 0.25 or $m = 4$ and Young's modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

Let δV = Increase in volume of the bar.

We know that original volume of the bar,

$$V = l.b.t = (2\times10^3)\times20\times15 = 600\times10^3\ \text{mm}^3$$

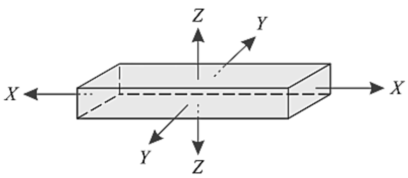
and

$$\frac{\delta V}{V} = \frac{P}{btE}\left(1-\frac{2}{m}\right)=\frac{30\times10^3}{20\times15\times(200\times10^3)}\left(1-\frac{2}{4}\right) = 0.000\,25$$

$$\therefore \quad \delta V = 0.000\,25\times V = 0.000\,25\times(600\times10^3) = 150\ \text{mm}^3 \qquad \textbf{Ans.}$$

Volumetric Strain of a Rectangular Body Subjected to Three Mutually Perpendicular Forces

Consider a rectangular body subjected to direct tensile stresses along three mutually perpendicular axes as shown



Let σ_x = Stress in x-x direction,

σ_y = Stress in y-y direction,

σ_z = Stress in z-z direction and

E = Young's modulus of elasticity.

$\therefore$ Strain in x-x direction due to stress σ_x,

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Similarly, $$\varepsilon_y = \frac{\sigma_y}{E} \quad \text{and} \quad \varepsilon_z = \frac{\sigma_z}{E}$$

The resulting strains in the three directions may be found out by the principle of superposition, *i.e., by adding algebraically the strains in each direction due to each individual stress.* For the three tensile stresses shown. (taking tensile strains as +ve and compressive strains as –ve) the resultant strain in *x-x direction,*

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_x - \frac{\sigma_y}{m} - \frac{\sigma_z}{m}\right]$$

Similarly,
$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_y - \frac{\sigma_x}{m} - \frac{\sigma_z}{m}\right]$$

and
$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_y}{mE} = \frac{1}{E}\left[\sigma_z - \frac{\sigma_x}{m} - \frac{\sigma_y}{m}\right]$$

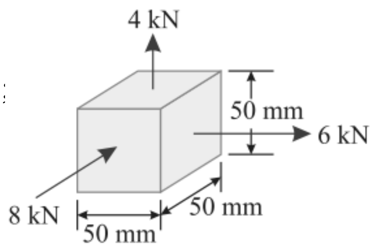
The volumetric strain may then be found by the relation;

$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Example *A steel cube block of 50 mm side is subjected to a force of 6 kN (Tension), 8kN (Compression) and 4 kN (Tension) along x, y and z direction respectively. Determine the change in volume of the block. Take E as 200 GPa and m as 10/3.*

Given:

Side of the cube = 50 mm ;
Force in *x*- direction (*Px*) = 6 kN = 6 × 10^3 N (Tension) ;
Force in *y-direction*(*Py*) = 8 kN = 8 × 10^3 N (Compression) ;
Force in *z-direction* (*Pz*) = 4 kN = 4 × 10^3 N (Tension) and
modulus of elasticity (*E*) = *200*GPa = 200 × 10^3 N/mm^2
and *m* = *10 / 3*



δV = Change in volume of the block.

original volume of the steel cube,

$$V = 50 \times 50 \times 50 = 125 \times 10^3 \text{ mm}^3$$

and stress in *x-x* direction,

$$\sigma_x = \frac{P_x}{A} = \frac{6\times10^3}{2500} = 2.4 \text{ N/mm}^2 \text{ (Tension)}$$

Similarly
$$\sigma_y = \frac{P_y}{A} = \frac{8\times10^3}{2500} = 3.2 \text{ N/mm}^2 \text{ (Compression)}$$

and
$$\sigma_z = \frac{P_z}{A} = \frac{4\times10^3}{2500} = 1.6 \text{ N/mm}^2 \text{ (Tension)}$$

We also know that resultant strain in *x-x* direction considering tension as positive and compression as negative,

$$\varepsilon_x = \frac{\sigma_x}{E} + \frac{\sigma_y}{mE} - \frac{\sigma_z}{mE} = \frac{2.4}{E} + \frac{3.2\times3}{10\,E} - \frac{1.6\times3}{10\,E} = \frac{2.88}{E}$$

Similarly,
$$\varepsilon_y = -\frac{\sigma_y}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_z}{mE} = -\frac{3.2}{E} - \frac{2.4\times3}{10\,E} - \frac{1.6\times3}{10\,E} = -\frac{4.4}{E}$$

and
$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\sigma_x}{mE} + \frac{\sigma_y}{mE} = \frac{1.6}{E} - \frac{2.4\times3}{10\,E} + \frac{3.2\times3}{10\,E} = \frac{1.84}{E}$$

volumetric strain,

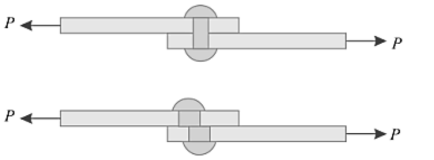
$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\frac{\delta V}{125\times10^3} = \frac{2.88}{E} - \frac{4.4}{E} + \frac{1.84}{E} = \frac{0.32}{E} = \frac{0.32}{200\times10^3}$$

$$\delta V = 125 \times 10^3 \times \frac{0.32}{200\times10^3} = 0.2 \text{ mm}^3 \qquad \textbf{Ans.}$$

Shear Stress

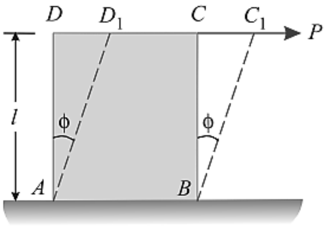
When a section is subjected to two equal and opposite forces, acting tangentially across the resisting section, as a result of which the body tends to shear off across the section as shown. The stress induced is called shear stress. The corresponding strain is called shear strain.



$$\text{Shear strain} = \frac{\text{Deformation}}{\text{Original length}}$$

$$= \frac{CC_1}{l} = \phi$$

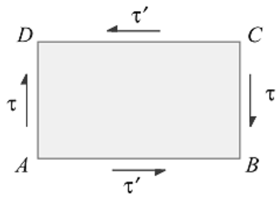
$$\tau = \frac{P}{AB}$$



Principle of Shear Stress

It states, "*A shear stress across a plane, is always accompanied by a balancing shear stress across the plane and normal to it.*

$$P = \tau \times .AD = \tau \times CB$$



Consider a rectangular block *ABCD, subjected to a shear stress of* intensity t on the faces *AD and CB as shown. Now consider* a unit thickness of the block. Therefore force acting on the faces *AD* and *CB,*

These forces will form a couple, whose moment is equal to $\tau \times AD \times AB$ *i.e., force* × *distance. If the* block is in equilibrium, there must be a restoring couple, whose moment must be equal to this couple. Let the shear stress of intensity t be set up on the faces *AB* and *CD* as shown. Therefore forces acting on the faces *AB* and *CD*,.

$$\tau \times AD \times AB = \tau' \times AD \times AB$$

$$\tau = \tau'$$

Relation between Modulus of Elasticity and Modulus of Rigidity

Consider a cube of length *l subjected to a shear stress of τ as shown.* due to these stresses the cube is subjected to some distortion, such that the diagonal *BD will be elongated and the diagonal AC will be shortened. Let this shear stress t cause* shear strain φ as shown. *We see that the diagonal BD is now distorted to BD*1.

$$\text{Strain of } BD = \frac{BD_1 - BD}{BD} \qquad \ldots\left(\because \text{ Strain} = \frac{\delta l}{l}\right)$$

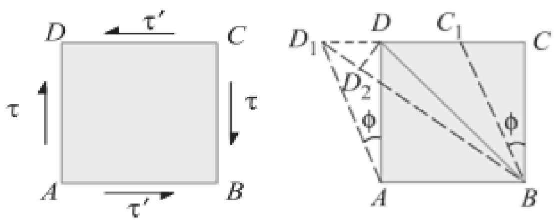
$$= \frac{D_1 D_2}{BD} = \frac{DD_1 \cos 45°}{AD\sqrt{2}} = \frac{DD_1}{2\,AD} = \frac{\phi}{2}$$

Linear strain of the diagonal *BD*

$$= \frac{\phi}{2} = \frac{\tau}{2C}$$

τ = Shear stress and

C = Modulus of rigidity.



(*a*) Before distortion (*b*) After distortion

Let us now consider this shear stress t acting on the sides *AB*, *CD*, *CB* and *AD*. We know that the effect of this stress is to cause tensile stress on the diagonal *BD* and compressive stress on the diagonal *AC*. Therefore tensile strain on the diagonal *BD* due to tensile stress on the diagonal *BD*

$$= \frac{\tau}{E} \qquad \text{...}(ii)$$

and the tensile strain on the diagonal *BD* due to compressive stress on the diagonal *AC*

$$= \frac{1}{m} \times \frac{\tau}{E} \qquad \text{...}(iii)$$

The combined effect of the above two stresses on the diagonal *BD*

$$= \frac{\tau}{E} + \frac{1}{m} \times \frac{\tau}{E} = \frac{\tau}{E}\left(1 + \frac{1}{m}\right) = \frac{\tau}{E}\left(\frac{m+1}{m}\right) \qquad \text{...}(iv)$$

Equating equations (*i*) and (*iv*),

$$\frac{\tau}{2C} = \frac{\tau}{E}\left(\frac{m+1}{m}\right) \qquad \text{or} \qquad C = \frac{mE}{2(m+1)}$$

Example *An alloy specimen has a modulus of elasticity of 120 GPa and modulus of rigidity of 45 GPa. Determine the Poisson's ratio of the material.*

Given:

Modulus of elasticity (*E) = 120 GPa*
Modulus of rigidity (C) = 45 GPa.

Let $\frac{1}{m}$ = Poisson's ratio of the material.

We know that modulus of rigidity (*C*),

$$45 = \frac{mE}{2(m+1)} = \frac{m \times 120}{2\,(m+1)} = \frac{120m}{2m+2}$$

$$90\,m + 90 = 120\,m \qquad \text{or} \qquad 30\,m = 90$$

$$\therefore \qquad m = \frac{90}{30} = 3 \qquad \text{or} \qquad \frac{1}{m} = \frac{1}{3} \qquad \textbf{Ans.}$$

Strain Energy and Impact Loading

When the load moves downwards, it loses its *potential energy. This energy is absorbed (or stored) in the stretched wire, which may be released by removing the load. On removing the load, the wire will spring back to its original position.

Resilience

It is a common term used for the total strain energy stored in a body. Sometimes the resilience is also defined as the capacity of a strained body for doing work (when it springs back) on the removal of the straining force.

Proof Resilience

It is also a common term, used for the maximum strain energy, which can be stored in a body. (This happens when the body is stressed up to the elastic limit). The corresponding stress is known as proof stress.

Modulus of Resilience

The proof resilience per unit volume of a material, is known as modulus of resilience and is a important property of the material.

A load may act in either of the following three ways:

1. Gradually 2. suddenly 3. with impact

Strain Energy Stored in a Body, when the Load is Gradually Applied

When loading a body, in which the loading starts from zero and increases gradually till the body is fully loaded. *e.g., when we lower a body with the help of a* crane, the body first touches the platform on which it is to be placed. On further releasing the chain, the platform goes on loading till it is fully loaded by the body. This is the case of a gradually applied load. Now consider a metallic bar subjected to a gradual load.

Let P = Load gradually applied,

A = Cross-sectional area of the bar,

l = Length of the bar,

E = Modulus of elasticity of the bar material and

d = Deformation of the bar due to load.

Since the load applied is gradual, and varies from zero to P, therefore the average load is equal to P/2

$$\therefore \text{ Work done } = \text{Force} \times \text{Distance}$$

$$= \text{Average load} \times \text{Deformation}$$

$$= \frac{P}{2} \times \delta l = \frac{P}{2}(\varepsilon . l) \qquad ...(\because \delta l = \varepsilon . l)$$

$$= \frac{1}{2} \sigma . \varepsilon A . l \qquad ...(\because P = \sigma A)$$

$$= \frac{1}{2} \times \text{Stress} \times \text{Strain} \times \text{Volume}$$

$$= \frac{1}{2} \times \sigma \times \frac{\sigma}{E} \times Al \qquad ...(\because \varepsilon = \frac{\sigma}{E})$$

$$= \frac{1}{2} \times \frac{\sigma^2}{E} \times Al$$

Since the strain energy stored is also equal to the work done, therefore strain energy stored,

$$U = \frac{\sigma^2}{2E} \times Al = \frac{\sigma^2}{2E} \times V \qquad ...(\because Al = \text{Volume} = V)$$

We also know that modulus of resilience

$$= \text{Strain energy per unit volume}$$

$$= \frac{\sigma^2}{2E}$$

Example *Calculate the strain energy stored in a bar 2 m long, 50 mm wide and 40 mm thick when it is subjected to a tensile load of 60kN. Take E as 200 GPa.*

Given:

Length of bar (*l*) = *2 m* = *2* × *10*³ mm

Width of bar (*b*) = *50 mm*

Thickness of bar (*t*) = *40 mm*

Tensile load on bar (P) = *60 kN* = *60* × *10*³ N and

Modulus of elasticity (*E*) = *200*GPa = 200 × 10³ N/mm²

We know that stress in the bar

$$\sigma = \frac{P}{A} = \frac{60 \times 10^3}{50 \times 40} = 30 \text{ N/mm}^2$$

$\therefore$ Strain energy stored in the bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(30)^2}{2 \times (200 \times 10^3)} \times 4 \times 10^6 \text{ N-mm}$$

$$= 9 \times 10^3 \text{ N-mm} = 9 \text{ kN-mm} \qquad \textbf{Ans.}$$

Strain Energy Stored in a Body when the Load is Suddenly Applied

The load is suddenly applied on a body. *e.g., when we* lower a body with the help of a crane, the body is, first of all, just above the platform on which it is to be placed. If the chain breaks at once at this moment the whole load of the body begins to act on the platform. This is the case of a suddenly applied load. Now consider a bar subjected to a sudden load.

P = Load applied suddenly,
A = Cross-sectional area of the bar,
l = Length of the bar,
E = Modulus of elasticity of the material,
δ = Deformation of the bar, and
σ = Stress induced by the application of the sudden load

Since the load is applied suddenly, therefore the load (P) is constant throughout the process of deformation of the bar.

$$\therefore \quad \text{Work done} = \text{Force} \times \text{Distance} = \text{Load} \times \text{Deformation} \qquad ...(i)$$
$$= P \times \delta l$$

We know that strain energy stored,

$$U = \frac{\sigma^2}{2E} \times Al \qquad ...(ii)$$

Since the strain energy stored is equal to the work done, therefore

$$\frac{\sigma^2}{2E} \times Al = P \times \delta l = P \times \frac{\sigma}{E} l \qquad ...\left(\delta l = \frac{\sigma}{E} l\right)$$

or

$$\sigma = 2 \times \frac{P}{A}$$

Example *An axial pull of 20 kN is suddenly applied on a steel rod 2.5 m long and 1000 mm2 in cross-section. Calculate the strain energy, which can be absorbed in the rod. Take E = 200 GPa.*

Given:

Axial pull on the rod (P) = 20 kN = 20×10^3 N;
Length of rod (l) = 2.5 m = 2.5×10^3 mm
Cross-sectional area of rod (A) =1000 mm^2
and modulus of elasticity (E) = 200GPa = 200×10^3 N/mm^2

We know that stress in the rod, when the load is suddenly applied

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{20 \times 10^2}{1000} = 440 \text{ N/mm}^2$$

and volume of the rod,

$$V = l \,.\, A = (2.5 \times 10^3) \times 1000 = 2.5 \times 10^6 \text{ mm}^3$$

$\therefore$ Strain energy which can be absorbed in the rod,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(40)^2}{2 \times (200 \times 10^3)} \times (2.5 \times 10^6) \text{ N-mm}$$
$$= 10 \times 10^3 \text{ N-mm} = 10 \text{ kN-mm} \qquad \textbf{Ans.}$$

Strain Energy Stored in a Body, when the Load is applied with Impact

The impact load is applied on a body *e.g., when we lower a body with the help of a crane, and the* chain breaks while the load is being lowered the load falls through a distance, before it touches the platform. This is the case of a load applied with impact. Now consider a bar subject to a load applied with impact as shown.

Let P = Load applied with impact,
A = Cross-sectional area of the bar,
E = Modulus of elasticity of the bar material,
l = Length of the bar,
δl = Deformation of the bar, as a result of this load,
σ = Stress induced by the application of this load with impact, and

h = Height through which the load will fall, before impacting on the collar of the bar.

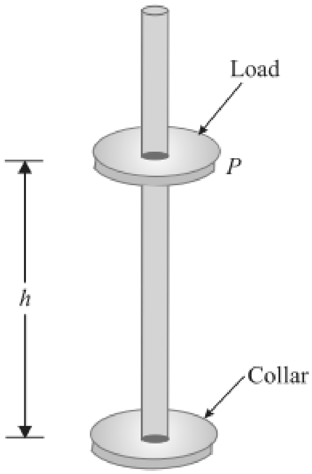
$\therefore$ Work done = Load × Distance moved

$$= P\,(h + \delta l)$$

and energy stored, $U = \dfrac{\sigma^2}{2E} \times Al$

Since energy stored is equal to the work done, therefore

$$\frac{\sigma^2}{2E} \times Al = P\,(h + \delta l) = P\left(h + \frac{\sigma}{E}.l\right) \qquad \left(\because \delta l = \frac{\sigma}{E}.l\right)$$



$$\frac{\sigma^2}{2E} \times Al = Ph + \frac{P\sigma l}{E}$$

$$\therefore \quad \sigma^2\left(\frac{Al}{2E}\right) - \sigma\left(\frac{Pl}{E}\right) - Ph = 0$$

Multiplying both sides by $\left(\dfrac{E}{Al}\right)$,

$$\frac{\sigma^2}{2} - \sigma\left(\frac{P}{A}\right) - \frac{PEh}{Al} = 0$$

This is a quadratic equation. We know that

$$\sigma = \frac{P}{A} \pm \sqrt{\left(\frac{P}{A}\right)^2 + \left(4 \times \frac{1}{2}\right)\left(\frac{PEh}{Al}\right)}$$

$$= \frac{P}{A}\left[1 \pm \sqrt{1 + \frac{2AEh}{Pl}}\right]$$

Once the stress (σ)is obtained, the corresponding instantaneous deformation (δl) or the strain energy stored may be found out as usual.

Cor. When δ is very small as compared to h, then

$$\text{Work done} = Ph$$

$$\therefore \quad \frac{\sigma^2}{2E}\,Al = Ph$$

$$\text{or} \quad \sigma^2 = \frac{2EPh}{Al}$$

$$\therefore \quad \sigma = \sqrt{\frac{2EPh}{Al}}$$

Example *A copper bar of 12 mm diameter gets stretched by 1 mm under a steady load of 4 kN. What stress would be produced in the bar by a weight 500 N, the weight falls through 80 mm before striking the collar rigidly fixed to the lower end of the bar? Take Young's modulus for the bar material as 100 GPa.*

Given :

Diameter of bar (d) = *12 mm*

Change in length of bar (dl) = 1 mm

Load on bar ($P1$) = 4 kN = 4×10^3 N

Weight falling on collar ($P2$) = 500 N

Height from which weight falls (h) = *80 mm*

Modulus of elasticity (E) = 100 GPa = 100×10^3 N/mm^2

Let l = Length of the copper bar.

We know that cross-sectional area of the bar,

$$A = \frac{\pi}{4} \times (d)^2 = \frac{\pi}{4} \times (12)^2 = 113.1 \text{ mm}^2$$

and stretching of the bar (δl),

$$l = \frac{P.l}{A.E} = \frac{(4 \times 10^3)}{113.1 \times (100 \times 10^3)} = \frac{l}{2.83 \times 10^3}$$

$$\therefore \quad l = 1 \times (2.83 \times 10^3) = 2.83 \times 10^3 \text{ mm}$$

We also know that stress produced in the bar by the falling weight.

$$\sigma = \frac{P_2}{A}\left(1 + \sqrt{1 + \frac{2AEh}{P_2 l}}\right)$$

$$= \frac{1500}{113.1}\left(1 + \sqrt{1 + \frac{2 \times 113.1 \times (100 \times 10^3) \times 80}{500 \times (2.83 \times 10^3)}}\right) \text{ N/mm}^2$$

$$= 4.2\,(1 + 35.77) = 162.52 \text{ N/mm}^2 = 162.52 \text{ MPa} \qquad \textbf{Ans.}$$

Strain Energy Stored in a Body of Varying Section

Sometimes, we come across bodies of varying section. The strain energy in such a body is obtained by adding the strain energies stored in different parts of the body. Mathematically total strain energy stored in a body.

$U = U_1 + U_2 + U_3 + ...$

Where U_1 = Strain energy stored in part 1,

U_2 = Strain energy stored in part 2,

U_3 = Strain energy stored in part 3

Example *A non-uniform tension bar 5 m long is made up of two parts as shown. Find the total strain energy stored in the bar, when it is subjected to a gradual load of 70 kN. Also find the total strain energy stored in the bar, when the bar is made of uniform cross-section of the same volume under the same load. Take E = 200 GPa.*

Given:

Total length of bar $(L) = 5\ m = 5 \times 10^3$ mm

Length of part 1 $(L1)$ = 3 m = 3×10^3 mm

Length of part 2 $(L2)$ = 2 m = 2×10^3 mm

Area of part 1 $(A1)$ = 1000 mm^2

Area of part 2 $(A2)$ = 2000 mm^2

Pull $(P) = 70\ kN = 70 \times 10^3$ N

Modulus of elasticity $(E) = 200\ Gpa = 200 \times 10^3$ N/mm^2

Total strain energy stored in the non-uniform bar

We know that stress in the first part,

$$\sigma_1 = \frac{P}{A_1} = \frac{70 \times 10^3}{1000} = 70 \text{ N/mm}^2$$

and volume of the first part,

$$V_1 = (3 \times 10^3) \times 1000 = 3 \times 10^6 \text{ mm}^3$$

$\therefore$ Strain energy stored in the first part,

$$U_1 = \frac{\sigma_1^2}{2E} \times V_1 = \frac{(70)^2}{2 \times (200 \times 10^3)} \times (3 \times 10^6) = 36.75 \times 10^3 \text{ N-mm} \quad ...(i)$$

Similarly, stress in the second part,

$$\sigma_2 = \frac{P}{A_2} = \frac{70 \times 10^3}{2000} = 35 \text{ N/mm}^2$$

and volume of the second part,

$$V_2 = (2 \times 10^3) \times 2000 = 4 \times 10^6 \text{ mm}^3$$

$\therefore$ Strain energy stored in the second part,

$$U_2 = \frac{\sigma_2^2}{2E} \times V_2 = \frac{(35)^2}{2 \times (200 \times 10^3)} \times (4 \times 10^6) = 12.25 \times 10^3 \text{ N-mm} \quad ...(ii)$$

and total strain energy stored in the non-uniform bar,

$$U = U_1 + U_2 = (36.75 \times 10^3) + (12.25 \times 10^3) = 49 \times 10^3 \text{ N=mm} = 49 \text{ N-m} \quad \textbf{Ans.}$$

Total strain energy in the uniform bar

We know that total volume of the bar,

$$V = V_1 + V_2 = (3 \times 10^6) + (4 \times 10^6) = 7 \times 10^6 \text{ mm}^3$$

and cross-sectional area of the circular bar,

$$A = \frac{\text{Volume of the bar}}{\text{Length of the bar}} = \frac{7 \times 10^6}{5 \times 10^3} = 1400 \text{ mm}^2$$

$\therefore$ Stress in the bar

$$\sigma = \frac{70 \times 10^3}{1400} = 50 \text{ N/mm}^2$$

and strain energy storad in the uniform bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(50)^2}{2 \times (200 \times 10^3)} \times (7 \times 10^6) = 43.75 \times 10^3 \text{ N-mm}$$

$$= 43.75 \text{ N-m} \quad \textbf{Ans.}$$

Strain Energy Stored in a Body due to Shear Stress

Consider a cube *ABCD* of length *l* fixed at the bottom face *AB* as shown in Fig 8.5.

Let P = Force applied tangentially on the face *DC*,

τ = Shear stress

ϕ = Shear strain, and

N = Modulus of rigidity or shear modulus.

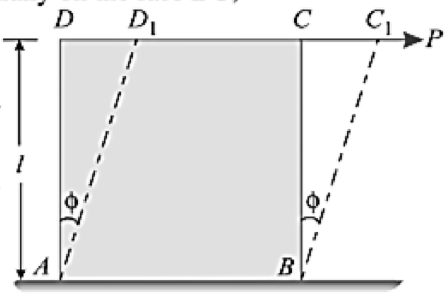


Fig. 8.5. Strain energy due to shear stress

If the force *P* is applied gradually then the average force is equal to *P*/2.

$\therefore$ Work done = Average force × Distance

$$= \frac{P}{2} \times DD_1$$

$$= \frac{1}{2} \times P \times AD \times \phi \quad ...(\because DD_1 = AD \times \phi)$$

$$= \frac{1}{2} \times \tau \times DC \times l \times AD \times \phi \quad ...(\because P = \tau \times DC \times l)$$

$$= \frac{1}{2} \times \tau \times \phi \times DC \times AD \times l$$

$$= \frac{1}{2} \text{ (stress} \times \text{strain} \times \text{volume)}$$

$$= \frac{1}{2} \times \tau \times \frac{\tau}{N} \times V \quad ...\left(\because \phi = \frac{\tau}{N}\right)$$

$$= \frac{\tau^2}{2N} \times V \qquad \text{...(where } V \text{ is the volume)}$$

Since energy stored is also equal to the work done, therefore energy stored,

$$U = \frac{\tau^2}{2N} \times V$$

We also know that modulus of resilience

$$= \text{Strain energy per unit volume}$$

$$= \frac{\tau^2}{2N}$$

Example *A rectangular body 500 mm long, 100 mm wide and 50 mm thick is subjected to a shear stress of 80 MPa. Determine the strain energy stored in the body. Take N = 85 GPa.*

Given:

Length of rectangular body (l) = *500 mm*
Width of rectangular body (b) = 100 mm
Thickness of rectangular body (t) = *50 mm*
Shear stress (t) = 80 MPa = 80 N/mm² and
modulus of rigidity (N) = *85 N/mm²*

We know that volume of the bar,

$$V = l.b.t = 500 \times 100 \times 50 = 2.5 \times 10^6 \text{ mm}^3$$

and strain energy stored in the body,

$$U = \frac{\tau^2}{2N} \times V = \frac{(80)^2}{2 \times (85 \times 10^3)} \times 2.5 \times 10^6 \text{ N-mm}$$

$$= 94.1 \times 10^3 \text{ N-mm} = 94.1 \text{ N-m} \qquad \textbf{Ans.}$$

Principal Stresses and Strains

At a time one type of stress, acting in one direction only. But the majority of engineering, component and structures are subjected to such loading conditions (or sometimes are of such shapes) that there exists a complex state of stresses; involving direct tensile and compressive stress as well as shear stress in various directions.

At any point in a strained material, there are three planes, mutually perpendicular to each other, which carry direct stresses only, and no shear stress. These three direct stresses one will be maximum, the other minimum, and the third and intermediate between the two. These particular planes, which have no shear stress, are known as ***principal planes.***

The magnitude of direct stress, across a principal plane, is known as **principal stress**. The determination of principal planes, and then principal stress is an important factor in the design of various structures and machine components.

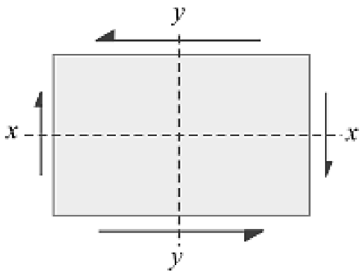
The following two methods for the determination of stresses on an oblique section of a strained body are important from the subject point of view: **1. Analytical method and 2. Graphical method.**

Analytical Method for the Stresses on an Oblique Section of a Body

The analytical method for the determination of stresses on an oblique section in the following cases, which are important from the subject point of view:

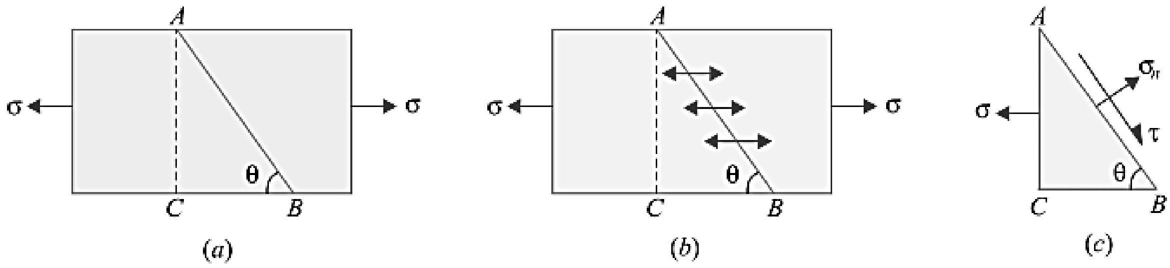
1. A body subjected to a direct stress in one plane.
2. A body subjected to direct stresses in two mutually perpendicular directions

In the element shown, the shear stress on the vertical faces (or *x-x axis) is taken as positive, whereas the shear stress on the* horizontal faces (or *y-y axis) is taken as negative*



Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a direct tensile stress along *x-x axis as shown. Now let us consider an oblique section AB* inclined with the *x-x axis.*



Let σ = Tensile stress across the face AC and

θ = Angle, which the oblique section AB makes with BC *i.e.* with the x-x axis in the clockwise direction.

First of all, consider the equilibrium of an element or wedge ABC whose free body diagram is shown in fig 7.2 (*b*) and (*c*). We know that the horizontal force acting on the face AC,

$$P = \sigma \,.\, AC \ (\leftarrow)$$

Resolving the force perpendicular or normal to the section AB

$$P_n = P \sin\theta = \sigma \,.\, AC \sin\theta \qquad \text{....}(i)$$

and now resolving the force tangential to the section AB,

$$P_t = P\cos\theta = \sigma \,.\, AC \cos\theta \qquad \text{....}(ii)$$

We know that normal stress across the section AB*,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma AC \sin\theta}{AB} = \frac{\sigma . AC \sin\theta}{\frac{AC}{\sin\theta}} = \sigma \sin^2\theta$$

$$= \frac{\sigma}{2}(1 - \cos 2\theta) = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta \qquad \text{...}(iii)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma . AC \cos\theta}{AB} = \frac{\sigma . AC \cos\theta}{\frac{AC}{\sin\theta}} = \sigma \sin\theta \cos\theta$$

$$= \frac{\sigma}{2}\sin 2\theta \qquad \text{...}(iv)$$

The face *AC will carry* the maximum direct stress. Similarly, the shear stress across the section *AB will be maximum when* $\sin 2\theta = 1$ or $2\theta = 90°$ or $270°$. Or in other words, the shear stress will be maximum on the planes inclined at 45° and 135° with the line of action of the tensile stress. Therefore maximum shear stress when θ is equal to 45°,

$$\tau_{max} = \frac{\sigma}{2}\sin 90° = \frac{\sigma}{2}\times 1 = \frac{\sigma}{2}$$

and maximum shear stress, when θ is equal to 135°,

$$\tau_{max} = -\frac{\sigma}{2}\sin 270° = -\frac{\sigma}{2}(-1) = \frac{\sigma}{2}$$

It is thus obvious that the magnitudes of maximum shear stress is half of the tensile stress. Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

NOTE : The planes of maximum and minimum normal stresses (*i.e. principal planes) may also be found out by* equating the shear stress to zero. This happens as the normal stress is either maximum or minimum on a plane having zero shear stress. Now equating the shear stress to zero, $\sigma \sin\theta \cos\theta = 0$

Example *Two wooden pieces 100 mm × 100 mm in cross-section are joined together along a line AB as shown. Find the maximum force (P), which can be applied if the shear stress along the joint AB is 1.3 MPa.*

Given:

Section = 100 mm × 100 mm ;
Angle made by section with the
Direction of tensile stress (θ) = 60° and
Permissible shear stress (t) = 1.3 MPa = 1.3 N/mm^2

Let σ = Safe tensile stress in the member

We know that cross- sectional area of the wooden member,

$$A = 100 \times 100 = 10\,000 \text{ mm}^2$$

and shear stress (τ),

$$1.3 = \frac{\sigma}{2}\sin 2\theta = \frac{\sigma}{2}\sin(2\times 60°) = \frac{\sigma}{2}\sin 120° = \frac{\sigma}{2}\times 0.866$$

$$= 0.433\,\sigma$$

or

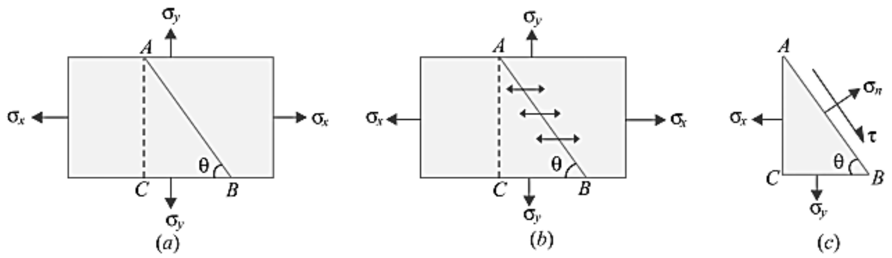
$$\sigma = \frac{1.3}{0.433} = 3.0 \text{ N/mm}^2$$

∴ Maximum axial force, which can be applied,

$$P = \sigma.A = 3.0 \times 10\,000 = 30\,000 \text{ N} = 30 \text{ kN} \qquad \textbf{Ans.}$$

Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along *x-x and y-y axes as shown*. Now let us consider an oblique section *AB inclined with x-x axis*



Let σ_x = Tensile stress along x-x axis (also termed as major tensile stress),

σ_y = Tensile stress along y-y axis (also termed as a minor tensile stress), and

θ = Angle which the oblique section AB makes with x-x axis in the clockwise direction.

First of all, consider the equilibrium of the wedge ABC. We know that horizontal force acting on the face AC (or x-x axis).

$$P_x = \sigma_x . AC \ (\leftarrow)$$

and vertical force acting on the face BC (or y-y axis),

$$P_y = \sigma_y . BC \ (\downarrow)$$

Resolving the forces perpendicular or normal to the section AB,

$$P_n = P_x \sin\theta + P_y \cos\theta = \sigma_x . AC \sin\theta + \sigma_y . BC \cos\theta \qquad ...(i)$$

and now resolving the forces tangential to the section AB,

$$P_t = P_x \cos\theta - P_y \sin\theta = \sigma_x . AC \cos\theta - \sigma_y . BC \sin\theta \qquad ....(ii)$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta + \sigma_y \ BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} = \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin^2\theta + \sigma_y . \cos^2\theta = \frac{\sigma_x}{2}(1 - \cos 2\theta) + \frac{\sigma_y}{2}(1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2}\cos 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad ...(iii)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta - \sigma_y . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{AB} - \frac{\sigma_y . BC \sin\theta}{AB} = \frac{\sigma_x . AC \cos\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x . \sin\theta\cos\theta - \sigma_y \sin\theta\cos\theta$$

$$= (\sigma_x - \sigma_y)\sin\theta\cos\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta \qquad ...(iv)$$

It will be interesting to know from equation (*iii*) the shear stress across the section AB will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $\theta = 45°$. Therefore maximum shear stress,

$$\tau_{max} = \frac{\sigma_x - \sigma_y}{2}$$

Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

Example: *The stresses at point of a machine component are 150 MPa and 50 Mpa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress. Also find the magnitude of the maximum shear stress in the component.*

Given: Tensile stress along *x-x axis* (s_x) = 150 MPa;

Tensile stress along *y-y axis* (s_y) = 50 MPa and

Angle made by the plane with the major tensile stress (θ) = 55°.

Normal stress on the inclined plane

We know that the normal stress on the inclined plane

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$= \frac{150+50}{2} - \frac{150-50}{2} \cos (2 \times 55°) \text{ MPa}$$

$$= 100 - 50 \cos 110° = 100 - 50 (-0.342) \text{ MPa}$$

$$= 10 + 17.1 = 117.1 \text{ MPa} \quad \textbf{Ans.}$$

Shear stress on the inclined plane

We know that the shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \frac{150-50}{2} \times \sin (2 \times 55°) \text{ MPa}$$

$$= 50 \sin 110° = 50 \times 0.9397 = 47 \text{ MPa} \quad \textbf{Ans.}$$

Resultant stress on the inclined plane

We know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(117.1)^2 + (47.0)^2} = 126.2 \text{ MPa} \quad \textbf{Ans.}$$

Maximum shear stress in the component

We also know that the magnitude of the maximum shear stress in the component,

$$\tau_{max} = \pm \frac{\sigma_x - \sigma_y}{2} = \pm \frac{150-50}{2} = \pm 50 \text{ MPa} \quad \textbf{Ans.}$$

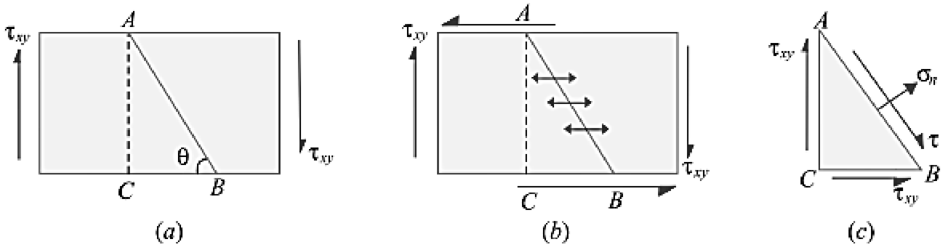
Stresses on an Oblique Section of a Body Subjected to a Simple Shear stress

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a positive (*i.e., clockwise) shear stress along x-x axis as shown. Now let us consider an* oblique section *AB inclined with x-x axis on which we are required to find out the stresses as shown.*

Let τ_{xy} = Positive (*i.e., clockwise) shear stress along x-x axis, and*

θ = Angle , which the oblique section *AB makes with x-x axis in* the anticlockwise direction.

First of all, consider the equilibrium of the wedge *ABC. We know that as per the principle of* simple shear, the face *BC, of the wedge will be subjected to an anticlockwise shear stress equal to* τ_{xy} as shown. *We know that vertical force acting on the face AC,*



$$P_1 = \tau_{xy} \cdot AC\ (\uparrow)$$

and horizontal force acting on the face BC,

$$P_2 = \tau_{xy} \,.\, BC \,(\rightarrow)$$

Resolving the forces perpendicular or normal to the AB,

$$P_n = P_1 \cos\theta + P_2 \sin\theta = \tau_{xy} \,.\, AC \cos\theta + \tau_{xy} \,.\, BC \sin\theta$$

and now resolving the forces tangential to the section AB,

$$P_t = P_2 \sin\theta - P_1 \cos\theta = \tau_{xy} \,.\, BC \sin\theta - \tau_{xy} \,.\, AC \cos\theta$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\tau_{xy} . AC \cos\theta + \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\tau_{xy} . AC \cos\theta}{AB} + \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \tau_{xy} \,.\, \sin\theta\cos\theta + \tau_{xy} \,.\, \sin\theta\cos\theta$$

$$= 2\,\tau_{xy} \,.\, \sin\theta\cos\theta = \tau_{xy} \,.\, \sin 2\theta$$

and shear stress (*i.e.* tangential stress) across the section AB

$$\tau = \frac{P_t}{AB} = \frac{\tau_{xy} \,.\, BC \sin\theta - \tau_{xy} \,.\, AC \cos\theta}{AB}$$

$$= \frac{\tau_{xy} . BC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} = \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\cos\theta}}$$

$$= \tau_{xy} \sin^2\theta - \tau_{xy} \cos^2\theta$$

$$= \frac{\tau_{xy}}{2}(1 - \cos 2\theta) - \frac{\tau_{xy}}{2}(1 + \cos 2\theta)$$

$$= \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta$$

$$= -\tau_{xy} \cos 2\theta$$ ...(Minus sign means that normal stress is opposite to that across AC)

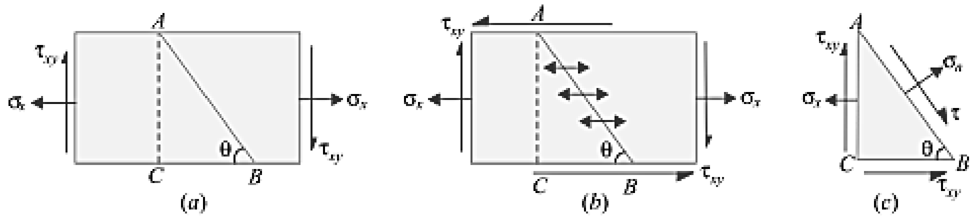
Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*

$$-\tau_{xy} \cos 2\theta = 0$$

The above equation is possible only if $2\theta = 90°$ or $270°$ (because $\cos 90°$ or $\cos 270° = 0$) or in other words, $\theta = 45°$ or $135°$.

Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane and Accompanied by a Simple Shear Stress

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a tensile stress along x-x axis accompanied by a positive (i.e. clockwise) shear stress along x-x axis as shown. Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses as shown in the figure.



Let σ_x = Tensile stress along x-x axis,

τ_{xy} = Positive (*i.e.* clockwise) shear stress along x-x axis, and

θ = Angle which the oblique section AB makes with x-x axis in clockwise direction.

