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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

UNIT – I – STRESS STRAIN DEFORMATION OF SOLIDS – SMEA1305

Mechanics of solids (SMEA1305) 2019 – Regulations

(B.E MECHANICAL ENGG / MECHATRONICS)

UNIT 1: STRESS STRAIN DEFORMATION OF SOLIDS

Rigid and Deformable bodies – Strength, Stiffness and Stability – Stresses; Tensile, Compressive and Shear – Deformation of simple and compound bars under axial load – Thermal stresses and strains. Elastic constants – Relation between Elastic constans-Strain energy and unit strain energy – Strain energy in uniaxial loads.

INTRODUCTION

The theory of strength of Materials was developed over several centuries by a judicious combination of mathematical analysis, scientific observations and experimental results. Ancient structures had been constructed based on thumb rules developed through experience and intuition of their builders.

A structure designed to carry loads comprises various members such as beams, columns and slabs. It is essential to know the load carrying capacity of various members of structure in order to determine their dimensions for the minimum rigidity and stability of isolated structural members such as beams and columns.

The theory of strength of materials is presented in this book in a systematic way to enable students understand the basic principles and prepare themselves to the tasks of designing large structures and systems subsequently. It should be appreciated that even awe inspiring structures such as bridges, high rise towers tunnels and space crafts, rely on these principle of their analysis and design

HISTORICAL REVIEW

Though ancient civilizations could boast of several magnificent structures, very little information is available on the analytical and design principles adopted by their builders. Most of the developments can be traced to the civilizations of Asia, Egypt, Greece and Rome. Greek philosophers Aristotle (384-322 BC) and Archimedes (287 – 212) who formulated significant fundamental principles of statics. Though Romans were generally excellent builders, they apparently had little knowledge about stress analysis. The strength of materials were formulated by Leonardo da Vinci (AD 1452 – 1549, Italy) arguably the greatest scientist and artist of all times. It was much later in the sixteenth century that Galileo Galilei (AD 1564 – 1642, Italy) commenced his studies on the strength of materials and behavior of structures. Robe Hooke (1635 – 1703) made one of the most significant observations in 1678 that materials displayed a certain relation between the stress applied and the strain induced. Mariotte (1620 – 1684), Jacob Bernoulli (1667 – 1748), Daniel Bernoulli (1700 – 1782), Euler (1707 – 1783), Lagrange (1736 – 1813), Parent (1666 – 1748), Columb (1736 – 1806) and Navier (1785 – 1836), among several others made the most significant contributions.

The first complete elastic analysis for flexure of beams was presented by Columb in 1773 but his paper failed to receive the attention it deserved until 1825 when Navier published a book on strength of materials. Rapid industrial growth of the nineteenth century gave a further impetus to scientific investigations; several researchers and scientist advanced the frontiers of knowledge to new horizons.

The simple theories formulated in the earlier centuries have been extended to complex structural configuration and load conditions. Engineers are expected not only to design but also to check the performance of structures under various limit states such a s collapse, deflection and crack widths. The emphasis is always on safety, economy, durability, nevertheless.

SIMPLE STRESSES AND STRAINS

INTRODUCTION

Within elastic stage, the resisting force equals applied load. This resisting force per unit area is called stress or intensity of stress.

STRESS

The force of resistance per unit area, offered by a body against deformation is known as stress. The external force acting on the body is called the load or force. The load is applied on the body while the stress is induced in the material of the body. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are equal.

Mathematically stress is written as, $\sigma = \frac{P}{A}$

where σ = Stress (also called intensity of stress),

P = Cross-Sectional or load, and

A = Cross-Sectional area.

In the S.I. Units, the force is expressed in newtons (Written as N) and area is expressed as m^2. Hence, unit stress becomes as N/m^2. The area is also expressed in millimetre square then unit of force becomes as N/mm^2.

$$1\ N/m^2 = 1\ N/(100cm)^2 = 1\ N/10^4\ cm^3$$

$$= 10^4\ N/cm^2 \text{ or } 10^{-6} N/mm^2 \qquad \left(\therefore = \frac{1}{cm^2} = \frac{1}{10^2 mm^2} \right)$$

STRAIN

When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to the original dimension is known as strain. Strain is dimensionless.

$e = \frac{s\ell}{\ell}$ Sl - Change in length in mm

l - original length in mm

Strain may be:-

1. Tensile strain, 2. Compressive strain
2. Volumetric strain, and 4. Shear strain

If there is some increase in length of a body due to external force, then the ratio of increase of length to the original length of body is known as tensile strain. But if there is some decrease in length of the body, then the ratio of decrease of the length of the body to the original length is known as compressive strain. The ratio of change of volume of the body to the original volume is known as volumetric strain. The strain produced by shear stress is known as shear strain.

TYPES OF STRESSES

The stress may be normal stress or a shear stress.

Normal stress is the stress which acts in a direction perpendicular to the areas. It is represented by σ (sigma). The normal stress is further divided into tensile stress and compressive stress.

Tensile Stress. The stress induced in a body, when subjected to two equal and opposite pulls as shown in Fig.1.1 (α) as a result of which there is an increase in length, is known as tensile stress. The ratio of increase in length to the original length is known as tensile strain. The tensile stress acts normal to the area and it pulls on the area.

Let P = Full (or force) acting on the body.
A = Cross-sectional area of the body.
L = Original length of the body
dL = Increase in length due to pull P acting on the body σ
= Stress induced in the body, and
e = Strain (i.e., tensile strain)

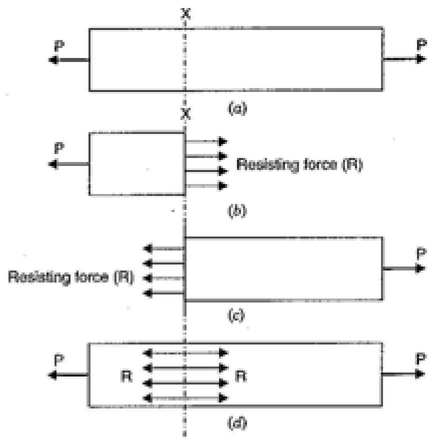


Fig. 1.1 Stress distributions during Tension

Fig.1.1 (∝) shown a bar subjected to a tensile force P as its ends. Consider χ-χ, which divides the bar into two parts. The part left to the section χ-χ, will be in equilibrium if P = resisting force (R). This is shown in Fig.1.1 (b). Similarly the part right to the sections χ-χ, will be in equilibrium if P = Resisting force as shown in Fig.1.1 (c). This relating force per unit area is known as stress or intensity of stress.

$$\therefore \text{Tensile} = \sigma \frac{\text{Reisting force (R)}}{\text{Cross - sectional area}} = \frac{\text{Tensile Load (P)}}{\text{A}} \qquad (\therefore \text{P= R})$$

or $$\sigma = \frac{P}{A} \qquad \text{... (1.1)}$$

And tensile strain is given by,

$$\text{e} = \frac{\text{Increase in length}}{\text{Original Length}} = \frac{\text{dL}}{\text{L}} \qquad \text{... (1.2)}$$

Compressive Stress

The stress induced in a body, when subjected to two equal and opposite pushes as shown in Fig.1.2. (α) as a result of which there is a decrease in length of the body, is shown as compressive stress. And the ratio of decrease in length to the original length is known as compressive strain. The compressive stress acts normal to the area and it pushes on the area.

Let an axial push *P* is acting on a body is cross-sectional area A. Due to external push *P*, let the original length *L* of the body decrease by *dL*.

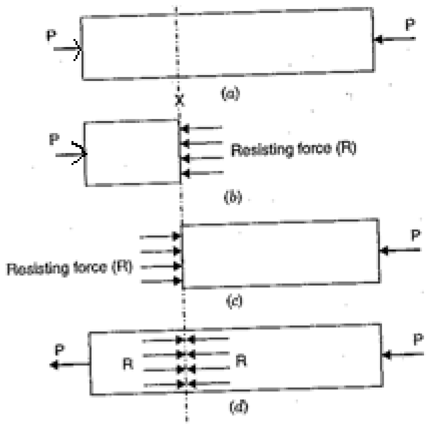


Fig. 1.2 Stress distributions during compression

The compressive stress is given by,

$$\sigma \frac{\text{Reisting force (R)}}{\text{Area (A)}} = \frac{\text{Push (P)}}{\text{Area (A)}} = \frac{\text{P}}{\text{A}}$$

And compressive strain is given by,

$$\text{e} \frac{\text{Decrease in length}}{\text{Original length}} = \frac{\text{dL}}{\text{L}}$$

1.4.2 Shear stress. The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as shown in Fig.1.3 as a result of which the body tends to shear off across the section, is known as shear stress. The corresponding strain is known as shear strain. The shear stress is the stress which acts tangential to the area. It is represented by τ.

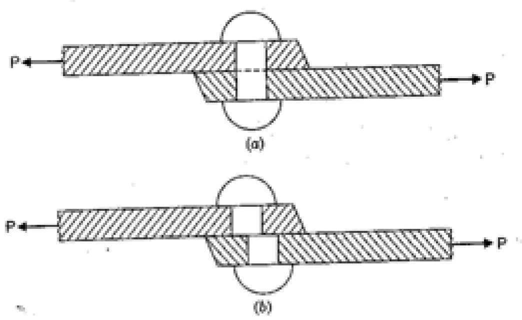


Fig. 1.4 Lap Joint in Shear

As the bottom face of the block is fixed, the face ABCD will be distorted to ABC, D through an angle ϕ as a result of force P as shown in Fig.1.4 (d).

And shear strain (ϕ) is given by

$$\phi = \frac{\text{Transversal displacement}}{\text{Distance AD}}$$

$$\text{or } \phi = \frac{DD_1 = \frac{dl}{h}}{AD} \qquad \text{...(1.4)}$$

ELASTICITY AND ELASTIC LIMIT

When an external force acts on a body tends to undergo some deformation. If the external force is removed and the body comes back to its origin shape and size (which means the deformation disappears completely), the body), the body is known as elastic body. The property by virtue of which certain materials return back to their original position after the removal of the external force, is called elasticity.

The body will regain its previous shape and size only when the deformation caused by the external force, is within a certain limit. Thus there is a limiting value of force upto and within which, the deformation completely disappears on the removal of the force. The value of stress corresponding to this limiting force is known as the elastic limit of the material.

If the external force is so large that the stress exceeds the elastic limit, the material loses to some extent its property of elasticity. If now the force is removed, the material will not return to the origin shape and size and there will be residual deformation in the material.

HOOKES LAW AND ELASTIC MODULII

Hooke's Law states that when a material is loaded within elastic limit, the stress is proportional to the strain produced by the stress. This means the ratio of the stress to the corresponding strain is a constant within the elastic limit. This constant is known as Module of Elasticity or Modulus of Rigidity or Elastic Modulii.

MODULUS OF ELASTICITY (OR YOUNG'S MODULUS)

The ratio of tensile or compressive stress to the corresponding strain is a constant. This ratio is known as Young's Modulus or Modulus of Elasticity and is denoted by E.

$$\therefore \quad E = \frac{\text{Tensile Stress}}{\text{Tensile Strain}} \text{ or } \frac{\text{Compressive Stress}}{\text{Compressive Strain}}$$

$$\text{or} \quad E = \frac{\sigma}{e} \qquad \text{... (1.5)}$$

Modulus of Rigidity or Shear Modulus. The ratio of shear stress to the corresponding shear strain within the elastic limit, is known as Modulus or Rigidity or Shear Modulus. This is denoted by C or G or N.

$$\therefore \quad C \text{ (or G or N)} = \frac{\text{Shear Stress}}{\text{Shear Strain}} = \frac{x}{\varphi} \qquad \text{... (1.6)}$$

Let us define factor of safety also.

FACTOR OF SAFETY

It is defined as the ratio of ultimate tensile stress to the working (or permissible) stress. Mathematically it is written as

$$\text{Factor of Safety} = \frac{\text{Ultimate Stress}}{\text{Pemissible Stress}} \qquad \text{... (1.7)}$$

CONSTITUTIVE RELATION BETWEEN STRESS AND STRAIN

For One Dimensional Stress System. The relationship between stress and strain for unidirectional stress (i.e., for normal stress in one direction only) is given by Hooke's law, which states that when a material is loaded within its elastic limit, the normal stress developed is proportional to the strain produced. This means that the ratio of the normal stress to the corresponding strain is a constant within the elastic limit. This constant is represented by E and is known as modulus of elasticity or Young's modulus of elasticity.

$$\therefore \quad = \frac{\text{Normal Stress}}{\text{Corresponding Strain}} = \text{Constant or } \frac{\sigma}{e} = E$$

where σ = Normal stress, e = strain and E = Young's Modulus

$$\text{or} \quad e = \frac{\sigma}{E} \qquad \text{... [1.7 (A)]}$$

The above equation gives the stress and strain relation for the normal stress in one direction.

For Two Dimensional Stress System. Before knowing the relationship between stress and strain for two-dimensional stress system, we shall have to define longitudinal strain, lateral strain, and Poisson's ratio.

Longitudinal Strain. When a body is subjected to an axial tensile load, there is an increase in the length of the body. But at the same time there is a decrease in other dimensions of the body at right angles to the line of action of the applied. Thus the body is having axial deformation and also deformation at right angles to the line of action of the applied load (i.e., lateral deformation).

The ratio of axial deformation to the original length of the body is known as longitudinal (or linear) strain. The longitudinal strain is also defined as the deformation of the body per unit length in the direction of the applied load.

Let L = Length of the body,
P = Tensile force acting on the body.
δL = Increase in the length of the body in the direction of P.

Then, longitudinal strain = $\frac{\delta L}{L}$

Lateral strain. The strain at right angles to the direction of applied load is known as lateral strain. Let a rectangular bar of length L, breadth b and depth δ is subjected to an axial tensile load *P* as shown in Fig.1.6. The length of the bar will increase while the breadth and depth will decrease.

Let L = Length of the body,
δb = Decrease in breadth, and
δd = Decrease in depth.

Then longitudinal strain = $\frac{\delta L}{L}$ ... [1.7 (B)]

and lateral strain = $\frac{\delta b}{b}$ or $\frac{\delta d}{d}$ ... [1.7 (C)]

Note:(i) If longitudinal strain is tensile, the lateral strains will be compressive.
(ii) If longitudinal strain is compressive then lateral strains will be tensile.
(iii) Hence every longitudinal strain in the direction of load is accompanied by lateral strains of the opposite kind in all directions perpendicular to the load.

Poisson's Ratio. The ratio is lateral strain to the longitudinal strain is a constant for a given material, when the material is stressed within the elastic limit. This ratio is called Poisson's ratio and it is generally denoted by μ. Hence mathematically.

Poisson's ratio, $\mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$ ... [1.7 (D)]

or Lateral strain = μ x Longitudinal strain

As lateral strain is opposite in sign to longitudinal strain, t\hence algebraically, lateral strain is written as

Relation between and strain. Consider a two dimensional figure ABCD, subjected to two mutually perpendicular stress σ_1 and σ_2

Longitudinal strain and will be equal to $\frac{\delta_1}{E}$ whereas the strain in

the direction of y will be lateral strain and will be equal to - μ x $\frac{\delta_1}{E}$. (∴ Lateral strain = - μx longitudinal strain)

The above two equations gives the stress and strain relationship for the two dimensional stress system. In the above equations, tensile stress is taken to be positive whereas the compressive stress negative.

For Three Dimensional Stress System. Fig. 1.5 (b) shows a three-dimensional body subjected to three orthogonal normal stress σ_1, σ_2, σ_3 acting in the directions of x, y and z respectively.

Consider the strains produced by each stress separately

Similarly the stress σ_2 will produced strain $\frac{\delta_2}{E}$ in the direction of y and strain of - μ $\frac{\delta_2}{E}$ in the direction of x and y each.

Also the stress σ_2 will produce strain $\frac{\delta_3}{E}$ in the direction of z and strain of - μ x $\frac{\delta_3}{E}$ in the direction of x and y.

$$e_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E} \quad \text{... [1.7 (H)]}$$

$$e_2 = \frac{\sigma_3}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} \quad \text{... [1.7 (J)]}$$

$$e_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} \quad \text{... [1.7 (J)]}$$

and The above three equations giver the stress and strain relationship for the three orthogonal normal stress system.

Problem 1.1 A rod 150cm long and of diameter 2.0cm is subjected to an axial pull of 20 kN. If the modulus of elasticity of the material of the rod is 2 x 10^5 N/mm^2, determine:

(i) the stress
(ii) the strain, and
(iii) the elongation of the rods.

Sol. Given : Length the rod, L = 150 cm
Diameter of rod, D = 2.0 cm = 20mm

$\therefore$ Area, A = $\frac{\pi}{4}(20)^2 - 100\pi$ mm^2

Axial pull, P = 20 kN = 20,000N

Modulus of elasticity E = 2.0 x 10^5 N/mm^2

(i) The stress (σ) is given equation (1.1) as

$$\sigma = \frac{P}{A} = \frac{2000}{100\pi} \text{ - } 63.662 \text{ N/mm}^2\text{, Ans.}$$

(ii) Using equation (1.5) the strain is obtained as

$$E\frac{\sigma}{e}$$

$$\therefore \text{Strain, e} = E\frac{\sigma}{E} = \frac{63.662}{2 \times 10^6} = 0.000318. \text{ Ans.}$$

(iii) Elongation is obtained by using equation (1.2) as

$$e = \frac{dL}{L}$$

$\therefore$ Elongation, dL = e x L

= 0.000318 x 150 = 0.0477cm. Ans

Problem 1.2. Find the minimum diameter of a steel wire, which is used to raise σ load of 4000 N if the stress in the rod is not to exceed 95MN/m^2.

Sol. Given : Load, P = 4000N

Stress, σ = 95MN/m^2 = 95 x 10^6 N/m^2 ($\therefore$ M=Mega=10^6)

= 95N/mm^2 ($\therefore$ 10^6 N/m^2 = 1N/mm^2)

Let D = Diameter of wire in mm

$\therefore$ Area, A = $\frac{\pi}{4}D^2$

Now Stress = $\frac{\text{Load}}{\text{Area}} = \frac{\text{P}}{\text{A}}$

$$95 = \frac{4000}{\frac{\pi}{4}D^2} = \frac{4000 \times 4}{\pi D^2} \quad \text{or} \quad D^2 = \frac{4000 \times 4}{\pi \times 95} = 53.61$$

D = 7.32mm Ans.

Problem 1.3. A tensile test was conducted on a mild steel bar. The following data was obtained from the test:

(i)	Diameter of the steel bar	=	3cm
(ii)	Gauge length of the bar	=	20cm
(iii)	Load at elastic limit	=	250 kN
(iv)	Extension at a load of 150 kN	=	0.21mm
(v)	Maximum load	=	380 kN
(vi)	Total extension	=	60mm
(vii)	Diameter of the rod at the failure	=	2.25cm

Determine : (a) the Young's Modulus, (b) the stress elastic limit
(c) the percentage elongation, and (d) the percentage decrease in area.

Sol. Area of rod, A = $\frac{\pi}{4}D^2 = \frac{\pi}{4}(3)^2 cm^2$

$$= \quad 7.06835\ cm^2 = 7.0685\ x\ 10^{-4}\ m^2 \left[\therefore cm^2 = \left(\frac{1}{100}m\right)^2\right]$$

(a) To find Young's modulus, first calculate the value of stress and strain within elastic limit. The load at elastic limit it given but the extension corresponding to the load of elastic limit is not given. But a load 150 kN (which is within elastic limit) and corresponding extension of 0.21mm are given. Hence these values are used for stress and strain within elastic limit

$$Stress = \frac{Load}{Area} = \frac{150\ x\ 1000}{7.0685\ x\ 10^{-4}} N/m^2 \qquad (\therefore\ 1\ kN = 1000\ N)$$

$= \quad 21220.9\ x\ 10^4\ N/m^2$

and $\quad Strain = \frac{Increase\ in\ length\ (or\ Extension)}{Original\ Length\ (or\ Guage\ length)}$

$$= \frac{0.21mm}{20\ x\ 10mm} = 0.00105$$

$\therefore$ Young's Modolus

$$E = \frac{Stress}{Strain} x \frac{21220.9x10^4}{0.00105} = 20209523\ x\ 10^4\ N/m^2$$

$= \quad 202.095\ x\ 10^9\ N/m^2 \qquad (\therefore\ 10^9 = Giga = G)$

= 202.095 x GN/m² Ans.

(b) The stress at the elastic limit is given by

$$\frac{Stress = Load\ at\ elastic\ limit}{Area} = \frac{250x1000}{7.0685x10^{-4}}$$

$= \quad 35368\ x\ 10^4\ N/m^2$

$= \quad 353.68\ x\ 10^6\ N/m^2 \qquad (\therefore\ 10^6 = Mega = M)$

= 353.68 MN/m². Ans.

(c) The percentage decrease is obtained as,
percentage elongation

$$= \frac{\text{Total Increase in length}}{\text{Original length (or guage length)}} \times 100$$

$$= \frac{60\text{mm}}{20 \times 10\text{mm}} \times 100 = 30\% \quad \text{Ans.}$$

(d) The percentage decrease in area is obtained as percentage decrease in area.

$$= \frac{(\text{Original area - Area at the failure})}{\text{Original area}} \times 100$$

$$= \frac{\left(\frac{\pi}{4} \times 3^2 - \frac{\pi}{4} \times 2.25^2\right)}{\frac{\pi}{4} \times 3^2} \times 100$$

$$= \left(\frac{3^2 - 2.25^2}{3^2}\right) \times 100 = \frac{(9 - 5.0625)}{9} \times 100 = 43.75\% \quad \text{Ans.}$$

ANALYSISZS OF BARS OF VARYING SECTIONS

A bar of different lengths and of different diameters (and hence of different cross-sectional areas) is shown in Fig.1.4 (α). Let this bar is subjected to an axial load *P*.

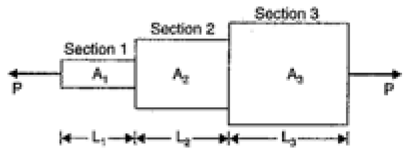


Fig. 1.5 Bar with varied cross sections and Axial load

Though each section is subjected to the same axial load *P*, yet the stresses, strains and change in length will be different. The total change in length will be obtained by adding the changes in length of individual section

Let P = Axial load acting on the bar,
L_1 = Length of section 1,
A_1 = Cross-Sectional area of section 1,
L_2, A_2 = Length and cross-sectional areas of section 2,
L_3, A_3 = Length and cross-sectional areas of section 3, and
E = Young's modulus for the bar.

Problem 1.4. An axial pull of 35000 N is acting on a bar consisting of three lengths as shown in Fig.1.6 (b). If the Young's modulus = 2.1×10^5 N/mm^2, determine.

(i) Stresses in each section and
(ii) total extension of the bar

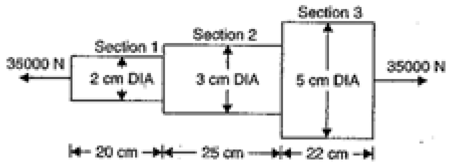


Fig. 1.6 Bar with varied cross sections and Axial load as 35000N

Sol. Given:

Axial pull, $P = 35000$ N
Length of section 1, $L_1 = 20cm = 220mm$
Dia. of Section 1, $D_1 = 2cm = 20mm$

$\therefore$ Area of Section 1, $A_1 = \frac{\pi}{4}(20^2) = 100\,\pi\ \text{mm}^2$

Length of section 2, $L_2 = 25cm = 250mm$
Dia. of Section 2, $D_2 = 3cm = 30mm$

$\therefore$ Area of Section 2, $A_2 = \frac{\pi}{4}(30^2) = 225\,\pi\ \text{mm}^2$

Length of section 3, $L_3 = 22cm = 220mm$
Dia. of Section 3, $D_3 = 5cm = 50mm$

$\therefore$ Area of Section 2, $A_3 = \frac{\pi}{4}(50^2) = 625\,\pi\ \text{mm}^2$

Young's Modulus, $E = 2.1 \times 10^5\ \text{N/mm}^2$

(i) Stress in each section

Stress in section 1, $\sigma_1 = \dfrac{\text{Axial load}}{\text{Area of Section 1}}$

$$= \frac{P}{A_1} = \frac{35000}{100\pi} = 111.408\text{N/mm}^2\text{. Ans.}$$

Stress in section 2, $= \dfrac{P}{A_2} = \dfrac{35000}{225 \times \pi} = 49.516\text{N/mm}^2\text{. Ans.}$

Stress in section 3, $= \dfrac{P}{A_3} = \dfrac{35000}{625 \times \pi} = 17.825\ \text{N/mm}^2\text{. Ans.}$

(ii) Total extension of the bar
Using equation (1.8), we get

Total Extension = $\frac{P}{E}\left(\frac{L_1}{A_1}+\frac{L_2}{A_2}+\frac{L_3}{A_3}\right)$

$$= \frac{35000}{2.1 \times 10^5}\left(\frac{200}{100\pi}+\frac{250}{225 \text{ x } \pi}+\frac{230}{625 \text{ x } \pi}\right)$$

$$= \frac{35000}{2.1 \times 10^5}(6.366 + 3.536 + 1.120) = 0.183\text{mm Ans.}$$

Problem 1.5. A member formed by connecting a steel bar to aluminium for bar is shown in Fig.1.7. Assuming that the bars are presented from buckling, sideways, calculate the magnitude of force P that will causes the total length of the member to decrease 0.25mm. The values of elastic modulus for steel and aluminium are 2.1 x 10^6 N/mm^2 and 7 x 10^4 N/mm^2 respectively.

Sol. Given

Length of Steel bar, L_1 = 30c m = 300mm
Area of Steel bar, A_1 = 5 x 5 = 25m^2 = 250mm^2
Elastic modulus for steel bar, E_1 = 2.1 x 10^5 N/mm^2
Length of Aluminium bar, L_2 = 38cm = 380mm
Area of Aluminium bar A_2 = 10 x 10 = 100cm^2 = 1000mm^2
Elastic modulus for aluminium bar E_2 = 7 x 10^4 N/mm^2
Total Decrease in length, dL = 0.25mm
Let P = Required force

As both the bars are made of different materials, hence total change in the lengths of the bar is given by equation (1.9)

$$dL = P\ \frac{L_1}{E_1A_1}+\frac{L_2}{E_2A_2}$$

or

$$0.25 = P\left(\frac{300}{2.1 \times 10^5 \text{ x } 2500}+\frac{380}{7 \times 10^4 \times 1000}\right)$$

$$= P\ (5.714 \times 10^{-7} + 5.428 \times 10^{-7}) = P \times 11.142 \times 10^{-7}$$

$$P = \frac{0.25}{11.142 \times 10^{-7}}+\frac{0.25 \times 10^7}{11.142} = 2.2437 \times 10^5 = 224.37 \text{ kN. Ans.}$$

Principle of Superposition. When a number of Loads are acting on a body, the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads.

While, using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body, first the free body diagram of individual section is drawn. Then the deformation of the each section is obtained.

The total deformation of the body will be then equal to the algebraic sum of deformation of the individual sections.

Problem 1.6 A brass bar, having cross-sectional area of 1000 mm^2 , is subjected to axial forces as shown in Fig.

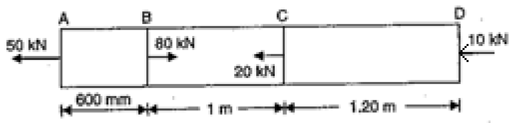


Fig. 1.7 Bar with same cross section and Axial loads

Find the total elongation of the bar, Take E = 1.05 x 10^5 N/mm^2

Sol. Given:

Area A = $1000mm^2$

Value of E = 1.05 x 10^5 N/mm^2

Let d = Total elongation of the bar

The force of 80 kN acting at B is split up into three forces of 50 kN, 20 kN and 10 kN. Then the part AB of the bar will be subjected to a tensile load of 50 kN, part BC is subjected to a compressive load of 20 kN and part BD is subjected to a compressive load of 10 kN as shown in Fig.

Part AB. This part is subjected to a tensile load of 50kN. Hence there will be increase in length of this part.,

∴ Increase in the length of AB

$$= \frac{P_1}{AE} \text{ x } L_1 \quad = \frac{500 \text{ x } 1000}{1000 \text{ x } 1.05 \text{ x } 10^5} \text{ x } 600$$

(∵ P_1=50,000 N,L_1 = 600mm)

= 0.2857

Part BC. This part is subjected to a compressive load of 20kN or 20,000 N. Hence there will be decrease in length of this part.

∴ Decrease in the length of BC

$$= \frac{P_2}{AE} \text{ x } L_2 = \frac{20{,}000}{1000 \text{ x } 1.05 \text{ x } 10^5} \text{ x } 1000$$

(∵ L_2=1m = 1000mm)

= 0.1904

Part BD. The part is subjected to a compressive load of 10kN or 10,000 N. Hence there will be decrease in length of this part.

∴ Decrease in the length of BC

$$= \frac{P_3}{AE} \text{ x } L_3 = \frac{10{,}000}{1000 \text{ x } 1.05 \text{ x } 10^5} \text{ x } 2200$$

(∵ L_2=1.2 + 1.22m or 2200m)

= 0.2095

∴ Total elongation of bar = 0.2857 – 0.1904 – 0.2095)

(Taking +ve sign for increase in length and –ve sign for decrease in length

$= -0.1142$mm. Ans.

Negative sign shows, that there will be decrease in length of the bar.

Problem 1.7. A Member ABCD is subjected to point loads P_1, P_2, P and P_4 as shown in Fig.

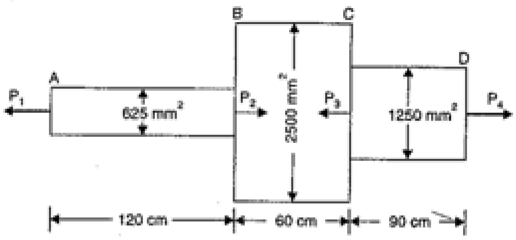


Fig. 1.8 Bar with varied cross section and Axial loads

Calculate the force P_2 necessary for equilibrium, if $P_1 = 45$ kN, $P_3 = 450$ kN and $P_4 = 130$ kN. Determine the total elongation of the member, assuming the modulus of elasticity to be 2.1×10^5 N/mm^2.

Set Given:

Part AB :	Area.	$A_1 = 625$ mm^2 and
	Length	$L_1 = 120$cm $= 1200$mm
Part BC :	Area	$A_2 = 2500$ mm^2 and
	Length	$L_2 = 60$cm $= 600$mm
Part CD :	Area	$A_3 = 12.0$mm^2 and
	Length	$L_3 = 90$cm $= 900$mm
Value of		$E = 2.1 \times 10^5$ N/mm^2

Value of P_2 necessary for equilibrium

Resolving the force on the rod along its (i.e., equating the forces acting towards right to those acting towards left) we get

$P_1 + P_3 = P_2 + P_4$

But $P_1 = 45$kN

$P_3 = 450$ kN and $P_4 = 130$kN

$\therefore \quad 45 + 450 = P_2 = 130$ or $P_2 = 495 - 130 =, 365$ kN

The force of 365 kN acting at B is split into two forces of 45 kN and 320 kN (i.e., 365 – 45 = 320 kN)

The force of 450 kN acting at C is split into two forces of 320 kN and 130 kN (i.e., 450 – 320 = 130 kN) as shown Fig.

It is clear that part AB is subjected to a tensile load of 45kN, part BC is subjected to a compressive load of 320 kN and par CD is subjected to a tensile load 130 kN.

Hence for part AB, there will be increase in length; for part BC there will be decrease in length and for past CD there will be increase in length.

$\therefore$ Increase in length of AB

$$= \frac{P}{A_1 E} \times L_1 = \frac{45000}{625 \times 2.1 \times 10^5} \times 1200 \qquad (\because P = 45 \text{ kN} = 45000 \text{ N})$$

= 0.4114 mm

$\therefore$ Decrease in length of BC

$$= \frac{P}{A_2 E} \times L_2 = \frac{320,000}{2500 \times 2.1 \times 10^5} \times 600 \qquad (\because P = 320 \text{ kN} = 320000 \text{ N})$$

= 0.3657 mm

Increase in length of CD

$$= \frac{P}{A_3 E} \times L_3 = \frac{130,000}{1250 \times 2.1 \times 10^5} \times 900 \qquad (\because P = 130 \text{ kN} = 130000 \text{ N})$$

Total change in the length of member

= 0.4114 – 0.3657 + 0.4457

(Taking +ve for increase in length and –ve sign for decrease in length)

= 0.4914mm (extension) Ans.

Problem 1.8. A rod, which tapers uniformly from 40mm diameter to 20mm diameter in a length of 400 mm is subjected to an axial load of 5000 N. If E = 2.1 x 10^6 N/mm^2, find the extension of rod.

Sol.Given

Larger diameter $D_1 = 40$mm
Smaller diameter $D_2 = 20$mm
Length of rod, L = 400mm
Axial load P = 5000 N
Young's modulus E – 2.1 x 10^5 N/mm^2
Let dL = Total extension of the rod

Using equation (1.10),

$$dL = \frac{4PL}{\pi E\, D_1 D_2} = \frac{4 \times 5000 \times 400}{\pi \times 2.1 \times 10^5 \times 40 \times 20}$$

= 0.01515mm Ans.

Problem 1.9. Find the modulus of elasticity for a rod, which tapers uniformly from 20mm, to 15mm diameter in a length of 350mm. The rod is subjected to an axial load of 5.5 kN and extension of the rod is 0.025mm.

Sol.Given

Larger diameter $D_1 = 30$mm
Smaller diameter $D_2 = 15$mm
Length of rod, L = 350mm
Axial load P = 5.5 kN = 5500 N
Extension dL = 0.025mm

Using equation (1.10), We get

$$dL = \frac{4PL}{\pi E\, D_1 D_2}$$

or $$E = \frac{4PL}{\pi\, D_1 D_2\, dL} = \frac{4 \times 5000 \times 350}{\pi \times 30 \times 15 \times 0.025}$$

= 217865 N/mm^2 or 2.17865 x 10^5 N/mm^2. Ans.

Problem 1.10. A rectangular bar made of steel is 2.8m long and 15mm thick. The rod is subjected to an axial tensile load of 40kN. The width of the rod varies from 75mm at one end to 30mm at the other. Find the extension of the rod if E = 2 x 10^5 N/mm^2.

Sol.Given

Larger L_1 = 2.8 m = 2800mm
Thickness t = 15mm
Axial load P = 40 kN = 40,000 N
Width at bigger end a = 75mm
Width at smaller end b = 30mm
Value of E = 2 x 10^5 N/mm^2
Let dL = Extension of the rod
Using equation (1.), We get

$$dL = \frac{PL}{Et(a - b)} \log_e \frac{a}{b}$$

$$= \frac{4000 \times 2800}{2 \times 10^5 \times 15(75 - 30)} \log_e \frac{75}{30}$$

= 0.8296 x 0.9163 = 0.76mm Ans.

Problem 1.11. The extension is a rectangular steel bar of length 400mm and thickness 10mm, is found to be 0.21 mm. The bar tapers uniformly in width from 100mm to 50mm. If E for the bar is 2 x 10^5 N/mm^2, determine the axial load on the bar.

Sol.Given

Extension dL = 0.21mm
Length L = 400mm
Thickness t = 10mm
Width at bigger end a = 100mm
Width at smaller end b = 50mm
Value of E = 2 x 10^5 N/mm^2
Let P = axial load
Using equation (1.), We get

$$dL = \frac{PL}{Et(a - b)} \log_e \left(\frac{a}{b}\right)$$

or $$0.21 = \frac{P \times 400}{2 \times 10^5 \times 10(100 - 50)} \log_e \frac{100}{50}$$

= 0.000004 P x 0.6931

$$\therefore P = \frac{0.21}{0.000004 \times 0.6931} 75746 \text{ N}$$

= 75.746 kN Ans.

ANALYSIS OF BARS OF COMPOSITE SECTIONS

A bar, made up two or more bars of equal lengths but of different materials rigidly fixed with each other and behaving as one unit for extension or compressive when subjected to an axial tensile or compressive loads, is called a composite bar. For the composite bar the following two points are important:

1. The extension or compression in each bar is equal. Hence determination per unit length i.e. strain in each bar is equal.
2. The total external load on the composite bar is equal to the sum of the loads carried by each different material.

Problem 1.12. A steel rod of 3cm diameter is enclosed centrally in a hollow copper tube of external diameter of 4cm. The composite bar is ten subjected to an axial pull of 45000 N. If the length of each bar is equal to 15cm, determine.

(i) The stresses in the rod and tube, and
(ii) Load carried by each bar

Take E for steel = 2.1×10^5 N/mm^2 and for copper = 1.1×10^5 N/mm^2

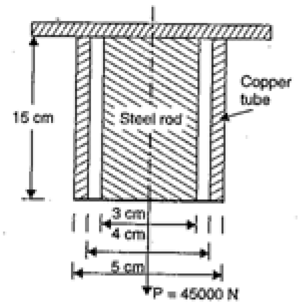


Fig. 1.9 Composite bar

Sol Given:

Dia of steel rod = 3cm = 30mm

$\therefore$ Area of steel rod,

$$A_e = \frac{\pi}{4}(30)^2 = 706.86 \text{mm}^2$$

External dia. of copper tube

= 5cm = 50mm

Internal dia. of copper tube

= 4cm = 40mm

$\therefore$ Area of copper tube,

$A_e = \frac{\pi}{4}(50^2-40^2)mm^2 = 706.86mm^2$

Axial pull on composite bar, P = 45000 N
Length of each bar L = 15cm
Young's modulus for steel, $E_S = 2.1 \times 10^5$ N/mm^2
Young's modulus for copper $E_c = 1.1 \times 10^5$ N/mm^2
(i) The stress in the rod and tube
Let σ_S = Stress in steel
P_S = Load carried by steel rod
σ_c = Stress in copper, and
P_c = Load carried by copper tube.

