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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT – I –FINITE ELEMENT ANALYSIS – SME1308

UNIT – I

1D FINITE ELEMENT METHOD

1. Introduction to Finite Element Analysis

1.1 Lecture Introduction

1.1.1 Introduction

The Finite Element Method (FEM) is a numerical technique to find approximate solutions of partial differential equations. It was originated from the need of solving complex elasticity and structural analysis problems in Civil, Mechanical and Aerospace engineering. In a structural simulation, FEM helps in producing stiffness and strength visualizations. It also helps to minimize materialweight and its cost of the structures. FEM allows for detailed visualization and indicates the distribution of stresses and strains inside the body of a structure. Many of FE software are powerful yet complex tool meant for professional engineers with the training and education necessary to properly interpret the results.

Several modern FEM packages include specific components such as fluid, thermal, electromagnetic and structural working environments. FEM allows entire designs to be constructed, refined and optimized before the design is manufactured. This powerful design tool has significantly improved both the standard of engineering designs and the methodology of the design process in many industrial applications. The use of FEM has significantly decreased the time to take products from concept to the production line. One must take the advantage of the advent of faster generation of personal computers for the analysis and design of engineering product with precision level of accuracy.

1.1.2 Background of Finite Element Analysis

The finite element analysis can be traced back to the work by Alexander Hrennikoff (1941)and Richard Courant(1942). Hrenikoff introduced the framework method, in which a plane elastic medium was represented as collections of bars and beams.These pioneers share one essential characteristic: mesh discretization of a continuous domain into a set of discrete sub-domains, usually called elements.

- In 1950s, solution of large number of simultaneous equations became possible because of the digitalcomputer.
- In 1960, Ray W. Clough first published a paper using term “Finite Element Method”.
- In 1965, First conference on “finite elements” was held.
- In 1967, the first book on the “Finite Element Method” was published by Zienkiewicz and Chung.
- In the late 1960s and early 1970s, the FEM was applied to a wide variety of engineering problems.

- In the 1970s, most commercial FEM software packages (ABAQUS, NASTRAN, ANSYS, etc.) originated.Interactive FE programs on supercomputer lead to rapid growth of CAD systems.
- In the 1980s, algorithm on electromagnetic applications, fluid flow and thermal analysis were developed with the use of FE program.
- Engineers can evaluate ways to control the vibrations and extend the use of flexible, deployablestructures in space using FE and other methods in the 1990s. Trends to solve fully coupled solution of fluid flows with structural interactions, bio-mechanics related problems with a higher level of accuracy were observed in this decade.

With the development of finite element method, together with tremendous increases in computing power and convenience, today it is possible to understand structural behavior with levels of accuracy. This was in fact the beyond of imagination before the computer age.

1.1.3 Numerical Methods

The formulation for structural analysis is generally based on the three fundamental relations: equilibrium, constitutive and compatibility. There are two major approaches to the analysis: Analytical and Numerical. Analytical approach which leads to closed-form solutions is effective in case of simple geometry, boundary conditions, loadings and material properties. However, in reality, such simple cases may not arise. As a result, various numerical methods are evolved for solving such problems which are complex in nature. For numerical approach, the solutions will be approximate when any of these relations are only approximately satisfied. The numerical method depends heavily on the processing power of computers and is more applicable to structures of arbitrary size and complexity. It is common practice to use approximate solutions of differential equations as the basis for structural analysis. This is usually done using numerical approximation techniques. Few numerical methods which are commonly used to solve solid and fluid mechanics problems are given below.

- Finite Difference Method
- Finite Volume Method
- Finite Element Method
- Boundary Element Method
- Meshless Method

The application of finite difference method for engineering problems involves replacing the governing differential equations and the boundary condition by suitable algebraic equations. For example in the analysis of beam bending problem the differential equation is reduced to be solution of algebraic equations written at every nodal point within the beam member. For example, the beam equation can be expressed as:

$$\frac{d^4 w}{dx^4} = \frac{q}{EI} \tag{1.1.1}$$

To explain the concept of finite difference method let us consider a displacement function variable namely $w = f(x)$

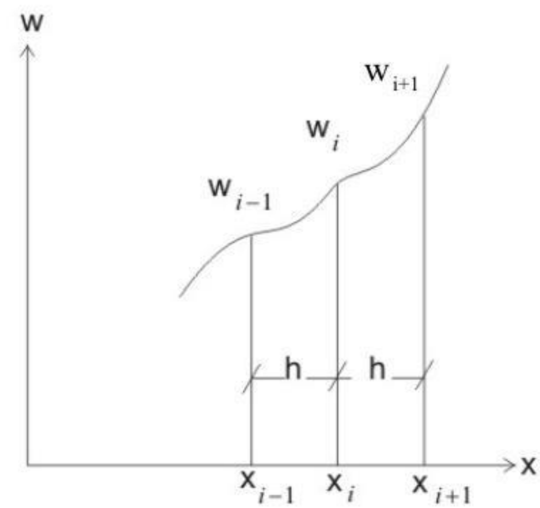


Fig. 1.1.1 Displacement Function

Now, $\Delta w = f(x + \Delta x) - f(x)$

$$\text{So, } \frac{dw}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta w}{\Delta x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{h}\left(w_{i+1} - w_i\right) \tag{1.1.2}$$

Thus,

$$\frac{d^2 w}{dx^2} = \frac{d}{dx}\left[\frac{1}{h}\left(w_{i+1} - w_i\right)\right] = \frac{1}{h^2}\left(w_{i+2} - w_{i+1} - w_{i+1} + w_i\right) = \frac{1}{h^2}\left(w_{i+2} - 2w_{i+1} + w_i\right) \tag{1.1.3}$$

$$\begin{aligned} \frac{d^3 w}{dx^3} &= \frac{1}{h^3}\left(w_{i+3} - w_{i+2} - 2w_{i+2} + 2w_{i+1} + w_{i+1} - w_i\right) \\ &= \frac{1}{h^3}\left(w_{i+3} - 3w_{i+2} + 3w_{i+1} - w_i\right) \end{aligned} \tag{1.1.4}$$

$$\frac{d^4w}{dx^4} = \frac{1}{h^4}\left(w_{i+4} - w_{i+3} - 3w_{i+3} + 3w_{i+2} + 3w_{i+2} - 3w_{i+1} - w_{i+1} + w_i\right)$$
$$= \frac{1}{h^4}\left(w_{i+4} - 4w_{i+3} + 6w_{i+2} - 4w_{i+1} + w_i\right) \tag{1.1.5}$$
$$= \frac{1}{h^4}\left(w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}\right)$$

Thus, eq. (1.1.1) can be expressed with the help of eq. (1.1.5) and can be written in finite difference form as:

$$(w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}) = \frac{q}{EI}h^4 \tag{1.1.6}$$

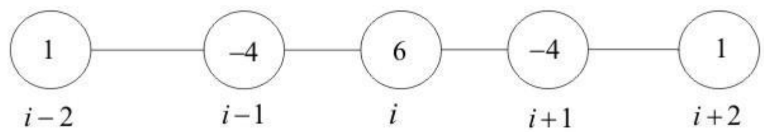


Fig. 1.1.2 Finite difference equation at node i

Thus, the displacement at node *i* of the beam member corresponds to uniformly distributed load can be obtained from eq. (1.1.6) with the help of boundary conditions. It may be interesting to note that, the concept of node is used in the finite difference method. Basically, this method has an array of grid points and is a point wise approximation, whereas, finite element method has an array of small interconnecting sub-regions and is a piece wise approximation.

Each method has noteworthy advantages as well as limitations. However it is possible to solve various problems by finite element method, even with highly complex geometry and loading conditions, with the restriction that there is always some numerical errors. Therefore, effective and reliable use of this method requires a solid understanding of its limitations.

1.1.4 Concepts of Elements and Nodes

Any continuum/domain can be divided into a number of pieces with very small dimensions. These small pieces of finite dimension are called ‘Finite Elements’ (Fig. 1.1.3). A field quantity in each element is allowed to have a simple spatial variation which can be described by polynomial terms. Thus the original domain is considered as an assemblage of number of such small elements. These elements are connected through number of joints which are called ‘Nodes’. While discretizing the structural system, it is assumed that the elements are attached to the adjacent elements only at the nodal points. Each element contains the material and geometrical properties. The material properties inside an element are assumed to be constant. The elements may be 1D elements, 2D elements or 3D elements. The physical object can be modeled by choosing appropriate element such as frame element, plate element, shell element, solid element, etc. All elements are then assembled to obtain the solution of the entire domain/structure under certain loading conditions. Nodes are assigned at a certain density throughout the continuum depending on the anticipated stress levels of a particular domain. Regions which will receive large amounts of stress variation usually have a higher node density than those which experience little or no stress.

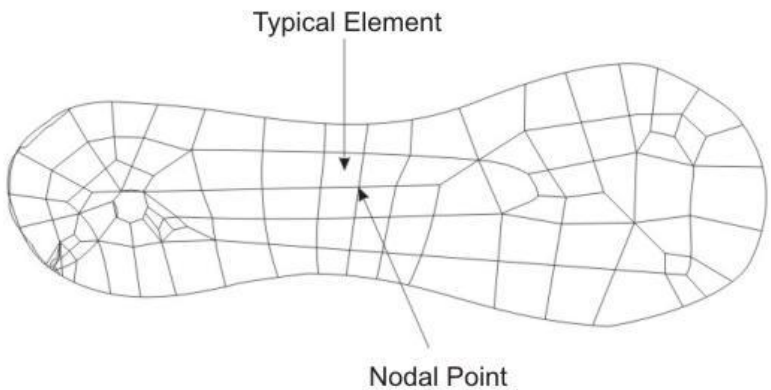


Fig. 1.1.3 Finite element discretization of a domain

1.1.5 Degrees of Freedom

A structure can have infinite number of displacements. Approximation with a reasonable level of accuracy can be achieved by assuming a limited number of displacements. This finite number of displacements is the number of degrees of freedom of the structure. For example, the truss member will undergo only axial deformation. Therefore, the degrees of freedom of a truss member with respect to its own coordinate system will be one at each node. If a two dimension structure is modeled by truss elements, then the deformation with respect to structural coordinate system will be two and therefore degrees of freedom will also become two. The degrees of freedom for various types of element are shown in Fig. 1.1.4 for easy understanding. Here (u, v, w) and $(\theta_x, \theta_y, \theta_z)$ represent displacement and rotation respectively.