First of all, consider the equilibrium of the wedge ABC. We know that as per the principle of simple shear, the face BC of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in Fig. 7.7 (*b*). We know that horizontal force acting on the face AC,

$$P_x = \sigma_x \,.\, AC\ (\leftarrow) \qquad ...(i)$$

Similarly, vertical force acting on the face AC,

$$P_y = \tau_{xy} \,.\, AC\ (\uparrow) \qquad ...(ii)$$

and horizontal force acting on the face BC,

$$P = \tau_{xy} \,.\, BC\ (\rightarrow) \qquad ...(iii)$$

Resolving the forces perpendicular to the section AB,

$$\begin{aligned} P_n &= P_x \sin\theta - P_y \cos\theta - P\sin\theta \\ &= \sigma_x \,.\, AC \sin\theta - \tau_{xy} \,.\, AC\cos\theta - \tau_{xy} \,.\, BC\sin\theta \end{aligned}$$

and now resolving the forces tangential to the section AB,

$$\begin{aligned} P_t &= P_x \cos\theta + P_y \sin\theta - P\cos\theta \\ &= \sigma_x \,.\, AC\cos\theta + \tau_{xy} \,.\, AC\sin\theta - \tau_{xy} \,.\, BC\cos\theta \end{aligned}$$

We know that normal stress across the section AB,

$$\begin{aligned} \sigma_n &= \frac{P_n}{AB} = \frac{\sigma_x . AC\sin\theta - \tau_{xy} . AC\cos\theta - \tau_{xy} . BC\sin\theta}{AB} \\ &= \frac{\sigma_x . AC\sin\theta}{AB} - \frac{\tau_{xy} . AC\cos\theta}{AB} - \frac{\tau_{xy} . BC\sin\theta}{AB} \\ &= \frac{\sigma_x . AC\sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC\cos\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC\sin\theta}{\frac{BC}{\cos\theta}} \\ &= \sigma_x \,.\, \sin^2\theta - \tau_{xy}\sin\theta\cos\theta - \tau_{xy}\sin\theta\cos\theta \\ &= \frac{\sigma_x}{2}(1-\cos 2\theta) - 2\tau_{xy}\sin\theta\cos\theta \\ &= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad ...(iv) \end{aligned}$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\begin{aligned} \tau &= \frac{P_t}{AB} = \frac{\sigma_x . AC\cos\theta + \tau_{xy} . AC\sin\theta - \tau_{xy} . BC\cos\theta}{AB} \\ &= \frac{\sigma_x . AC\cos\theta}{AB} + \frac{\tau_{xy}\ AC\sin\theta}{AB} - \frac{\tau_{xy} . BC\cos\theta}{AB} \\ &= \frac{\sigma_x . AC\cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy}\ AC\sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC\cos\theta}{\frac{BC}{\cos\theta}} \\ &= \sigma_x\sin\theta\cos\theta + \tau_{xy}\sin^2\theta - \tau_{xy}\cos^2\theta \\ &= \frac{\sigma_x}{2}\sin 2\theta + \frac{\tau_{xy}}{2}(1-\cos 2\theta) - \frac{\tau_{xy}}{2}(1+\cos 2\theta) \\ &= \frac{\sigma_x}{2}\sin 2\theta + \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta \\ &= \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad ...(v) \end{aligned}$$

Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*, from the above equation, we find that the shear stress on any plane is a function of σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_x and τ_{xy} are constant and thus the shear stress varies with the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

$$\therefore \quad \frac{\sigma_x}{2}\sin 2\theta_p - \tau_{xy}\cos 2\theta_p = 0 \quad \text{or} \quad \frac{\sigma_x}{2}\sin 2\theta_p = \tau_{xy}\cos 2\theta_p$$

$$\therefore \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x}$$

From the above equation we find that the following two cases satisfy this condition as shown in Fig 7.8 (*a*) and (*b*)

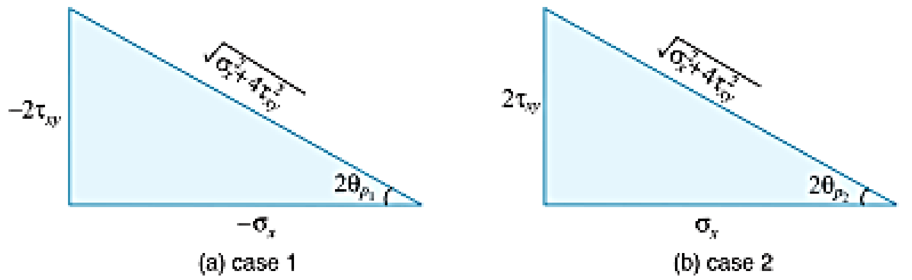


Fig. 7.8

Thus we find that these are two principal planes at right angles to each other, their inclination with *x-x* axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{\sigma_x^2+4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_1} = \frac{-\sigma_x}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{\sigma_x^2+4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_2} = \frac{\sigma_x}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (*iv*).

$$\text{Maximum principal stress,} \quad \sigma_{p_1} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\times\frac{-\sigma_x}{\sqrt{\sigma_x^2+4\tau_{xy}^2}} - \tau_{xy}\times\frac{-2\tau_{xy}}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2}{2\sqrt{\sigma_x^2+4\tau_{xy}^2}} + \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2+4\tau_{xy}^2}{2\sqrt{\sigma_x^2+4\tau_{xy}^2}} = \frac{\sigma_x}{2} + \frac{\sqrt{\sigma_x^2+4\tau_{xy}^2}}{2}$$

$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x^2}{2}\right)+\tau_{xy}^2}$$

Minimum principal stress, $\sigma_{p_2} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \times \frac{\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \tau_{xy} \times \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}{2} = \frac{\sigma_x}{2} - \frac{\sigma_x^2 + 4\tau_{xy}^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Example *An element in a strained body is subjected to a tensile stress of 150 MPa and a shear stress of 50 MPa tending to rotate the element in an anticlockwise direction. Find (i) the magnitude of the normal and shear stresses on a section inclined at 40° with the tensile stress; and (ii) the magnitude and direction of maximum shear stress that can exist on the element.*

Given:

Tensile stress along horizontal *x-x* axis (σx) = 150 MPa

Shear stress (τxy) – 50 MPa (Minus sign due to anticlockwise) and angle made by section with the tensile stress (θ) = 40°.

Normal and Shear stress on the inclined section

We know that magnitude of the normal stress on the section

$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{150}{2} - \frac{150}{2}\cos(2 \times 40°) - (-50)\sin(2 \times 40°) \text{ MPa}$$

$$= 75 - (75 \times 0.1736) + (50 \times 0.9848) \text{ MPa}$$

$$= 75 - 13.02 + 49.24 = 111.22 \text{ MPa} \qquad \textbf{Ans.}$$

and shear stress on the section

$$\tau = \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{150}{2}\sin(2 \times 40°) - (-50)\cos(2 \times 40°) \text{ MPa}$$

$$= (75 \times 0.9848) + (50 \times 0.1736) \text{ MPa}$$

$$= 73.86 + 8.68 = 82.54 \text{ MPa} \qquad \textbf{Ans.}$$

(*ii*) *Maximum shear stress and its direction that can exist on the element*

We know that magnitude of the maximum shear stress.

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{150}{2}\right)^2 + (-50)^2} = \pm 90.14 \text{ MPa} \quad \textbf{Ans.}$$

Let θ_s = Angle which plane of maximum shear stress makes with *x-x* axis.

We know that, $\tan 2\theta_s = \frac{\sigma_x}{2\tau_{xy}} = \frac{150}{2 \times 50} = 1.5$ or $2\theta_s = 56.3°$

$\therefore$ θ_s = 28.15° or 118.15° **Ans.**

Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress

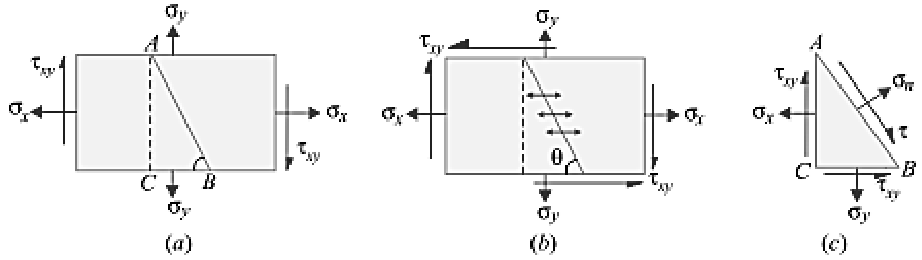


Fig. 7.9

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to tensile stresses along *x-x* and *y-y* axes and accompanied by a positive (*i.e.*, clockwise) shear stress along *x-x* axis as shown in Fig.7.9 (*b*). Now let us consider an oblique section *AB* inclined with *x-x* axis on which we are required to find out the stresses as shown in the figure.

Let σ_x = Tensile stress along *x-x* axis,

σ_y = Tensile stress along *y-y* axis,

τ_{xy} = Positive (*i.e.* clockwise) shear stress along *x-x* axis, and

θ = Angle, which the oblique section *AB* makes with *x-x* axis in an anticlockwise direction.

First of all, consider the equilibrium of the wedge *ABC*. We know that as per the principle of simple shear, the face *BC* of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in Fig. 7.9 (*b*). We know that horizontal force acting on the face *AC*,

$$P_1 = \sigma_x . AC\ (\leftarrow) \quad \text{... }(i)$$

and vertical force acting on the face *AC*,

$$P_2 = \tau_{xy} . AC\ (\uparrow) \quad \text{...}(ii)$$

Similarly, vertical force acting on the face *BC*,

$$P_3 = \sigma_y . BC\ (\downarrow) \quad \text{...}(iii)$$

and horizontal force on the face *BC*,

$$P_4 = \tau_{xy} . BC\ (\rightarrow) \quad \text{...}(iv)$$

Now resolving the forces perpendicular to the section *AB*,

$$P_n = P_1 \sin\theta - P_2 \cos\theta + P_3 \cos\theta - P_4 \sin\theta$$
$$= \sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta + \sigma_y . BC \cos\theta - \tau_{xy} . BC \sin\theta$$

and now resolving the forces tangential to *AB*,

$$P_t = P_1 \cos\theta + P_2 \sin\theta - P_3 \sin\theta - P_4 \cos\theta$$
$$= \sigma_x . AC \cos\theta + \tau_{xy} . AC \sin\theta - \sigma_y . BC \sin\theta - \tau_{xy} . BC \cos\theta$$

Normal Stress (across the inclined section *AB*)

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta + \sigma_y . BC \cos\theta - \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} - \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\frac{BC}{\cos\theta}} - \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x . \sin^2\theta - \tau_{xy} \sin\theta \cos\theta + \sigma_y . \cos^2\theta - \tau_{xy} . \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} (1 - \cos 2\theta) + \frac{\sigma_y}{2} (1 + \cos 2\theta) - 2\,\tau_{xy} . \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

or

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad \text{...}(v)$$

Shear Stress or Tangential Stress (across inclined the section AB)

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta + \tau_{xy} . AC \sin\theta - \sigma_y . BC \sin\theta - \tau_{xy}\, BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{AB} + \frac{\tau_{xy} . AC \sin\theta}{AB} - \frac{\sigma_y . BC \sin\theta}{AB} - \frac{\tau_{xy}\, BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy} . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y . BC \sin\theta}{\frac{BC}{\cos\theta}} - \frac{\tau_{xy} . BC \cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin\theta \cos\theta + \tau_{xy} \sin^2\theta - \sigma_y \sin\theta\cos\theta - \tau_{xy}\cos^2\theta$$

$$= (\sigma_x - \sigma_y)\sin\theta\cos\theta + \frac{\tau_{xy}}{2}(1 - \cos 2\theta) - \frac{\tau_{xy}}{2}(1 + \cos 2\theta)$$

or

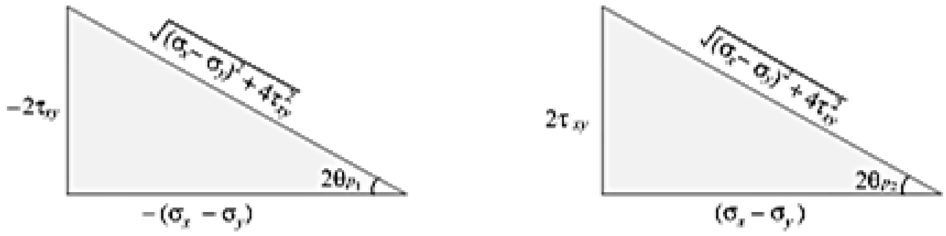
$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad ...(vi)$$

Now the planes of maximum and minimum normal stresses (*i.e.* principal planes) may be found out by equating the shear stress to zero. From the above equations, we find that the shear stress to any plane is a function of σ_y, σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_y, σ_x and τ_{xy} are constant and thus the shear stress varies in the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

$$\therefore \quad \frac{\sigma_x - \sigma_y}{2}\sin 2\theta_p - \tau_{xy}\cos 2\theta_p = 0$$

or $\quad \dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta_p = \tau_{xy}\cos 2\theta_p \quad$ or $\quad \tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$

From the above equation, we find that the following two cases satisfy this condition as shown in Fig 7.10 (a) and (b).



(a) Case 1 (b) Case 2

Thus we find that there are two principal planes, at right angles to each other, their inclinations with x-x axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_1} = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_2} = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (v).

Maximum Principal Stress,

$$\sigma_{p_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2} \times \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y^2) + 4\tau_{xy}^2}}\right) - \left(\tau_{xy} \times \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\right)$$

or $$\sigma_{p_1} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum Principal Stress

$$\sigma_{p2} = \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x+\sigma_y}{2} - \left(\frac{\sigma_x-\sigma_y}{2} \times \frac{(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right) - \left(\tau_{xy} \times \frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right)$$

$$= \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}{2\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} = \frac{\sigma_x-\sigma_y}{2} - \frac{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}{2}$$

or $$\sigma_{p_2} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Example *A point is subjected to a tensile stress of 250 MPa in the horizontal direction and another tensile stress of 100 MPa in the vertical direction. The point is also subjected to a simple shear stress of 25 MPa, such that when it is associated with the major tensile stress, it tends to rotate the element in the clockwise direction. What is the magnitude of the normal and shear stresses on a section inclined at an angle of 20° with the major tensile stress?*

Given:

Tensile stress in horizontal *x-x* direction (σx) = 250 MPa

Tensile stress in vertical *y-y* direction (σy) = 100 MPa

Shear stress (τxy) = 25 MPa and angle made by section with the major tensile stress (θ) = 20°.

Magnitude of normal stress

We know that magnitude of normal stress,

$$\begin{aligned}\sigma_n &= \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ &= \frac{250+100}{2} - \frac{250-100}{2}\cos(2\times 20°) - 25\sin(2\times 20°) \\ &= 175 - 75\cos 40° - 25\sin 40° \text{ MPa} \\ &= 175 - (75\times 0.766) - (25\times 0.6428) \text{ MPa} \\ &= 175 - 57.45 - 16.07 = 101.48 \text{ MPa} \quad \textbf{Ans.}\end{aligned}$$

Magnitude of shear stress

We also know that magnitude of shear stress,

$$\begin{aligned}\tau &= \frac{\sigma_x-\sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \\ &= \frac{250-100}{2}\sin(2\times 20°) - 25\cos(2\times 20°) \\ &= 75\sin 40° - 25\cos 40° \text{ MPa} \\ &= (75\times 0.6428) - (25\times 0.766) \text{ MPa} \\ &= 48.21 - 19.15 = 29.06 \text{ MPa} \quad \textbf{Ans.}\end{aligned}$$

Graphical Method for the Stresses on an Oblique Section of a Body

The Mohr's Circle of Stresses for the following cases:

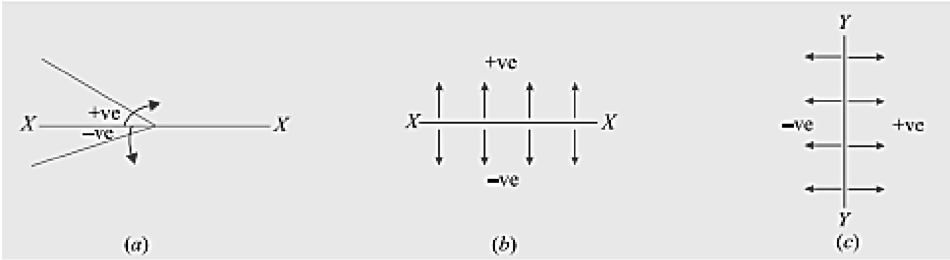
1. A body subjected to a direct stress in one plane.

2. A body subjected to direct stresses in two mutually perpendicular directions.

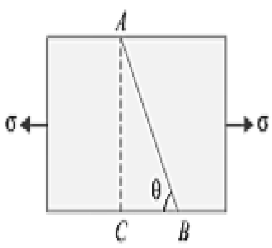
3. A body subjected to a simple shear stress.

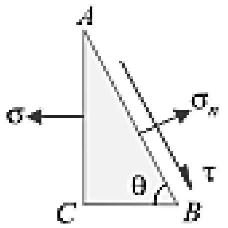
4. A body subjected to a direct stress in one plane accompanied by a simple shear stress.

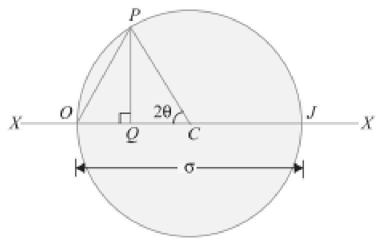
5. A body subjected to direct stresses in two mutually perpendicular directions accompanied by a simple shear stress.



Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane







Proof

From the geometry of the Mohr's Circle of Stresses, we find that,

$$OC = CJ = CP = \sigma/2 \qquad \text{... (Radius of the circle)}$$

∴ Normal Stress.

$$\sigma_n = OQ = OC - QC = \left(\frac{\sigma}{2}\right) - \left(\frac{\sigma}{2}\right)\cos 2\theta \qquad \text{...(Same as in Art. 7.7)}$$

and shear stress

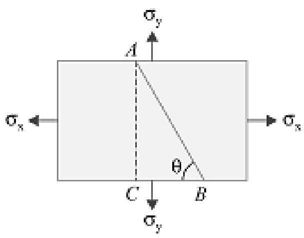
$$\tau = QP = CP \sin 2\theta = \frac{\sigma}{2}\sin 2\theta \qquad \text{...(Same as in Art. 7.7)}$$

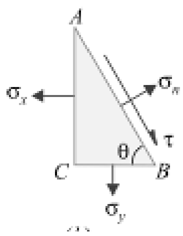
We also find that maximum shear stress will be equal to the radius of the Mohr's Circle of Stresses *i.e.*, $\frac{\sigma}{2}$. It will happen when 2θ is equal to 90° or 270° *i.e.*, θ is equal to 45° or 135°.

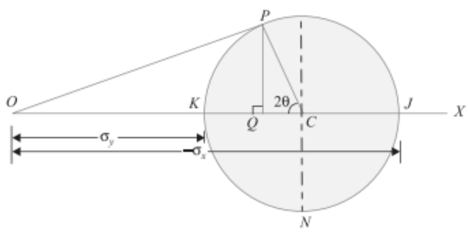
However when $\theta = 45°$ then the shear stress is equal to $\frac{\sigma}{2}$.

And when $\theta = 135°$ then the shear stress is equal to $-\frac{\sigma}{2}$.

Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Direction







Proof

From the geometry of the Mohr's Circle of Stresses, we find that

$$KC = CJ = CP = \frac{\sigma_x - \sigma_y}{2}$$

or

$$OC = OK + KC = \sigma_y + \frac{\sigma_x - \sigma_y}{2} = \frac{2\sigma_y + \sigma_x - \sigma_y}{2} = \frac{\sigma_x + \sigma_y}{2}$$

∴ Normal stress,

$$\sigma_n = OQ = OC - CQ = \frac{\sigma_x - \sigma_y}{2} - CP \cos 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta \qquad \text{...(Same as Art. 7.8)}$$

and shear stress,

$$\tau = QP = CP \sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} \sin 2\theta \qquad \text{...(Same as Art. 7.8)}$$

We also find that the maximum shear stress will be equal to the radius of the Mohr's Circle of Stresses. *i.e.*, $\frac{\sigma_x - \sigma_y}{2}$. It will happen when 2θ is equal to 90° or 270° *i.e.*, when θ is equal to 45° or 135°.

However when $\theta = 45°$ then the shear stress is equal to $\frac{\sigma_x - \sigma_y}{2}$

And when $\theta = 135°$ then the shear stress will be equal to $\frac{-(\sigma_x - \sigma_y)}{2}$ or $\frac{\sigma_y - \sigma_x}{2}$.

Example The *stresses at a point of a machine component are 150 MPa and 50 MPa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress. Also find the magnitude of the maximum shear stresses in the component.*

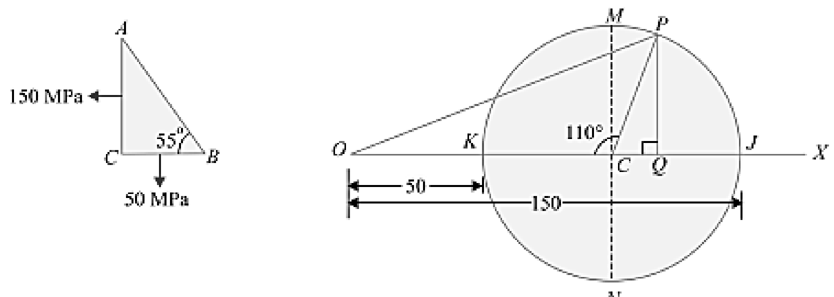
Given:

Tensile stress along horizontal *x-x* axis (s*x*) = 150 MPa

Tensile stress along vertical *y-y* axis (s*y*) = 50 MPa and

Angle made by the plane with the axis of major tensile stress (θ) = 55°.

The given stresses on the planes *AC* and *BC* in the machine component are shown.

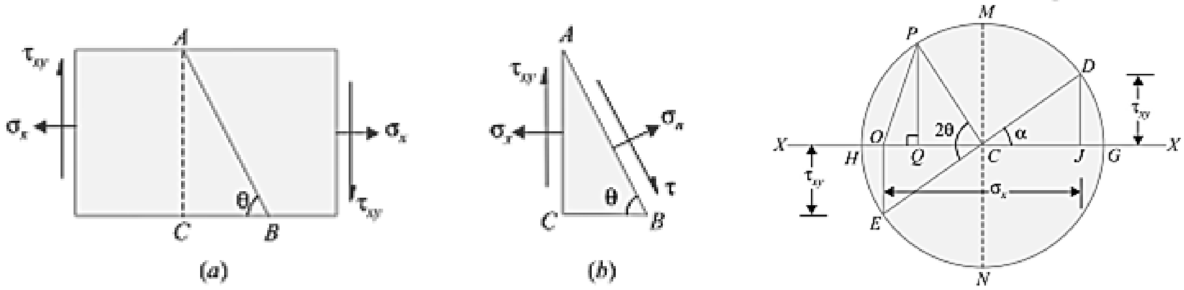


(*a*) (*b*)

1. First of all, take some suitable point *O* and draw a horizontal line *OX*.
2. Cut off *OJ* and *OK* equal to the tensile stresses σ_x and σ_y respectively (*i.e.* 150 MPa and 50 MPa) to some suitable scale towards right. The point *J* represents the stress system on the plane *AC* and the point *K* represents the stress system on the plane *BC*. Bisect *KJ* at *C*.
3. Now with *C* as centre and radius equal to *CJ* or *CK* draw the Mohr's Circle of Stresses.
4. Now through *C* draw two lines *CM* and *CN* at right angles to the line *OX* meeting the circle at *M* and *N*. Also through *C* draw a line *CP* making an angle of 2 × 55° = 110° with *CK* in clockwise direction meeting the circle at *P*. The point *P* represents the stress system on the plane *AB*.
5. Through *P*, draw *PQ* perpendicular to the line *OX*. Join *OP*.

   By measurement, we find that the normal stress (σ_n) = *OQ* = 117.1 MPa ; Shear stress (τ) = *QP* = 47.0 MPa ; Resultant stress (σ_R) = *OP* = 126.2 MPa and maximum shear stress (τ_{max}) = *CM* = ± 50 MPa **Ans.**

Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to a Direct Stresses in One Plane Accompanied by a Simple Shear Stress



Proof

From the geometry of the Mohr's Circle of Stresses, we find that

$$OC = \frac{\sigma_x}{2}$$

and radius of the circle,

$$R = EC = CD = CP = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Now in the right angled triangle DCJ,

$$\sin\alpha = \frac{DJ}{CD} = \frac{\tau_{xy}}{R} \qquad \text{and} \qquad \cos\alpha = \frac{JC}{CD} = \frac{\sigma_x}{2} \times \frac{1}{R} = \frac{\sigma_x}{2R}$$

and similarly in right angled triangle CPQ,

$$\angle PCQ = (2\theta - \alpha)$$

$$\therefore \quad CQ = CP\cos(2\theta - \alpha) = R\,[\cos(2\theta - \alpha)]$$

$$= R\,[\cos\alpha\cos 2\theta + \sin\alpha\sin 2\theta]$$

$$= R\cos\alpha\cos 2\theta + R\sin\alpha\sin 2\theta$$

$$= R \times \frac{\sigma_x}{2R}\cos 2\theta + R \times \frac{\tau_{xy}}{R}\sin 2\theta$$

$$= \frac{\sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

We know that normal stress across the section AB,

$$\sigma_n = OQ = OC - CQ = \frac{\sigma_x}{2} - \left(\frac{\sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right)$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad \text{...(Same as in Art. 7.10)}$$

and shear stress,

$$\tau = QP = CP\sin(2\theta - \alpha) = R\sin(2\theta - \alpha)$$

$$= R\,(\cos\alpha\sin 2\theta - \sin\alpha\cos 2\theta)$$

$$= R\cos\alpha\sin 2\theta - R\sin\alpha\cos 2\theta$$

$$= R \times \frac{\sigma_x}{2R}\sin 2\theta - R \times \frac{\tau_{xy}}{2}\cos 2\theta$$

$$= \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad \text{...(Same as in Art. 7.10)}$$

We also know that maximum stress,

$$\sigma_{max} = OG = OC + CG = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

and minimum stress

$$\sigma_{min} = OH = OC - CH = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

We also find that the maximum shear stress will be equal to the radius of the Mohr's circle of stresses *i.e.*, $\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$. It will happen when $(2\theta - \alpha)$ is equal to 90° or 270°.

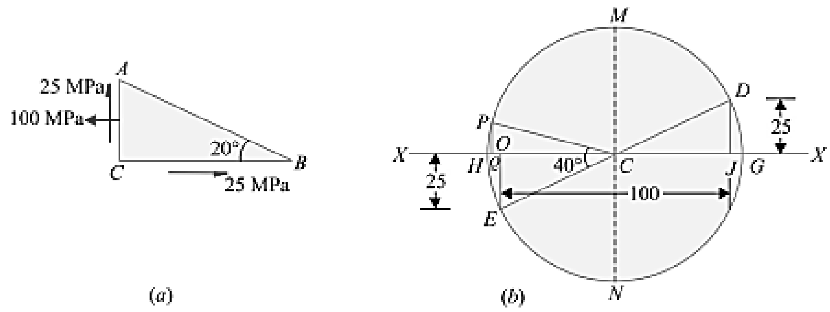
However when $(2\theta - \alpha)$ is equal to 90° then the shear stress is equal to $+\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$.

And when $(2\theta - \alpha) = 270°$ then the shear stress is equal to $-\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$.

Example *A plane element in a body is subjected to a tensile stress of 100 MPa accompanied by a clockwise shear stress of 25 MPa. Find (i) the normal and shear stress on a plane inclined at an angle of 20° with the tensile stress; and (ii) the maximum shear stress on the plane.*

Given:

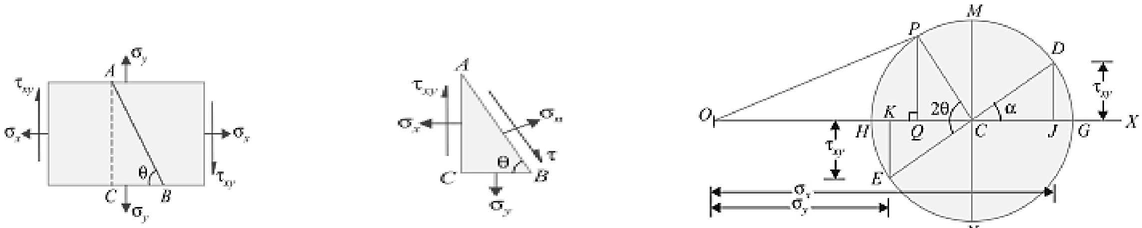
Tensile stress along horizontal *x-x* axis (σx) = 100 MPa
Shear stress (τxy) = 25 MPa and
angle made by plane with tensile stress (θ) = 20°



1. First of all, take some suitable point *O*, and through it draw a horizontal line *XOX*.
2. Cut off *OJ* equal to the tensile stress on the plane *AC* (*i.e.*, 100 MPa) to some suitable scale towards right.
3. Now erect a perpendicular at *J* above the line *X-X* and cut off *JD* equal to the positive shear stress on the plane *BC* (*i.e.*, 25 MPa) to the scale. The point *D* represents the stress system on the plane *AC*. Similarly erect a perpendicular at *O* below the line *X-X* and cut off *OE* equal to the negative shear stress on the plane *BC* (*i.e.*, 25 MPa) to the scale. The point *E* represents the stress system on the plane *BC*. Join *DE* and bisect it at *C*.
4. Now with *C* as centre and radius equal to *CD* or *CE* draw the Mohr's Circle of Stresses.
5. Now through *C*, draw two lines *CM* and *CN* at right angle to the line *OX* meeting the circle at *M* and *N*. Also through *C*, draw a line *CP* making an angle of 2 × 20° = 40° with *CE* in clockwise direction meeting the circle at *P*. The point *P* represents the stress system on the section *AB*.
6. Through *P*, draw *PQ* perpendicular to the line *OX*.

   By measurement, we find that the normal stress (σ_n) = *OQ* = 4.4 MPa (compression) ; Shear stress (τ) = *QP* = 13.0 MPa and maximum shear stress (τ_{max}) = *CM* = 55.9 MPa **Ans.**

Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress



Proof

From the geometry of the Mohr's Circle of Stresses, we find that

$$OC = \frac{\sigma_x + \sigma_y}{2}$$

and radius of the circle

$$R = EC = CD = CP = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Now in the right angled triangle *DCJ*

$$\sin\alpha = \frac{JD}{DC} = \frac{\tau_{xy}}{R} \quad \text{and} \quad \cos\alpha = \frac{JD}{DC} = \frac{\sigma_x - \sigma_y}{2} \times \frac{1}{R} = \frac{\sigma_x - \sigma_y}{2R}$$

Similarly in right angled triangle *CPQ*

$$\therefore \quad \angle PCQ = (2\theta - \alpha)$$

$$\begin{aligned} CQ &= CP \cos 2\theta - \alpha \\ &= R[\cos(2\theta - \alpha)] \\ &= R[\cos\alpha\cos 2\theta + \sin\alpha \sin 2\theta] \\ &= R\cos\alpha\cos 2\theta + R\sin\alpha\sin 2\theta \\ &= R \times \frac{\sigma_x - \sigma_y}{2R}\cos 2\theta + R \times \frac{\tau_{xy}}{R}\sin 2\theta \\ &= \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \end{aligned}$$

Normal Stress (across the inclined section *AB*)

$$\sigma_n = OQ = OC - CQ$$

or

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \quad \text{...(Same as in Art. 7.11)}$$

Shear Stress or Tangential Stress (across the inclined section *AB*)

$$\begin{aligned} \tau &= QP = CP\sin[(2\theta - \alpha)] = R\sin(2\theta - \alpha) \\ &= R(\cos\alpha\sin 2\theta - \sin\alpha\cos 2\theta) \\ &= R\cos\alpha\sin 2\theta - R\sin\alpha\cos 2\theta \\ &= R \times \frac{\sigma_x - \sigma_y}{2R}\sin 2\theta - R \times \frac{\tau_{xy}}{R}\cos 2\theta \end{aligned}$$

or

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \quad \text{...(Same as in Art. 7.11)}$$

Maximum Principal Stress

$$\sigma_{max} = OG = OC + CG = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum Principal Stress

$$\sigma_{min} = OH = OC - CH = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

We also find the maximum shear stress will be equal to the radius of the Mohr's circle of Stresses.

i.e., $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$. It will happen when $(2\theta - \alpha)$ is equal to 90° or 270°.

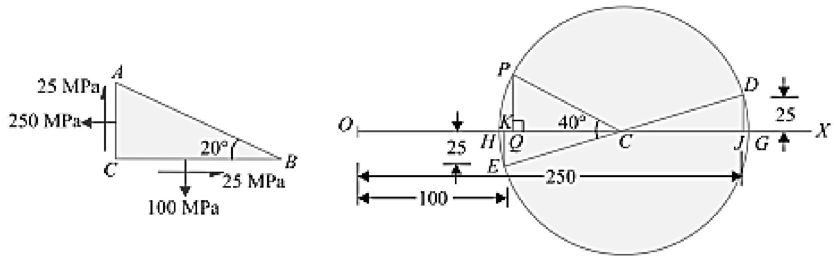
However when $(2\theta - \alpha) = 90°$ then the shear stress is equal to $+\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

And when $(2\theta - \alpha) = 270°$ then the shear stress is equal to $-\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

Example *A point is subjected to a tensile stress of 250 MPa in the horizontal direction and another tensile stress of 100 MPa in the vertical direction. The point is also subjected to a simple shear stress of 25 MPa, such that when it is associated with the major tensile stress, it tends to rotate the element in the clockwise direction. What is the magnitude of the normal and shear stresses inclined on a section at an angle of 20° with the major tensile stress?*

Given:

Tensile stress in horizontal direction (σx) = 250 MPa
Tensile stress in vertical direction (σy) = 100 MPa
Shear stress (τ) = 25 MPa and
angle made by section with major tensile stress (θ) = 20°



(*a*) (*b*)

The given stresses on the face *AC* of the point alongwith a tensile stress on the plane *BC* and a complimentary shear stress on the plane *BC* are shown in Fig 7.27 (*a*). Now draw the Mohr's Circle of Stresses as shown in Fig. 7.27 (*b*) and as discussed below :

1. First of all, take some suitable point *O*, and through it draw a horizontal line *OX*.
2. Cut off *OJ* and *OK* equal to the tensile stresses σ_x and σ_y respectively (*i.e.*, 250 MPa and 100 MPa) to some suitable scale towards right.
3. Now erect a perpendicular at *J* above the line *OX* and cut off *JD* equal to the positive shear stress on the plane *AC* (*i.e.*, 25 MPa) to the scale. The point *D* represents the stress system on the plane *AC*. Similarly, erect a perpendicular at *K* below the *OX* and cut off *KE* equal to the negative shear stress on the plane *BC* (*i.e.*, 25 MPa) to the scale. The point *E* represents the stress system on the plane *BC*. Join *DE* and bisect it at *C*.
4. Now with *C* as centre and radius equal to *CD* or *CE* draw the Mohr's Circle of Stresses.
5. Now through *C* draw a line *CP* making an angle of 2 × 20° = 440° with *CE* in clockwise direction meeting the circle at *P*. The point *P* represents the stress system on the section to *AB*.
6. Through *P*, draw *PQ* perpendicular to the line *OX*.

   By measurement, we find that the normal stress, (σ_x) = *OQ* = 101.5 MPa and shear stress τ = *QP* = 29.0 MPa **Ans.**

SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

SMEA1306 SOLID AND FLUID MECHANICS

UNIT 2 BENDING MOMENT IN BEAMS AND TORSION OF SHAFTS

BEAM

Classification of Beams:

Beams are classified on the basis of their geometry and the manner in which they are supported.

Cantilever Beam: A beam which is supported on the fixed support is termed as a cantilever beam: Now let us understand the meaning of fixed support. Such a support is obtained by building a beam into a brick wall, casting it into concrete or welding the end of the beam. Such a support provides both the translational and rotational constrainment to the beam, therefore the reaction as well as the moments appears, as shown in the figure below

Simply Supported Beam: The beams are said to be simply supported if their supports creates only the translational constraints.

Some times the translational movement may be allowed in one direction with the help of rollers and can be represented like this

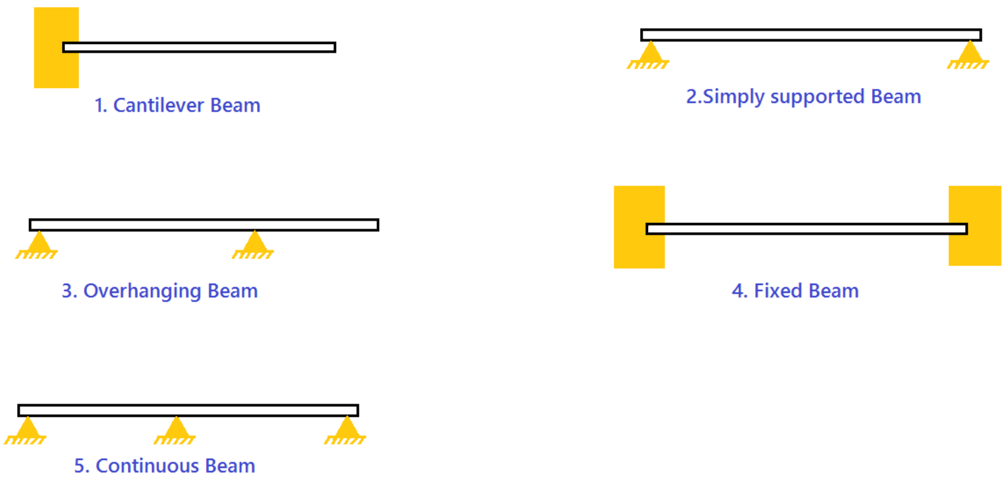


Fig 2.1 Classification of Beams

Supports and Loads

Types of beams: Supports and Loads

In many engineering structures members are required to resist forces that are applied laterally or transversely to their axes. These type of members are ter ed as beam. There are various ways to define the beams such as

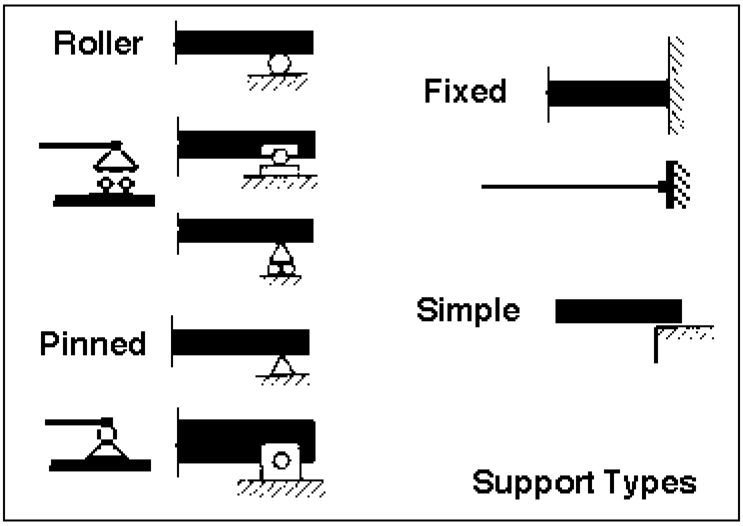


Fig 2.2 Types of Supports

Definition I: A beam is a laterally loaded member, wh se cr ss-sectional dimensions are small as compared to its length.
Definition II: A beam is nothing simply bar which is subjected to forces or couples that lie in a plane containing the longitudnal axis of the bar. The forces are understood to act perpendicular to the longitudnal axis of the b r.

Definition III: A bar working under bending is generally termed as a beam.

2.3 Materials for Beam:

The beams may be made from several usable engineering materials such commonly among them are as follows:

Metal

Wood

Concrete

Plastic

Types of loads acting on beams:

A beam is normally horizontal where as he ex ern l loads acting on the beams is generally in the vertical directions. In order to study he behaviors of beams under flexural loads. It becomes pertinent that one must be f mili r with the various types of loads acting on the beams as well as their physical manifestations.

A. Concentrated Load: It is a kind of oad which is considered to act at a point. By this we mean that the length of beam over wh ch the force acts is so small in comparison to its total length that one can model the force as though applied at a point in two dimensional view of beam. Here in this case, force or load may be made to act on a beam by a hanger or though other means

B. Distributed Load: The distributed load is a kind of load which is made to spread over a entire span of beam or over particular portion of the beam in some specific manner

In the above figure, the rate of loading „q' is a function of x i.e. span of the beam, hence this is a non uniformly distributed load.

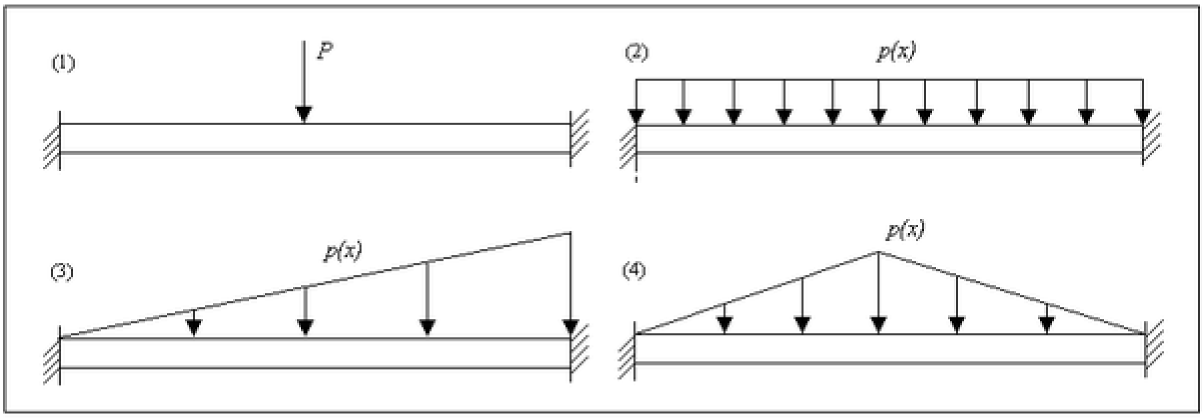


Fig 2.3 Types of Loads

The rate of loading „q' over the length of the beam may be uniform over the entire span of beam, then we cell this as a uniformly distributed load (U.D.L). The U.D.L may be represented in either of the way on the beams

some times the load acting on the beams may be the uniformly varying as in the case of dams or on inclind wall of a vessel containing liquid, then this may be represented on the beam as below:

The U.D.L can be easily realized by making idealization of the ware house load, where the

Shear force and Bending Moment in beams

Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms

Now let us consider the beam as shown in fig 1(a) which is supporting the loads P1, P2, P3 and is simply supported at two points creating the reactions R1 and R2respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is „F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, a ting downwards. This forces „F' is as a shear force. The shearing for ceat any x-section of a beam represents the tendency for the portion of the beam to one side of the section to lide or hear laterally relative to the other portion.

Therefore, now we are in a position to define he shear force „F' to as follows:

At any x-section of a beam, the she r force „F' is the algebraic sum of all the lateral components of the forces cting on either si e of the x-section.