Now strain in steel = Strain in copper $\left(\because \frac{\sigma}{E} = \text{Strain}\right)$

or $\frac{\sigma_S}{E_S} = \frac{\sigma_c}{E_c}$

$\therefore \sigma_S = \frac{E_S}{E_c} \times \sigma_c = \frac{2.1 \times 10^6}{11 \times 106} \times \sigma_c =- 1.900\ \sigma_c$

Now Stress = $\frac{\text{Load}}{\text{Area}}$, $\therefore$ Load = Stress x Area
Load on steel + load on copper = Total load
$\sigma_S \times A_S + \sigma_c \times A_c = P$ $(\because \text{Total Load} = P)$
or 1.909 σ_c x 706.86 + 706.86 = 45000
or σ_c (1.909 x 706.86 + 706.86) = 45000
or 2056.25 σ_c = 45000
$\therefore\ \sigma_c \frac{45000}{2056.25} = 21.88\text{N/mm}^2\text{Ans}$
Substituting the value of σ_c in equation (i), we get
σ_c = 1.909 x 21.88 N/mm^2
= 41.77 N/mm^2. Ans

(ii) Load carried by each bar
As Load = Stress x Area
$\therefore$ Load carried by steel rod
$P_S = \sigma_S \times A_S$
= 41.77 x 706.86 = 29525.5 N. Ans
Load Carried by copper tube,
$P_c =$ 45000 – 29525.5
= 15474.5 N. Ans

Problem 1.13. A compound tube consists of a steel tube 140mm internal diameter and 160mm external diameter and an out brass tube 160mm internal diameter and 180mm external diameter. The two tubes are of the same length. The compound tube carries an axial load of 900 kN. Find the stresses and the load carried by each tube and the amount if shortens. Length of each tube is 140mm. Take E for Steel as 2×10^5 N/mm^2 and for brass as 1×10^5 N/mm^2.

Sol Given:

Internal dia. of steel tube = 140mm

External dia. of steel tube = 160mm

$\therefore$ Area of steel tube, $A_a = \frac{\pi}{4}(160^2 - 140^2) = 4712.4 mm^2$

Internal dia. of brass tube = 160mm

External dia. of brass tube= 180mm

$\therefore$ Area of steel tube, $A_b = \frac{\pi}{4}(180^2 - 160^2) = 5340.7.4 mm^2$

Axial load carried by compound tube,

P = 900 kN = 900 x 1000 = 900000N

Length of each tube L = 140mm

E for steel $E_a = 2 \times 10^5$ N/mm^2

E for brass $E_b = 1 \times 10^5$ N/mm^2

Let σ_a = Stress in steel in N/mm^2 and

σ_b = Stress in brass in N/mm^2

Now strain in steel = Strain in brass $\left(\because \text{Strain } \frac{\text{Stress}}{E}\right)$

or $\frac{\sigma_S}{E_S} = \frac{\sigma_b}{E_b}$

$\therefore \sigma_S = \frac{E_a}{E_b} \times \sigma_b = \frac{2 \times 10^6}{1 \times 10^5} \times \sigma_b = 2\sigma_b$

Now load on steel + Load on brass = Total load

or $\sigma_S \times A_a + \sigma_b \times A_b$ = 900000 ($\because$ Load = Stress x Area)

or $2\sigma_b \times 4712.4 + \sigma_b \times 5340.7 = 900000$ ($\because \sigma_S = 2\sigma_b$)

or $147655\ \sigma_b$ = 900000

$\therefore$ $\sigma_b = \frac{900000}{14765.5} = 60.95 N/mm^2$.Ans

Substituting the value of P_b in equation (i), we get

$\sigma_s = 2 \times 60.95 = 121.9$ N/mm^2. Ans.

Load carried by brass tube

= Stress x Area

= $\sigma_b \times A_b = 60.95 \times 5340.7$N

= 325515 N = 325.515 kN Ans.

Load carried by steel tube

= 900 – 325.515 = 574.485 kN. Ans.

Decrease in the length of the compound tube

= Decrease in length of either of the tubes

= Decrease in length of brass tube

= Strain in brass tube x original length

$= \frac{\sigma_b}{E_b} \times L = \frac{60.95}{1 \times 10^5} \times 140 = 0.0853 mm.$ Ans

Thermal Stresses

A solid structure is changes in original shape due to change in temperature its might expand or contract.

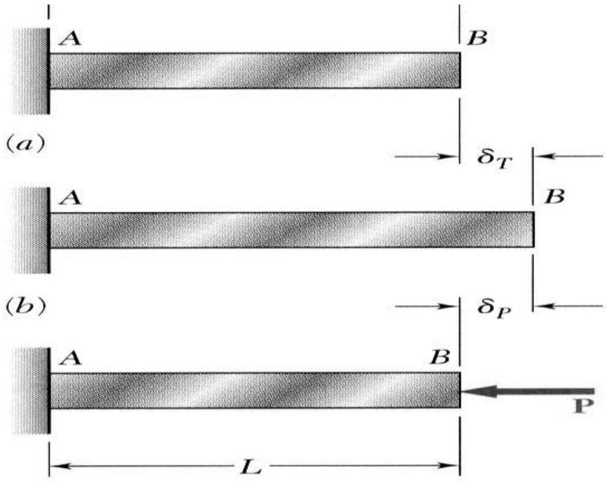


Fig. 1.10 Thermal expansion and contraction

Definition: A temperature change results in a change in length or thermal strain. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.

Raise at temperature ∝ materials is expands (elongate)

Decreases at temperature ∝ materials is contract (shorten)

$\delta_T = \alpha(\Delta T)L$ $\qquad\qquad\qquad$ $\delta_P = \dfrac{PL}{AE}$

Thermal strain e = α .T and thermal stress p= α .T.E

α = thermal expansion coef. T=Rise or fall of temperature E= young's modulus

Solved problem

Problem: 1A steel rod of 50m long and 3cm diameter is connected to two grips and the rod is maintained at a temperature of 95oC. Find out the force exerted by the rod after it has been cooled to 30oC, if (a) the ends do not yield, and (b). The ends yield by .12cm. Take E = 2.1 x 105 N/mm2; α = 12 x 10-6/ oC.

Given Data

Diameter	length	T1	T2	T=T1-T2	E	α
3cm	5m	95	30	65	2.00E+05	1.20E-05
30	5000					
mm	m	˚C	˚C	˚C	N/mm^2	/˚C
To find	I)when the ends do not yield ii)when the ends yield.12cm					

Required formula

The rod not yield	stress α.T.E	Area (∏/4) d^2	pi 3.14	4 4	d^2 900
	stressXArea				
The ends yield by 0.12cm	stress (α.T.L-δ)/L X E stressXArea				
1.2	δ				

Sloution	The rod not yield	α.T.E 1.56E+02	(∏/4) d^2 706.5	stressXArea 1.10E+05 N (Ans)				
	The ends yield by 0.12cm	α.T.L 3.90E+00	δ 1.2	L 50	E 2.00E+05	α.T.L-δ 2.70E+00	α.T.L-δ/L 5.40E-04	(α.T.L-δ/L)XE 1.08E+02
				stressXArea 7.63E+04 N (Ans)				

Problem: 2 A copper rods of 10cm diameter and 1.5m long is connected to two grips and the rod is maintained at a temperature of 125oC. Find out the force exerted by the rod after it has been cooled to 45oC, if (a) the ends do not yield, and (b). The ends yield by 1.7mm.Take E = 120Gpa; α = 1.7 x 10-6/ oC.

Given Data

Diameter	length	T1	T2	T=T1-T2	E	α
10cm	1.5m	125	45	80	1.20E+05	1.70E-05
100	1500					
mm	m	˚C	˚C	˚C	N/mm^2	/˚C
To find	I)when the ends do not yield ii)when the ends yield.12cm					

Required formula

The rod not yield	stress α.T.E	Area (∏/4) d^2	pi 3.14	4 4	d^2 10000
	stressXArea				
The ends yield by 0.15cm	stress (α.T.L-δ)/L X E stressXArea				
1.5	δ				

Sloution	The rod not yield	α.T.E 1.63E+02	(∏/4) d^2 7850	stressXArea 1.28E+06 N (Ans)				
	The ends yield by 0.12cm	α.T.L 2.04E+00	δ 1.5	L 50	E 1.20E+05	α.T.L-δ 5.40E-01	α.T.L-δ/L 3.60E-04	(α.T.L-δ/L)XE 4.32E+01
				stressXArea 3.39E+05 N (Ans)				

Thermal stress in composite bar

In certain application it is necessary to use a combination of elements or bars made from different materials, each material performing a different function. Temperature remains the same for all the materials but strain rate is different due to thermal expansion of materials. The blow figure shows the thermal expansion on composite bar.

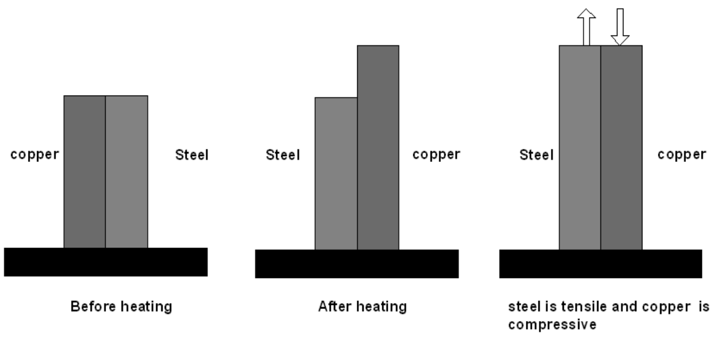


Fig. 1.11 Thermal expansion on Composite bar

The Expression for thermal stress is Load on the brass = load on the steel

From the stress equation

$\sigma_c \, X A_c = \sigma_s \, X A_s$

Thermal stress for copper $\sigma_c = \frac{\sigma_s \, X A_s}{A_c}$

Thermal stress for steel $\sigma_s = \frac{\sigma_c \, x A_c}{A_s}$

Actual expansion of copper = Actual expansion of steel

Free expansion of copper – contraction due to compressive stress = Free expansion of steel – expansion due to tensile stress

$\left\{\alpha_s x\ T\ x\ L\right\} + \left\{\frac{\sigma_s}{E_s} x\ L\right\} = \left\{\alpha_c x\ T\ x\ L\right\} + \left\{\frac{\sigma_c}{E_c}\ x\ L\right\}$

"L" is the common for both the sides therefore rewriting the above equation

$\alpha_s x\ T + \frac{\sigma_s}{E_s} = \alpha_c x\ T + \frac{\sigma_c}{E_c}$

Problem: A copper rod of 15 mm diameter passes centrally through a steel tube of 30 mm outer diameter and 20 mm internal diameter. The tube is closed at each end by rigid plates of negligible thickness. Calculate the stress developed in copper and steel when the temperature of the assembly is raised from 10oC to200oC. Take E for steel = 2 .1 x 105 N/mm2, E for copper = 1 x 105N/mm2, αs = 11 x 10-6/ oC , αc = 18 x 10-6/ oC

<u>Given</u>

Diameter of copper rod	dc =15 mm
Steel tube OD	do = 30 mm
Steel tube ID	di = 20 mm
T_1 and T_2 respectively	10 °C and 200 °C {$T = T_2 - T_1$}
Young's modules for steel	$E_s = 2.1 \times 10^5$ N/mm^2
Young's modules for copper	$E_c = 1 \times 10^5$ N/mm^2

$\alpha_s = 11 \times 10^{-6}$/ °C and $\alpha_c = 18 \times 10^{-6}$/ °C

<u>To find</u>

Thermal stress in copper [α_c]andsteel [α_s]

<u>Solution</u>

For temperature is the same for both the materials

Compressive load on copper = tensile load on steel

$\alpha_s x\ T + \frac{\sigma_s}{E_s} = \alpha_c x\ T + \frac{\sigma_c}{E_c}$

Area of Steel (hollow tube) = $\frac{\pi}{4}\{30^2 - 20^2\} = 125\,\pi$ mm^2

Area of copper = $\frac{\pi}{4} 15^2 = 56.25\,\pi$ mm^2

$$\sigma_c = \frac{\sigma_s \, x\, 56.25\,\pi}{125\,\pi} = 2.22\,\sigma_s$$

$$\sigma_c = 2.22\,\sigma_s$$

$\alpha_s x\ T + \frac{\sigma_s}{E_s} = \alpha_c x\ T - \frac{\sigma_c}{E_c}$

$11 \times 10^{-6} \times 190 + \frac{\sigma_s}{2.1 \times 10^5} = 18 \times 10^{-6} \times 190 - \frac{\sigma_c}{1 \times 10^5}$

$\frac{\sigma_s}{2.1 \times 10^5} + \frac{\sigma_c}{1 \times 10^5} = 18 \times 10^{-6} \times 190 - 11 \times 10^{-6} \times 190$

Substitute $\sigma_c = 2.22\,\sigma_s$

$\frac{\sigma_s}{2.1 \times 10^5} + \frac{2.22\,\sigma_s}{1 \times 10^5} = 5 \times 10^{-6} \times 190$

$5.662\,\sigma_s = 1.995$

$\sigma_s = 35.235$ N/mm^2 and $\sigma_s = 78.22$ N/mm^2

Elastic constants

When the structural stressed by axial load it's under goes the deformation and it's comes back to original shape or structural stressed by within the elastic limit then there is the changes in length along x-direction, y-direction and z - direction.

Types of elastic constant related to isotropic materials

1.Elasticity Modulus (E)0r Young's Modulus
2. Poisson's Ratio (μ)
3. Shear Modulus (G)
4. Bulk Modulus (K)

Elasticity Modulus or Young's Modulus(E)

$$E = \frac{Tensile\ Stress}{Tensile\ Strain}$$

$$E = \frac{\sigma}{\varepsilon}$$

$$E = \frac{F/A}{e/L}$$

$$\frac{FL}{eA} = E$$

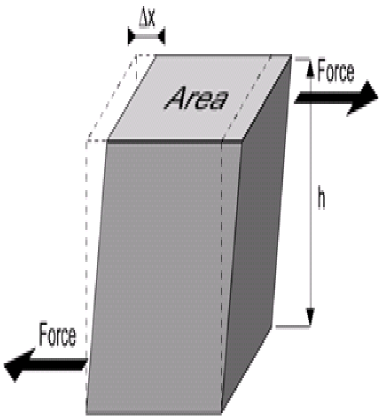


Fig. Before applied load and after applied load

2. Poisson's Ratio (μ)

(μ) (or) $\frac{1}{m}$ $= \frac{lateral\ strain}{longitudinal\ strain}$

Lateral strain (e_t) $= \partial d/d$ (or) $\partial t/t$

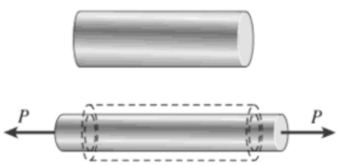


Fig. 1.12 Load applied on rod Fig. 1.13 linear change and lateral change

Longitudinal strain (e_l) $= \partial l/L$

Shear Modulus (G)

Shear modulus G $= \frac{Shear\ Stress}{Shear\ Strain}$

G $= \frac{\gamma}{\tau}$

Volumetric Strain e_v

$$e_v = \frac{Change\ in\ volume}{Original\ volume} = \frac{dV}{v}$$

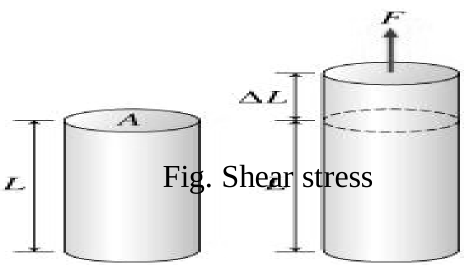


Fig. Shear stress

Fig. 1.14 Shear force applied situation

The volumetric strain is defined as materials tends to change in volume at three direction by external load within the elastic limit

$$e_v = \{\frac{\delta L}{L} - \frac{\delta b}{b} - \frac{\delta d}{d}\}$$

Volume of uniform rectangular section = L X b X d

Here b=d

$$e_v = \{\frac{\delta L}{L} - 2\frac{\delta d}{d}\}$$

$$\mu = \frac{lateral\ strain}{Longitudinal\ strain}$$

μ X $Longitudinal\ strain$ = lateral strain

$$e_v = \{\frac{\delta L}{L} - 2\mu\frac{\delta L}{L}\}$$

Rewriting the above equation

$$e_v = \frac{\delta L}{L}\{1 - 2\mu\}$$

Volumetric strain of rectangular structural subjected to three forces which are mutually perpendicular

$$e_x = \{\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}\}$$

$$e_x = \frac{\sigma_x}{E} - \frac{\mu}{E}\{\sigma_y + \sigma_z\}$$

$$e_x = \frac{1}{E}\{\sigma_x - \mu\{\sigma_y + \sigma_z\}$$

Similarly for e_y and e_y

$$e_y = \frac{1}{E}\{\sigma_y - \mu\{\sigma_x + \sigma_z\}$$

$$e_z = \frac{1}{E}\{\sigma_z - \mu\{\sigma_x + \sigma_y\}$$

$$\frac{dV}{v} = \{e_x + e_y + e_z\}$$

$$\{e_x + e_y + e_z\} = \frac{1}{E}\{\sigma_x + \sigma_y + \sigma_z\} - \frac{2\mu}{E}\{\sigma_x + \sigma_y + \sigma_z\}$$

$$\{e_x + e_y + e_z\} = \frac{1}{E}\{\sigma_x + \sigma_y + \sigma_z\}\{1 - 2\mu\}$$

Volumetric strain of cylindrical rod

$$e_v = \{\frac{\delta L}{L} - 2\frac{\delta d}{d}\}$$

Bulk modulus [K]

$$[K] = \frac{Direct\ stress}{volimetric\ strain}$$

$$[K] = \frac{\sigma}{\frac{dV}{v}}$$

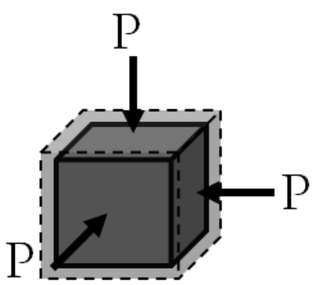


Fig. 1.15 Change in volume

Relation between young’s modulus and bulk modulus

Volume = L x L x L

$V = L^3$ $\qquad\qquad$ $dV = 3L^2 \times dL$

$$\frac{dL}{L} = \{\frac{\sigma}{E} - \mu\frac{\sigma}{E} - \mu\frac{\sigma}{E}\}$$

$$= \frac{\sigma}{E}\{1 - 2\mu\}$$

$$dL = \frac{\sigma}{E}\{1 - 2\mu\} \times L$$

$$dV = 3L^2 \times dL$$

$$dV = 3L^2 \times \frac{\sigma}{E}\{1 - 2\mu\} \times L$$

$$\frac{dV}{v} = \frac{3\,dL}{L}$$

$$dV = \frac{3}{L}\frac{\sigma}{E} \times \{1 - 2\mu\} \times L$$

$$[K] = \frac{\sigma}{\frac{dV}{v}}$$

$$[K] = \frac{\sigma}{\left[\frac{3\sigma}{E}\right][1 - 2\mu]}$$

$$[K] = \frac{E}{3[1 - 2\mu]}$$

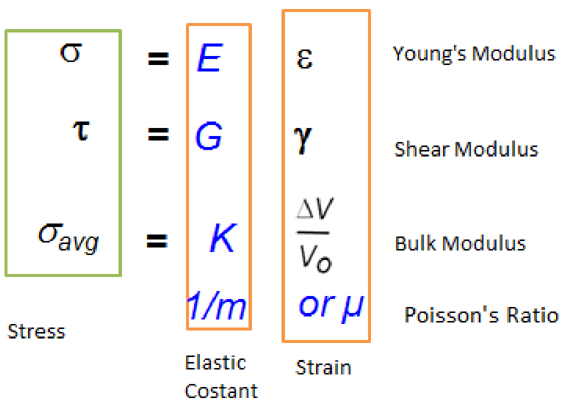
$3K[1 - 2\mu] = E$ $\qquad$ From this equation

$$\mu = \frac{3K - E}{6K}$$

Relationship between modulus of elasticity and modulus of rigidity {E and G}

$$G = \frac{E}{2\{1 + \mu\}}$$

Easy to identify with the four elastic constant are calculated by single module as shown in fig.



Relation between modulus of elasticity (E) and bulk modulus (K):

$$E = 3K(1 - 2\mu)$$

Relations between modulus of elasticity (E) and modulus of rigidity (G):

$$E = 2G(1 + \mu)$$

Relation among three elastic constants:

$$E = \frac{9KG}{G + 3K}$$

Problem:

Determine the changes in length, breadth and thickness of a steel bar which 5cm long, 40mm wide and 30mm thick and is subjected to an axial pull of 35KN in the direction in length take the young 'modulus and position's ratio 200Gpa and 0.32 respectively .

Given:

L = 5cm=50mm

b=40mm

d=30mm

E=200Gpa= 2 x 10^5N/mm^2

μ = 0.32

To find: $\delta L, \delta b \text{ and } \delta d$

Solultion:

$\mu = \frac{\delta L}{L} / \frac{\delta b}{b}$

$\mu = \frac{\delta L}{L}$

$E = \frac{\sigma}{e}$

$\sigma = \frac{P}{A}\sigma \quad = \frac{35 \times 10^3}{40 \times 30} = \frac{350}{12} = 29.16 \ N / mm^2$

(i) Change in length (δL)

$= \frac{PL}{AE} = \frac{35 \times 10^3 \times 50}{40 \times 30 \times 2 \times 10^5} = \frac{35 \times 5}{40 \times 30 \times 2 \times 10} = \frac{175}{24000}$ =7.29x10^3 mm

ii) Change in breadth (δb)

$\mu = \frac{lateral\ strain}{Longitudinal\ strain}$

μ X $Longitudinal\ strain$ = lateral strain

μ X $\frac{\delta L}{L} = \frac{\delta b}{b}$

$\delta b = \mu$ X $\frac{\delta L}{L}$ X b = 0.32 x 40 x $\frac{72.29 \, X 10^{-3}}{50}$= 1.866 x 10 $^{-3}$ mm

ii) Change in diameter (δd)

μ X $\frac{\delta L}{L} = \frac{\delta d}{d}$

$\delta d = \mu$ X $\frac{\delta L}{L}$ X d=0.32 x $\frac{72.29 \, X 10^{-3}}{50}$ x 40 =1.39 X10 $^{-3}$ mm

Problem:

Calculate the modulus of rigidity and bulk modulus of cylindrical bar of diameter of 25mm and of length 1.6m. if the longitudinal strain in a bar during a tensile test is four times the lateral strain find the change in volume when the bar subjected to hydrostatic pressure of 100 N / mm^2 the young's modulus of cylindrical bar E is 100 GPa

Given:

D=25 mm

L=1.6m=1600 mm

Longitudinal strain = 4 X lateral strain

E=100Gpa=1 x 10^5N/mm^2

To find:

(i) Modulus of Rigidity (ii) Bulk modulus (iii) Change in volume

(i) Modulus of Rigidity[G]

E= 2G (1+μ) ---------- Relationship between E, G & μ

Longitudinal strain = 4 X lateral strain

$\frac{1}{4} = \frac{lateral \, strain}{Longitudinal \, strain}$

E= 2G (1+μ) =2G (1+$\frac{1}{4}$) =2G (1+0.25)

E=2G (1+0.25)

G = = $\frac{E}{2(1.25)} = \frac{1 \, X 10^5}{2(1.25)}$ = 4 x 10 ^{4} N/mm^2

(ii) Bulk modulus [K]

E = 3K [1-2 μ]

= K x 3[1-0.5]

1 x 10^5 = 1.5 x K

$\frac{1 \, X 10^5}{1.5}$ = K

K =0.666 X 10^5 N/mm^2

(iii)Change in volume [dV]

$[K] = \frac{Direct \, stress}{volimetric \, strain}$

$[K] = \frac{\sigma}{\frac{dV}{v}}$

$\frac{dV}{v} = \frac{\sigma}{K} = \frac{100}{0.666 \, X 10^5}$=1.5 X 10 $^{-3}$

$$V = \frac{\pi}{4} X\ d^2\ X\ L$$

$$V = \frac{\pi}{4} X\ 25^2\ X\ 1600 = 785000$$

$$\frac{dV}{v} = 1177.5\ mm^3$$

Strain energy

When material is deformed by external loading, energy is stored *internally* throughout its volume the stored energy is called strain energy.

Strain energy = work done

Resilience: the total strain energy stored in a volume or capacity of work after removing straining force is called Resilience

Proof Resilience:
The maximum strain energy stored in the volume or quantity of strain energy stored in volume in a body when strained up to elastic limit its called Proof Resilience.

$$\text{Proof Resilience} = \frac{\sigma^2}{2E}\ X\ Volume$$

Modulus of Resilience

$$= \frac{\text{Proof Resilience}}{Volume\ of\ the\ body}$$

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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

UNIT – II– ANALYSIS OF STRESSES IN TWO DIMENSIONS – SMEA1305

MECHANICS OF SOLIDS (SMEA1305)

UNIT 2: ANALYSIS OF STRESSES IN TWO DIMENSIONS

Principal planes and stresses – Mohr's circle for biaxial stresses – Maximum shear stress - simple problems- Stresses on inclined plane

Biaxial state of stresses – Thin cylindrical and spherical shells – Deformation in thin cylindrical and spherical shells – Efficiency of joint- Effect of Internal Pressure

Introduction: Principal planes and stresses

The planes, which have no shear stress, are known as principal planes. Hence principal planes are the planes of zero shear stress. These planes carry only normal stresses. The normal stresses, acting on a principal plane, are known as principal stresses.

Methods for determining principal planes and stresses

- Analytical method
- Graphical method

Analytical method on oblique section

The following are the two cases considered

1. A member subjected to a direct stress in one plane
2. A member subjected to like direct stresses in two mutually perpendicular directions.

Direct stress in one plane

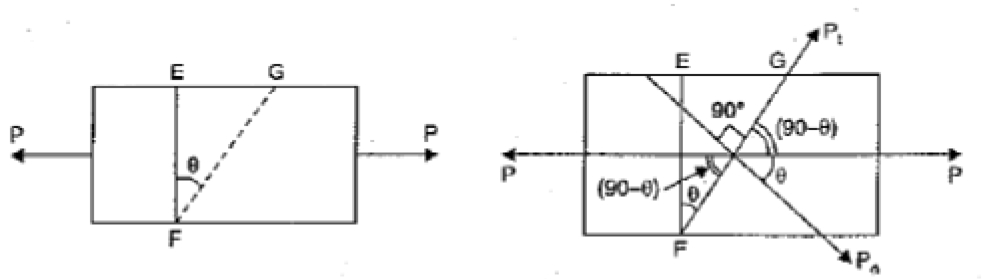


Fig. 2.1 Direct Stress in one plane

Normal stress, $\sigma_n = \sigma \cos^2 \theta$

Tangential stress, $\sigma_t = \dfrac{\sigma}{2} \sin 2\theta$

σ_n will be maximum, when $\cos^2 \theta$ (or) $\cos\theta$ is maximum.

Cos θ will be maximum when θ = 0° as cos 0° = 1

Therefore, max. normal stress = $\sigma \cos^2\theta = \sigma$

σ_t will be max, when sin 2θ is maximum.
Sin2θ be max. when sin2θ = 1 or 2θ = 90° (or) 270°
θ = 45° (or) 135°

Max. value of shear stress = $\frac{\sigma}{2}\sin 2\theta$

$$= \frac{\sigma}{2}$$

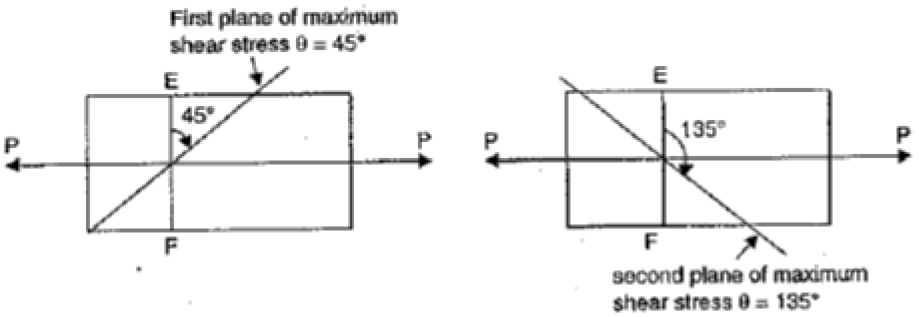


Fig. 2.2 Position of planes

Member subjected to direct stresses in two mutually perpendicular directions

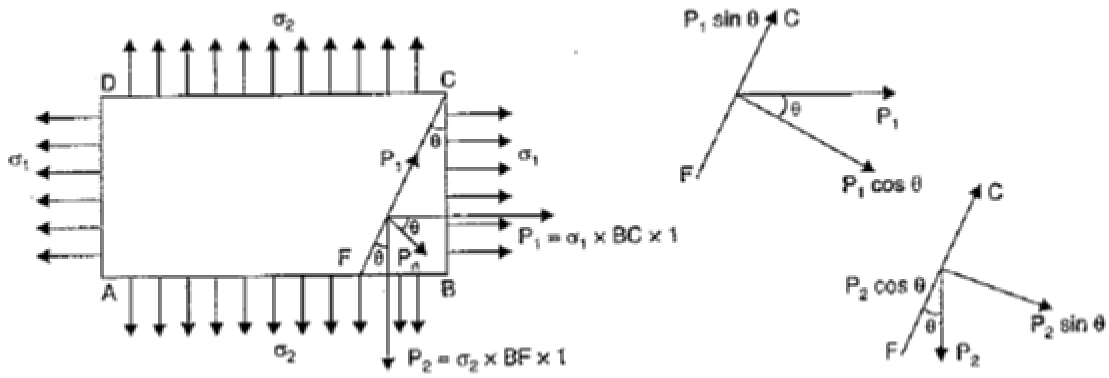


Fig. 2.3 Member subjected to Direct stress in two perpendicular directions

Normal Stress, $\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta$

Tangential Stress, $\sigma_t = \frac{(\sigma_1 - \sigma_2)}{2}\sin 2\theta$

Resultant Stress, $\sigma_R = \sqrt{\sigma_{n^2} + \sigma_{t^2}}$

Obliquity, $\tan\phi = \frac{\sigma_t}{\sigma_n}$

Problem

A small block is 4 cm long, 3 cm high and 0.5 cm thick. It is subjected to uniformly distributed tensile forces of resultants 1200 N and 500 N as shown in Fig. below. Compute the normal and shear stresses developed along the diagonal AB.

Given

Length = 4 cm, Height = 3 cm and Width = 0.5 cm

Force along x-axis = 1200 N Force along y-axis = 500 N

Area of cross-section normal to x-axis = 3 x 0.5 = 1.5 cm^2

Area of cross-section normal to y-axis = 4 x 0.5 = 2 cm^2

Stress along x - axis, $\sigma_1 = \frac{F_x}{A_x} = 800 \text{N/cm}^2$

Stress along y - axis, $\sigma_2 = \frac{F_y}{A_y} = 250 \text{N/cm}^2$

$\tan\theta = \frac{4}{3} = 1.33$

$\theta = \tan^{-1}(1.33) = 53.06°$

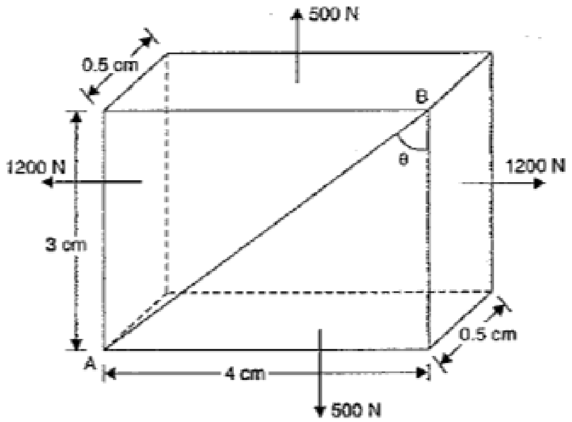


Fig. 2.4 Cube at loading condition

$$\text{Normal Stress, } \sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta$$

$$= \frac{800 + 250}{2} + \frac{800 - 250}{2}\cos(2 \times 53.06)$$

$$= 448.65 \text{N/cm}^2$$

$$\text{Tangential Stress, } \sigma_t = \frac{(\sigma_1 - \sigma_2)}{2}\sin 2\theta$$

$$= \frac{800 - 250}{2}\sin(2 \times 53.06)$$

$$= 264.18 \text{N/cm}^2$$

Members subjected to direct stresses in two mutually perpendicular directions accompanied by simple shear stress

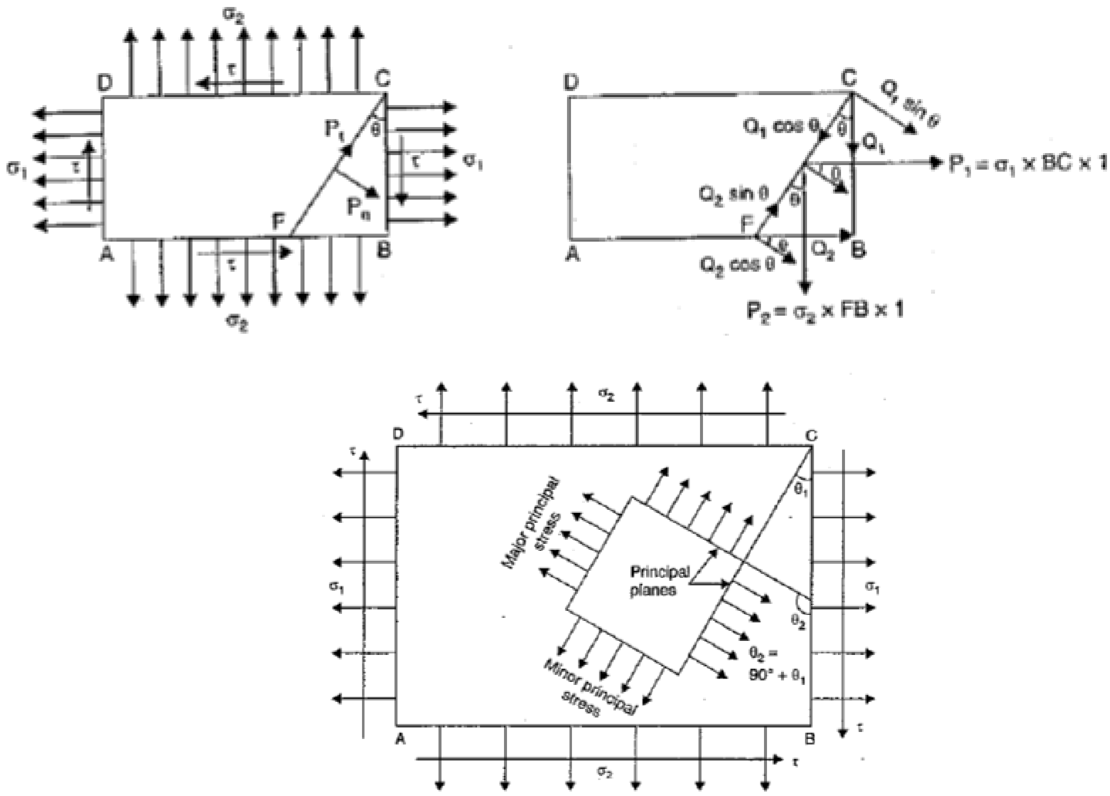


Fig. 2.5 Principal Planes Identification diagram

Normal Stress, $\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$

Tangential Stress, $\sigma_t = \frac{(\sigma_1 - \sigma_2)}{2}\sin 2\theta - \tau \cos 2\theta$

$$\tan 2\theta = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

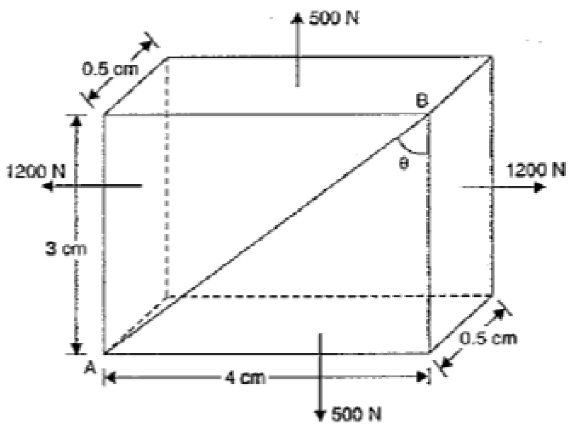


Fig. 2.6 Loaded Cube

$$\text{Major principal stress} = \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$\text{Minor principal stress} = \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$\text{Max. Shear stress} = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

Problem

A rectangular block of material is subjected to a tensile stress of 110 N/mm2 on one plane and a tensile stress of 47 N/mm2 on the plane at right angles to the former. Each of the above stresses is accompanied by a shear stress of 63 N/mm2 and that associated with the former tensile stress tends to rotate the block anticlockwise. Find:

(i) The direction and magnitude of each of the principal stress and

(ii) Magnitude of the greatest shear stress

Given

$\sigma_1 = 110\ \text{N/mm}^2$

$\sigma_2 = 47\ \text{N/mm}^2$

$\tau = 63\ \text{N/mm}^2$

$\theta = 45°$

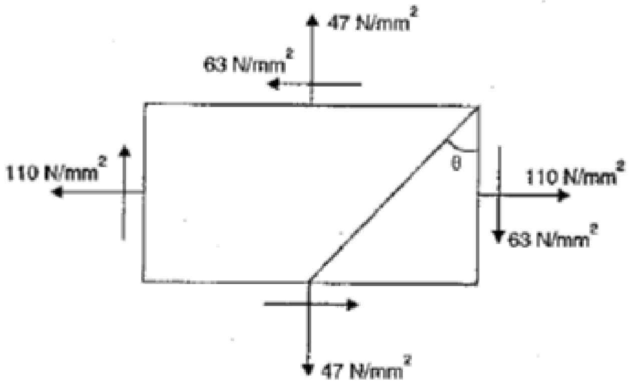


Fig. 2.7 Stress acting on Rectangular block

$$\text{Major principal stress} = \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{110 + 47}{2} + \sqrt{\left(\frac{110 - 47}{2}\right)^2 + 63^2}$$

$$= \frac{157}{2} + \sqrt{\left(\frac{63}{2}\right)^2 + 63^2}$$

$$= 78.5 + \sqrt{31.5^2 + 63^2}$$

$$= 78.5 + 70.436$$

$$= 148.936\ \text{N/mm}^2$$

$$\text{Minor principal stress} = \frac{\sigma_1+\sigma_2}{2} - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{110+47}{2} - \sqrt{\left(\frac{110-47}{2}\right)^2 + 63^2}$$

$$= 78.5 - 70.436$$

$$= 8.064 \text{ N/mm}^2$$

$$\tan 2\theta = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

$$2\theta = \tan^{-1}(2)$$

$$\theta = 31°43'$$

Magnitude of the greatest shear stress

$$(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(100-47)^2 + 4\times 63^2}$$

$$(\sigma_t)_{max} = 70.436 \text{N/mm}^2$$

Mohr's circle method

It is a graphical method of finding normal, tangential and resultant stresses or an oblique plane. It is drawn for following cases

1. A body subjected to two mutually perpendicular principal stresses of unequal intensities
2. A body subjected to two mutually perpendicular stresses which are unequal and unlike (one is tension and other is compression)
3. A body subjected to two mutually perpendicular tensile stresses accompanied by a simple shear stress.