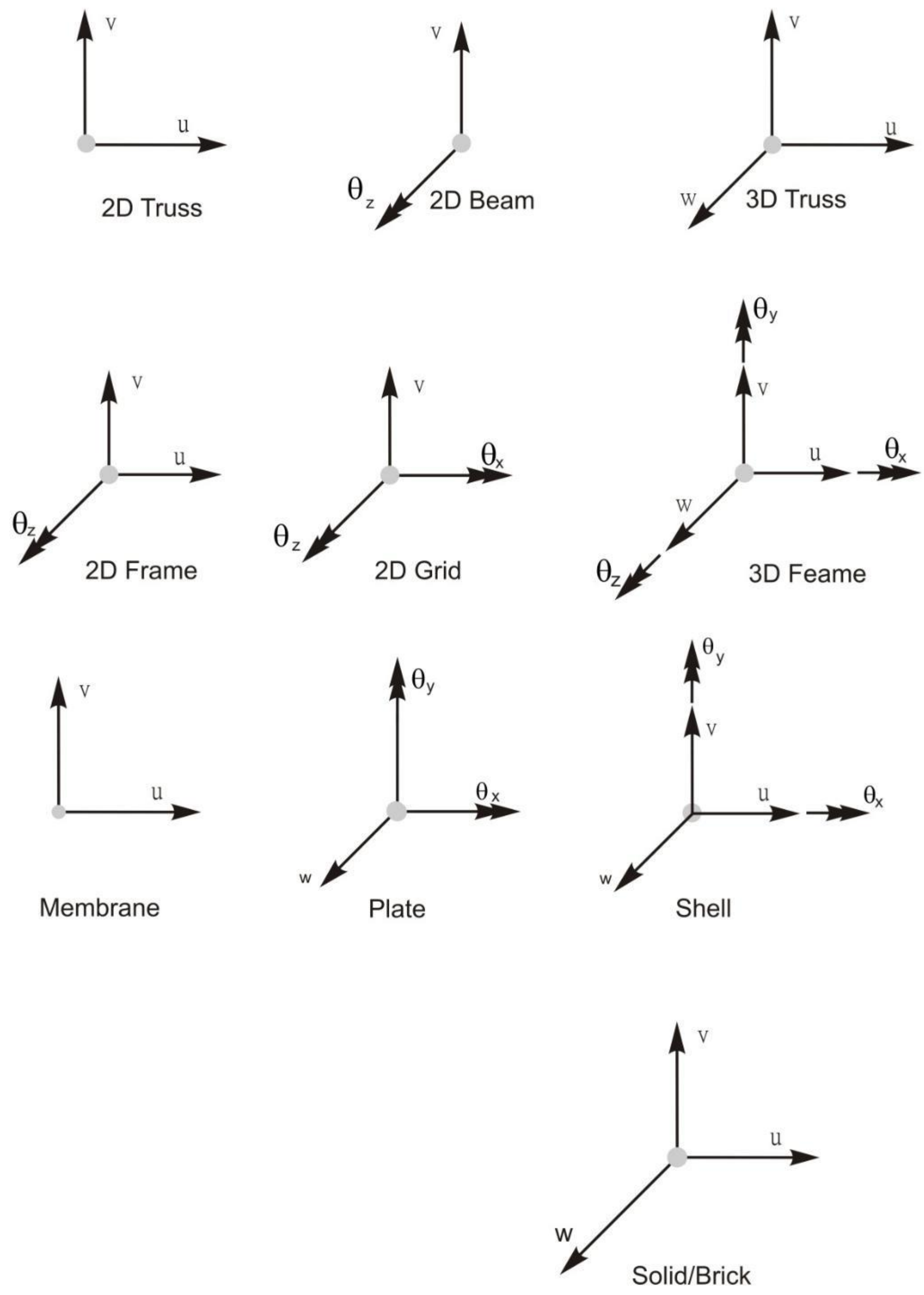


Fig. 1.1.4 Degrees of Freedom for Various Elements

1.2. Basic Concepts of Finite Element Analysis

1.2.1 Idealization of a Continuum

A continuum may be discretized in different ways depending upon the geometrical configuration of the domain. Fig. 1.2.1 shows the various ways of idealizing a continuum based on the geometry.

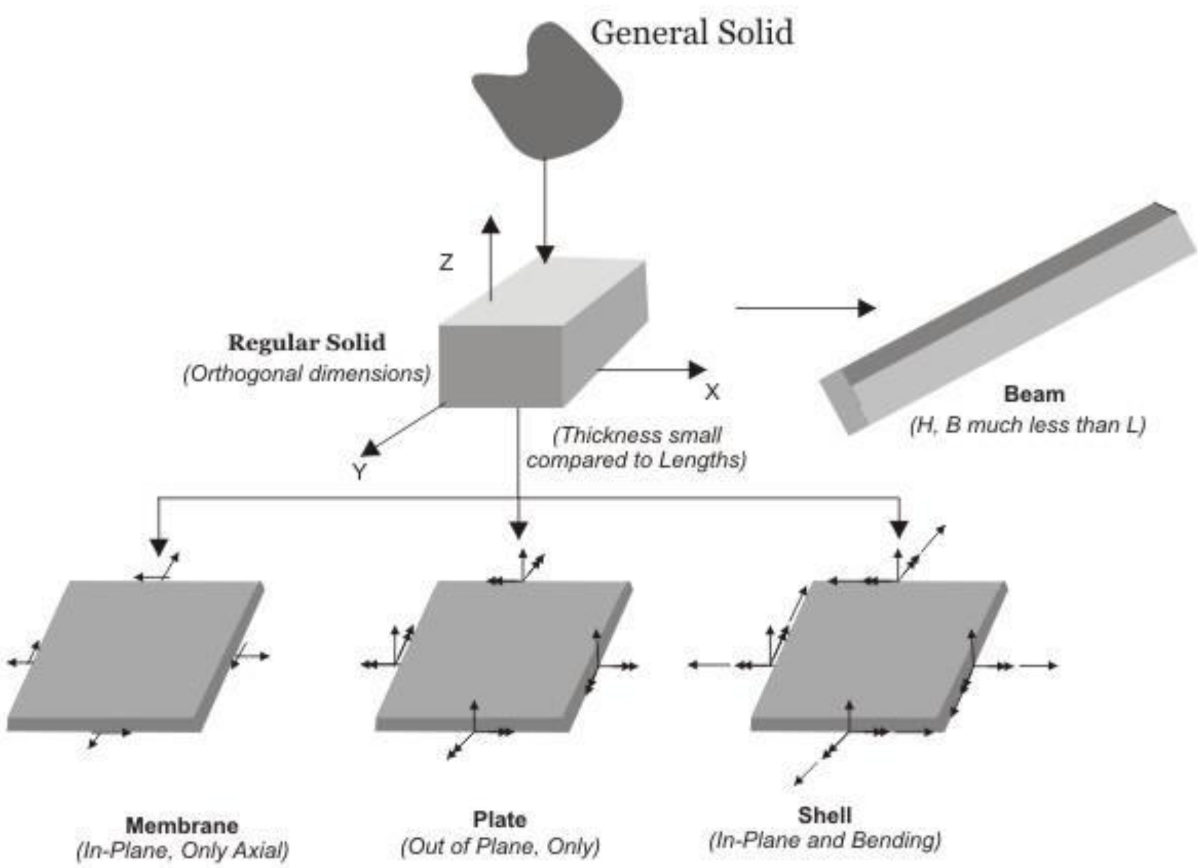
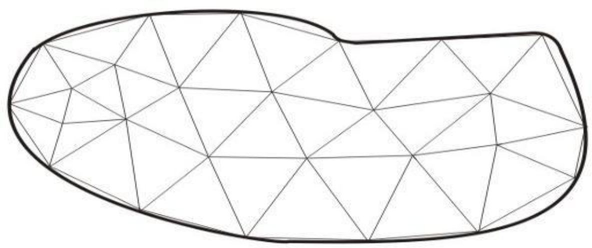


Fig. 1.2.1 Various ways of Idealization of a Continuum

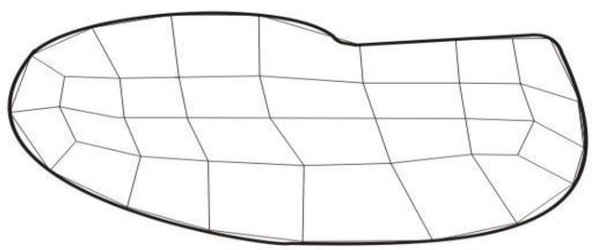
1.2.2 Discretization of Technique

The need of finite element analysis arises when the structural system in terms of its either geometry, material properties, boundary conditions or loadings is complex in nature. For such case, the whole structure needs to be subdivided into smaller elements. The whole structure is then analyzed by the assemblage of all elements representing the complete structure including its all properties.

The subdivision process is an important task in finite element analysis and requires some skill and knowledge. In this procedure, first, the number, shape, size and configuration of elements have to be decided in such a manner that the real structure is simulated as closely as possible. The discretization is to be in such that the results converge to the true solution. However, too fine mesh will lead to extra computational effort. Fig. 1.2.2 shows a finite element mesh of a continuum using triangular and quadrilateral elements. The assemblage of triangular elements in this case shows better representation of the continuum. The discretization process also shows that the more accurate representation is possible if the body is further subdivided into some finer mesh.



(a) Triangular mesh



(b) Quadrilateral mesh

Fig. 1.2.2 Discretization of a continuum

1.2.3 Concepts of Finite Element Analysis

FEA consists of a computer model of a continuum that is stressed and analyzed for specific results. A continuum has infinite particles with continuous variation of material properties. Therefore, it needs to simplify to a finite size and is made up of an assemblage of substructures, components and members. Discretization process is necessary to convert whole structure to an assemblage of members/elements for determining its responses. Fig. 1.2.3 shows the process of idealization of actual structure to a finite element form to obtain the response results. The assumptions are required to be made by the experienced engineer with finite element background for getting appropriate response results. On the basis of assumptions, the appropriate constitutive model can be constructed.

For the linear-elastic-static analysis of structures, the final form of equation will be made in the form of $F=Kd$where F, K and d are the nodal loads, global stiffness and nodal displacements respectively.

Classical

Actual Structure

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\sigma_z}{\partial z}+F_\Omega=0$$

(Partial Differential Equations)

Assumptions

Equilibrium Equations

Stress-Strain Law

Compatibility Conditions

$$\int_\Omega \delta\{\varepsilon\}^T\{\sigma\}d\Omega=\int_\Gamma \delta\{d\}^T\{F_\Gamma\}d\Gamma+\int_\Omega \delta\{d\}^T\{F_\Omega\}d\Omega$$

(Principles of Virtual Work)

FEM

Structural Model

$$\{F\}=[K]\{d\}$$

(Algebraic Equations)

Fig. 1.2.3Fromclassical to FE solution

Varieties of engineering problem like solid and fluid mechanics, heat transfer can easily be solved by the concept of finite element technique. The basic form of the equation will become as follows where action, property and response parameter will vary for case to case as outlined in Table 1.2.1.

$$\{F\} = [\,K\,]\{d\} \; OR \; \{d\} = [K]^{-1}\{F\}$$

Table 1.2.1 Response parameters for different cases

	Property	**Action**	**Response**
Solid	Stiffness	Load	Displacement
Fluid	Viscosity	Body force	Pressure/Velocity
Thermal	Conductivity	Heat	Temperature

1.2.4 Advantages of FEA

1. The physical properties, which are intractable and complex for any closed bound solution, can be analyzed by this method.
2. It can take care of any geometry (may be regular or irregular).
3. It can take care of any boundary conditions.
4. Material anisotropy and non-homogeneity can be catered without much difficulty.
5. It can take care of any type of loading conditions.
6. This method is superior to other approximate methods like Galerkine and Rayleigh-Ritz methods.
7. In this method approximations are confined to small sub domains.
8. In this method, the admissible functions are valid over the simple domain and have nothing to do with boundary, however simple or complex it may be.
9. Enable to computer programming.

1.2.5 Disadvantages of FEA

1. Computational time involved in the solution of the problem is high.
2. For fluid dynamics problems some other methods of analysis may prove efficient than the FEM.

1.2.6 Limitations of FEA

1. Proper engineering judgment is to be exercised to interpret results.
2. It requires large computer memory and computational time to obtainintend results.
3. There are certain categories of problems where other methods are more effective, e.g., fluid problems having boundaries at infinity are better treated by the boundary element method.
4. For some problems, there may be a considerable amount of input data. Errors may creep up in their preparation and the results thus obtained may also appear to be acceptable which indicates deceptive state of affairs. It is always desirable to make a visual check of the input data.
5. In the FEM,many problems lead to round-off errors. Computer works with a limited number of digits and solving the problem with restricted number of digits may not yield the desired degree of accuracy or it may give total erroneous results in some cases. For many problems the increase in the number of digits for the purpose of calculation improves the accuracy.

1.2.7 Errors and Accuracy in FEA

Every physical problem is formulated by simplifying certain assumptions. Solution to the problem, classical or numerical, is to be viewed within the constraints imposed by these simplifications. The material may be assumed to be homogeneous and isotropic; its behavior may be considered as linearly elastic; the prediction of the exact load in any type of structure is next to impossible. As such the true behavior of the structure is to be viewed with in these constraints and obvious errors creep in engineering calculations.