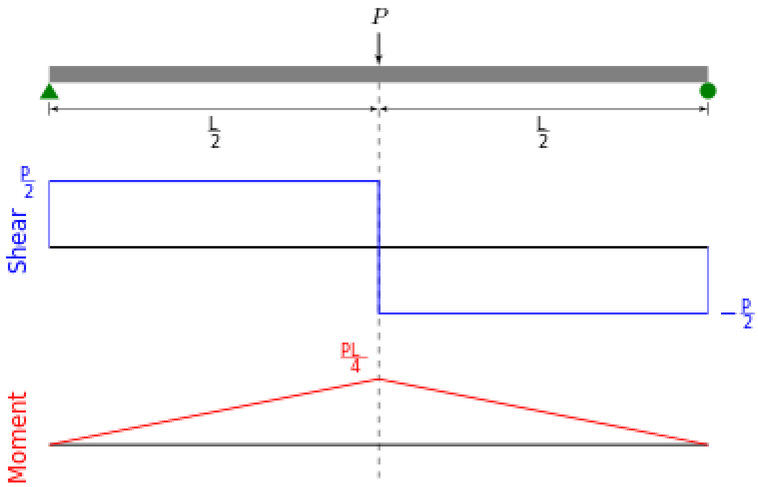


Fig 2.4 Shear force and Bending Moment Diagram

Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further.

Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force „F' varies along the length of beam. If x dentotes the length f the beam, then F is function x i.e. F(x).
Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment „M' varies along the length of the beam Again M is a function x i.e. M(x).

Construction of shear force and bending moment diagrams:

A shear force diagram an be constructed from the loading diagram of the beam. In order to draw this, first the reactions must be determined always. Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.
It may also be observed that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. It may also further observe that dm/dx= F therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero. In order to check the validity of the bending moment diagram, the terminal conditions for the moment must be satisfied. If the end is free or pinned, the computed sum must be equal to zero. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction. These conditions must always be satisfied.

Cantilever beams – problems

Cantilever with a point load at the free end:

$M_X = - w.x$

W.K.T $\quad M = EI.\frac{d^2}{dx^2}$

$EI.\frac{d^2y}{dx^2} = - w.x$

on integrating we get

$EI.\frac{dy}{dx} = \frac{-wx^2}{2} + {}_1$

Integrating again

$EI.y = - \frac{wx^3}{6} + {}_1x + c_2$

Boundary conditions

i) when x = L , slope dy/dx = 0

ii) when x = L, deflection y = 0

Applying the first B C to eqn (1)

$0 = - \frac{l^2}{} + c_1 \qquad c_1 = \frac{wl^2}{}$

Applying the second B.C to eqn (2)

$0 = \frac{- \; l^3}{6} + c_1l + c_2$

$C_2 = \frac{-wl3}{3}$

Sub c_1,c_2 values in slope eqn we get

$EI.\frac{dy}{dx} = \frac{-wx^2}{2} + wl^2 \frac{}{2}$

Max. slope eqn can be obtained by x = 0

$EI.\frac{dy}{dx} = 0 + \frac{wl^2}{2} \qquad ?_B = \frac{wl^2}{2EI}$

Sub c_1,c_2 values in deflection eqn we get

$EI.y = -wx^3 + wl^2.x – wl^3$

$\quad 2 \quad 2 \quad 6$

Max. deflection can be obtained by x = 0

$EI.y_B = 0 – 0 – \frac{wl^3}{3} \qquad y_B = \frac{wl^3}{3EI}$

Cantilever with a point load at a distance of 'a' from free end:

$$?_B = ?_c = \frac{w(l-a)^2}{2EI}$$

$$y_B = \frac{w(l-a)^3}{3EI} + \frac{w(l-a)^2}{2EI}.a \qquad y_c = \frac{w(l-a)^3}{3EI}$$

When the load acts at mid span:

$$y_B = \frac{5wl^3}{48EI}$$

Cantilever with UDL:

$$?_B = \frac{wl^3}{2EI} \qquad y_B = \frac{wl^4}{8EI}$$

Cantilever with UDL from f xed end:

$$?_B = ?_c = \frac{w(l-a)^3}{6EI}$$

$$y_B = \frac{w(l-a)^4}{8EI} + \frac{(l-a)^3}{6EI}.a \qquad y_c = \frac{w(l-a)^4}{8EI}$$

When a = l/2 ie. UDL acting half of the length

$$y_B = \frac{7\ \ l^3}{384EI}$$

Cantilever with UDL from free end:

$$?_B = \frac{wl^3}{6EI} - \frac{w(l-a)^3}{6EI}$$

$$y_B = \frac{wl^4}{8EI} - \frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI}.a$$

Cantilever with UVL:

$?_B = \frac{wl^3}{24EI}$ $y_B = \frac{wl^4}{30EI}$

A cantilever of length carries a concentrated load ‘W' at its free end.

Draw shear force and bending moment.

Solution:

At a section a distancecivildatasxfromfreeendconsiderthe forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward

direction and therefore negative

Taking moments about the section gives (obviously to the left of the section)

M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention)

so that the maximum bending moment occurs at the fixed end i.e. M = -W l

Simplysupported beam -problems

Simply supported beam subjected to central load (i.e. load acting at the mid-way)

By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.

.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

If e consider another section Y-Y which is beyond l/2 then

for all values greater = l/2

SSB with central point load:

$?_B = -wl^3$ $y_B = wl^4$

16EI 30EI

SSB with eccentric point load:

$?_B = \frac{-wab}{6EIL}(b+2a)$ $y_{max} = \frac{-wa}{9v3\ EIL}(b^2 + 2ab)^{3/2}$

If a >b then

$y_{max} = \frac{-wb}{9v3\ EIL}(a^2 + 2ab)^{3/2}$

SSB with UDL:

$?_B = \frac{wl^3}{24EI}$ $y_B = \frac{5wl^4}{384EI}$

Overhanging beams - problems

In the problem given below, the intensity of loading varies from q1 kN/m at one end to the q2 kN/m at the other end.This problem c n be treated by considering a U.d.i of intensity q1 kN/m over the entire span and a uniformly varying load of 0 to (q2- q1)kN/m over the entire span and then super impose teh two loadings.

Point of Contraflexure:

Consider the loaded beam a shown be ow along with the shear force and Bending moment diagrams for It may be observed that this case, the bending moment diagram is completely positive so that the cur ature of the beam varies along its length, but it is always concave upwards or sagging.However f we consider again a loaded beam as shown below along with the S.F and B.M diagrams, then

It may be noticed that for the beam loaded in this case,

The bending moment diagram is partly positive and partly negative.If we plot the deflected shape of the beam just below the bending moment

This diagram shows that L.H.S of the beam „sags' while the R.H.S of the beam „hogs'

The point C on the beam where the curvature changes from sagging to hogging is a point of contraflexure.

OR

It corresponds to a point where the bending moment changes the sign, hence in order to find the point of contraflexures obviously the B.M would change its sign when it cuts the X-axis

therefore to get the points of contraflexure equate the bending moment equation equal to zero.The fibre stress is zero at such sections

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam

is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final

position of the sections, are still straight lines, they then subtend some angle q.

Consider now fiber AB in the material, at adistance y from the N.A, when the beam bends this will stretch to A'B'

Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance „y' from the N.A, is given by the expression

Now the termis the property of the material and is called as a sec nd moment of area of the cross-section and is denoted by a symbol I.

Therefore M/I = sigma/y = E/R

This equation is known as the Bending Theory Equation. The above proof has involved

the assumption of pure bending without any she r force being present. Therefore this termed

as the pure bending equation. This equa ion gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Stress variation along the length and in the beam section Bending Stress and Deflection Equation

In this section, we consider the case of pure bending; i.e., where only bending stresses exist as a result of applied bend ng moments. To develop the theory, we will take the phenomenological approach to de elop what is called the "**Euler-Bernoulli theory of beam bending**." Geometry: Cons der long slender straight beam of length L and cross-sectional area A. We assume the beam is prismatic or nearly so. The length dimension is large compared to the dimensions of the cross-section. While the cross-section may be any shape, we will assume that it is symmetric about the y axis

Loading: For our purposes, we will consider shear forces or distributed loads that are applied in the y direction only (on the surface of the beam) and moments about the z-axis. We have consider examples of such loading in ENGR 211 previously and some examples are shown belo :

Kinematic Observations: In order to obtain a “feel” for the kinematics (deformation) of a beam subjected to pure bending loads, it is informative to conduct an experiment. Consider a rectangular lines have been scribed on the beam’s surface, which are parallel to the top and bottom surfaces (and thus parallel to a centroidally placed x-axis along the length of the beam). Lines are also scribed around the circumference of the beam so that they are perpendicular to the longitudinals (these circumferential lines form flat planes as shown). The longitudinal and circumferential lines form a square grid on the surface. The beam is now bent by moments at each end as shown in the lower photograph. After loading, we note

that the top line has stretched and the bottom line has shortened (implies that there is strain *exx*). If measured carefully, we see that the longitudinal line at the center has not changed length (implies that *exx* = 0 at *y* = 0). The longitudinal lines now appear to form concentric circular lines.

We also note that the vertical lines originally perpendicular to the longitudinal lines remain straight

and perpendicular to the longitudinal lines. If measured carefully, we will see that the vertical lines remain approximately the same length (implies *eyy* = 0). Each of the vertical lines (as well as the planes they form) has rotated and, if extended downward, they will pass through a common point that forms the center of the concentric l ngitudinal lines (with some radius *?*). The flat planes originally normal to the longitudinal axis remain essentially flat planes and remain normal to the deformed longitudinal lines. The squares on the surface are

now quadrilaterals and each appears to have tension (or compression) stress in the longitudinal direction (since the horizontal lines of a quare have changed length). However,

in pure bending we make the assumption that. If the *x*-axis is along the length of beam and the *y*-axis is normal to the beam, this suggests th t we have an axial normal stress *sxx* that is tension above the *x*-axis and compression below the *y*-axis. The remaining normal stresses *syy* and *szz* will generally be negligible for pure bending about the *z*-axis. For pure bending, all shear stresses are assumed to be zero. Consequently, for pure bending, the stress matrix reduces to zero.

Effect of shape of beam section on stress induced

CIRCULAR SECTION :

For a circular x-section, the polar moment of inertia may be computed in the following manner

Consider any circular strip of thickness dr located a radius 'r'.

Than the area of the circular strip would be dA = 2pr. dr

Thus

Parallel Axis Theorem:

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the area times the square of the distance between the axes.

If „ZZ' is any axis in the plane of cross-section and „XX' is a parallel axis through the centroid G, of the cross-section, then

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed as below :

Consider the rectangular beam cross-section as shown above and an element of area **dA** , thickness **dy** , breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

Similarly, the second moment of area of the rectangular secti n about an axis through the lower edge of the section would be found using the same pr cedure but with integral limits of **0** to **D** .
Therefore

These standards formulas prove very convenient in the determination of INA for build up sections which can be conveniently divided in o rect ngles. For instance if we just want to find out the Moment of Inertia of an I - sec ion, hen we can use the above relation.

Let us consider few examples to determaine the sheer stress distribution in a given X-sections

Rectangular x-section:

Consider a rectangular x-section of dimension b and d

A is the area of the x-sect on cut off by line parallel to the neutral axis. is the distance of the centroid of A from the neutral axis

This shows that there is a parabolic distribution of shear stress with y.

The maximum value of shear stress would obviously beat the location y = 0.

Therefore the shear stress distribution is shown as below.

It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.

I - section :

Consider an I - section of the dimension shown below.

The shear stress distribution for any arbitrary shape is given as

Let us evaluate the quantity, thequantity for this case comprise the contribution due to flange area and web area

Flange area

Web Area

To get the maximum and minimum values of t substitute in the above relation.

y = 0 at N. A. And y = d/2 at the tip.

The maximum shear stress is at the neutral axis. i.e. for the ondition y = 0 at N. A. Hence, ...(2)

The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression

...........(3)

The distribution of shear stress may be rawn as below, which clearly indicates a parabolic distribution

Note: from the above distribut on we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2.Ob ously than this will have some constant value and than onwards this will have parabol c str bution.

In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is neglible however if we have the concrete analysis i.e. if e analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation.

This distribution is known as the "top – hat" distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

Shear stress distribution in beams of circular cross-section:

Let us find the shear stress distribution in beams of circular cross-section. In a beam of circular cross-section, the value of Z width depends on y.

Using the expression for the determination of shear stresses for any arbitrary shape or a

arbitrary section.

Where òy dA is the area moment of the shaded portion or the first moment of area.

Here in this case „dA' is to be found out using the Pythagoras theorem

The distribution of shear stresses is shown below, which indicates a parabolic distribution

Principal Stresses in Beams

It becomes clear that the bending stress in beam sx is not a principal stress, since at any distance y from the neutral axis; there is a shear stress t (r txy we are assuming a plane stress situation)

In general the state of stress at a distance y from the neutral axis will be as

follows. At some point „P' in thecivildatasbeam,thevalueofbendingtre es is

given as

After substituting the appropriate values in he bove expression we may get the inclination of the principal planes.

Illustrative examples: Let us study some illustrative examples, pertaining to determination of principal stresses in beam

1. Find the principal stress at a po nt A in a uniform rectangular beam 200 mm deep and 100 mm wide, simply supported at each end over a span of 3 m and carrying a uniformly distributed load of 15,000 N/m.

Solution: The reaction can be determined by

symmetry R1 = R2 = 22,500 N

consider any cross-section X-X located at a distance x from the left end.

Hence,

S. F at XX =22,500 – 15,000 x

B.M at XX = 22,500 x – 15,000 x (x/2) = 22,500 x – 15,000 . x2 / 2

Therefore,

S. F at X = 1 m = 7,500 N

B. M at X = 1 m = 15,000 N

Now substituting these values in the principal stress equation,

We get s1 = 11.27 MN/m2

s2 = - 0.025 MN/m2

Bending Of Composite or Flitched Beams

A composite beam is defined as the one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting t gether two timber joists and a reinforcing steel plate, then it is termed as a flitched beam.
The bending theory is valid when a constant value of Young's modulus applies across a section it cannot be used directly to solve the compo ite-beam problems where two different materials, and therefore different values of E, exi ts. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.

Consider, a beam as shown in figure in which a s eel plate is held centrally in an appropriate recess/pocket between two blocks of wood .Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength. i.e. the moment at any section must be the same in the equivalent section as in the original section so that the force at any given dy in the equivalent beam must be equal to that at the strip it replaces.

Hence to replace a steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'.

The equivalent section s then one of the same materials throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value the same point in the equivalent material as follows by utilizing the given relations.

Stress in steel = modular ratio x stress in equivalent wood

The above procedure of course is not limited to the two materials treated above but applies well for any material combination. The wood and steel flitched beam was nearly chosen as a just for the sake of convenience.

Assumption

In order to analyze the behavior of composite beams, we first make the assumption that the materials are bonded rigidly together so that there can be no relative axial movement between them. This means that all the assumptions, which were valid for homogenous

beams are valid except the one assumption that is no longer valid is that the Young's Modulus is the same throughout the beam.

The composite beams need not be made up of horizontal layers of materials as in the earlier example. For instance, a beam might have stiffening plates as shown in the figure below.

Again, the equivalent beam of the main beam material can be formed by scaling the breadth of the plate material in proportion to modular ratio. Bearing in mind that the strain at any level is same in both materials, the bending stresses in them are in proportion to the Young's modulus.

Shear stresses in beams

When a beam is subjected to non uniform bending, both bending moments, M, and shear forces, V, act on the cross section. The normal stresses, sx, associated with the bending
moments are obtained from the flexure formula. We will now consider the distribution of shear stresses, t, associated with the shear force, V Let us begin by examining a beam of rectangular cross section. We can reasonably a ume that the shear stresses t act parallel to

the shear force V. Let us also assume that the distribution of shear stresses is uniform across the width of the beam.

Solved Problems

<u>Problem 1</u>

A Beam of Total length 8m is freely supported at a left end & at a point 6m from left end. It carries 2 points floats of 15KN & 18KN. In which one is at the free end and another is 3m from the left support. Draw the shear force and bending moment diagram. Locate the point of contraflexture.

Solution :

To fine the support reactions:

Taking moment about A,

(Rc 6) –(18 3) –(10 8) = 0

6 Rc = 54+120

Rc = 174/6

Rc = 29 KN

RA + Rc = 18+15

RA + 29 = 33

RA = 4 KN

To fine Shear force:

Shear force at D = 15 KN

Shear force at C = 29 KN +15 = -14 KN

Shear force at B = -14+18 = 4 KN

Shear force at A = 4 KN

To find bending moment

Bending Moment at D = 0

Bending Moment at C = -(15 2) = - 30 KNm

Bending Moment at B = -(15 5)+(29 3)

Bending Moment at A = -(15 8) +(29 6)-(18 3)

= -120-54+174

= 0

Problem-2:

The cross section of the beam is shown is beam is cantiliver type &carries a UDL of 16KN/m. If the span of beam is 2.5m. Determine the maximum tension & Compressible stress in the beam.

solution: Section (1) :

Area (a_1) = l×b

= $500mm^2$

Section(2):

Area (a_2) = l×b

To find the centroid distance:

$y_1 = 35+\frac{10}{2} = 40mm$

$y_2 = \frac{35}{2} = 17.5mm$

$\bar{y} = \frac{a1y1+a2y2}{a_1+a_2} = \frac{500(40)+(525)(17.5)}{1025}$

$\bar{y} = 28.47mm$

To find the moment of inertia:

$$I = \left[\frac{b_1 d_1{}^3}{12} + a_1(\bar{y} - y_1)^2 + \frac{b_2 d_2{}^3}{12} + a_2(\bar{y} - y_2)^2\right]$$

$$= \left[\frac{50(10)^3}{12} + (500)(28.47 - 40)^2 + \frac{15(35)^3}{12} + (525) + (28.47 - 17.5)2\right.$$

$$= \left[\frac{50000}{12} + (6879.64) + \frac{15(35)^3}{12} + (525)(1203409)\right]$$

$$= 187409.8392mm^4$$

To find moment (M):

$M = 16\times 2.5\times \frac{2.5}{2}$

$= 50KNm$

w.k.t,

$$\frac{M}{I} = \frac{\sigma}{J}$$

$$\sigma = \frac{M \times \bar{y}}{I}$$

The maximum compressive bending stress is the topmost layer of the beam.

The distance from y to top layer is

$$\bar{y} = 45\text{-}28.47$$

$$= 16.53\text{mm}$$

Compressive stress $\sigma = \frac{50 \times (16.53)\,(mm^2)}{187409.83\,(mm^2)}$

$$= 4.410 \times 10^{-3}\text{mm}^{-2}.\ \text{KN}$$

to find the maximum tensile stress:

$$\frac{M}{I} = \frac{\sigma}{\bar{y}}$$

$$\sigma = \frac{M \times \bar{y}}{I} \qquad \bar{y} = 28.47$$

$$= \frac{50 \times 28.47}{187409.83} = 7.595 \times 10^{-3}\text{KN}$$

$$= 7.59 \times 10^{3}\ \text{KN/m}^2$$

Problem-3:

The cast iron bracket subjected to bending has a cross section of I-shaped with unequal flanges as shown. If the compressive force on the top of the flanges is not to exceed 17mega pa. What is the bending moment of the section can take if the section is subjected to a shear force of 90KN. Draw the shear stress distribution over the depth of the section.

Solution:

Area of section (1) = lb

= 250 50

Solution:

Area of section (1) = lb

= 250×50

= $12750mm^2$

Area of section (2) = lb

= 50×250 = $12500mm^2$

Area of section (3) = lb

= 150×50 = $7500mm^2$

To find centroid distance:

$y_1 = 50+250+\frac{50}{2} = 325mm$

$y_2 = \frac{250}{2}+50 = 175mm$

$y_3 = \frac{50}{2} = 25mm$

$$\bar{y} = \frac{a_1y_1+a_2y_2+a_3y_3}{a_1a_2a_3} = \frac{12500(325)+(12500)(175)+(7500)(25)}{12500+}$$

$$= \frac{6437500}{32500} = 199.076mm$$

To find moment of inertia:

$$I = \left[\frac{b_1 {d_1}^3}{12} + a_1(\bar{y} - y_1)^2 + \frac{b_2 {d_2}^3}{12} + a_2(\bar{y} - y_2)^2 + a_3\frac{b_3 {d_3}^3}{12} + a3(y-y3)2\right.$$

$$= \left[\frac{250(50)^3}{12} + (12500(199.08 - 325)^2 + \frac{50(250)^3}{12} + (12500(199.08 - 175)2+150(50)312\,7500(199.08-25)2\right.$$

$$= 2604166.667 + 198198080 + 65104166.67 + 7248080 + 1562500 + 22727848$$

$$= 501995840.7$$

$$I = 5.01\times10^8 mm^4.$$

W.K.T,

$$\frac{M}{I} = \frac{\sigma}{\bar{y}}$$

$$= \frac{\sigma}{\bar{y}} \times I$$

$$= \frac{17}{199.08}(5.01) \times 10^8$$

$$= 0.4278\times10^8$$

$$= 42.78\times10^8 Nmm$$

To find shear stress:

shear force (τ) at top of the

top flange = 0

shear force (τ) at bottom of the

bottom flange = 0

τ at the bottom of the top flanges

$$= \frac{FA.\bar{y}}{Ib}$$

where,

$$\bar{y} = 152 - \frac{50}{2} = 127mm$$

A-Area-12500, I-moment of inertia, B-breath

$$F = \frac{90\times10^3(12500)(127)}{5.01\times10^8(250)}$$

$$= 1.14071N/mm^2$$

τ at junction of flange and web:

$$\tau = \tau \times \frac{B}{T}$$

Where, B-breath, T-Thickness

$$= 1.14\times \frac{250}{50} = 5.7N/mm^2.$$

τ at the Neutral axis:

$$= \frac{FA.\bar{y}}{Ib} \qquad \bar{y} = 152 - \frac{50}{2} = 127$$

$$A\bar{y} = (\overset{(A)}{250 \times 50} \times \overbrace{127}^{\bar{y}}) + (\overset{(A)}{102 \times 50} \times \overbrace{51}^{\bar{y}})$$

$$= 1847600 = 1.8\times10^6 mm^3$$

$$b = 50.$$

τ at Neutral axis:

$$= \frac{(90\times10^3)(1.8\times10^6)}{(5.01\times10^8)(50)}$$

$$= 6.4670N/mm^2.$$

To find shear stress:

τ at the bottom of the bottom flange is 0.

τ at the top of the bottom flange

$$= \frac{FA.\bar{y}}{Ib}$$

$$= \frac{90\times10^3\,(7500)(173)}{5.01\times10^8(150)}$$

$$= 1.55\text{N/mm}^2.$$

τ at junction of flange & web:

$$= \tau \times \frac{B}{T}$$

$$= 1.55\times\frac{150}{50} = 4.65\text{N/mm}^2.$$

$$i_C = \frac{A1}{EI} = \frac{5}{20} = 0.25 \text{ radians}$$

$$i_D = \frac{A2}{EI} = \frac{5}{20} = 0.25 \text{ radians}$$

To find Centroid distance:

$$\bar{x}1 = 1/3 \times 1 = 0.33\text{m}$$

$$\bar{x}2 = 0.33\text{m}$$

$$y_C = \frac{A\bar{x}}{EI} = \frac{5(0.33)}{20} = 0.825\text{m}$$

Torsion

In solid mechanics, **torsion** is the twisting of an object due to an applied torque. In sections perpendicular to the torque axis, the resultant shear stress in this section is perpendicular to the radius.

For solid shafts of uniform circular cross-section or hollow circular shafts with constant wall thickness, the torsion relations are:

where: R is the outer radius of the shaft. e.m,ft.s

- t is the maximum shear stress at the ou er surface.
- *f* is the angle of twist in radians.
- *T* is the torque (N·m or ft·lbf).
- *l* is the length of the object the torque is being applied to or over.
- *G* is the shear modulus or more commonly the modulus of rigidity and is usually given in gigapascals (GPa), bf/in^2 (psi), or lbf/ft^2.
- *J* is the torsion constant for the section. It is identical to the polar moment of inertia for a round shaft or concentric tube only. For other shapes J must be determined by other means. For solid shafts the membrane analogy is useful, and for thin walled tubes of arbitrary shape the shear flow approximation is fairly good, if the section is not re-entrant. For thick walled tubes of arbitrary shape there is no simple solution, and finite element analysis (FEA) may be the best method.
- The product *GJ* is called the torsion.

Stepped shaft ,Twist and torsion stiffness – Compound shafts – Fixed and simply supported shafts

Shaft: The shafts are the machine elements which are used to transmit power in machines.

Twisting Moment: The twisting moment for any section along the bar / shaft is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section under consideration. The choice of the side in any case is of course arbitrary.

Shearing Strain: If a generator a ?? b is marked on the surface of the unloaded bar, then after the twisting moment 'T' has been applied this line moves to ab'. The angle ???'

measured in radians, between the final and original positions of the generators is defined as the shearing strain at the surface of the bar or shaft. The same definition will hold at any interior point of the bar.

Modulus of Elasticity in shear: The ratio of the shear stress to the shear strain is called the modulus of elasticity in shear OR Modulus of Rigidity and in represented by the symbol

Angle of Twist: If a shaft of length L is subjected to a c nstant twisting moment T along its length, than the angle ? through which one end of the bar will twist relative to the other is known is the angle of twist.

Despite the differences in the forms of loading, we that there are number of similarities between bending and torsion, including for example, a linear variation of stresses and strain with position.

In torsion the members are subjected to momen s (couples) in planes normal to their axes.

For the purpose of desiging a circular shaft to withstand a given torque, we must develop an equation giving the relation between twisting moment, maximum shear stress produced, and a quantity representing the size and shape of the cross-sectional area of the shaft.

Not all torsion problems, involve rotating machinery, however, for example some types of vehicle suspension system employ torsional springs. Indeed, even coil springs are really curved members in tors on shown in figure.

Many torque carrying engineering members are cylindrical in shape. Examples are drive shafts, bolts and screw drivers.

Simple Torsion Theory or Development of Torsion Formula : Here we are basically interested to derive an equation between the relevant parameters

Assumption:

(i) The materiel is homogenous of uniform elastic properties exists throughout the material.

(ii) The material is elastic, follows Hook's l w, with shear stress proportional to shear strain.

(iii) The stress does not exceed the elastic limit.

(iv) The circular section remains circular

(v) Cross section remain plane.

(vi) Cross section rotate as if r g .e. every diameter rotates through the same angle.

Consider now the sol d c rcular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the act on of this torque a radial line at the free end of the shaft twists through an angle , point A moves to B, and AB subtends an angle ' at the fixed end. This is then the angle of distortion of the shaft .e the shear strain.

Since angle in radius = arc / Radius

arc AB = R?

= L ? [since L and ? also constitute the

arc AB] Thus, ? = R? / L (1)

From the definition of Modulus of rigidity or Modulus of elasticity in shear.

T = applied external Torque, which is cons ant over Length L;

J = Polar moment of Inertia

[D = Outside diameter ; d = insi e iameter]

G = Modules of rigidity (or Modulus of elasticity in shear)

? = It is the angle of tw st n radians on length L.

Problem 1

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T0 at the shoulder as shown in the figure. Determine the angle of rotation ?0 of the shoulder section where T0 is applied ?

Solution: This is a statically indeterminate system because the shaft is built in at both ends. All that e can find from the statics is that the sum of two reactive torque TA and TB at the built ?? in ends of the shafts must be equal to the applied torque T0

Thus TA+ TB = T0 ------ (1)

[from static principles]

Where TA ,TB are the reactive torque at the built in ends A and B. wheeras T0 is the applied torque

From consideration of consistent deformation, we see that the angle of twist in each portion of the shaft must be same.

i.e ?a = ? b = ? 0

using the relation for angle of twist

N.B: Assuming modulus of rigidity G to be same for the two p rti ns

So the defines the ratio of TA and TB So by

solving (1) & (2) we get

Non Uniform Torsion: The pure torsion refers o torsion of a prismatic bar subjected to torques acting only at the ends. While the non uniform torsion differs from pure torsion in a sense that the bar / shaft need not to be prism tic and the applied torques may vary along the length.

Here the shaft is made up of two ifferent segments of different diameters and having torques applied at several cross sect ons. Each region of the bar between the applied loads between changes in cross sect on n pure torsion, hence the formula's derived earlier may be applied. Then form the internal torque, maximum shear stress and angle of rotation for each region can be calculated from the relation

The total angle to twist of one end of the bar with respect to the other is obtained by summation using the formula

If either the torque or the cross section changes continuously along the axis of the bar, then the ? (summation can be replaced by an integral sign (?). i.e We will have to consider a differential element.

After considering the differential element, we can write

Substituting the expressions for Tx and Jx at a distance x from the end of the bar, and then integrating between the limits 0 to L, find the value of angle of twist may be determined.

SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

SMEA1306 SOLID AND FLUID MECHANICS

UNIT 3 FLUID PROPERTIES – SMEA1306

1. FLUID PROPERTIES

Fluid Properties: Density - Specific Weight - Specific Gravity - Viscosity - Surface tension - Capillarity - compressibility. Fluid Statics: Hydrostatic Law - Pressure Variation in static fluid - Hydrostatic force on a submerged plane surface - Location of hydrostatic force. Manometers - Simple U tube and differential manometers - Buoyancy - Meta-centric height - determination of stability of floating bodies and submerged bodies.

Fluids*:* Substances capable of flowing are known as fluids. Flow is the continuous deformation of substances under the action of shear stresses.

Fluids have no definite shape of their own, but confirm to the shape of the containing vessel. Fluids include liquids and gases.

Fluid Mechanics:

Fluid mechanics is the branch of science that deals with the behavior of fluids at rest as well as in motion. Thus, it deals with the static, kinematics and dynamic aspects of fluids.

The study of fluids at rest is called fluid statics. The study of fluids in motion, where pressure forces are not considered, is called fluid kinematics and if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics.

1. Density (or) Mass Density:

Density or mass density of a fluid is defined as the ratio of the mass of the fluid to its volume. Thus, Mass per unit volume of a fluid is called density.

$$\boxed{Mass\ density,\ \rho = \frac{Mass\ of\ fluid}{Volume\ of\ fluid}}$$

S.I unit of density is kg/m^3. The value of density for water is 1000 kg/m^3

2. Specific weight (or) Weight Density (w):

Specific weight or weight density of a fluid is the ratio between the weights of a fluid to its volume.

$$Weight\ density = \frac{Weight\ of\ fluid}{Volume\ of\ fluid}$$

$$w = \frac{Mass\ of\ fluid\ X\ g}{Volume\ of\ fluid}$$

$$w = \rho g$$

S.I unit of specific weight is **N/m^3** .
The value of specific weight or weight density of water is **9810N/m^3 or 9.81 kN/m^3**.

3. Specific Volume (v):

Specific volume of a fluid is defined as the volume of a fluid occupied by unit mass.
Volume per unit mass of a fluid is called Specific volume.

$$\text{Specific volume} = \frac{\text{Volume of a fluid}}{\text{Mass of fluid}} = \frac{1}{\rho}$$

Thus specific volume is the reciprocal of mass density. S.I unit: m^3 /kg

4.Specific Gravity (s):

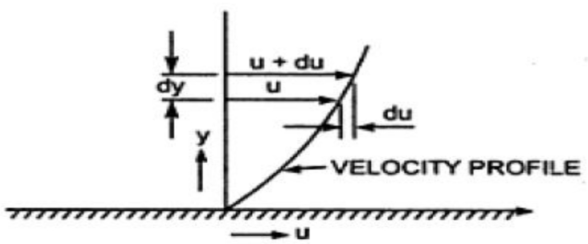
Specific gravity is defined as the ratio of the specific weight of a fluid to the specific weight of a standard fluid.

$$\text{Specific gravity} = \frac{\text{Specific weight or density of liquid}}{\text{Specific weight or density of water}}$$

Specific gravity of water=1
Specific gravity of mercury=13.6

5.Viscosity:

Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over adjacent layer of the fluid. When two layers of a fluid, at distance 'dy' apart, move one over the other at different velocities, say u and u+du as shown in figure. The viscosity together with relative velocity causes a shear stress acting between the fluid layers.The top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer.This shear stress is proportional to the rate of change of velocity with respect to y.



Velocity variation near a solid boundary.

Fig.3.1 Velocity distribution curve

6. Newtons law of viscosity: The shear stress between two layers is proportional to the rate of change of velocity with respect to y.

$$\tau \alpha \frac{du}{dy}$$

$$\tau = \mu \frac{du}{dy}$$

6. Compressibility: Compressibility is the reciprocal of the bulk modulus of elasticity, K, which is defined as the ratio of compressive stress to volumetric strain.

$$\text{Bulk modulus K} = \frac{\textit{Increase of pressure}}{\textit{Volumetric Strain}}$$

$$= \frac{dp}{\frac{-dV}{V}}$$

$$\text{Compressibility} = \frac{1}{K}$$

Cohesion is due to the force of attraction between molecules of same liquid

Adhesion is defined as the force of attraction between the molecules of two different liquids or between the molecules of the liquid and molecules of the solid boundary surface.

7. Surface tension: Surface tension is defined as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension.

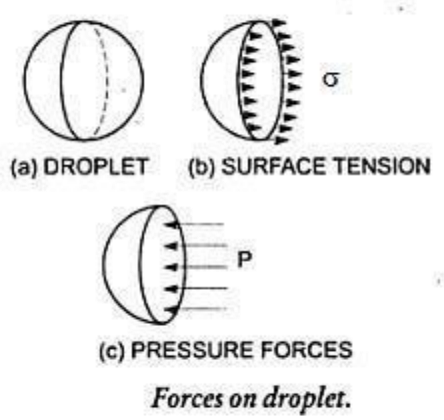


Fig.3.2 Forces on droplet

Surface Tension on Liquid Droplet:

Consider a small spherical droplet of a liquid of diameter ‘d’. On the entire surface of the droplet, the tensile force due to surface tension will be acting.
Let σ = Surface tension of the liquid,*p* = Pressure intensity inside the droplet (in excess of the outside pressure intensity~~) d~~= Dia. of droplet.Let the droplet is cut into two halves. The forces acting on one half will be i) Tensile force (FT)due to surface tension acting around the circumference of the cut portion as shown in fig. and this is equal to = σ x Circumference = σ x π d Pressure force (Fp) on the area C= p x (π/4) d^2 as shown in the

figure. These two forces are equal under equilibrium conditions. i.e

Surface Tension on a Hollow Bubble*:*

A hollow bubble like a soap bubble in air has two surfaces in contact with air, one inside and other outside. Thus two surfaces arc subjected to surface tension. In that case,

$$p \times \frac{\pi d^2}{4} = 2 \times (\sigma \times \pi d)$$

$$\text{Therefore, } p = \frac{8\sigma}{d}$$

8. Capillarity:

Capillarity is defined as a phenomenon of rise or fall of a liquid surface in a small tube relative to the adjacent general level of liquid when the tube is held vertically in the liquid. The rise of liquid surface is known as capillary rise while the fall of the liquid surface is known as capillary depression. It is expressed in terms of cm or mm of liquid. Its value depends upon the specific weight of the liquid, diameter of the tube and surface tension of the liquid.

Expression for Capillary Rise:

Consider a glass tube of small diameter'd' opened at both ends and is inserted in a liquid. The liquid will rise in the lube above the level of the liquid.

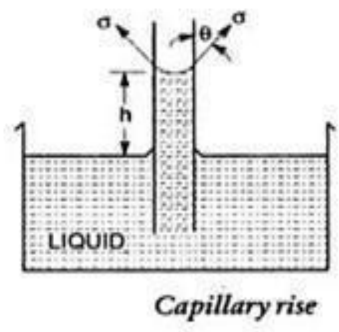


Fig.3.3 Capillary Rise

Under a state of equilibrium,

The weight of liquid of height h = Vertical component of surface tension force

(Area of tube x h) x ρ x g = σ x Circumference x cos θ

$$\frac{\pi d^2}{4} \times h \times \rho \times g = \sigma \times \pi d \times \cos\theta$$

$$h = \frac{4\sigma \cos\theta}{\rho \times g \times d} = \frac{4\sigma \cos\theta}{wd}$$

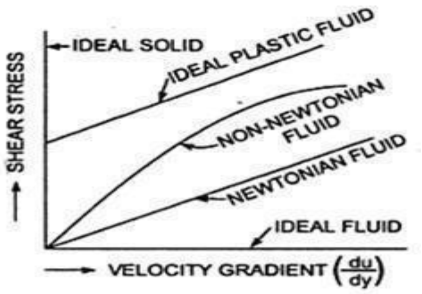
9. Vapour pressure:

Vapour pressure is the pressure of the vapor over a liquid which is confined in a closed vessel at equilibrium. Vapour pressure increases with temperature. All liquids exhibit this phenomenon**.**

10. Types of fluid

i. Ideal Fluid: A fluid, which is incompressible and is having no viscosity, is known as an ideal fluid.

ii. Real Fluid: A fluid, which possesses viscosity, is known as real fluid. All the fluids, are real fluids in actual practice.



Types of fluids.

Fig.3.4 Types of fluid

iii. Newtonian Fluid: A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or) velocity gradient, is known as a Newtonian fluid

iv. Non-Newtonian Fluid: A real fluid, in which the shear stress is not proportional to the rate of shear strain (or) velocity gradient, is known as a Non-Newtonian fluid.

v. Ideal Plastic Fluid: A fluid, in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain (or) velocity gradient, is known as ideal plastic fluid

Fluid pressure

Fluid pressure is the force exerted by the fluid per unit area. Fluid pressure is transmitted with equal intensity in all directions and acts normal to any plane.

$$\boxed{p = \frac{F}{A}}$$

S.I unit of fluid pressure are N/m^2 or Pa,
where 1 N/m^2 = 1 Pa.
Many other pressure units are commonly used:
1 bar = 105 N/m^2
1 atmosphere = 101325 N/m^2 = 101.325kN/m^2 = 1.01325 bar= 760mm of mercury = 10.336m of water
Pressure Head: The pressure intensity exerted at the base of a column of homogenous fluid of a given height in metres.
Atmospheric Pressure: The pressure at the surface of the earth exerted by the head of air above the surface
Gauge Pressure: The pressure measured by a pressure gauge above or below atmospheric pressure

Vacuum pressure: The gauge pressure less than atmospheric is called Vacuum pressure or negative pressure

Absolute Pressure: The pressure measured above absolute zero or vacuum.

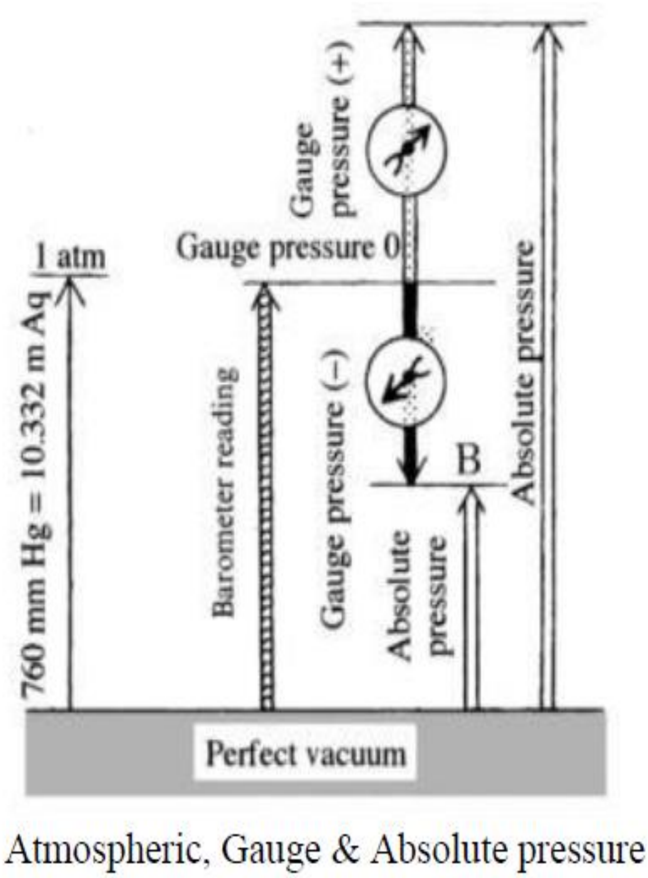


Fig.3.5 Barometer, Atmospheric, Gauge and Absolute Pressure

Fluid Pressure

Fluid pressure is the force exerted by the fluid per unit area.

Fluid pressure or Intensity of pressure or pressure, = Fluids exert pressure on surfaces with which they are in contact.

Fluid pressure is transmitted with equal intensity in all directions and acts normal to any plane. In the same horizontal plane the pressure intensities in a liquid are equal.

Hydrostatic law

The **hydrostatic law** is a principle that identifies the amount of pressure exerted at a specific point in a given area of fluid.

It states that, "The rate of increase of pressure in the vertically downward direction, at a point in a static fluid, must be equal to the specific weight of the fluid."

Pressure Variation in static fluid

Consider a small vertical cylinder of static fluid in equilibrium.

Pressure Variation in static fluid

Consider a small vertical cylinder of static fluid in equilibrium.

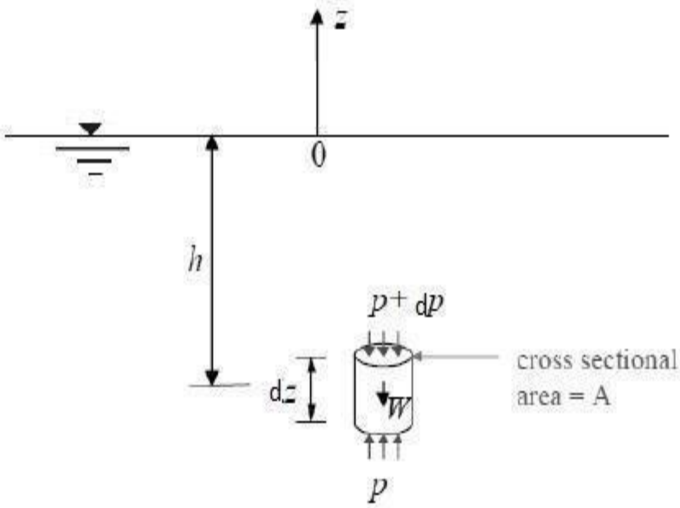


Fig 3.6 Pressure variation in static fluid

Assume that the sectional area is "**A**" and the pressure acting upward on the bottom surface is **p** and the pressure acting downward on the upper surface (dz above bottom surface) is (p + dp)dz.