Case 1: A body subjected to two mutually perpendicular principal stresses of unequal intensities

Let σ_1= Major tensile stress

σ_2 = Minor tensile stress

θ = Angle made by the oblique plane with the axis of minor tensile stress

Mohr's Circle procedure

Take any point A and draw a horizontal line through A. Take AB = σ_1 and AC = σ_2 towards right from A to some suitable scale. With BC as diameter draw a circle. Let O is the centre of circle. Now through O, draw a line OE marking an angle 2θ with OB. From E, draw ED perpendicular on AB. Join AE. Then the normal and tangential stresses on the oblique plane are given by AD and ED respectively. The resultant stress on the oblique plane is given by AE.

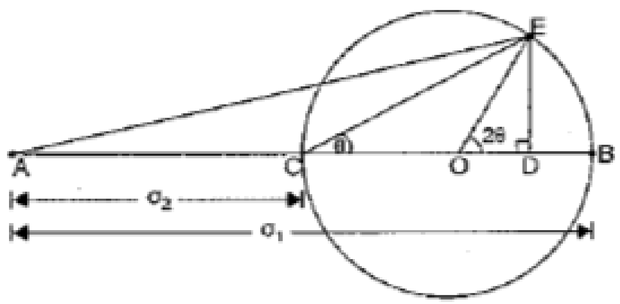


Fig. 2.8 Mohr's Circle

From Figure, we have

Length AD = Normal stress on oblique plane; Length ED = Tangential stress on oblique plane; Length AE = Resultant stress on oblique plane; Angle φ = obliquity

Case 2: Mohr's circle when a body is subjected to two mutually perpendicular principal stresses which are unequal and unlike (one is tensile and other is compressive)

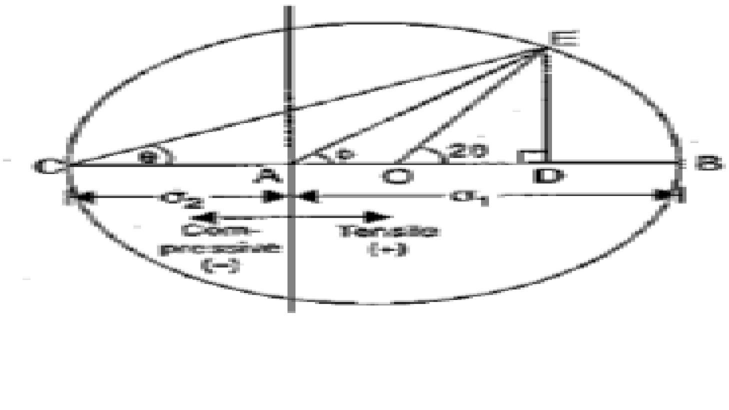


Fig. 2.9 Mohr Circle Position

Take any point A and draw a horizontal line through A on both sides of A as shown in Fig. Take AB = $\sigma_{1(+)}$ towards right of A and AC = σ_2(-) towards left of A to some suitable scale. Bisect BC at O. With O as centre and radius equal to CO or OB, draw a circle. Through O draw a line OE making an angle 2θ with OB. From E, draw ED perpendicular to AB. Join AE and CE. Then normal and shear stress on the oblique plane are given by AD and ED. Length AE represents the resultant stress on the oblique plane.

Case 3: Mohr's circle when a body subjected to two mutually perpendicular tensile stresses accompanied by a simple shear stress.

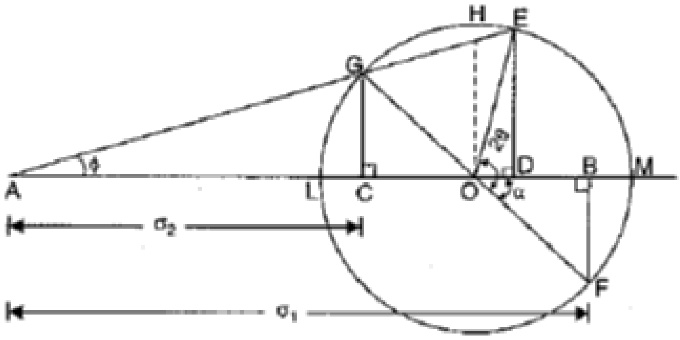


Fig. 2.10 Mohr Circle

Take any point A and draw a horizontal line through A. Take AB = $\sigma_1$and AC = σ_2 towards right of A to some suitable scale. Draw perpendiculars at B and C and cut off BF and CG equal to shear stress to the same scale. Bisect BC at O. Now with O as centre and radius equal to OG or OF draw a circle. Through O, draw a line OE making an angle of 2θ with OF as shown in Fig. From E, draw ED perpendicular to CB. Join AE. Then length AE represents the resultant stress on the oblique plane. And lengths AD and ED represents the normal stress and tangential stress respectively.

Problems

1. A point in a strained material is subjected to stresses shown in Fig. Using Mohr's circle method, determine the normal and tangential stress across the oblique plane.

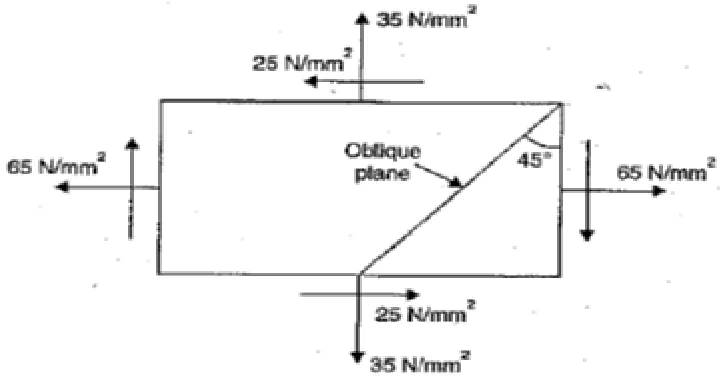


Fig. 2.11 Rectangular bar Stress state

Given:

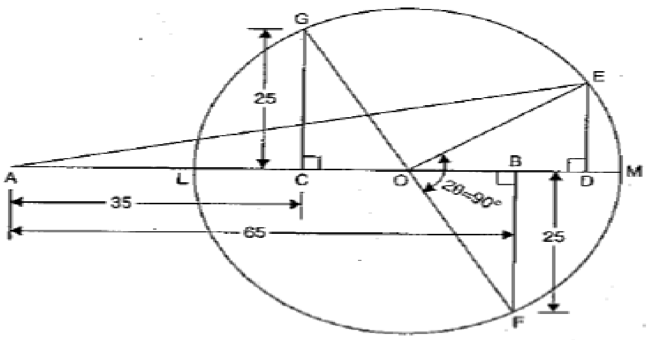


Fig. 2.12 Mohr Circle Diagram

$\sigma_1 = 65$ N/mm^2

$\sigma_2 = 35$ N/mm^2

$\tau = 25$ N/mm^2

$\theta = 45°$

Let 1 cm = 10 N/mm^2

$$\sigma_1 = \frac{65}{10} = 6.5\text{cm}$$

$$\sigma_2 = \frac{35}{10} = 3.5\text{cm}$$

$$\tau = \frac{25}{10} = 2.5\text{cm}$$

By measurements, Length AD = 7.5 cm and
Length ED = 1.5 cm

Normal stress (σ_n) = Length AD x Scale = 7.5 x 10 = 75 N/mm^2

Tangential stress (σ_t) = Length ED x Scale = 1.5 x 10 = 15 N/mm^2

2. An elemental cube is subjected to tensile stresses of 30 N/mm2 and 10 N/mm2 acting on two mutually perpendicular planes and a shear stress of 10 N/mm2 on these planes. Draw the Mohr's circle of stresses and hence or otherwise determine the magnitudes and directions of principal stresses and also the greatest shear stress.

Given:

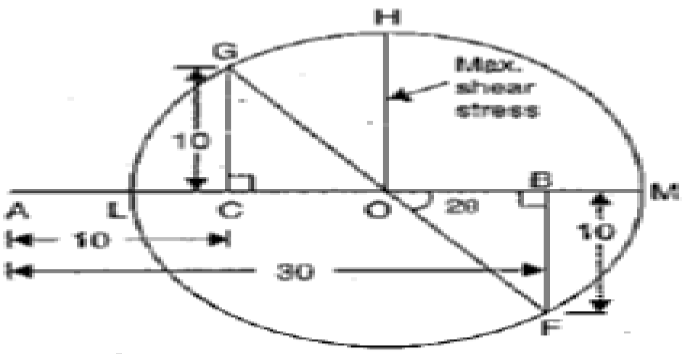


Fig. 2.13 Mohr Circle diagram

$\sigma_1 = 30$ N/mm^2

$\sigma_2 = 10$ N/mm^2

$\tau = 10$ N/mm^2

Let 1 cm = 2 N/mm^2

$$\sigma_1 = \frac{30}{2} = 15\text{cm}$$

$$\sigma_2 = \frac{10}{2} = 5\text{cm}$$

$$\tau = \frac{10}{2} = 5\text{cm}$$

By measurements,

Length AM = 17.1 cm; Length AL = 2.93 cm; Length OH = Radius of Mohr's circle = 7.05 cm;

$$\angle FOB(or)2\theta = 45°$$

Major Principal stress = Length AM x Scale = 17.1 x 2 = 34.2 N/mm^2
Minor principal stress = Length AL x Scale = 2.93 x 2 = 5.86 N/mm^2

$$\theta = \frac{45}{2} = 22.5°$$

The second principal plane is given by θ+90°

= 22.5 + 90

= 112.5°

Greatest shear stress = Length OH x Scale

= 7.05 x 20
= 14.1 N/mm^2

BIAXIAL STRESS SYSTEMS

A biaxial stress system has a stress state in two directions and a shear stress typically showing in Fig..

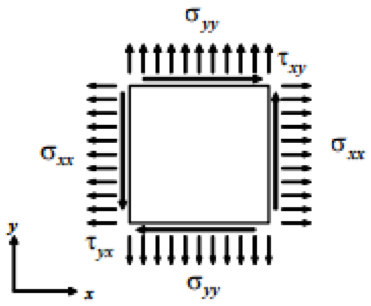


Fig. 2.14 Element of a structure showing a biaxial stress system

When a Biaxial Stress state occurs in a thin metal, all the stresses are in the plane of the material. Such a stress system is called PLANE STRESS. We can see plane stress in pressure vessels, aircraft skins, car bodies, and many other structures.

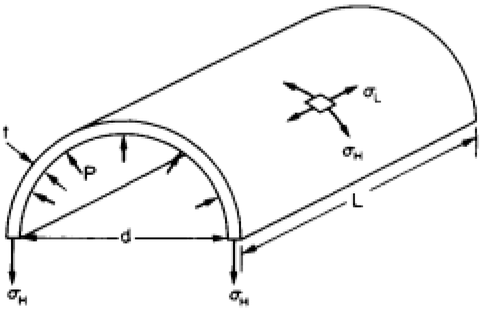
THIN CYLINDERS AND SPHERICAL SHELLS

The stresses set up in the walls of a thin cylinder owing to an internal pressure p are:
circumferential or hoop stress = pd/2t and
longitudinal or axial stress = pd/4t

DEFORMATION IN THIN CYLINDRICAL AND SPHERICAL SHELLS

Hoop or circumferential stress

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of half of the cylinder as shown in Fig.



.. Half of a thin cylinder subjected to internal pressure showing the hoop and longitudinal stresses acting on any element in the cylinder surface.

Fig. 2.15 Hoop stress failure

Total force on half-cylinder owing to internal pressure = $p \times$ projected area = $p \times dL$

Total resisting force owing to hoop stress σ_H set up in the cylinder walls

$$= 2\sigma_H \times Lt$$

$$\therefore \quad 2\sigma_H Lt = pdL$$

$$\therefore \quad \textbf{circumferential or hoop stress } \sigma_H = \frac{pd}{2t}$$

Longitudinal stress or axial stress

Consider now the cylinder shown in the fig..

Total force on the end of the cylinder owing to internal pressure

$$= \text{pressure} \times \text{area} = p \times \frac{\pi d^2}{4}$$

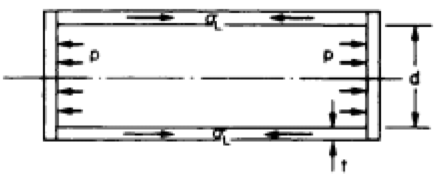


Fig. 2.16 Longitudinal failure.

Area of metal resisting this force = πdt (approximately)

$\therefore$ $$\text{stress set up} = \frac{\text{force}}{\text{area}} = p \times \frac{\pi d^2/4}{\pi dt} = \frac{pd}{4t}$$

i.e. $$\textbf{longitudinal stress } \sigma_L = \frac{pd}{4t}$$

Problem 1

A thin cylindrical pipe of diameter 1.5 mm and thickness 1.5 cm is subjected to an internal fluid pressure of 1.2 N/mm^2. Determine:

i)Longitudinal stress developed in the pipe and

ii)Circumferential stress developed in the pipe.

Solution:

Given:

Dia of pipe d=1.5 m

Thickness, t=1.5 cm = 1.5x10^{-2}m

Internal fluid pressure, p=1.2 N/mm^2

i) The longitudinal stress is given by

σ =pd/2t

= (1.2x1.5)/(4x1.5x10^{-2})

=30 N/mm^2

ii) The circumferential stress is given by

σ =pd/4t

=(1.2x1.5) / (2x1.5x10^{-2})

=60 N/mm^2

Problem 2

A cylinder of internal diameter 2.5 m and of thickness 5cm contains a gas.If the tensile stress in the material is not to exceed 80 N/mm^2, determine the internal pressure of the gas.

Solution:

Given:

Internal dia of cylinder d=2.5 cm

Thickness of cylinder t= 5cm=5x10^{-2}m

Maximum permissible stress =80 N/mm^2

As maximum permissible stress is given, hence this should be equal to circumferential stress σ

σ =80 N/mm^2

σ =pd/2t

P=(2t x σ)/d

$=(2x5x10^{-2}x80) / 2.5$

$=3.2\ N/mm^2$

Efficiency of a joint

The cylindrical shells are having two types of joints namely longitudinal joint and circumferential joint.

Let η_l = efficiency of a longitudinal joint and

η_c = efficiency of a circumferential joint……

the circumferential stress(σ_1) is given by,

$$\sigma_1 = (p \times d) / (2t \times \eta_l) \quad \text{and}$$

longitudinal stress(σ_2) is given by.,

$$\sigma_2 = (p \times d) / (4t \times \eta_c)$$

In longitudinal joint, the circumferential stress is developed whereas in circumferential joint the longitudinal stress is developed.

Problem 3:

A boiler is subjected to an internal steam pressure of 2 N/mm^2, the thickness of a boiler plate is 2cm and permissible tensile stress is 120 N/mm^2 , find out the maximum diameter when efficiency of longitudinal joint is 90% and that of circumferential joint is 40%.

Solution:

Given

Internal steam pressure, p = 2 N/mm^2

Thickness of boiler plate, t =2cm

Permissible tensile stress = 120 N/mm^2

In case of a joint, the permissible stress may be circumferential stress or longitudinal stress.

efficiency of longitudinal joint = η_l = 90% = 0.90

efficiency of circumferential joint = η_c = 40% = 0.40

max. diameter for circumferential stress is given by,

$\sigma_1 = (p \times d) /(2t \times \eta_l)$

where σ_1 = given Permissible tensile stress = 120 N/mm^2

$$120 = (2 \times d) / (2x\ 0.90\ x2)$$

$$d= (120x2x0.9x2) / 2$$
$$= 216\ cm.$$

Max.diameter for longitudinal stress is given by,

$\sigma_2 = (p \times d) / (4t \times \eta_c)$

where σ_2 = given Permissible tensile stress = 120 N/mm^2

$$120 = (2 \times d) / (4 \times 0.40 \times 2)$$

$$d = (120 \times 4 \times 0.4 \times 2) / 2$$

$$d = 192 \text{ cm.}$$

the longitudinal or circumferential stresses included in the material are directly proportional to the diameter (d), and hence stress induced will be less if the value of d is less. Hence minimum value of d is taken…..so, max.diameter = 192 cm

Effect of internal pressure on the dimensions of a thin cylindrical shell

When a fluid having internal pressure (p) is stored in a thin cylindrical shell, due to internal pressure of the fluid the stresses set up at any point of the material of the shell are :

(i) Hoop or circumferential stress (σ_1), acting on longitudinal section.

(ii) Longitudinal stress (σ_2) acting on the circumferential section.

These stresses are principal stresses, as they are acting on principal planes. The stress in the third principal plane is zero as the thickness (t) of the cylinder is very small. Actually the stress in the third principal plane is radial stress which is very small for thin cylinders and can be neglected.

Let p = Internal pressure of fluid

L = Length of cylindrical shell

d = Diameter of the cylindrical shell

t = Thickness of the cylindrical shell

E = Modulus of Elasticity for the material of the shell

σ_1 = Hoop stress in the material

σ_2 = Longitudinal stress in the material

μ = Poisson's ratio

δd = Change in diameter due to stresses set up in the material

δL = Change in length

δV = Change in volume.

Then, circumferential strain,

$$e_1 = (\sigma_1 / E) - (\mu \sigma_2 / E)$$

$$= \frac{pd}{2tE}(1 - \mu/2)$$

and longitudinal strain,

$$e_2 = (\sigma_2 / E) - (\mu \sigma_1 / E)$$

$$= \frac{pd}{2tE}(1/2 - \mu)$$

Change in diameter, $\delta d/d = \frac{pd}{2tE}(1 - \mu/2)$

Change in length, $\delta L/L = \frac{pd}{2tE}(1/2 - \mu)$

Change in volume, $\delta V/V = (2e_1 + e_2)$

$= V(2\,\delta d/d + \delta L/L)$

Problem 4:
Calculate change in diameter, change in length and change in volume of a thin cylindrical shell 100cm diameter, 1cm thickness and 5m long when subjected to internal pressure of $3 N/mm^2$, take the value of $E = 2 \times 10^5 N/mm^2$ and poisson's ratio $\mu = 0.3$

Solution:
Given: diameter of shell, d=100cm
Thickness of shell, t= 1cm
Length of shell, L= 5m= 500cm
Internal pressure, $p = 3 N/mm^2$
Young's modulus, $E = 2 \times 10^5 N/mm^2$
And Poisson's ratio $\mu = 0.3$

(*i*) **Change in diameter (δd) is given by equation**

$$\delta d = \frac{pd^2}{2tE}\left(1 - \frac{1}{2} \times \mu\right)$$

$$= \frac{2.5 \times 80^2}{2 \times 1 \times 2 \times 10^5}\left[1 - \frac{1}{2} \times 0.25\right]$$

$= 0.04\,[1 - 0.125] = \mathbf{0.035\ cm.}$

(*ii*) **Change in length (δL) is given by equation**

$$\delta L = \frac{pdL}{2tE}\left[\frac{1}{2} - \mu\right]$$

$$= \frac{2.5 \times 80 \times 300}{2 \times 1 \times 2 \times 10^5}\left[\frac{1}{2} - 0.25\right] = \mathbf{0.0375\ cm.}$$

(iii) change in volume $\delta V/V$ is given by,

$$\frac{\delta V}{V} = 2\frac{\delta d}{d} + \frac{\delta L}{L}$$

$$= 2 \times \frac{0.035}{80} + \frac{0.0375}{300} \quad \left(\because \begin{array}{ll} \delta d = 0.035, & \delta L = 0.0375 \\ d = 80, & L = 300 \end{array}\right)$$

$$= 0.000875 + 0.000125 = 0.001$$

$$\therefore \quad \delta V = 0.001 \times V$$

where volume $V = \frac{\pi}{4} d^2 \times L = \frac{\pi}{4} \times 80^2 \times 300 = 1507964.473 \text{ cm}^3$

$\therefore$ Change in volume, $\delta V = 0.001 \times 1507964.473 = \mathbf{1507.96 \text{ cm}^3}$. Ans.

Thin spherical shells

The figure shows a thin spherical shell of internal diameter d and thickness t and subjected to internal fluid pressure p , the fluid inside the shell has a tendency to split the shell into two hemispheres along x-x axis.

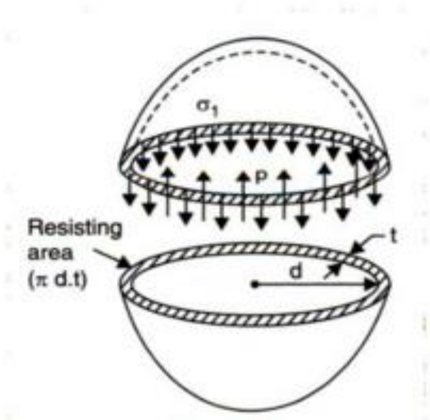


Fig. 2.17 Spherical shell

Circumferential or hoop stress(σ_1) is given by,

$$\sigma_1 = pd/4t$$

circumferential stress when the joint efficiency is given by,

$$\sigma_1 = pd/4t.\ \eta$$

Problem 5

A vessel in the shape of a spherical shell of 1.20m internal diameter and 12mm shell thickness is subjected to pressure of 1.6 N/mm^2, determine the stress induced in the material of the vessel.

Solution

Given.

Internal diameter , d = 1.2m = 1200mm

Shell thickness, t = 12mm and

Fluid pressure, p = 1.6 N/mm^2

The stress induced in the material of the spherical shell is given by,

$$\sigma_1 = pd/4t$$

$$= (1.6 \times 1200) / (4 \times 12)$$

$$= 40 \text{ N/mm}^2$$

Problem 6

A spherical vessel 1.5m diameter is subjected to an internal fluid pressure of 2 N/mm^2, find the thickness of the plate required if maximum stress is not to exceed 150 N/mm^2 and joint efficiency is 75%

Solution

Given

Diameter of shell, d = 1.5m = 1500mm,

Fluid pressure, p = 2 N/mm^2

Stress in the material, $\sigma_1 = 150$ N/mm^2

Joint efficiency, $\eta = 75\% = 0.75$

Let t = thickness of the plate and

Stress induced is given by,

$$\sigma_1 = pd/4t.\ \eta$$

$$t = (p \times d) / (4 \times \eta \times \sigma_1)$$

$$= (2 \times 1500) / (4 \times 0.75 \times 150)$$

$$= 6.67\text{mm}$$

Change in dimension of a thin spherical shell due to an internal pressure

Strain in any direction is also noted as δd/d which is given by the equation

$$\delta d/d = \frac{pd}{4tE}(1-\mu)$$

and volumetric strain δV/V is given by,

$$\delta V/V = 3 \times (\delta d/d)$$

$$= \frac{3pd}{4tE}(1-\mu)$$

Problem 7

A spherical shell of internal diameter 0.9m and of thickness 10mm is subjected to an internal pressure of 1.4 N/mm^2, determine the increase in diameter and increase in volume, take E = 2 x 10^5 N/mm^2 and μ = 0.33

Solution.

Given.

Internal diameter, d = 0.9m=900mm

Thickness of the shell, t=10mm

Fluid pressure, p = 1.4 N/mm^2

And E = 2 x 10^5 N/mm^2

μ = 0.33

using the relation

$$\delta d/d = \frac{pd}{4tE}(1-\mu)$$

$$= \frac{1.4 \times 0.9 \times 1000}{4 \times 10 \times 2 \times 10000}(1-0.33)$$

$$= 105 \times 10^{-6}$$

increase in diameter, $\delta d = 105 \times 10^{-6} \times 900$

$= 94.5 \times 10^{-3}$mm

= 0.0945mm.

Now,

Volumetric strain = $\delta V/V = 3 \times (\delta d/d)$

$= 3 \times 105 \times 10^{-6}$

$\delta V/V = 315 \times 10^{-6}$

increase in volume , $\delta V = 315 \times 10^{-6} \times V$

$= 315 \times 10^{-6} \times (\pi/6\ d^3)$

$= 315 \times 10^{-6} \times (\pi/6 \times 900^3)$

$= 12028.5\ mm^3$

Normal and shear stresses on inclined sections

To obtain a complete picture of the stresses in a bar, we must consider the stresses acting on an "inclined" (as opposed to a "normal") section through the bar.

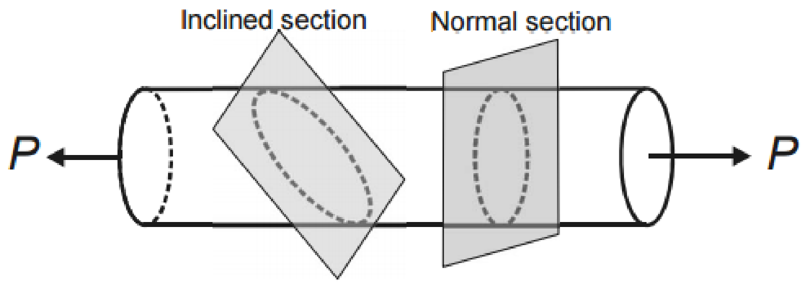


Fig. 2.18 Normal and Shear stresses on inclined planes

Because the stresses are the same throughout the entire bar, the stresses on the sections are uniformly distributed.

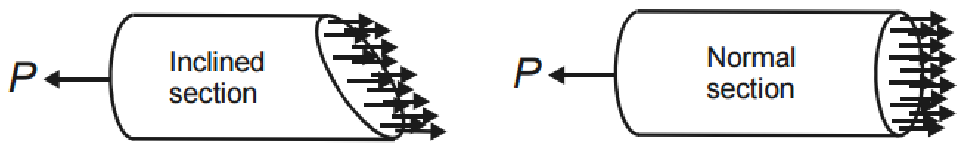


Fig. 2.19 Normal and Shear stresses pattern

2D view of the normal section

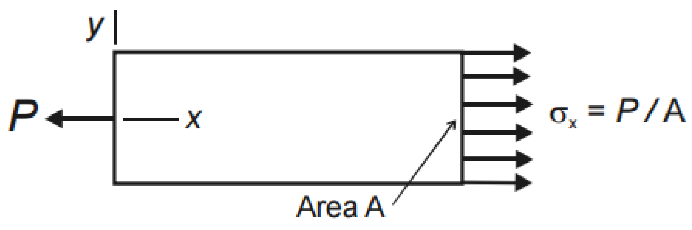


Fig. 2.20 '2D' view of Normal section

2D view of the inclined section

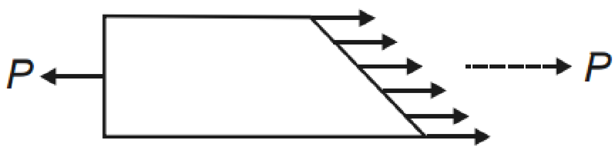


Fig. 2.21 '2D' view of inclined section

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SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

UNIT – III – BEAMS - LOADS AND STRESSES – SMEA1305

MECHANICS OF SOLIDS (SMEA1305)

UNIT 3: BEAMS - LOADS AND STRESSES

Types of beams - Supports and Loads – Shear force and Bending Moment in beams – Cantilever, Simply supported and Overhanging beams – SFD and BMD for inclined loads and couples

Stresses in beams – Theory of simple bending – Stress variation along the length and in the beam section – Effect of shape of beam section on stress induced.

Introduction: Types of beams

There are 5 most important beams. They are

- Simple supported beam
- Cantilever beam
- Overhanging beam
- Fixed beam
- Continuous beam

Simple supported beam: A beam supported or resting freely on the supports at its both ends, is known as simply supported beam.

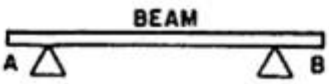


Fig. 3.1 Simply Supported Beam

Cantilever beam: A beam which is fixed at one end and free at the other end is known as cantiver beam.

Fig. 3.2 Cantilever Beam

Over hanging beam: If the end portion of a beam is extended beyond the support such beam is known as Overhanging beam

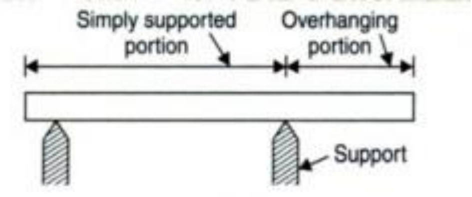


Fig. 3.3 Over hanging Beam

Fixed beam: A beam whose both ends are fixed or built in walls is known as fixed beam.

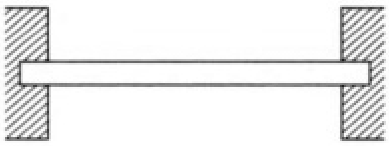


Fig. 3.4 Fixed Beam

Continuous beam: A beam which is provided more than two supports is known as continuous beam.

Fig. 3.5 Continuous Beam

Types of supports

There are 5 most important supports. They are

- Simple supports or knife edged supports
- Roller support
- Pin-joint or hinged support
- Smooth surface support
- Fixed or built-in support

Simple supports or knife edged support: in this case support will be normal to the surface of the beam. If AB is a beam with knife edges A and B, then R_A and R_B will be the reaction.

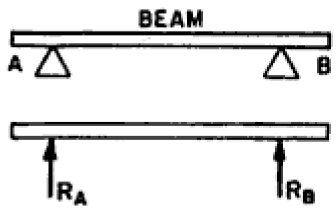


Fig. 3.6 simple/Knife edge Support

Roller support: here beam AB is supported on the rollers. The reaction will be normal to the surface on which rollers are placed.

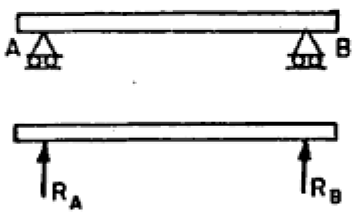


Fig. 3.7 Roller Support

Pin joint (or hinged) support: here the beam AB is hinged at point A. the reaction at the hinged end may be either vertical or inclined depending upon the type of loading. If load is vertical, then the reaction will also be vertical. But if the load is inclined, then the reaction at the hinged end will also be inclined.

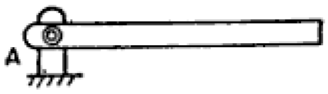


Fig. 3.8 Hinged Support

Fixed or built-in support: in this type of support the beam should be fixed. The reaction will be inclined. Also the fixed support will provide a couple.

Types of loading

There are 3 most important type of loading:

- Concentrated or point load
- Uniformly distributed load
- Uniformly varying load

Concentrated or point load: A concentrated load is one which is considered to act at a point.

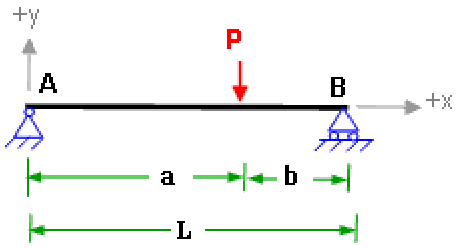


Fig. 3.9Concentrated or point load

Uniformly distributed load: A uniformly distributed load is one which is spread over a beam in such a manner that rate of loading is uniform along the length.

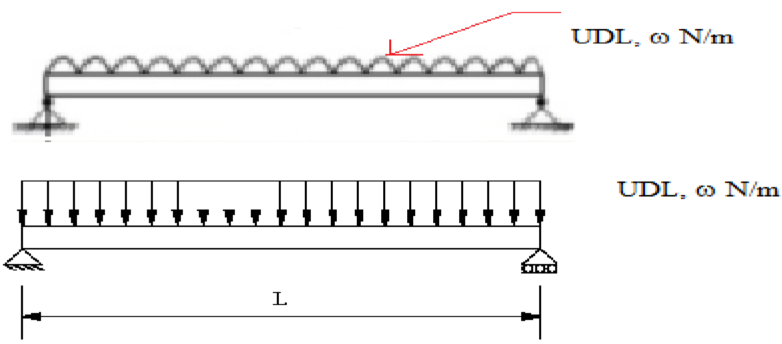


Fig. 3.10 Uniformly distributed load

Uniformly varying load: A uniformly varying load is one which is spread over a beam in such a manner that rate of loading varies from point to point along the beam.

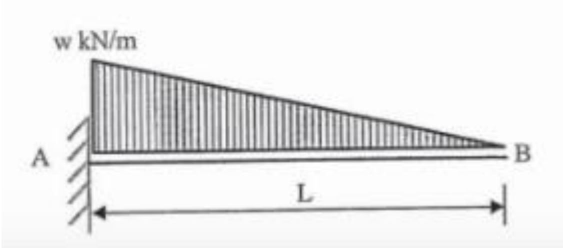


Fig. 3.11 Uniformly varying load

CONCEPT AND SIGNIFICANCE OF SHEAR FORCE AND BENDING MOMENT SIGN CONVENTIONS FOR SHEAR FORCE AND BENDING MOMENT

(i) Shear force: Fig. 1 shows a simply supported beam AB. carrying a load of 1000 N at its middle point. The reactions at the supports will be equal to 500 N. Hence R_A= R_B= 500 N.

Now imagine the beam to be divided into two portions by the section X-X. The resultant of the load and reaction to the left of X-X is 500 N vertically upwards. And the resultant of the load and reaction to the right of X-X is (1000↓ -500 ↑= 500↓N) 500 N downwards. The resultant force acting on any one of the parts normal to the axis of the beam is called the shear force at the section X-X is 500N.

The shear force at a section will be considered positive when the resultant of the forces to the left to the section is upwards, or to the right of the section is downwards. Similarly the shear force at a the section will be considered negative if the resultant of the forces to the left of the section is downward, or to the right of the section is upwards. Here the resultant force to the left of the section is upwards and hence the shear force will be positive.

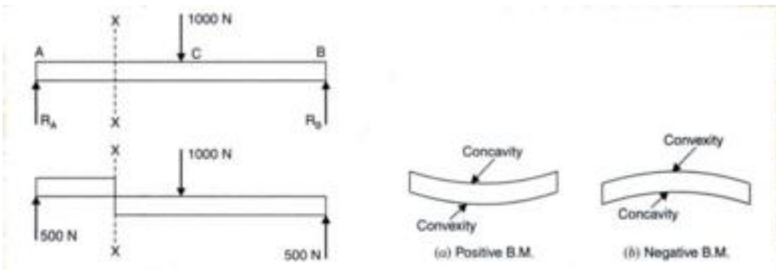


Fig. 3.12 Shear force and Bending Moment Sign Convention

(ii) Bending moment. The bending moment at a section is considered positive if the bending moment at that section is such that it tends to bend the beam to a curvature having concavity at the top as shown in Fig. 2. Similarly the bending moment at a section is considered negative if the bending moment at that section is such that it tends to bend the beam to a curvature haling convexity at the top. The positive B.M. is often called sagging moment and negative B.M. as hogging Moment.

IMPORTANT POINTS FOR DRAWING SHEAR FORCE AND BENDING MOMENT DIAGRAMS

The shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which show the variation of the bending moment along the length of beam. In these diagrams, the shear force or bending moment are represented by ordinates whereas the length of the beam represents abscissa.

The following are the important points for drawing shear force and bending moment diagrams

1. Consider the left or the right portion of the section.
2. Add the forces (including reaction) normal to the beam on one of the portion. If right portion of the section is chosen, a force on the right portion acting downwards is positive while force acting upwards is negative.
If the left portion of the section is chosen, a force on the left portion acting upwards is positive while force acting downwards is negative.
3. The positive values of shear force and bending moments are plotted above the base line, and negative values below the base line.
4. The shear force diagram will increase or decrease suddenly i.e., by a vertical straight line at a section where there is a vertical point load.
5. The shear force between any two vertical loads will be constant and hence the shear force diagram between two vertical loads will be horizontal.
6. The bending moment at the two supports of a simply supported beam and at the free end of a cantilever will be zero.

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER BEAM WITH A POINT LOAD

A cantilever beam of length 2m carries the point loads as shown in fig. draw the shear force and B.M diagrams for the cantilever beam.

Shear force diagram:

The shear force at D is +800N. this shear force remains constant between D and C. At C, due to point load the force becomes 1300N. between C and D, the shear force remains 1300N. At B again, the shear force becomes 1600N. the shear force between B and A remains constant and equal to 1600N. hence the shear force at different points will be as follows:

S.F. at D, FD= + 800 N
S.F. at C. Fe.=+ 800 + 500= 1300N
S.F. at B, Fa= + 800 + 500 +300 =1600N

S.F. at A, FA = + 1600 N.