1. The results will be erroneous if any mistake occurs in the input data. As such, preparation of the input data should be made with great care.
2. When a continuum is discretised, an infinite degrees of freedom system is converted into a model having finite number of degrees of freedom. In a continuum, functions which are continuous are now replaced by ones which are piece-wise continuous within individual elements. Thus the actual continuum is represented by a set of approximations.
3. The accuracy depends to a great extent on the mesh grading of the continuum. In regions of high strain gradient, higher mesh grading is needed whereas in the regions of lower strain, the mesh chosen may be coarser. As the element size decreases, the discretisation error reduces.
4. Improper selection of shape of the element will lead to a considerable error in the solution. Triangle elements in the shape of an equilateral or rectangular element in the shape of a square will always perform better than those having unequal lengths of the sides. For very long shapes, the attainment of convergence is extremely slow.
5. In the finite element analysis, the boundary conditions are imposed at the nodes of the element whereas in an actual continuum, they are defined at the boundaries. Between the

nodes, the actual boundary conditions will depend on the shape functions of the element forming the boundary.

6. Simplification of the boundary is another source of error. The domain may be reduced to the shape of polygon. If the mesh is refined, then the error involved in the discretized boundary may be reduced.
7. During arithmetic operations, the numbers would be constantly round-off to some fixed working length. These round–off errors may go on accumulating and then resulting accuracy of the solution may be greatly impaired.

1.4. Steps in FEA:

1.4.1 Loading Conditions

There are multiple loading conditions which may be applied to a system. The load may be internal and/or external in nature. Internal stresses/forces and strains/deformations are developed due to the action of loads.Most loads are basically "Volume Loads" generated due to mass contained in a volume. Loads may arise from fluid-structure interaction effects such as hydrodynamic pressure of reservoir on dam, waves on offshore structures, wind load on buildings, pressure distribution on aircraft etc. Again, loads may be static, dynamic or quasi-static in nature. All types of static loads can be represented as:

3. Point loads
4. Line loads
5. Area loads
6. Volume loads

The loads which are not acting on the nodal points need to be transferred to the nodes properly using finite element techniques.

1.4.2 Support Conditions

In finite element analysis, support conditions need to be taken care in the stiffness matrix of the structure. For fixed support, the displacement and rotation in all the directionswill be restrained and accordingly, the global stiffness matrix has to modify. If the support prevents translation only in one direction, it can be modeled as 'roller' or 'link supports'. Such link supports are commonly used in finite element software to represent the actual structural state. Sometimes, the support itself undergoes translation under loadings. Such supports are called as 'elastic support' and are modeled with 'spring'. Such situation arises if the structures are resting on soil. The supports may be represented in finite element modeling as:

6. Point support
7. Line support
8. Area support
9. Volume support

1.4.3 Type of Engineering Analysis

Finite element analysis consists of linear and non-linear models. On the basis of the structural system and its loadings, the appropriate type of analysis is chosen. The type of analysis to be carried out depends on the following criteria:

6. Type of excitation (loads)
7. Type of structure (material and geometry)
8. Type of response

Considering above aspects, types of engineering analysis are decided. FEA is capable of using multiple materials within the structure such as:

Isotropic (i.e., identical throughout)

Orthotropic (i.e., identical at 90^0)

General anisotropic (i.e., different throughout)

The Equilibrium Equations for different cases are as follows:

8. Linear-Static:

$$Ku = F \qquad (1.4.1)$$

2. Linear-Dynamic

$$Mu(t) + Cu(t) + Ku(t) = F(t) \qquad (1.4.2)$$

3. **Nonlinear - Static**

$$Ku + F_{NL} = F \qquad (1.4.3)$$

1. Nonlinear-Dynamic

$$Mu(t) + Cu(t) + Ku(t) + F(t)_{NL} = F(t) \qquad (1.4.4)$$

Here, M, C, K, F and U are mass, damping, stiffness, force and displacement of the structure respectively. Table 1.4.1 shows various types of analysis which can be performed according to engineering judgment.

Table 1.4.1 Types of analysis

Excitation	Structure	Response	Basic analysis type
Static	Elastic	Linear	Linear-Elastic-Static Analysis
Static	Elastic	Nonlinear	Nonlinear-Elastic-Static Analysis
Static	Inelastic	Linear	Linear-Inelastic-Static Analysis
Static	Inelastic	Nonlinear	Nonlinear-Inelastic-Static Analysis
Dynamic	Elastic	Linear	Linear-Elastic-Dynamic Analysis
Dynamic	Elastic	Nonlinear	Nonlinear-Elastic-Dynamic Analysis
Dynamic	Inelastic	Linear	Linear-Inelastic-Dynamic Analysis
Dynamic	Inelastic	Nonlinear	Nonlinear-Inelastic-Dynamic Analysis

1.4.4 Basic Steps in Finite Element Analysis

The following steps are performed for finite element analysis.

Discretisation of the continuum: The continuum is divided into a number of elements by imaginary lines or surfaces. The interconnected elements may have different sizes and shapes.

Identification of variables: The elements are assumed to be connected at their intersecting points referred to as nodal points. At each node, unknown displacements are to be prescribed.

Choice of approximating functions: Displacement function is the starting point of the mathematical analysis. This represents the variation of the displacement within the element. The displacement function may be approximated in the form a linear function or a higher-order function. A convenient way to express it is by polynomial expressions. The shape or geometry of the element may also be approximated.

Formation of the element stiffness matrix: After continuum is discretised with desired element shapes, the individual element stiffness matrix is formulated. Basically it is a minimization procedure whatever may be the approach adopted. For certain elements, the form involves a great deal of sophistication. The geometry of the element is defined in reference to the global frame. Coordinate transformation must be done for elements where it is necessary.

Formation of overall stiffness matrix: After the element stiffness matrices in global coordinates are formed, they are assembled to form the overall stiffness matrix. The assembly is done through the nodes which are common to adjacent elements. The overall stiffness matrix is symmetric and banded.

Formation of the element loading matrix: The loading forms an essential parameter in any structural engineering problem. The loading inside an element is transferred at the nodal points and consistent element matrix is formed.

Formation of the overall loading matrix: Like the overall stiffness matrix, the element loading matrices are assembled to form the overall loading matrix. This matrix has one column per loading case and it is either a column vector or a rectangular matrix depending on the number of loading cases.

Incorporation of boundary conditions: The boundary restraint conditions are to be imposed in the stiffness matrix. There are various techniques available to satisfy the boundary conditions. One is the size of the stiffness matrix may be reduced or condensed in its final form. To ease computer programming aspect and to elegantly incorporate the boundary conditions, the size of overall matrix is kept the same.

Solution of simultaneous equations: The unknown nodal displacements are calculated by the multiplication of force vector with the inverse of stiffness matrix.

Calculation of stresses or stress-resultants: Nodal displacements are utilized for the calculation of stresses or stress-resultants. This may be done for all elements of the continuum or it may be limited to some predetermined elements. Results may also be obtained by graphical means. It may desirable to plot the contours of the deformed shape of the continuum.

The basic steps for finite element analysis are shown in the form of flow chart below:

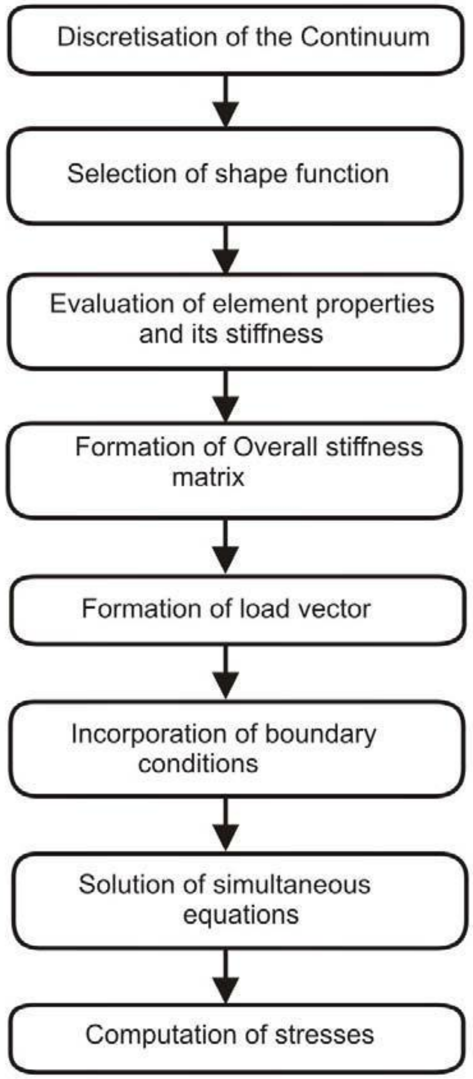


Fig. 1.4.1 Flowchart for steps in FEA

1.4.5 Element Library in FEA Software

A real structure can be modeled with various ways with appropriate assumptions. The structure may be divided into following categories:

Cable or tension structures

Skeletal or framed structures

Surface or spatial structures

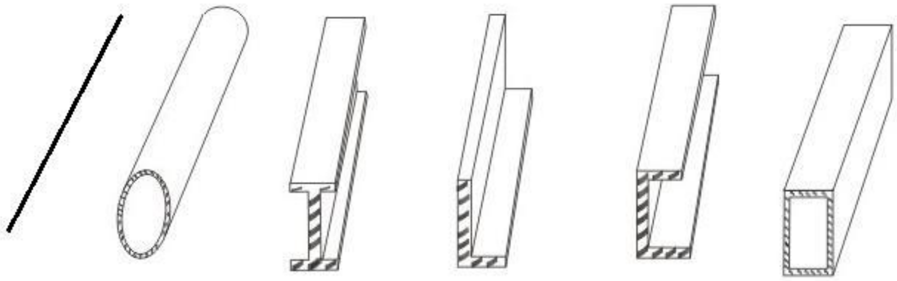
Solid structures

Mixed structures

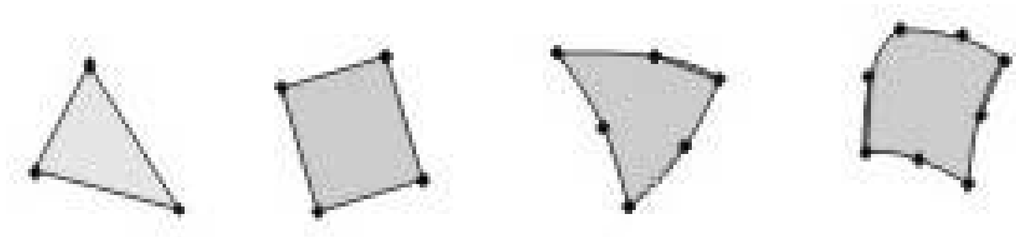
The configuration of structural elements depends upon the geometry of the structural system and the number of independent space coordinates (i.e., x, y and z) required to describe the problem. Thus, the element can be categorized as one, two or three dimensional element. One dimensional element can be represented by a straight line whose ends will be nodal points. The skeletal structures are generally modeled by this type of elements. The pin jointed bar or truss element is the simplest structural element. This element undergoes only axial deformation. The beam element is another type of element which undergoes in-plane transverse displacements and rotations. The frame element is the combination of truss and beam element. Thus, the frame element has axial and in-plane transverse displacements and rotations. This element is generally used to model 1D, 2D and 3D skeletal structural systems. Two-dimensional elements are generally used to model 2D and 3D continuum. These elements are of constant thickness and material properties. The shapes of these elements are triangular or rectangular and it consists of 3 to 9 or even more nodes. These elements are used to solve many problems in solid mechanics such as plane stress, plane strain, plate bending. Three-dimensional element is the most cumbersome which is generally used to model the 3-D continuum. The elements have 6 to 27 numbers of nodes or more. Because of large degrees of freedom, the analysis is time consuming using 3-D elementsand difficult to interpret its results. However, for accurate analysis of the irregular continuum, 3-D elements are useful. To analyze any real structure, appropriate elements are to be assigned for the finite element analysis. In standard FEA software, following types of element library are used to discretize the domain.