Let the free surface of the fluid be the origin, i.e., Z = 0. Then the pressure variation at a depth Z = -h below the free surface is governed by

$(p + dp) A + W = pA$

$\Rightarrow$ $dpA + \rho gAdz = 0$ [W= w x volume = ρg Adz] $dp = -\rho gdz$

$\Rightarrow$ $= - \rho g = - w$

Therefore, the hydrostatic pressure increases linearly with depth at the rate of the specific weight, $w = \rho g$ of the fluid.

If fluid is homogeneous, ρ is constant. By simply integrating the above equation, $\int dp = - \int \rho g\, dz$ =>

$p = - \rho g Z + C$ Where C is constant of integration.

When z = 0 (on the free surface), p = C = po = the atmospheric pressure. Hence, $p = - \rho gZ + p_o$

Pressure given by this equation is called absolute pressure, i.e., measured above perfect vacuum. However, it is more convenient to measure the pressure as gauge pressure by setting atmospheric pressure as datum pressure. By setting po = 0,

$p = -\rho gz+0 = -\rho gz = \rho gh$

p = *w*h

The equation derived above shows that when the density is constant, the pressure in a liquid at rest increases linearly with depth from the free surface.

Here, **h** is known as ***pressure head*** or simply ***head*** of fluid.

In fluid mechanics, fluid pressure is usually expressed in height of fluids or head of fluids.

Hydrostatic force

Hydrostatic pressure is the force exerted by a static fluid on a plane surface, when the static fluid comes in contact with the surface. This force will act normal to the surface. It is also known as ***Total Pressure***.

The point of application of the hydrostatic or total pressure on the surface is known as ***Centre of pressure***.

The vertical distance between the free surface of fluid and the centre of pressure is called depth of centre of pressure or location of hydrostatic force.

Total Pressure on a Horizontally Immersed Surface

Consider a plane horizontal surface immersed in a liquid as shown in figure.

Let, w = Specific weight of the liquid, kN/m³ A = Area of the immersed surface in m²
= Depth of the horizontal surface from the liquid level in m We know that,
Total pressure on the surface, **P** = Weight of the liquid above the immersed surface
P = Specific weight of liquid x Volume of liquid
= Specific weight of liquid x Area of surface x Depth of liquid P = wA kN

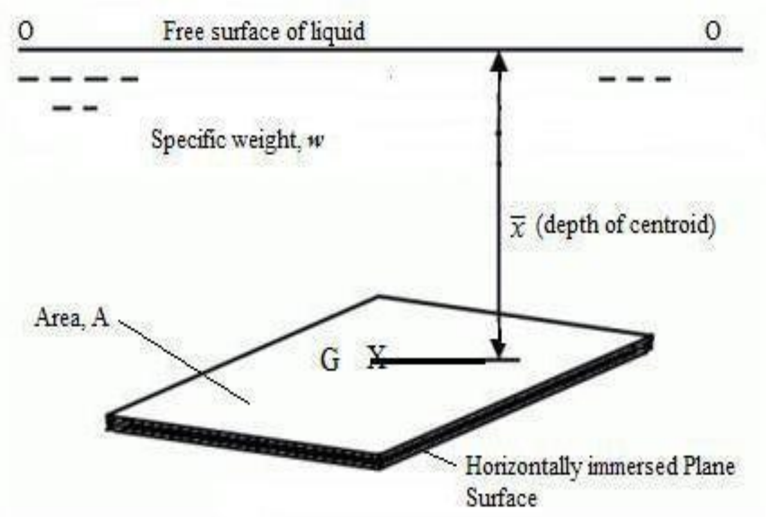


Fig:3.7 Horizontal Plane surface submerged in liquid

Total Pressure and depth of centre of pressure on a Vertically Immersed Surface

Consider an irregular plane vertical surface immersed in a liquid as shown in figure. Let,
w = Specific weight of liquid
A = Total area of the immersed surface
= Depth of the center of gravity of the immersed surface from the liquid surface
Now. consider a strip of width 'b', thickness 'dx' and at a depth x from the free surface of the liquid

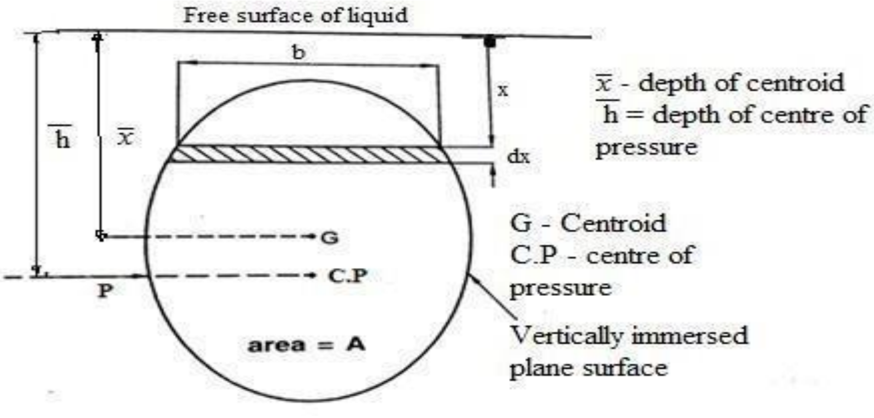


Fig: 3.8 Vertical Plan immersed in liquid

Moment of pressure on the strip about the free surface of liquid = w $x\ b\ dx$ X x = w $x^2\ b\ dx$ Total moment on the entire plane immersed surface = $\int w\ x^2\ b\ dx$
M = $w \int {}^2$
But, $\int {}^2$ = second moment of area about free liquid surface = Io
therefore, M = w Io
Io = IG + A x², according to parallel axis theorem.

Therefore, M = w (IG + A x²) (1)

Also = x h = A x xh (2)
Since equations 1 & 2 are equal,

A x h = (IG + A x²)

Depth of centre of pressure, h= (IG + A x²) / A

Total Pressure and depth of Centre of Pressure on an Inclined Immersed Surface

Consider a plane inclined surface, immersed in a liquid as shown in figure. Let,
w = Specific weight of the liquid
A = Total area of the immersed surface
x = Depth of the centroid of the immersed plane surface from the free surface of liquid. θ = Angle at which the immersed surface is inclined with the liquid
Surface h= depth of centre of pressure from the liquid surface
b = width of the considered thin strip dx = thickness of the strip
O = the reference point obtained by projecting the plane surface with the free surface of liquid
x = distance of the strip from O

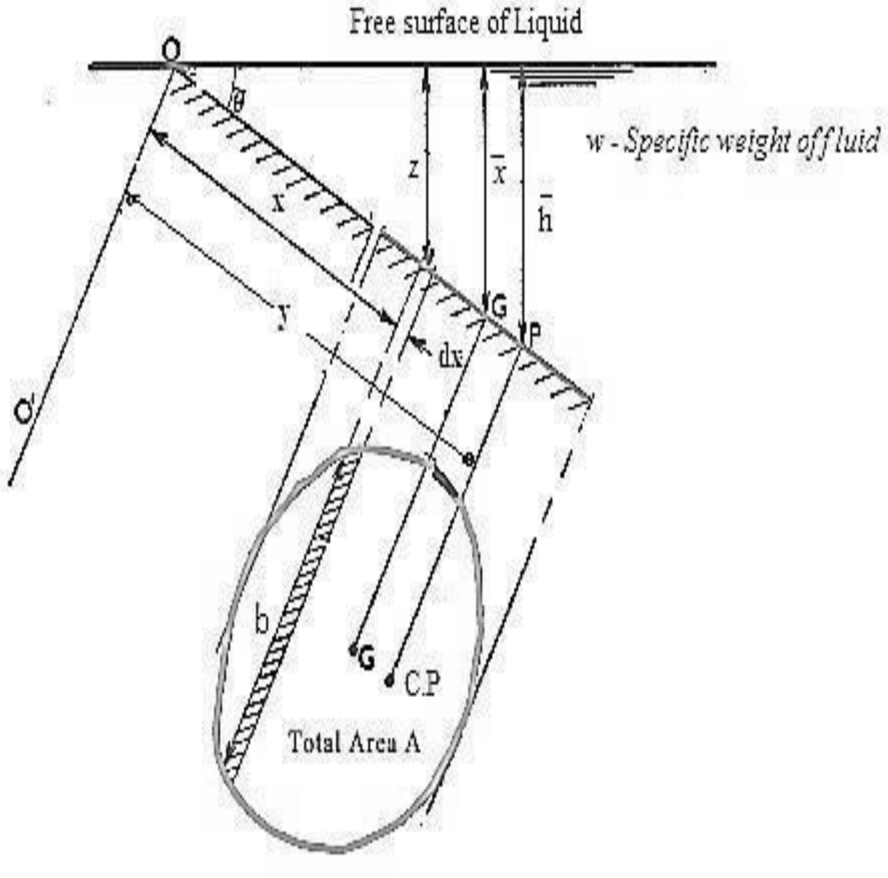


Fig: 3.10 Inclined Immersed Plain

Let the plane of the surface, if produced meet the free liquid surface at O. Then O-O is the axis perpendicular to the plane of the surface.

Let $\bar{y}$ = distance of the C.G. of the inclined surface from O-O

y^* = distance of the centre of pressure from O-O.

Consider a small strip of area dA at a depth 'h' from free surface and at a distance y from the axis O-O as shown in Fig. 3.18.

Pressure intensity on the strip, $p = \rho gh$

$\therefore$ Pressure force, dF, on the strip, $dF = p \times \text{Area of strip} = \rho gh \times dA$

Total pressure force on the whole area, $F = \int dF = \int \rho ghdA$

But from Fig. 3.18, $\dfrac{h}{y} = \dfrac{\bar{h}}{\bar{y}} = \dfrac{h^*}{y^*} = \sin\theta$

$$\therefore \quad h = y \sin\theta$$

$$\therefore \quad F = \int \rho g \times y \times \sin\theta \times dA = \rho g \sin\theta \int ydA$$

But $\int ydA = A\bar{y}$

where $\bar{y}$ = Distance of C.G. from axis O-O

$$\therefore \quad F = \rho g \sin\theta\, \bar{y} \times A$$

$$= \rho g A\bar{h} \qquad (\because \quad \bar{h} = \bar{y}\sin\theta) \quad ...(3.6)$$

Centre of Pressure (h*)

Pressure force on the strip, $dF = \rho ghdA$

$$= \rho gy \sin\theta\, dA \qquad [h = y\sin\theta]$$

Moment of the force, dF, about axis O-O

$$= dF \times y = \rho gy \sin\theta\, dA \times y = \rho g \sin\theta\, y^2 dA$$

Sum of moments of all such forces about O-O

$$= \int \rho g \sin\theta\, y^2\, dA = \rho g \sin\theta \int y^2\, dA$$

But $\int y^2\, dA$ = M.O.I. of the surface about O-O = I_0

$\therefore$ Sum of moments of all forces about O-O = $\rho g \sin\theta\, I_0$

Moment of the total force, F, about O-O is also given by

$$= F \times y^*$$

where y^* = Distance of centre of pressure from O-O.

Equating the two values given by equations (3.7) and (3.8)

$$F \times y^* = \rho g \sin\theta\, I_0$$

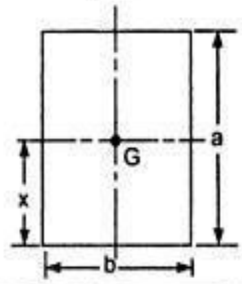
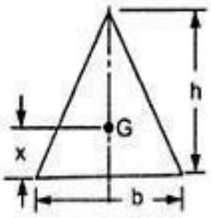
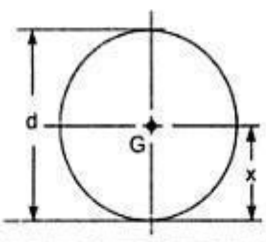
or

$$y^* = \frac{\rho g \sin\theta\, I_0}{F}$$

Now $y^* = \dfrac{h^*}{\sin\theta}$, $F = \rho g A\bar{h}$

and I_0 by the theorem of parallel axis = $I_G + A\bar{y}^2$.

Table: M.I and Geometric Properties of some plane surfaces

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base (I_G)*
1. Rectangle 	$x = \frac{d}{2}$	bd	$\frac{bd^3}{12}$
2. Triangle 	$x = \frac{h}{3}$	$\frac{bh}{2}$	$\frac{bh^3}{36}$
3. Circle 	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$

Pascal's law

The basic property of a static fluid is pressure.

Pressure is the surface force exerted by a fluid against the walls of its container. Pressure also exists at every point within a volume of fluid.

For a static fluid, as shown by the following analysis, pressure turns to be independent direction.

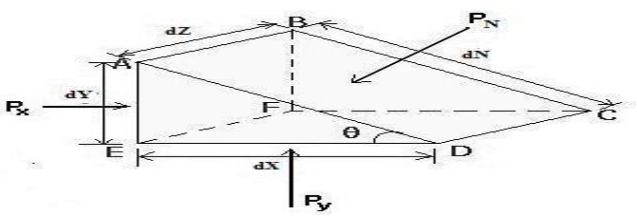


Fig: 3.11. Pascal Law

Consider a triangular prism of small fluid element ABCDEF in equilibrium. Let P*x* is the intensity of pressure in the X direction acting at right angle on the face ABFE, Py is the intensity of pressure in the Y direction acting at right angle on the face CDEF, and Ps is the intensity of pressure normal to inclined plane at an angle θ as shown in figure at right angle to ABC ..
For a fluid at rest there will be no shear stress, there will be no accelerating forces, and therefore the sum of the forces in any direction must be zero.
Thus the forces acting on the fluid element are the pressures on the surrounding and the gravity force. Force due to p*x* = px x Area ABFE = px dydz
Horizontal component of force due to pN = - (pN x Area ABC) sin(θ) = - pNdNdz dy/ds = - PNdydz As Py has no component in the x direction, the element will be in equilibrium, if
px dydz + (-pNdydz) = 0
i.e. p*x* = pN
Similarly in the y direction, force due to py = pydxdz
Component of force due to pN = - (pN x Area ABC) cos(θ) = - pNdsdz d*x*/ds = - pNd*x*dz
Force due to weight of element is negligible and the equation reduces to, py = pN
Therefore, p*x* = py = pN
Thus, Pressure at a point in a fluid at rest is same in all directions.

Manometers:

Manometer is an instrument for measuring the pressure of a fluid, consisting of a tube filled with a heavier gauging liquid, the level of the liquid being determined by the fluid pressure and the height of the liquid being indicated on a scale. A U-tube manometer consists of a glass tube bent in U-Shape, one end of which is connected to gauge point and the other end is exposed to atmosphere.

Manometric liquids:

1. Manometric liquids should neither mix nor have any chemical reaction with the liquid whose pressure intensity is to be measured.
2. It should not undergo any thermal variation.
3. Manometric liquid should have very low vapour pressure.
4. Manometric liquid should have pressure sensitivity depending upon the magnitude of pressure to be measured and accuracy requirement.

Simple U-Tube Manometer: It consist of glass tube in U shape one end of which is connected to a point at which pressure is to be measured and other end remains open to the atmosphere as shown in fig. The tube generally contains mercury or any other liquid whose specific gravity is greater than the specific gravity of the liquid whose pressure is to be measured.

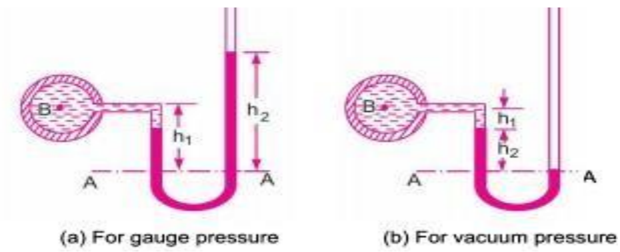


Fig: 3.12 Simple U tube Manometer

For Gauge Pressure. Let B is the point at which pressure is to be measured, whose value is p. The datum line is A-A Let, H_1 = Height of light liquid above the datum line
H_2 = Height of heavier liquid above the datum line S_1 = Specific gravity of light liquid
ρ_1 = Density of light liquid = 1000 x S_1 S_2 = Specific gravity of heavy liquid
ρ_2 = Density of heavy liquid = 1000 x S_2

As the pressure is the same for the horizontal surface. Hence pressure above the horizontal datum line *A-A* in the left column and in the right column of U-tube manometer should be same.

Pressure above *A-A* in the left column $= p + \rho_1 \times g \times h_1$

Pressure above *A-A* in the right column $= \rho_2 \times g \times h_2$

Hence equating the two pressures $p + \rho_1 g h_1 = \rho_2 g h_2$

$$\therefore \quad p = (\rho_2 g h_2 - \rho_1 \times g \times h_1).$$

(*b*) **For Vacuum Pressure.** For measuring vacuum pressure, the level of the heavy liquid in the manometer will be as shown in Fig. 2.9 (*b*). Then

Pressure above *A-A* in the left column $= \rho_2 g h_2 + \rho_1 g h_1 + p$

Pressure head in the right column above *A-A* $= 0$

$$\therefore \quad \rho_2 g h_2 + \rho_1 g h_1 + p = 0$$

$$\therefore \quad p = -(\rho_2 g h_2 + \rho_1 g h_1).$$

Differential U-Tube Manometer:

Let, A and B are the two pipes carrying liquids of specific gravity s1 and s3 & s2 = specific gravity of manometer liquid.

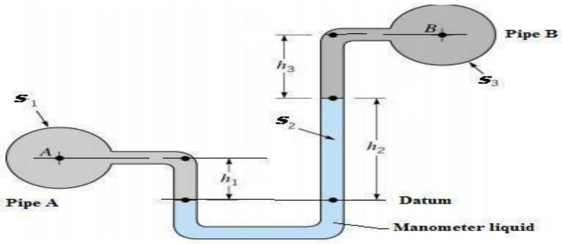


Fig:3.13 Differential U-tube Manometer

Let two point A & B are at different level and also contains liquids of different sp.gr. These points are connected to the U-tube differential manometer. Let the pressure at A and B are P_A and P_B

Let h = Difference of mercury level in the U-tube.
y = Distance of the centre of *B*, from the mercury level in the right limb.
x = Distance of the centre of *A*, from the mercury level in the right limb.
ρ_1 = Density of liquid at *A*.
ρ_2 = Density of liquid at *B*.
ρ_g = Density of heavy liquid or mercury.

Taking datum line at *X-X*.

Pressure above *X-X* in the left limb $= \rho_1 g(h + x) + p_A$

where p_A = pressure at *A*.

Pressure above *X-X* in the right limb $= \rho_g \times g \times h + \rho_2 \times g \times y + p_B$

where p_B = Pressure at *B*.

Equating the two pressure, we have

$$\rho_1 g(h + x) + p_A = \rho_g \times g \times h + \rho_2 g y + p_B$$

$$\therefore \quad p_A - p_B = \rho_g \times g \times h + \rho_2 g y - \rho_1 g(h + x)$$

$$= h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x$$

$\therefore$ Difference of pressure at *A* and *B* $= h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x$

In Fig. 2.18 (*b*), the two points *A* and *B* are at the same level and contains the same liquid of density ρ_1. Then

Pressure above *X-X* in right limb $= \rho_g \times g \times h + \rho_1 \times g \times x + p_B$

Pressure above *X-X* in left limb $= \rho_1 \times g \times (h + x) + p_A$

Equating the two pressure

$$\rho_g \times g \times h + \rho_1 g x + p_B = \rho_1 \times g \times (h + x) + p_A$$

$$\therefore \quad p_A - p_B = \rho_g \times g \times h + \rho_1 g x - \rho_1 g(h + x)$$

$$= g \times h(\rho_g - \rho_1).$$

Buoyant force: The upward force exerted by a liquid on a body when the body is immersed in the liquid is known as buoyancy or buoyant force.

The point through which force of buoyancy is supposed to act is called centre of buoyancy. The buoyant force acting on a body is equal to the weight of the liquid displaced by the body. For a fluid with constant density, the buoyant force is independent of the distance of the body from the free surface. It is also independent of the density of the solid body.

Archimedes principle: The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume. For floating bodies, the weight of the entire body must be equal to the buoyant force, which is the weight of the fluid whose volume is equal to the volume of the submerged portion of the floating body.

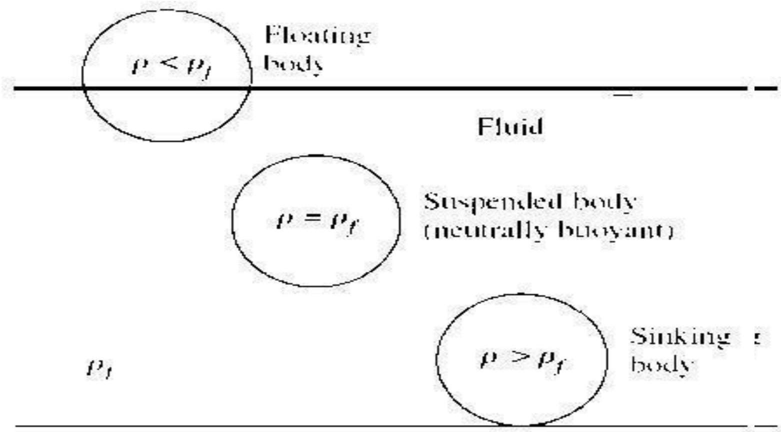


Fig:3.14. Floating Body

Stability of immersed and floating bodies

A floating body possesses vertical stability, while an immersed neutrally buoyant body is neutrally stable since it does not return to its original position after a disturbance.

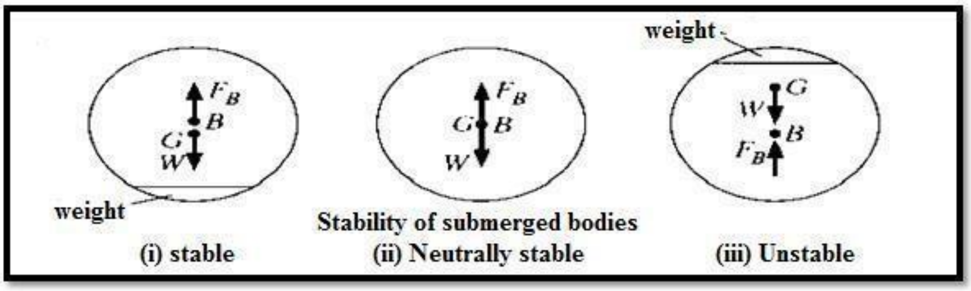


Fig:3.15. An immersed neutrally buoyant body is (a) stable if the center of gravity G is directly below the center of buoyancy B of the body, (b) neutrally stable if G and B are coincident, and (c) unstable if G is directly above B.

Metacentre: The point about which a body starts oscillating when the body is tilted is known meta- centre.

Metacentric height GM: The distance between the center of gravity G and the metacenter M is known as Meta centric height. It is the point of intersection of line of action of buoyant force with the line passing through centre of gravity, when the body is slightly tilted.

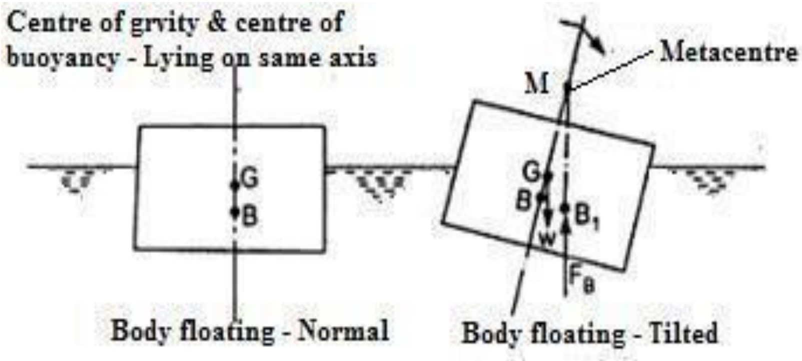


Fig.3.16. Metacentric Height

The length of the metacentric height GM above G is a measure of the stability: If the metacentric height increases, then the floating body will be more.. The meta-centric height (GM) is.given by, GM = V - BGWhere, I = Moment of Inertia of the floating body (in plan) at water surface about the axis Y- Y V = Volume of ihe body sub merged in waterBG = Distance between centre of gravity and centre of buoyancy. Conditions of equilibrium of a floating and submerged body are :

Table.2. Condition of Equilibrium of a Floating bodies

Equilibrium	Floating Body	Sub-merged Body
(i) Stable Equilibrium	M is above G	B is above G
(a) Unstable Equilibrium	M is below G	B is below G
(Hi) Neutral Equilibrium	Af and G coincide	B and G coincide

Stability of floating bodies .A floating body is stable if the body is bottom-heavy and thus the center of gravity G is below the centroid B of the body, or if the metacentre M is above point G. However, the body is unstable if point M is below point G.

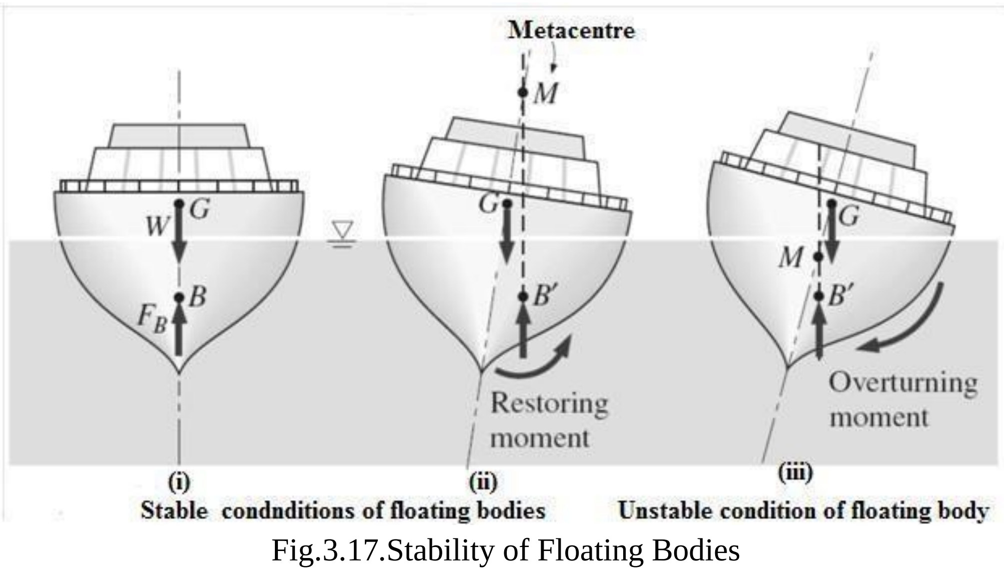


Fig.3.17.Stability of Floating Bodies

Problems:

1.Calculate the sp.weight, density and sp.gravity of one litre of liquid which weights 7N.

$$\text{Volume} = 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \quad \left(\because \ 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \text{ or } 1 \text{ litre} = 1000 \text{ cm}^3\right)$$

$$\text{Weight} = 7 \text{ N}$$

$$(i) \text{ Specific weight } (w) = \frac{\text{Weight}}{\text{Volume}} = \frac{7\text{N}}{\left(\frac{1}{1000}\right)\text{m}^3} = \mathbf{7000 \ N/m^3. \ Ans.}$$

$$(ii) \text{ Density } (\rho) = \frac{w}{g} = \frac{7000}{9.81} \text{ kg/m}^3 = \mathbf{713.5 \ kg/m^3. \ Ans.}$$

$$(iii) \text{ Specific gravity} = \frac{\text{Density of liquid}}{\text{Density of water}} = \frac{713.5}{1000} \quad \{\because \ \text{Density of water} = 1000 \text{ kg/m}^3\}$$

$$= \mathbf{0.7135. \ Ans.}$$

2.Calculate the density, sp.weight and weight of one litre of petrol of specific gravity = 0.7

Solution. Given : Volume $= 1 \text{ litre} = 1 \times 1000 \text{ cm}^3 = \frac{1000}{10^6} \text{ m}^3 = 0.001 \text{ m}^3$

Sp. gravity $S = 0.7$

(*i*) *Density* (ρ)

Using equation (1.1*A*),

Density (ρ) $= S \times 1000 \text{ kg/m}^3 = 0.7 \times 1000 = \mathbf{700 \ kg/m^3}$. **Ans.**

(*ii*) *Specific weight* (w)

Using equation (1.1), $w = \rho \times g = 700 \times 9.81 \text{ N/m}^3 = \mathbf{6867 \ N/m^3}$. **Ans.**

(*iii*) *Weight (W)*

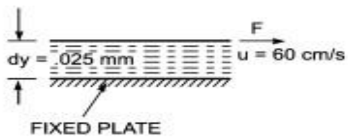
We know that specific weight $= \frac{\text{Weight}}{\text{Volume}}$

or $w = \frac{W}{0.001}$ or $6867 = \frac{W}{0.001}$

$\therefore$ $W = 6867 \times 0.001 = \mathbf{6.867 \ N}$. **Ans.**

3.A plate 0.023 mm distant from a fixed plate moves at 60 cm/s and requires a force of 2N per unit area i.e 2 N/m^2 to maintain this speed. Determine the fluid viscosity between the plates.

Solution. Given :



Distance between plates, $dy = .025 \text{ mm} = .025 \times 10^{-3} \text{ m}$

Velocity of upper plate, $u = 60 \text{ cm/s} = 0.6 \text{ m/s}$

Force on upper plate, $F = 2.0 \ \frac{\text{N}}{\text{m}^2}$.

This is the value of shear stress *i.e.*, τ

Let the fluid viscosity between the plates is μ.

Using the equation (1.2), we have $\tau = \mu \ \frac{du}{dy}$.

where du = Change of velocity $= u - 0 = u = 0.60$ m/s

dy = Change of distance $= .025 \times 10^{-3}$ m

τ = Force per unit area $= 2.0 \ \frac{\text{N}}{\text{m}^2}$

$$\therefore \quad 2.0 = \mu \ \frac{0.60}{.025 \times 10^{-3}} \quad \therefore \quad \mu = \frac{2.0 \times .025 \times 10^{-3}}{0.60} = 8.33 \times 10^{-5} \ \frac{\text{Ns}}{\text{m}^2}$$

$$= 8.33 \times 10^{-5} \times 10 \text{ poise} = \mathbf{8.33 \times 10^{-4} \ poise. \ Ans.}$$

4.The dynamic viscosity of oil used for lubrication between a shaft and sleeve is 6 poise. The shaft is of diameter 0.4 m and rotates at 190 rpm. Calculate the power lost in the bearing for a sleeve length of 90mm. The thickness of the oil film is 1.5mm.

Solution. Given :

Viscosity $\mu = 6$ poise

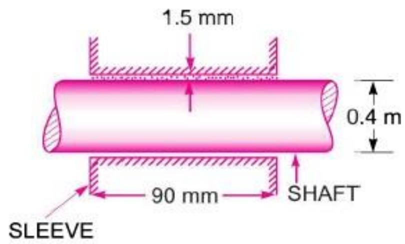
$$= \frac{6}{10}\frac{\text{N s}}{\text{m}^2} = 0.6\ \frac{\text{N s}}{\text{m}^2}$$

Dia. of shaft, $D = 0.4$ m

Speed of shaft, $N = 190$ r.p.m

Sleeve length, $L = 90 \text{ mm} = 90 \times 10^{-3}$ m

Thickness of oil film, $t = 1.5 \text{ mm} = 1.5 \times 10^{-3}$ m



Tangential velocity of shaft, $u = \dfrac{\pi D N}{60} = \dfrac{\pi \times 0.4 \times 190}{60} = 3.98$ m/s

Using the relation $\tau = \mu \dfrac{du}{dy}$

where du = Change of velocity = $u - 0 = u = 3.98$ m/s

dy = Change of distance = $t = 1.5 \times 10^{-3}$ m

$$\tau = 10 \times \frac{3.98}{1.5 \times 10^{-3}} = 1592 \text{ N/m}^2$$

This is shear stress on shaft

$\therefore$ Shear force on the shaft, F = Shear stress × Area

$$= 1592 \times \pi D \times L = 1592 \times \pi \times .4 \times 90 \times 10^{-3} = 180.05 \text{ N}$$

Torque on the shaft, $T = \text{Force} \times \dfrac{D}{2} = 180.05 \times \dfrac{0.4}{2} = 36.01$ Nm

$\therefore$ *Power lost $= \dfrac{2\pi NT}{60} = \dfrac{2\pi \times 190 \times 36.01}{60}$ = **716.48 W. Ans.**

5.The surface tension of water in contact with air at 20˚C is 0.0725N/m. The pressure inside a droplet of water is to be 0.02 N/cm^2 greater then the outside pressure. Calculate the diameter of the droplet of water.

Solution. Given :

Surface tension, $\sigma = 0.0725$ N/m

Pressure intensity, p in excess of outside pressure is

$$p = 0.02 \text{ N/cm}^2 = 0.02 \times 10^4\ \frac{\text{N}}{\text{m}^2}$$

Let d = dia. of the droplet

we get $p = \dfrac{4\sigma}{d}$ or $0.02 \times 10^4 = \dfrac{4 \times 0.0725}{d}$

$$d = \frac{4 \times 0.0725}{0.02 \times (10)^4} = .00145 \text{ m} = .00145 \times 1000 = \mathbf{1.45\ mm.\ Ans.}$$

6.Calculate the capillary rise in a glass tube of 2.5mm diameter when immersed vertically in a) water b) Mercury. Take surface tension of 2.5 mm diameter when immersed vertically in contact with air. The specific gravity for mercury is given as 13.6 and angle of contact = 130∘

∴ Density $= 13.6 \times 1000 \text{ kg/m}^3$.

(a) Capillary rise for water ($\theta = 0°$)

Using equation (1.20), we get $h = \dfrac{4\sigma}{\rho \times g \times d} = \dfrac{4 \times 0.0725}{1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= .0118 \text{ m} = \mathbf{1.18 \text{ cm. Ans.}}$

(b) For mercury

Angle of contact between mercury and glass tube, $\theta = 130°$

Using equation (1.21), we get $h = \dfrac{4\sigma \cos\theta}{\rho \times g \times d} = \dfrac{4 \times 0.52 \times \cos 130°}{13.6 \times 1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= -.004 \text{ m} = -\mathbf{0.4 \text{ cm. Ans.}}$

The negative sign indicates the capillary depression.

7.The right limb of a single U-tube manometer containing mercury is open to the atmosphere while the left limb is connected to a pipe in which a fluid of sp.gravity is 0.9 is flowing. The centre of the pipe is 12cm below the level of mercury in the right limb. Find the pressure of fluid in the pipe if the difference of mercury in the two limbs is 20cm.

Solution. Given :

Sp. gr. of fluid, $S_1 = 0.9$

∴ Density of fluid, $\rho_1 = S_1 \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3$

Sp. gr. of mercury, $S_2 = 13.6$

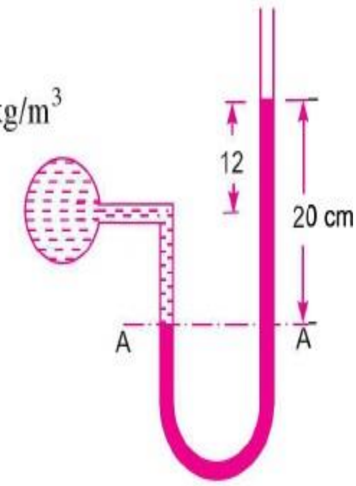
∴ Density of mercury, $\rho_2 = 13.6 \times 1000 \text{ kg/m}^3$

Difference of mercury level, $h_2 = 20 \text{ cm} = 0.2 \text{ m}$

Height of fluid from A-A, $h_1 = 20 - 12 = 8 \text{ cm} = 0.08 \text{ m}$

Let p = Pressure of fluid in pipe

Equating the pressure above A-A, we get



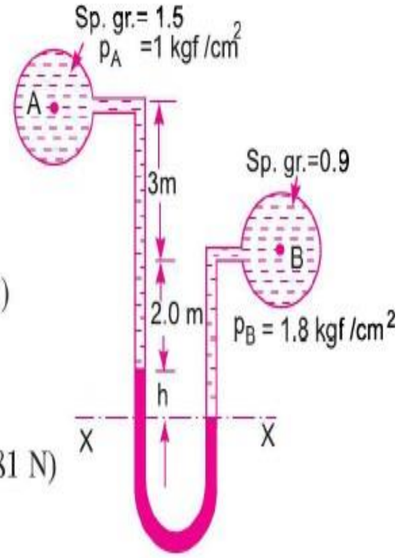
$$p + \rho_1 g h_1 = \rho_2 g h_2$$

$$p + 900 \times 9.81 \times 0.08 = 13.6 \times 1000 \times 9.81 \times .2$$

$$p = 13.6 \times 1000 \times 9.81 \times .2 - 900 \times 9.81 \times 0.08$$

$$= 26683 - 706 = 25977 \text{ N/m}^2 = \mathbf{2.597 \text{ N/cm}^2}. \textbf{ Ans.}$$

8.A differential manometer is connected at the two points A and B of two pipes as shown in fig. The pipe A contains a liquid of Sp.gravity = 1.5 while pipe B contains a liquid of sp.gravity = 0.9. The pressure at A and B are 1 Kgf/cm^2 and1.80 Kgf/cm^2 respectively. Find the difference in mercury level in the differential manometer.



Solution. Given :

Sp. gr. of liquid at A, $S_1 = 1.5$ $\therefore$ $\rho_1 = 1500$

Sp. gr. of liquid at B, $S_2 = 0.9$ $\therefore$ $\rho_2 = 900$

Pressure at A, $p_A = 1 \text{ kgf/cm}^2 = 1 \times 10^4 \text{ kgf/m}^2$

$= 10^4 \times 9.81 \text{ N/m}^2$ ($\because$ 1 kgf = 9.81 N)

Pressure at B, $p_B = 1.8 \text{ kgf/cm}^2$

$= 1.8 \times 10^4 \text{ kgf/m}^2$

$= 1.8 \times 10^4 \times 9.81 \text{ N/m}^2$ ($\because$ 1 kgf = 9.81 N)

Density of mercury $= 13.6 \times 1000 \text{ kg/m}^3$

Taking X-X as datum line.

Pressure above X-X in the left limb

$$= 13.6 \times 1000 \times 9.81 \times h + 1500 \times 9.81 \times (2 + 3) + p_A$$

$$= 13.6 \times 1000 \times 9.81 \times h + 7500 \times 9.81 + 9.81 \times 10^4$$

Pressure above X-X in the right limb $= 900 \times 9.81 \times (h + 2) + p_B$

$$= 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Equating the two pressure, we get

$$13.6 \times 1000 \times 9.81h + 7500 \times 9.81 + 9.81 \times 10^4$$

$$= 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Dividing by 1000×9.81, we get

$$13.6h + 7.5 + 10 = (h + 2.0) \times .9 + 18$$

$$13.6h + 17.5 = 0.9h + 1.8 + 18 = 0.9h + 19.8$$

$$(13.6 - 0.9)h = 19.8 - 17.5 \text{ or } 12.7h = 2.3$$

$\therefore$ $h = \dfrac{2.3}{12.7} = 0.181 \text{ m} = \textbf{18.1 cm. Ans.}$

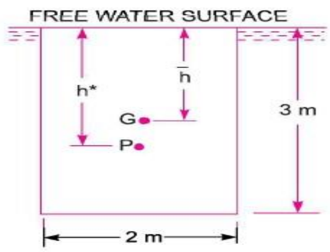
9.A rectangular plane surface is 2m wide and 3m deep. It lies in vertical plane in water. Determine the total pressure and position of centre of pressure on the plane surface when its upper edge is horizontal and a) coincide with water surfaceb) 2.5 m below the free water surface.

Solution. Given :

Width of plane surface, $b = 2$ m

Depth of plane surface, $d = 3$ m

(*a*) **Upper edge coincides with water surface**



$$F = \rho g A \bar{h}$$

where $\rho = 1000$ kg/m^3, $g = 9.81$ m/s^2

$$A = 3 \times 2 = 6 \text{ m}^2, \ \bar{h} = \frac{1}{2}(3) = 1.5 \text{ m}.$$

$\therefore$ $$F = 1000 \times 9.81 \times 6 \times 1.5$$
$$= \mathbf{88290 \ N. \ Ans.}$$

Depth of centre of pressure is given by equation (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where I_G = M.O.I. about C.G. of the area of surface

$$= \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

$\therefore$ $$h^* = \frac{4.5}{6 \times 1.5} + 1.5 = 0.5 + 1.5 = \mathbf{2.0 \ m. \ Ans.}$$

(*b*) **Upper edge is 2.5 m below water surface**

Solution. Given :

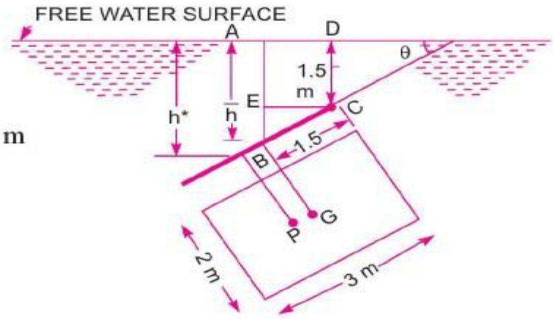
Width of plane surface, $b = 2$ m

Depth, $d = 3$ m

Angle, $\theta = 30°$

Distance of upper edge from free water surface = 1.5 m

(*i*) **Total pressure** force is given by equation



$$F = \rho g A \bar{h}$$

where $\rho = 1000$ kg/m^3

$A = b \times d = 3 \times 2 = 6$ m^2

$\therefore$ $\bar{h}$ = Depth of C.G. from free water surface

$= 1.5 + 1.5 \sin 30°$

$\{\because \ \bar{h} = AE + EB = 1.5 + BC \sin 30° = 1.5 + 1.5 \sin 30°\}$

$= 1.5 + 1.5 \times \frac{1}{2} = 2.25$ m

$\therefore$ $$F = 1000 \times 9.81 \times 6 \times 2.25 = \mathbf{132435 \ N. \ Ans.}$$

(*ii*) **Centre of pressure (h*)**

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A\bar{h}} + \bar{h}, \quad \text{where } I_G = \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

$\therefore$ $$h^* = \frac{4.5 \times \sin^2 30°}{6 \times 2.25} + 2.25 = \frac{4.5 \times \frac{1}{4}}{6 \times 2.25} + 2.25$$
$$= 0.0833 + 2.25 = \mathbf{2.3333 \ m. \ Ans.}$$

10.A rectangular plane surface 2m wide and 3m deep lies in water in such a way that its plane makes an angle of 30◦ with the free surface of water. Determine the total surface and position of centre of pressure when the upper edge is 1.5m below the free water surface.