The shear force, diagram is shown in Fig. which is drawn as: Draw a horizontal line AD as base line. On the base line mark the points B and C below the point loads. Take the ordinate DE = 800 N in the upward direction. Draw a line EF parallel to AD. The point F is vertically above C. Take vertical line FG is 500 N. Through G, draw a horizontal line GH in which point H is vertically above B. Draw vertical line HI = 300 N. From I, draw a horizontal line IJ. The point J is vertically above A. This completes the shear force diagram.

Bending Moment Diagram

The bending moment at D is zero:

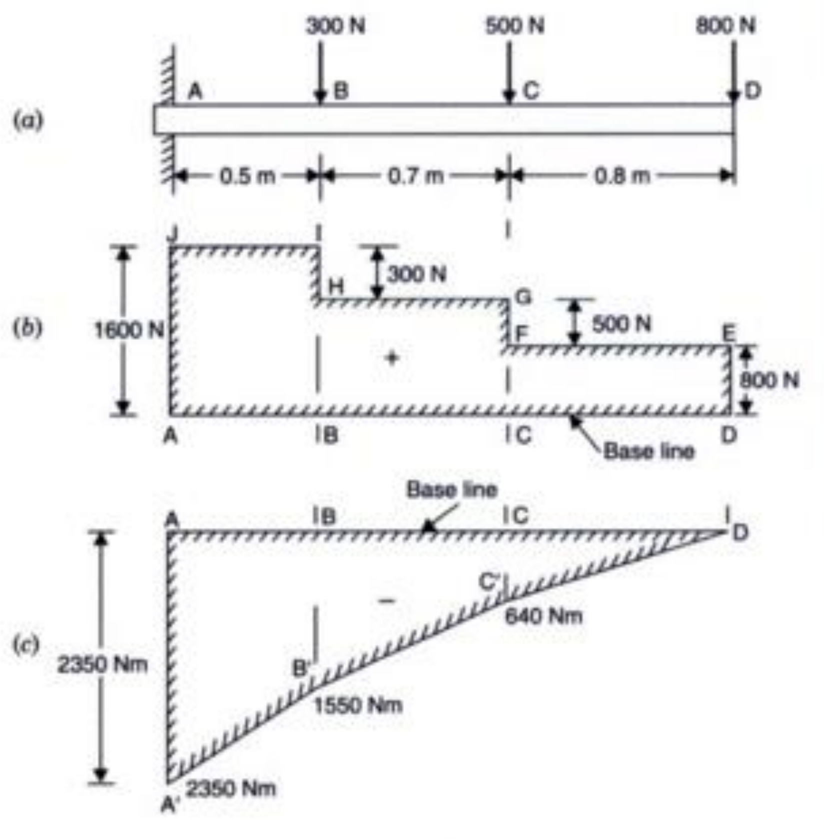


Fig. 3.13 SF & BM Diagram

(i) The bending moment at any section between C and Data distance: and D is given by,
M_x = - 800 X x which follows a straight line law.
At C, the value of x = 0.8 m. B.M. at C, = - 800 X 0.8 = - 640 Nm.

(ii) The B.M. at any section between B and C at a distance x from D is given by (At C, x = 0.8 and at B, x = 0.8 + 0.7 = 1.5 m. Hence here varies from 0.8 to 1.5).
M_x = - 800x - 500(x- 0.8)
Bending moment between B and C also varies by a straight line law.
B.M. at B is obtained by substituting x = 1.5 m in equation (i).
M_B = -800 X 1.5 - 500 (1.5 - 0.8)
= 1200 – 350 = 1550 Nm.

(iii) The B.M. at any section between A and B at a distance x from D is given by

(At B, x = 1.5 and at A, x = 2.0 m. Hence here x varies from 1.5m to 2.0 m

$M_x = - 800x - 500 (x - 0.8) – 300 (x- 1.5)$

Bending moment between A and B varies by a straight line law.

B.M. at A is obtained by substituting x = 2.0 m in equation (ii),

$M_A = - 800 X 2 - 500 (2 - 0.8) - 300 (2 - 1.5)$

$= - 800 X 2 - 500 X 1.2 - 300 X 0.5$

= - 1600 - 600 - 160 = - 2350 Nm. Hence the bending moments at different points will be as given below : MD = 0 Mc = - 640 Nm M_B= - 1550 Nm, M_A= - 2350 Nm

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER BEAM WITH A UNIFORMLY DISTRIBUTED LOAD

A cantilever beam of length 2m carries a uniformly distributed load of 2kN/m length over the whole length and a point load of 3kN at the free end. draw the shear force and B.M diagrams for the cantilever beam.

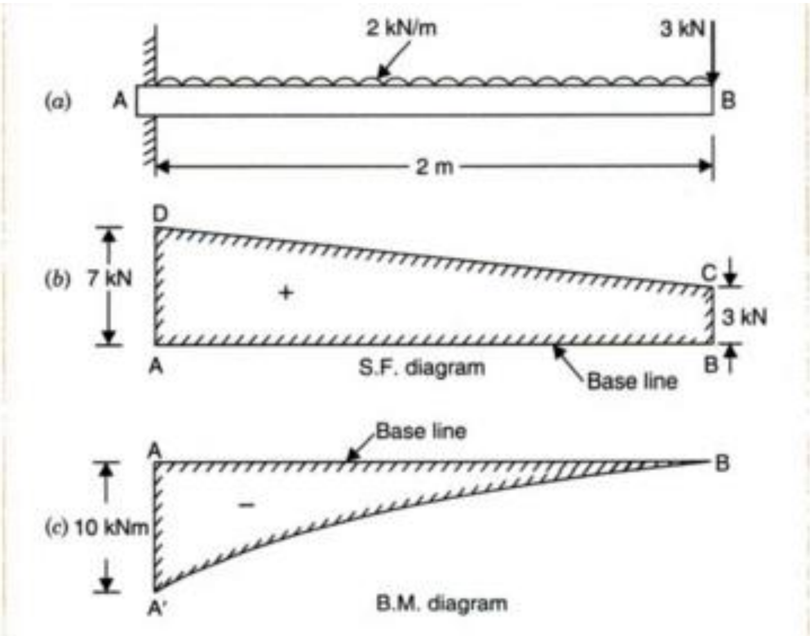


Fig. 3.14 SF & BM Diagram

Shear Force diagram

The shear force at B = 3 kN

Consider any section at a distance x from the free end B. The shear force at the section is given by.

$F_x = 3.0 + w.x$ (+ve sign is due to downward force on right portion of the section)

$= 3.0 + 2 X x$

The above equation shows that shear force follows a straight line law.

At B, x = 0 hence $F_B = 3.0$ kN

At A. x = 2 m hence FA = 3 + 2 x 2 = 7 kN.

The shear force diagram is shown in Fig. 6.18 (b), in which F_B = BC = 3 kN and FA = AD = 7 kN. The points C and D are joined by a straight line.

Bending Moment Diagram

The bending moment at any section at a distance x from the free end B is given by.

$M_x = - (3x + wx . x/2)$

$= - (3x + 2x^2/2)$

$= - (3x + x^2)$

(The bending moment will be negative as for the right portion of the section. the moment of loads at x is clockwise)

Equation (i) shows that the B. M. varies according to the parabolic law. From equation (i) we have At B. x = 0 hence $M_B = -(3x0 + 0^2) = 0$

At A, x = 2 m hence $M_A = - (3 \times 2 + 2^2) = - 10$ kN/m

Now the bending moment diagram is drawn In this diagram.

AA' = 10 kNm and points A' and B are joined by a parabolic curve.

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER CARRYING A GRADUALLY VARYING LOAD

A cantilever of length 4 m carries a gradually varying load, zero at the free end to 2 Kn/m. at the fixed end. Draw the S.F. and B.M. diagrams for the cantilever.

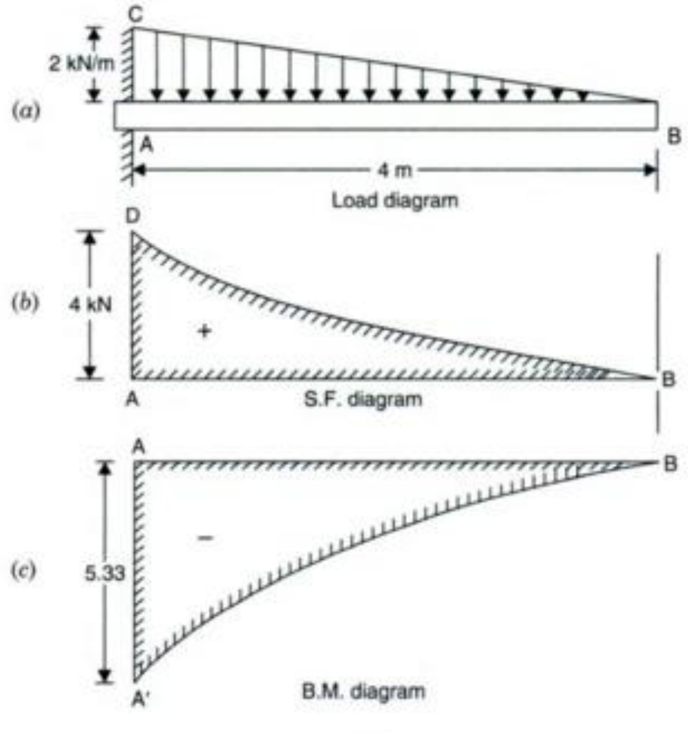


Fig. 3.15 SF & BM Diagram

Shear Force Diagram

The shear force is zero at B.
The shear force at C will be equal to the area of load diagram ABC.

Shear force at C = (4 x 2) / 2 = 4 kN

The shear force between A and B varies according to parabolic law.
Bending Moment Diagram
The B.M. at B is zero.
The bending moment at A is equal to $M_A = -w. l^2 / 6 = -2 \times 4^2 / 6 = -5.33$ kNm.
The B.M. between A and B varies according to cubic law.

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A SIMPLY SUPPORTED BEAM WITH POINT LOAD

A simply supported beam of length 6 m, carries point load of 3 kN and 6 kN at distances of 2 m and 4 m from the left end. Draw the shear force and bending moment diagrams for the beam.
Sol.
First calculate the reactions RA and RB.
Taking moments of the force about A, we get
RB X 6 = 3 X 2 + 6 X 4 = 30
RB = 30/ 6 = 5 kN
RA = Total load on beam - RB = (3 + 6) – 5 = 4 kN

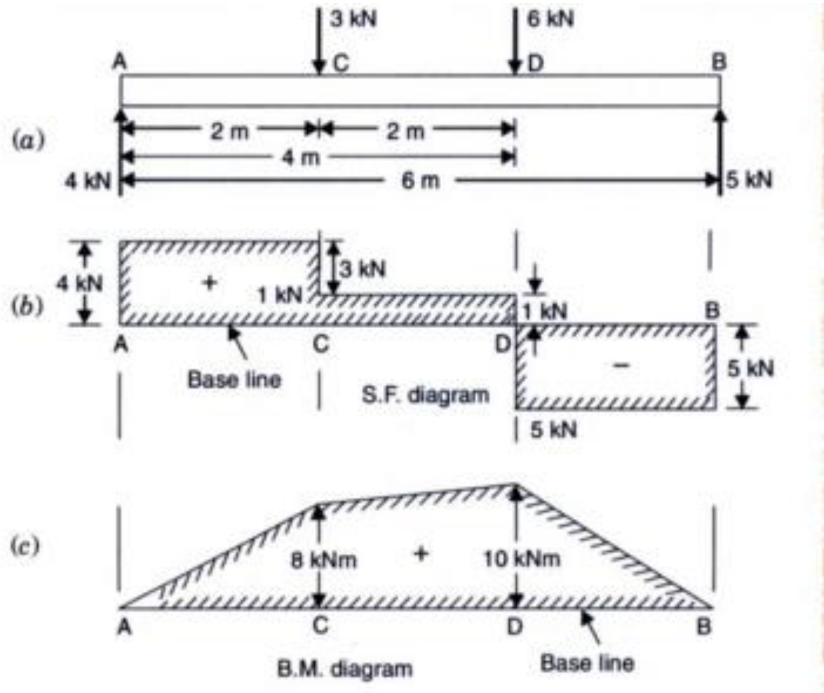


Fig. 3.16 SF & BM Diagram

Shear Force Diagram
Shear force at A, FA= + RA= + 4 kN
Shear force between A and C is constant and equal to + 4 kN
Shear force at C, Fc = + 4 - 3.0 = + 1 kN

Shear force between C and D is constant and equal to + 1 kN.
Shear force at D, FD= + 1 - 6 = - 5 kN
The shear force between D and B is constant and equal to - 5 kN.
Shear force at B, FB= - 5 kN
Bending Moment Diagram
B.M. at A, MA = 0
B.M. at C, MC = RA X 2 = 4 X 2 = +8kNm
B.M. at D, MD = RA X 4 -3 x 2 = 4 x 4 -3 x 2 =+ 10 kNm
B.M. at B, MB= 0

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A SIMPLY SUPPORTED BEAM WITH A UNIFORMLY DISTRIBUTED LOAD

Draw the S.F. and B.M. diagrams of a simply supported beam of length 7 m carrying uniformly distributed load

Sol. First calculate the reactions RA and RB,
Taking moments of all forces about A, we get
RB X 7 = 10 X 3 X (3/2) + 5 X 2 X (3+2+(2/2)
= 45 + 60 = 105
RB = 105 /7 = 15 kN
and RA =Total load on beam - RB
= (10X 3 +5 X 2) - 15 = 40- 15 = 25kN
S.F. Diagram
The shear force at A is + 25 kN
The shear force at C = RA - 3 X 10 = + 25 - 30 = - 5 kN
The shear force varies between A and C by a straight line law.
The shear force between C and D is constant and equal to - 5 kN.
The shear force at B is - 15 kN The shear force between D and B varies by a straight line law.
The shear force is zero at point E between A and C. Let us find the location of E from A. Let the point E be at a distance x from A.
The shear force at E = RA - 10 x = 25 - 10x
But shear force at E = 0

25-10x = 0 or
10x = 25
x = 2.5m

B.M. Diagram
B.M. at A is zero
B.M. at B is zero
B.M. at C,
M_C = R_A X 3 -10 X 3 x 3/2
= 25 X 3 – 45 = 75 -45 = 30 kNm
At E, x = 2.5 and hence
B.M. at E, M_E = R_A X 2.5 - 10 X 2.5 X (2.5 /2) = 25 X 2.5 - 5 X 6.25 = 62.5 - 31.25
= 31.25 kNm

B.M. at D. $M_D = 25(3 + 2) - 10 \times 3 \times ((3/2) + 2) = 125 - 105 = 20$ kNm

The B.M. between AC and between BD varies according to parabolic law. But B.M. between C and D varies according to straight line law.

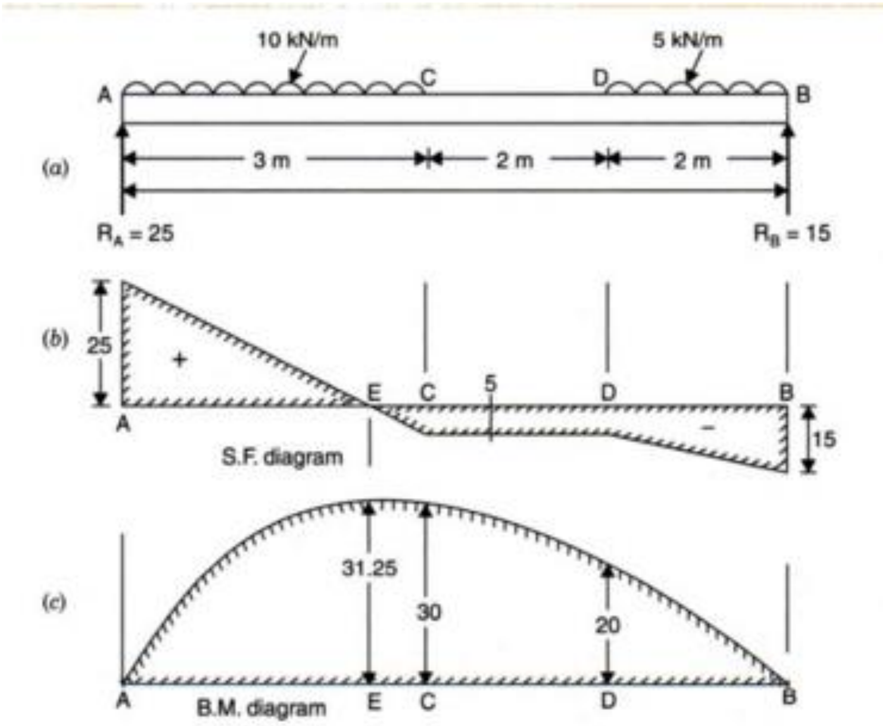


Fig. 3.17 SF & BM Diagram

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR OVER HANGING BEAM

A beam of length 12 m is simply supported at two supports which are 8m apart, with an overhang of 2 m on each side as shown in Fig. The beam carries a concentrated load of 1000 N at each end. Draw S.F. and B.M. diagrams.

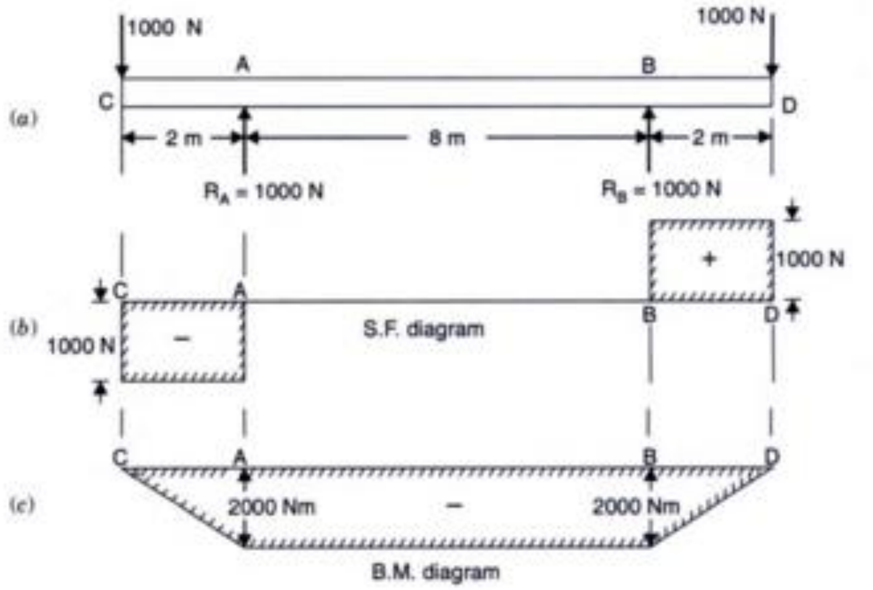


Fig. 3.18 SF & BM Diagram

As the loading on the beam is symmetrical. Hence reactions R_A and R_B will be equal and their magnitude will be half of the total load.

$$R_A = R_B = (1000 + 1000)/2 = 1000N$$

S.F. at C = -1000 N

S.F. remains constant (i.e., = - 1000 N) between C and A

S.F. at A = $1000+R_A$ = - 1000 + 1000 = 0

S.F. remains constant (i.e., = 0) between A and B

S.F. at B = 0+ 1000 =+ 1000N

S.F. remains constant (i.e., 1000 N) between B and D

B.M. Diagram

B.M. at C = 0

B.M. at A = - 1000 x 2 = - 2000 Nm

B.M. between C and A varies according to straight line law.

The B.M. at any section in AB at a distance of x from C is given by,

$M_X = -1000 \text{ X } x + R_A (x - 2)$

$= -1000 \text{ X } x + 1000(x - 2) = -2000$ Nm

Hence B.M. between A and B is constant and equal to - 2000 Nm.

B.M. at D = 0

STRESSES IN BEAMS

When some external load acts on a beam, the shear force and bending moments are set up at all sections of the beam. Due to the shear force and bending moment, the beam undergoes certain deformation. The material of the beam will offer resistance or stresses against these deformations. These stresses with certain assumptions can be calculated. The stresses introduced by bending moment are known as bending stresses.

If a length of a beam is subjected to a constant bending moment and no shear force (i.e., zero shear force), then the stresses will be set up in that length of the beam due to B.M. only and that length of the beam is said to be in pure bending or simple bending. The stresses set up in that length of beam are known as bending stresses.

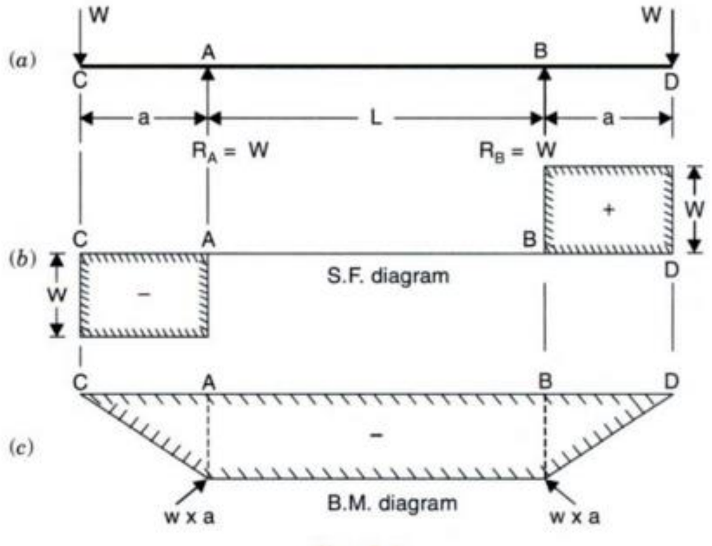


Fig. 3.19 SF & BM Diagram

A beam simply supported at A and B and overhanging by same length at each support is shown in Fig. 7.1. A point load W is applied at each end of the overhanging portion. The S.F. and B.M. for the beam are drawn as shown in Fig. 7.1 (b) and Fig. 7.1 (c) respectively. From these diagrams, it is clear that there is no shear force between A and B but the B.M. between A and B is constant. This means that between A and B, the beam is subjected to a constant bending moment only. This condition of the beam between A and B is known as pure bending or simple bending.

THEORY OF SIMPLE BENDING

THEORY OF SIMPLE BENDING WITH ASSUMPTIONS MADE

Before discussing the theory of simple bending, let us see the assumptions made in the theory of simple bending. The following are the important assumptions:

1. The material of the beam is homogeneous* and isotropic**.
2. The value of Young's modulus of elasticity is the same in tension and compression.

3. The transverse sections which were plane before bending, remain plane after bending
4. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common centre of curvature.
5. The radius of curvature is large compared with the dimensions of the cross-section. 6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.

A beam subjected to simple bending. Consider a small length fit of this part of beam. Consider two sections AB and CD which are normal to the axis of the beam N - N. Due to the action of the bending moment, the part of length &x will be deformed as shown in Fig.(b). From this figure, it is clear that all the layers of the beam, which were originally of the same length, do not remain of the same length any more. The top layer such as AC has deformed to the shape NC. This layer has been shortened in its length. The bottom layer BD has deformed to the shape B'D'. This layer has been elongated. From the Fig. 7.2 (b), it is clear that some of the layers have been shortened while some of them are elongated. At a level between the top and bottom of the beam, there will be a layer which is neither shortened nor elongated. This layer is known as neutral layer or neutral surface. This layer in Fig.(b) is shown by N' — N' and in Fig.(a) by N — N. The line of intersection of the neutral layer on a cross-section of a beam is known as neutral axis (written as N.A.).

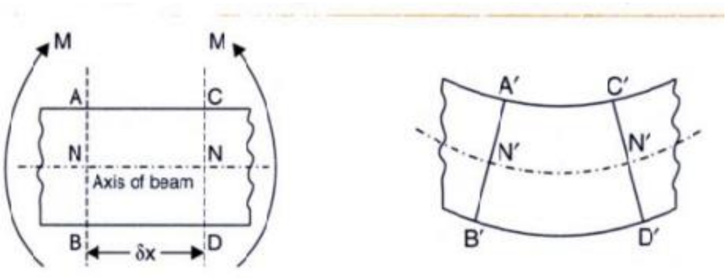


Fig. 3.20 Before and after Bending of Beam

The layers above N — N (or N' — N') have been shortened and those below, have been elongated. Due to the decrease in lengths of the layers above N— N, these layers will be subjected to compressive stresses. Due to the increase in the lengths of layers below N — N, these layers will be subjected to tensile stresses. We also see that the top layer has been shortened maximum. As we proceed towards the layer N— N, the decrease in length of the layers decreases. At the layer N— N, there is no change in length. This means the compressive stress will be maximum at the top layer. Similarly the increase in length will be maximum at the bottom layer. As we proceed from bottom layer towards the layer N — N, the increase in length of layers decreases. Hence the amount by which a layer increases or decreases in length, depends upon the position of the layer with respect to N - N. This theory of bending is known as theory of simple bending.

Simple Bending Theory OR Theory of Flexure for Initially Straight Beams
(The normal stress due to bending are called flexure stresses)

Preamble:

When a beam having an arbitrary cross section is subjected to a transverse loads the beam will bend. In addition to bending the other effects such as twisting and buckling may occur, and to investigate a problem that includes all the combined effects of bending, twisting and buckling could become a complicated one. Thus we are interested to investigate the bending effects alone, in order to do so; we have to put certain constraints on the geometry of the beam and the manner of loading.

Assumptions:

The constraints put on the geometry would form the **assumptions:**

1. Beam is initially **straight,** and has a **constant cross-section.**
2. Beam is made of **homogeneous material** and the beam has a **longitudinal plane of symmetry.**
3. Resultant of the applied loads lies in the plane of symmetry.
4. The geometry of the overall member is such that bending not buckling is the primary cause of failure.
5. Elastic limit is nowhere exceeded and **'E'** is same in tension and compression.
6. Plane cross - sections remains plane before and after bending.

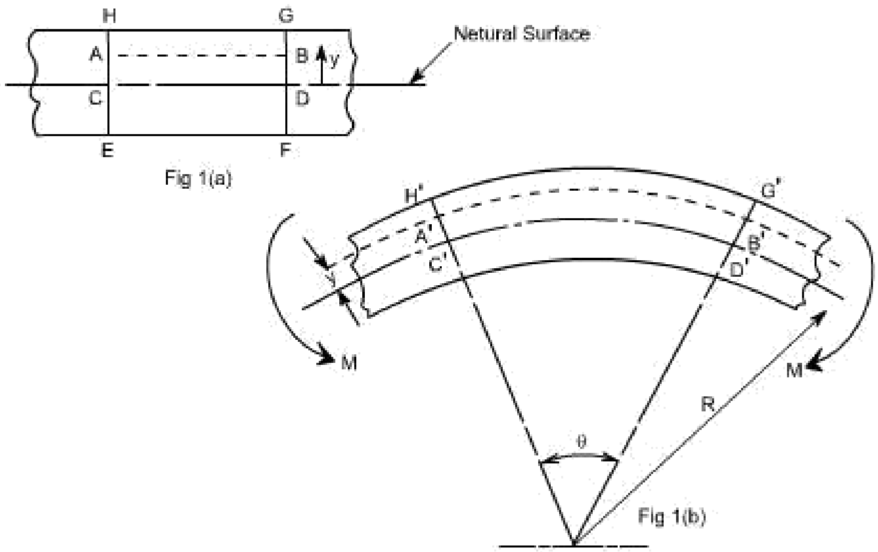


Fig. 3.21 Before and after Bending to an arc

Let us consider a beam initially unstressed as shown in fig 1(a). Now the beam is subjected to a constant bending moment (i.e. „Zero Shearing Force') along its length as would be obtained by applying equal couples at each end. The beam will bend to the radius R as shown in Fig 1(b)

As a result of this bending, the top fibers of the beam will be subjected to tension and the bottom to compression it is reasonable to suppose, therefore, **that somewhere between the two there are points at which the stress is zero. The locus of all such points is known as neutral axis.** The radius of curvature R is then measured to this axis. For symmetrical sections the N. A. is the axis of symmetry but whatever the section N. A. will always pass through the centre of the area or centroid.

The above restrictions have been taken so as to eliminate the possibility of 'twisting' of the beam.

Concept of pure bending: Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

Since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam

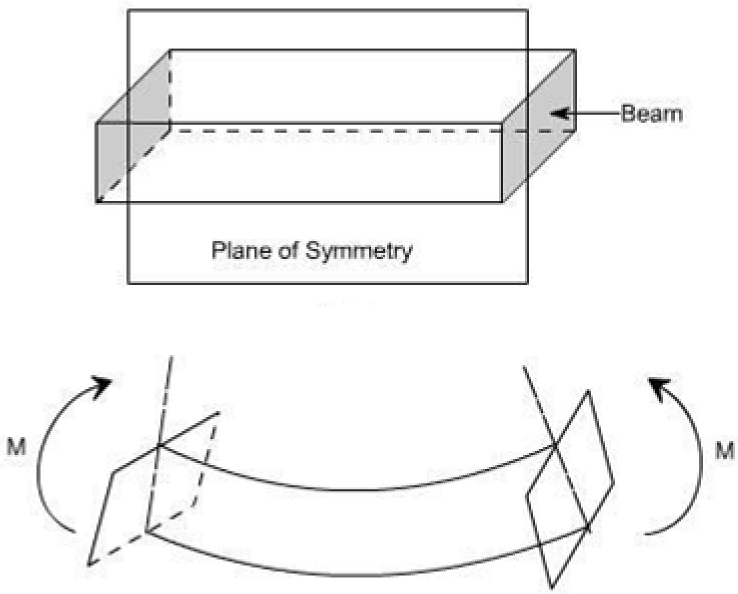


Fig.3.22 Plane of Bending

When a member is loaded in such a fashion it is said to be in **pure bending.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below:

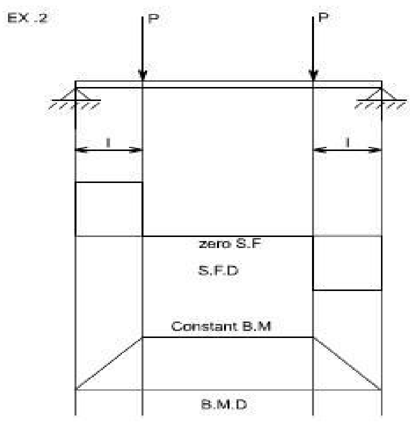


Fig. 3.23 Pure bending State for SSB

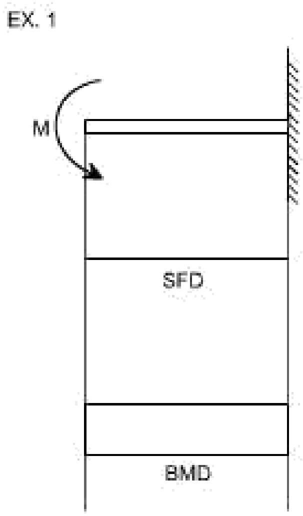


Fig. 3.24 Pure bending State for Cantilever

When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,
1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross-section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.
2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.

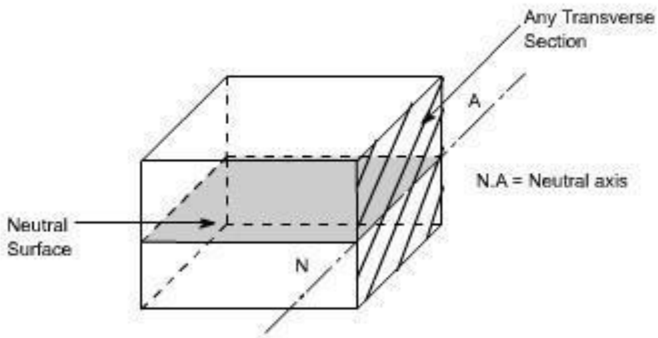


Fig. 3.25 Position of Neutral Surface/Axis

We know that when a beam is under bending the fibers at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibers which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibers is called neutral surface.The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axis Neutral axis **(N A).**

.

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle q.

Consider now fiber AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \text{..............(1)}$$

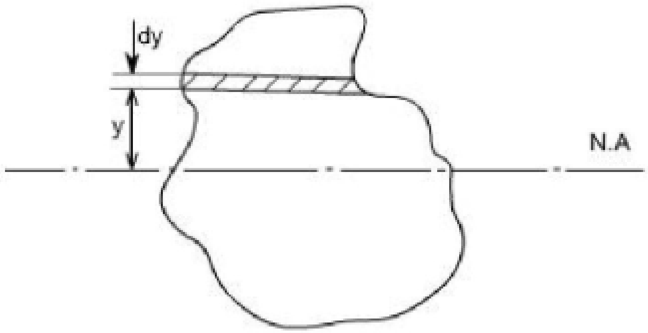


Fig. 3.26 Area MI consideration

Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance „y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area 'dA'

then the force on the strip is

$$F = \sigma\ \delta A = \frac{E}{R} y\ \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \sum \frac{E}{R} y^2\ \delta A = \frac{E}{R} \sum y^2 \delta A$$

Now the term is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation.

The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max}^{m} = \frac{M}{T} y_{max}^{m}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max}^{m}} \sigma_{max}^{m}$$

For ready comparison of the strength of various beam cross-section this relationship is sometimes written in the form

$$M = Z \sigma_{max}^{m} \text{ where } Z = \frac{I}{y_{max}^{m}}$$

Is termed as section modulus

STRESSES IN BEAMS

In previous chapter concern was with shear forces and bending moment in beams. Focus in this chapter is on the stresses and strains associated with those shear forces and bending moments.

Loads on a beam will cause it to bend or flex.

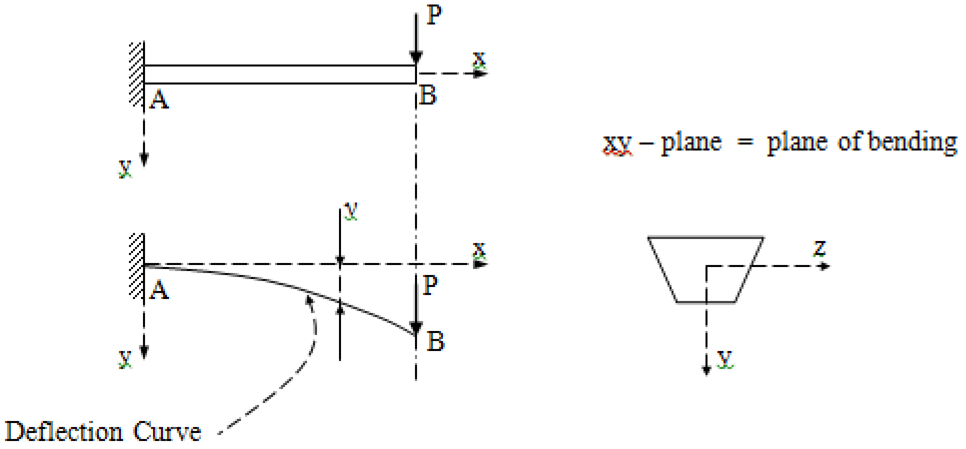


Fig. 3.27 Cantilever Beam with a point load at free end

PURE BENDING AND NONUNIFORM BENDING

Pure Bending = flexure of a beam under constant bending moment

$\Rightarrow$ shear force = 0 (V = 0 = dM / dx); no change in moment.

Non uniform Bending = flexure of a beam in the presence of shear forces

$\Rightarrow$ bending moment is no longer constant

Moment Diagram example:

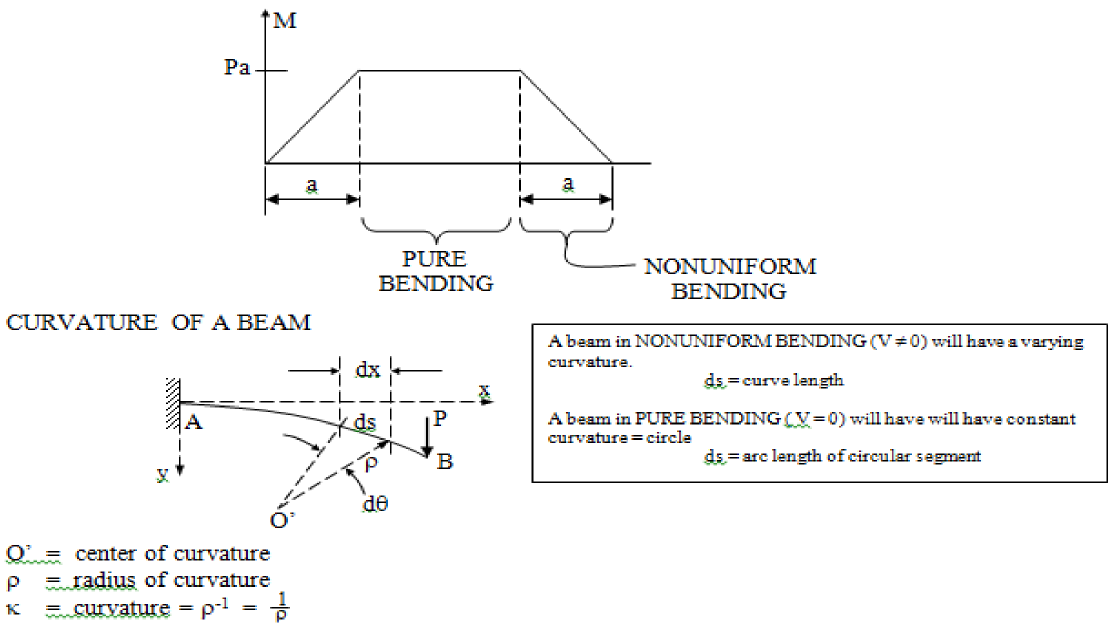


Fig. 3.28 Moment diagram

$\rho \, d\theta = ds$

For small deflections: $ds \approx dx$

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} \qquad (1)$$

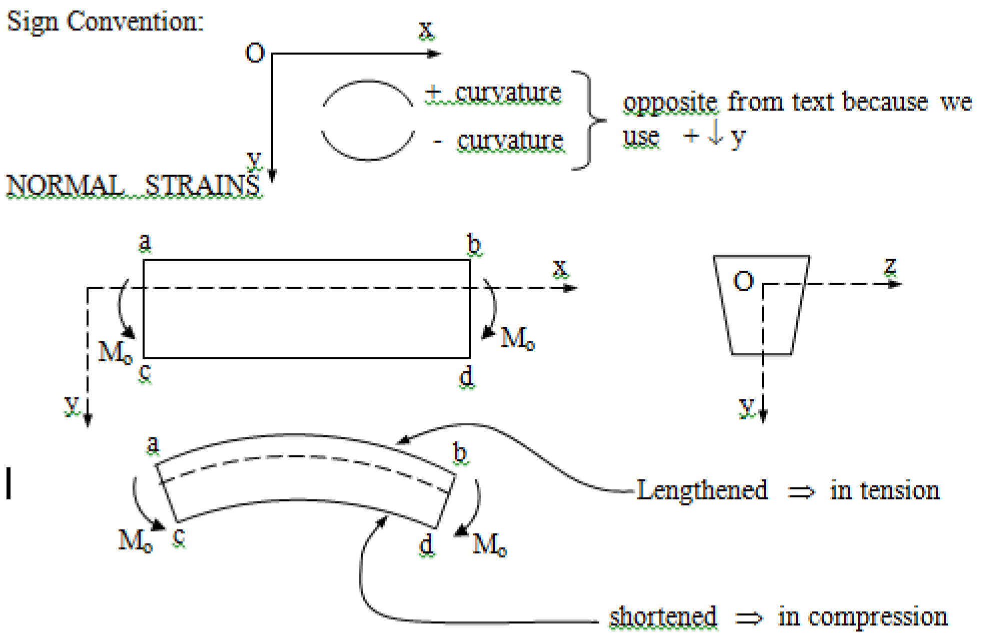


Fig. 3.29 Sign Convention

Somewhere between the top and bottom of the beam is a place where the fibers are neither in tension or compression.