Truss element
Beam element
Frame element
Membrane/ Plate/Shell element
Solid element
Composite element
Shear panel
Spring element
Rigid/Link element
Viscous damping element

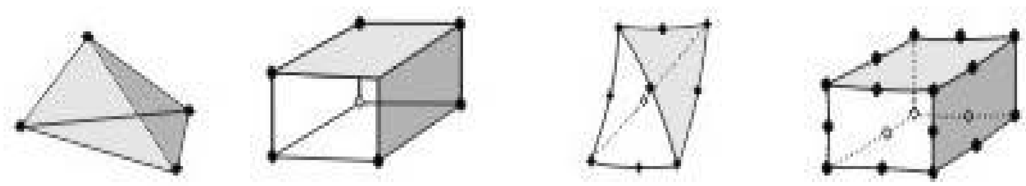
The different types of elements available in standard finite element software are shown in Fig. 1.4.2.



1D Elements (Truss,beam, grid and frame)



2D Elements(Plane stress, Plane strain, Axisymmetric, Plate and Shell)



3D Elements

Fig. 1.4.2Varioustypes of elements for computer modeling

Module: 2 Finite Element Formulation Techniques
Lecture 1: Virtual Work and Variational Principle

2.1.1 Introduction

Finite element formulation can be constructed from governing differential equations over a domain. This can be formulated by various ways like Virtual Work Method, VariationalMethod, Weighted Residual Method etc.

2.1.2 Principle of Virtual Work

The principle of virtual work is a very useful approach for solving varieties of structural mechanics problem. When the force and displacement are unrelated to the cause and effect relation, the work is called virtual work. Therefore, the virtual work may be caused by true force moving through imaginary displacements or vice versa. Thus, the principle of virtual work can be divided into two categories: (a) principle of virtual forces and (b) principle of virtual displacements. The principle of virtual forces establishes the compatibility conditions. The principle of virtual displacements establishes the conditions of equilibrium and is used in the displacement model of the finite element technique.

The external virtual work is the work done by real load moving through imaginary displacements in a structure. These loads include both the load distributed over the entire surface and volume. Thus, the virtual work done by the external force is:

$$\delta W_E = \int_\Gamma \{\delta u \quad \delta v \quad \delta w\} \begin{Bmatrix} F_{\Gamma x} \\ F_{\Gamma y} \\ F_{\Gamma z} \end{Bmatrix} d\Gamma + \int_\Omega \{\delta u \quad \delta v \quad \delta w\} \begin{Bmatrix} F_{\Omega x} \\ F_{\Omega y} \\ F_{\Omega z} \end{Bmatrix} d\Omega \tag{2.1.1}$$

Where, δu, δv and δw are the components of the virtual displacements in x, y and z direction respectively. $F_{\Gamma x}$, $F_{\Gamma y}$ and $F_{\Gamma z}$ are the surface forces and $F_{\Omega x}$, $F_{\Omega y}$and $F_{\Omega z}$ are the body forces in x, y and z direction respectively. In the above equation, the integration is carried out over the entire surface in the first term and over the entire volume in the second term. The above expression can be rewritten as:

$$\delta W_E = \int_\Gamma \delta\{d\}^T \{F_\Gamma\} d\Gamma + \int_\Omega \delta\{d\}^T \{F_\Omega\} d\Omega \tag{2.1.2}$$

Here, $\{d\}^T = \{u \quad v \quad w\}$.For the three dimensional stress-strain condition, there are six components of stresses $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$ and six components of strains in virtual displacement fields $(\delta\varepsilon_x, \delta\varepsilon_y, \delta\varepsilon_z, \delta\gamma_{xy}, \delta\gamma_{yz}, \delta\gamma_{zx})$.Therefore, the virtual internal work can be expressed as follows:

$$\delta U=\int_{\Omega}\left\{\delta\varepsilon_x \quad \delta\varepsilon_y \quad \delta\varepsilon_z \quad \delta\gamma_{xy} \quad \delta\gamma_{yz} \quad \delta\gamma_{zx}\right\}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}d\Omega \qquad (2.1.3)$$

Or

$$\delta U=\int_{\Omega}\delta\{\varepsilon\}^T\{\sigma\}d\Omega \qquad (2.1.4)$$

According to principle of virtual work, the work done by external forces due to the virtual displacement of a structure in equilibrium is equal to the work done by the internal forces for the virtual internal displacement. Therefore, $\delta W_E=\delta U$ Thus eqs. (2.1.2) and (2.1.4) can be made equal and can be related as follows:

$$\int_{\Gamma}\delta\{d\}^T\{F_\Gamma\}d\Gamma+\int_{\Omega}\delta\{d\}^T\{F_\Omega\}d\Omega=\int_{\Omega}\delta\{\varepsilon\}^T\{\sigma\}d\Omega \qquad (2.1.5)$$

2.1.3 Variational Principle

Variational formulation is the generalized method of formulating the element stiffness matrix and load vector using the variational principle of solid mechanics. The strain energy in a structural body is given by the relation

$$U=\frac{1}{2}\iiint_{\Omega}\{\varepsilon\}^T\{\sigma\}d\Omega \qquad (2.1.6)$$

For a 3D structural problem, stress has six components: $\{\sigma\}^T=\{\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{yz},\tau_{zx}\}$. Similarly, there are six components of strains: $\{\varepsilon\}^T=\{\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx}\}$. Now the strain-displacement relationship can be expressed as $\{\varepsilon\}=[B]\{d\}$, where $\{d\}$ is the displacement vector in x, y and z directions and $[B]$ is called as the strain displacement relationship matrix. Again, the stress can be represented in terms of its constitutive relationship matrix: $\{\sigma\}=[D]\{\varepsilon\}$. Here $[D]$ is called as the constituent relationship matrix.Using the above relationship in the strain energy equation one can arrive

$$U=\frac{1}{2}\iiint_{\Omega}[[B]\{d\}]^T[D]\{B\}\{d\}d\Omega \qquad (2.1.7)$$

Applying the variational principle one can express

$$\{F\} = \frac{\partial U}{\partial\{d\}} = \iiint_{\Omega} [B]^T [D][B] d\Omega \{d\} \tag{2.1.8}$$

Now, from the relationship of $\{F\} = [K]\{d\}$, one can arrive at the element stiffness matrix as:

$$[K] = \iiint_{\Omega} [B]^T [D][B] d\Omega \tag{2.1.9}$$

Thus, by the use of variational principle, the stiffness matrix of a structural element can be obtained as expressed in the above equation.

2.1.4Weighted ResidualMethod

Virtual work and Variational method are applicable and adequate for most of the problems. However, in some cases functional analogous to potential energy cannot be written because of not having clear physical meaning. For some applications, such as in fluid mechanics problem, functional needed for a variational approach cannot be expressed. For some types of fluid flow problems, only differential equations and boundary conditions are available. For Such problems weighted residual method can be used for obtaining the solutions. Approximate solutions of differential equation satisfy only part of conditions of the problem. For example a differential equation may be satisfied only at few points, rather than at each. The strategy used in weighted residual method is to first take an approximate solution and then its validity is assessed. The different methods in weighted Residual Method are

- Collocation method
- Least square method
- Method of moment
- Galerkin method

The mathematical statement of a physical problem can be defined as:

In domainΩ,

$$\mathrm{Du} - \mathrm{f} = 0 \tag{2.1.10}$$

Where,

D is the differential operator

u = u(x) = dependent variables such as displacement, pressure, velocity, potential function

x = independent variables such as coordinates of a point

f = a function of x which may be constant or zero

If $\overline{\mathrm{u}}$ is an approximate solution then residual in domainΩ,

$$\mathrm{R} = \mathrm{D}\overline{\mathrm{u}} - \mathrm{f} \tag{2.1.11}$$

According to the weighted residual method, the weak form of above equation will become

$$\int_{\Omega} w_i \; R \; d\Omega = 0 \quad \text{for } i = 1, 2, 3, \ldots, n$$

or (2.1.12)

$$\int_{\Omega} w_i \left(D\overline{u} - f \right) d\Omega = 0$$

Where weighting function $w_i = w_i(x)$ is chosen from the approximate basis function used for constructing approximated solution $\overline{u}$.

Lecture 2: Galerkin Method

2.2.1 Introduction

Galerkin method is the most widely used among the various weighted residual methods. Galerkin method incorporates differential equations in their weak form, i.e., before starting integration by parts it is in strong form and after by parts it will be in weak form, so that they are satisfied over a domain in an integral. Thus, in case of Galerkin method, the equations are satisfied over a domain in an integral or average sense, rather than at every point. The solution of the equations must satisfy the boundary conditions. There are two types of boundary conditions:

- Essential or kinematic boundary condition
- Non essential or natural boundary condition

For example, in case of a beam problem ($EI\dfrac{\partial^4 y}{\partial x^4} - q = 0$) differential equation is of fourth order. As a result, displacement and slope will be essential boundary condition where as moment and shear will be non-essential boundary condition.

2.2.2 Galerkin Method for2D Elasticity Problem

For a two dimensional elasticity problem, equation of equilibrium can be expressed as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_{\Omega x} = 0 \qquad (2.2.1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_{\Omega y} = 0 \qquad (2.2.2)$$

Where, $F_{\Omega x}$ and $F_{\Omega y}$ are the body forces in X and Y direction respectively. Let assume, $\Gamma_{\Gamma x}$ and $\Gamma_{\Gamma y}$ are surface forces in X and Y direction and α as angle made by normal to surface with X– axis (Fig. 2.2.1). Therefore, force equilibrium of element can be written as:

$$F_{\Gamma x}(PQ)t = \sigma_x(OP)t + \tau_{xy}(OQ)t$$

$$F_{\Gamma x} = \sigma_x\frac{OP}{PQ} + \tau_{xy}\frac{OQ}{PQ} = \sigma_x\cos\alpha + \tau_{xy}\sin\alpha = \sigma_x\cos\alpha + \tau_{xy}\mathrm{Cos}(90-\alpha)$$

$$\text{Thus, } F_{\Gamma x} = \sigma_x\ell + \tau_{xy}m \qquad (2.2.3)$$

Where, ℓ and m are direction cosines of normal to the surface. Similarly,

$$F_{\Gamma y} = \tau_{xy}\ell + \sigma_y m \qquad (2.2.4)$$

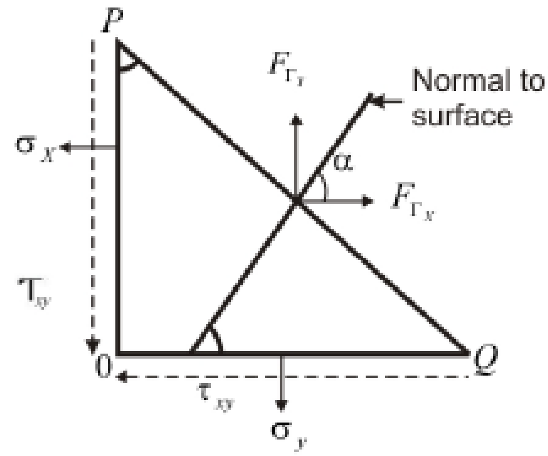


Fig. 2.2.1 Elemental stresses in 2D

AdoptingGalerkin'sapproach using eq. (2.2.2 and 2.2.3)

$$\left[\iint\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_{\Gamma x}\right)\delta u+\iint\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_{\Gamma y}\right)\delta v\right]dxdy=0 \qquad (2.2.5)$$

Where δu and δv are weighting functions i.e elemental displacements in X and Y directions respectively. Now one can expand above equation by using Green's Theorem.