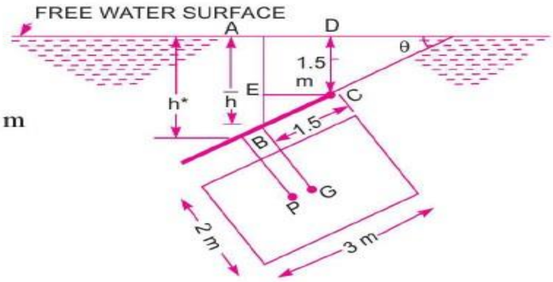
Solution. Given :

Width of plane surface, $b = 2$ m

Depth, $d = 3$ m

Angle, $\theta = 30°$

Distance of upper edge from free water surface = 1.5 m



(*i*) **Total pressure** force is given by equation

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$\therefore$ $\bar{h}$ = Depth of C.G. from free water surface

$= 1.5 + 1.5 \sin 30°$

$\{\because \bar{h} = AE + EB = 1.5 + BC \sin 30° = 1.5 + 1.5 \sin 30°\}$

$= 1.5 + 1.5 \times \frac{1}{2} = 2.25$ m

$\therefore$ $F = 1000 \times 9.81 \times 6 \times 2.25 =$ **132435 N. Ans.**

(*ii*) **Centre of pressure (h*)**

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}, \quad \text{where } I_G = \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

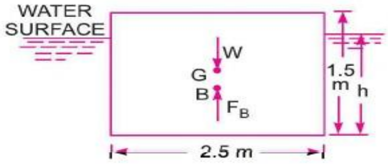
$$\therefore \quad h^* = \frac{4.5 \times \sin^2 30°}{6 \times 2.25} + 2.25 = \frac{4.5 \times \frac{1}{4}}{6 \times 2.25} + 2.25$$

$= 0.0833 + 2.25 =$ **2.3333 m. Ans.**

11. Find the volume of the water displaced and position of centre of buoyancy for a wooden block of width 2.5m and depth 1.5m. When it floats horizontally in water. The density of wooden block is 650 kg/m^3 and its length 6m.

Solution. Given :

Width	= 2.5 m
Depth	= 1.5 m
Length	= 6.0 m
Volume of the block	$= 2.5 \times 1.5 \times 6.0 = 22.50$ m^3
Density of wood,	$\rho = 650$ kg/m^3
$\therefore$ Weight of block	$= \rho \times g \times$ Volume
	$= 650 \times 9.81 \times 22.50$ N $= 143471$ N



For equilibrium the weight of water displaced = Weight of wooden block

= 143471 N

$\therefore$ Volume of water displaced

$$= \frac{\text{Weight of water displaced}}{\text{Weight density of water}} = \frac{143471}{1000 \times 9.81} = \mathbf{14.625\ m^3.\ Ans.}$$

($\because$ Weight density of water = 1000×9.81 N/m^3)

Position of Centre of Buoyancy. Volume of wooden block in water

= Volume of water displaced

$2.5 \times h \times 6.0 = 14.625$ m^3, where h is depth of wooden block in water

$$\therefore \quad h = \frac{14.625}{2.5 \times 6.0} = 0.975 \text{ m}$$

$$\therefore \quad \text{Centre of Buoyancy} = \frac{0.975}{2} = \mathbf{0.4875\ m\ from\ base.\ Ans.}$$

12.A rectangular pontoon is 5m long, 3m wide and 1.20m high. The depth of immersion of the position is 0.80 m in sea water. If the centre of gravity is 0.6m above the bottom of the position, determine the meta centric height. The density for sea water is 1025 kg/m^3.

Solution. Given :

Dimension of pontoon $= 5 \text{ m} \times 3 \text{ m} \times 1.20 \text{ m}$

Depth of immersion $= 0.8 \text{ m}$

Distance $AG = 0.6 \text{ m}$

Distance $AB = \frac{1}{2} \times \text{Depth of immersion}$

$= \frac{1}{2} \times .8 = 0.4 \text{ m}$

Density for sea water $= 1025 \text{ kg/m}^3$

Meta-centre height GM, given by equation

$$GM = \frac{I}{\forall} - BG$$

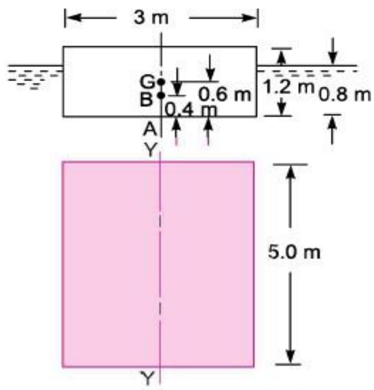
where I = M.O. Inertia of the plan of the pontoon about Y-Y axis

$= \frac{1}{12} \times 5 \times 3^3 \text{ m}^4 = \frac{45}{4} \text{ m}^4$

$\forall$ = Volume of the body sub-merged in water

$= 3 \times 0.8 \times 5.0 = 12.0 \text{ m}^3$

$BG = AG - AB = 0.6 - 0.4 = 0.2 \text{ m}$



PLAN AT WATER SURFACE

$$\therefore \quad GM = \frac{45}{4} \times \frac{1}{12.0} - 0.2 = \frac{45}{48} - 0.2 = 0.9375 - 0.2 = \mathbf{0.7375 \text{ m.}} \quad \textbf{Ans.}$$

TEXT / REFERENCE BOOKS

1. Bansal.R.K, Fluid Mechanics & Hydraulics Machines, 9th Edition, Laxmi Publications, 2015.
2. Modi P.N., Seth S.M., Hydraulics and Fluid Mechanics Including Hydraulic Machines, 21st Edition, Standard Book House, 2017.
3. Goyal, Manish Kumar, Fluid Mechanics and Hydraulic Machines, PHI Learning Pvt. Ltd., 2015.
4. Kumar K.L., Engineering Fluid Mechanics, 8thEdition, Eurasia Publication House (P) Ltd, 2014.
5. R. K. Rajput, Fluid Mechanics & Hydraulics Machines, 4th Edition, S. Chand Limited, 2008.
6. Yunus.A.Cengel, John.M.Cimbala, Fluid Mechanics Fundamentals and Application, 3rd Edition, Tata McGraw Hill, 2013.
7. Robert W. Fox, Alan T. McDonald, Philip J.Pritchard, John W.Mitchell, Fluid Mechanics, 9th Edition, Wiley India Edition, 2016.

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SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

SMEA1306 SOLID AND FLUID MECHANICS

UNIT 4 EQUATIONS OF MOTION

4. EQUATIONS OF MOTION

Basic equations of motion: Types of fluid flow-Concept of Control Volume- Control Volume Analysis of mass, momentum and energy. Differential equation of continuity and momentum - Euler's and Bernoulli's Equation and its applications. Flow Measurement: Orifice meter, Venturimeter, Piezometer.

Types of Fluid Flow

- Steady and Unsteady.
- Uniform and Non-Uniform.
- Laminar and Turbulent.
- Compressible and In-compressible.
- Rotational and Irrotational Flow.
- One, Two, and Three -dimensional Fluid Flow.

Steady flow

A steady flow is one in which all conditions at any point in a stream remain constant with respect to time. Or A steady flow is the one in which the quantity of liquid flowing per second through any section, is constant. This is the definition for the ideal case. True steady flow is present only in Laminar flow. In turbulent flow, there are continual fluctuations in velocity. Pressure also fluctuates at every point. But if this rate of change of pressure and velocity are equal on both sides of a constant average value, the flow is steady flow. The exact term use for this is mean steady flow. Steady flow may be uniform or non-uniform.

Uniform flow

A truly uniform flow is one in which the velocity is same at a given instant at every point in the fluid. This definition holds for the ideal case. Whereas in real fluids velocity varies across the section. But when the size and shape of cross section are constant along the length of channels under consideration, the flow is said to be uniform.

Unsteady Flow

A flow, in which quantity of liquid flowing per second is not constant, is called unsteady flow. Unsteady flow is a transient phenomenon. It may be in time become steady or zero flow. For example when a valve is closed at the discharge end of the pipeline. Thus, causing the velocity in the pipeline to decrease to zero.

One, Two and Three Dimensional Flows

Term one, two or three dimensional flow refers to the number of space coordinated required to describe a flow. It appears that any physical flow is generally three-dimensional. But these are difficult to calculate and call for as much simplification as possible. This is achieved by ignoring changes to flow in any of the directions, thus reducing the complexity. It may be possible to reduce a three-dimensional problem to a two-dimensional one, even an one dimensional one at times.

Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} = 0$$

where ∂V = Change of velocity
∂s = Length of flow in the direction S.

Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant.}$$

Rotational and Irrotational Flows. Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$ called the Reynold number,

where D = Diameter of pipe
V = Mean velocity of flow in pipe
and ν = Kinematic viscosity of fluid.

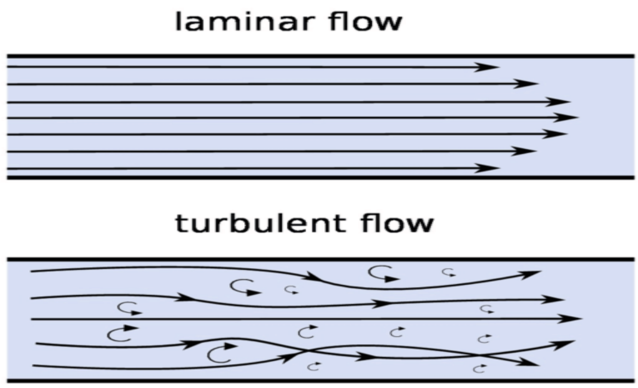


Figure 4.1. Laminar and Turbulent.

Rate of Flow or Discharge (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second.

For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus (i) For liquids the units of Q are m^3 /s or litres/s (ii) For gases the units of Q are kgf/s or Newton/s

Consider a fluid flowing through a pipe in which A= Cross-sectional area of pipe. V= Average area of fluid across the section Then discharge $Q=A \times V$

Continuity Equation

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Figure 4.2.

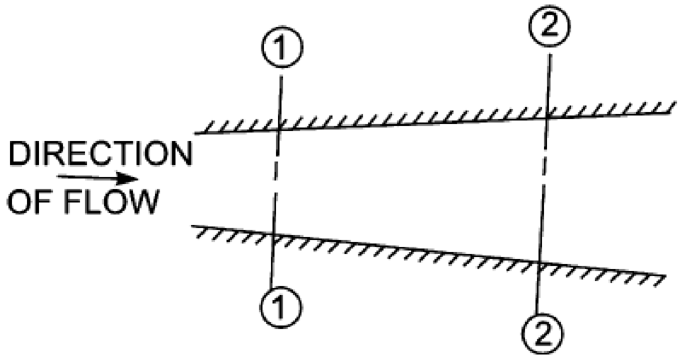


Figure 4.2. Flow through Pipe

According to law of conservation of mass

Rate of flow at section 1-1 =Rate of flow at section 2-2

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

The above equation is applicable to the compressible as well as incompressible fluids is called Continuity Equation.

If the fluid is incompressible $\rho_1 = \rho_2$and continuity equation reduces to

$$A_1 V_1 = A_2 V_2$$

Equation of motion

The dynamics of fluid flow is the study of fluid motion with forces causing flow. The dynamic behaviours of the fluid flow is analyzed by the Newton's law of motion (F=ma), which relates the acceleration with the forces. The fluid is assumed to be incompressible and non-viscous.

Mathematically, $F_x = m.a_x$

In the fluid flow, following forces are present:

• Pressure force 'F_p'

• Gravity force 'F_g'

• Viscous force 'F_v'

• Turbulent flow 'F_t'

• Compressibility force 'F_e'

- The pressure force 'Fp' is exerted on the fluid mass, if there exists a pressure gradient between the 2 parts in the direction of flow.
- The gravity force 'Fg' is due to the weight of the fluid and it is equal to 'Mg'. The gravity force for unit volume is equal to 'ρg'.
- The viscous force 'Fv' is due to the viscosity of the flowing fluid and thus exists in the case of all real fluid.
- The turbulent flow 'Ft' is due to the turbulence of the flow. In the turbulent flow, the fluid particles move from one layer to other and therefore, there is a continuous momentum transfer between adjacent layer, which results in developing additional stresses(called Reynolds stresses) for the flowing fluid.
- The compressibility force 'Fe' is due to elastic property of fluid and it is important only either for compressible fluids or in the cases of flowing fluids in which the elastic properties of fluids are significant.

In the fluid flow, the following forces are present :

(*i*) F_g, gravity force.

(*ii*) F_p, the pressure force.

(*iii*) F_v, force due to viscosity.

(*iv*) F_t, force due to turbulence.

(*v*) F_c, force due to compressibility.

the net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x.$$

(*i*) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion.**

(*ii*) For flow, where (F_t) is negligible, the resulting equations of motion are known as **Navier-Stokes Equation.**

(*iii*) If the flow is assumed to be ideal, viscous force (F_v) is zero and equation of motions are known as **Euler's equation of motion.**

Euler's Equation of motions

In an ideal incompressible fluid, when the flow is steady and continuous, sum of the velocity head, pressure head and datum head along a stream line is constant.

Assumptions:

- The fluid is ideal and incompressible.
- Flow is steady and continuous.
- Flow is along streamline and it is 1-D.
- The velocity is uniform across the section and is equal to the mean velocity.
- Flow is Irrotational.
- The only forces acting on the fluid are gravity and the pressure forces.

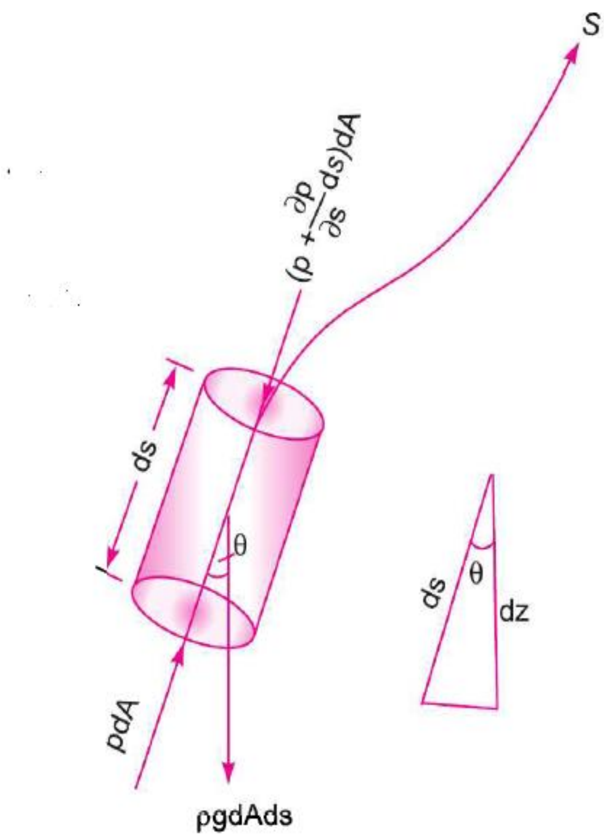


Figure 4.3. Euler's Equation of motions

1. Pressure force pdA in the direction of flow.

2. Pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow.

3. Weight of element $\rho g dA ds$.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element $\times$ acceleration in the direction s.

$$pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos\theta$$
$$= \rho dA ds \times a_s$$

where a_s is the acceleration in the direction of s.

$$a_s = \frac{dv}{dt}, \text{ where } v \text{ is a function of } s \text{ and } t.$$

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{\because \frac{ds}{dt} = v\right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\therefore \qquad a_s = \frac{v\partial v}{\partial s}$$

Substituting the value of a_s

$$-\frac{\partial p}{\partial s} dsdA - \rho g\, dAds \cos\theta = \rho dAds \times \frac{\partial v}{\partial s}$$

Dividing by $\rho dsdA$, $-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{v\partial v}{\partial s}$

$$\text{or} \qquad \frac{\partial p}{\rho\partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0$$

we have $\cos\theta = \dfrac{dz}{ds}$

$$\therefore \qquad \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + \frac{vdv}{ds} = 0$$

or
$$\frac{dp}{\rho} + gdz + vdv = 0$$

This is the required Euler's equation for motion.

Bernoulli's Equation from Euler's equation for motion:

Bernoulli's equation is obtained by integrating the Euler's equation of motion

$$\int \frac{dp}{\rho} + \int gdz + \int vdv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\therefore \qquad \frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

$$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}$$

$\dfrac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

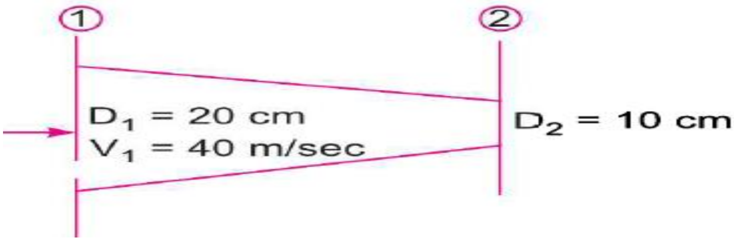
$v^2/2g$ = kinetic energy per unit weight or kinetic head.

z = potential energy per unit weight or potential head.

Statement of Bernoulli's Theorem. It states that in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant. The total energy consists of pressure energy, kinetic energy and potential energy or datum energy.

Problem 1:

A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 4.0 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.



$D_1 = 20 \text{ cm} = 0.2 \text{ m}$

$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$

$V_1 = 4.0 \text{ m/s}$

$D_2 = 0.1 \text{ m}$

$A_2 = \frac{\pi}{4} (.1)^2 = .00785 \text{ m}^2$

(*i*) Velocity head at section 1

$$= \frac{V_1^2}{2g} = \frac{4.0 \times 4.0}{2 \times 9.81} = \mathbf{0.815 \text{ m.}}$$

(*ii*) Velocity head at section 2 = $V_2^2/2g$

To find V_2, apply continuity equation at 1 and 2

$$A_1V_1 = A_2V_2 \quad \text{or} \quad V_2 = \frac{A_1V_1}{A_2} = \frac{.0314}{.00785} \times 4.0 = 16.0 \text{ m/s}$$

$$\text{Velocity head at section 2} = \frac{V_2^2}{2g} = \frac{16.0 \times 16.0}{2 \times 9.81} = \mathbf{83.047 \text{ m.}}$$

(*iii*) Rate of discharge

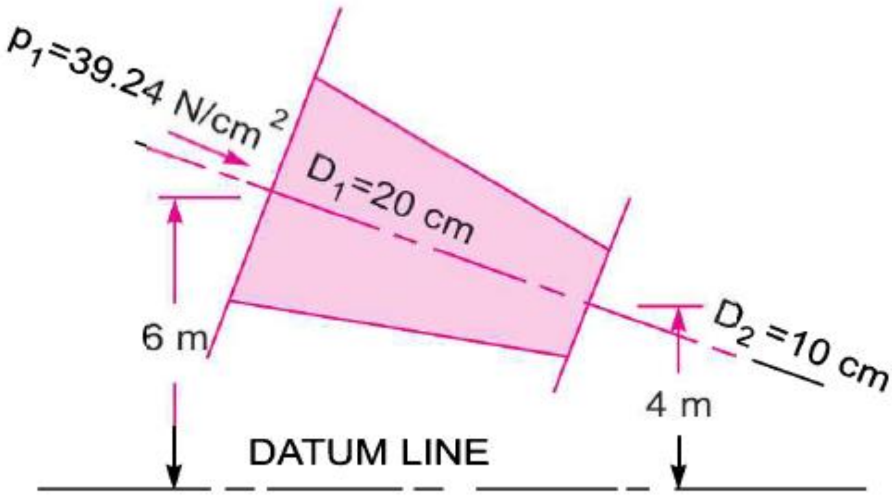
$= A_1V_1 \quad \text{or} \quad A_2V_2$

$= 0.0314 \times 4.0 = 0.1256 \text{ m}^3\text{/s}$

$=$ **125.6 litres/s. Ans.** $\{\because 1 \text{ m}^3 = 1000 \text{ litres}\}$

Problem 2:

The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through pipe is 35 litres/s. The section 1 is 6 m above datum and section 2 is 4 m above datum. If the pressure at section 1 is 39.24 N/cm², find the intensity of pressure at section 2.



At section 1,

$$D_1 = 20 \text{ cm} = 0.2 \text{ m}$$

$$A_1 = \frac{\pi}{4}(.2)^2 = .0314 \text{ m}^2$$

$$p_1 = 39.24 \text{ N/cm}^2$$
$$= 39.24 \times 10^4 \text{ N/m}^2$$

$$z_1 = 6.0 \text{ m}$$

At section 2,

$D_2 = 0.10$ m

$$A_2 = \frac{\pi}{4}(0.1)^2 = .00785 \text{ m}^2$$

$z_2 = 4$ m

$p_2 = ?$

$$Q = 35 \text{ lit/s} = \frac{35}{1000} = .035 \text{ m}^3/\text{s}$$

$$Q = A_1V_1 = A_2V_2$$

$$V_1 = \frac{Q}{A_1} = \frac{.035}{.0314} = 1.114 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{.035}{.00785} = 4.456 \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

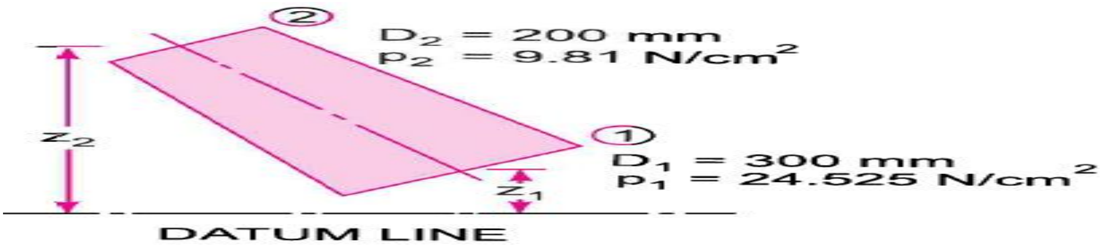
$$\frac{39.24 \times 10^4}{1000 \times 9.81} + \frac{(1.114)^2}{2 \times 9.81} + 6.0 = \frac{p_2}{1000 \times 9.81} + \frac{(4.456)^2}{2 \times 9.81} + 4.0$$

$$p_2 = 41.051 \times 9810 \text{ N/m}^2$$

$$= \frac{41.051 \times 9810}{10^4} \text{ N/cm}^2 = \mathbf{40.27\ N/cm^2}.$$

Problem 3:

Water is flowing through a pipe having diameter 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm^2 and the pressure at the upper end is 9.81 N/cm^2. Determine the difference in datum head if the rate of flow through pipe is 40 lit/s.



Section 1,

D_1 = 300 mm = 0.3 m

p_1 = 24.525 N/cm^2 = 24.525 × 10^4 N/m^2

Section 2,

D_2 = 200 mm = 0.2 m

p_2 = 9.81 N/cm^2 = 9.81 × 10^4 N/m^2

Rate of flow = 40 lit/s

$$Q = \frac{40}{1000} = 0.04 \text{ m}^3/\text{s}$$

$$A_1V_1 = A_2V_2 = \text{rate of flow} = 0.04$$

$$V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4}D_1^2} = \frac{0.04}{\frac{\pi}{4}(0.3)^2} = 0.5658 \text{ m/s}$$

$$\simeq 0.566 \text{ m/s}$$

$$V_2 = \frac{.04}{A_2} = \frac{.04}{\frac{\pi}{4}(D_2)^2} = \frac{0.04}{\frac{\pi}{4}(0.2)^2} = 1.274 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\frac{24.525\times10^4}{1000\times9.81}+\frac{.566\times.566}{2\times9.81}+z_1=\frac{9.81\times10^4}{1000\times9.81}+\frac{(1.274)^2}{2\times9.81}+z_2$$

$$25 + .32 + z_1 = 10 + 1.623 + z_2$$

$$25.32 + z_1 = 11.623 + z_2$$

$$z_2 - z_1 = 25.32 - 11.623 = 13.697 = 13.70 \text{ m}$$

Difference in datum head $= z_2 - z_1 =$ **13.70 m. Ans.**

Problem 4:

The water is flowing through a taper pipe of length 100 m having diameters 600 mm at the upper end and 300 mm at the lower end, at the rate of 50 litres/s. The pipe has a slope of 1 in 30. Find the pressure at the lower end if the pressure at the higher level is 19.62 N/cm².

$L = 100$ m

$D_1 = 600$ mm $= 0.6$ m

$$A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}\times(.6)^2$$

$= 0.2827 \text{ m}^2$

p_1 = pressure at upper end

$= 19.62 \text{ N/cm}^2$

$D_2 = 300$ mm $= 0.3$ m

$$A_2 = \frac{\pi}{4}D_2^2 = \frac{\pi}{4}(.3)^2 = 0.07068 \text{ m}$$

$$Q = \text{rate of flow} = 50 \text{ litres/s} = \frac{50}{1000} = 0.05 \text{ m}^3/\text{s}$$

Let the datum line passes through the centre of the lower end.

Then $z_2 = 0$

As slope is 1 in 30 means $\quad z_1 = \frac{1}{30} \times 100 = \frac{10}{3}$ m

Also we know $\quad Q = A_1 V_1 = A_2 V_2$

$\therefore \quad V_1 = \frac{Q}{A} = \frac{0.05}{.2827} = 0.1768$ m/sec $= 0.177$ m/s

and $\quad V_2 = \frac{Q}{A_2} = \frac{0.5}{.07068} = 0.7074$ m/sec $= 0.707$ m/s

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{.177^2}{2 \times 9.81} + \frac{10}{3} = \frac{p_2}{\rho g} + \frac{.707^2}{2 \times 9.81} + 0$$

$$20 + 0.001596 + 3.334 = \frac{p_2}{\rho g} + 0.0254$$

$$23.335 - 0.0254 = \frac{p_2}{1000 \times 9.81}$$

$$p_2 = 23.3 \times 9810 \text{ N/m}^2 = 228573 \text{ N/m}^2 = \mathbf{22.857 \text{ N/cm}^2}.$$

Practical applications of Bernoulli's equation:

Although Bernoulli's equation is applicable in all problems of incompressible flow where there is involvement of energy considerations. But we shall consider its application to the following measuring devices. 1) Venturimeter 2) Orifice meter 3) Pitot tube

Venturimeter: is a device used for measuring the rate of flow of a fluid flowing through a pipe. It consists of three parts:

- A short converging part
- Throat
- Diverging part

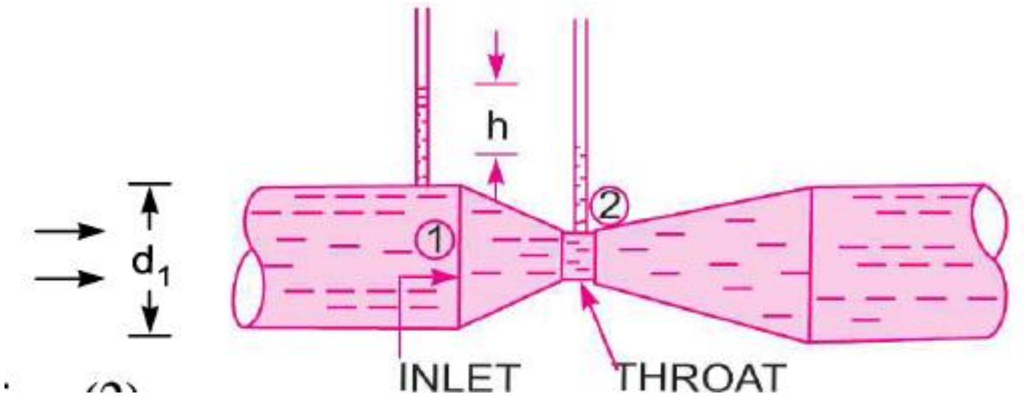


Figure 4.4. Venturimeter

Let d_1 = diameter at inlet or at section (1),

p_1 = pressure at section (1)

v_1 = velocity of fluid at section (1),

a = area at section (1) = $\frac{\pi}{4} d_1^2$

d_2, p_2, v_2, a_2 are corresponding values at section (2).

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

As pipe is horizontal, hence $z_1 = z_2$

$$\therefore \quad \frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} \quad \text{or} \quad \frac{p_1 - p_2}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But $\frac{p_1 - p_2}{\rho g}$ is the difference of pressure heads at sections 1 and 2 and it is equal to h or $\frac{p_1 - p_2}{\rho g} = h$

Substituting this value of $\frac{p_1 - p_2}{\rho g}$ in the above equation, we get

$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

Now applying continuity equation at sections 1 and 2

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2 v_2}{a_1}$$

Substituting this value of v_1

$$h = \frac{v_2^2}{2g} - \frac{\left(\dfrac{a_2 v_2}{a_1}\right)^2}{2g} = \frac{v_2^2}{2g}\left[1 - \frac{a_2^2}{a_1^2}\right] = \frac{v_2^2}{2g}\left[\frac{a_1^2 - a_2^2}{a_1^2}\right]$$

$$v_2{}^2 = 2gh\,\frac{a_1^2}{a_1^2 - a_2^2}$$

$$v_2 = \sqrt{2gh\frac{a_1^2}{a_1^2 - a_2^2}} = \frac{a_1}{\sqrt{a_1^2 - a_2^2}}\sqrt{2gh}$$

Discharge, $Q = a_2 v_2$

$$= a_2\,\frac{a_1}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

Equation gives the discharge under ideal conditions and is called, theoretical discharge. Actual discharge will be less than theoretical discharge.

$$\therefore \qquad Q_{\text{act}} = C_d \times \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

where C_d = Co-efficient of venturimeter and its value is less than 1.

Value of 'h' given by differential U-tube manometer

Case I. Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let

S_h = Sp. gravity of the heavier liquid
S_o = Sp. gravity of the liquid flowing through pipe
x = Difference of the heavier liquid column in U-tube

Then
$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

Case II. If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

$$h = x\left[1 - \frac{S_l}{S_o}\right]$$

where S_l = Sp. gr. of lighter liquid in U-tube
S_o = Sp. gr. of fluid flowing through pipe
x = Difference of the lighter liquid columns in U-tube.

Case III. Inclined Venturimeter with Differential U-tube manometer. The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_h}{S_o} - 1\right]$$

Case IV. Similarly, for inclined venturimeter in which differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[1 - \frac{S_l}{S_o}\right]$$

Problem 5:

A horizontal venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to the inlet and the throat is 20 cm of mercury. Determine the rate of flow. Take C_d = 0.98.

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ Area at inlet, $a_1 = \frac{\pi}{4} d_1^{\ 2} = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \frac{\pi}{4} \times 15^2 = 176.7 \text{ cm}^2$

$C_d = 0.98$

Reading of differential manometer = x = 20 cm of mercury.

$\therefore$ Difference of pressure head is given by (6.9)

or

$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

where S_h = Sp. gravity of mercury = 13.6, S_o = Sp. gravity of water = 1

$$= 20\left[\frac{13.6}{1} - 1\right] = 20 \times 12.6 \text{ cm} = 252.0 \text{ cm of water.}$$

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 9.81 \times 252}$$

$$= \frac{86067593.36}{\sqrt{499636.9 - 31222.9}} = \frac{86067593.36}{684.4}$$

$$= 125756 \text{ cm}^3\text{/s} = \frac{125756}{1000} \text{ lit/s} = \mathbf{125.756 \text{ lit/s.}}$$

Problem 6:

A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of oil of sp. gr. 0.8. The discharge of oil through venturimeter is 60 litres/s. Find the reading of the oil-mercury differential manometer. Take C_d = 0.98.

$$d_1 = 20 \text{ cm}$$

$$a_1 = \frac{\pi}{4}\, 20^2 = 314.16 \text{ cm}^2$$

$$d_2 = 10 \text{ cm}$$

$$a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$$

$$C_d = 0.98$$

$$Q = 60 \text{ litres/s} = 60 \times 1000 \text{ cm}^3\text{/s}$$

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$60 \times 1000 = 9.81 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times h} = \frac{1071068.78\sqrt{h}}{304}$$

$$\sqrt{h} = \frac{304 \times 60000}{1071068.78} = 17.029$$

$$h = (17.029)^2 = 289.98 \text{ cm of oil}$$

$$289.98 = x\left[\frac{13.6}{0.8} - 1\right] = 16x$$

$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

$$x = \frac{289.98}{16} = 18.12 \text{ cm.}$$

Reading of oil-mercury differential manometer = **18.12 cm.**

Problem 7:

The inlet and throat diameters of a horizontal venturimeter are 30 cm and 10 cm respectively. The liquid flowing through the meter is water. The pressure intensity at inlet is 13.734 N/cm^2 while the vacuum pressure head at the throat is 37 cm of mercury. Find the rate of flow. Assume that 4% of the differential head is lost between the inlet and throat. Find also the value of C_d for the venturimeter.

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \frac{\pi}{4}(30)^2 = 706.85\ \text{cm}^2$

Dia. at throat, $d_2 = 10$ cm

$\therefore$ $a_2 = \frac{\pi}{4}(10)^2 = 78.54\ \text{cm}^2$

Pressure, $p_1 = 13.734\ \text{N/cm}^2 = 13.734 \times 10^4\ \text{N/m}^2$

$\therefore$ Pressure head, $\frac{p_1}{\rho g} = \frac{13.734 \times 10^4}{1000 \times 9.81} = 14$ m of water

$$\frac{p_2}{\rho g} = -\ 37 \text{ cm of mercury}$$

$$= \frac{-37 \times 13.6}{100} \text{ m of water} = -\ 5.032 \text{ m of water}$$

Differential head,

$$h = p_1/\rho g - p_2/\rho g$$
$$= 14.0 - (-\ 5.032) = 14.0 + 5.032$$
$$= 19.032 \text{ m of water} = 1903.2 \text{ cm}$$

Head lost, $h_f = 4\%$ of $h = \frac{4}{100} \times 19.032 = 0.7613$ m

$$\therefore \quad C_d = \sqrt{\frac{h - h_f}{h}} = \sqrt{\frac{19.032 - .7613}{19.032}} = 0.98$$

$$\therefore \quad \text{Discharge} = C_d \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$\therefore$ Discharge

$$= C_d \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$= \frac{0.98 \times 706.85 \times 78.54 \times \sqrt{2 \times 981 \times 1903.2}}{\sqrt{(706.85)^2 - (78.54)^2}}$$

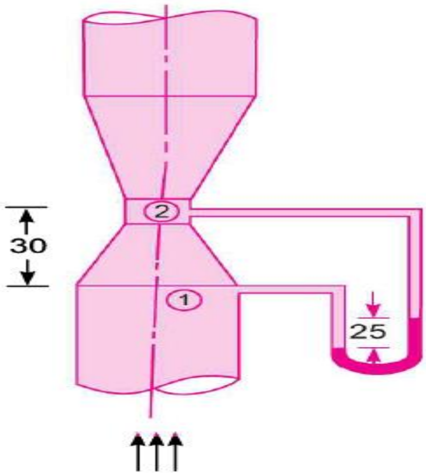
$$= \frac{105132247.8}{\sqrt{499636.9 - 6168}} = 149692.8 \text{ cm}^3/\text{s} = \mathbf{0.14969\ m^3/s.}$$

Problem 8:

A 30 cm × 15 cm venturimeter is provided in a vertical pipe line carrying oil of specific gravity 0.9, the flow being upwards. The difference in elevation of the throat section and entrance section of the venturimeter is 30 cm. The differential U-tube mercury manometer shows a gauge deflection of 25 cm. Calculate :

(i) the discharge of oil, and

(ii) the pressure difference between the entrance section and the throat section. Take the co-efficient of discharge as 0.98 and specific gravity of mercury as 13.6.



Dia. at inlet, $d_1 = 30$ cm

$\therefore$ Area, $a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ Area, $a_2 = \frac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

Sp. gr. of oil, $S_o = 0.9$

Sp. gr. of mercury, $S_g = 13.6$

Reading of diff. manometer, $x = 25$ cm

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right)$$

$$= x\left[\frac{S_g}{S_o} - 1\right] = 25\left[\frac{13.6}{0.9} - 1\right] = 352.77 \text{ cm of oil}$$

(i) The discharge, Q of oil

$$= C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= \frac{0.98 \times 706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} = \sqrt{2 \times 981 \times 352.77}$$

$$= \frac{101832219.9}{684.4} = 148790.5 \text{ cm}^3\text{/s}$$

= 148.79 litres/s. Ans.

(ii) Pressure difference between entrance and throat section

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = 352.77$$

$$\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) + z_1 - z_2 = 352.77$$

$$z_2 - z_1 = 30 \text{ cm}$$

$$\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) - 30 = 352.77$$

$\frac{p_1}{\rho g} - \frac{p_2}{\rho g}$ = 352.77 + 30 = 382.77 cm of oil = **3.8277 m of oil.**

$$(p_1 - p_2) = 3.8277 \times \rho g$$

$$= \text{Sp. gr. of oil} \times 1000 \text{ kg/m}^3$$

$$= 0.9 \times 1000 = 900 \text{ kg/cm}^3$$

$$(p_1 - p_2) = 3.8277 \times 900 \times 9.81 \ \frac{\text{N}}{\text{m}^2}$$

$$= \frac{33795}{10^4} \text{ N/cm}^2 = \mathbf{3.3795 \ N/cm^2.}$$

Orifice Flow Measurement – History:

The first record of the use of orifices for the measurement of fluids was by Giovanni B.Venturi, an Italian Physicist, who in 1797 did some work that led to the development of the modern Venturi Meter by Clemons Herschel in 1886. It has been reported that an orifice meter, designed by Professor Robinson of Ohio State University was used to measure gas near Columbus, Ohio, about 1890. About 1903 Mr. T.B. Weymouth began a series of tests in Pennsylvania leading to the publication of coefficients for orifice meters with flange taps. At the same time Mr. E.O. Hickstein made a similar series of tests at Joplin, Missouri, from which he developed data for orifice meters with pipe taps. An orifice in a pipeline is shown in Figure 4.5 with a manometer for measuring the drop in pressure (differential) as the fluid passes thru the orifice. The minimum cross sectional area of the jet is known as the "vena contracta."

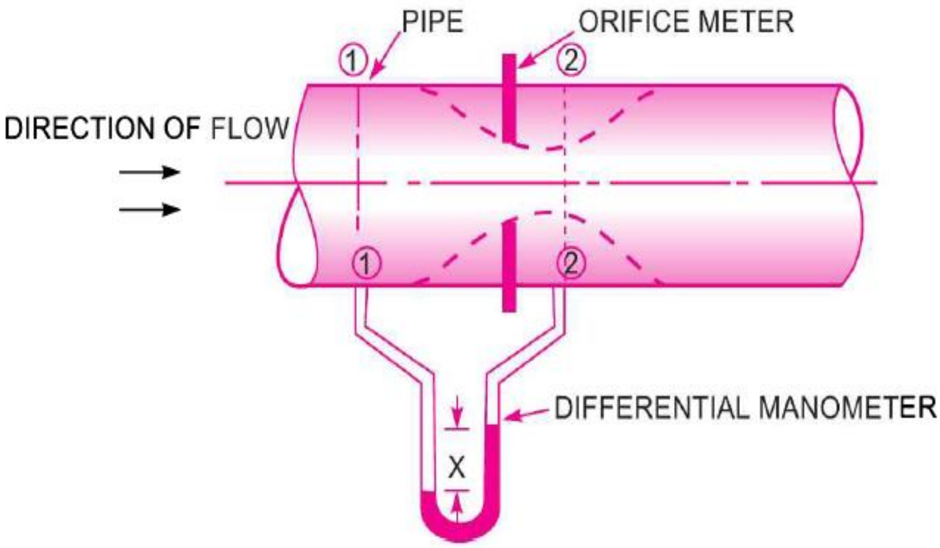


Figure 4.5. Orificemeter

The discharge, Q is given by equation

$$Q = C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

What is an Orifice Meter?

An orifice meter is a conduit and a restriction to create a pressure drop. An hour glass is a form of orifice. A nozzle, venturi or thin sharp edged orifice can be used as the flow restriction. In order to use any of these devices for measurement it is necessary to empirically calibrate them. That is, pass a known volume through the meter and note the reading in order to provide a standard for measuring other quantities. Due to the ease of duplicating and the simple construction, the thin sharp edged orifice has been adopted as a standard and extensive calibration work has been done so that it is widely accepted as a standard means of measuring fluids. Provided the standard mechanics of construction are followed no further calibration is required.

Major Advantages of Orifice Meter Measurement

Flow can be accurately determined without the need for actual fluid flow calibration. Well established procedures convert the differential pressure into flow rate, using empirically derived coefficients. These coefficients are based on accurately measurable dimensions of the orifice plate and pipe diameters as defined in standards, combined with easily measurable characteristics of the fluid, rather than on fluid flow calibrations. With the exception of the orifice meter, almost all flow meters require a fluid flow calibration at flow and temperature conditions closely approximating service operation in order to establish accuracy.

Problem 9:

An orifice meter with orifice diameter 10 cm is inserted in a pipe of 20 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter gives readings of 19.62 N/cm^2 and 9.81 N/cm^2 respectively. Co-efficient of discharge for the orifice meter is given as 0.6. Find the discharge of water through pipe.

Dia. of orifice, $d_0 = 10 \text{ cm}$

$\therefore$ Area, $a_0 = \frac{\pi}{4}(10)^2 = 78.54 \text{ cm}^2$

Dia. of pipe, $d_1 = 20 \text{ cm}$

$\therefore$ Area, $a_1 = \frac{\pi}{4}(20)^2 = 314.16 \text{ cm}^2$

$p_1 = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2$

$$\frac{p_1}{\rho g} = \frac{19.62 \times 10^4}{1000 \times 9.81} = 20 \text{ m of water}$$

$$\frac{p_2}{\rho g} = \frac{9.81 \times 10^4}{1000 \times 9.81} = 10 \text{ m of water}$$

$$h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 20.0 - 10.0 = 10 \text{ m of water} = 1000 \text{ cm of water}$$

$$Q = C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.6 \times \frac{78.54 \times 314.16}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 1000}$$

$$= \frac{20736838.09}{304} = 68213.28 \text{ cm}^3\text{/s} = \mathbf{68.21 \text{ litres/s.}}$$

Problem 10:

An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure difference measured by a mercury oil differential manometer on the two sides of the orifice meter gives a reading of 50 cm of mercury. Find the rate of flow of oil of sp. gr. 0.9 when the co-efficient of discharge of the orifice meter = 0.64.