Neutral axis of the cross section

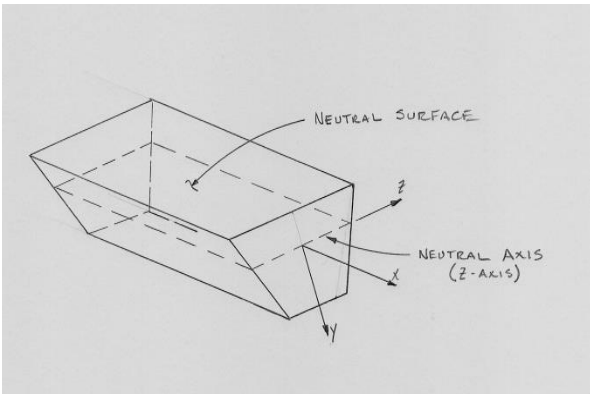


Fig. 3.30 Neutral Axis and Neutral Layer

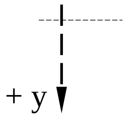
dashed line = neutral surface of the beam

when bent: a b lengthens, c d shortens } causes normal strains, ε_x

The normal strain is: $\boxed{\varepsilon_x = -\dfrac{y}{\rho} = -\kappa y}$ ———————— (2)

Where, y = distance from neutral axis

From Eqn (2):



- y = + ε_x (elongation)
+ y = - ε_x (shortening)

} for + κ

Transverse Strains: $\varepsilon_z = -\nu\varepsilon_x = \nu\kappa y$

Where ν = Poisson's Ratio

NORMAL STRESSES IN BEAMS

If material is elastic with linear stress-strain diagram, THEN:

$\sigma = E\varepsilon$ (Hooke's Law)

$\boxed{\sigma_x = E\varepsilon_x = -E\kappa y}$ ———— (3) { varies linearly with y

Where x is longitudinal axis of beam and σ_x is the normal stresses in this direction acting on the cross section. These stresses varies linearly with the distance y from the neutral surface.

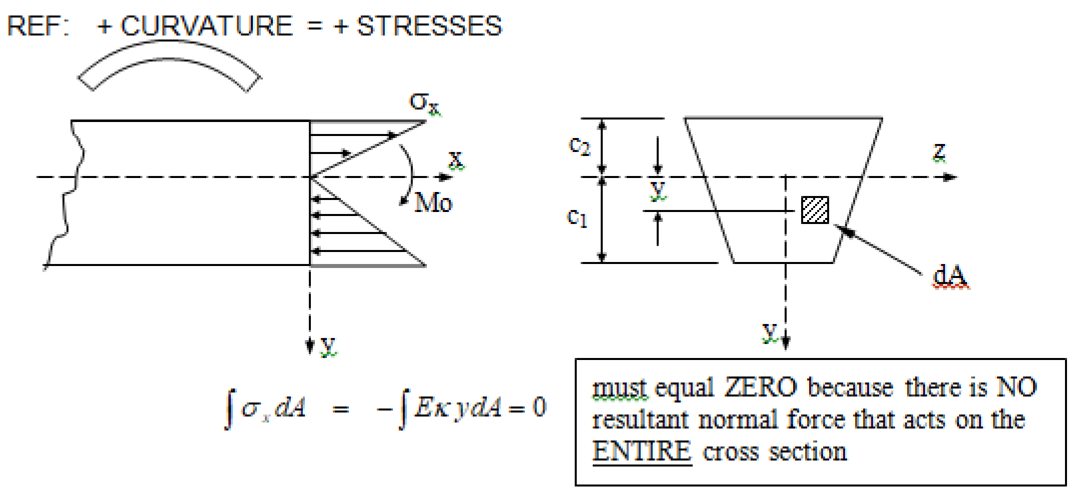


Fig. 3.31 Stress Distribution Diagram

$$\int \sigma_x dA \quad = \quad -\int E\kappa y dA = 0$$

$\boxed{\int y dA = 0}$ ———————— (4)

Eqn (4) is the 1st Moment of the Area of the cross section w.r.t. z-axis and it is zero

⇒ z-axis must pass thru the centroid of the cross section.
⇒ z-axis is also the neutral axis
⇒ neutral axis passes thru the centroid of the cross section

Limited to beams where y-axis is the axis of symmetry.
y, z –axes are the PRINCIPAL CENTROIDAL AXES.

Consider the Moment Resultant of σ_x :

RECALL Eqn (3):

$$\sigma_x = -E\kappa y$$

$$dM_O = -\sigma_x y dA$$

$$M_O = -\int \sigma_x y dA = \kappa E \int y^2 dA \qquad M = -M_0$$

$$M = -\kappa EI$$

$$I = \int y^2 dA$$

⇐ where, I = Moment of Inertia of cross sectional area w.r.t. z-axis (neutral axis)

$$\kappa = \frac{1}{\rho} = -\frac{M}{EI}$$

⇐ EI = FLEXURAL RIGIDITY

$$-E\kappa = \frac{M}{I}$$

⇐ substitute into Eqn (3)

$$\sigma_x = \left(\frac{M}{I}\right) y$$

$$\sigma_x = \frac{My}{I}$$

⇐ Flexure Formula
σ_x = Bending Stress

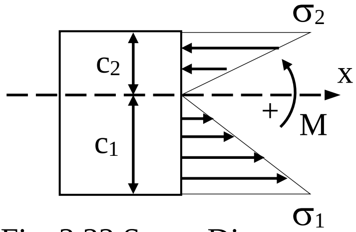


Fig. 3.32 Stress Diagram

MAXIMUM STRESSES:

$$\sigma_1 = \frac{Mc_1}{I} \qquad \sigma_2 = -\frac{Mc_2}{I}$$

Text defines Section Moduli as:

$$S_1 = \frac{I}{c_1} \qquad S_2 = \frac{I}{c_2}$$

$$\sigma_1 = \frac{M}{S_1} \qquad \sigma_2 = -\frac{M}{S_2}$$

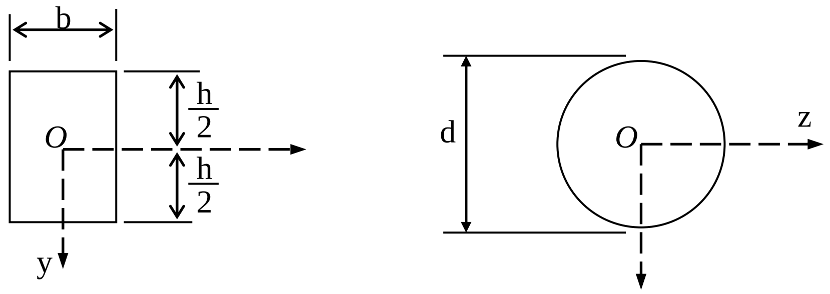
Section Modulus is handy to use when evaluating bending stress w.r.t. to moment which varies along length of a beam.

If cross section is symmetrical w.r.t. z-axis, then:

$$c_1 = c_2 = c$$

$$\boxed{\sigma_1 = -\sigma_2 = \frac{Mc}{I}}$$

Moments of Inertia to know:



$$I = \frac{bh^3}{12} \qquad I = \frac{\pi d^4}{64}$$

Fig.3.33 Area MI for different section

Problems for Practice

A high-strength steel wire of diameter d = 4 mm, modulus of elasticity E = 200 GPa, proportional limit σ_{pl} = 1200 MPa is bent around a cylindrical drum of radius R_0 = 0.5 m .

FIND:

a. bending moment, M
b. maximum bending stress, σ_{max}

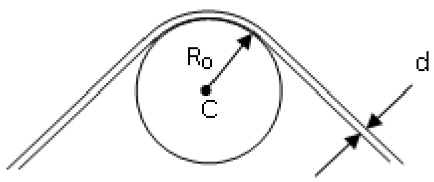


Fig. 3.34 Loading situation

Problems for Practice

The beam shown which is constructed of glued laminated wood. The uniform load includes the weight of the beam.

FIND:

a. Maximum Tensile Stress in the beam due to bending.
b. Maximum compressive stress in the beam due to bending.

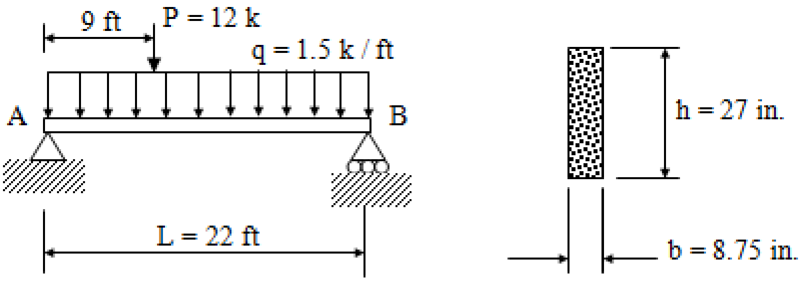


Fig. 3.35 SSB with load

DESIGN of BEAMS for BENDING STRESSES

After all factors have been considered (i.e., materials, environmental conditions) it usually boils down to

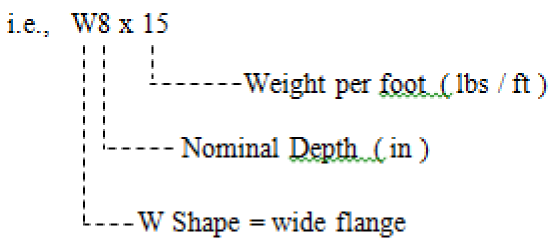
$$\sigma_{Allow} > \sigma_{Beam}$$

$$\sigma_{Allow} > \frac{M_{max}\, c}{I}$$

Here is where the section modulus is useful.

RECALL: $\sigma = \dfrac{M}{S}$ thus, $\boxed{S = \dfrac{M_{max}}{\sigma_{allow}}}$

Appendix E and F give properties of beams.



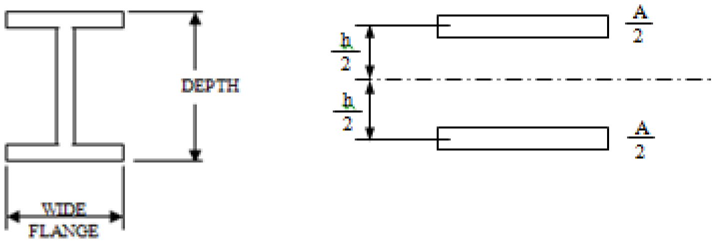


Fig. 3.36 I Section considerations

Wood Beams - 2 x 4 $\Rightarrow$ really is: 1.5” x 3.5” net dimensions (should always use net dims.)

$$I = \frac{A}{2}\left(\frac{h}{2}\right)^2 + \frac{A}{2}\left(\frac{h}{2}\right)^2$$

$$I = \frac{Ah^2}{4}$$

$$S = \frac{I}{c} = \frac{\frac{Ah^2}{4}}{\frac{h}{2}} = \frac{1}{2}Ah$$

For W Shapes; $S \approx 0.35\,Ah$

You want as much material as possible, as far from the neutral axis as possible because this is where the greatest stress is occurring.

However, you have to be careful because if the web is too thin, it could fail by:

1.) being overstressed in shear
2.) buckling

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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

UNIT – IV – TORSION – SMEA1305

MECHANICS OF SOLIDS (SMEA1305)

UNIT 4: TORSION

Analysis of torsion of circular bars – Shear stress distribution – Bars of Solid and hollow circular section – Stepped shaft – Twist and torsion stiffness – Composite shafts

Springs - Laminated springs, axial load and twisting moment acting simultaneously both for open and closed coiled springs– Deflection of helical coil springs under axial loads – stresses in helical coil springs under torsion.

INTRODUCTION: TORSION

In machinery, the general term “shaft” refers to a member, usually of circular cross section, which supports gears, sprockets, wheels, rotors, etc., and which is subjected to torsion and to transverse or axial loads acting singly or in combination. An “axle” is a rotating/non-rotating member that supports wheels, pulley and carries no torque. A “spindle” is a short shaft. Terms such as line shaft, head shaft, stub shaft, transmission shaft, countershaft, and flexible shaft are names associated with special usage.

Analysis of torsion

In a slender member under the action of a torsional moment (also called twisting moment or torque) shearing stresses appear, whose moment about the bar axis is equal to the applied torque. In the same way as the shearing stresses caused by the shear force, these stresses must be tangent to the contour in the points lying close the boundary of the cross-section. These two conditions are not sufficient to determine the distribution of shearing stresses in the cross-section. Furthermore, the twisting moment is not a symmetrical loading with respect to the middle cross-section of a piece of bar.

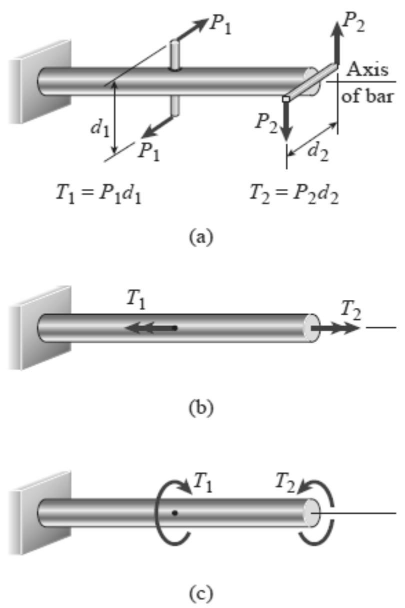


Fig. 4.1 Shaft subjected to Torsion

An idealized case of torsional loading is a straight bar supported at one end and loaded by two pairs of equal and opposite forces. The first pair consists of the forces P1 acting near the midpoint of the bar and the second pair consists of the forces P2 acting at the end. Each pair of forces forms a couple that tends to twist the bar about its longitudinal axis. As we know from statics, the moment of a couple is equal to the product of one of the forces and the perpendicular distance between the lines of action of the forces; thus, the first couple has a moment T1 = P1d1 and the second has a moment T2 = P2d2.

Torsion refers to the twisting of a straight bar when it is loaded by moments (or torques) that tends to produce rotation about the longitudinal axis of the bar. For instance, when you turn a screwdriver, your hand applies a torque *T* to the handle and twists the shank of the screwdriver. Other examples of bars in torsion are drive shafts in automobiles, axles, propeller shafts, steering rods, and drill bits.

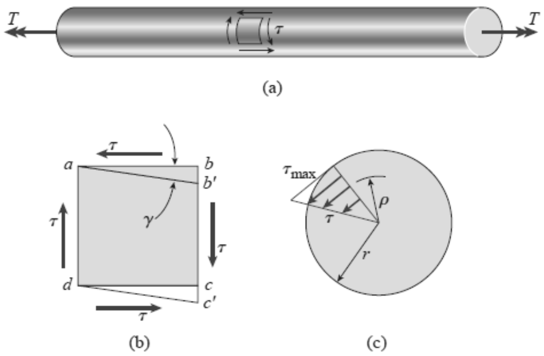
The moment of a couple may be represented by a **vector** in the form of a double-headed arrow. The arrow is perpendicular to the plane containing the couple, and therefore in this case both arrows are parallel to the axis of the bar. The direction (or *sense*) of the moment is indicated by the *right-hand rule* for moment vectors—namely, using your right hand, let your fingers curl in the direction of the moment, and then your thumb will point in the direction of the vector. An alternative representation of a moment is curved arrow acting in the direction of rotation. The choice depends upon convenience and personal preference. Moments that produce twisting of a bar, such as the moments $T1$ and $T2$, are called **torques** or **twisting moments**. Cylindrical members that are subjected to torques and transmit power through rotation are called **shafts**; for instance, the drive shaft of an automobile or the propeller shaft of a ship. Most shafts have circular cross sections, either solid or tubular. In this chapter we begin by developing formulas for the deformations and stresses in circular bars subjected to torsion. We then analyze the state of stress known as *pure shear* and obtain the relationship between the moduli of elasticity *E* and *G* in tension and shear, respectively. Next, we analyze rotating shafts and determine the power they transmit. Finally, we cover several additional topics related to torsion, namely, statically indeterminate members, strain energy, thin-walled tubes of noncircular cross section, and stress concentrations.

Torsional deformations of a circular bar

A prismatic bar with a circular cross-section has a symmetrical geometry with respect to any plane passing through the bar axis. If, in addition, the material also has symmetrical rheological properties with respect to these planes, which happens if the material is isotropic or monotropic with the monotropy direction parallel to the bar axis, the bar is totally symmetric with respect to the bar axis, i.e., it is axisymmetric. As a consequence of this type of symmetry, all the points of a cross-section lying on a circumference with the centre in the bar axis, are in the same conditions with respect to the centre of the cross-section. If we consider a vector applied at the centre of the cross-section, representing the torque acting on the bar, all the points of that circumference are also in the same conditions with respect to that vector. As a consequence, all the points will undergo the same displacement in relation to the bar axis, i.e., the radial, circumferential and longitudinal components of the displacement will be the same in all points of the circumference. This means that the circumference will remain on a plane perpendicular to the bar axis and that its centre will remain on that axis.

The shear strains in a circular bar in torsion, we are ready to determine the directions and magnitudes of the corresponding shear stresses. The directions of the stresses can be determined by inspection. We observe that the torque T tends to rotate the right-hand end of the bar counterclockwise when viewed from the right. The magnitudes of the shear stresses can be determined from the strains by using the stress-strain relation for the material of the bar. If the material is linearly elastic, we can use Hooke's law in shear, in which G is the shear modulus of elasticity and γ is the shear strain in radians. Combining this equation with the equations for the shear strains, in which τ max is the shear stress at the outer surface of the bar (radius r), τ is the shear stress at an interior point (radius r), and θ is the rate of twist. (In these equations, θ has units of radians per unit of length.)

$$\tau = G\gamma$$



$$\tau_{max} = Gr\theta \qquad \tau = G\rho\theta = \frac{\rho}{r}\tau_{max}$$

Fig. 4.2 Torsion Analysis

Equations show that the shear stresses vary linearly with the distance from the center of the bar, illustrated by the triangular stress diagram. This linear variation of stress is a consequence of Hooke's law. If the stress-strain relation is nonlinear, the stresses will vary nonlinearly and other methods of analysis will be needed.

The shear stresses acting on a cross-sectional plane are accompanied by shear stresses of the same magnitude acting on longitudinal planes. This conclusion follows from the fact that equal shear stresses always exist on mutually perpendicular planes. If the material of the bar is weaker in shear on longitudinal planes than on cross-sectional planes, as is typical of wood when the grain runs parallel to the axis of the bar, the first cracks due to torsion will appear on the surface in the longitudinal direction. The state of pure shear at the surface of a bar is equivalent to equal tensile and compressive stresses acting on an element oriented at an angle of 45. Therefore, a rectangular element with sides at 45° to the axis of the shaft will be subjected to tensile and compressive stresses. If a torsion bar is made of a material that is weaker in tension than in shear, failure will occur in tension along a helix inclined at 45° to the axis.

Torsion of circular shafts

Equation for shafts subjected to torsion "T"

$$\frac{\tau}{R} = \frac{T}{J} = \frac{G\theta}{L}$$

Torsion Equation

Where J = Polar moment of inertia,τ = Shear stress induced due to torsion T.
G = Modulus of rigidity,θ = Angular deflection of shaft, R, L = Shaft radius & length respectively.

Assumptions

- The bar is acted upon by a pure torque.
- The section under consideration is remote from the point of application of the load and from a change in diameter.
- Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.
- The material obeys Hooke's law
- Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle

Polar moment of Inertia

As stated above, the polar second moment of area, J is defined as

$$J = \int_0^R 2\pi r^3 \, dr$$

For a solid shaft

$$J = 2\pi \left[\frac{r^4}{4} \right]_0^R = \frac{2\pi R^4}{4} = \frac{\pi D^4}{32}$$

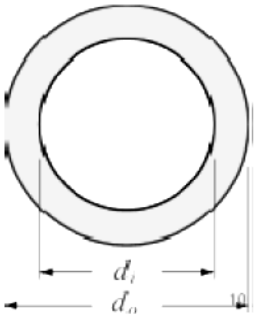


Fig. 4.3 Hollow Shaft

For a hollow shaft of internal radius r:

$$J = \int_0^R 2\pi r^3 \, dr = 2\pi \left[\frac{r^4}{4} \right]_r^R = \frac{\pi}{2}(R^4 - r^4) = \frac{\pi}{32}\left(D^4 - d^4\right)$$

Where D is the external and d is the internal diameter.

- Solid shaft "J" = $\frac{\pi d^4}{32}$
- Hollow shaft, "J" = $\frac{\pi}{32}(d_o^4 - d_i^4)$

Polar section Modulus

$$Z_p = J / c, \text{ where } c = r = D/2$$

For a solid circular cross-section, $Z_p = \pi D^3 / 16$

For a hollow circular cross-section, $Z_p = \pi (D_o^4 - D_i^4) / (16D_o)$

Then, $\tau_{max} = T / Z_p$

If design shears stress, τ_d is known, required polar section modulus can be calculated from:

$Z_p = T / \tau_d$

Polar Moment of Inertia and Section Modulus.

The polar moment of inertia, J, of a cross-section with respect to a polar axis, that is, an axis at right angles to the plane of the cross-section, is defined as the moment of inertia of the cross-section with respect to the point of intersection of the axis and the plane. The polar moment of inertia may be found by taking the sum of the moments of inertia about two perpendicular axes lying in the plane of the cross-section and passing through this point. Thus, for example, the polar moment of inertia of a circular or a square area with respect to a polar axis through the center of gravity is equal to two times the moment of inertia with respect to an axis lying in the plane of the cross-section and passing through the center of gravity. The polar moment of inertia with respect to a polar axis through the center of gravity is required for problems involving the torsional strength of shafts since this axis is usually the axis about which twisting of the shaft takes place.

The polar section modulus

(also called section modulus of torsion), Zp, for circular sections may be found by dividing the polar moment of inertia, J, by the distance c from the center of gravity to the most remote fiber. This method may be used to find the approximate value of the polar section modulus of sections that are nearly round. For other than circular cross-sections, however, the polar section modulus does not equal the polar moment of inertia divided by the distance c.

Power Transmission

$$P \text{ (in Watt)} = \frac{2\pi NT}{60}$$

$$P \text{ (in hp)} = \frac{2\pi NT}{4500} \qquad \text{(1 hp = 75 Kgm/sec).}$$

[Where N = rpm; T = Torque in N-m.]

Safe diameter of a shaft (d)

- Stiffness consideration

$$\frac{T}{J} = \frac{G\theta}{L}$$

- Shear Stress consideration

$$\frac{T}{J} = \frac{\tau}{R}$$

We take higher value of diameter of both cases above for overall safety if other parameters are given.

In Twisting

- Solid shaft, $\tau_{max} = \frac{16T}{\pi d^3}$
- Hollow shaft, $\tau_{max} = \frac{16Td_o}{\pi(d_o^4 - d_i^4)}$
- Diameter of a shaft to have a maximum deflection "α" $\quad d = 4.9 \times \sqrt[4]{\frac{TL}{G\alpha}}$

  [Where T in N-mm, L in mm, G in N/mm²]

Problems on Solid and hollow circular section

1. What torque, applied to a hollow circular shaft of 25 cm outside diameter and 17.5 cm inside
   diameter will produce a maximum shearing stress of 75 MN/m2 in the material.

We have

$$r_1 = 12.5 \text{ cm}, \qquad r_2 = 8.75 \text{ cm}$$

Then

$$J = \frac{\pi}{2}\left[(0.125)^4 - (0.0875)^4\right] = 0.292 \times 10^{-3} \text{ m}^4$$

If the shearing stress is limited to 75 MN/m², the torque is

$$T = \frac{J\tau}{r_1} = \frac{(0.292 \times 10^{-3})(75 \times 10^6)}{(0.125)} = 175.5 \text{ kNm}$$

2. A ship's propeller shaft has external and internal diameters of 25 cm and 15 cm. What power can be
   transmitted at 1 10 rev/minute with a maximum shearing stress of 75 MN/m2, and what will then
   be the twist in degrees of a 10 m length of the shaft? G = 80 GN/m2

$$r_1 = 0.125 \text{ m}, \qquad r_2 = 0.075 \text{ m}, \qquad l = 10 \text{ m}$$

$$J = \frac{\pi}{2}\left[(0.125)^4 - (0.075)^4\right] = 0.335 \times 10^{-3} \text{ m}^4$$

and

$$\tau = 75 \text{ MN/m}^2$$

Then

$$T = \frac{J\tau}{r_1} = \frac{(0.335 \times 10^{-3})(75 \times 10^6)}{0.125} = 201 \text{ kNm}$$

At 110 rev/min the power generated is

$$(201 \times 10^3)\left(2\pi \times \frac{110}{60}\right) = 2.32 \times 10^6 \text{ Nm/s}$$

The angle of twist is

$$\theta = \frac{TL}{GJ} = \frac{(201 \times 10^3)(10)}{(80 \times 10^9)(0.335 \times 10^{-3})} = 0.075 \text{ radians} = 4.3°$$

3. A solid circular shaft of 25 cm diameter is to be replaced by a hollow shaft, the ratio of the external to internal diameters being 2 to 1. Find the size of the hollow shaft if the maximum shearing stress is to be the same as for the solid shaft. What percentage economy in mass will this change effect?

Let r be the inside radius of the new shaft; then $= 2r$ the outside radius of the new shaft

$$J \text{ for the new shaft} = \frac{\pi}{2}(16r^4 - r^4) = 7.5\pi r^4$$

$$J \text{ for the old shaft} = \frac{\pi}{2} \times (0.125)^4 = 0.384 \times 10^{-3} \text{ m}^4$$

If T is the applied torque, the maximum shearing stress for the old shaft is

$$\frac{T(0.125)}{0.384 \times 10^{-3}}$$

and that for the new one is

$$\frac{T(2r)}{7.5\pi r^4}$$

If these are equal,

$$\frac{T(0.125)}{0.384 \times 10^{-3}} = \frac{T(2r)}{7.5\pi r^4}$$

Then

$$r^3 = 0.261 \times 10^{-3} \text{ m}^3$$

or $$r = 0.640 \text{ m}$$

Hence the internal diameter will be 0.128 m and the external diameter 0.256 m.

$$\frac{\text{area of new cross-section}}{\text{area of old cross-section}} = \frac{(0.128)^2 - (0.064)^2}{(0.125)^2} = 0.785$$

Thus, the saving in mass is about 21%.

4. A ship's propeller shaft transmits 7.5 x 106 W at 240 rev/min. The shaft has an internal diameter of 15 cm. Calculate the minimum permissible external diameter if the shearing stress in the shaft is to be limited to 150 MN/m2.

If T is the torque on the shaft, then

$$T\left(\frac{2\pi \times 240}{60}\right) = 7.5 \times 10^6$$

Thus

$$T = 298 \text{ kNm}$$

If d_1 is the outside diameter of the shaft, then

$$J = \frac{\pi}{32}\left(d_1^4 - 0.150^4\right) \text{ m}^4$$

If the shearing stress is limited to 150 MN/m^2, then

$$\frac{Td_1}{2J} = 150 \times 10^6$$

Thus,

$$Td_1 = \left(300 \times 10^6\right)J$$

On substituting for J and T

$$(298 \times 10^3)d_1 = (300 \times 10^6)\left(\frac{\pi}{32}\right)(d_1^4 - 0.150^4)$$

This gives

$$\left(\frac{d_1}{0.150}\right)^4 - 3\left(\frac{d_1}{0.150}\right) - 1 = 0$$

On solving this by trial-and-error, we get

$$d_1 = 1.54(0.150) = 0.231 \text{ m}$$

or $\quad d_1 = 23.1$ cm

Problems for practice

1. A solid steel bar of circular cross section has diameter d =1.5 in., length L =54 in., and shear modulus of elasticity G = 11.5 x10^6 psi. The bar is subjected to torques T acting at the ends.
(a) If the torques has magnitude T =250 lb-ft, what is the maximum shear stress in the bar? What is the angle of twist between the ends?
(b) If the allowable shear stress is 6000 psi and the allowable angle of twist is 2.5°, what is the maximum permissible torque?

2. A steel shaft is to be manufactured either as a solid circular bar or as a circular tube. The shaft is required to transmit a torque of 1200 N_m without exceeding an allowable shear stress of 40 MPa nor an allowable rate of twist of 0.75°/m. (The shear modulus of elasticity of the steel is 78 GPa.)
(a) Determine the required diameter d0 of the solid shaft.
(b) Determine the required outer diameter d2 of the hollow shaft if the thickness t of the shaft is specified as one-tenth of the outer diameter.
(c) Determine the ratio of diameters (that is, the ratio d2/d0) and the ratio of weights of the hollow and solid shafts.

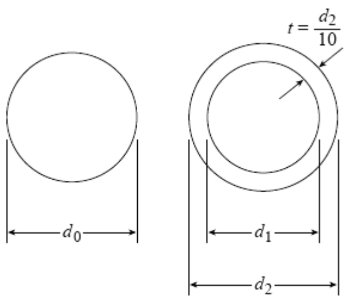


Fig. 4.4 Solid and Hollow shaft

3. A hollow shaft and a solid shaft constructed of the same material have the same length and the same outer radius R . The inner radius of the hollow shaft is 0.6R. (a) Assuming that both shafts are subjected to the same torque, compare their shear stresses, angles of twist, and weights. (b) Determine the strength-to-weight ratios for both shafts.

Stepped shafts

When a shaft is made of different lengths and of different diameters, it is termed as shaft as varying cross section. For such a shaft, the torque induced in its individual sections should be calculated first. The strength of the shaft is the minimum of all these torques.

Problems

A stepped shaft has the appearance as shown in figure. The region AB is aluminum, having G = 28 GPa, and the region BC is steel, having G = 84 GPa. The aluminum portion is of solid circular cross section 45 mm in diameter, and the steel region is circular with 60-mm outside diameter and 30-mm inside diameter. Determine the maximum shearing stress in each material as well as the angle of twist at B where a torsional load of 4000 N - m is applied. Ends A and C are rigidly clamped.

SOLUTION: The free-body diagram of the system is shown. The applied load of 4000 N-m as well as the unknown end reactive torques are as indicated. The only equation of static equilibrium is

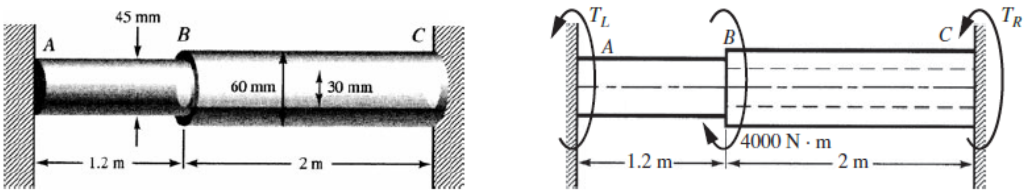


Fig. 4.5 Stepped shaft subjected to Torque

$$\sum M_x = T_L + T_R - 4000 = 0$$

Since there are two unknowns TL and TR, another equation (based upon deformations) is required. This is set up by realizing that the angular rotation at B is the same if we determine it at the right end of AB or the left end of BC. We thus have

$$\frac{T_L \times 1.2}{(28\times 10^9)\pi \times 0.045^4/32} = \frac{T_R \times 2.0}{(84\times 10^9)\pi(0.06^4 - 0.03^4)/32} \quad \text{or} \quad T_L = 0.1875\, T_R$$

Solving for T_L and T_R, we find

$$T_L = 632 \text{ N} \cdot \text{m} \quad \text{and} \quad T_R = 3368 \text{ N} \cdot \text{m}$$

The maximum shearing stress in AB is given by

$$\tau_{AB} = \frac{T\rho}{J} = \frac{(632)(0.0225)}{\pi(0.045)^4/32} = 35.6 \text{ MPa}$$

and in BC by

$$\tau_{BC} = \frac{T\rho}{J} = \frac{(3370)(0.030)}{\pi(0.06^4 - 0.03^4)/32} = 85.0 \text{ MPa}$$

The angle of twist at B, using parameters of the region AB, is

$$\theta_B = \frac{TL}{GJ} = \frac{(632)(1.2)}{(28\times 10^9)(\pi \times 0.045^4/32)} = 0.0673 \text{ rad} \quad \text{or} \quad 3.86°$$

Problems for practice

A circular cross-section steel shaft is of diameter 50 mm over the left 150 mm of length and of diameter 100 mm over the right 150 mm, as shown in Fig. 5-21. Each end of the shaft is loaded by a twisting moment of 1000 N · m (as indicated by the double-headed arrows). If $G = 80$ GPa, determine the angle of twist between the ends of the shaft as well as the peak shearing stress. *Ans.* 1.09°, 40.7 MPa

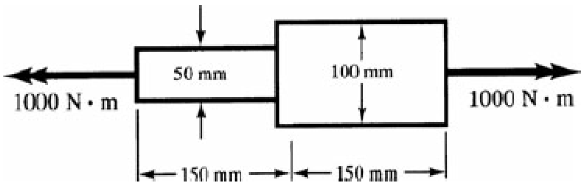


Fig. 4.6 Stepped shaft

A compound shaft is composed of a 70-cm length of solid copper 10 cm in diameter, joined to 90-cm length of solid steel 12 cm in diameter. A torque of 14 kN · m is applied to each end of the shaft. Find the maximum shear stress in each material and the total angle of twist of the entire shaft. For copper $G = 40$ GPa, for steel $G = 80$ GPa. *Ans.* In the copper, 71.3 MPa; in the steel, 41.3 MPa; $\theta = 0.0328$

Compound shafts – fixed and simply supported shafts

A compound shaft is made of two or more different materials joined together in such a way that the shaft is elongated or compressed as a single shaft. The total torque transmitted by a compound shaft is the sum of the torques transmitted by each individual shaft and the angle of twist in each shaft will be equal.

1. A compound shaft consisting of a steel segment and an aluminum segment is acted upon by two torques as shown. Determine the maximum permissible value of T subject to the following conditions: $\tau_{st} = 83$ MPa, $\tau_{al} = 55$ MPa, and the angle of rotation of the free end is limited to 6°. For steel, G = 83 GPa and for aluminum, G = 28 GPa.

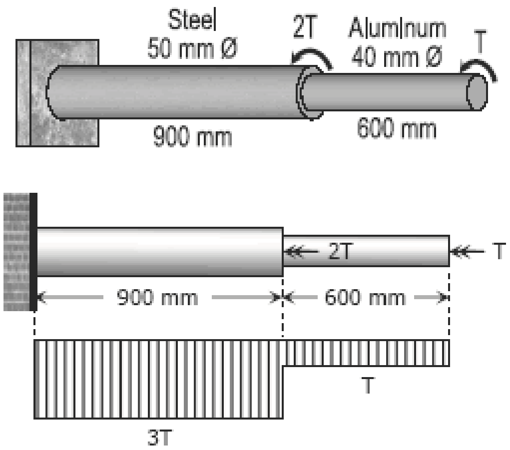


Fig. 4.7 Stepped shaft with Torque applied

Based on maximum shearing stress $\tau_{max} = 16T / \pi d^3$:

$$\tau_{st} = \frac{16(3T)}{\pi(50^3)} = 83$$

$T = 679\,042.16$ N·mm

$T = 679.04$ N·m

$$\tau_{al} = \frac{16T}{\pi(40^3)} = 55$$

$T = 691\,150.38$ N·mm

$T = 691.15$ N·m

Based on maximum angle of twist:

$$\theta = \left(\frac{TL}{JG}\right)_{st} + \left(\frac{TL}{JG}\right)_{al}$$

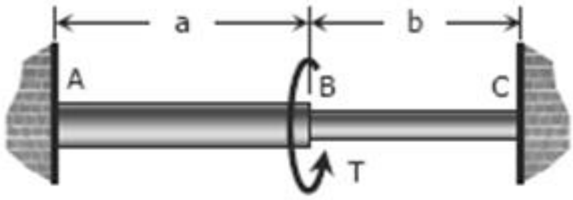
$$6^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{3T(900)}{\frac{1}{32}\pi(50^4)(83\,000)} + \frac{T(600)}{\frac{1}{32}\pi(40^4)(28\,000)}$$

$T = 757\,316.32$ N·mm

$T = 757.32$ N·m

Use $T =$ **679.04 N·m**

2. The compound shaft shown is attached to rigid supports. For the bronze segment AB, the diameter is 75 mm, $\tau \le 60$ MPa, and G = 35 GPa. For the steel segment BC, the diameter is 50 mm, $\tau \le 80$ MPa, and G = 83 GPa. If a = 2 m and b = 1.5 m, compute the maximum torque T that can be applied.