Green Theorem states that if $\phi(x,y)$ and $\psi(x,y)$ are continuous functions then their first and second partial derivatives are also continuous. Therefore,

$$\iint\left[\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}\right]dxdy=-\iint\phi\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]dxdy+\int\phi\left[\frac{\partial\psi}{\partial x}\ell+\frac{\partial\psi}{\partial y}m\right]ds \qquad (2.2.6)$$

Assuming, $\phi=\sigma_x;\frac{\partial\psi}{\partial x}=\delta u;\ \frac{\partial\psi}{\partial y}=0$ one can rewrite with the use of above relationas

$$\iint\frac{\partial\sigma_x}{\partial x}\delta u\ dx\ dy=-\iint\sigma_x\frac{\partial(\delta u)}{\partial x}dx\ dy+\int\sigma_x\ell\ \delta u\ ds \qquad (2.2.7)$$

Similarly, assuming $\phi=\sigma_y;\frac{\partial\psi}{\partial x}=0$ and $\frac{\partial\psi}{\partial y}=\delta v$

$$\iint\frac{\partial\sigma_y}{\partial y}\delta v\ dx\ dy=-\iint\sigma_y\frac{\partial(\delta v)}{\partial y}dx\ dy+\int\sigma_y m\ \delta v\ ds \qquad (2.2.8)$$

Again, assuming $\phi = \tau_{xy}; \dfrac{\partial \psi}{\partial x} = \delta v; \dfrac{\partial \psi}{\partial y} = 0$

$$\iint \frac{\partial \tau_{xy}}{\partial y} \delta v \, dx \, dy = -\iint \tau_{xy} \frac{\partial (\delta v)}{\partial x} dx \, dy + \int \tau_{xy} \ell \, \delta v \, ds \tag{2.2.9}$$

And assuming, $\phi = \tau_{xy}; \dfrac{\partial \psi}{\partial x} = 0; \dfrac{\partial \psi}{\partial y} = \delta u$

$$\iint \frac{\partial \tau_{xy}}{\partial y} \delta u \, dx \, dy = -\iint \tau_{xy} \frac{\partial (\delta u)}{\partial y} dx \, dy + \int \tau_{xy} m \, \delta u \, ds$$

Putting values of eqs.(2.2.7), (2.2.8) and (2.2.9), in eq. (2.2.5), one can get the following relation:

$$\begin{aligned} &-\iint \left[\sigma_x \frac{\partial}{\partial x}(\delta u) + \sigma_y \frac{\partial}{\partial y}(\delta v) + \tau_{xy} \frac{\partial}{\partial x}(\delta v) + \tau_{xy} \frac{\partial}{\partial y}(\delta u) \right] dx \, dy \\ &+ \int \left[\sigma_x \ell \delta u + \sigma_y m \delta v + \tau_{xy} \ell \delta v + \tau_{xy} m \delta u \right] ds + \iint F_{\Omega x} \delta u \, dx \, dy + \iint F_{\Omega y} \delta v \, dx \, dy = 0 \end{aligned} \tag{2.2.10}$$

Rearranging the terms of above expression, the following relations are obtained.

$$\begin{aligned} &-\iint \left[\sigma_x \frac{\partial}{\partial x}(\delta u) + \sigma_y \frac{\partial}{\partial y}(\delta v) + \tau_{xy} \frac{\partial}{\partial x}(\delta v) + \tau_{xy} \frac{\partial}{\partial y}(\delta u) \right] dx \, dy + \iint \left(F_{\Omega x} \delta u + F_{\Omega y} \delta v \right) dx \, dy \\ &+ \int \left(\sigma_x \ell + \tau_{xy} m \right) \delta u ds + \int \left(\tau_{xy} \ell + \sigma_y m \right) \delta v ds = 0 \end{aligned} \tag{2.2.11}$$

Here, $F_{\Omega x}$ and $F_{\Omega y}$ are the body forces and δu & δv are virtual displacements in X and Ydirections respectively.

Considering firstterm of eq. (2.2.11), virtual displacement δu is given to the element of unit thickness. Dotted position in Fig. 2.2.2 shows the virtual displacement. Thus, work done by σ_x :

$$\sigma_x dy \left[\delta u + \frac{\partial}{\partial x}(\delta u) dx \right] - \sigma_x dy \delta u = \sigma_x \frac{\partial}{\partial x}(\delta u) dx dy \tag{2.2.12}$$

Similarly, considering secondterm of eq. (2.2.11), virtual work done by body forces is

$$\iint \left(F_{\Omega x}\delta u + F_{\Omega y}\delta v\right)dx\ dy$$

Putting eqs.(2.2.3) &(2.2.4) in third term of eq. (2.2.11) we get the virtual work done by surface forces as:

$$\int F_{\Gamma x}\delta u ds + \int F_{\Gamma y}\delta v ds$$

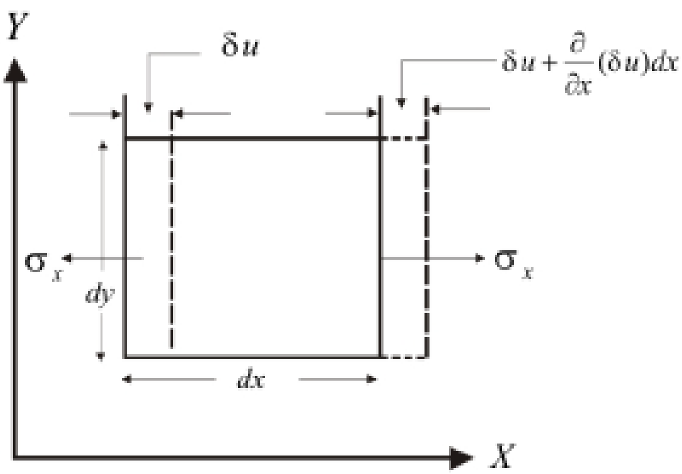


Fig. 2.2.2 Element subjected to stresses

Due to virtual displacement δu, change in strain $\delta \in_x$ is given by:

$$\delta \in_x = \frac{\left[\delta u + \frac{\partial}{\partial x}(\delta u)dx\right] - \delta u}{dx} = \frac{\partial}{\partial x}(\delta u) \qquad (2.2.13)$$

The virtual work doneby σ_x is $\sigma_x.\delta \in_x .dxdy$. Similarly all the individual term in the first term of eq. (2.2.11) can be derived from eq. (2.2.13) which will be as follows:

$$\iint \sigma_x \frac{\partial}{\partial x}(\delta u)dxdy = \iint \sigma_x \delta \in_x dxdy$$

$$\iint \sigma_y \frac{\partial}{\partial y}(\delta v)dxdy = \iint \sigma_y \delta \in_y dxdy \qquad (2.2.14)$$

$$\iint \tau_{xy}\left\{\frac{\partial}{\partial x}(\delta v) + \frac{\partial}{\partial y}(\delta u)\right\} = \iint \tau_{xy}\delta\gamma_{xy}dxdy$$

Now, the work done by internal forces will be

$$\delta U = \iint \left(\sigma_x \delta \in_x + \sigma_y \delta \in_y + \tau_{xy} \delta \gamma_{xy} \right) dxdy \qquad (2.2.15)$$

If external work done is represented by W_E and U is the internal work done then,

$$-\delta U + \delta w_E = 0 \text{ or } \delta U = \delta w_E \qquad (2.2.16)$$

Thus in elasticity problems, Galerkin's method turns out to be the principle of virtual work, which can be stated that "A Deformable body is said to be in equilibrium,if the total work done by external forces is equal to the total work done by internal forces." The work done above is virtual as either forces or deformations are also virtual. Thus, Galerkin'sapproach can be followed in all problems involving solution of a set of equations subjected to specified boundary values.

2.2.3 Galerkin Method for 2D Fluid Flow Problem

Let consider the two dimensional incompressible fluid equation which can be expressed by pressure variable only as follows.

$$\nabla^2 p = 0 \qquad (2.2.17)$$

Where p is the pressure inside the fluid domain. The above equation can be expressed in 2D form as:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

$$\text{or} \qquad (2.2.18)$$

$$p_{,ii} = 0$$

Applying weighted residual method, the weak form of the above equation will become

$$\int_\Omega w_i \; p_{,ii} \; d\Omega = 0 \qquad (2.2.19)$$

Integrating by parts of the above expression, the following relation can be obtained.

$$\int_\Gamma w_i \; p_{,i} \; d\Gamma - \int_\Omega w_{i,i} \;\; p_{,i} \;\; d\,\Omega = 0$$

$$\text{or} \int_\Omega w_{i,i} \;\; p_{,i} \;\; d\,\Omega = \int_\Gamma w_i \; p_{,i} \;\; d\Gamma \qquad (2.2.20)$$

If the nodal pressure and interpolation functions are denoted by $\overline{p}$ and N respectively, then the pressure at any point inside the fluid domain can be expressed as

$$p = [N]\{\overline{p}\}$$

Similarly, the weighted function can also be written with the help of interpolation function as

$$w = [N]\{\overline{w}\}$$

Thus, $p_{i,i} = [L]\{p\} = [L][N]\{\overline{p}\} = [B]\{\overline{p}\}$, where, $[L] = \left[\dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y}\right]$ = differential operator.

Similarly, $w_{i,i} = [L]\{W\} = [L][N]\{\overline{w}\} = [B]\{\overline{w}\}$

$$\text{Thus, } \int_{\Omega} w_{i,i}\, p_{,i}\, d\,\Omega = \int [\overline{w}]^T [B]^T [B][\overline{p}] d\Omega \tag{2.2.21}$$

$$\int_{\Gamma} w_i\, p_{,i}\, d\,\Gamma = \int_{\Gamma} \{\overline{w}\}^T [N]^T \frac{\partial p}{\partial n} d\Gamma \tag{2.2.22}$$

Here, Γ denotes the surface of the fluid domain and n represents the direction normal to the surface. Thus, from eq. (2.2.20), one can write the expression as:

$$\text{Thus, } \int_{\Omega} \{\overline{w}\}^T [B]^T [B] \{\overline{p}\} d\Omega = \int_{\Gamma} \{\overline{w}\}^T [N]^T \frac{\partial p}{\partial n} d\Gamma$$

$$\text{Or, } [G]\{\overline{p}\} = \{S\} \tag{2.2.23}$$

Where,

$$[G] = \int_{\Omega} [B]^T [B] d\Omega = \int_{\Omega} \left(\frac{\partial}{\partial x}[N]^T \frac{\partial}{\partial x}[N] + \frac{\partial}{\partial y}[N]^T \frac{\partial}{\partial y}[N] \right) d\Omega$$

$$\text{and} \{S\} = \int_{\Gamma} [N]^T \frac{\partial p}{\partial n} d\Gamma \tag{2.2.24}$$

Here, n is the direction normal to the surface. Thus, solving the above equationwith the prescribed boundary conditions, one can find out the pressure distribution inside the fluid domain by the use of finite element technique.

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UNIT – II –FINITE ELEMENT ANALYSIS – SME1308

UNIT – II

2D FINITE ELEMENT METHOD

2.3.1 Choice of Displacement Function

Displacement function is the beginning point for the structural analysis by finite element method. This function represents the variation of the displacement within the element. On the basis of the problem to be solved, the displacement function needs to be approximated in the form of either linear or higher-order function. A convenient way to express it is by the use of polynomial expressions.

2.3.1.1 Convergence criteria

The convergence of the finite element solution can be achieved if the following three conditions are fulfilled by the assumed displacement function.

a. The displacement function must be continuous within the elements. This can be ensured by choosing a suitable polynomial. For example, for an n degrees of polynomial, displacement function in I dimensional problem can be chosen as:

$$u = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + ... + \alpha_n x^n \qquad (2.3.1)$$

b. The displacement function must be capable of rigid body displacements of the element. The constant terms used in the polynomial (α_0 to α_n) ensure this condition.

c. The displacement function must include the constant strains states of the element. As element becomes infinitely small, strain should be constant in the element. Hence, the displacement function should include terms for representing constant strain states.

2.3.1.2 Compatibility

Displacement should be compatible between adjacent elements. There should not be any discontinuity or overlapping while deformed. The adjacent elements must deform without causing openings, overlaps or discontinuous between the elements.

Elements which satisfy all the three convergence requirements and compatibility condition are called Compatible or Conforming elements.