Dia. of orifice, $d_0 = 15$ cm

$\therefore$ Area, $a_0 = \frac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

Dia. of pipe, $d_1 = 30$ cm

$\therefore$ Area, $a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Sp. gr. of oil, $S_o = 0.9$

Reading of diff. manometer, $x = 50$ cm of mercury

$\therefore$ Differential head, $h = x\left[\frac{S_g}{S_o} - 1\right] = 50\left[\frac{13.6}{0.9} - 1\right]$ cm of oil

$$= 50 \times 14.11 = 705.5 \text{ cm of oil}$$

$$Q = C_d \cdot \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.64 \times \frac{176.7 \times 706.85}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 705.5}$$

$$= \frac{94046317.78}{684.4} = 137414.25 \text{ cm}^3\text{/s} = \mathbf{137.414 \text{ litres/s.}}$$

Pitot tube for Flow Measurement Construction:

The principle of flow measurement by Pitot tube was adopted first by a French Scientist Henri Pitot in 1732 for measuring velocities in the river. A right angled glass tube, large enough for capillary effects to be negligible, is used for the purpose. One end of the tube faces the flow while the other end is open to the atmosphere as shown in Fig.4.6.

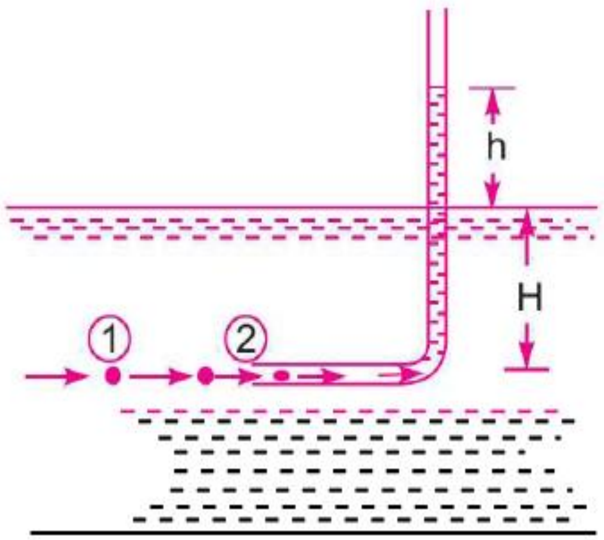


Figure 4.6. Pitot tube

Consider two points (1) and (2) at the same level in such a way that point (2) is just as the inlet of the pitot-tube and point (1) is far away from the tube.

Let p_1 = intensity of pressure at point (1)

v_1 = velocity of flow at (1)

p_2 = pressure at point (2)

v_2 = velocity at point (2), which is zero

H = depth of tube in the liquid

h = rise of liquid in the tube above the free surface.

Applying Bernoulli's equation at points (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $z_1 = z_2$ as points (1) and (2) are on the same line and $v_2 = 0$.

$$\frac{p_1}{\rho g} = \text{pressure head at (1)} = H$$

$$\frac{p_2}{\rho g} = \text{pressure head at (2)} = (h + H)$$

Substituting these values, we get

$$\therefore \quad H + \frac{v_1^2}{2g} = (h + H) \quad \therefore \quad h = \frac{v_1^2}{2g} \quad \text{or} \quad v_1 = \sqrt{2gh}$$

This is theoretical velocity. Actual velocity is given by

$$(v_1)_{act} = C_v \sqrt{2gh}$$

where C_v = Co-efficient of pitot-tube

Velocity of flow in a pipe by pitot-tube. For finding the velocity at any point in a pipe by pitot-tube, the following arrangements are adopted :

1. Pitot-tube along with a vertical piezometer tube
2. Pitot-tube connected with piezometer tube
3. Pitot-tube and vertical piezometer tube connected with a differential *U*-tube manometer

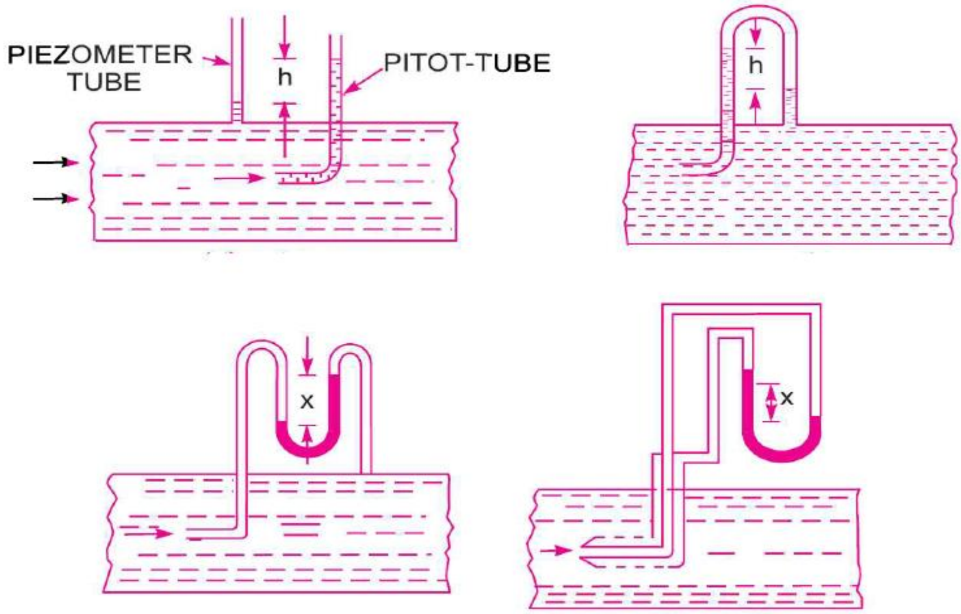


Figure 4.7. Velocity of flow in a pipe by Pitot tube

Problem 11:

Find the velocity of the flow of an oil through a pipe, when the difference of mercury level in a differential U-tube manometer connected to the two tappings of the pitot-tube is 100 mm. Take co-efficient of pitot-tube 0.98 and sp. gr. of oil = 0.8.

Diff. of mercury level, $x = 100 \text{ mm} = 0.1 \text{ m}$

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_g = 13.6$

$C_v = 0.98$

Diff. of pressure head, $$h = x\left[\frac{S_g}{S_o} - 1\right] = .1\left[\frac{13.6}{0.8} - 1\right] = 1.6 \text{ m of oil}$$

$\therefore$ Velocity of flow $= C_v\sqrt{2gh} = 0.98\sqrt{2 \times 9.81 \times 1.6} =$ **5.49 m/s. Ans.**

Problem 12:

A sub-marine moves horizontally in sea and has its axis 15 m below the surface of water. A pitot-tube properly placed just in front of the sub-marine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 170 mm. Find the speed of the sub-marine knowing that the sp. gr. of mercury is 13.6 and that of sea-water is 1.026 with respect of fresh water.

Diff. of mercury level, $x = 170 \text{ mm} = 0.17 \text{ m}$

Sp. gr. of mercury, $S_g = 13.6$

Sp. gr. of sea-water, $S_o = 1.026$

$$\therefore \quad h = x\left[\frac{S_g}{S_o} - 1\right] = 0.17\left[\frac{13.6}{1.026} - 1\right] = 2.0834 \text{ m}$$

$$\therefore \quad V = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2.0834} = 6.393 \text{ m/s}$$

$$= \frac{6.393 \times 60 \times 60}{1000} \text{ km/hr} = \mathbf{23.01\ km/hr.\ Ans.}$$

Problem 13:

A pitot-tube is inserted in a pipe of 300 mm diameter. The static pressure in pipe is 100 mm of mercury (vacuum). The stagnation pressure at the centre of the pipe, recorded by the pitot-tube is 0.981 N/cm². Calculate the rate of flow of water through pipe, if the mean velocity of flow is 0.85 times the central velocity. Take $C_v = 0.98$.

Dia. of pipe, $d = 300 \text{ mm} = 0.30 \text{ m}$

$\therefore$ Area, $a = \frac{\pi}{4} d^2 = \frac{\pi}{4}(.3)^2 = 0.07068 \text{ m}^2$

Static pressure head $= 100$ mm of mercury (vacuum)

$$= -\frac{100}{1000} \times 13.6 = -1.36 \text{ m of water}$$

Stagnation pressure $= .981 \text{ N/cm}^2 = .981 \times 10^4 \text{ N/m}^2$

$\therefore$ Stagnation pressure head $= \frac{.981 \times 10^4}{\rho g} = \frac{.981 \times 10^4}{1000 \times 9.81} = 1 \text{ m}$

$\therefore$ h = Stagnation pressure head – Static pressure head

$= 1.0 - (-1.36) = 1.0 + 1.36 = 2.36$ m of water

$\therefore$ Velocity at centre $= C_v \sqrt{2gh}$

$$= 0.98 \times \sqrt{2 \times 9.81 \times 2.36} = 6.668 \text{ m/s}$$

Mean velocity, $\bar{V} = 0.85 \times 6.668 = 5.6678$ m/s

$\therefore$ Rate of flow of water $= \bar{V} \times$ area of pipe

$= 5.6678 \times 0.07068\ \text{m}^3/\text{s} =$ **$0.4006\ \text{m}^3/\text{s}$. Ans.**

Force exerted by a flowing fluid on a pipe bend

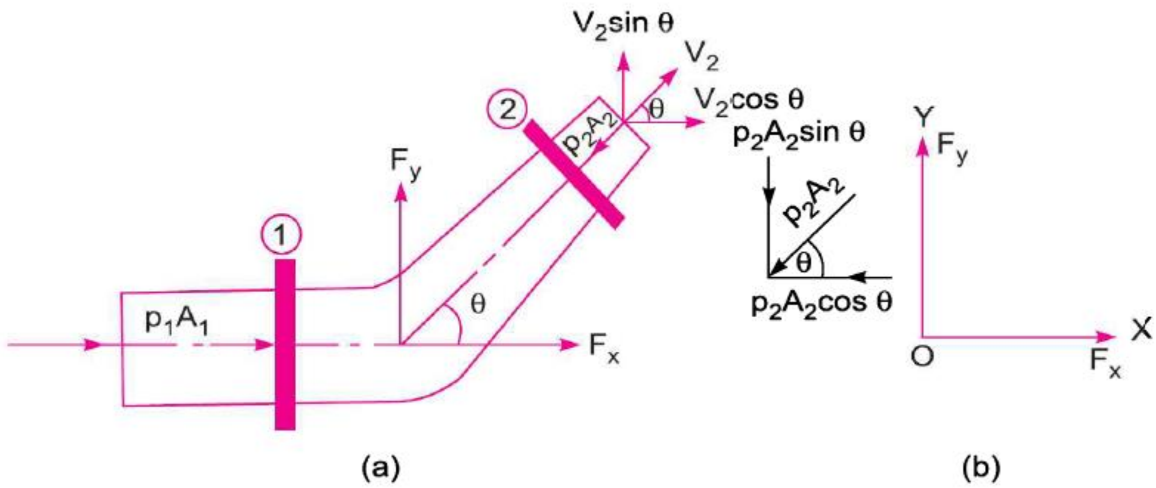


Figure4.8. Forces on bend

Let v_1 = velocity of flow at section (1),

p_1 = pressure intensity at section (1),

A_1 = area of cross-section of pipe at section (1) and

v_2, p_2, A_2 = corresponding values of velocity, pressure and area at section (2)

Net force acting on fluid in the direction of x = Rate of change of momentum in x-direction

$\therefore \quad p_1A_1 - p_2A_2 \cos\theta - F_x$ = (Mass per sec) (change of velocity)

$= \rho Q$ (Final velocity in the direction of x – Initial velocity in the direction of x)

$$= \rho Q\,(V_2 \cos\theta - V_1)$$

$$F_x = \rho Q\,(V_1 - V_2 \cos\theta) + p_1A_1 - p_2A_2 \cos\theta$$

Similarly the momentum equation in y-direction gives

$$0 - p_2A_2 \sin\theta - F_y = \rho Q\,(V_2 \sin\theta - 0)$$

$$\therefore \quad F_y = \rho Q\,(-V_2 \sin\theta) - p_2A_2 \sin\theta$$

Now the resultant force (F_R) acting on the bend

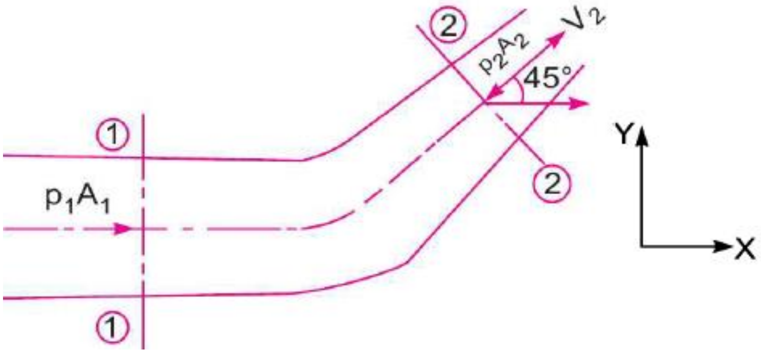
$$= \sqrt{F_x^2 + F_y^2}$$

And the angle made by the resultant force with horizontal direction is given by

$$\tan \theta = \frac{F_y}{F_x}$$

Problem 14:

A 45° reducing bend is connected in a pipe line, the diameters at the inlet and outlet of the bend being 600 mm and 300 mm respectively. Find the force exerted by water on the bend if the intensity of pressure at inlet to bend is 8.829 N/cm² and rate of flow of water is 600 litres/s.



Angle of bend, $\theta = 45°$

Dia. at inlet, $D_1 = 600 \text{ mm} = 0.6 \text{ m}$

∴ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.6)^2 = 0.2827 \text{ m}^2$

Dia. at outlet, $D_2 = 300 \text{ mm} = 0.30 \text{ m}$

∴ Area, $A_2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$

Pressure at inlet, $p_1 = 8.829 \text{ N/cm}^2 = 8.829 \times 10^4 \text{ N/m}^2$

$$Q = 600 \text{ lit/s} = 0.6 \text{ m}^3/\text{s}$$

$$V_1 = \frac{Q}{A_1} = \frac{0.6}{.2827} = 2.122 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.6}{.07068} = 8.488 \text{ m/s.}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$z_1 = z_2$$

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \quad \text{or} \quad \frac{8.829\times 10^4}{1000\times 9.81} + \frac{2.122^2}{2\times 9.81} = \frac{p_2}{\rho g} + \frac{8.488^2}{2\times 9.81}$$

$$9 + .2295 = p_2/\rho g + 3.672$$

$$\frac{p_2}{\rho g} = 9.2295 - 3.672 = 5.5575 \text{ m of water}$$

$$p_2 = 5.5575 \times 1000 \times 9.81 \text{ N/m}^2 = 5.45 \times 10^4 \text{ N/m}^2$$

Forces on the bend in x- and y-directions are given by equations

$$\begin{aligned} F_x &= \rho Q\,[V_1 - V_2 \cos\theta] + p_1 A_1 - p_2 A_2 \cos\theta \\ &= 1000 \times 0.6\,[2.122 - 8.488 \cos 45°] \\ &\qquad + 8.829 \times 10^4 \times .2827 - 5.45 \times 10^4 \times .07068 \times \cos 45° \\ &= -2327.9 + 24959.6 - 2720.3 = 24959.6 - 5048.2 \\ &= 19911.4 \text{ N} \end{aligned}$$

TEXT / REFERENCE BOOKS

1. Bansal.R.K, Fluid Mechanics & Hydraulics Machines, 9th Edition, Laxmi Publications, 2015.
2. Modi P.N., Seth S.M., Hydraulics and Fluid Mechanics Including Hydraulic Machines, 21st Edition, Standard Book House, 2017.
3. Goyal, Manish Kumar, Fluid Mechanics and Hydraulic Machines, PHI Learning Pvt. Ltd., 2015.
4. Kumar K.L., Engineering Fluid Mechanics, 8thEdition, Eurasia Publication House (P) Ltd, 2014.
5. R. K. Rajput, Fluid Mechanics & Hydraulics Machines, 4th Edition, S. Chand Limited, 2008.

6. Yunus.A.Cengel, John.M.Cimbala, Fluid Mechanics Fundamentals and Application, 3rd Edition, Tata McGraw Hill, 2013.
7. Robert W. Fox, Alan T. McDonald, Philip J.Pritchard, John W.Mitchell, Fluid Mechanics, 9th Edition, Wiley India Edition, 2016.
8. Franck M.White, Fluid Mechanics, 8th Edition, Tata McGraw Hill Publication, 2015

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SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

SMEA1306 SOLID AND FLUID MECHANICS

UNIT 5 FLOW THROUGH ORIFICE, NOTCHES AND WEIR AND PIPES

1. Flow through Orifice, Notches and Weir and Pipes

Hydraulic co-efficient-Flow through orifice, Notches and weirs. Laminar and Turbulent flow-Reynolds experiment-laminar flow through circular pipe (Hagen poiseulle's)-Major and minor losses in pipes-Darcy weisbach's equation, chezy's formula-friction factor- moody diagram-pipes in series and pipes in parallel-total energy line-hydraulic gradient line-Equivalent pipe. Concept of Boundary Layer-Types of boundary layer thickness-drag on flat plate.

Orifice

Orifice is a small opening on the side or at the bottom of a tank, through which a fluid is flowing. The orifices are classified according to the size, shape, nature of discharge and shape of the edge.

1. According to the size of orifice and head of liquid from the centre of the orifice: Small orifice and Large orifice.

Small Orifice: If the head of liquid from the centre of orifice is more than five times the depth of orifice, the orifice is called small orifice.
Large Orifice: If the head of liquid is less than five times the depth of orifice, it is known as large orifice.

2. According to shape of orifice: (i) Circular orifice, (ii) Triangular orifice,(iii) Rectangular orifice and (iv) Square orifice
3. According to their cross-sectional area or edge: (i) Sharp-edged orifice and (ii) Bell mouthed orifice

According to the discharge condition: (i) Free discharging orifices (ii) Fully drowned or submerged orifices and (iii) Partially submerged orifices

Flow through a Small Orifice

Flow from a tank through a hole in the side.

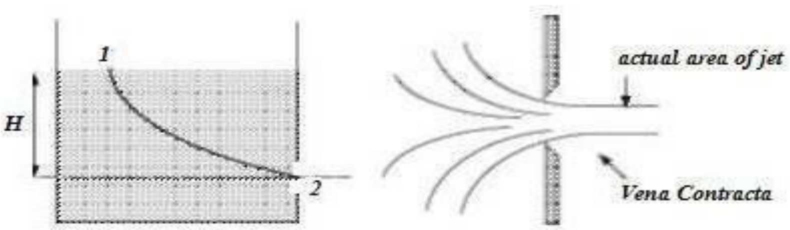


Fig.5.1. Flow through a small Orifice

The edges of the hole are sharp to minimize frictional losses by minimizing the contact between the hole and the liquid. The streamlines at the orifice contract reducing the area of flow. This contraction is called the vena contracta.
The amount of contraction must be known to calculate the flow.
Applying Bernoulli's equation along the streamline joining point 1 on the surface to point 2 at the centre of the orifice.
At the surface velocity is negligible (v1 = 0) and the pressure atmospheric (p1 = 0). At the orifice the jet is open to the atmosphere so again the pressure is atmospheric (p2 = 0).

If we take the datum line through the orifice then Z1 = H and Z2 = 0 leaving h = 2Z2 = h=√2
This theoretical value of velocity is an overestimate as friction losses have not been taken into account.
Each orifice has its own coefficient of velocity, they usually lie in the range 0.97 - 0.99
The discharge through the orifice = jet area X jet velocity
The area of the jet is the area of the vena contracta and not the area of the orifice. We use a Coefficient of contraction to get the area of the
jet,Aa.
Aa = Cc x area of orifice

Discharge through the Orifice Q = Area x Velocity Actual Discharge Qa = C_d x Q_{th}
Q_{th} = Area of Orifice x V_{th} Hydraulic Coefficient
The following three coefficients are known as hydraulic coefficients or orifice coefficient
Coefficient of Contraction
Coefficient of Velocity
Coefficient of Discharge

Coefficient of Contraction:

The ratio of the area of the jet, at vena-contracta, to the area of the orifice is known as *coefficient of contraction*. Mathematically coefficient of contraction,The value of Coefficient of contraction varies slightly with the available head of theliquid, size and shape of the orifice. The average value ofis 0.64.

$$C_c = \frac{Area\ of\ jet\ at\ vena\ contracta}{Area\ of\ the\ orifice}$$

C_c

Coefficient of Velocity:

The ratio of actual velocity of the jet, at vena-contracta, to the theoretical velocity is known as *coefficient of velocity*.
The theoretical velocity of jet at vena-contracta is given by the relation, *h* = √2
, where *H* is the head of water at vena-contracta. Mathematically coefficient of velocity.

$$C_v = \frac{Actual\ velocity\ of\ the\ jet\ at\ vena\ contracta}{Theoretical\ velocity\ of\ the\ jet}$$

The difference between the velocities is due to friction of the orifice. The value of Coefficient of velocity varies slightly with the different shapes of the edges of the orifice. This value is very small for sharp-edged orifices. For a sharp edged orifice, the value of increases with the head of water.

Coefficient of Discharge:

The ratio of a actual discharge through an orifice to the theoretical discharge is known as *coefficient of discharge*. Mathematically coefficient of discharge,

$$C_d = \frac{Actual\ discharge}{Theoretical\ discharge}$$

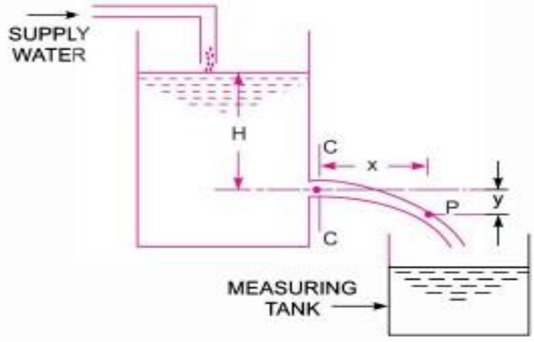
$$= \frac{Actual\ velocity \times Actual\ area}{Theoretical\ velocity \times Theoretical\ area}$$
$$= C_v \times C_c$$

Determination of Coefficient of Discharge (Cd):

The water is allowed to flow through an orifice provided in a tank under a constant head H. The water is collected in a collecting tank for a known height. The time of collection of water in the collecting tank is noted down

$$Q = \frac{\text{Area of measuring tank} \times \text{Height of water in measuring tank}}{\text{Time } (t)}$$

and theoretical discharge = area of orifice $\times \sqrt{2gH}$



$$\therefore \quad C_d = \frac{Q}{a \times \sqrt{2gH}}$$

Determination of Coefficient of Velocity (Cv): Let C-C represents the vena – contracta of a jet water coming out from an orifice under constant head H as shown in fig. Consider a liquid particle which is at vena contracta at any time and takes the position at P along the jet time t.

Let x = horizontal distance travelled by the particle in time 't'

y = vertical distance between P and C-C

V = actual velocity of jet at vena-contracta.

Then horizontal distance, $x = V \times t$

and vertical distance, $y = \frac{1}{2} gt^2$

From equation (*i*), $t = \frac{x}{V}$

Substituting this value of 't' in (*ii*), we get

$$y = \frac{1}{2} g \times \frac{x^2}{V^2}$$

$$V^2 = \frac{gx^2}{2y}$$

$$\therefore \quad V = \sqrt{\frac{gx^2}{2y}}$$

But theoretical velocity,

$$V_{th} = \sqrt{2gH}$$

$$\therefore \text{Co-efficient of velocity, } C_v = \frac{V}{V_{th}} = \sqrt{\frac{gx^2}{2y}} \times \frac{1}{\sqrt{2gH}} = \sqrt{\frac{x^2}{4yH}}$$

$$= \frac{x}{\sqrt{4yH}}.$$

Determination of Coefficient of Contraction (C_C):

The coefficient of contraction is determined from the

equation $C_d = C_V \times C_C$ $C_C = C_d / C_V$

Flow through Large Orifices:

If the head of liquid is less than 5 times the depth of the orifice, the orifice is called large orifice. In case of small orifice, the velocity in the entire cross-section of the jet is considered to be constant and discharge can be calculated by $Q = C_d \times a \times \sqrt{2gh}$. But in case of a large orifice, the velocity is not constant over the entire cr by $Q = C_d \times a \times \sqrt{2gh}$.

$$= C_d \times b \times \sqrt{2g}\int_{H_1}^{H_2} \sqrt{h}\, dh = C_d \times b \times \sqrt{2g}\left[\frac{h^{3/2}}{3/2}\right]_{H_1}^{H_2}$$

$$= \frac{2}{3} C_d \times b\ \sqrt{2g}\left[H_2^{3/2} - H_1^{3/2}\right].$$

Discharge through Large Rectangular Orifice:

Consider a large rectangular orifice in one side of the tank discharging freely in to atmosphere under a constant head H as shown in fig

Let H_1 = height of liquid above top edge of orifice

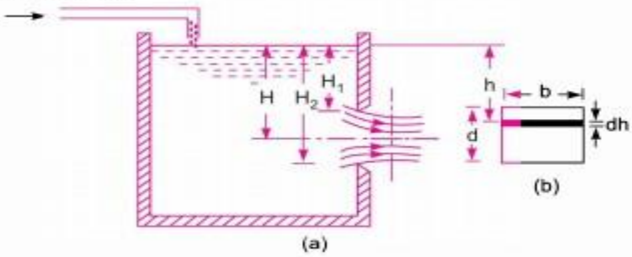
H_2 = height of liquid above bottom edge of orifice

b = breadth of orifice

d = depth of orifice = $H_2 - H_1$

C_d = co-efficient of discharge.

Consider an elementary horizontal strip of depth 'dh' at a depth of 'h' below the free surface of the liquid in the tank as shown in Fig.



Large rectangular orifice.

$\therefore$ Area of strip = $b \times dh$

and theoretical velocity of water through strip = $\sqrt{2gh}$.

$\therefore$ Discharge through elementary strip is given

$$dQ = C_d \times \text{Area of strip} \times \text{Velocity}$$

$$= C_d \times b \times dh \times \sqrt{2gh} = C_d\, b \times \sqrt{2gh}\ dh$$

By integrating the above equation between the limits H_1 and H_2, the total discharge through the whole orifice is obtained

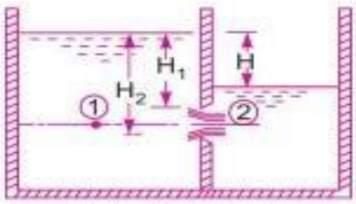
$$\therefore \quad Q = \int_{H_1}^{H_2} C_d \times b \times \sqrt{2gh}\ dh$$

$$= C_d \times b \times \sqrt{2g}\int_{H_1}^{H_2} \sqrt{h}\, dh = C_d \times b \times \sqrt{2g}\left[\frac{h^{3/2}}{3/2}\right]_{H_1}^{H_2}$$

$$= \frac{2}{3}C_d \times b\ \sqrt{2g}\left[H_2^{3/2} - H_1^{3/2}\right].$$

Discharge through Fully Sub-Merged Orifice:

Fully sub-merged orifice is one which has its whole of the outlet side sub merged under liquid so that it discharges a jet of liquid in to the liquid of the same kind. It is also called



Fully sub-merged orifice.

totally drowned orifice as shown in Fig. Consider two points (1) & (2). Point 1 being in the reservoir on the upstream side of the orifice and point 2 being at vena contracta.

Fig.4.Fully Sub-merged Orifice

Let H_1 = Height of water above the top of the orifice on the upstream side,

H_2 = Height of water above the bottom of the orifice,

H = Difference in water level,

b = Width of orifice,

C_d = Co-efficient of discharge.

Height of water above the centre of orifice on upstream side

$$= H_1 + \frac{H_2 - H_1}{2} = \frac{H_1 + H_2}{2}$$

Height of water above the centre of orifice on downstream side

$$= \frac{H_1 + H_2}{2} - H$$

Applying Bernoulli's equation at (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g}$$

Now $\quad \dfrac{p_1}{\rho g} = \dfrac{H_1 + H_2}{2}, \dfrac{p_2}{\rho g} = \dfrac{H_1 + H_2}{2} - H$ and V_1 is negligible

$$\therefore \qquad \frac{H_1 + H_2}{2} + 0 = \frac{H_1 + H_2}{2} - H + \frac{V_2^2}{2g}$$

$$\therefore \qquad \frac{V_2^2}{2g} = H$$

$$\therefore \qquad V_2 = \sqrt{2gH}$$

$$\therefore \qquad V_2 = \sqrt{2gH}$$

Area of orifice $= b \times (H_2 - H_1)$

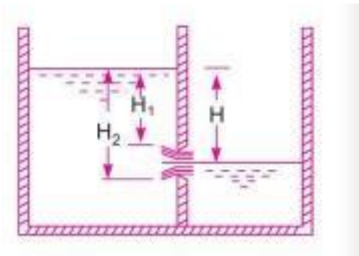
$\therefore$ Discharge through orifice $= C_d \times$ Area $\times$ Velocity

$$= C_d \times b\,(H_2 - H_1) \times \sqrt{2gH}$$

$$\therefore \qquad Q = C_d \times b\,(H_2 - H_1) \times \sqrt{2gH}.$$

Discharge through Partially Sub-Merged Orifice:

Partially sub-merged orifice is one which has its outlet side partially sub-merged under liquid as shown in Fig. It is also known as partially drowned orifice. Thus the partially sub-merged orifice has two portions. The upper portion behaves as an orifice discharging free while the lower portion behaves as a sub-merged orifice. Only a large orifice can behave as a partially sub-merged orifice. The total discharge Q through partially sub-merged orifice is equal to the discharges through free and the sub-merged portions.



Partially sub-merged orifice.

Fig.5.2 Partially sub-merged orifice

Discharge through the free portion is given by equation (7.8) as

$$Q_2 = \frac{2}{3} C_d \times b \times \sqrt{2g}\,[H_2^{3/2} - H_1^{3/2}]$$

$\therefore$ Total discharge $\qquad Q = Q_1 + Q_2$

$$= C_d \times b \times (H_2 - H) \times \sqrt{2gH} + \frac{2}{3} C_d \times b \times \sqrt{2g}\,[H_2^{3/2} - H^{3/2}]. ..$$

Time of Emptying a Tank through an Orifice at its Bottom:

Consider a tank containing some liquid up to a height of H_1. Let an orifice is fitted at the bottom of the tank. It is required to find the time for the liquid surface to fall from the height H_1 to a height H_2.

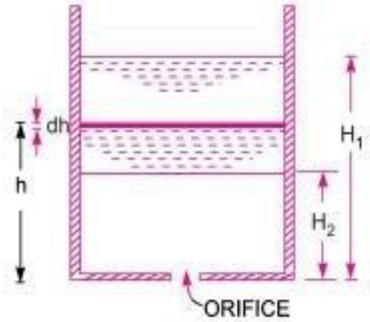


Fig5.3.Time of Emptying a Tank

Let A = Area of the tank

a = Area of the orifice

H_1 = Initial height of the liquid

H_2 = Final height of the liquid

T = Time in seconds for the liquid to fall from H_1 to H_2.

Let at any time, the height of liquid from orifice is h and let the liquid surface fall by a small height dh in time dT. Then

Volume of liquid leaving the tank in time, $dT = A \times dh$

Also the theoretical velocity through orifice, $V = \sqrt{2gh}$

$\therefore$ Discharge through orifice/sec,

$$dQ = C_d \times \text{Area of orifice} \times \text{Theoretical velocity} = C_d \,.\, a \,.\, \sqrt{2gh}$$

$\therefore$ Discharge through orifice in time interval

$$dT = C_d \,.\, a \,.\, \sqrt{2gh} \,.\, dT$$

As the volume of liquid leaving the tank is equal to the volume of liquid flowing through orifice in time dT, we have

$$A(-dh) = C_d \,.\, a \,.\, \sqrt{2gh} \,.\, dT$$

$-$ ve sign is inserted because with the increase of time, head on orifice decreases.

$$\therefore \qquad -Adh = C_d \,.\, a \,.\, \sqrt{2gh} \,.\, dT \text{ or } dT = \frac{-A\,dh}{C_d \,.\, a \,.\, \sqrt{2gh}} = \frac{-A(h)^{-1/2}}{C_d \,.\, a \,.\, \sqrt{2g}} dh$$

By integrating the above equation between the limits H_1 and H_2, the total time, T is obtained as

$$\int_0^T dT = \int_{H_1}^{H_2} \frac{-Ah^{-1/2}\,dh}{C_d \,.\, a \,.\, \sqrt{2g}} = \frac{-A}{C_d \,.\, a \,.\, \sqrt{2g}} \int_{H_1}^{H_2} h^{-1/2}\,dh$$

or

$$T = \frac{-A}{C_d \,.\, a \,.\, \sqrt{2g}} \left[\frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_{H_1}^{H_2} = \frac{-A}{C_d \,.\, a \,.\, \sqrt{2g}} \left[\frac{\sqrt{h}}{\frac{1}{2}}\right]_{H_1}^{H_2}$$

$$= \frac{-2A}{C_d \,.\, a \,.\, \sqrt{2g}} \left[\sqrt{H_2} - \sqrt{H_1}\right] = \frac{2A\left[\sqrt{H_1} - \sqrt{H_2}\right]}{C_d \,.\, a \,.\, \sqrt{2g}}$$

For emptying the tank completely, $H_2 = 0$ and hence

$$T = \frac{2A\sqrt{H_1}}{C_d \,.\, a \,.\, \sqrt{2g}}.$$

Time of Emptying a Hemispherical Tank

Consider a hemispherical tank of radius R fitted with an orifice of area "a" at its bottom as shown in Fig. The tank contains some liquid whose initial height is H_1 and in time T, the height of liquid falls to H2. It is required to find the time T.

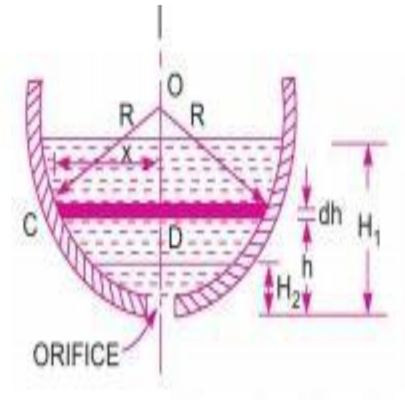


Fig.7. Hemispherical Tank

Let at any instant of time, the head of liquid over the orifice is h and at this instant let x be the radius of the liquid surface. Then

Area of liquid surface, $A = \pi x^2$

and theoretical velocity of liquid $= \sqrt{2gh}$.

Let the liquid level falls down by an amount of dh in time dT.

$\therefore$ Volume of liquid leaving tank in time $dT = A \times dh$

$$= \pi x^2 \times dh$$

Also volume of liquid flowing through orifice

$$= C_d \times \text{area of orifice} \times \text{velocity} = C_d.a.\ \sqrt{2gh}\ \text{second}$$

$\therefore$ Volume of liquid flowing through orifice in time dT

$$= C_d.a.\ \sqrt{2gh} \times dT$$

From equations (i) and (ii), we get

$$\pi x^2 (-\,dh) = C_d.a.\ \sqrt{2gh}\ .\ dT$$

$-$ve sign is introduced, because with the increase of T, h will decrease

$$\therefore \qquad -\,\pi x^2\, dh = C_d.a.\ \sqrt{2gh}\ .\ dT$$

But from Fig. for ΔOCD, we have $OC = R$

$$DO = R - h$$

$$\therefore \quad CD = x = \sqrt{OC^2 - OD^2} = \sqrt{R^2 - (R-h)^2}$$

$$\therefore \quad x^2 = R^2 - (R-h)^2 = R^2 - (R^2 + h^2 - 2Rh) = 2Rh - h^2$$

Substituting x^2 in equation (*iii*), we get

$$-\pi(2Rh - h^2)dh = C_d.a.\sqrt{2gh}\ .\ dT$$

or

$$dT = \frac{-\pi\left(2Rh - h^2\right)dh}{C_d\ .\ a\ .\sqrt{2gh}} = \frac{-\pi}{C_d\ .\ a\ .\sqrt{2g}}\left(2Rh - h^2\right)h^{-1/2}\ \ dh$$

$$= \frac{-\pi}{C_d\ .\ a\ .\sqrt{2g}}\,(2Rh^{1/2} - h^{3/2})dh$$

The total time T required to bring the liquid level from H_1 to H_2 is obtained by integrating the above equation between the limits H_1 and H_2.

$$\therefore \quad T = \int_{H_1}^{H_2}\frac{-\pi}{C_d\ .\ a\ .\sqrt{2g}}\,(2Rh^{1/2} - h^{3/2})dh$$

$$= \frac{-\pi}{C_d\ .\ a\ .\sqrt{2g}}\int_{H_1}^{H_2}(2Rh^{1/2} - h^{3/2})dh$$

$$= \frac{-\pi}{C_d \times a \times \sqrt{2g}}\left[2R\frac{h^{1/2+1}}{\frac{1}{2}+1} - \frac{h^{3/2}+1}{\frac{3}{2}+1}\right]_{H_1}^{H_2}$$

$$= \frac{-\pi}{C_d \times a \times \sqrt{2g}}\left[2 \times \frac{2}{3}Rh^{3/2} - \frac{2}{5}h^{5/2}\right]_{H_1}^{H_2}$$

$$= \frac{-\pi}{C_d \times a \times \sqrt{2g}}\left[\frac{4}{3}R\left(H_2^{3/2} - H_1^{3/2}\right) - \frac{2}{5}\left(H_2^{5/2} - H_1^{5/2}\right)\right]$$

$$= \frac{\pi}{C_d \times a \times \sqrt{2g}}\left[\frac{4}{3}R\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right)\right]$$

For completely emptying the tank, $H_2 = 0$ and hence

$$T = \frac{\pi}{C_d\ .\ a\ .\sqrt{2g}}\left[\frac{4}{3}RH_1^{3/2} - \frac{2}{5}H_1^{5/2}\right].$$

Classification of Mouthpieces:

1. The mouthpieces are classified as (*i*) External mouthpiece or (*ii*) Internal mouthpiece depending upon their position with respect to the tank or vessel to which they are fitted.

2. The mouthpiece are classified as (*i*) Cylindrical mouthpiece or (*ii*) Convergent mouthpiece or (*iii*) Convergent-divergent mouthpiece depending upon their shapes.

3. The mouthpieces are classified as (*i*) Mouthpieces running full or (*ii*) Mouthpieces running free, depending upon the nature of discharge at the outlet of the mouthpiece. This classification is only for internal mouthpieces which are known Borda's or Re-entrant mouthpieces. A mouthpiece is said to be running free if the jet of liquid after contraction does not touch the sides of the mouthpiece. But if the jet after contraction expands and fills the whole mouthpiece it is known as running full.

NOTCHES:

A **notch** is a device used for measuring the rate of flow of a liquid through a small channel or a tank. It may be defined as an opening in the side of a tank or a small channel in such a way that the liquid surface in the tank or channel is below the top edge of the opening.

A **weir** is a concrete or masonary structure, placed in an open channel over which the flow occurs. It is generally in the form of vertical wall, with a sharp edge at the top, running all the way across the open channel. The notch is of small size while the weir is of a bigger size. The notch is generally made of metallic plate while weir is made of concrete or masonary structure.

1. Nappe or Vein. The sheet of water flowing through a notch or over a weir is called Nappe or Vein.

2. Crest or Sill. The bottom edge of a notch or a top of a weir over which the water flows, is known as the sill or crest.

Classification of Notches and Weirs:

The notches are classified as :

1. According to the shape of the opening :
   - (*a*) Rectangular notch,
   - (*b*) Triangular notch,
   - (*c*) Trapezoidal notch, and
   - (*d*) Stepped notch.
2. According to the effect of the sides on the nappe :
   - (*a*) Notch with end contraction.
   - (*b*) Notch without end contraction or suppressed notch.

Weirs are classified according to the shape of the opening, the shape of the crest, the effect of the sides on the nappe and nature of discharge. The following are important classifications.

(*a*) According to the shape of the opening :
- (*i*) Rectangular weir,
- (*ii*) Triangular weir, and
- (*iii*) Trapezoidal weir (Cipolletti weir)

(*b*) According to the shape of the crest :
- (*i*) Sharp-crested weir,
- (*ii*) Broad-crested weir,
- (*iii*) Narrow-crested weir, and
- (*iv*) Ogee-shaped weir.

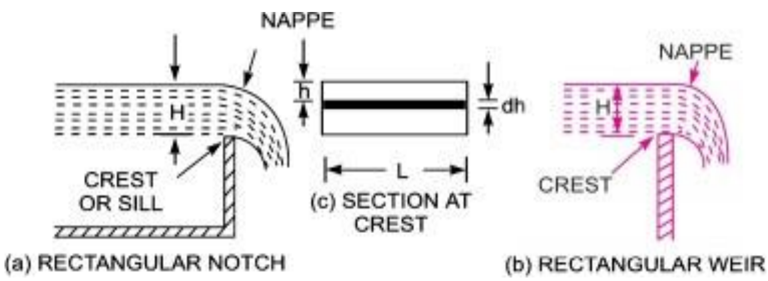


Fig.5.11.RectangulaNotch and Weir

Consider a rectangular notch or weir provided in a channel carrying water as shown in Fig.

Let H = Head of water over the crest

L = Length of the notch or weir

For finding the discharge of water flowing over the weir or notch, consider an elementary horizontal strip of water of thickness dh and length L at a depth h from the free surface of water as shown in Fig.

The area of strip $= L \times dh$

and theoretical velocity of water flowing through strip = $\sqrt{2gh}$

The discharge dQ, through strip is

$$dQ = C_d \times \text{Area of strip} \times \text{Theoretical velocity}$$

$$= C_d \times L \times dh \times \sqrt{2gh} \qquad ...(i)$$

where C_d = Co-efficient of discharge.