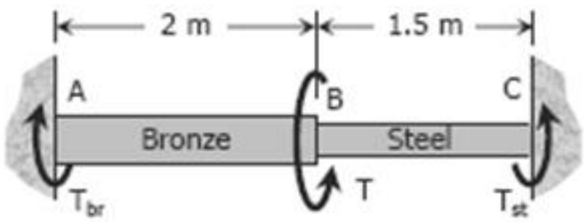


Fig. 4.8 Stepped shaft with Torque

$$\Sigma M = 0$$

$$T = T_{br} + T_{st} \quad \rightarrow \text{Equation (1)}$$

$$\theta_{br} = \theta_{st}$$

$$\left(\frac{TL}{JG}\right)_{br} = \left(\frac{TL}{JG}\right)_{st}$$

$$\frac{T_{br}(2)(1000)}{\frac{1}{32}\pi(75^4)(35000)} = \frac{T_{st}(1.5)(1000)}{\frac{1}{32}\pi(50^4)(83000)}$$

$$\left.\begin{aligned} T_{br} &= 1.6011T_{st} \\ T_{st} &= 0.6246T_{br} \end{aligned}\right\} \text{Equations (2)}$$

$$\tau_{max} = \frac{16T}{\pi D^3}$$

Based on $\tau_{br} \leq 60$ MPa

$$60 = \frac{16T_{br}}{\pi(75^3)}$$

$T_{br} = 4\,970\,097.75$ N·mm

$T_{br} = 4.970$ kN·m → Maximum allowable torque for bronze

$T_{st} = 0.6246(4.970)$ → From one of Equations (2)

$T_{st} = 3.104$ kN·m

Based on $\tau_{st} \leq 80$ MPa

$$80 = \frac{16T_{st}}{\pi(50^3)}$$

$T_{st} = 1\,963\,495.41$ N·mm

$T_{st} = 1.963$ kN·m → maximum allowable torque for steel

$T_{br} = 1.6011(1.963)$ → From Equations (2)

$T_{br} = 3.142$ kN·m

Use $T_{br} = 3.142$ kN·m and $T_{st} = 1.963$ kN·m

$T = 3.142 + 1.963$ → From Equation (1)

$T = 5.105$ kN·m

3. The compound shaft shown is attached to rigid supports. For the bronze segment AB, the maximum shearing stress is limited to 8000 psi and for the steel segment BC, it is limited to 12 ksi. Determine the diameters of each segment so that each material will be simultaneously stressed to its permissible limit when a torque T = 12 kip·ft is applied. For bronze, G = 6 × 106 psi and for steel, G = 12 × 106 psi.

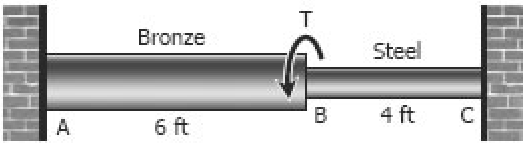


Fig. 4.9 Shaft with Torque applied

$$\tau_{max} = \frac{16T}{\pi D^3}$$

For bronze:

$$8000 = \frac{16T_{br}}{\pi D_{br}^3}$$

$$T_{br} = 500\pi\, D_{br}^3 \text{ lb·in}$$

For steel:

$$12\ 000 = \frac{16T_{st}}{\pi D_{st}^3}$$

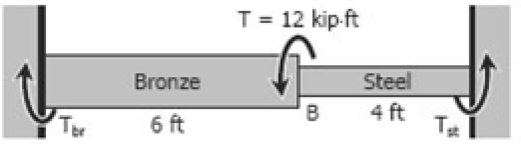


Fig. 4.10 Stepped shaft

$$T_{st} = 750\pi\, D_{st}^3 \text{ lb·in}$$

$$\Sigma M = 0$$

$$T_{br} + T_{st} = T$$

$$T_{br} + T_{st} = 12(1000)(12)$$

$$T_{br} + T_{st} = 144\ 000 \text{ lb·in}$$

$$500\pi\, D_{br}^3 + 750\pi\, D_{st}^3 = 144\ 000$$

$$D_{br}^3 = 288/\pi - 1.5\, D_{st}^3 \quad \rightarrow \text{equation (1)}$$

$$\theta_{br} = \theta_{st}$$

$$\left(\frac{TL}{JG}\right)_{br} = \left(\frac{TL}{JG}\right)_{st}$$

$$\frac{T_{br}(6)}{\frac{1}{32}\pi D_{br}^4 (6\times10^6)} = \frac{T_{st}(4)}{\frac{1}{32}\pi D_{st}^4 (12\times10^6)}$$

$$\frac{T_{br}}{D_{br}^4} = \frac{T_{st}}{3D_{st}^4}$$

$$\frac{500\pi D_{br}^3}{D_{br}^4} = \frac{750\pi D_{st}^3}{3D_{st}^4}$$

$$D_{st} = 0.5D_{br}$$

From Equation (1)

$$D_{br}^3 = 288/\pi - 1.5(0.5D_{br})^3$$

$$1.1875\,D_{br}^3 = 288/\pi$$

D_{br} = **4.26 in.**

D_{st} = 0.5(4.26) = **2.13 in.**

4. A shaft composed of segments AC, CD, and DB is fastened to rigid supports and loaded as shown. For bronze, G = 35 GPa; aluminum, G = 28 GPa, and for steel, G = 83 GPa. Determine the maximum shearing stress developed in each segment.

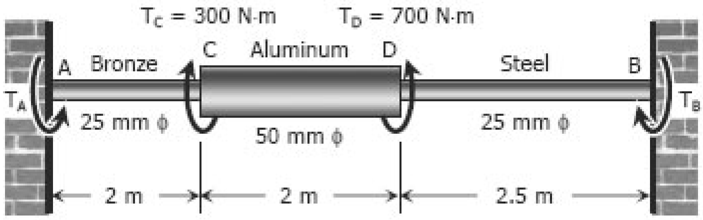


Fig. 4.11 Stress developed in each segment with respect to T_A

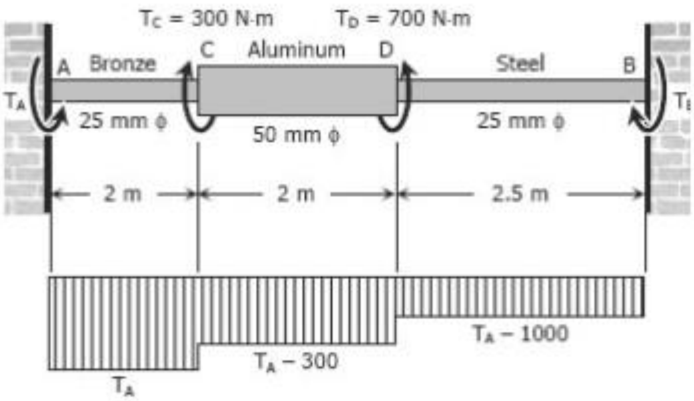


Fig. 4.12 Stress developed in each segment

The rotation of B relative to A is zero.

$\theta_{A/B} = 0$

$$\left(\sum \frac{TL}{JG}\right)_{A/B} = 0$$

$$\frac{T_A(2)(1000^2)}{\frac{1}{32}\pi(25^4)(35000)} + \frac{(T_A - 300)(2)(1000^2)}{\frac{1}{32}\pi(50^4)(28000)} + \frac{(T_A - 1000)(2.5)(1000^2)}{\frac{1}{32}\pi(25^4)(83000)} = 0$$

$$\frac{2T_A}{(25^4)(35)} + \frac{2(T_A - 300)}{(50^4)(28)} + \frac{2.5(T_A - 1000)}{(25^4)(83)} = 0$$

$$\frac{16T_A}{35} + \frac{T_A - 300}{28} + \frac{20(T_A - 1000)}{83} = 0$$

$$\tfrac{16}{35}T_A + \tfrac{1}{28}T_A - \tfrac{75}{7} + \tfrac{20}{83}T_A - \tfrac{20000}{83} = 0$$

$$\tfrac{8527}{11620}T_A = 251.678$$

$$T_A = 342.97 \text{ N·m}$$

$$\Sigma M = 0$$
$$T_A + T_B = 300 + 700$$
$$342.97 + T_B = 1000$$
$$T_B = 657.03 \text{ N·m}$$

$$\tau_{max} = \frac{16T}{\pi D^3}$$

$$\tau_{br} = \frac{16(342.97)(1000)}{\pi(25^3)} = 111.79 \text{ MPa}$$

$$\tau_{al} = \frac{16(42.97)(1000)}{\pi(50^3)} = 1.75 \text{ MPa}$$

$$\tau_{st} = \frac{16(657.03)(1000)}{\pi(25^3)} = 214.16 \text{ MPa}$$

5. A hollow bronze shaft of 3 in. outer diameter and 2 in. inner diameter is slipped over a solid steel shaft 2 in. in diameter and of the same length as the hollow shaft. The two shafts are then fastened rigidly together at their ends. For bronze, G = 6 × 106 psi, and for steel, G = 12 × 106 psi. What torque can be applied to the composite shaft without exceeding a shearing stress of 8000 psi in the bronze or 12 ksi in the steel?

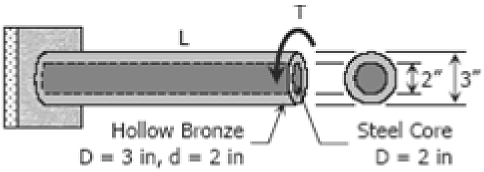


Fig. 4.13 Composite shaft

$$\theta_{st} = \theta_{br}$$

$$\left(\frac{TL}{JG}\right)_{st} = \left(\frac{TL}{JG}\right)_{br}$$

$$\frac{T_{st}L}{\frac{1}{32}\pi(2^4)(12\times10^6)} = \frac{T_{br}L}{\frac{1}{32}\pi(3^4-2^4)(6\times10^6)}$$

$$\frac{T_{st}}{192\times10^6} = \frac{T_{br}}{390\times10^6} \quad \rightarrow \text{Equation (1)}$$

Applied Torque = Resisting Torque

$$T = T_{st} + T_{br} \quad \rightarrow \text{Equation (2)}$$

Equation (1) with T_{st} in terms of T_{br} and Equation (2)

$$T = \frac{192\times10^6}{390\times10^6}T_{br} + T_{br}$$

$$T_{br} = 0.6701T$$

Equation (1) with T_{br} in terms of T_{st} and Equation (2)

$$T = T_{st} + \frac{390\times10^6}{192\times10^6}T_{st}$$

$$T_{st} = 0.3299T$$

Based on hollow bronze ($T_{br} = 0.6701T$)

$$\tau_{max} = \left[\frac{16TD}{\pi(D^4 - d^4)}\right]_{br}$$

$$8000 = \frac{16(0.6701T)(3)}{\pi(3^4 - 2^4)}$$

$$T = 50\,789.32 \text{ lb·in}$$

$$T = 4232.44 \text{ lb·ft}$$

Based on steel core ($T_{st} = 0.3299T$):

$$\tau_{max} = \left[\frac{16T}{\pi D^3}\right]_{st}$$

$$12\,000 = \frac{16(0.3299T)}{\pi(2^3)}$$

$$T = 57\,137.18 \text{ lb·in}$$

$$T = 4761.43 \text{ lb·ft}$$

Use T = **4232.44 lb·ft**

6. The two steel shaft shown in Fig. P-325, each with one end built into a rigid support have flanges rigidly attached to their free ends. The shafts are to be bolted together at their flanges. However, initially there is a 6° mismatch in the location of the bolt holes as shown in the figure. Determine the maximum shearing stress in each shaft after the shafts are bolted together. Use G = 12 × 106 psi and neglect deformations of the bolts and flanges.

$$\theta_{\text{of 6.5' shaft}} + \theta_{\text{of 3.25' shaft}} = 6°$$

$$\left(\frac{TL}{JG}\right)_{\text{of 6.5' shaft}} + \left(\frac{TL}{JG}\right)_{\text{of 3.25' shaft}} = 6°\left(\frac{\pi}{180°}\right)$$

$$\frac{T(6.5)(12^2)}{\frac{1}{32}\pi(2^4)(12\times10^6)} + \frac{T(3.25)(12^2)}{\frac{1}{32}\pi(1.5^4)(12\times10^6)} = \frac{\pi}{30}$$

$T = 817.32$ lb·ft

$$\tau_{max} = \frac{16T}{\pi D^3}$$

$$\tau_{\text{of 6.5' shaft}} = \frac{16(817.32)(12)}{\pi(2^3)} = \mathbf{6243.86\ psi}$$

$$\tau_{\text{of 3.25' shaft}} = \frac{16(817.32)(12)}{\pi(1.5^3)} = \mathbf{14\,800.27\ psi}$$

Closed Coiled helical springs subjected to axial loads:

Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released. Also Springs are energy absorbing units whose function is to store energy and to restore it slowly or rapidly depending on the particular application.

Important types of springs are:

There are various types of springs such as

(i) helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is Torsional shear stress due to twisting. They are both used in tension and compression.

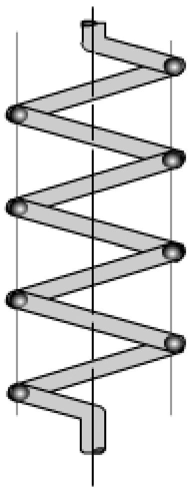


Fig.4.14 Helical Spring

(ii) Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.
In this the major stresses are tensile and compression due to bending.

(iii) Leaf springs: They are composed of flat bars of varying lengths clamped together so as to obtain greater efficiency. Leaf springs may be full elliptic, semi elliptic or cantilever types, In these type of springs the major stresses which come into picture are tensile & compressive.

Uses of springs:
(a) To apply forces and to control motions as in brakes and clutches.
(b) To measure forces as in spring balance.
(c) To store energy as in clock springs.
(d) To reduce the effect of shock or impact loading as in carriage springs.
(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

Derivation of the Formula :
In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load W.

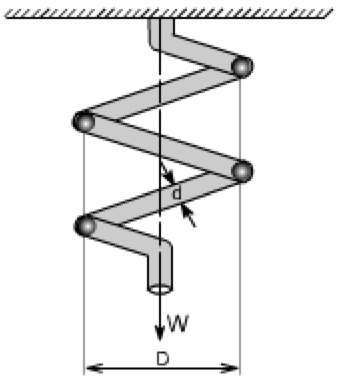


Fig.4.15 Helical Spring major notation

Let
W = axial load
D = mean coil diameter
d = diameter of spring wire
n = number of active coils
C = spring index = D / d For circular wires
l = length of spring wire
G = modulus of rigidity
x = deflection of spring
q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that

x = D / 2 . q
again l = p D n [consider ,one half turn of a close coiled helical spring]

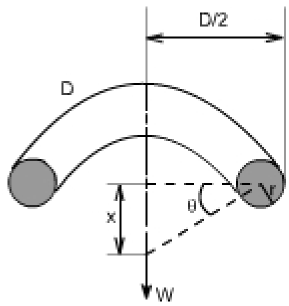


Fig.4.16 Helical Spring wire

Assumptions: (1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a spring will be assumed to lie in a plane which is nearly perpendicular to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical. Hence to

maintain equilibrium of a segment of the spring, only a shearing force V = F and Torque T = F. r are required at any X – section. In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is
uniformly distributed and is negligible
so applying the torsion formula. Using the torsion formula i.e

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

$$\text{and substitituting } J = \frac{\pi d^4}{32}; T = w.\frac{d}{2}$$

$$\theta = \frac{2.x}{D}; l = \pi D.x$$

SPRING DEFLECTION

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{G.2x/D}{\pi D.n}$$

Thus,

$$x = \frac{8w.D^3.n}{G.d^4}$$

Spring striffness: The stiffness is defined as the load per unit deflection therefore

$$k = \frac{w}{x} = \frac{w}{\frac{8w.D^3.n}{G.d^4}}$$

Therefore

$$k = \frac{G.d^4}{8.D^3.n}$$

Shear stress

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{\tau_{max^m}}{d/2}$$

$$\text{or } \tau_{max^m} = \frac{8wD}{\pi d^3}$$

WAHL'S FACTOR :

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

K = Wahl' s factor and is defined as

$$K = \frac{4c-1}{4c-4} + \frac{0.615}{c}$$

Where C = spring index
= D/d

if we take into account the Wahl's factor than the formula for the shear stress becomes

$$\tau_{max^m} = \frac{16.T.k}{\pi d^3}$$

Strain Energy : The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion

$$U = \frac{T^2 L}{2EI}$$

$$L = \pi D n$$

$$I = \frac{\pi d^4}{64}$$

so after substitution we get

$$U = \frac{32T^2 Dn}{E.d^4}$$

Worked examples:

1. A close coiled helical spring is to carry a load of 5000N with a deflection of 50 mm and a maximum shearing stress of 400 N/mm2 if the number of active turns or active coils is 8.Estimate the following:

(i) wire diameter
(ii) mean coil diameter
(iii) weight of the spring.
Assume G = 83,000 N/mm2 ; r = 7700 kg/m3

solution :

(i) for wire diametre if W is the axial load, then

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{\tau_{max^m}}{d/2}$$

$$D = \frac{400}{d/2}.\frac{\pi d^4}{32}.\frac{2}{W}$$

$$D = \frac{400.\pi d^3.2}{5000.16}$$

$$D = 0.0314\, d^3$$

Further, deflection is given as

$$x = \frac{8wD^3.n}{G.d^4}$$

on substituting the relevant parameters we get

$$50 = \frac{8.5000.(0.0314d^3)^3.8}{83{,}000.d^4}$$

$$d = 13.32mm$$

Therefore,
D = .0314 x (13.317)3mm
=74.15mm
D = 74.15 mm

2. Determine the maximum shearing stress and elongation in a helical steel spring composed of 20 turns of 20-mm-diameter wire on a mean radius of 90 mm when the spring is supporting a load of 1.5 kN. G = 83 GPa.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(\frac{4m-1}{4m-4} + \frac{0.615}{m}\right) \quad \rightarrow$$

Where P = 1.5 kN = 1500 N; R = 90 mm
d = 20 mm; n = 20 turns
m = 2R/d = 2(90)/20 = 9

$$\tau_{max} = \frac{16(1500)(90)}{\pi(20^3)}\left[\frac{4(9)-1}{4(9)-4} + \frac{0.615}{9}\right]$$

$$\tau_{max} = \mathbf{99.87\ MPa}$$

$$\delta = \frac{64PR^3n}{Gd^4} = \frac{64(1500)(90^3)(20)}{83\,000(20^4)}$$

$$\delta = \mathbf{105.4\ mm}$$

3. Determine the maximum shearing stress and elongation in a bronze helical spring composed of 20 turns of 1.0-in.-diameter wire on a mean radius of 4 in. when the spring is supporting a load of 500 lb. G = 6 × 106 psi.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(\frac{4m-1}{4m-4} + \frac{0.615}{m}\right)$$

Where P = 500 lb; R = 4 in
d = 1 in; n = 20 turns
m = 2R/d = 2(4)/1 = 8

$$\tau_{max} = \frac{16(500)(4)}{\pi(1^3)}\left[\frac{4(8)-1}{4(8)-4} + \frac{0.615}{8}\right] \qquad \delta = \frac{64PR^3n}{Gd^4} = \frac{64(500)(4^3)(20)}{(6\times10^6)(1^4)}$$

$$\tau_{max} = 12\,060.3\text{ psi} = \mathbf{12.1\ ksi} \qquad \delta = \mathbf{6.83\ in}$$

4. A helical spring is fabricated by wrapping wire ¾ in. in diameter around a forming cylinder 8 in. in diameter. Compute the number of turns required to permit an elongation of 4 in. without exceeding a shearing stress of 18 ksi. G = 12 × 106 psi.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(1+\frac{d}{4R}\right)$$

$$18000 = \frac{16P(4)}{\pi(3/4)^3}\left[1+\frac{3/4}{4(4)}\right]$$

$$P = 356.07 \text{ lb}$$

$$\delta = \frac{64PR^3n}{Gd^4}$$

$$4 = \frac{64(356.07)(4^3)n}{(12\times10^6)(3/4)^3}$$

$n = 13.88$ **say 14 turns**

Weight

mass or weight = volume. density

= area.length of the spring.density of spring material

$$= \frac{\pi d^2}{4}.\pi Dn.\rho$$

On substituting the relevant parameters we get

Weight = 1.996 kg

= 2.0kg

Close – coiled helical spring subjected to axial torque T or axial couple.

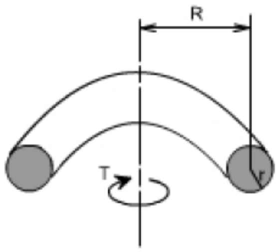


Fig.4.17 Helical Spring wire under Torque 'T'

In this case the material of the spring is subjected to pure bending which tends to reduce Radius R of the coils. In this case the bending moment is constant through out the spring and is equal to the applied axial Torque T. The stresses i.e. maximum bending stress may

$$\sigma_{max} = \frac{M.y}{I} = \frac{T.d/2}{\frac{\pi d^4}{64}}$$

$$\sigma_{max} = \frac{32T}{\pi d^3}$$

thus be determined from the bending theory.

Springs in Series: If two springs of different stiffness are joined endon and carry a common load W, they are said to be connected in series and the combined stiffness and deflection are given by the following equation

.

$$\frac{W}{k} = x_1 + x_2 = \frac{W}{k_1} + \frac{W}{k_2}$$

or

$$\boxed{\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}}$$

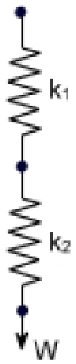


Fig.4.18 Springs in Series

Springs in parallel: If the two spring are joined in such a way that they have a common deflection ‘x' ; then they are said to be connected in parallel. In this care the load carried is shared between the two springs and total load W = W1 + W2

$$x = \frac{W}{k} = \frac{W_1}{k_1} = \frac{W_2}{k_2}$$

Thus $W_1 = \frac{Wk_1}{k}$

$W_2 = \frac{Wk_2}{k}$

Futher

$W=W_1+W_2$

thus $\boxed{k = k_1 + k_2}$

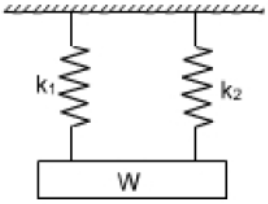
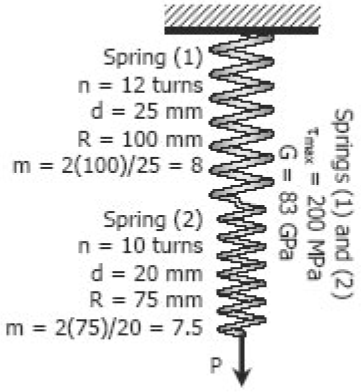


Fig.4.19 Springs in Parallel

1. Two steel springs arranged in series as shown supports a load P. The upper spring has 12 turns of 25-mm-diameter wire on a mean radius of 100 mm. The lower spring consists of 10 turns of 20-mmdiameter wire on a mean radius of 75 mm. If the maximum shearing stress in either spring must not exceed 200 MPa, compute the maximum value of P and the total elongation of the assembly. G = 83 GPa. Compute the equivalent spring constant by dividing the load by the total elongation.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(\frac{4m-1}{4m-4} + \frac{0.615}{m}\right)$$



For Spring (1)

$$200 = \frac{16P(100)}{\pi(25^3)}\left[\frac{4(8)-1}{4(8)-4} + \frac{0.615}{8}\right]$$

$$P = 5182.29 \text{ N}$$

For Spring (2)

$$200 = \frac{16P(75)}{\pi(20^3)}\left[\frac{4(7.5)-1}{4(7.5)-4} + \frac{0.615}{7.5}\right]$$

$$P = 3498.28 \text{ N}$$

Use P = **3498.28 N**

Fig.4.20 Helical Spring in Series

Total elongation:

$$\delta = \delta_1 + \delta_2$$

$$\delta = \left(\frac{64PR^3n}{Gd^4}\right)_1 + \left(\frac{64PR^3n}{Gd^4}\right)_2$$

$$\delta = \frac{64(3498.28)(100^3)12}{83\,000(25^4)} + \frac{64(3498.28)(75^3)(10)}{83\,000(20^4)}$$

δ = **153.99 mm**

Equivalent spring constant, $k_{equivalent}$:

$$k_{equivalent} = \frac{P}{\delta} = \frac{3498.28}{153.99}$$

$k_{equivalent}$ = **22.72 N/mm**

Design of helical coil springs – stresses in helical coil springs under torsion loads

Worked problems

Design a close-coiled helical compression spring with a following data :

- Service load range. = 2250N to 2750N
- Axial deflection of spring for load range = 6mm
- Spring index = 5
- Permissible shear stress for spring = 420N/mm^2
- Modulus of rigidity for spring material = 84 KN/mm^2
- Neglect the effect of stress concentration. Draw a dimensioned sketch of the spring

Given : $F_{min} = 2250N$; $F_{max} = 2750N$;

$\delta = 6$ mm ; $C = 5$;

$\tau = 420$ N/mm^2 ; $G = 84 \times 10^3$ N/mm^2.

- **Wire diameter :**
- Neglecting effect of stress concentration,

$$K_s = \left[1+\frac{0.5}{C}\right] = \left[1+\frac{0.5}{5}\right] = 1.1$$

$$\text{Now,} \quad \tau = K_s\left[\frac{8F_{max}C}{\pi d^2}\right]$$

$$\therefore \quad 420 = 1.1 \times \left[\frac{8\times 2750 \times 5}{\pi\, d^2}\right]$$

$$\therefore \quad d = 9.58 \text{ mm or } 9.6 \text{ mm}$$

$$\mathbf{d = 9.6\ mm}$$

- **Mean coil diameter :**

$$D = C \cdot d = 5 \times 9.6$$

$$\text{or} \quad \mathbf{D = 48\ mm}$$

- **Number of coils :**

Spring stiffness,

$$K = \frac{F_{max} - F_{min}}{\delta}$$

$$= \frac{2750 - 2250}{6}$$

or,

$$K = 83.33 \text{ N/mm}$$

Now,

$$K = \frac{Gd}{8C^3 n}$$

$$\therefore \quad 83.33 = \frac{84 \times 10^3 \times 9.6}{8 \times 5^3 \times n}$$

$$\therefore \quad n = 9.68 \text{ or } 9.7 \text{ turns}$$

$$\mathbf{n = 9.7}$$

Assuming square and ground ends,

$$n' = n + 2 = 9.7 + 2 = 11.7 \text{ turns}$$

$$\mathbf{n' = 11.7}$$

- **Solid length :**

$$L_s = (n + 2)\, d = (9.7 + 2) \times 9.6$$

or

$$\mathbf{L_s = 112.32\ mm}$$

- **Free length :**

maximum deflection,

$$\delta_{max} = \frac{F_{max}}{K} = \frac{2750}{83.33}$$

$$\therefore \quad \delta_{max} = 33 \text{ mm}$$

Free length, L_F = solid length + maximum deflection + total clearance

$$= L_s + \delta_{max} + 0.15\ \delta_{max}$$

- (Assume total clearance as 15 % of maximum deflection)

$$= 112.32 + 33 + 0.15 \times 33$$

or $\quad \mathbf{L_F = 150.27\ mm}$ **...Ans.**

- **Pitch of coil :**

Now, $\quad L_F = p\,n + d$

$$150.27 = p \times 9.7 + 9.6$$

$\therefore \quad \mathbf{p = 14.5\ mm}$ **...Ans.**

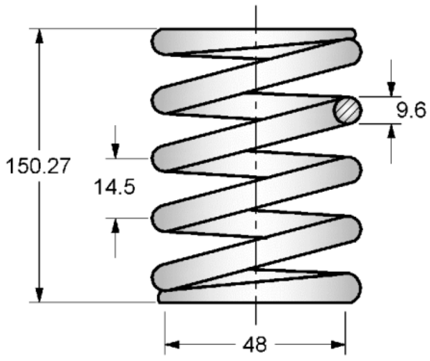


Fig.4.21 Helical Spring

The following data refers to a helical compression spring :

- Mean coil diameter = 125 mm
- Maximum axial load = 8000 N
- Spring rate = 72 kN/m
- Allowable shear stress for string = 275 N/mm^2
- Modulus of rigidity for spring material = $80 \times 10^3\ N/mm^2$

Determine :

(i) Wire diameter; and

(ii) Number of active turns.

Given : $D = 125$ mm ; $F_{max} = 8000$ N;
$K = 72$ kN/m = 72 N/mm ; $\tau = 275\ N/mm^2$;
$G = 80 \times 10^3\ N/mm^2$.

(i) Wire diameter :

- **Trial 1 :**

As spring index is not known, initially assuming $K_w = 1$,

$$\tau = K_w \left[\frac{8\ F_{max} D}{\pi d^3} \right]$$

$$275 = \frac{1 \times 8 \times 8000 \times 125}{\pi d^3}$$

$$\therefore \quad d = 21\ mm$$

- **Trial 2 :**

The initial value of wire diameter d = 21 mm is used to estimate C and K_w . Taking the new value of K_w, the wire diameter is determined as follows :

$$C = D/d = \frac{125}{21} = 5.95$$

$$K_w = \frac{4C-1}{4C-4} + \frac{0.615}{C} = \frac{4 \times 5.95 - 1}{4 \times 5.95 - 4} + \frac{0.615}{5.95}$$

or $$K_w = 1.255$$

$$\tau = K_w \left[\frac{8\, F_{max}\, D}{\pi d^3}\right]$$

$$275 = \frac{1.255 \times 8 \times 8000 \times 125}{\pi d^3}$$

$$\therefore \quad d = 22.65 \text{ mm or } 23 \text{ mm}$$

- **Check for shear stress induced in spring wire :**

$$C = D/d = \frac{125}{23} = 5.43$$

$$K_w = \frac{4C-1}{4C-4} + \frac{0.615}{C} = \frac{4 \times 5.43 - 1}{4 \times 5.43 - 4} + \frac{0.615}{5.43}$$

or $$K_w = 1.2826$$

$$\tau = K_{w\,G} \left[\frac{8\, F_{max}\, D}{\pi d^3}\right] = \frac{1.2826 \times 8 \times 8000 \times 125}{\pi \times (23)^3}$$

or $$\tau = 268.44 \text{ N/mm}^2 < 275 \text{ N/mm}^2$$

design is safe $$\mathbf{d = 23 \text{ mm}}$$

Hence, $$\mathbf{C = 5.43}$$

(ii) Number of active coils :

Now, $$K = \frac{Gd}{8\, C^3\, n}$$

$$72 = \frac{80 \times 10^3 \times 23}{8 \times (5.43)^3 \times n}$$

$$\therefore \quad n = 19.95 \text{ or } 20 \text{ turns}$$

$$\mathbf{n = 20}$$

Design a helical compression for a spring operated pressure relief valve with following data :

- Operating pressure = 1.25 N/mm^2
- Valve lift = 3.5 mm at 10% pressure rise over operating pressure
- Diameter of valve = 25 mm
- Limiting mean coil diameter = 40 mm

- Permissible shear stress for spring = 500 N/mm^2
- Modulus of rigidity for spring material = 834 Pa
- The available standard spring wire diameters are : 2, 3, 4, 5, 6, 7, 8 and 10 mm.

Given : $p_o = 1.25$ N/mm^2 ; $\delta_o = 3.5$ mm;

$p_{max} = 1.1\, p_o = 1.25 \times 1.1 = 1.375$ N/mm^2 ; $dv = 25$ mm;

$D = 40$ mm ; $\tau = 500$ N /mm^2;

$G = 83 \times 10^3$ N/mm^2.

- **Maximum spring force :**

Cross-sectional area of valve, $A_V = \pi dv^2/4 = \pi \times (25)^2/4$

or $A_V = 490.87 \text{ mm}^2$

The spring force at operating pressure,

$$F_o = p_o A_V = 1.25 \times 490.87 = 613.59 \text{ N}$$

The maximum spring force,

$$F_{max} = p_{max} A_V = 1.375 \times 490.87 = 674.95 \text{ N}$$

- **Wire diameter :**

$$\tau = \frac{K_w\, 8\, F_{max} C}{\pi d^2}$$

$$C = D/d$$

$$\therefore \quad d = D/C$$

Substituting value of **'d'** from Equation (b) in Equation (a),

$$\tau = \frac{K_w\, 8F_{max}\, C}{\pi\, (D/C)^2} = \frac{K_w\, 8\, F_{max}\, C^3}{\pi D^2}$$

$$500 = \frac{K_w \times 8 \times 674.95 \times C^3}{\pi \times (40)^2}$$

$$\therefore \quad K_w C^3 = 465.45$$

$$\left[\frac{4C-1}{4C-4} + \frac{0.615}{C}\right] C^3 = 465.45$$

Solving Equation (c) by trial and error, we get,

$$C = 7.3$$

$$\therefore \quad d = D/C = 40/7.3 = 5.48 \text{ mm or } 5.5 \text{ mm}$$

The next standard wire diameter selected is,

$$\mathbf{d = 6 \text{ mm}}$$

$$\therefore \quad C = D/d = 40/6 = 6.667$$

$$\mathbf{C = 6.667}$$

- **Number of coils :**

The spring stiffness is, $K = \dfrac{F_{max} - F_o}{\delta_o} = \dfrac{674.95 - 613.59}{3.5}$

or $K = 17.53 \text{ N/mm}$

Now, $K = \dfrac{Gd}{8C^3 n}$

$$17.53 = \frac{83 \times 10^3 \times 6}{8 \times (6.677)^3 \times n}$$

$$\therefore \quad n = 11.98 \text{ or } 12 \text{ turns}$$

$$\mathbf{n = 12}$$

Assuming square and ground ends,

$$n' = n + 2 = 12 + 2 = 14 \text{ turns}$$

$$\mathbf{n' = 14}$$

- **Solid length :**

$$\text{Solid length, } L_s = (n + 2)\, d = (12 + 2) \times 6 = 84 \text{ mm}$$

$$\text{or} \quad \mathbf{L_s = 84 \text{ mm}}$$

- **Free length :**

$$\text{Maximum deflection, } \delta_{max} = \frac{F_{max}}{K} = \frac{675.95}{17.53} = 38.5 \text{ mm}$$

Free length, L_F = solid length + maximum deflection + total clearance

(Assuming total clearance as 15% of δ_{max})

$$L_F = L_s + \delta_{max} + 0.15\, \delta_{max}$$
$$= 84 + 38.5 + 0.15 \times 38.5 = 128.275 \text{ mm}$$

$$\text{or} \quad \mathbf{L_F = 128.275 \text{ mm}}$$

- **Pitch of coil :**

$$\text{Now,} \quad L_F = pn + 2d$$
$$128.275 = p \times 12 + 2 \times 6$$
$$\mathbf{p = 9.69 \text{ mm}}$$

Two helical springs are arranged in a concentric manner, with one inside the other. Both the springs have same free length and carry a total load of 5500 N. The outer spring has 8 coils with mean coil diameter of 128 mm and wire diameter of 16 mm. The inner spring has 12 coils with mean coil diameter of 84 mm and wire diameter of 12 mm. Determine :

(i) the maximum load carried by each spring;

(ii) the total deflection of each spring; and

(iii) the maximum stress in each spring.

Assume G = 81 GPa.

Given :

	$L_{F1} = L_{F2}$;		$F = 5500$ N ;
	$n_1 = 8$;		$D_1 = 128$ mm ;
	$d_1 = 16$ mm ;		$n_2 = 12$;
	$D_2 = 84$ mm ;		$d_2 = 12$ mm ;
	$G = 81 \times 10^3$ N/mm^2.		

1. Stiffness of outer spring :

$$C_1 = \frac{D_1}{d_1} = \frac{128}{16} = 8$$

$$\therefore \quad K_1 = \frac{Gd_1}{8\, C_1^3\, n_1} = \frac{81 \times 10^3 \times 16}{8 \times (8)^3 \times 8} = 39.55 \text{ N/mm}$$

2. Stiffness of inner spring :

$$C_2 = \frac{D_2}{d_2} = \frac{84}{12} = 7$$

$$\therefore \quad K_2 = \frac{Gd_2}{8\, C_2^3\, n_2} = \frac{81 \times 10^3 \times 12}{8 \times (7)^3 \times 12} = 29.52 \text{ N/mm}$$

3. Load shared by each spring :

$$F_1 + F_2 = F$$
$$F_1 + F_2 = 5500 \text{ N}$$

$$\delta_1 = \delta_2 \qquad \frac{F_1}{K_1} = \frac{F_2}{K_2}$$

$$\frac{F_1}{39.55} = \frac{F_1}{29.52} \qquad \therefore\ F_1 = 1.34\ F_2$$

Substituting Equation (b) in Equation (a),

$$1.34\, F_2 + F_2 = 5500 \qquad 2.34\, F_2 = 5500$$

$$\therefore\quad F_2 = 2350.61 \text{ N} \qquad \therefore\quad F_1 = 5500 - 2350.61$$

or $\quad F_1 = 3149.39$ N

The load shared by outer spring, $\mathbf{F_1 = 3149.39\ N}$

The load shared by inner spring, $\mathbf{F_2 = 2350.61\ N}$

4. Deflection of each spring :

$$\delta_1 = \frac{F_1}{K_1} = \frac{3149.39}{39.55} = 79.63 \text{ mm}$$

$$\delta_2 = \frac{F_2}{K_2} = \frac{2350.61}{29.52} = 79.63 \text{ mm}$$

$$\therefore\ \boldsymbol{\delta_1 = \delta_2 = 79.63\ \text{mm}}$$

5. Maximum stress in each spring :

$$K_{w1} = \frac{4\,C_1 - 1}{4\,C_1 - 4} + \frac{0.615}{C_1} = \frac{4 \times 8 - 1}{4 \times 8 - 4} + \frac{0.615}{8} = 1.184$$

$$\tau_1 = K_{w1}\left[\frac{8\,F_1\,C_1}{\pi\, d_1^2}\right] = \frac{1.184 \times 8 \times 3149.39 \times 8}{\pi \times (16)^3}$$

The Maximum stress in outer spring, $\boldsymbol{\tau_1 = 296.73\ \text{N/mm}^2}$

$$K_{w2} = \frac{4\,C_2 - 1}{4\,C_2 - 4} + \frac{0.615}{C_2} = \frac{4 \times 7 - 1}{4 \times 7 - 4} + \frac{0.615}{7} = 1.2128$$

$$\tau_2 = K_{w2}\left[\frac{8\,F_2\,C_2}{\pi\, d_2^2}\right] = \frac{1.2128 \times 8 \times 2350.61 \times 7}{\pi \times (12)^2}$$

The Maximum stress inner spring, $\boldsymbol{\tau_2 = 352.91\ \text{N/mm}^2}$

A composite compression spring has two closed coil helical springs and is subjected to an axial load of 400 N. The outer spring is 15 mm longer than the inner spring. The outer spring has 10 coils of 40 mm mean diameter and 5 mm wire diameter. The inner spring has 8 coils of 30 mm mean diameter and 4 mm wire diameter. If the modulus of rigidity for spring material is 84 GPa, determine :

(i) the compression of each spring;
(ii) the load carried by each spring; and
(iii) the shear stress induced in each spring.