2.3.1.3 Geometric invariance

Displacement shape should not change with a change in local coordinate system. This can be achieved if polynomial is balanced in case all terms cannot be completed. This 'balanced' representation can be achieved with the help of Pascal triangle in case of two-dimensional polynomial. For example, for a polynomial having four terms, the invariance can be obtained if the following expression is selected from the Pascal triangle.

$$u = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy \tag{2.3.2}$$

The geometric invariance can be ensured by the selection of the corresponding order of terms on either side of the axis of symmetry.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & x & & y & & & \\
 & & x^2 & & xy & & y^2 & & \\
 & x^3 & & x^2y & & xy^2 & & y^3 & \\
x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4
\end{array}$$

Fig. 2.3.1 Pascal'sTriangle

2.3.2 Shape Function

In finite element analysis, the variations of displacement within an element are expressed by its nodal displacement ($u = \sum N_i u_i$) with the help of interpolation function since the true variation of displacement inside the element is not known. Here, u is the displacement at any point inside the element and u_i are the nodal displacements. This interpolating function is generally a polynomial with n degree which automatically provides a single-valued and continuous field. In finite element literature, this interpolation function (N_i) is referred to "Shape function" as well. For linear interpolation,n will be 1 and for quadratic interpolation n will become 2 and so on. There are two types of interpolation functions namely (i) Lagrange interpolationand (ii) Hermitian interpolation. Lagrange interpolation function is widely used in practice. Here the assumed function takes on the same values as the given function at specified points. In case of Hermitian interpolation function, the slopes of the function also take the same values as the given function at specified points. The derivation of shape function for varieties of elements will be discussed in subsequent lectures.

2.3.3 Degree of Continuity

Let consider ϕas an interpolation function in a piecewise fashion over finite element mesh. While such interpolation function ϕ can be ensured to vary smoothly within the element, the transition between adjacent elements may not be smooth. The term C^m is considered to define the continuity of a piecewise displacement. A function C^m is continuous if its derivative up to and including degree m are inter-element continuous. For example, for one dimensional problem, $\phi=\phi(x)$ is C^0 continuous if ϕ is continuous, but $\phi_{,x}$is not. Similarly, $\phi=\phi(x)$ is C^1 continuous if $\phi$and $\phi_{,x}$ are continuous, but $\phi_{,xx}$is not. In general, C^0 element is used to model plane and solid body and C^1 element is used to model beam, plate and shell like structure, where inter-element continuity of slope is necessary to ensure.Let assume a linear function for bar like element: $\phi_1 = \alpha_0 + \alpha_1 x$ This function is C^0 continuous as $\phi_{1,x}$is discontinuous. If the interpolation function is considered as $\phi_2 = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ then $\phi_{2,x} = \alpha_1 + 2\alpha_2 x$ is also continuous but $\phi_{2,xx} = 2\alpha_2$ is discontinuous. As a result, this function ϕ_2 will become C^1 continuous.

2.3.4 Isoparametric Elements

If the shape functions (N_i) used to represent the variation of geometry of the element are the same as the shape functions (N'_i) used to represent the variation of the displacement then the elements are called isoparametric elements. For example, the coordinates (x,y) inside the element are defined by the shape functions (N_i) and displacement (u,v) inside the element are defined by the shape functions (N'_i) as below.

$$\begin{aligned} x &= N_i x_i \qquad & u &= N_i' u_i \\ y &= N_i y_i \qquad & v &= N_i' v_i \end{aligned} \qquad \text{# # # # # # #} \qquad (2.3.3)$$

If $N_i = N'_i$, then the element is called isroparametric. Fig. 2.3.2(a) shows the two dimensional 8 node isoparametric element.

If the geometry of element is defined by shape functions of order higher than that for representing the variation of displacements, then the elements are called superparametric (Fig. 2.3.2(b)).

If the geometry of element is defined by shape functions of order lower than that for representing the variation of displacements then the elements are called subparametric (Fig. 2.3.2(c)).

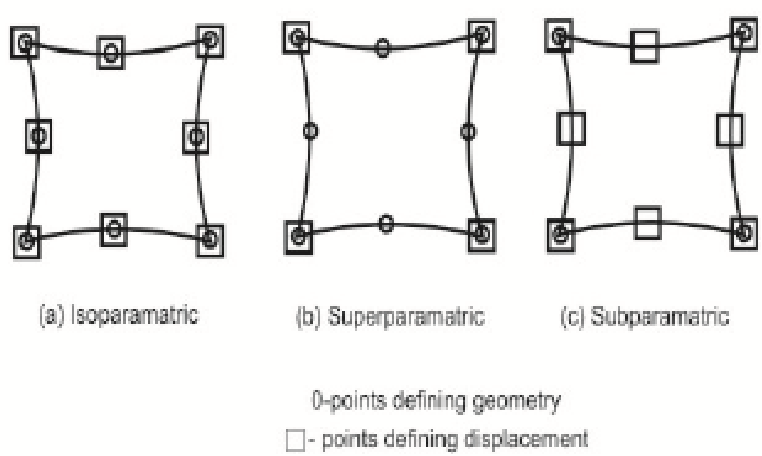

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Fig. 2.3.2Shape functions for geometry and displacements

2.3.5 Various Elements

Selection of the order of the polynomial depends on the type of elements. For example, in case of one dimensional element having single degrees of freedom with two nodes, the displacement function can be chosen as $u = \alpha_0 + \alpha_1 x$. However, if the same has two degrees of freedom at each node, then the chosen displacement function shouldbe $u = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$. Various types of elements used in finite element analysis are given below:

1. <u>One dimensional elements.</u>
   (a) Two node element
   (b) Three node element

Fig. 2.3.3One dimensional elements

2. <u>Two dimensional elements</u>
   (a) Triangular element
   (b) Rectangular element
   (c) Quadrilateral element
   (d) Quadrilateral formed by two triangles
   (e) Quadrilateral formed by four triangles

Few of the elements with number of nodes are shown in Fig. 2.3.4.

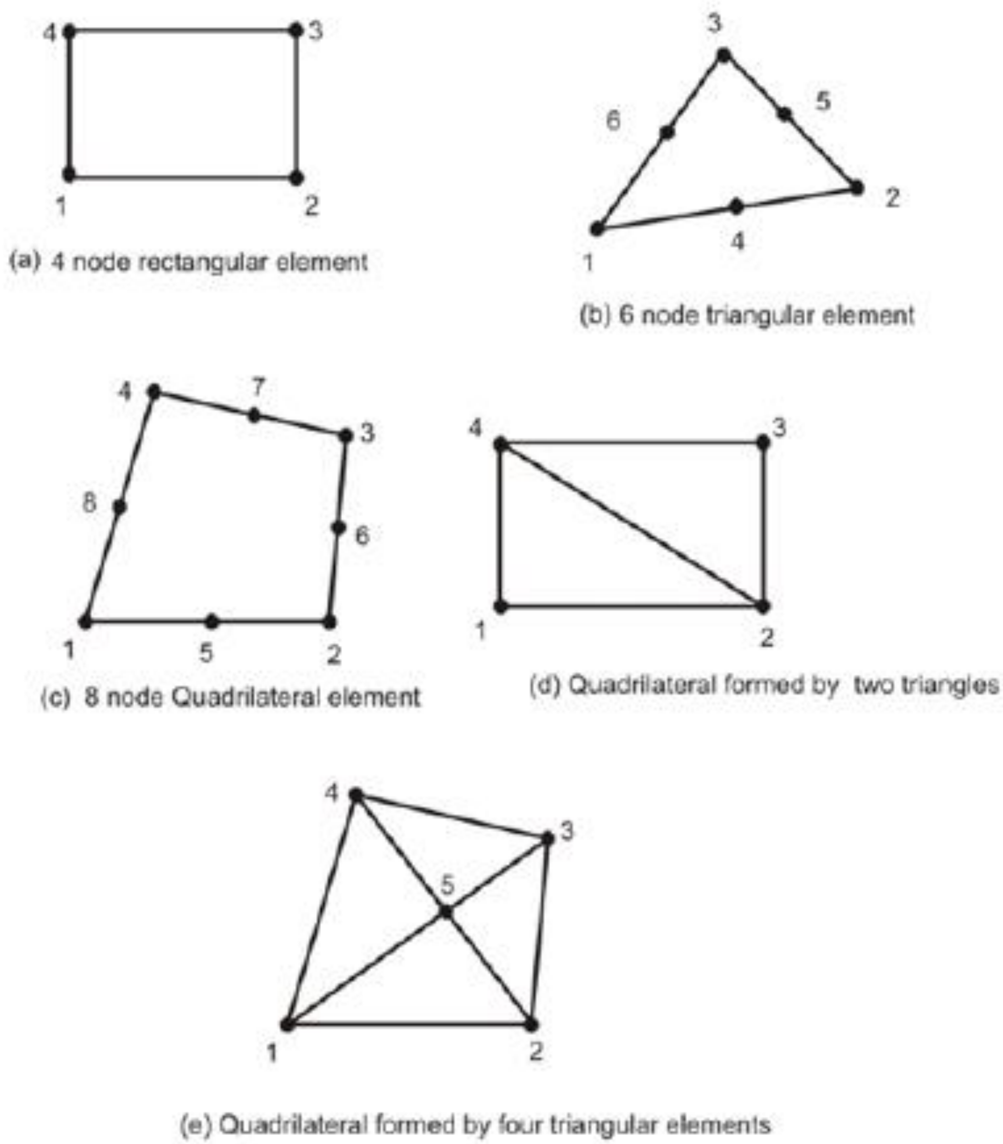


Fig. 2.3.4Two dimensional elements

3. Three dimensional elements.
   (a) Tetrahedron
   (b) Rectangular brick element

Few of the three dimensional solid elements are shown in Fig. 2.3.5.

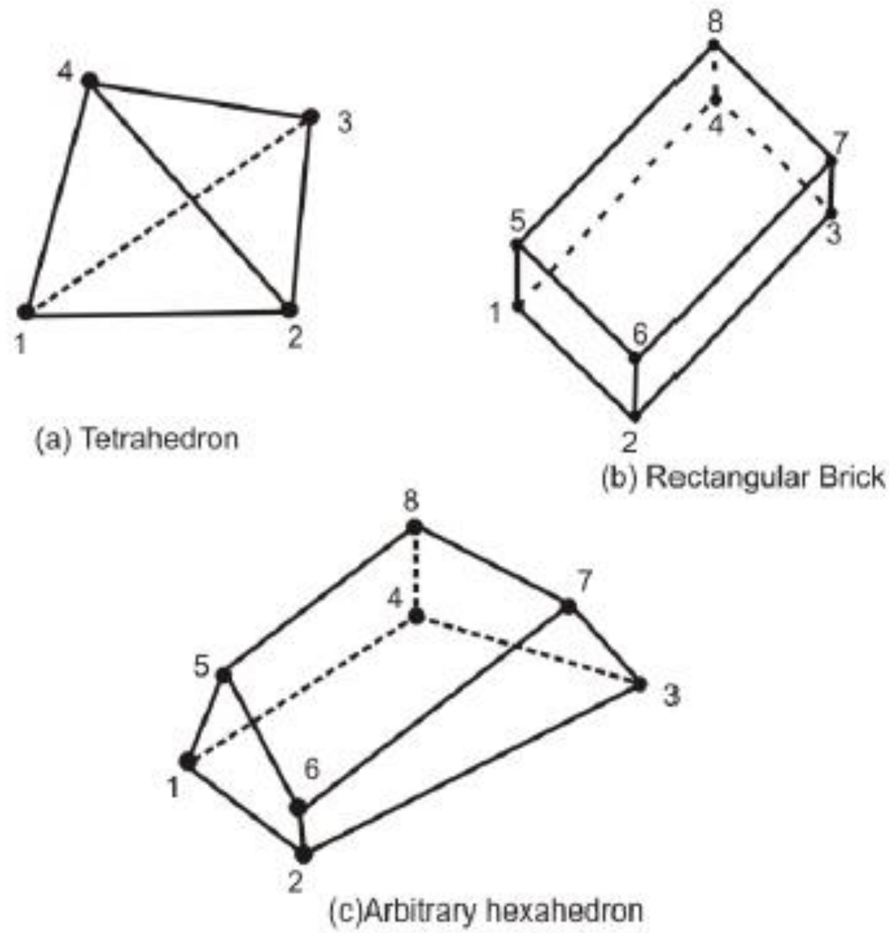


Fig. 2.3.5Three dimensional elements

2.4.1Element Stiffness Matrix

The stiffness matrix of a structural system can be derived by various methods like variationalprinciple, Galerkin method etc. The derivation of an element stiffness matrix has already been discussed in earlier lecture. The stiffness matrix is an inherent property of the structure. Element stiffness is obtained with respect to its axes and then transformed this stiffness to structure axes.The properties of stiffness matrix are as follows:

- Stiffness matrix issymmetric and square.
- In stiffness matrix, all diagonal elements are positive.
- Stiffness matrix is positive definite

For example, if$K$ is a symmetric $n \times n$ real matrix and x is non-zero column vector, then$K$ will bepositive definitewhile$x^T Kx$is positive.