The total discharge, Q, for the whole notch or weir is determined by integrating equation (i) between the limits 0 and H.

$$\therefore \qquad Q = \int_0^H C_d \,.\, L \,.\, \sqrt{2gh}\ .\ dh = C_d \times L \times \sqrt{2g} \int_0^H h^{1/2}\, dh$$

$$= C_d \times L \times \sqrt{2g} \left[\frac{h^{1/2+1}}{\frac{1}{2}+1} \right]_0^H = C_d \times L \times \sqrt{2g} \left[\frac{h^{3/2}}{3/2} \right]_0^H$$

$$= \frac{2}{3}\, C_d \times L \times \sqrt{2g}\ [H]^{3/2}.$$

Discharge over a Triangular Notch or Weir:

The expression for the discharge over a triangular notch or weir is the same. It is derived as :

Let H = head of water above the V- notch

θ = angle of notch

Consider a horizontal strip of water of thickness 'dh' at a depth of h from the free surface of water as shown in Fig.

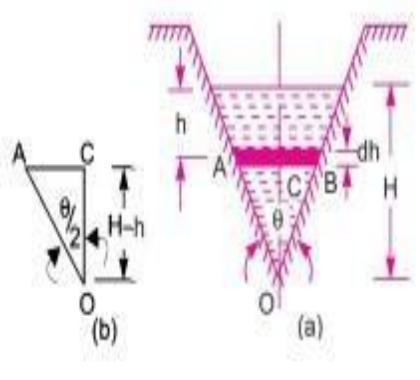


Fig.5.12.Triangular Notch or

Weir

From Fig. we have

$$\tan\frac{\theta}{2} = \frac{AC}{OC} = \frac{AC}{(H-h)}$$

$\therefore$ $$AC = (H-h)\tan\frac{\theta}{2}$$

Width of strip $$= AB = 2AC = 2\,(H-h)\tan\frac{\theta}{2}$$

$\therefore$ Area of strip $$= 2\,(H-h)\tan\frac{\theta}{2}\times dh$$

The theoretical velocity of water through strip = $\sqrt{2gh}$

$\therefore$ Discharge, through the strip,

$$dQ = C_d \times \text{Area of strip} \times \text{Velocity (theoretical)}$$

$$= C_d \times 2\,(H-h)\tan\frac{\theta}{2}\times dh \times \sqrt{2gh}$$

$$= 2C_d\,(H-h)\tan\frac{\theta}{2}\times\sqrt{2gh}\times dh$$

$\therefore$ Total discharge, $$Q = \int_0^H 2C_d\,(H-h)\tan\frac{\theta}{2}\times\sqrt{2gh}\times dh$$

$$= 2C_d\times\tan\frac{\theta}{2}\times\sqrt{2g}\int_0^H (H-h)h^{1/2}\,dh$$

$$= 2\times C_d\times\tan\frac{\theta}{2}\times\sqrt{2g}\int_0^H (Hh^{1/2}-h^{3/2})\,dh$$

$$= 2\times C_d\times\tan\frac{\theta}{2}\times\sqrt{2g}\left[\frac{Hh^{3/2}}{3/2}-\frac{h^{5/2}}{5/2}\right]_0^H$$

$$= 2\times C_d\times\tan\frac{\theta}{2}\times\sqrt{2g}\left[\frac{4}{15}H^{5/2}\right]$$

$$= \frac{8}{15}\,C_d\times\tan\frac{\theta}{2}\times\sqrt{2g}\times H^{5/2}$$

For a right-angled *V*-notch, if $C_d = 0.6$

$$\theta = 90^\circ,\quad \therefore\quad \tan\frac{\theta}{2} = 1$$

Discharge, $$Q = \frac{8}{15}\times 0.6\times 1\times\sqrt{2\times 9.81}\times H^{5/2}$$

$$= 1.417\,H^{5/2}.$$

Discharge over a Trapezoidal Notch or Weir:

As shown in Fig. a trapezoidal notch or weir is a combination of a rectangular and triangular notch or weir. Thus the total discharge will be equal to the sum of discharge through a rectangular weir or notch and discharge through a triangular notch or weir.

Let H = Height of water over the notch

L = Length of the crest of the notch

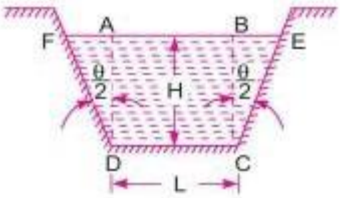


Fig 5.13. Trapezoidal Notch

C_{d_1} = Co-efficient of discharge for rectangular portion $ABCD$ of Fig.

C_{d_2} = Co-efficient of discharge for triangular portion [FAD and BCE]

The discharge through rectangular portion $ABCD$ is given by

or $$Q_1 = \frac{2}{3} \times C_{d_1} \times L \times \sqrt{2g} \times H^{3/2}$$

The discharge through two triangular notches FDA and BCE is equal to the discharge through a single triangular notch of angle θ and it is given by equation

$$Q_2 = \frac{8}{15} \times C_{d_2} \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

∴ Discharge through trapezoidal notch or weir $FDCEF = Q_1 + Q_2$

$$= \frac{2}{3} C_{d_1} L \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan\theta/2 \times \sqrt{2g} \times H^{5/2}.$$

Viscous Flow

This chapter deals with the flow of fluids which are viscous and flowing at very low velocity. At low velocity the fluid moves in layers. Each layer of fluid slides over the adjacent layer. Due to relative velocity between two layers the velocity gradient $\frac{du}{dy}$ exists and hence a shear stress $\tau = \mu \frac{du}{dy}$ acts on the layers.

The following cases will be considered in this chapter :

1. Flow of viscous fluid through circular pipe.
2. Flow of viscous fluid between two parallel plates.
3. Kinetic energy correction and momentum correction factors.
4. Power absorbed in viscous flow through
   (*a*) Journal bearings, (*b*) Foot-step bearings, and (*c*) Collar bearings.

Flow of Viscous Fluid through Circular Pipe:

For the flow of viscous fluid through circular pipe, the velocity distribution across a section, the ratio of maximum velocity to average velocity, the shear stress distribution and drop of pressure for a given length is to be determined. The flow through the circular pipe will be viscous or laminar, if the Reynolds number (R_e*) is less than 2000. The expression for Reynold number is given by

$$R_e = \frac{\rho V D}{\mu}$$

where ρ = Density of fluid flowing through pipe

V = Average velocity of fluid

D = Diameter of pipe and

μ = Viscosity of fluid.

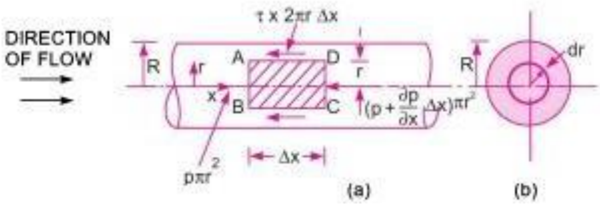


Fig.5.16.Viscous flow through a pipe

Consider a horizontal pipe of radius R. The viscous fluid is flowing from left to right in the pipe as shown in Fig. 9.1 (*a*). Consider a fluid element of radius r, sliding in a cylindrical fluid element of radius $(r + dr)$. Let the length of fluid element be Δx. If 'p' is the intensity of pressure on the face AB, then the intensity of pressure on face CD will be $\left(p + \frac{\partial p}{\partial x}\Delta x\right)$. Then the forces acting on the fluid element are :

1. The pressure force, $p \times \pi r^2$ on face AB.
2. The pressure force, $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2$ on face CD.
3. The shear force, $\tau \times 2\pi r \Delta x$ on the surface of fluid element. As there is no acceleration, hence the summation of all forces in the direction of flow must be zero *i.e.*,

$$p\pi r^2 - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or
$$-\frac{\partial p}{\partial x}\Delta x \pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or
$$-\frac{\partial p}{\partial x} \cdot r - 2\tau = 0$$

$\therefore$
$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2}$$

The shear stress τ across a section varies with 'r' as $\frac{\partial p}{\partial x}$ across a section is constant. Hence shear stress distribution across a section is linear as shown in Fig.

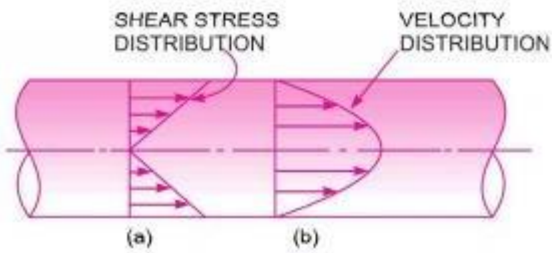


Fig.5.17.Shear stress and velocity distribution across a section

(*i*) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu \frac{du}{dy}$ is substituted in equation (9.1).

But in the relation $\tau = \mu \frac{du}{dy}$, y is measured from the pipe wall. Hence

$$y = R - r \quad \text{and} \quad dy = -dr$$

$\therefore$
$$\tau = \mu \frac{du}{-dr} = -\mu \frac{du}{dr}$$

Substituting this value in , we get

$$-\mu \frac{du}{dr} = -\frac{\partial p}{\partial x}\frac{r}{2} \quad \text{or} \quad \frac{du}{dr} = \frac{1}{2\mu}\frac{\partial p}{\partial x} r$$

Integrating this above equation w.r.t. 'r', we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 + C$$

where C is the constant of integration and its value is obtained from the boundary condition that at $r = R$, $u = 0$.

$$\therefore \quad 0 = \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 + C$$

$$\therefore \quad C = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

Substituting this value of C in equation (9.2), we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2]$$

In equation values of μ, $\frac{\partial p}{\partial x}$ and R are constant, which means the velocity, u varies with the square of r. Thus equation is a equation of parabola. This shows that the velocity distribution across the section of a pipe is parabolic. This velocity distribution is shown in Fig.

(*ii*) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum, when $r = 0$ in equation . Thus maximum velocity, U_{max} is obtained as

$$U_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge of the fluid across the section by the area of the pipe (πR^2). The discharge (Q) across the section is obtained by considering the flow through a circular ring element of radius r and thickness dr as shown in Fig. (*b*). The fluid flowing per second through this elementary ring

$$dQ = \text{velocity at a radius } r \times \text{area of ring element}$$

$$= u \times 2\pi r\, dr$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] \times 2\pi r\, dr$$

$$\therefore \quad Q = \int_0^R dQ = \int_0^R -\frac{1}{4\mu}\frac{\partial p}{\partial x}(R^2 - r^2) \times 2\pi r\, dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \int_0^R (R^2 - r^2)\, r dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \int_0^R (R^2 r - r^3)\, dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \times \frac{R^4}{4} = \frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^4$$

$$\therefore \quad \text{Average velocity,} \qquad \bar{u} = \frac{Q}{\text{Area}} = \frac{\frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^4}{\pi R^2}$$

$$\text{or} \qquad \bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^2$$

Dividing equation (9.4) by equation (9.5),

$$\frac{U_{max}}{\bar{u}} = \frac{-\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2}{\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2} = 2.0$$

$\therefore$ Ratio of maximum velocity to average velocity = 2.0.

(*iii*) **Drop of Pressure for a given Length (*L*) of a pipe**

From equation (9.5), we have

$$\bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^2 \quad \text{or} \quad \left(\frac{-\partial p}{\partial x}\right) = \frac{8\mu\bar{u}}{R^2}$$

Integrating the above equation w.r.t. x, we get

$$-\int_2^1 dp = \int_2^1 \frac{8\mu\bar{u}}{R^2}\,dx$$

$$\therefore \qquad -[p_1 - p_2] = \frac{8\mu\bar{u}}{R^2}[x_1 - x_2] \text{ or } (p_1 - p_2) = \frac{8\mu\bar{u}}{R^2}[x_2 - x_1]$$

$$= \frac{8\mu\bar{u}}{R^2}L$$

$$= \frac{8\mu\bar{u}L}{(D/2)^2} \qquad \left\{\because \; R = \frac{D}{2}\right\}$$

or $\qquad (p_1 - p_2) = \dfrac{32\mu\bar{u}L}{D^2}$, where $p_1 - p_2$ is the drop of pressure.

$$\therefore \quad \text{Loss of pressure head} = \frac{p_1 - p_2}{\rho g}$$

$$\therefore \qquad \frac{p_1 - p_2}{\rho g} = h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$$

Equation (9.6) is called **Hagen Poiseuille Formula.**

Flow in Pipes:

In this chapter, however, a method of expressing the loss using an average flow velocity is stated. Studies will be made on how to express losses caused by a change in the cross sectional area of a pipe, a pipe bend and a valve, in addition to the frictional loss of a pipe. Consider a case where fluid runs from a tank into a pipe whose entrance section is fully rounded. At the entrance, the velocity distribution is roughly uniform while the pressure head is lower by V2/2g./The section from the entrance to just where the boundary layer develops to the tube centre is called the inlet or entrance region, whose length is called the inlet or entrance length. For steady flow at a known flow rate, these regions exhibit the following: Laminar flow: A local velocity constant with time, but which varies spatially due to viscous shear and geometry. Turbulent flow: A local velocity which has a constant mean value but also has a statistically random fluctuating component due to turbulence in the flow. Typical plots of velocity time histories for laminar flow, turbulent flow, and the region of transition between the two are shown below.

$$\mathrm{Re} = \frac{\rho V D}{\mu} = \frac{V D}{\nu}$$

V- Flow velocity

D – Flow dimension

μ - Dynamic Viscosity Ʊ – Kinematic Viscosity

Frictional Loss in Pipe flow

When a liquid is flowing through a pipe, the velocity of the liquid layer adjacent to the pipe wall is zero. The velocity of liquid goes on increasing from the wall and thus velocity gradient and hence shear stresses are produced in the whole liquid due to viscosity. This viscous action causes loss of energy which is usually known as frictional loss.

On the basis of his experiments, William Froude gave the following laws of fluid fraction for turbulent flow.

The frictional resistance for turbulent flow is :

(*i*) proportional to V^n, where n varies from 1.5 to 2.0,

(*ii*) proportional to the density of fluid,

(*iii*) proportional to the area of surface in contact,

(*iv*) independent of pressure,

(*v*) dependent on the nature of the surface in contact.

Expression for Loss of Head due to friction in pipes:

Consider a uniform horizontal pipe having steady flow as shown in fig 18. Let 1-1 and 2-2 are two sections of pipe.
Let P_1 = pressure intensity at section 1-1 V_1 = Velocity of flow at section 1-1

L = length of the pipe between sections 1-1 and 2-2,

d = diameter of pipe,

f' = frictional resistance per unit wetted area per unit velocity,

h_f = loss of head due to friction,

and p_2, V_2 = are values of pressure intensity and velocity at section 2-2.

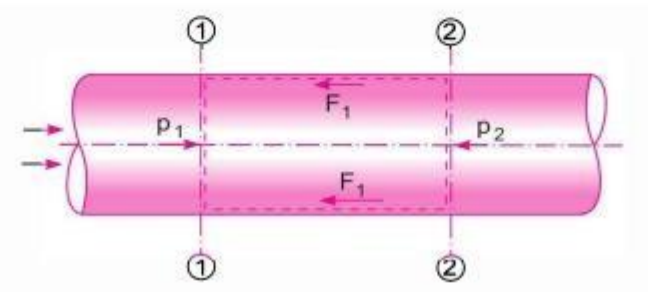


Fig.5.18.Uniform Horizontal

Pipe

Applying Bernoulli's equations between sections 1-1 and 2-2,

Total head at 1-1 = Total head at 2-2 + loss of head due to friction between 1-1 and 2-2

or
$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2+h_f$$

But $z_1 = z_2$ as pipe is horizontal

$V_1 = V_2$ as dia. of pipe is same at 1-1 and 2-2

$$\therefore \quad \frac{p_1}{\rho g}=\frac{p_2}{\rho g}+h_f \text{ or } h_f=\frac{p_1}{\rho g}-\frac{p_2}{\rho g} \qquad \ldots(i)$$

But h_f is the head lost due to friction and hence intensity of pressure will be reduced in the direction of flow by frictional resistance.

Now frictional resistance = frictional resistance per unit wetted area per unit velocity × wetted area × velocity^2

or
$$F_1 = f' \times \pi d L \times V^2 \quad [\because \text{ wetted area} = \pi d \times L, \text{ velocity} = V = V_1 = V_2]$$
$$= f' \times P \times L \times V^2 \quad [\because \pi d = \text{Perimeter} = P] \ldots(ii)$$

The forces acting on the fluid between sections 1-1 and 2-2 are :

1. pressure force at section 1-1 = $p_1 \times A$

where A = Area of pipe

2. pressure force at section 2-2 = $p_2 \times A$
3. frictional force F_1 as shown in Fig. 10.3.

Resolving all forces in the horizontal direction, we have

$$p_1A - p_2A - F_1 = 0 \qquad \ldots(10.1)$$

or
$$(p_1 - p_2)A = F_1 = f' \times P \times L \times V^2 \quad [\because \text{ From } (ii), F_1 = f'PLV^2]$$

or
$$p_1 - p_2 = \frac{f' \times P \times L \times V^2}{A}$$

Equating the value of $(p_1 - p_2)$, we get

$$\rho g h_f = \frac{f' \times P \times L \times V^2}{A}$$

or
$$h_f = \frac{f'}{\rho g} \times \frac{P}{A} \times L \times V^2 \qquad \ldots(iii)$$

In equation (iii), $\frac{P}{A} = \frac{\text{Wetted perimeter}}{\text{Area}} = \frac{\pi d}{\frac{\pi}{4}d^2} = \frac{4}{d}$

$$\therefore \quad h_f = \frac{f'}{\rho g} \times \frac{4}{d} \times L \times V^2 = \frac{f'}{\rho g} \times \frac{4LV^2}{d} \qquad \ldots(iv)$$

Putting $\frac{f'}{\rho} = \frac{f}{2}$, where f is known as co-efficient of friction.

Equation (*iv*), becomes as $h_f = \frac{4 \cdot f}{2g} \cdot \frac{LV^2}{d} = \frac{4f \cdot L \cdot V^2}{d \times 2g}$

is known as Darcy-Weisbach equation. This equation is commonly used for finding loss of head due to friction in pipes.

Sometimes equation (10.2) is written as

$$h_f = \frac{f \cdot L \cdot V^2}{d \times 2g}$$

Then f is known as friction factor.

Loss of Energy in Pipes:

When a fluid is flowing through a pipe, the fluid experiences some resistance due to which some of the energy of fluid is lost. This loss of energy is classified as :

Energy Losses

1. Major Energy Losses

This is due to friction and it is calculated by the following formulae :

(*a*) Darcy-Weisbach Formula

(*b*) Chezy's Formula

2. Minor Energy Losses

This is due to

(*a*) Sudden expansion of pipe

(*b*) Sudden contraction of pipe

(*c*) Bend in pipe

(*d*) Pipe fittings etc.

(*e*) An obstruction in pipe.

Loss of Energy due to friction:

(*a*) **Darcy-Weisbach Formula.** The loss of head (or energy) in pipes due to friction is calculated from Darcy-Weisbach equation which has been derived in chapter and is given by

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$$

where h_f = loss of head due to friction

f = co-efficient of friction which is a function of Reynolds number

$= \frac{16}{R_e}$ for $R_e < 2000$ (viscous flow)

$= \frac{0.079}{R_e^{1/4}}$ for R_e varying from 4000 to 10^6

L = length of pipe,

V = mean velocity of flow,

d = diameter of pipe.

(*b*) **Chezy's Formula for loss of head due to friction in pipes.** Refer to chapter article in which expression for loss of head due to friction in pipes is derived. Equation (*iii*) of article

$$h_f = \frac{f'}{\rho g} \times \frac{P}{A} \times L \times V^2$$

where h_f = loss of head due to friction, P = wetted perimeter of pipe,
A = area of cross-section of pipe, L = length of pipe,
and V = mean velocity of flow.

Now the ratio of $\frac{A}{P}\left(= \frac{\text{Area of flow}}{\text{Perimeter (wetted)}}\right)$ is called hydraulic mean depth or hydraulic radius and is denoted by m.

$\therefore$ Hydraulic mean depth, $m = \frac{A}{P} = \frac{\frac{\pi}{4}d^2}{\pi d} = \frac{d}{4}$

Substituting $\frac{A}{P} = m$ or $\frac{P}{A} = \frac{1}{m}$ in equation , we get

$$h_f = \frac{f'}{\rho g} \times L \times V^2 \times \frac{1}{m} \text{ or } V^2 = h_f \times \frac{\rho g}{f'} \times m \times \frac{1}{L} = \frac{\rho g}{f'} \times m \times \frac{h_f}{L}$$

$$\therefore \quad V = \sqrt{\frac{\rho g}{f'} \times m \times \frac{h_f}{L}} = \sqrt{\frac{\rho g}{f'}}\sqrt{m\frac{h_f}{L}}$$

Let $\sqrt{\frac{\rho g}{f'}} = C$, where C is a constant known as Chezy's constant and $\frac{h_f}{L} = i$, where i is loss of head per unit length of pipe.

Substituting the values of $\sqrt{\frac{\rho g}{f'}}$ and $\sqrt{\frac{h_f}{L}}$ in equation (11.3), we get

$$V = C\sqrt{mi}$$

Minor Energy Losses

The loss of head or energy due to friction in a pipe is known as major loss while the loss of energy due to change of velocity of the following fluid in magnitude or direction is called minor loss of energy. The minor loss of energy (or head) includes the following cases :

1. Loss of head due to sudden enlargement,
2. Loss of head due to sudden contraction,
3. Loss of head at the entrance of a pipe,
4. Loss of head at the exit of a pipe,
5. Loss of head due to an obstruction in a pipe,
6. Loss of head due to bend in the pipe,
7. Loss of head in various pipe fittings.

In case of long pipe the above losses are small as compared with the loss of head due to friction and hence they are called minor losses and even may be neglected without serious error. But in case of a short pipe, these losses are comparable with the loss of head due to friction.

Loss of Head Due to Sudden Enlargement. Consider a liquid flowing through a pipe which has sudden enlargement as shown in Fig. Consider two sections (1)-(1) and (2)-(2) before and after the enlargement.

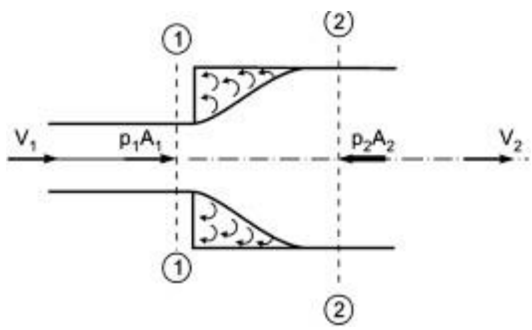
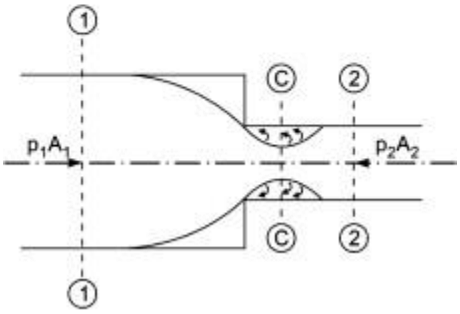


Fig.519. Sudden Enlargement

$$h_e = \frac{(V_1 - V_2)^2}{2g}.$$

Loss of Head due to Sudden Contraction. Consider a liquid flowing in a pipe which has a sudden contraction in area as shown in Fig. Consider two sections 1-1 and 2-2 before and after contraction. As the liquid flows from large pipe to smaller pipe, the area of flow goes on decreasing and becomes minimum at a section *C-C* as shown in Fig. This section *C-C* is called Vena-contracta. After section *C-C*, a sudden enlargement of the area takes place. The loss of head due to sudden contraction is actually due to sudden enlargement from Vena-contracta to smaller pipe.

Fig.20.Sudden Contraction



Loss of Head Due to an Obstruction in a Pipe. Whenever there is an obstruction in a pipe, the loss of energy takes place due to reduction of the area of the cross-section of the pipe at the place where obstruction is present. There is a sudden enlargement of the area of flow beyond the obstruction due to which loss of head takes place as shown in Fig.

Loss of Head at the Exit of Pipe. This is the loss of head (or energy) due to the velocity of liquid at outlet of the pipe which is dissipated either in the form of a free jet (if outlet of the pipe is free) or it is lost in the tank or reservoir (if the outlet of the pipe is connected to the tank or reservoir). This loss is equal to $\frac{V^2}{2g}$, where V is the velocity of liquid at the outlet of pipe. This loss is denoted h_o.

$$\therefore \qquad h_o = \frac{V^2}{2g}$$

where V = velocity at outlet of pipe.

Loss of Head at the Entrance of a Pipe. This is the loss of energy which occurs when a liquid enters a pipe which is connected to a large tank or reservoir. This loss is similar to the loss of head due to sudden contraction. This loss depends on the form of entrance. For a sharp edge entrance, this loss is slightly more than a rounded or bell mouthed entrance. In practice the value of loss of head at the entrance (or inlet) of a pipe with sharp cornered entrance is taken = $0.5 \frac{V^2}{2g}$, where V = velocity of liquid in pipe.

This loss is denoted by h_i

$$\therefore \qquad h_i = 0.5 \frac{V^2}{2g}$$

Fig.21. Obstruction in a pipe

Consider a pipe of area of cross-section A having an obstruction as shown in Fig.

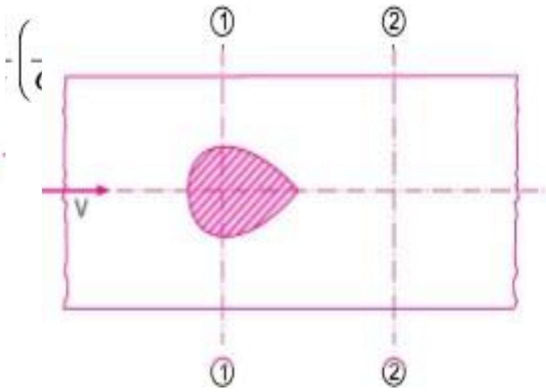
Let a = Maximum area of obstruction

A = Area of pipe

V = Velocity of liquid in pipe

Then $(A - a)$ = Area of flow of liquid at section 1-1.

As the liquid flows and passes through section 1-1, a vena-contracta is formed beyond section 1-1, after which the stream of liquid widens again and velocity of flow at section 2-2 becomes uniform and equal to velocity, V in the pipe. This situation is similar to the flow of liquid through sudden enlargement.



Loss of Head due to Bend in Pipe. When there is any bend in a pipe, the velocity of flow changes, due to which the separation of the flow from the boundary and also formation of eddies takes place. Thus the energy is lost. Loss of head in pipe due to bend is expressed as

$$h_b = \frac{kV^2}{2g}$$

where h_b = loss of head due to bend, V = velocity of flow, k = co-efficient of bend

The value of k depends on

(*i*) Angle of bend, (*ii*) Radius of curvature of bend, (*iii*) Diameter of pipe.

Loss of Head in Various Pipe Fittings. The loss of head in the various pipe fittings such as valves, couplings etc., is expressed as

$$= \frac{kV^2}{2g}$$

where V = velocity of flow, k = co-efficient of pipe fitting.

HYDRAULIC GRADIENT AND TOTAL ENERGY LINE

The concept of hydraulic gradient line and total energy line is very useful in the study of flow of fluids through pipes. They are defined as :

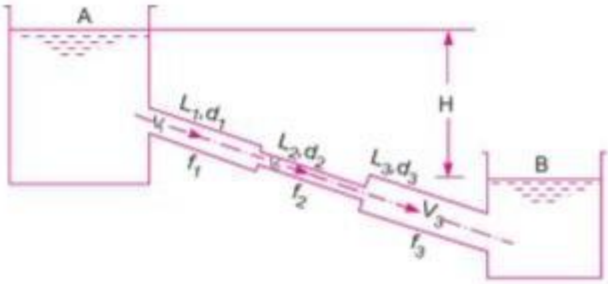
Hydraulic Gradient Line. It is defined as the line which gives the sum of pressure head $\left(\frac{p}{w}\right)$ and datum head (z) of a flowing fluid in a pipe with respect to some reference line or it is the line which is obtained by joining the top of all vertical ordinates, showing the pressure head (p/w) of a flowing fluid in a pipe from the centre of the pipe. It is briefly written as H.G.L. (Hydraulic Gradient Line).

Total Energy Line. It is defined as the line which gives the sum of pressure head, datum head and kinetic head of a flowing fluid in a pipe with respect to some reference line. It is also defined as the line which is obtained by joining the tops of all vertical ordinates showing the sum of pressure head and kinetic head from the centre of the pipe. It is briefly written as T.E.L. (Total Energy Line).

FLOW THROUGH PIPES IN SERIES OR FLOW THROUGH COMPOUND PIPES

Pipes in series or compound pipes are defined as the pipes of different lengths and different diameters connected end to end (in series) to form a pipe line as shown in Fig.

Let, L_1, L_2, L_3 = length of pipes 1, 2 and 3 respectively
d_1, d_2, d_3 = diameter of pipes 1, 2, 3 respectively
V_1, V_2, V_3 = velocity of flow through pipes 1, 2, 3
f_1, f_2, f_3 = co-efficient of frictions for pipes 1, 2, 3
H = difference of water level in the two tanks.



The discharge passing through each pipe is same.

$$\therefore \quad Q = A_1V_1 = A_2V_2 = A_3V_3$$

The difference in liquid surface levels is equal to the sum of the total head loss in the pipes.

$$\therefore \quad H = \frac{0.5\,V_1^2}{2g} + \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{0.5\,V_2^2}{2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{(V_2 - V_3)^2}{2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g} + \frac{V_3^2}{2g}.$$

If minor losses are neglected, then above equation becomes as

$$H = \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g}.$$

If the co-efficient of friction is same for all pipes

i.e., $f_1 = f_2 = f_3 = f$, then equation becomes as

$$H = \frac{4fL_1V_1^2}{d_1 \times 2g} + \frac{4fL_2V_2^2}{d_2 \times 2g} + \frac{4fL_3V_3^2}{d_3 \times 2g}$$

$$= \frac{4f}{2g}\left[\frac{L_1V_1^2}{d_1} + \frac{L_2V_2^2}{d_2} + \frac{L_3V_3^2}{d_3}\right].$$

Equivalent Pipe

This is defined as the pipe of uniform diameter having loss of head and discharge equal to the loss of head and discharge of a compound pipe consisting of several pipes of different lengths and diameters. The uniform diameter of the equivalent pipe is called equivalent size of the pipe. The length of equivalent pipe is equal to sum of lengths of the compound pipe consisting of different pipes.

Let L_1 = length of pipe 1 and d_1 = diameter of pipe 1

L_2 = length of pipe 2 and d_2 = diameter of pipe 2

L_3 = length of pipe 3 and d_3 = diameter of pipe 3

H = total head loss

L = length of equivalent pipe

d = diameter of the equivalent pipe

Then $L = L_1 + L_2 + L_3$

Total head loss in the compound pipe, neglecting minor losses

$$H = \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g}$$

Assuming $f_1 = f_2 = f_3 = f$

Discharge, $Q = A_1V_1 = A_2V_2 = A_3V_3 = \frac{\pi}{4} d_1^2V_1 = \frac{\pi}{4} d_2^2V_2 = \frac{\pi}{4} d_3^2V_3$

$\therefore$ $V_1 = \frac{4Q}{\pi d_1^2}$, $V_2 = \frac{4Q}{\pi d_2^2}$ and $V_3 = \frac{4Q}{\pi d_3^2}$

Substituting these values in equation

$$H = \frac{4fL_1 \times \left(\frac{4Q}{\pi d_1^2}\right)^2}{d_1 \times 2g} + \frac{4fL_2\left(\frac{4Q}{\pi d_2^2}\right)^2}{d_2 \times 2g} + \frac{4fL_3\left(\frac{4Q}{\pi d_3^2}\right)^2}{d_3 \times 2g}$$

$$= \frac{4 \times 16fQ^2}{\pi^2 \times 2g}\left[\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right]$$

Head loss in the equivalent pipe, $H = \frac{4f \,.\, L \,.\, V^2}{d \times 2g}$ [Taking same value of f as in compound pipe]

where $V = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}d^2} = \frac{4Q}{\pi d^2}$

$$\therefore \quad H = \frac{4\,f \,.\, L \,.\left(\frac{4Q}{\pi d^2}\right)^2}{d \times 2g} = \frac{4 \times 16Q^2f}{\pi^2 \times 2g}\left[\frac{L}{d^5}\right]$$

Head loss in compound pipe and in equivalent pipe is same hence equating equations

$$\frac{4 \times 16fQ^2}{\pi^2 \times 2g}\left[\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right] = \frac{4 \times 16Q^2f}{\pi^2 \times 2g}\left[\frac{L}{d^5}\right]$$

or

$$\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5} = \frac{L}{d^5} \quad \text{or} \quad \frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$$

Equation is known as Dupuit's equation. In this equation $L = L_1 + L_2 + L_3$ and d_1, d_2 and d_3 are known. Hence the equivalent size of the pipe, *i.e.*, value of d can be obtained.

Flow through Parallel Pipes:

Consider a main pipe which divides into two or more branches as shown in Fig. and again join together downstream to form a single pipe, then the branch pipes are said to be connected in parallel. The discharge through the main is increased by connecting pipes in parallel.

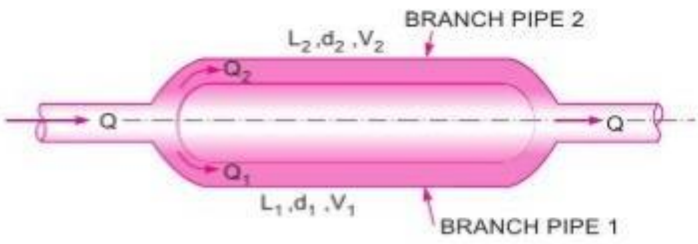


Fig.24.Parallel Pipes

The rate of flow in the main pipe is equal to the sum of rate of flow through branch pipes. Hence from Fig.

$$Q = Q_1 + Q_2$$

In this, arrangement, the loss of head for each branch pipe is same.

∴ Loss of head for branch pipe 1 = Loss of head for branch pipe 2

$$\frac{4f_1L_1V_1^2}{d_1 \times 2g} = \frac{4f_2L_2V_2^2}{d_2 \times 2g}$$

$$f_1 = f_2, \text{ then } \frac{L_1V_1^2}{d_1 \times 2g} = \frac{L_2V_2^2}{d_2 \times 2g}$$

Flow through Branched Pipes:

When three or more reservoirs are connected by means of pipes, having one or more junctions, the system is called a branching pipe system. Fig. shows three reservoirs at different levels connected to a single junction, by means of pipes which are called branched pipes. The lengths, diameters and co-efficient of friction of each pipes is given. It is required to find the discharge and direction of flow in each pipe. The basic equations used for solving such problems are :

1. **Continuity equation** which means the inflow of fluid at the junction should be equal to the outflow of fluid.
2. **Bernoulli's equation, and**
3. **Darcy-Weisbach equation**

Also it is assumed that reservoirs are very large and the water surface levels in the reservoirs are constant so that steady conditions exist in the pipes. Also minor losses are assumed very small. The flow from reservoir *A* takes place to junction *D*. The flow from junction *D* is towards reservoirs *C*. Now the flow from junction *D* towards reservoir *B* will take place only when piezometric head at *D* $\left(\text{which is equal to } \frac{p_D}{\rho g} + Z_D\right)$ is more than the piezometric head at *B* (*i.e.*, Z_B). Let us consider that flow is from *D* to reservoir *B*.

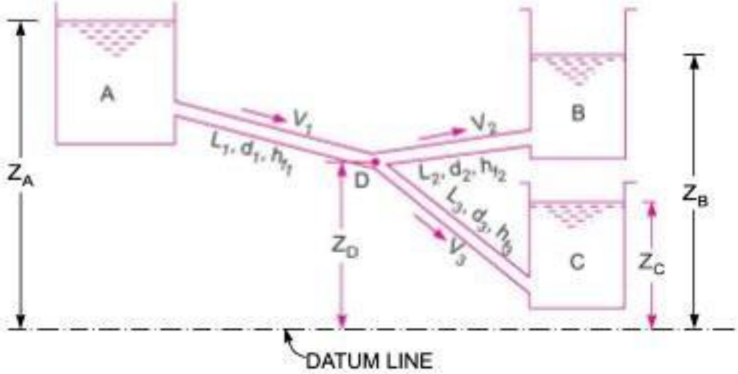


Fig.25.Branched Pipes

For flow from A to D from Bernoulli's equation

$$Z_A = Z_D + \frac{p_D}{\rho g} + h_{f_1} \quad ...(i)$$

For flow from D to B from Bernoulli's equation

$$Z_D + \frac{p_D}{\rho g} = Z_B + h_{f_2} \quad ...(ii)$$

For flow from D to C from Bernoulli's equation

$$Z_D + \frac{p_D}{\rho g} = Z_C + h_{f_3} \quad ...(iii)$$

From continuity equation,

Discharge through AD = Discharge through DB + Discharge through DC

$$\therefore \quad \frac{\pi}{4} d_1^2 V_1 = \frac{\pi}{4} d_2^2 \times V_2 + \frac{\pi}{4} d_3^2 V_3$$

or

$$d_1^2 V_1 = d_2^2 V_2 + d_3^2 V_3 \quad ...(iv)$$

There are four unknowns *i.e.*, V_1, V_2, V_3 and $\frac{p_D}{\rho g}$ and there are four equations (*i*), (*ii*), (*iii*) and (*iv*).

Hence unknown can be calculated.

Water Hammer in Pipes:

Consider a long pipe AB as shown in Fig. connected at one end to a tank containing water at a height of H from the centre of the pipe. At the other end of the pipe, a valve to regulate the flow of water is provided. When the valve is completely open, the water is flowing with a velocity, V in the pipe. If now the valve is suddenly closed, the momentum of the flowing water will be destroyed and consequently a wave of high pressure will be set up. This wave of high pressure will be transmitted along the pipe with a velocity equal to the velocity of sound wave and may create noise called knocking. Also this wave of high pressure has the effect of hammering action on the walls of the pipe and hence it is also known as water hammer.