Given : $F = 400$ N ; $G = 84 \times 10^3$ N/mm^2 ;

For outer spring : For inner spring :

$n_1 = 10$; $n_2 = 8$;

$D_1 = 240$ mm ; $D_2 = 30$ mm ;

$d_1 = 5$ mm ; $d_2 = 4$ mm ;

$L_{F1} = h + 15$ mm ; $L_{F2} = h$ mm

Referring Fig. 12.22.1;

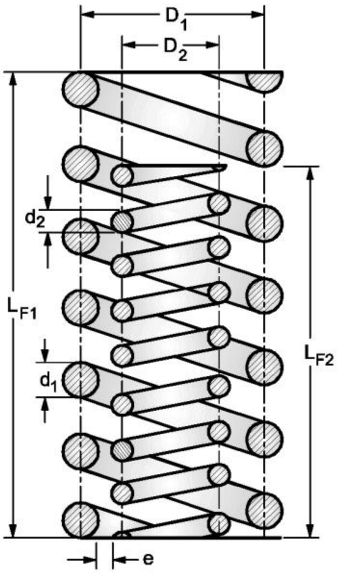


Fig.4.22 Helical Spring in parallel

- **Deflection of outer spring :**

$$\therefore \quad C_1 = \frac{D_1}{d_1} = \frac{40}{5} = 8$$

$$\frac{F_1}{\delta_1} = \frac{Gd_1}{8\,C_1^3\,n_1}$$

$$\frac{F_1}{\delta_1} = \frac{84 \times 10^3 \times 5}{8 \times 8^3 \times 10}$$

$$\therefore \quad \delta_1 = 9.75 \times 10^{-2}\,F_1,\ \text{mm}$$

- **Deflection of inner spring :**

$$\therefore \quad C_2 = \frac{D_2}{d_2} = \frac{30}{4} = 7.5$$

$$\frac{F_2}{\delta_2} = \frac{Gd_2}{8\,C_2^3\,n_2} = \frac{84 \times 10^3 \times 4}{8 \times 7.5^3 \times 8}$$

$$\therefore \quad \delta_2 = 8.04 \times 10^{-2}\,F_2,\ \text{mm}$$

$$\text{Now, } \delta_1 = \delta_2 + 15$$

- **Load carried by each spring :**

Substituting Equations (a) and (b) in Equation (c),

$$9.75 \times 10^{-2}\,F_1 = 8.04 \times 10^{-2}\,F_2 + 15$$

$$F_1 = 0.824\,F_2 + 153.846$$

$$\text{Now,} \quad F_1 + F_2 = 400$$

Substituting Equation (d) in Equation (e),

$$0.824\,F_2 + 153.846 + F_2 = 400$$

$$1.824\,F_2 = 246.15$$

$$\therefore \quad F_2 = 134.95\ \text{N}$$

$$\text{and} \quad F_1 = 400 - F_2 = 400 - 134.95 = 265.05\ \text{N} \quad \text{[from Equation (e)]}$$

$$F_1 = 265.05\ \text{N}$$

$$\mathbf{F_2 = 134.95\ N}$$

- **Compression of each spring :**

From Equation (a),

$$\delta_1 = 9.75 \times 10^{-2}\, F_1 = 9.75 \times 10^{-2} \times 265.05$$

or $$\boldsymbol{\delta_1 = 25.84 \text{ mm}}$$

From Equation (b),

$$\delta_2 = 8.04 \times 10^{-2}\, F_2 = 8.04 \times 10^{-2} \times 134.95$$

or $$\boldsymbol{\delta_2 = 10.85 \text{ mm}}$$

- **Shear stress in outer spring :**

$$K_{w1} = \frac{4C_1 - 1}{4C_1 - 4} + \frac{0.615}{C_1} = \frac{4 \times 8 - 1}{4 \times 8 - 4} + \frac{0.615}{8}$$

$$K_{w1} = 1.16$$

$$\tau_1 = \frac{K_{W1}\, 8\, F_1\, C_1}{\pi\, d_1^2} = \frac{1.16 \times 8 \times 265.05 \times 8}{\pi \times (5)^2}$$

or $$\boldsymbol{\tau_1 = 250.58 \text{ N/mm}^2}$$

- **Shear stress in inner spring :**

$$K_{w2} = \frac{4C_2 - 1}{4C_2 - 4} + \frac{0.615}{C_2} = \frac{4 \times 7.5 - 1}{4 \times 7.5 - 4} + \frac{0.615}{7.5}$$

$$K_{w2} = 1.2$$

$$\tau_2 = \frac{K_{w2}\, 8\, F_2\, C_2}{\pi\, d_2^2} = \frac{1.2 \times 8 \times 134.95 \times 7.5}{\pi \times (4)^2}$$

or $$\boldsymbol{\tau_2 = 193.3 \text{ N/mm}^2}$$

A composite compression spring has two closed coil helical springs. The outer spring is 15 mm longer than the inner spring. The outer spring has 10 coils of mean diameter 40 mm and wire diameter 5 mm. The inner spring has 8 coils of mean diameter 30 mm and wire diameter 4 mm. When the spring is subjected to an axial load of 400 N, find :

(i) Compression of each spring;

(ii) Load shared by each spring;

(iii) Shear stress induced in each spring

Modulus of rigidity may be taken as 84 kN/mm^2.

Given : $L_{F1} = L_{F2} + 15$; $n_1 = 10$;

$D_1 = 40$ mm ; $d_1 = 5$ mm;

$n_2 = 8$; $D_2 = 30$ mm;

$d_2 = 4$ mm ; $F = 400$ N;

$G = 84 \times 10^3$ N/mm^2.

- **Stiffness of inner spring :**

$$C_2 = \frac{D_2}{d_2} = \frac{30}{4} = 7.5$$

$$\therefore \quad K_2 = \frac{Gd_2}{8\,C_2^3\,n_2} = \frac{84 \times 10^3 \times 4}{8 \times (7.5)^3 \times 8} = 12.444 \text{ N/mm}^2$$

$$\mathbf{K_2 = 12.444 \text{ N/mm}^2}$$

- **Load shared by each spring :**

$$F_1 + F_2 = F$$

$$\therefore \quad F_1 + F_2 = 400$$

$$\therefore \quad F_2 = 400 - F_1$$

Again,

$$\delta_1 = \delta_2 + 15$$

$$\therefore \quad \frac{F_1}{K_1} = \frac{F_2}{K_2} + 15$$

$$\frac{F_1}{10.2539} = \frac{F_2}{12.444} + 15$$

$$1.2136\,F_1 = F_2 + 186.66$$

Substituting Equation (a) in Equation (b),

$$1.2136\,F_1 = 400 - F_1 + 186.66$$

$$2.2136\,F_1 = 586.66$$

$$\therefore \quad F_1 = 265 \text{ N}$$

$$\text{and} \quad F_1 + F_2 = 400$$

$$\therefore \quad F_2 = 400 - 265 = 135 \text{ N}$$

$$F_1 = 265 \text{ N}$$

$$\mathbf{F_2 = 135 \text{ N}}$$

- **Compression of each spring :**

$$\delta_1 = \frac{F_1}{K_1} = \frac{265}{10.2539} = 25.84 \text{ mm}$$

$$\text{and} \quad \delta_2 = \frac{F_2}{K_2} = \frac{135}{12.4444} = 10.84 \text{ mm}$$

$$\delta_1 = 25.84 \text{ mm}$$

$$\mathbf{\delta_2 = 10.84 \text{ mm}}$$

- **Shear stress induced in each spring :**

$$K_{W1} = \frac{4C_1 - 1}{4C_1 - 4} + \frac{0.615}{C_1} = \frac{4 \times 8 - 1}{4 \times 8 - 4} + \frac{0.615}{8} = 1.184$$

$$\tau_1 = K_{W1}\left[\frac{8\,F_1\,C_1}{\pi d_1^2}\right] = \frac{1.184 \times 8 \times 265 \times 8}{\pi\,(5)^2}$$

$$\therefore \quad \tau_1 = 255.67 \text{ N/mm}^2$$

$$K_{W2} = \frac{4C_2 - 1}{4C_2 - 4} + \frac{0.615}{C_2} = \frac{4 \times 7.5 - 1}{4 \times 7.5 - 4} + \frac{0.615}{7.5} = 1.1974$$

$$\tau_2 = K_{W2}\left[\frac{8\,F_2\,C_2}{\pi d_2^2}\right] = \frac{1.1974 \times 8 \times 135 \times 7.5}{\pi\,(4)^2}$$

$$\therefore \quad \tau_2 = 192.65\ \text{N/mm}^2$$

$$\boldsymbol{\tau_1 = 255.67\ \text{N/mm}^2}$$

$$\boldsymbol{\tau_2 = 192.65\ \text{N/mm}^2}$$

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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

UNIT – V – BEAM DEFLECTION – SMEA1305

MECHANICS OF SOLIDS (SMEA1305)

UNIT 5: BEAM DEFLECTION

UNIT 5: BEAM DEFLECTION **Columns – End conditions – Equivalent length of a column – Euler equation – Slenderness ratio – Rankine Gordon formula for columns**

Elastic curve of Neutral axis of the beam under normal loads – Evaluation of beam deflection and slope: Double integration method, Macaulay Method, and Moment-area Method

Introduction: Elastic Stability of Columns

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions. The analysis and design of compression members can differ significantly from that of members loaded in tension or in torsion. If you were to take a long rod or pole, such as a meter stick, and apply gradually increasing compressive forces at each end, nothing would happen at first, but then the stick would bend (buckle), and finally bend so much as to fracture. Try it. The other extreme would occur if you were to saw off, say, a 5-mm length of the meter stick and perform the same experiment on the short piece. You would then observe that the failure exhibits itself as a mashing of the specimen, that is, a simple compressive failure. For these reasons it is convenient to classify compression members according to their length and according to whether the loading is central or eccentric. The term column is applied to all such members except those in which failure would be by simple or pure compression.

General comments

The critical load of a column is proportional to the flexural rigidity EI and inversely proportional to the square of the length. Of particular interest is the fact that the strength of the material itself, as represented by a quantity such as the proportional limit or the The flexural rigidity can be increased by using a "stiffer" material (that is, a material with larger modulus of elasticity E) or by distributing the material in such a way as to increase the moment of inertia I of the cross section, just as a beam can be made stiffer by increasing the moment of inertia. The moment of inertia is increased by distributing the material farther from the centroid of the cross section. Hence, a hollow tubular member is generally more economical for use as a column than a solid member having the same cross-sectional area. Reducing the wall thickness of a tubular member and increasing its lateral dimensions (while keeping the cross-sectional area constant) also increases the critical load because the moment of inertia is increased. This process has a practical limit, however, because eventually the wall itself will become unstable. When that happens, localized buckling occurs in the form of small corrugations or wrinkles in the walls of the column. Thus, we must distinguish between overall buckling of a column, which is discussed in this chapter, and local buckling of its parts. yield stress, does not appear in the equation for the critical load. Therefore, increasing a strength property does not raise the critical load of a slender column. It can only be raised by increasing the flexural rigidity, reducing the length, or providing additional lateral support.

we assumed that the xy plane was a plane of symmetry of the column and that buckling took place in that plane. The latter assumption will be met if the column has lateral supports perpendicular to the plane of the figure, so that the column is constrained to buckle in the xy plane. If the column is supported only at its ends and is free to buckle in any direction, then bending will occur about the principal centroidal axis having the smaller moment of inertia. If the cross section is square or circular, all centroidal axes have the same moment of inertia and buckling may occur in any longitudinal plane.

Limitations

In addition to the requirement of small deflections, the Euler buckling theory used in this section is valid only if the column is perfectly straight before the load is applied, the column and its supports have no imperfections, and the column is made of a linearly elastic material that follows Hooke's law.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded. Columns can be categorized then as:

- Long columns with central loading
- Intermediate-length columns with central loading
- Columns with eccentric loading
- Struts or short columns with eccentric loading

Struts:

Long, slender columns are generally termed as struts; they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons. A short bar loaded in pure compression by a force P acting along the centroidal axis will shorten in accordance with Hooke's law, until the stress reaches the elastic limit of the material. At this point, permanent set is introduced and usefulness as a machine member may be at an end. If the force P is increased still more, the material either becomes "barrel-like" or fractures. When there is eccentricity in the loading, the elastic limit is encountered at smaller loads.

(a) The strut may not be perfectly straight initially.

(b) The load may not be applied exactly along the axis of the Strut.

(c) One part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties throughout the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as $I = Ak^2$,

Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k. There will be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

is called the slenderness ratio. Its numerical value indicates whether the member falls into the class of columns or struts.

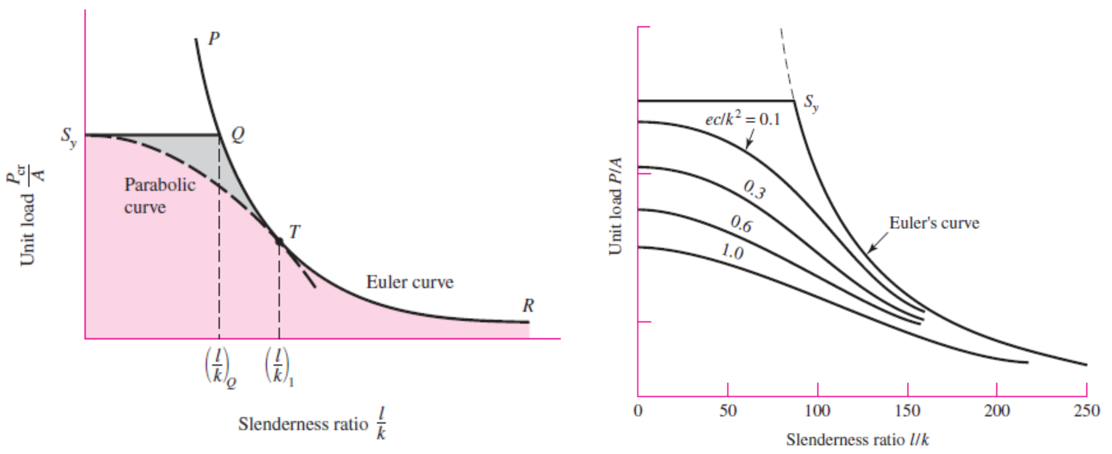

Fig. 5.1 Slenderness ratio against Stress

Euler's Theory: The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

Consider an axially loaded strut, shown below, and is subjected to an axial load ‘P' this load ‘P' produces a deflection ‘y' at a distance ‘x' from one end.

Assume that the ends are either pin jointed or rounded so that there is no moment at either end.

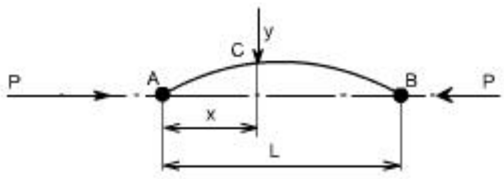

Fig. 5.2 Strut with Pinned Ends

Assumption:
The strut is assumed to be initially straight, the end load being applied axially through centroid.

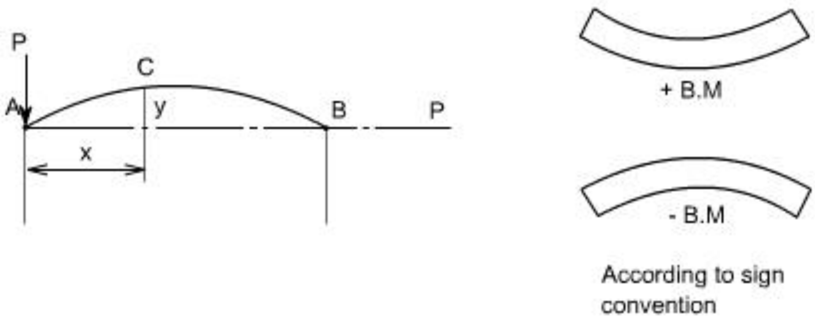

Fig. 5.3 Strut with Pinned Ends sign convention

$$B.M|_C = -Py$$

Futher,we know that

$$EI\,\frac{d^2y}{dx^2} = M$$

$$EI\,\frac{d^2y}{dx^2} = -P.y = M$$

In this equation ‘M' is not a function ‘x'. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI \frac{d^2y}{dx^2} + Py = 0$$

Though this equation is in 'y' but we can't say at this stage where the deflection would be maximum or minimum.

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

So the above differential equation can be arranged in the following form

Let us define a operator

D = d/dx

$(D^2 + n^2)\ y = 0$ where $n^2 = P/EI$

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus y = A cos (nx) + B sin (nx)

Where A and B are some constants.

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at x = 0; y = 0

(ii) at x = L ; y = 0

Applying the first boundary condition yields A = 0 and applying the second boundary condition gives

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Thus either $B = 0$, or $\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$

if B=0,that y0 for all values of x hence the strut has not buckled yet.Therefore,the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \text{ or } \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \text{ or } P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the " **Euler Crippling Load** " P_e from which w obtain.

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}} \right) = 0 \qquad \text{i.e. } \sin\ nL = 0$$

The interpretation of the above analysis is that for all the values of the load P, other than those which make sin nL = 0; the strut will remain perfectly straight since

y = B sin nL = 0

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

$$\sin nL = 0 \quad \text{or } nL = \pi$$

$$\text{Therefore} \quad n = \frac{\pi}{L}$$

$$\text{Hence } y = B \sin nx = B \sin \frac{\pi x}{L}$$

Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that 'L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; like wise it will be found that the maximum stress is not proportional to load.

The solution chosen of nL = p is just one particular solution; the solutions nL= 2p, 3p, 5p etc are equally valid mathematically and they do, infact, produce values of 'P_e' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling.

The value selected above is so called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.

The solution nL = 2p produces buckling in two half – waves, 3p in three half-waves etc.

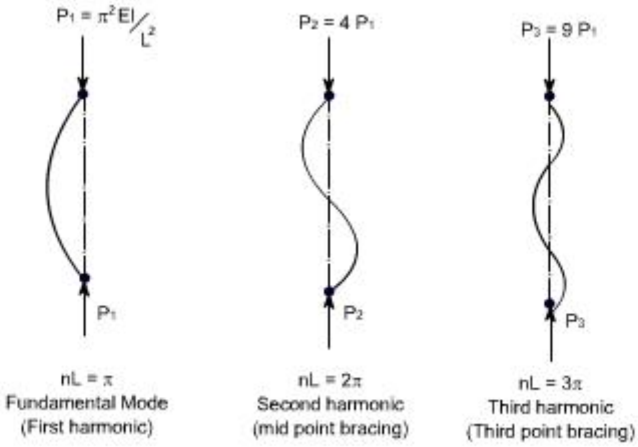

Fig. 5.4 Member subjected with different modes

$$L\sqrt{\frac{P}{EI}} = \pi \quad \text{or} \quad P_1 = \frac{\pi^2 EI}{L^2}$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \text{ or } P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \text{ or } P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse.

<u>struts and columns with other end conditions:</u> Let us consider the struts and columns having different end conditions

Case b: One end fixed and the other free:

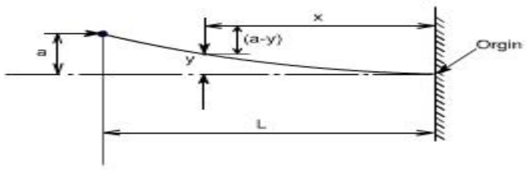

Fig. 5.5 One End fixed and other is free End condition

writing down the value of bending moment at the point C

$B.M|_c = P(a - y)$

Hence, the differential equation becomes,

$$EI \frac{d^2y}{dx^2} = P(a - y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

Let $\frac{P}{EI} = n^2$

Hence in operator form, the differential equation reduces to $(D^2 + n^2) y = n^2a$

The solution of the above equation would consist of complementary solution and particular solution, therefore

$y_{gen} = A \cos(nx) + \sin(nx) + P.I$

where

P.I = the P.I is a particular value of y which satisfies the differential equation

Hence $y_{P.I} = a$

Therefore the complete solution becomes

$Y = A \cos(nx) + B \sin(nx) + a$

Now imposing the boundary conditions to evaluate the constants A and B

(i) at $x = 0$; $y = 0$

This yields $A = -a$

(ii) at $x = 0$; $dy/dx = 0$

This yields $B = 0$

Hence

$y = -a \cos(nx) + a$

Futher, at $x = L$; $y = a$

Therefore $a = - a \cos(nx) + a$ or $0 = \cos(nL)$

Now the fundamental mode of buckling in this case would be

$$nL = \frac{\pi}{2}$$

$$\sqrt{\frac{P}{EI}}\,L = \frac{\pi}{2}$$, Therefore, the Euler's crippling load is given as

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3

Strut with fixed ends:

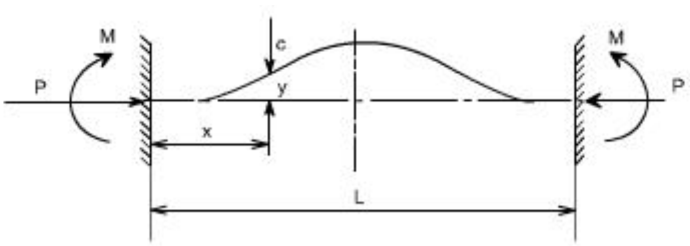

Fig. 5.6 Both Ends in fixed End condition

Due to the fixed end supports bending moment would also appears at the supports, since this is the property of the support.

Bending Moment at point C = M – P.y

$$EI\frac{d^2y}{dx^2} = M - Py$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$$

$$n^2 = \frac{P}{EI}$$, Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\,integral}$$

$$y|_{P.I} = \frac{M}{n^2 EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\,Cosnx + A\,Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at x=0;y=0

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

A=0

Therefore,

$$y = -\frac{M}{P}\cos nx + \frac{M}{P}$$

$$y = \frac{M}{P}(1 - \cos nx)$$

Futher, it may be noted that at $x = L; y = 0$

Then $0 = \frac{M}{P}(1 - \cos nL)$

Thus, either $\frac{M}{P} = 0$ or $(1 - \cos nL) = 0$

obviously, $(1 - \cos nL) = 0$

$\cos nL = 1$

Hence the least solution would be

$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\,L = 2\pi$, Thus, the buckling load or crippling load is

$$P_e = \frac{4\pi^2 . EI}{L^2}$$

Case 4

<u>One end fixed, the other pinned</u>

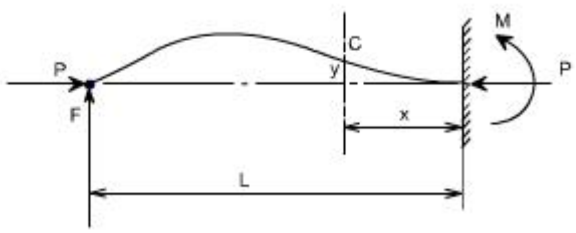

Fig. 5.7 One end Fixed and oer pinned End conditions

In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin. The moment of F about the built in end then balances the fixing moment.

With the origin at the built in end, the B,M at C is given as

$$EI\frac{d^2y}{dx^2} = -Py + F(L - x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L - x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L - x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L - x)$$

$$y_{particular} = \frac{F}{n^2EI}(L - x) \text{ or } y = \frac{F}{P}(L - x)$$

The full solution is therefore

$$y = A\,Cos\,mx + B\,Sin\,nx + \frac{F}{P}(L - x)$$

The boundry conditions relevants to the problem are at x=0;y=0

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x = 0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P}Cos\,nx + \frac{F}{nP}Sin\,nx + \frac{F}{P}(L - x)$$

$$y = \frac{F}{nP}\left[Sin\,nx - nL\,Cos\,nx + n(L - x)\right]$$

Also when x = L ; y = 0

Therefore

nL Cos nL = Sin nL   or tan nL = nL

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is nL = 4.49radian

$$\text{or} \sqrt{\frac{P}{EI}}\,L = 4.49$$

$$\frac{P_e}{EI}L^2 = 20.2$$

$$P_e = \frac{2.05\pi^2\,EI}{L^2}$$

Equivalent Strut Length:

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.

$$\text{i.e. } P_e = \frac{\pi^2 EI}{L^2}$$

Where L is the equivalent length of the strut and can be related to the actual length of the strut depending on the end conditions.

The equivalent length is found to be the length of a simple bow(half sine wave) in each of the strut deflection curves shown. The buckling load for each end condition shown is then readily obtained. The use of equivalent length is not restricted to the Euler's theory and it will be used in other derivations later.

The critical load for columns with other end conditions can be expressed in terms of the critical load for a hinged column, which is taken as a fundamental case.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the freebody diagram would indicates that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we have considered so far are:

(a) Both ends pinned (c) One end fixed, other free

(b) Both ends fixed (d) One end fixed and other pinned

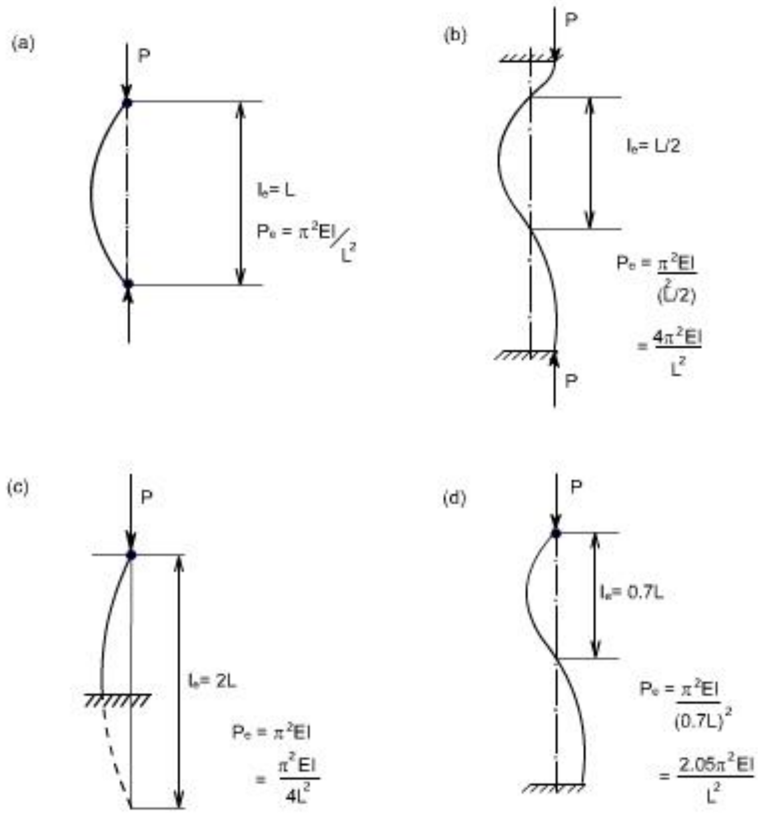

Fig. 5.8 Different End conditions loading

Solved Problems on deflection of beams

1. Determine the deflection at every point of the cantilever beam subject to the single concentrated force P, as shown in Figure shown below

SOLUTION: The x-y coordinate system shown is introduced, where the x-axis coincides with the original unbent position of the beam. The deformed beam has the appearance indicated by the heavy line in Fig It is first necessary to find the reactions exerted by the supporting wall upon the bar, and these are easily found from statics to be a vertical force reaction P and a moment PL, as shown.

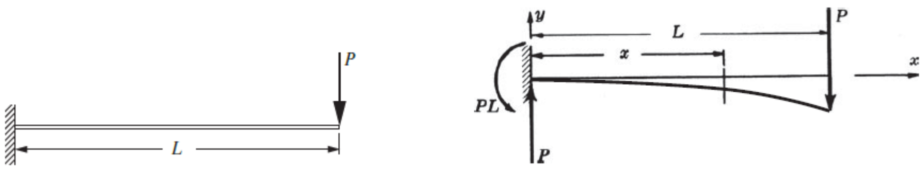

Fig. 5.9 Cantilever beam subjected to a point load at free end.

According to the sign convention of Chap. 6, the bending moment M at the section x is

$$M = -PL + Px$$

The differential equation (8.4) of the bent beam is then

$$EI\frac{d^2y}{dx^2} = -PL + Px \tag{1}$$

This equation is readily integrated once to yield

$$EI\frac{dy}{dx} = -PLx + \frac{Px^2}{2} + C_1 \tag{2}$$

which represents the equation of the slope, where C_1 denotes a constant of integration. This constant may be evaluated by use of the condition that the slope dy/dx of the beam at the wall is zero since the beam is rigidly clamped there. Equation (2) is true for all values of x and y, and if the condition $x = 0$ is substituted we obtain $0 = 0 + 0 + C_1$ or $C_1 = 0$.

Next, integration of Eq. (2) yields

$$EIy = -PL\frac{x^2}{2} + \frac{Px^3}{6} + C_2 \tag{3}$$

where C_2 is a second constant of integration. Again, the condition at the supporting wall will determine this constant. At $x = 0$, the deflection y is zero since the bar is rigidly clamped. We find $0 = 0 + 0 + C_2$ or $C_2 = 0$.

Thus Eqs. (2) and (3) with $C_1 = C_2 = 0$ give the slope dy/dx and deflection y at any point x in the beam. The deflection is maximum at the right end of the beam ($x = L$), under the load P, and from Eq. (3),

$$EIy_{max} = \frac{-PL^3}{3} \quad (4)$$

where the negative value denotes that this point on the deflection curve lies below the x-axis. If only the magnitude of the maximum deflection at $x = L$ is desired, it is usually denoted by Δ_{max} and we have

$$\Delta_{max} = \frac{PL^3}{3EI} \quad (5)$$

2. The cantilever beam AB is of uniform cross section and carries a load P at its free end A). Determine the equation of the elastic curve and the deflection and slope at A.

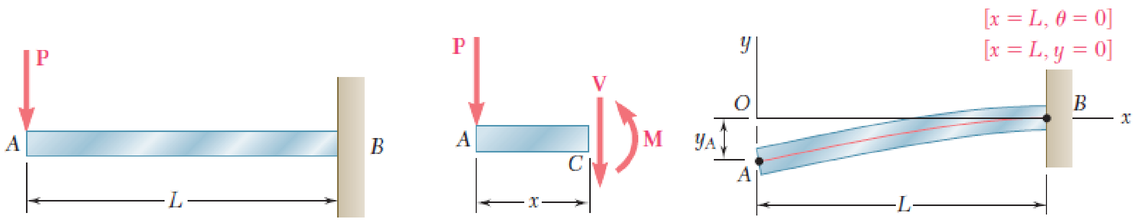

Fig. 5.10 Cantilever beam subjected to a point load and reactions

Using the free-body diagram of the portion AC of the beam , where C is located at a distance x from end A, we find

$$M = -Px$$

Substituting for M and multiplying both members by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating in x, we obtain

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1$$

We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ Substituting these values and solving for C_1, we nave

$$C_1 = \tfrac{1}{2}PL^2$$

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2$$

Integrating both members we write

$$EI\, y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2$$

But, at B we have $x = L$, $y = 0$. Substituting we have

$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$
$$C_2 = -\tfrac{1}{3}PL^3$$

Carrying the value of C_2 , we obtain the equation of the elastic curve:

$$EI\, y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

The deflection and slope at A are obtained by letting $x = 0$ We find

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

3. The simply supported prismatic beam AB carries a uniformly distributed load w per unit length. Determine the equation of the elastic curve and the maximum deflection of the beam.

Drawing the free-body diagram of the portion AD of the beam and taking moments about D, we find that

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$$

Substituting for M and multiplying both members of this equation by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

Integrating twice in x, we have

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1$$
$$EI\, y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2$$

Observing that $y = 0$ at both ends of the beam we first let $x = 0$ and $y = 0$ and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_1L$$
$$C_1 = -\tfrac{1}{24}wL^3$$

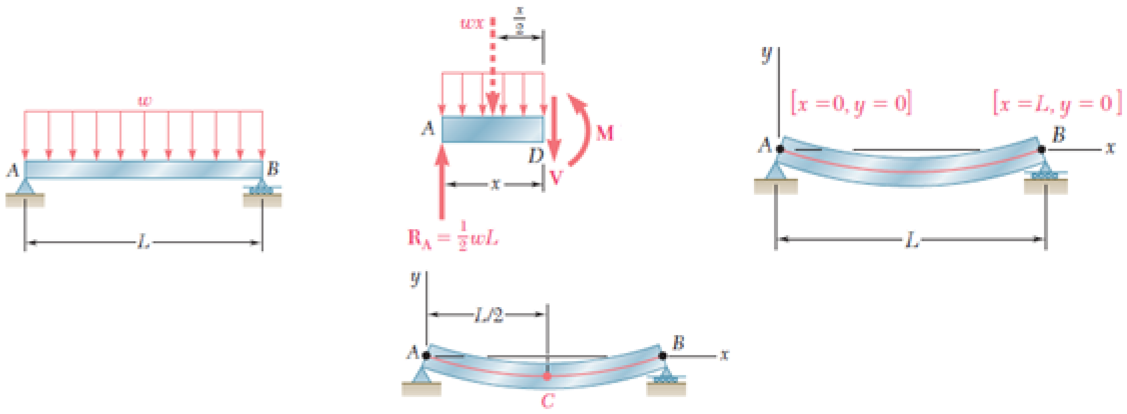

Fig. 5.11 Simply supported beam subjected to UDL

Carrying the values of C_1 and C_2 we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 - \tfrac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x)$$

Substituting the value obtained for C_1, we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam . Letting $x = L/2$, we have

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{max} = \frac{5wL^4}{384EI}$$

4. A steel rod 5 cm diameter protrudes 2 m horizontally from a wall. (i) Calculate the deflection due to a load of **1 kN** hung on the end of the rod. The weight of the rod may be neglected. (ii) If **a** vertical steel wire 3 m long, 0.25 cm diameter, supports the end of the cantilever, being taut but unstressed before the load is applied, calculate the end deflection on application of the load. TakeE = 200GN/m2.

The second moment of are of the cross-section is

$$I_x = \frac{\pi}{64}(0.050)^4 = 0.307 \times 10^{-6}\ \text{m}^4$$

The deflection at the end is then

$$v = \frac{PL^3}{3EI} = \frac{(1000)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = 0.0434\ \text{m}$$

Let T = tension in the wire; the area of cross-section of the wire is $4.90 \times 10^{-6}\ \text{m}^2$. The elongation of the wire is then

$$e = \frac{Tl}{EA} = \frac{T(3)}{(200 \times 10^9)(4.90 \times 10^{-6})}$$

The load on the end of the cantilever is then $(1000 - T)$, and this produces a deflection of

$$v = \frac{(1000 - T)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})}$$

If this equals the stretching of the wire, then

$$\frac{(1000 - T)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = \frac{T(3)}{(200 \times 10^9)(4.90 \times 10^{-6})}$$

This gives $T = 934$ N, and the deflection of the cantilever becomes

$$v = \frac{(66)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = 0.00276 \text{ m}$$

5. **A** steel beam rests on two supports 6 m apart, and carries a uniformly distributed load of **10 kN** per metre run. The second moment of area of the cross-section is **1 x 10-3 m4** and E = 200 **GN/m2.** Estimate the maximum deflection.

The greatest deflection occurs at mid-length and has the value given by equation

$$v = \frac{5wL^4}{384EI} = \frac{5(100 \times 10^3)(6)^4}{384(200 \times 10^9)(1 \times 10^{-3})} = 0.00844 \text{ m}$$

Solved Problems on columns

1. A 2-m-long pin-ended column of square cross section is to be made of wood. Assuming E = 13 GPa, σ =12 MPa, and using a factor of safety of 2.5 in computing Euler's critical load for buckling, determine the size of the cross section if the column is to safely support (a) a 100-kN load, (b) a 200-kN load.

(a) For the 100-kN Load. Using the given factor of safety, we make

$$P_{cr} = 2.5(100 \text{ kN}) = 250 \text{ kN} \qquad L = 2 \text{ m} \qquad E = 13 \text{ GPa}$$

in Euler's formula (10.11) and solve for I. We have

$$I = \frac{P_{cr}L^2}{\pi^2 E} = \frac{(250 \times 10^3 \text{ N})(2 \text{ m})^2}{\pi^2(13 \times 10^9 \text{ Pa})} = 7.794 \times 10^{-6} \text{ m}^4$$

Recalling that, for a square of side a, we have $I = a^4/12$, we write

$$\frac{a^4}{12} = 7.794 \times 10^{-6} \text{ m}^4 \qquad a = 98.3 \text{ mm} \approx 100 \text{ mm}$$

We check the value of the normal stress in the column:

$$\sigma = \frac{P}{A} = \frac{100 \text{ kN}}{(0.100 \text{ m})^2} = 10 \text{ MPa}$$

Since σ is smaller than the allowable stress, a 100 × 100-mm cross section is acceptable.