2.4.2Global Stiffness Matrix

A structural system is an assemblage of number of elements. These elements are interconnected together to form the whole structure. Therefore, the element stiffness of all the elementsarefirst need to be calculated and then assembled together in systematic manner. It may be noted that the stiffness at a joint is obtained by adding the stiffness of all elements meeting at that joint.

To start with, the degrees of freedom of the structure are numberedfirst. This numbering will start from 1 to n where n is the total degrees of freedom. These numberings are referred to as degrees of freedom corresponding to global degrees of freedom. The element stiffness matrix of each element should be placed in their proper position in the overall stiffness matrix. The following steps may be performed to calculate the global stiffness matrix of the whole structure.

a. Initialize global stiffness matrix $\left[K\right]$ as zero. The size of global stiffness matrix will be equal to the total degrees of freedom of the structure.
b. Compute individual element properties and calculate local stiffness matrix $\left[k\right]$ of that element.
c. Add local stiffness matrix$\left[k\right]$to global stiffness matrix$\left[K\right]$ using proper locations
d. Repeat the Step b. and c. till all local stiffness matrices are placed globally.

The steps to be followed in the computer program are shown in the form of flow chart in Fig. 2.4.1 for assembling the local stiffness matrix to global stiffness matrix.

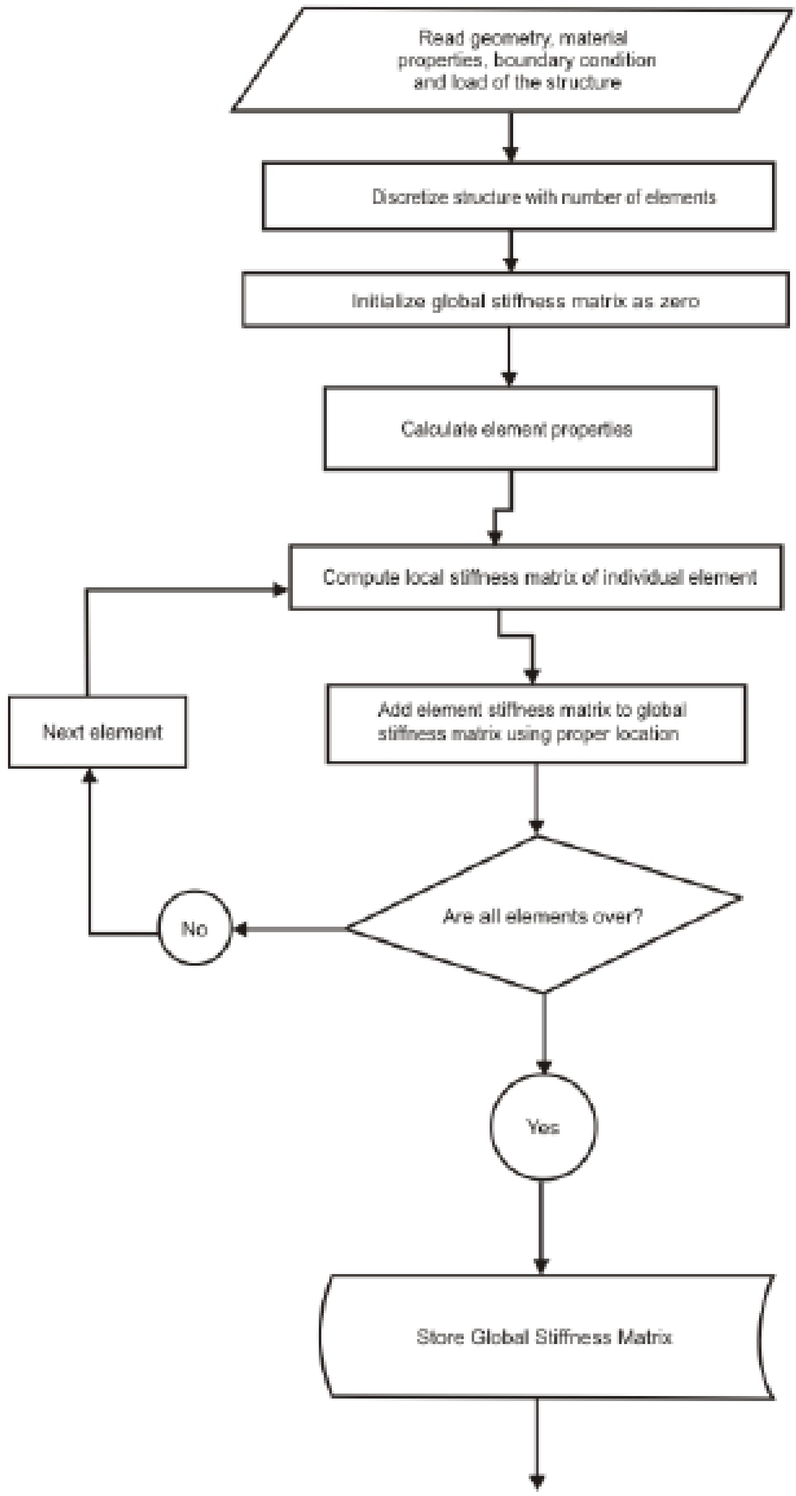


Fig. 2.4.1Assembly of stiffness matrix from local to global

2.4.3Boundary Conditions

Under this section, procedure to include the effect of boundarycondition in the stiffness matrix for the finite element analysis will be discussed. The solution cannot be obtained unless support conditions are included in the stiffness matrix. This is because, if all the nodes of the structure are included in displacement vector, the stiffness matrix becomes singular and cannot be solved if the structure is not supported amply, and it cannot resist the applied loads.A solution cannot be achieved until the boundary conditions *i.e.*, the known displacements are introduced.

In finite element analysis, the partitioning of the global matrix is carried out in a systematic way for the hand calculation as well as for the development of computer codes. In partitioning, normally the equilibrium equations can be partitioned by rearranging corresponding rows and columns, so that prescribed displacements are grouped together. For example, let considerthe equation of equilibrium is expressed in compact form as:

$$\{F\}=[K]\{d\} \tag{2.4.1}$$

Where,

$[K]$ is the global stiffness matrix,

$\{d\}$ is the displacement vector consisting of global degrees of freedom, and

$\{F\}$ is the load vector corresponding to degrees of freedom.

By the method of partitioning the above equation can be partitioned in the following manner.

$$\begin{Bmatrix} \{F_\alpha\} \\ \{F_\beta\} \end{Bmatrix} = \begin{bmatrix} [K_{\alpha\alpha}] & [K_{\alpha\beta}] \\ [K_{\beta\alpha}] & [K_{\beta\beta}] \end{bmatrix} \begin{Bmatrix} \{d_\alpha\} \\ \{d_\beta\} \end{Bmatrix} \tag{2.4.2}$$

Where,subscripts α refers to the displacements free to move andβ refers to the prescribed support displacements.As the prescribed displacements $\{d_\beta\}$ are known,eq. (2.4.2) may be written in expanded form as:

$$\{F_\alpha\}=[K_{\alpha\alpha}]\{d_\alpha\}+[K_{\alpha\beta}]\{d_\beta\} \tag{2.4.3}$$

Thus it is possible to obtain the free displacement of the structure $\{d_\alpha\}$ as

$$\{d_\alpha\} = [K_{\alpha\alpha}]^{-1}\left\{\{F_\alpha\} - [K_{\alpha\beta}]\{d_\beta\}\right\} \tag{2.4.4}$$

If the displacements at supports $\{d_\beta\}$are zero, then the above equation can be simplified to the following expression.

$$\{d_\alpha\} = [K_{\alpha\alpha}]^{-1}\{F_\alpha\} \tag{2.4.5}$$

Thus, by rearranging assembled matrix, the portion corresponding to the unknown displacements in eq.(2.4.4) can be taken out for the solution purpose. This is possible as the known displacements $\{d_\beta\}$are restrained, i.e., displacementsare zero. If the support has some known displacements, then eq. (2.4.4) can be used to find the solution. If the few supports of the structures yield, then the above method may be modified by partitioning the stiffness matrix into three parts as shown below:

$$\begin{Bmatrix} \{F_\alpha\} \\ \{F_\beta\} \\ \{F_\gamma\} \end{Bmatrix} = \begin{bmatrix} [K_{\alpha\alpha}] & [K_{\alpha\beta}] & [K_{\alpha\gamma}] \\ [K_{\beta\alpha}] & [K_{\beta\beta}] & [K_{\beta\gamma}] \\ [K_{\gamma\alpha}] & [K_{\gamma\beta}] & [K_{\gamma\gamma}] \end{bmatrix} \begin{Bmatrix} \{d_\alpha\} \\ \{d_\beta\} \\ \{d_\gamma\} \end{Bmatrix} \tag{2.4.6}$$

Here, α refers to unknown displacement; β refers to known displacement ($\neq 0$) and γ refers to zero displacement. Thus, the above equation can be separated and solved independently to find required unknown results as shown below.

$$\{F_\alpha\} = [K_{\alpha\alpha}]\{d_\alpha\} + [K_{\alpha\beta}]\{d_\beta\} + [K_{\alpha\gamma}]\{d_\gamma\}$$

$$\text{or, } [K_{\alpha\alpha}]\{d_\alpha\} = \{F_\alpha\} - [K_{\alpha\beta}]\{d_\beta\} \ \text{ as } \{d_\gamma\} = \{0\}$$

$$\text{Thus, } \{d_\alpha\} = [K_{\alpha\alpha}]^{-1}\left\{\{F_\alpha\} - [K_{\alpha\beta}]\{d_\beta\}\right\} \tag{2.4.7}$$

For computer programming, several techniques are available for handling boundary conditions. One of the approachesis to make the diagonal element of stiffness matrix corresponding to zero displacement as unity and corresponding all off-diagonal elements as zero. For example,let consider a 3×3 stiffness matrix with following force-displacement relationship.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{2.4.8}$$

Now, if the third node has zero displacement (i.e., $d_3 = 0$) then the matrix will be modified as follows to incorporate the boundary condition.

$$\begin{Bmatrix} F_1 \\ F_2 \\ 0 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{2.4.9}$$

Thus, while inverting whole matrix, d_3 will become zero automatically.