Fig.28.Water Hammer

The pressure rise due to water hammer depends upon : (*i*) the velocity of flow of water in pipe, (*ii*) the length of pipe, (*iii*) time taken to close the valve, (*iv*) elastic properties of the material of the pipe. The following cases of water hammer in pipes will be considered :

1. Gradual closure of valve,
2. Sudden closure of valve and considering pipe rigid, and

Practice Problems:

Problem .1 *The head of water over an orifice of diameter 40 mm is 10 m. Find the actual discharge and actual velocity of the jet at vena-contracta. Take $C_d = 0.6$ and $C_v = 0.98$.*

Solution. Given :

Head, $H = 10$ cm

Dia. of orifice, $d = 40$ mm $= 0.04$ m

$\therefore$ Area, $a = \frac{\pi}{4}(.04)^2 = .001256 \text{ m}^2$

$C_d = 0.6$

$C_v = 0.98$

(*i*) $$\frac{\text{Actual discharge}}{\text{Theoretical discharge}} = 0.6$$

But Theoretical discharge $= V_{th} \times$ Area of orifice

V_{th} = Theoretical velocity, where $V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 10} = 14$ m/s

$\therefore$ Theoretical discharge $= 14 \times .001256 = 0.01758 \ \frac{\text{m}^2}{\text{s}}$

$\therefore$ Actual discharge $= 0.6 \times$ Theoretical discharge

$= 0.6 \times .01758 =$ **$0.01054 \text{ m}^3/\text{s}$. Ans.**

(*ii*) $$\frac{\text{Actual velocity}}{\text{Theoretical velocity}} = C_v = 0.98$$

$\therefore$ Actual velocity $= 0.98 \times$ Theoretical velocity

$= 0.98 \times 14 =$ **13.72 m/s. Ans.**

Problem .2 *The head of water over the centre of an orifice of diameter 20 mm is 1 m. The actual discharge through the orifice is 0.85 litre/s. Find the co-efficient of discharge.*

Solution. Given :

Dia. of orifice, $d = 20$ mm $= 0.02$ m

$\therefore$ Area, $a = \frac{\pi}{4}(0.02)^2 = 0.000314 \text{ m}^2$

Head, $H = 1$ m

Actual discharge, $Q = 0.85$ litre/s $= 0.00085 \text{ m}^3/\text{s}$

Theoretical velocity, $V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 1} = 4.429$ m/s

$\therefore$ Theoretical discharge, $Q_{th} = V_{th} \times$ Area of orifice

$= 4.429 \times 0.000314 = 0.00139 \text{ m}^3/\text{s}$

$$\therefore \quad \text{Co-efficient of discharge} = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{0.00085}{0.00139} = \mathbf{0.61.\ Ans.}$$

Problem .3 *A jet of water, issuing from a sharp-edged vertical orifice under a constant head of 10.0 cm, at a certain point, has the horizontal and vertical co-ordinates measured from the vena-contracta as 20.0 cm and 10.5 cm respectively. Find the value of C_v. Also find the value of C_c if $C_d = 0.60$.*

Solution. Given :

Head,	$H = 10.0$ cm
Horizontal distance,	$x = 20.0$ cm
Vertical distance,	$y = 10.5$ cm
	$C_d = 0.6$

The value of C_v is given by equation (7.6) as

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{20.0}{\sqrt{4 \times 10.5 \times 10.0}} = \frac{20}{20.493} = 0.9759 = \mathbf{0.976.\ Ans.}$$

The value of C_c is given by equation (7.7) as

$$C_c = \frac{C_d}{C_v} = \frac{0.6}{0.976} = 0.6147 = \mathbf{0.615.\ Ans.}$$

Problem .4 *The head of water over an orifice of diameter 100 mm is 10 m. The water coming out from orifice is collected in a circular tank of diameter 1.5 m. The rise of water level in this tank is 1.0 m in 25 seconds. Also the co-ordinates of a point on the jet, measured from vena-contracta are 4.3 m horizontal and 0.5 m vertical. Find the co-efficients, C_d, C_v and C_c.*

Solution. Given :

Head,	$H = 10$ m
Dia. of orifice,	$d = 100$ mm $= 0.1$ m
∴ Area of orifice,	$a = \frac{\pi}{4}(.1)^2 = 0.007853\ \text{m}^2$
Dia. of measuring tank,	$D = 1.5$ m
∴ Area,	$A = \frac{\pi}{4}(1.5)^2 = 1.767\ \text{m}^2$
Rise of water,	$h = 1$ m
Time,	$t = 25$ seconds
Horizontal distance,	$x = 4.3$ m
Vertical distance,	$y = 0.5$ m

Now theoretical velocity, $V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 10} = 14.0$ m/s

∴ Theoretical discharge, $Q_{th} = V_{th} \times$ Area of orifice $= 14.0 \times 0.007854 = 0.1099\ \text{m}^3/\text{s}$

Actual discharge, $$Q = \frac{A \times h}{t} = \frac{1.767 \times 1.0}{25} = 0.07068$$

∴ $$C_d = \frac{Q}{Q_{th}} = \frac{0.07068}{0.1099} = \mathbf{0.643.\ Ans.}$$

The value of C_v is given by equation (7.6) as

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{4.3}{\sqrt{4 \times 0.5 \times 10}} = \frac{4.3}{4.472} = \mathbf{0.96.\ Ans.}$$

C_c is given by equation (7.7) as $C_c = \dfrac{C_d}{C_v} = \dfrac{0.643}{0.96} = \mathbf{0.669.\ Ans.}$

Problem .5 *Water discharge at the rate of 98.2 litres/s through a 120 mm diameter vertical sharp-edged orifice placed under a constant head of 10 metres. A point, on the jet, measured from the vena-contracta of the jet has co-ordinates 4.5 metres horizontal and 0.54 metres vertical. Find the co-efficient C_v, C_c and C_d of the orifice.*

Solution. Given :

Discharge, $Q = 98.2 \text{ lit/s} = 0.0982 \text{ m}^3\text{/s}$

Dia. of orifice, $d = 120 \text{ mm} = 0.12 \text{ m}$

∴ Area of orifice, $a = \frac{\pi}{4}(0.12)^2 = 0.01131 \text{ m}^2$

Head, $H = 10 \text{ m}$

Horizontal distance of a point on the jet from vena-contracta, $x = 4.5$ m and vertical distance, $y = 0.54$ m

Now theoretical velocity, $V_{th} = \sqrt{2g \times H} = \sqrt{2 \times 9.81 \times 10} = 14.0 \text{ m/s}$

Theoretical discharge, $Q_{th} = V_{th} \times \text{Area of orifice}$
$= 14.0 \times 0.01131 = 0.1583 \text{ m}^3\text{/s}$

The value of C_d is given by, $C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{Q}{Q_{th}} = \frac{0.0982}{0.1583} =$ **0.62. Ans.**

The value of C_c is given by equation (7.6),

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{4.5}{\sqrt{4 \times 0.54 \times 10}} = \mathbf{0.968.\ Ans.}$$

The value of C_c is given by equation (7.7) as

$$C_c = \frac{C_d}{C_v} = \frac{0.62}{0.968} = \mathbf{0.64.\ Ans.}$$

Problem *Find the discharge through a rectangular orifice 2.0 m wide and 1.5 m deep fitted to a water tank. The water level in the tank is 3.0 m above the top edge of the orifice. Take $C_d = 0.62$.*

Solution. Given :

Width of orifice, $b = 2.0$ m

Depth of orifice, $d = 1.5$ m

Height of water above top edge of the orifice, $H_1 = 3$ m

Height of water above bottom edge of the orifice,

$$H_2 = H_1 + d = 3 + 1.5 = 4.5 \text{ m}$$
$$C_d = 0.62$$

Discharge Q is given by equation (7.8) as

$$Q = \frac{2}{3} C_d \times b \times \sqrt{2g}\ [H_2^{3/2} - H_1^{3/2}]$$
$$= \frac{2}{3} \times 0.62 \times 2.0 \times \sqrt{2 + 9.81}\,[4.5^{1.5} - 3^{1.5}] \text{ m}^3\text{/s}$$
$$= 3.66[9.545 - 5.196] \text{ m}^3\text{/s} = \mathbf{15.917\ m^3/s.\ Ans.}$$

Problem *A rectangular orifice, 1.5 m wide and 1.0 m deep is discharging water from a tank. If the water level in the tank is 3.0 m above the top edge of the orifice, find the discharge through the orifice. Take the co-efficient of discharging for the orifice = 0.6.*

Solution. Given :

Width of orifice, $b = 1.5$ m

Depth of orifice, $d = 1.0$ m

$H_1 = 3.0$ m

$H_2 = H_1 + d = 3.0 + 1.0 = 4.0$ m

$C_d = 0.6$

Discharge, Q is given by the equation (7.8) as

$$Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g}\ [H_2^{3/2} - H_1^{3/2}]$$

$$= \frac{2}{3} \times 0.6 \times 1.5 \times \sqrt{2 + 9.81}\,[4.0^{1.5} - 3.0^{1.5}] \text{ m}^3/\text{s}$$

$$= 2.657\ [8.0 - 5.196] \text{ m}^3/\text{s} = \mathbf{7.45\ m^3/s.\ Ans.}$$

Problem *A rectangular orifice, 1.5 m wide and 1.0 m deep is discharging water from a tank. If the water level in the tank is 3.0 m above the top edge of the orifice, find the discharge through the orifice. Take the co-efficient of discharging for the orifice = 0.6.*

Solution. Given :

Width of orifice, $b = 1.5$ m

Depth of orifice, $d = 1.0$ m

$H_1 = 3.0$ m

$H_2 = H_1 + d = 3.0 + 1.0 = 4.0$ m

$C_d = 0.6$

Discharge, Q is given by the equation (7.8) as

$$Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g}\ [H_2^{3/2} - H_1^{3/2}]$$

$$= \frac{2}{3} \times 0.6 \times 1.5 \times \sqrt{2 + 9.81}\,[4.0^{1.5} - 3.0^{1.5}] \text{ m}^3/\text{s}$$

$$= 2.657\ [8.0 - 5.196] \text{ m}^3/\text{s} = \mathbf{7.45\ m^3/s.\ Ans.}$$

Problem *Find the discharge through a fully sub-merged orifice of width 2 m if the difference of water levels on both sides of the orifice be 50 cm. The height of water from top and bottom of the orifice are 2.5 m and 2.75 m respectively. Take $C_d = 0.6$.*

Solution. Given :

Width of orifice, $b = 2$ m

Difference of water level, $H = 50$ cm $= 0.5$ m

Height of water from top of orifice, $H_1 = 2.5$ m

Height of water from bottom of orifice, $H_2 = 2.5$ m

$C_d = 0.6$

Discharge through fully sub-merged orifice is given by equation (7.9)

or $$Q = C_d \times b \times (H_2 - H_1) \times \sqrt{2gH}$$

$$= 0.6 \times 2.0 \times (2.75 - 2.5) \times \sqrt{2 \times 9.81 \times 0.5} \text{ m}^3/\text{s}$$

$$= \mathbf{0.9396\ m^3/s.\ Ans.}$$

Problem *Find the discharge through a totally drowned orifice 2.0 m wide and 1 m deep, if the difference of water levels on both the sides of the orifice be 3 m. Take $C_d = 0.62$.*

Solution. Given :

Width of orifice, $b = 2.0$ m

Depth of orifice, $d = 1$ m.

Difference of water level on both the sides

$H = 3$ m

$C_d = 0.62$

Discharge through orifice is $Q = C_d \times \text{Area} \times \sqrt{2gH}$

$$= 0.62 \times b \times d \times \sqrt{2gH}$$

$$= 0.62 \times 2.0 \times 1.0 \times \sqrt{2 \times 9.81 \times 3} \text{ m}^3\text{/s} = \mathbf{9.513 \text{ m}^3\text{/s. Ans.}}$$

Problem *A circular tank of diameter 4 m contains water upto a height of 5 m. The tank is provided with an orifice of diameter 0.5 m at the bottom. Find the time taken by water (i) to fall from 5 m to 2 m (ii) for completely emptying the tank. Take $C_d = 0.6$.*

Solution. Given :

Dia. of tank, $D = 4$ m

$\therefore$ Area, $A = \frac{\pi}{4}(4)^2 = 12.566 \text{ m}^2$

Dia. of orifice, $d = 0.5$ m

$\therefore$ Area, $a = \frac{\pi}{4}(.5)^2 = 0.1963 \text{ m}^2$

Initial height of water, $H_1 = 5$ m

Final height of water, (*i*) $H_2 = 2$ m (*ii*) $H_2 = 0$

First Case. When $H_2 = 2$ m

Using equation we have $T = \frac{2A}{C_d \cdot a \cdot \sqrt{2g}}\left[\sqrt{H_1} - \sqrt{H_2}\right]$

$$= \frac{2 \times 12.566}{0.6 \times .1963 \times \sqrt{2 \times 9.81}}\left[\sqrt{5} - \sqrt{2.0}\right] \text{ seconds}$$

$$= \frac{20.653}{0.5217} = \mathbf{39.58 \text{ seconds. Ans.}}$$

Second Case. When $H_2 = 0$

$$T = \frac{2A}{C_d \cdot a \cdot \sqrt{2g}}\sqrt{H_1} = \frac{2 \times 12.566 \times \sqrt{5}}{0.6 \times .1963 \times \sqrt{2 \times 9.81}}$$

$$= \mathbf{107.7 \text{ seconds. Ans.}}$$

Problem *A rectangular orifice of 2 m width and 1.2 m deep is fitted in one side of a large tank. The water level on one side of the orifice is 3 m above the top edge of the orifice, while on the other side of the orifice, the water level is 0.5 m below its top edge. Calculate the discharge through the orifice if $C_d = 0.64$.*

Solution. Given : Width of orifice, $b = 2$ m

Depth of orifice, $d = 1.2$ m

Height of water from top edge of orifice, $H_1 = 3$ m

Difference of water level on both sides, $H = 3 + 0.5 = 3.5$ m

Height of water from the bottom edge of orifice, $H_2 = H_1 + d = 3 + 1.2 = 4.2$ m

The orifice is partially sub-merged. The discharge through sub-merged portion,

$$Q_1 = C_d \times b \times (H_2 - H) \times \sqrt{2gH}$$

$$= 0.64 \times 2.0 \times (4.2 - 3.5) \times \sqrt{2 \times 9.81 \times 3.5} = 7.4249 \text{ m}^3/\text{s}$$

The discharge through free portion is

$$Q_2 = \frac{2}{3} C_d \times b \times \sqrt{2g}\,[H^{3/2} - H_1^{3/2}]$$

$$= \frac{2}{3} \times 0.64 \times 2.0 \times \sqrt{2 \times 9.81}\ [3.5^{3/2} - 3.0^{3/2}]$$

$$= 3.779\,[6.5479 - 5.1961] = 5.108 \text{ m}^3/\text{s}$$

∴ Total discharge through the orifice is

$$Q = Q_1 + Q_2 = 7.4249 + 5.108 = \mathbf{12.5329\ m^3/s.\ Ans.}$$

Problem *A hemispherical tank of diameter 4 m contains water upto a height of 1.5 m. An orifice of diameter 50 mm is provided at the bottom. Find the time required by water (i) to fall from 1.5 m to 1.0 m (ii) for completely emptying the tank. Tank $C_d = 0.6$.*

Solution. Given :

Dia. of hemispherical tank, $D = 4$ m

∴ Radius, $R = 2.0$ m

Dia. of orifice, $d = 50$ mm $= 0.05$ m

∴ Area, $a = \frac{\pi}{4}(.05)^2 = 0.001963 \text{ m}^2$

Initial height of water, $H_1 = 1.5$ m

$C_d = 0.6$

First Case. $H_2 = 1.0$

Time T is given by equation

$$\therefore \quad T = \frac{\pi}{C_d \times a \times \sqrt{2g}}\left[\frac{4}{3}R\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right)\right]$$

$$= \frac{\pi}{0.6 \times .001963 \times \sqrt{2 \times 9.81}} \times \left[\frac{4}{3} \times 2.0\left(1.5^{3/2} - 1.0^{3/2}\right) - \frac{2}{5}\left(1.5^{5/2} - 1.0^{5/2}\right)\right]$$

$$= 602.189\,[2.2323 - 0.7022] = 921.4 \text{ second}$$

$$= \mathbf{15\ min\ 21.4\ sec.\ Ans.}$$

Second Case. $H_2 = 0$ and hence time T is given by equation

$$\therefore \quad T = \frac{\pi}{C_d \,.\, a \,.\, \sqrt{2g}}\left[\frac{4}{3}RH_1^{3/2} - \frac{2}{5}H_1^{5/2}\right]$$

$$= \frac{\pi}{0.6 \times .001963\sqrt{2 \times 9.81}}\left[\frac{4}{3} \times 2.0 \times 1.5^{3/2} - \frac{2}{5} \times 1.5^{5/2}\right]$$

$$= 602.189\,[4.8989 - 1.1022] \text{ sec} = 2286.33 \text{ sec}$$

$$= \mathbf{38\ min\ 6.33\ sec.\ Ans.}$$

Problem *An orifice of diameter 150 mm is fitted at the bottom of a boiler drum of length 8 m and of diameter 3 metres. The drum is horizontal and contains water upto a height of 2.4 m. Find the time required to empty the boiler. Take $C_d = 0.6$.*

Solution. Given :

Dia. of orifice, $d = 150 \text{ mm} = 0.15 \text{ m}$

$\therefore$ Area, $a = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$

Length, $L = 8.0 \text{ m}$

Dia. of boiler, $D = 3.0 \text{ m}$

$\therefore$ Radius, $R = 1.5 \text{ m}$

Initial height of water, $H_1 = 2.4 \text{ m}$

Find height of water, $H_2 = 0$

$C_d = 0.6.$

For completely emptying the tank, T is given by equation

$$T = \frac{4L}{3C_d \times a \times \sqrt{2g}}[(2R)^{3/2} - (2R - H_1)^{3/2}]$$

$$= \frac{4 \times 8.0}{3 \times .6 \times .01767 \times \sqrt{2 \times 9.81}}[(2 \times 1.5)^{3/2} - (2 \times 1.5 - 2.4)^{3/2}]$$

$$= 227.14\ [5.196 - 0.4647] = 1074.66 \text{ sec}$$

$$= \mathbf{17\ min\ 54.66\ sec.\ Ans.}$$

Problem *Find the discharge from a 100 mm diameter external mouthpiece, fitted to a side of a large vessel if the head over the mouthpiece is 4 metres.*

Solution. Given :

Dia. of mouthpiece = 100 m = 0.1 m

$\therefore$ Area, $a = \frac{\pi}{4}(0.1)^2 = 0.007854 \text{ m}^2$

Head, $H = 4.0 \text{ m}$

C_d for mouthpiece $= 0.855$

$\therefore$ Discharge $= C_d \times \text{Area} \times \text{Velocity} = 0.855 \times a \times \sqrt{2gH}$

$= .855 \times .007854 \times \sqrt{2 \times 9.81 \times 4.0} = \mathbf{.05948\ m^3/s.\ Ans.}$

Problem *Find the discharge of water flowing over a rectangular notch of 2 m length when the constant head over the notch is 300 mm. Take $C_d = 0.60$.*

Solution. Given :

Length of the notch, $L = 2.0 \text{ m}$

Head over notch, $H = 300 \text{ m} = 0.30 \text{ m}$

$C_d = 0.60$

Discharge, $Q = \frac{2}{3}\ C_d \times L \times \sqrt{2g}\left[H^{3/2}\right]$

$= \frac{2}{3} \times 0.6 \times 2.0 \times \sqrt{2 \times 9.81} \times [0.30]^{1.5} \text{ m}^3/\text{s}$

$= 3.5435 \times 0.1643 = \mathbf{0.582\ m^3/s.\ Ans.}$

Problem *Determine the height of a rectangular weir of length 6 m to be built across a rectangular channel. The maximum depth of water on the upstream side of the weir is 1.8 m and discharge is 2000 litres/s. Take $C_d = 0.6$ and neglect end contractions.*

Solution. Given :

Length of weir, $L = 6$ m

Depth of water, $H_1 = 1.8$ m

Discharge, $Q = 2000$ lit/s $= 2\ m^3/s$

$C_d = 0.6$

Let H is height of water above the crest of weir, and H_2 = height of weir

The discharge over the weir is given by the equation

$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g}\ H^{3/2}$$

or

$$2.0 = \frac{2}{3} \times 0.6 \times 6.0 \times \sqrt{2 \times 9.81} \times H^{3/2}$$

$$= 10.623\ H^{3/2}$$

$$\therefore \quad H^{3/2} = \frac{2.0}{10.623}$$

$$\therefore \quad H = \left(\frac{2.0}{10.623}\right)^{2/3} = 0.328 \text{ m}$$

$\therefore$ Height of weir, $H_2 = H_1 - H$

= Depth of water on upstream side $- H$

$= 1.8 - .328 =$ **1.472 m. Ans.**

Problem *Find the discharge over a triangular notch of angle 60° when the head over the V-notch is 0.3 m. Assume $C_d = 0.6$.*

Solution. Given :

Angle of *V*-notch, $\theta = 60°$

Head over notch, $H = 0.3$ m

$C_d = 0.6$

Discharge, Q over a *V*-notch is given by equation

$$Q = \frac{8}{15} \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$$= \frac{8}{15} \times 0.6 \tan\frac{60^\circ}{2} \times \sqrt{2 \times 9.81} \times (0.3)^{5/2}$$

$= 0.8182 \times 0.0493 =$ **0.040 m³/s. Ans.**

Problem *Water flows over a rectangular weir 1 m wide at a depth of 150 mm and afterwards passes through a triangular right-angled weir. Taking C_d for the rectangular and triangular weir as 0.62 and 0.59 respectively, find the depth over the triangular weir.*

Solution. Given :

For rectangular weir, length, $L = 1$ m

Depth of water, $H = 150$ mm $= 0.15$ m

$C_d = 0.62$

For triangular weir, $\theta = 90°$

$C_d = 0.59$

Let depth over triangular weir $= H_1$

The discharge over the rectangular weir is given by equation

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$= \frac{2}{3} \times 0.62 \times 1.0 \times \sqrt{2 \times 9.81} \times (.15)^{3/2} \text{ m}^3/\text{s} = 0.10635 \text{ m}^3/\text{s}$$

The same discharge passes through the triangular right-angled weir. But discharge, Q, is given by equation for a triangular weir as

$$Q = \frac{8}{15} \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$\therefore$
$$0.10635 = \frac{8}{15} \times .59 \times \tan\frac{90^\circ}{2} \times \sqrt{2g} \times H_1^{5/2} \qquad \{\because \theta = 90^\circ \text{ and } H = H_1\}$$

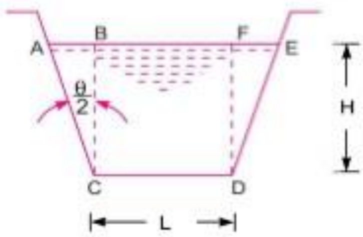
$$= \frac{8}{15} \times .59 \times 1 \times 4.429 \times H_1^{5/2} = 1.3936\, H_1^{5/2}$$

$\therefore$
$$H_1^{5/2} = \frac{0.10635}{1.3936} = 0.07631$$

$\therefore$
$$H_1 = (.07631)^{0.4} = \mathbf{0.3572 \text{ m. Ans.}}$$

Problem *Find the discharge through a trapezoidal notch which is 1 m wide at the top and 0.40 m at the bottom and is 30 cm in height. The head of water on the notch is 20 cm. Assume C_d for rectangular portion = 0.62 while for triangular portion = 0.60.*

Solution. Given :



Top width, $AE = 1$ m

Base width, $CD = L = 0.4$ m

Head of water, $H = 0.20$ m

For rectangular portion, $C_{d_1} = 0.62$

For triangular portion, $C_{d_2} = 0.60$

From ΔABC, we have

$$\tan\frac{\theta}{2} = \frac{AB}{BC} = \frac{(AE - CD)/2}{H}$$

$$= \frac{(1.0 - 0.4)/2}{0.3} = \frac{0.6/2}{0.3} = \frac{0.3}{0.3} = 1$$

Discharge through trapezoidal notch is given by equation

$$Q = \frac{2}{3} C_{d_1} \times L \times \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$$= \frac{2}{3} \times 0.62 \times 0.4 \times \sqrt{2 \times 9.81} \times (0.2)^{3/2} + \frac{8}{15} \times .60 \times 1 \times \sqrt{2 \times 9.81} \times (0.2)^{5/2}$$

$$= 0.06549 + 0.02535 = 0.09084 \text{ m}^3/\text{s} = \mathbf{90.84 \text{ litres/s. Ans.}}$$

Problem *(a) A broad-crested weir of 50 m length, has 50 cm height of water above its crest. Find the maximum discharge. Take $C_d = 0.60$. Neglect velocity of approach. (b) If the velocity of approach is to be taken into consideration, find the maximum discharge when the channel has a cross-sectional area of 50 m^2 on the upstream side.*

Solution. Given :

Length of weir, $L = 50$ m

Head of water, $H = 50$ cm $= 0.5$ m

$C_d = 0.60$

(*i*) **Neglecting velocity of approach.** Maximum discharge is given by equation

$$Q_{max} = 1.705 \times C_d \times L \times H^{3/2}$$
$$= 1.705 \times 0.60 \times 50 \times (.5)^{3/2} = \mathbf{18.084\ m^3/s.\ Ans.}$$

(*ii*) **Taking velocity of approach into consideration**

Area of channel, $A = 50\ m^2$

Velocity of approach, $V_a = \dfrac{Q}{A} = \dfrac{18.084}{50} = 0.36$ m/s

$\therefore$ Head due to V_a, $h_a = \dfrac{V_a^2}{2g} = \dfrac{0.36 \times .36}{2 \times 9.81} = .0066$ m

Maximum discharge, Q_{max} is given by

$$Q_{max} = 1.705 \times C_d \times L \times [(H + h_a)^{3/2} - h_a^{3/2}]$$
$$= 1.705 \times 0.6 \times 50 \times [(.50 + .0066)^{1.5} - (.0066)^{1.5}]$$
$$= 51.15[0.3605 - .000536] = \mathbf{18.412\ m^3/s.\ Ans.}$$

Problem *Find the head lost due to friction in a pipe of diameter 300 mm and length 50 m, through which water is flowing at a velocity of 3 m/s using (i) Darcy formula, (ii) Chezy's formula for which C = 60.*

Take ν for water = 0.01 stoke.

Solution. Given :

Dia. of pipe, $d = 300$ mm $= 0.30$ m

Length of pipe, $L = 50$ m

Velocity of flow, $V = 3$ m/s

Chezy's constant, $C = 60$

Kinematic viscosity, $\nu = 0.01$ stoke $= 0.01\ cm^2/s$

$= 0.01 \times 10^{-4}\ m^2/s.$

(*i*) **Darcy Formula** is given by equation

$$h_f = \frac{4 . f . L . V^2}{d \times 2g}$$

where 'f' = co-efficient of friction is a function of Reynolds number, R_e

But R_e is given by $$R_e = \frac{V \times d}{\nu} = \frac{3.0 \times 0.30}{.01 \times 10^{-4}} = 9 \times 10^5$$

$\therefore$ Value of $$f = \frac{0.079}{R_e^{1/4}} = \frac{0.079}{\left(9 \times 10^5\right)^{1/4}} = .00256$$

$\therefore$ Head lost, $$h_f = \frac{4 \times .00256 \times 50 \times 3^2}{0.3 \times 2.0 \times 9.81} = \mathbf{.7828\ m.\ Ans.}$$

(*ii*) **Chezy's Formula.** Using equation

$$V = C\sqrt{mi}$$

where $C = 60$, $m = \frac{d}{4} = \frac{0.30}{4} = 0.075$ m

$\therefore$ $$3 = 60\sqrt{.075 \times i} \text{ or } i = \left(\frac{3}{60}\right)^2 \times \frac{1}{.075} = 0.0333$$

But $$i = \frac{h_f}{L} = \frac{h_f}{50}$$

Equating the two values of i, we have $\frac{h_f}{50} = .0333$

$\therefore$ $$h_f = 50 \times .0333 = \mathbf{1.665\ m.\ Ans.}$$

Problem *Find the diameter of a pipe of length 2000 m when the rate of flow of water through the pipe is 200 litres/s and the head lost due to friction is 4 m. Take the value of C = 50 in Chezy's formulae.*

Solution. Given :

Length of pipe, $L = 2000$ m

Discharge, $Q = 200$ litre/s $= 0.2\ m^3/s$

Head lost due to friction, $h_f = 4$ m

Value of Chezy's constant, $C = 50$

Let the diameter of pipe $= d$

Velocity of flow, $$V = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{\frac{\pi}{4}d^2} = \frac{0.2}{\frac{\pi}{4}d^2} = \frac{0.2 \times 4}{\pi d^2}$$

Hydraulic mean depth, $$m = \frac{d}{4}$$

Loss of head per unit length, $i = \frac{h_f}{L} = \frac{4}{2000} = .002$

Chezy's formula is given by equation as $V = C\sqrt{mi}$

Substituting the values of V, m, i and C, we get

$$\frac{0.2 \times 4}{\pi d^2} = 50\sqrt{\frac{d}{4} \times .002} \text{ or } \sqrt{\frac{d}{4} \times .002} = \frac{0.2 \times 4}{\pi d^2 \times 50} = \frac{.00509}{d^2}$$

Squaring both sides, $\frac{d}{4} \times .002 = \frac{.00509^2}{d^4} = \frac{.0000259}{d^4}$ or $d^5 = \frac{4 \times .0000259}{.002} = 0.0518$

$\therefore$ $$d = \sqrt[5]{0.0518} = (.0518)^{1/5} = 0.553 \text{ m} = \mathbf{553\ mm.\ Ans.}$$

Problem *An oil of sp. gr. 0.7 is flowing through a pipe of diameter 300 mm at the rate of 500 litres/s. Find the head lost due to friction and power required to maintain the flow for a length of 1000 m. Take* ν *= .29 stokes.*

Solution. Given :

Sp. gr. of oil, $S = 0.7$

Dia. of pipe, $d = 300 \text{ mm} = 0.3 \text{ m}$

Discharge, $Q = 500 \text{ litres/s} = 0.5 \text{ m}^3\text{/s}$

Length of pipe, $L = 1000 \text{ m}$

Velocity, $$V = \frac{Q}{\text{Area}} = \frac{0.5}{\frac{\pi}{4}d^2} = \frac{0.5 \times 4}{\pi \times 0.3^2} = 7.073 \text{ m/s}$$

$\therefore$ Reynolds number, $$R_e = \frac{V \times d}{\nu} = \frac{7.073 \times 0.3}{0.29 \times 10^{-4}} = 7.316 \times (10)^4$$

$\therefore$ Co-efficient of friction, $$f = \frac{.079}{R_e^{1/4}} = \frac{0.79}{\left(7.316 \times 10^4\right)^{1/4}} = .0048$$

$\therefore$ Head lost due to friction, $$h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .0048 \times 1000 \times 7.073^2}{0.3 \times 2 \times 9.81} = 163.18 \text{ m}$$

Power required $$= \frac{\rho g \,.\, Q \,.\, h_f}{1000} \text{ kW}$$

where ρ = density of oil = $0.7 \times 1000 = 700 \text{ kg/m}^3$

$\therefore$ Power required $$= \frac{700 \times 9.81 \times 0.5 \times 163.18}{1000} = \textbf{560.28 kW. Ans.}$$

Problem *Find the loss of head when a pipe of diameter 200 mm is suddenly enlarged to a diameter of 400 mm. The rate of flow of water through the pipe is 250 litres/s.*

Solution. Given :

Dia. of smaller pipe, $D_1 = 200 \text{ mm} = 0.20 \text{ m}$

$\therefore$ Area, $$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4}(.2)^2 = 0.03141 \text{ m}^2$$

Dia. of large pipe, $D_2 = 400 \text{ mm} = 0.4 \text{ m}$

$\therefore$ Area, $$A_2 = \frac{\pi}{4} \times (0.4)^2 = 0.12564 \text{ m}^2$$

Discharge, $Q = 250 \text{ litres/s} = 0.25 \text{ m}^3\text{/s}$

Velocity, $$V_1 = \frac{Q}{A_1} = \frac{0.25}{.03141} = 7.96 \text{ m/s}$$

Velocity, $$V_2 = \frac{Q}{A_2} = \frac{0.25}{.12564} = 1.99 \text{ m/s}$$

Loss of head due to enlargement is given by equation

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2g} = \textbf{1.816 m of water. Ans.}$$

Problem *The rate of flow of water through a horizontal pipe is 0.25 m³/s. The diameter of the pipe which is 200 mm is suddenly enlarged to 400 mm. The pressure intensity in the smaller pipe is 11.772 N/cm². Determine :*

(*i*) *loss of head due to sudden enlargement,* (*ii*) *pressure intensity in the large pipe,*
(*iii*) *power lost due to enlargement.*

Solution. Given :

Discharge, $Q = 0.25 \text{ m}^3/\text{s}$

Dia. of smaller pipe, $D_1 = 200 \text{ mm} = 0.20 \text{ m}$

$\therefore$ Area, $A_1 = \frac{\pi}{4} (.2)^2 = 0.03141 \text{ m}^2$

Dia. of large pipe, $D_2 = 400 \text{ mm} = 0.40 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4} (0.4)^2 = 0.12566 \text{ m}^2$

Pressure in smaller pipe, $p_1 = 11.772 \text{ N/cm}^2 = 11.772 \times 10^4 \text{ N/m}^2$

Now velocity, $V_1 = \frac{Q}{A_1} = \frac{0.25}{.03141} = 7.96 \text{ m/s}$

Velocity, $V_2 = \frac{Q}{A_2} = \frac{0.25}{.12566} = 1.99 \text{ m/s}$

(*i*) Loss of head due to sudden enlargement,

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2 \times 9.81} = \mathbf{1.816 \text{ m. Ans.}}$$

(*ii*) Let the pressure intensity in large pipe $= p_2$.

Then applying Bernoulli's equation before and after the sudden enlargement,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_e$$

But $z_1 = z_2$ (Given horizontal pipe)

$$\therefore \quad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_e \text{ or } \frac{p_2}{\rho g} = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - h_e$$

$$= \frac{11.772 \times 10^4}{1000 \times 9.81} + \frac{7.96^2}{2 \times 9.81} - \frac{1.99^2}{2 \times 9.81} - 1.816$$

$$= 12.0 + 3.229 - 0.2018 - 1.8160$$

$$= 15.229 - 2.0178 = 13.21 \text{ m of water}$$

$$\therefore \quad p_2 = 13.21 \times \rho g = 13.21 \times 1000 \times 9.81 \text{ N/m}^2$$

$$= 13.21 \times 1000 \times 9.81 \times 10^{-4} \text{ N/cm}^2 = \mathbf{12.96 \text{ N/cm}^2\text{. Ans.}}$$

(*iii*) Power lost due to sudden enlargement,

$$P = \frac{\rho g \cdot Q \cdot h_e}{1000} = \frac{1000 \times 9.81 \times 0.25 \times 1.816}{1000} = \mathbf{4.453 \text{ kW. Ans.}}$$

Problem *A horizontal pipe of diameter 500 mm is suddenly contracted to a diameter of 250 mm. The pressure intensities in the large and smaller pipe is given as 13.734 N/cm² and 11.772 N/cm² respectively. Find the loss of head due to contraction if $C_c = 0.62$. Also determine the rate of flow of water.*

Solution. Given :

Dia. of large pipe, $D_1 = 500 \text{ mm} = 0.5 \text{ m}$

Area, $A_1 = \frac{\pi}{4}(0.5)^2 = 0.1963 \text{ m}^2$

Dia. of smaller pipe, $D_2 = 250 \text{ mm} = 0.25 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4}(.25)^2 = 0.04908 \text{ m}^2$

Pressure in large pipe, $p_1 = 13.734 \text{ N/cm}^2 = 13.734 \times 10^4 \text{ N/m}^2$

Pressure in smaller pipe, $p_2 = 11.772 \text{ N/cm}^2 = 11.772 \times 10^4 \text{ N/m}^2$

$C_c = 0.62$

$$\text{Head lost due to contraction} = \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1.0\right]^2 = \frac{V_2^2}{2g}\left[\frac{1}{0.62} - 1.0\right]^2 = 0.375\frac{V_2^2}{2g}$$

From continuity equation, we have $A_1V_1 = A_2V_2$

or
$$V_1 = \frac{A_2V_2}{A_1} = \frac{\frac{\pi}{4}D_2^2 \times V_2}{\frac{\pi}{4}D_1^2} = \left(\frac{D_2}{D_1}\right)^2 \times V_2 = \left(\frac{0.25}{0.50}\right)^2 V_2 = \frac{V_2}{4}$$

Applying Bernoulli's equation before and after contraction,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_c$$

But $z_1 = z_2$ (pipe is horizontal)

$$\therefore \quad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_c$$

But $h_c = 0.375\frac{V_2^2}{2g}$ and $V_1 = \frac{V_2}{4}$

Substituting these values in the above equation, we get

$$\frac{13.734 \times 10^4}{9.81 \times 1000} + \frac{(V_2/4)^2}{2g} = \frac{11.772 \times 10^4}{1000 \times 9.81} + \frac{V_2^2}{2g} + 0.375\frac{V_2^2}{2g}$$

or
$$14.0 + \frac{V_2^2}{16 \times 2g} = 12.0 + 1.375\frac{V_2^2}{2g}$$

or
$$14 - 12 = 1.375\frac{V_2^2}{2g} - \frac{1}{16}\frac{V_2^2}{2g} = 1.3125\frac{V_2^2}{2g}$$

or
$$2.0 = 1.3125 \times \frac{V_2^2}{2g} \text{ or } V_2 = \sqrt{\frac{2.0 \times 2 \times 9.81}{1.3125}} = 5.467 \text{ m/s.}$$

(*i*) Loss of head due to contraction, $h_c = 0.375\frac{V_2^2}{2g} = \frac{0.375 \times (5.467)^2}{2 \times 9.81}$ = **0.571 m. Ans.**

(*ii*) Rate of flow of water, $Q = A_2V_2 = 0.04908 \times 5.467 = 0.2683 \text{ m}^3/\text{s}$ = **268.3 lit/s. Ans.**

Problem *The difference in water surface levels in two tanks, which are connected by three pipes in series of lengths 300 m, 170 m and 210 m and of diameters 300 mm, 200 mm and 400 mm respectively, is 12 m. Determine the rate of flow of water if co-efficient of friction are .005, .0052 and .0048 respectively, considering : (i) minor losses also (ii) neglecting minor losses.*

Solution. Given :

Difference of water level, $H = 12$ m

Length of pipe 1, $L_1 = 300$ m and dia., $d_1 = 300$ mm $= 0.3$ m

Length of pipe 2, $L_2 = 170$ m and dia., $d_2 = 200$ mm $= 0.2$ m

Length of pipe 3, $L_3 = 210$ m and dia., $d_3 = 400$ mm $= 0.4$ m

Also, $f_1 = .005$, $f_2 = .0052$ and $f_3 = .0048$

(*i*) **Considering Minor Losses.** Let V_1, V_2 and V_3 are the velocities in the 1st, 2nd and 3rd pipe respectively.

From continuity, we have $A_1V_1 = A_2V_2 = A_3V_3$

$$\therefore \quad V_2 = \frac{A_1V_1}{A_2} = \frac{\frac{\pi}{4}d_1^2}{\frac{\pi}{4}d_2^2}V_1 = \frac{d_1^2}{d_2^2}V_1 = \left(\frac{0.3}{.2}\right)^2 \times V_1 = 2.25\ V_1$$

and
$$V_3 = \frac{A_1V_1}{A_3} = \frac{d_1^2}{d_3^2}V_1 = \left(\frac{0.3}{0.4}\right)^2 V_1 = 0.5625\ V_1$$

Now using equation (11.12), we have

$$H = \frac{0.5\,V_1^2}{2g} + \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{0.5\,V_2^2}{2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{(V_2 - V_3)^2}{2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g} + \frac{V_3^2}{2g}$$

Substituting V_2 and V_3,
$$12.0 = \frac{0.5\,V_1^2}{2g} + \frac{4 \times .005 \times 300 \times V_1^2}{0.3 \times 2g} + \frac{0.5 \times \left(2.25 V_1^2\right)^2}{2g}$$

$$+ 4 \times 0.0052 \times 170 \times \frac{(2.25\,V_1)^2}{0.2 \times 2g} + \frac{(2.25\,V_1 - .562\,V_1)^2}{2g} + \frac{4 \times .0048 \times 210 \times (.5625\,V_1)^2}{0.4 \times 2g} + \frac{(.5625\,V_1)^2}{2g}$$

$$\therefore \quad V_1 = \sqrt{\frac{12 \times 2 \times 9.81}{118.887}} = 1.407 \text{ m/s}$$

$\therefore$ Rate of flow, Q = Area × Velocity $= A_1 \times V_1$

$$= \frac{\pi}{4}(d_1)^2 \times V_1 = \frac{\pi}{4}(.3)^2 \times 1.407 = 0.09945 \text{ m}^3\text{/s}$$

= 99.45 litres/s. Ans.

(*ii*) **Neglecting Minor Losses.** Using equation we have

$$H = \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g}$$

or
$$12.0 = \frac{V_1^2}{2g}\left[\frac{4 \times .005 \times 300}{0.3} + \frac{4 \times .0052 \times 170 \times (2.25)^2}{0.2} + \frac{4 \times .0048 \times 210 \times (.5625)^2}{0.4}\right]$$

$$= \frac{V_1^2}{2g}[20.0 + 89.505 + 3.189] = \frac{V_1^2}{2g} \times 112.694$$

$$V_1 = \sqrt{\frac{2 \times 9.81 \times 12.0}{112.694}} = 1.445 \text{ m/s}$$

Discharge, $Q = V_1 \times A_1 = 1.445 \times \frac{\pi}{4}(.3)^2 = 0.1021 \text{ m}^3\text{/s} =$ **102.1 litres/s. Ans.**

Problem *Three pipes of lengths 800 m, 500 m and 400 m and of diameters 500 mm, 400 mm and 300 mm respectively are connected in series. These pipes are to be replaced by a single pipe of length 1700 m. Find the diameter of the single pipe.*

Solution. Given :

Length of pipe 1, $L_1 = 800$ m and dia., $d_1 = 500$ mm $= 0.5$ m

Length of pipe 2, $L_2 = 500$ m and dia., $d_2 = 400$ mm $= 0.4$ m

Length of pipe 3, $L_3 = 400$ m and dia., $d_3 = 300$ mm $= 0.3$ m

Length of single pipe, $L = 1700$ m

Let the diameter of equivalent single pipe $= d$

Applying equation $$\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$$

or $$\frac{1700}{d^5} = \frac{800}{.5^5} + \frac{500}{.4^5} + \frac{400}{0.3^5} = 25600 + 48828.125 + 164609 = 239037$$

$\therefore$ $$d^5 = \frac{1700}{239037} = .007118$$

$\therefore$ $$d = (.007188)^{0.2} = 0.3718 = \textbf{371.8 mm. Ans.}$$

Problem 11.32 *A main pipe divides into two parallel pipes which again forms one pipe as shown in Fig. The length and diameter for the first parallel pipe are 2000 m and 1.0 m respectively, while the length and diameter of 2nd parallel pipe are 2000 m and 0.8 m. Find the rate of flow in each parallel pipe, if total flow in the main is 3.0 m³/s. The co-efficient of friction for each parallel pipe is same and equal to .005.*

Solution. Given :

Length of pipe 1, $L_1 = 2000$ m

Dia. of pipe 1, $d_1 = 1.0$ m

Length of pipe 2, $L_2 = 2000$ m

Dia. of pipe 2, $d_2 = 0.8$ m

Total flow, $Q = 3.0 \text{ m}^3\text{/s}$

$f_1 = f_2 = f = .005$

Let Q_1 = discharge in pipe 1

Q_2 = discharge in pipe 2

From equation $Q = Q_1 + Q_2 = 3.0$ ...(*i*)

Using equation , we have

$$\frac{4 f_1 L_1 V_1^2}{d_1 \times 2g} = \frac{4 f_2 L_2 V_2^2}{d_2 \times 2g}$$

$$\frac{4 \times .005 \times 2000 \times V_1}{1.0 \times 2 \times 9.81} = \frac{4 \times .005 \times 2000 \times V_2^2}{0.8 \times 2 \times 9.81}$$

or $$\frac{V_1^2}{1.0} = \frac{V_2^2}{0.8} \text{ or } V_1^2 = \frac{V_2^2}{0.8}$$

$\therefore$ $$V_1 = \frac{V_2}{\sqrt{0.8}} = \frac{V_2}{.894} \qquad ...(ii)$$

Now $$Q_1 = \frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4}(1)^2 \times \frac{V_2}{.894} \qquad \left[\because V_1 = \frac{V_2}{.894}\right]$$

and $$Q_2 = \frac{\pi}{4} d_2^2 \times V_2 = \frac{\pi}{4}(.8)^2 \times V_2 = \frac{\pi}{4} \times .64 \times V_2$$

Substituting the value of Q_1 and Q_2 in equation (*i*), we get

$$\frac{\pi}{4} \times \frac{V_2}{0.894} + \frac{\pi}{4} \times .64\, V_2 = 3.0 \text{ or } 0.8785\, V_2 + 0.5026\, V_2 = 3.0$$

$$V_2[.8785 + .5026] = 3.0 \text{ or } V = \frac{3.0}{1.3811} = 2.17 \text{ m/s.}$$

Substituting this value in equation (*ii*),

$$V_1 = \frac{V_2}{.894} = \frac{2.17}{0.894} = 2.427 \text{ m/s}$$

Hence $$Q_1 = \frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4} \times 1^2 \times 2.427 = \mathbf{1.906 \text{ m}^3\text{/s. Ans.}}$$

$\therefore$ $$Q_2 = Q - Q_1 = 3.0 - 1.906 = \mathbf{1.094 \text{ m}^3\text{/s. Ans.}}$$

TEXT / REFERENCE BOOKS

1. Bansal.R.K, Fluid Mechanics & Hydraulics Machines, 9th Edition, Laxmi Publications, 2015.
2. Modi P.N., Seth S.M., Hydraulics and Fluid Mechanics Including Hydraulic Machines, 21st Edition, Standard Book House, 2017.
3. Goyal, Manish Kumar, Fluid Mechanics and Hydraulic Machines, PHI Learning Pvt. Ltd., 2015.
4. Kumar K.L., Engineering Fluid Mechanics, 8thEdition, Eurasia Publication House (P) Ltd, 2014.
5. R. K. Rajput, Fluid Mechanics & Hydraulics Machines, 4th Edition, S. Chand Limited, 2008.
6. Yunus.A.Cengel, John.M.Cimbala, Fluid Mechanics Fundamentals and Application, 3rd Edition, Tata McGraw Hill, 2013.
7. Robert W. Fox, Alan T. McDonald, Philip J.Pritchard, John W.Mitchell, Fluid Mechanics, 9th Edition, Wiley India Edition, 2016.