(b) For the 200-kN Load. Solving again Eq. (10.11) for I, but making now $P_{cr} = 2.5(200) = 500$ kN, we have

$$I = 15.588 \times 10^{-6} \text{ m}^4$$

$$\frac{a^4}{12} = 15.588 \times 10^{-6} \qquad a = 116.95 \text{ mm}$$

The value of the normal stress is

$$\sigma = \frac{P}{A} = \frac{200 \text{ kN}}{(0.11695 \text{ m})^2} = 14.62 \text{ MPa}$$

Since this value is larger than the allowable stress, the dimension obtained is not acceptable, and we must select the cross section on the basis of its resistance to compression. We write

$$A = \frac{P}{\sigma_{all}} = \frac{200 \text{ kN}}{12 \text{ MPa}} = 16.67 \times 10^{-3} \text{ m}^2$$

$$a^2 = 16.67 \times 10^{-3} \text{ m}^2 \qquad a = 129.1 \text{ mm}$$

A 130 × 130-mm cross section is acceptable.

Deflection of Beams: Problems for practice

1. A cantilever steel beam has a free length of 3m. The moment of inertia of the section is 30x10^6 mm4. A concentrated load of 50kN at the free end. Find the deflection at the free end using
   a. Double integration method
   b. Macauley's Method
   c. Moment Area Method
   d. Conjugate Beam Method, Take E= 2x10^5 N/mm^2

2. A cantilever Beam of 8m carries a UDL of 5kN/m run and a load of W at the free end. If the deflection at the free end is 30mm, calculate the magnitude of the load W, and the slope at the free end. Take E= 2x10^5 N/mm^2, I = 5x10^7 mm^4.

3. A cantilever beam of 6m long carries a UDL of 5kN/m throughout its length and a concentrated load of 80 kN. Determine the slope and deflection at the free eng by using moment area method. Take E= 2x10^5 N/mm^2, I = 2x10^9 mm^4.

4. A SSB of 6m span carries a concentrated load of 50 kN at 3m from left support. Find the slope at the supports and deflection under the load. EI = 2000 kN-m^2.

5. A SSB of 10 m span carries a concentrated load of 10 kN at its center. It carries a UDL of 2 kN/m over its length. Find the maximum Deflection of beam by
   a. Double integration method
   b. Macauley's Method
   c. Moment Area Method
   d. Conjugate Beam Method, Take E= 2x10^5 N/mm^2, I = 200x106 mm^4.

6. A beam is simply supported at its ends over a span of 10 m and carries two concentrated loads of 100 kN and 60 kN at a distance of 2 m and 5 m respectively

from the left support. Calculate (i) slope at the left support (ii) slope and deflection under the 100 kN load. Assume EI = 36 × 104 kN-m2.

7. (i) State Moment-Area Mohr's theorem.
(ii) A simply supported beam AB uniform section, 4 m span is subjected to a clockwise moment of 10 kNm applied at the right hinge B. Derive the equation to the deflected shape of the beam. Locate the point of maximum deflection and find the maximum deflection.

Columns: Problems for practice

1. Find the Euler critical load for a hollow cylindrical cast iron column 150mm external diameter, 20 mm wall thickness if it is 6 m long with hinged at both ends. Assume Young's modulus of cast iron as 80 kN/mm2. Compare this load with that given by Rankine formula. Using Rankine constants α = 1/1600 and 567 N/mm2.

2. A column of solid circular section, 12 cm diameter, 3.6 m long is hinged at both ends. Rankine's constant is 1 / 1600, σc = 54 KN/cm2. Find the buckling load. ii) If another column of the same length, end conditions and rankine constant but of 12 cm X 12 cm square cross-section, and different material, has the same buckling load, find the value of σc of its material.

3. Determine the section of a hollow C.I. cylindrical column 5 m long with ends firmly built in. The column has to carry an axial compressive load of 588.6 KN. The internal diameter of the column is 0.75 times the external diameter. Use Rankine's constants. a = 1 / 1600, σc = 57.58 KN/cm2 and F.O.S = 6.

4. Find the euler critical load for a hollow cylindrical cast iron column 150mm external diameter, 20mm wall thick ness if it is 6m long with hinged at both ends. Assume young's modulus of cast iron as 80 KN/mm2.compare this load with that given by rankine constants. a=1/1600 and 567N/mm2.

5. A 1.2m long column has a cross section of 45mm diameter one of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using. I. Rankine's formula, take yield stress=560N/mm2 and a=1/1600 for pinned ends. II. Euler's formula Young's modulus for cast iron = 1.2X105 N/mm2.

6. The external and internal diameters of a hollow cast iron column are 50mm and 40mm respectively. If the length of this column is 3m and both of its ends are fixed, determine the crippling load using Euler formula taking E=100Gpa. Also determine the rankine load for the column assuming fc=550Mpa and α=1/1600.

7. An I section joists 400mmx200mmx20mm and 6m long is used as a strut with both ends fixed. What is Euler's crippling load for the column? Take E=200Gpa.

Deflection of Beams

In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value. In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumptions: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

2. The curvature is always small.

3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.

Equation of the Elastic curve

We first recall from elementary calculus that the curvature of a plane curve at a point Q(x,y) of the curve can be expressed as

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

where dy/dx and d^2y/dx^2 are the first and second derivatives of the function y(x) represented by that curve. But, in the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity. We write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

It should be noted that, in this chapter, y represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve. The product EI is known as the flexural rigidity and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integrate. However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Equations by EI and integrate in x. We write

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1$$

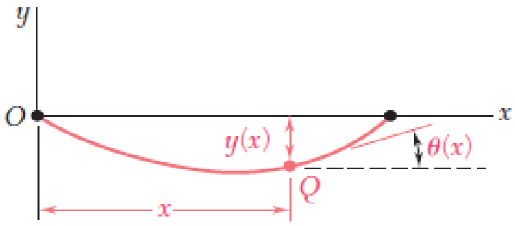

Fig. 5.12 Deflection and Slope

where C_1 is a constant of integration. Denoting by u(x) the angle, measured in radians, that the tangent to the elastic curve at Q forms with the horizontal, and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x)$$

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1$$

Integrating both members of Eq. (9.5) in x, we have

$$EI\,y = \int_0^x \left[\int_0^x M(x)\,dx + C_1\right]dx + C_2$$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

where C_2 is a second constant, and where the first term in the right hand member represents the function of x obtained by integrating twice in x the bending moment M(x). If it were not for the fact that the constants C_1 and C_2 are as yet undetermined, would define the deflection of the beam at any given point Q, and define the slope of the beam at Q.

The constants C_1 and C_2 are determined from the boundary conditions or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to statically determinate beams, i.e., to beams supported in such a way that the reactions at the supports can be obtained by the methods of statics, we note that only three types of beams need to be considered here (a) the simply supported beam, (b) the overhanging beam, and (c) the cantilever beam.

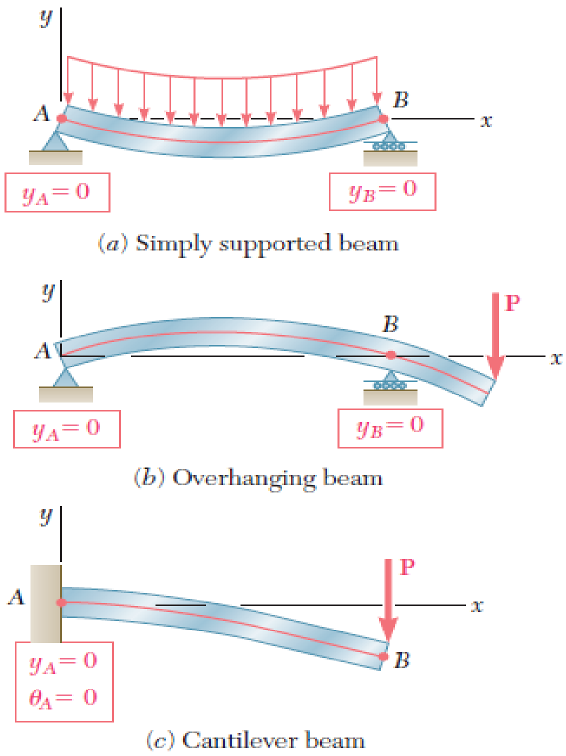

Fig. 5.13 SSB with UDL

In the first two cases, the supports consist of a pin and bracket at A and of a roller at B, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in the Equation, and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations that can be solved for C_1 and C_2. In the case of the cantilever beam, we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Equation and $x = x_A$, $u = u_A = 0$ in Equation, we obtain again two equations that can be solved for C_1 and C_2.

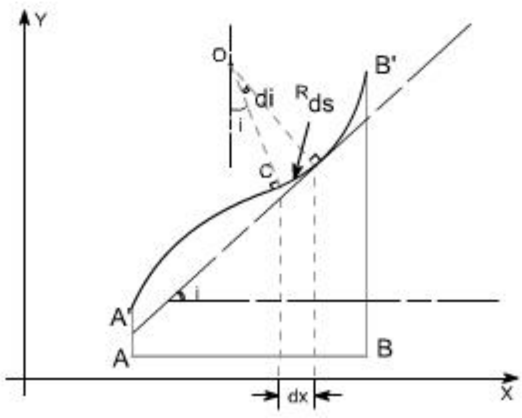

Fig. 5.14 Deflection pattern

Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflects to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points. To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

or $\boxed{\frac{di}{dx} = \frac{1}{R}}$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force, bending moment and deflection of the beam may be obtained as differentiating the equation as derived

$$\frac{dM}{dx}=EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx}=F$$

Thus,

$$F=EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{slope = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods. The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int\frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

Integrate once again to get the deflection.

$$y = \int\int\frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples: let us consider few illustrative examples to have a familiarity with the direct integration method

Case 1: Cantilever Beam with Concentrated Load at the end:-_A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam

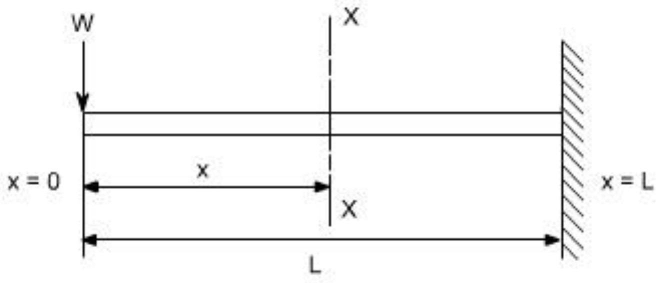

Fig. 5.15 Cantilever Beam with Concentrated Load at the free end

In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force and the bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI} dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI} dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(Slope)_{max}m = +\frac{WL^2}{2EI}}$$

Case 2: A Cantilever with Uniformly distributed Loads:- In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length. The same procedure can also be adopted in this case

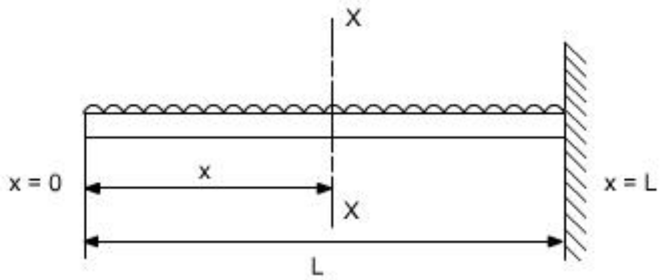

Fig. 5.16 Cantilever Beam with UDL

$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int \frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0

2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

So y_{max^m} will be at $x = 0$

$$\boxed{y_{max^m} = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{max^m} = \frac{wL^3}{6EI}}$$

Case 3: Simply Supported beam with uniformly distributed Loads:- In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.

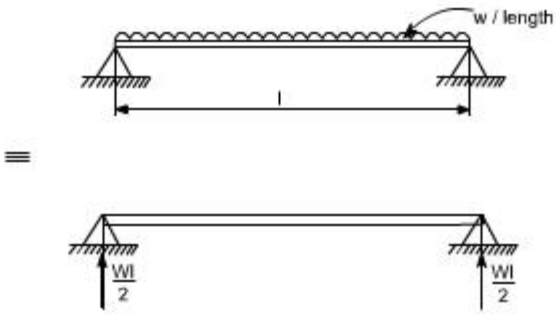

Fig. 5.17 SSB with UDL

In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.

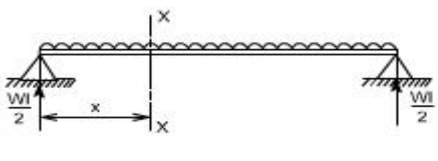

Fig. 5.18 SSB with UDL a section at X-X

$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----}(1)$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

$$\text{Then} \quad y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) The value of maximum deflection would be at the centre i.e. at x = L/2.

The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Slope (dy/dx)

$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$

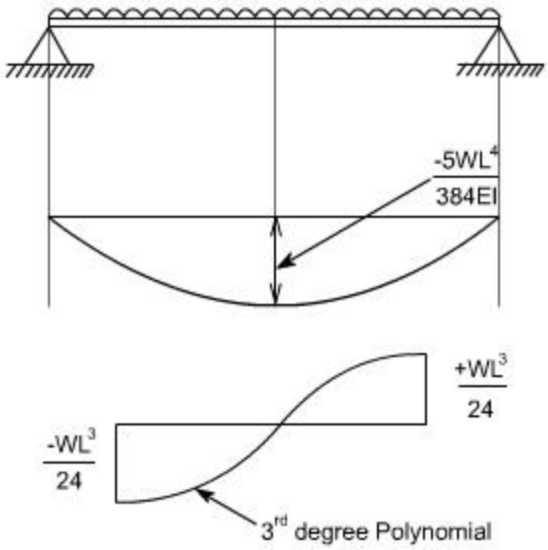

Fig. 5.19 SSB with UDL Deflection and Slope

So the bending moment diagram would be

Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$

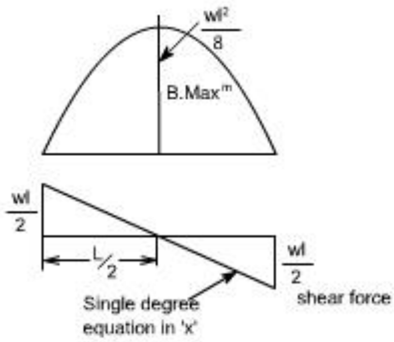

Fig. 5.20 SSB with UDL SF & BM

Shear Force

Shear force is obtained by taking third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.

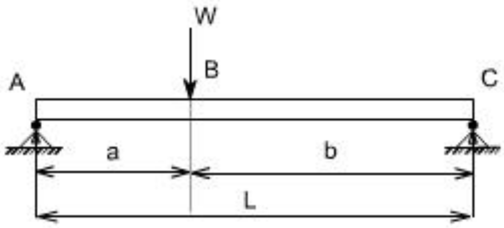

Fig.5.21 SSB with a point load acting elsewhere in the beam

Let R_1 & R_2 be the reactions then,

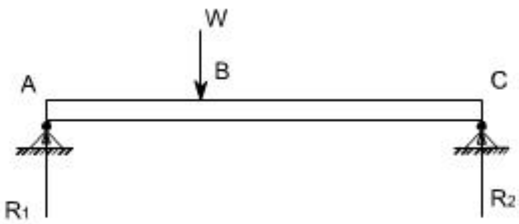

Fig.5.22 SSB with a point load reaction marked diagram

B.M for the portion AB

$$M|_{AB} = R_1.x \quad 0 \le x \le a$$

B.M for the portion BC

$$M|_{BC} = R_1.x - W(x-a) \quad a \le x \le l$$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1\,x$$

$$EI\frac{d^2y}{dx^2} = R_1\,x - W\,(x - a)$$

These two equations can be integrated in the usual way to find ‘y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at x = 0; y = 0 in the portion AB i.e. $0 \le x \le a$

(b) at x = l; y = 0 in the portion BC i.e. $a \le x \le l$

(c) at x = a; dy/dx, the slope is same for both portion

(d) at x = a; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting $K_1 = K_2 = K$, Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{ ------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{ ------- (4)}$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{ ------(5)}$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{ ------(6)}$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a+b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y\big|_{\text{from equation 5}} = y\big|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{----} \mathbf{for\ 0 \le x \le a} \text{-----(7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{For } \mathbf{for\ a \le x \le l} \text{-----(8)}$$

so either of the equation (7) or (8) may be used to find the deflection at x = a

hence substituting x = a in either of the equation we get

$$Y\big|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD: There is also an alternative way to attempt this problem in a simpler way. Let us considering the origin at the point of application of the load,

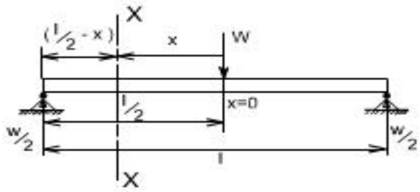

Fig.5.23 SSB with a Mid-point load

$$S.F|_{xx} = \frac{W}{2}$$
$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$
$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$
$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$
$$B = -\frac{WL^3}{48}$$

Hence he equation which governs the deflection would be

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$Y_{max^m}\Big	_{at\,x=0} = -\frac{WL^3}{48EI}$	At the centre
$\left(\frac{dy}{dx}\right)_{max^m}\Big	_{at\,x=\pm\frac{L}{2}} = \pm\frac{WL^2}{16EI}$	At the ends

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,

Fig.5.24 Member with varied cross section

i.e. it is having different cross-section then this method also fails. So there are other methods by which we find the deflection like

1. Macaulay's method in which we can write the different equation for bending moment for different sections.

2. Area moment methods

3. Energy principle methods

THE AREA-MOMENT / MOMENT-AREA METHODS

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

The moment-area method provides a semigraphical technique for finding the slope and displacement at specific points on the elastic curve of a beam or shaft. Application of the method requires calculating areas associated with the beam’s moment diagram; and so if this diagram consists of simple shapes, the method is very convenient to use. Normally this is the case when the beam is loaded with concentrated forces and couple moments. To develop the moment-area method we will make the same assumptions we used for the method of integration: The beam is initially straight, it is
elastically deformed by the loads, such that the slope and deflection of the elastic curve are very small, and the deformations are only caused by bending. The moment-area method is based on two theorems, one used to determine the slope and the other to determine the displacement at a point on the elastic curve.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.

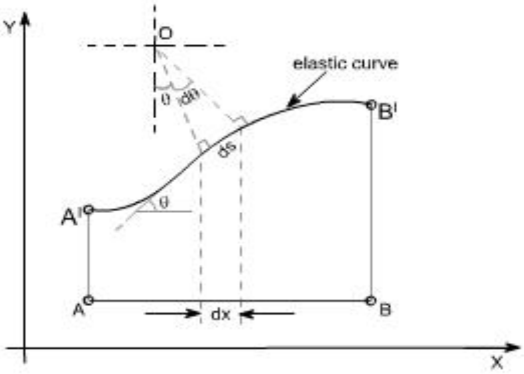

Fig.5.25 Area moment Method

It may be noted that dq is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected. We can assume, ds = dx [since the curvature is small]

hence, R dq = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle, slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$, hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting ds ≈ dx]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \quad ----- (1)$$

The relation as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram

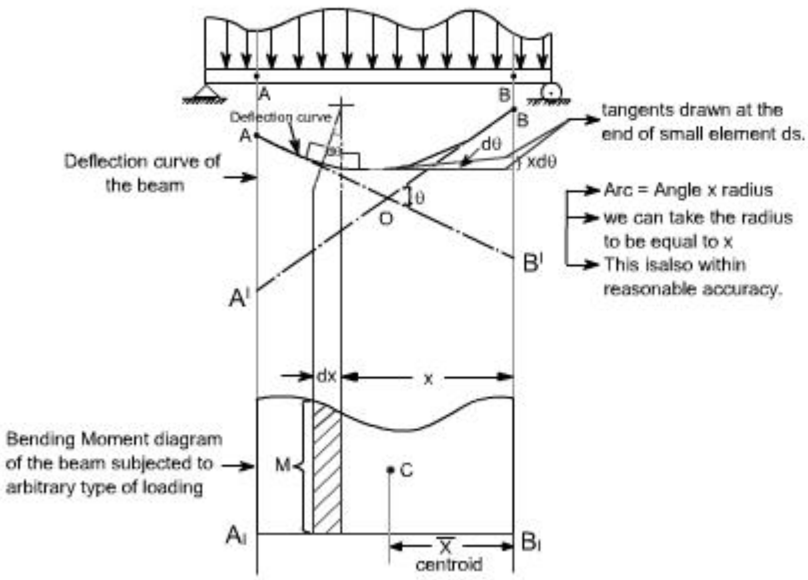

Fig.5.26 Area moment Method

Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and A1B1is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are futher understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be dq. Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.

From the above relationship the total angle q between the tangents A and B may be determined as

$$\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or } \theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI} \times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x dq [each of this intercept may be considered as the arc of a circle of radius x subtended by the angle q

Hence the total distance B'B becomes $\delta = \int_A^B x d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of dq = M dx / EI as derived earlier

$$\delta = \int_A^B x \frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI} . x$$

[This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Therefore, we are in a position to state the above conclusion in the form of theorem as follows:

Theorem II:

Deflection of point 'B' relative to point A $= \frac{1}{EI} \times \left\{\begin{matrix}\text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram}\end{matrix}\right\}$

Futher, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

$$\delta_A = \frac{1}{EI} A\bar{X}$$

Therefore,the first moment of area may be obtained simply as a product of the total area of the B.M diagram betweenthe points A and B multiplied by the distance $\bar{x}$ to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,

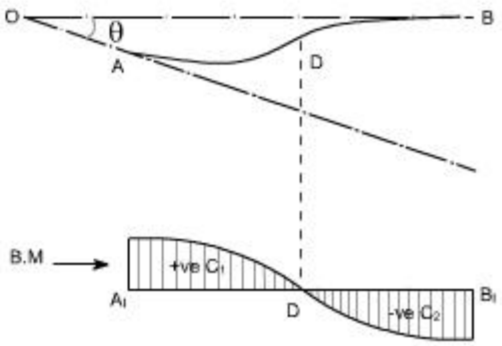

Fig.5.27 Contraflexure point

Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids C1and C2. Then to find an angle q between the tangentsat the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Illustrative Examples: Let us study few illustrative examples, pertaining to the use of these theorems

Example 1:

1. A cantilever is subjected to a concentrated load at the free end.It is required to find out the deflection at the free end.

Fpr a cantilever beam, the bending moment diagram may be drawn as shown below

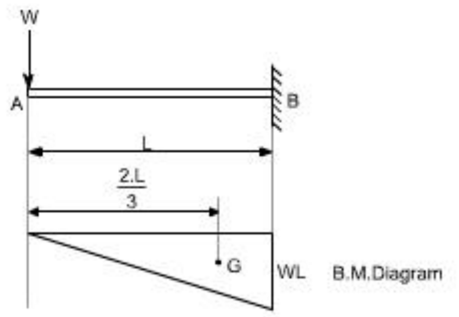

Fig.5.28 Cantilever point load at free end BM diagram

Let us workout this problem from the zero slope condition and apply the first area - moment theorem

$$\text{slope at A} = \frac{1}{EI}\left[\text{Area of B.M diagram between the points A and B}\right]$$
$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$
$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

NOTE: In this case the point B is at zero slope.

Thus,

$$\delta = \frac{1}{EI}\left[\text{first moment of area of B.M diagram between A and B about A}\right]$$
$$= \frac{1}{EI}\left[A\bar{y}\right]$$
$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$
$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid span determine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.

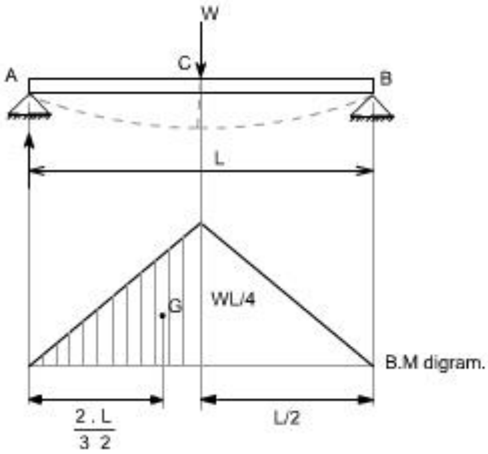

Fig.5.29 SSB and Mid-point Load - BM diagram

Again working relative to the zero slope at the centre C.

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B.M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, with a intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam, the value of maximum B.M is equal to Wl2 / 8

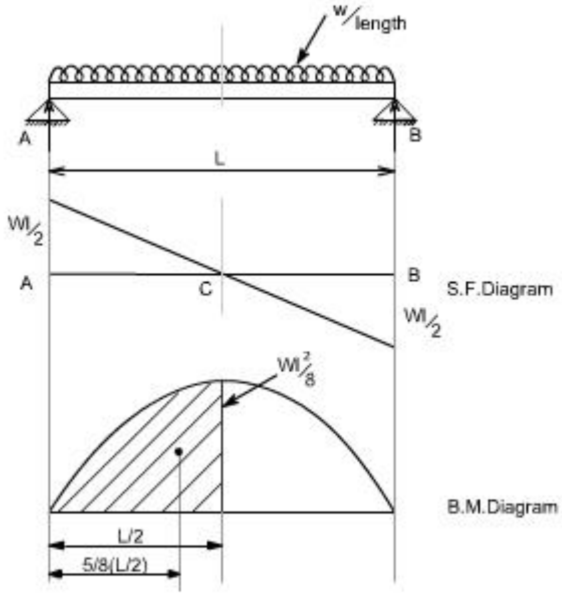

Fig.5.30 SSB with UDL – SF & BM diagram

So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}[\text{Area of B.M diagram between point A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}[A\,\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integrations be made for each such moment equation. Evaluation of the constants introduced each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e.

Laplace transform) in order to illustrate this method, however both are essentially the same.

For example consider the beam shown in fig below:

Let us write the general moment equation using the definition M = (∑ M)L, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows

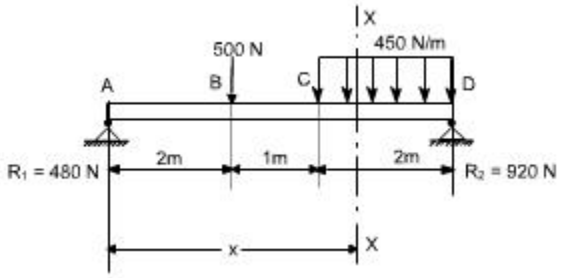

Fig.5.31 SSB with different load intensities

$$M_{AB} = 480\,x \text{ N.m}$$
$$M_{BC} = \left[480\,x - 500\,(x-2)\right] \text{N.m}$$
$$M_{CD} = \left[480\,x - 500\,(x-2) - \frac{450}{2}(x-3)^2\right] \text{N.m}$$

It may be observed that the equation for MCD will also be valid for both MAB and MBC provided that the terms (x - 2) and (x - 3)2are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and (x - 3)2 are nonexistent for values of x for which the terms in parentheses are negative.

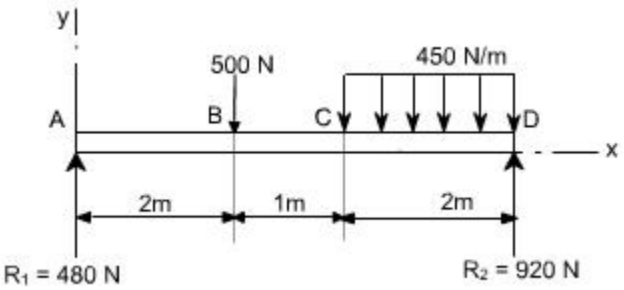

Fig.5.32 SSB with different loads

As an clear indication of these restrictions,one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

$$M = \left(480x - 500(x-2) - \frac{450}{2}(x-3)^2\right)N.m$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As an another example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward-distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:

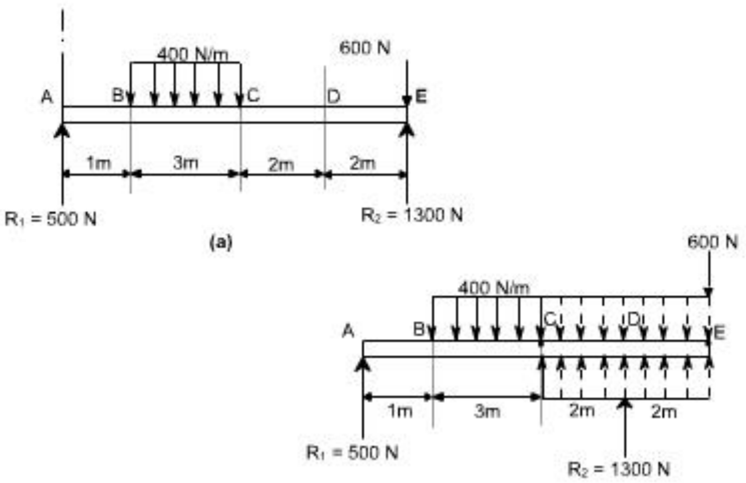

Fig.5.33 SSB with different loads

$$M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right)N.m$$

It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratary just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

llustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig. Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.

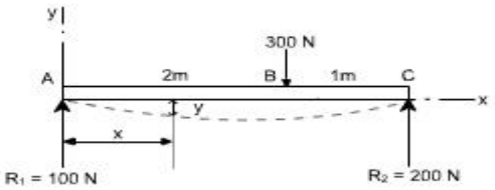

Fig.5.34 SSB with point load

Solution : writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \qquad ......(3)$$

To evaluate the two constants of integration. Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find C2 = 0.keep in mind that < x -2 >3 is to be neglected for negative values.

2. At the other support where x = 3m, the value of deflection y is also zero.

substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50\langle 3-2\rangle^3 + 3.C_1\right) \text{or } C_1 = -133N.m^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB ($0 \le x \le 2m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right) N.m^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right) N.m^3 \qquad ....(5)$$

segment BC ($2m \le x \le 3m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 150\left(x-2\right)^2 - 133x\right) N.m^2 ....(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\left(x-2\right)^3 - 133x\right) N.m^3 \; ....(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

We obtain

50 x2– 133 = 0 or x = 1.63 m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max} m = -145 N.m^3 \qquad ....(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by d, the use of y may be reserved to indicate a directed value of deflection.

if E = 30 Gpa and I = 1.9 x 106 mm4 = 1.9 x 10 -6 m4 , Eq. (h) becomes

$$y\big|_{max} m = \left(30 \times 10^9\right)\left(1.9 \times 10^{-6}\right)$$
$$= -2.54 mm$$

Example 2:

It is required to determine the value of EIy at the position midway between the supports and at the overhanging end for the beam shown in figure below.

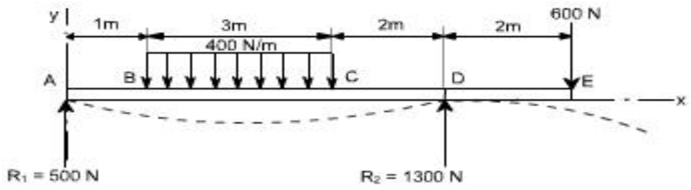

Fig.5.35 Overhanging Beam with different loads

Solution:

Writing down the moment equation which is valid for the entire span of the beam and applying the differential equation of the elastic curve, and integrating it twice, we obtain

$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right)N.m$$

$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}(x-1)^3 + \frac{200}{3}(x-4)^3 + 650(x-6)^2 + C_1\right)N.m$$

$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}(x-1)^4 + \frac{50}{3}(x-4)^4 + \frac{650}{3}(x-6)^3 + C_1x + C_2\right)N.m^3$$

To determine the value of C2, It may be noted that EIy = 0 at x = 0,which gives C2 = 0.Note that the negative terms in the pointed brackets are to be ignored Next,let us use the condition that EIy = 0 at the right support where x = 6m.This gives

$$0 = \frac{250}{3}(6)^3 - \frac{50}{3}(5)^4 + \frac{50}{3}(2)^4 + 6C_1 \quad \text{or} \quad C_1 = -1308N.m^2$$

Finally, to obtain the midspan deflection, let us substitute the value of x = 3m in the deflection equation for the segment BC obtained by ignoring negative values of the bracketed terms á x - 4 ñ4 and á x - 6 ñ3. We obtain

$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941 \ N.m^3$$

For the overhanging end where x=8 m,we have

$$EIy = \left(\frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)\right)$$

$$= -1814N.m^3$$

Example 3:

A simply supported beam carries the triangularly distributed load as shown in figure. Determine the deflection equation and the value of the maximum deflection.

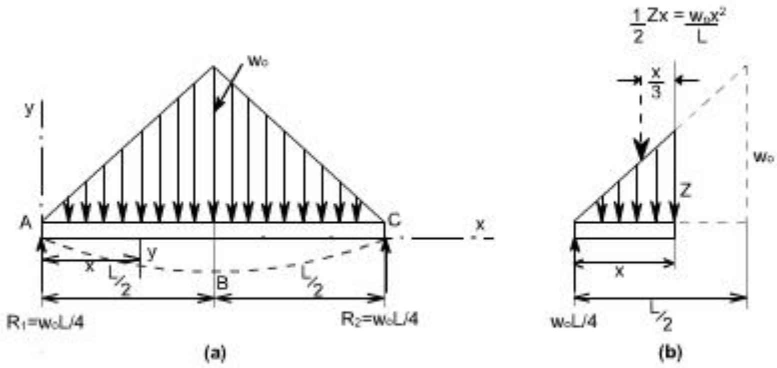

Fig.5.36 SSB with Triangular load

Solution:

Due to symmetry, the reactionsis one half the total load of 1/2w0L, or R1 = R2 = 1/4w0L.Due to the advantage of symmetry to the deflection curve from A to B is the mirror image of that from C to B. The condition of zero deflection at A and of zero slope at B do not require the use of a general moment equation. Only the moment equation for segment AB is needed, and this may be easily written with the aid of figure(b).

Taking into account the differential equation of the elastic curve for the segment AB and integrating twice, one can obtain

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{w_0L}{4}x - \frac{w_0x^2}{L}.\frac{x}{3} \qquad ......(1)$$

$$EI\frac{dy}{dx} = \frac{w_0Lx^2}{8} - \frac{w_0x^4}{12L} + C_1 \qquad ......(2)$$

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} + C_1x + C_2......(3)$$

In order to evaluate the constants of integration,let us apply the B.C'swe note that at the support A, y = 0 at x = 0.Hence from equation (3), we get C2 = 0. Also,because of symmetry, the slope dy/dx = 0 at midspan where x = L/2.Substituting these conditions in equation (2) we get

$$0 = \frac{w_0L}{8}\left(\frac{L}{2}\right)^2 - \frac{w_0}{12L}\left(\frac{L}{2}\right)^4 + C_1C_1 = -\frac{5w_0L^3}{192}$$

Hence the deflection equation from A to B (and also from C to B because of symmetry) becomes

$$EIy = \frac{w_0 L x^3}{24} - \frac{w_0 x^5}{60L} - \frac{5 w_0 L^3 x}{192}$$

Whichreducesto

$$EIy = -\frac{w_0 x}{960L}\left(25L^4 - 40L^2x^2 + 16x^4\right)$$

The maximum deflection at midspan,where $x = L/2$ is then found to be

$$\boxed{EIy = -\frac{w_0 L^4}{120}}$$

Example 4: couple acting

Consider a simply supported beam which is subjected to a couple M at adistance 'a' from the left end. It is required to determine using the Macauley's method.

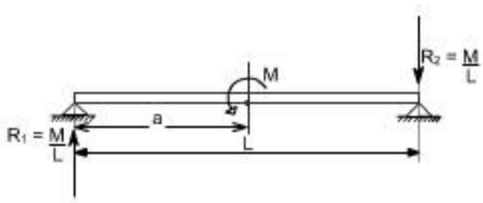

Fig.5.37 SSB with force and Couple

To deal with couples, only thing to remember is that within the pointed brackets we have to take some quantity and this should be raised to the power zero.i.e. M á x - a ñ0 . We have taken the power 0 (zero) ' because ultimately the term M á x - a ñ0 Should have the moment units.Thus with integration the quantity á x - a ñ becomes either á x - a ñ1or á x - a ñ2 Or

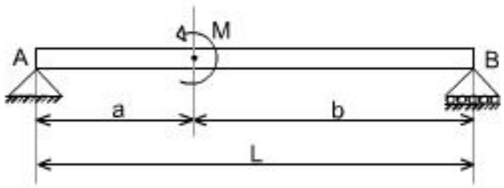

Fig.5.38 SSB with Couple

Therefore, writing the general moment equation we get

$$M = R_1 x - M\langle x - a\rangle \text{ or } EI\frac{d^2y}{dx^2} = M$$

Integrating twice we get

$$EI\frac{dy}{dx} = R_1.\frac{x^2}{2} - M\langle x - a\rangle^1 + C_1$$

$$EI.y = R_1.\frac{x^3}{6} - \frac{M}{2}\langle x - a\rangle^2 + C_1 x + C_2$$

Example 5:

A simply supported beam is subjected to U.d.l in combination with couple M. It is required to determine the deflection.

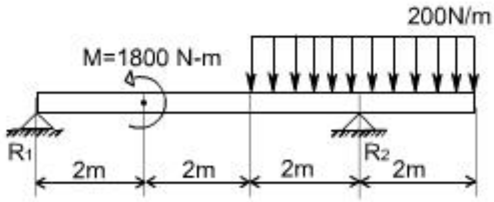

Fig.5.39 Overhanging beam with different loads

This problem may be attemped in the some way. The general moment equation my be written as

$$M(x) = R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle\langle x - 4\rangle}{2} + R_2\langle x - 6\rangle$$

$$= R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle$$

Thus,

$$EI\frac{d^2y}{dx^2} = R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle$$

Integrate twice to get the deflection of the loaded beam.

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