To incorporate known support displacement in computer programming following procedure may be adopted. Considering the displacement $d_2$has known value of δ, 1st row of eq. (2.4.8) can be written as:

$$F_1 = k_{11} \times d_1 + k_{12} \times d_2 + k_{13} \times d_3 \tag{2.4.10}$$

Or

$$F_1 - k_{12} \times \delta = k_{11} \times d_1 + k_{13} \times d_3 \tag{2.4.11}$$

Now the 2nd row of eq. (2.4.8) has to become:

$$\{\delta\} = \{d_2\} \tag{2.4.12}$$

Similarly 3rd row will be:

$$F_3 - k_{32} \times \delta = k_{31} \times d_1 + k_{33} \times d_3 \tag{2.4.13}$$

Thus above three equations can be written in a combined form as

$$\begin{Bmatrix} F_1 - k_{12}\delta \\ \delta \\ F_3 - k_{32}\delta \end{Bmatrix} = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & 1 & 0 \\ k_{31} & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{2.4.14}$$

Another approach may also be followed to take care the known restrained displacementsby assigning a higher value δ(say $\delta = 10^{20}$) in the diagonal element corresponding to that displacement.

$$\begin{Bmatrix} F_1 \\ \delta \times 10^{20} \times k_{22} \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} \times 10^{20} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{2.4.15}$$

$$\therefore \delta \times 10^{20} \times k_{22} = k_{21} d_1 + k_{22} \times 10^{20} \times d_2 + k_{23} \times d_3$$

As d_3 is corresponding to zero displacement, the above equation can be simplified to the following.

$$\therefore \delta \times 10^{20} \times k_{22} = k_{21} d_1 + k_{22} \times 10^{20} \times d_2$$

$$\text{or } \delta \times 10^{20} \times k_{22} = k_{22} \times 10^{20} \times d_2$$

$$\Rightarrow d_2 = \delta \rightarrow \text{known displacement is ensured}$$

If the overall stiffness matrix is to be formed in half band form then the numbering of nodes should be such that the bandwidth is minimum. For this the labels are put in a systematic manner irrespective of whether the joint displacements are unknowns or restraints. However, if the unknown displacements are labeled first then the matrix operations can be restricted up to unknown displacement labels and beyond that the overall stiffness matrix may be ignored.

Natural coordinate system is basically a local coordinate system which allows the specification of a point within the element by a set of dimensionless numbers whose magnitude never exceeds unity. This coordinate system is found to be very effective in formulating the element properties in finite element formulation. This system is defined in such that the magnitude at nodal points will have unity or zero or a convenient set of fractions. It also facilitates the integration to calculate element stiffness.

3.1.1 One Dimensional Line Elements

The line elements are used to represent spring, truss, beam like members for the finite element analysis purpose. Such elements are quite useful in analyzing truss, cable and frame structures. Such structures tend to be well defined in terms of the number and type of elements used. For example, to represent a truss member, a two node linear element is sufficient to get accurate results. However, three node line elements will be more suitable in case of analysis of cable structure to capture the nonlinear effects. The natural coordinate system for one dimensional line element with two nodes is shown in Fig. 3.1.1. Here, the natural coordinates of any point P can be defined as follows.

$$N_1 = 1 - \frac{x}{l} \quad and \quad N_2 = \frac{x}{l} \tag{3.1.1}$$

Where, x is represented in Cartesian coordinate system. Similarly, x/lcan be represented as ξ in natural coordinate system. Thus the above expression can be rewritten in the form of natural coordinate system as given below.

$$N_1 = 1 - \xi \;\; and \;\; N_2 = \xi \tag{3.1.2}$$

Now, the relationship between natural and Cartesian coordinates can be expressed from eq. (3.1.1) as

$$\begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & l \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \tag{3.1.3}$$

Here, N_1 and $N_2$is termed as shape function as well. The variation of the magnitude of two linear shape functions (N_1 and N_2) over the length of bar element are shown in Fig. 3.1.2. This example displays the simplest form of interpolation function. The linear interpolation used for field variable ϕ can be written as

$$\phi(\xi) = \phi_1 N_1 + \phi_2 N_2 \tag{3.1.4}$$

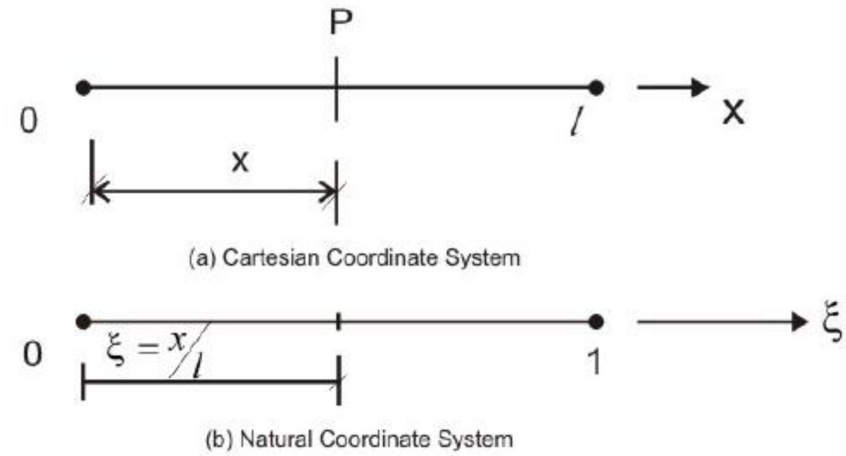


Fig. 3.1.1 Two node line element

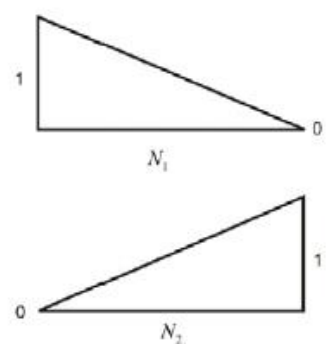


Fig. 3.1.2 Linear interpolation function for two node line element

Similarly, for three node line element, the shape function can be derived with the help of natural coordinate system which may be expressed as follows:

$$\{N\}=\begin{Bmatrix}N_1\\N_2\\N_3\end{Bmatrix}=\begin{Bmatrix}1-\frac{3x}{l}+\frac{2x^2}{l^2}\\ \frac{4x}{l}-\frac{4x^2}{l^2}\\ -\frac{x}{l}+\frac{2x^2}{l^2}\end{Bmatrix}=\begin{Bmatrix}1-3\xi+2\xi^2\\4\xi-4\xi^2\\-\xi+2\xi^2\end{Bmatrix} \tag{3.1.5}$$

The detailed derivation of the interpolation function will be discussed in subsequent lecture. The variation of the shape functions over the length of the three node element are shown in Fig. 3.1.3

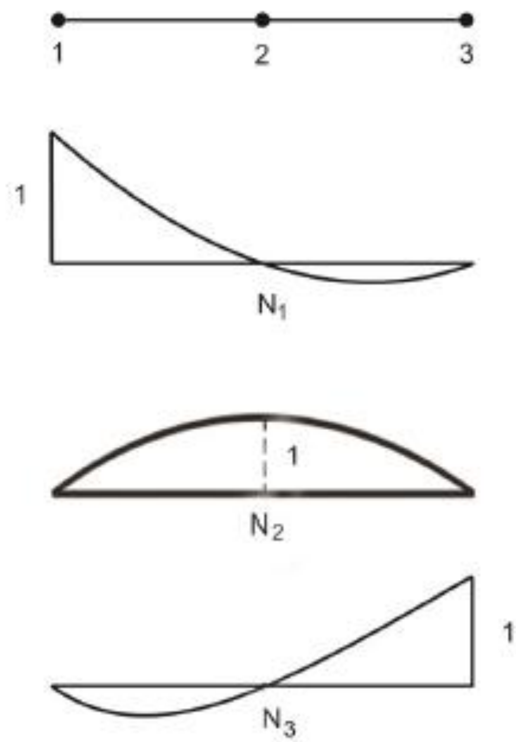


Fig. 3.1.3 Variation of interpolation function for three node line element

Now, if ϕ is considered to be a function of L_1 and L_2, the differentiation of $\phi$with respect to $x$for two node line element can be expressed by the chain rule formula as

$$\frac{d\phi}{dx}=\frac{\partial\phi}{\partial L_1}.\frac{\partial L_1}{\partial x}+\frac{\partial\phi}{\partial L_2}.\frac{\partial L_2}{\partial x} \tag{3.1.6}$$

Thus, eq.(3.1.4) can be written as

$$\frac{\partial L_1}{\partial x}=-\frac{1}{l} \text{ and } \frac{\partial L_2}{\partial x}=\frac{1}{l} \tag{3.1.7}$$

Therefore,

$$\frac{d}{dx}=\frac{1}{l}\left(\frac{\partial}{\partial L_2}-\frac{\partial}{\partial L_1}\right) \tag{3.1.7}$$

The integration over the length *l*in natural coordinate system can be expressed by

$$\int_l L_1^p L_2^q dl=\frac{p!q!}{(p+q+1)!}l \tag{3.1.9}$$

Here, p! is the factorial product p(p-1)(p-2)….(1) and 0! is defined as equal to unity.

3.1.2 Two Dimensional Triangular Elements

The natural coordinate system for a triangular element is generally called as triangular coordinate system. The coordinate of any point *P*inside the triangle is *x,y* in Cartesian coordinate system. Here, three coordinates, L_1, L_2 and L_3 can be used to define the location of the point in terms of natural coordinate system. The point *P* can be defined by the following set of area coordinates:

$$L_1=\frac{A_1}{A} \;\; ; \;\; L_2=\frac{A_2}{A} \;\; ; \;\; L_3=\frac{A_3}{A} \tag{3.1.10}$$

Where,

A_1= Area of the triangle P23

A_2= Area of the triangle P13

A_3= Area of the triangle P12

A=Area of the triangle 123

Thus,

$$A=A_1+A_2+A_3$$

and

$$L_1+L_2+L_3=1 \tag{3.1.11}$$

Therefore, the natural coordinate of three nodes will be: node 1 (1,0,0); node 2 (0,1,0); and node 3 (0,0,1).

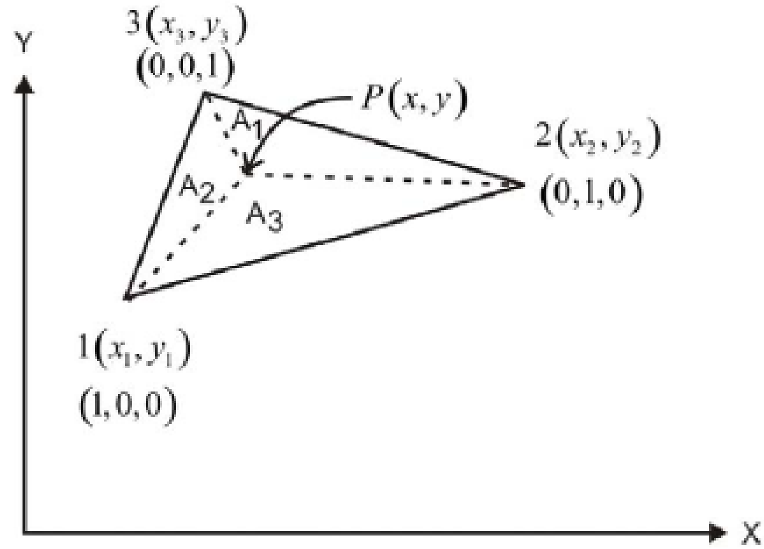


Fig. 3.1.4 Triangular coordinate system

The area of the triangles can be written using Cartesian coordinates considering *x*, *y* as coordinates of an arbitrary point *P* inside or on the boundaries of the element:

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A_2 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{vmatrix}$$

$$A_3 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}$$

The relation between two coordinate systems to define point P can be established by their nodal coordinates as

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \tag{3.1.12}$$

Where,

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3$$

$$y = L_1 y_1 + L_2 y_2 + L_3 y_3$$

The inverse between natural and Cartesian coordinates from eq.(3.1.12) may be expressed as

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{3.1.13}$$

The derivatives with respect to global coordinates are necessary to determine the properties of an element. The relationship between two coordinate systems may be computed by using the chain rule of partial differentiation as

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial L_1}.\frac{\partial L_1}{\partial x} + \frac{\partial}{\partial L_2}.\frac{\partial L_2}{\partial x} + \frac{\partial}{\partial L_3}.\frac{\partial L_3}{\partial x} \\ &= \frac{b_1}{2A}.\frac{\partial}{\partial L_1} + \frac{b_2}{2A}.\frac{\partial}{\partial L_2} + \frac{b_3}{2A}.\frac{\partial}{\partial L_3} \\ &= \sum_{i=1}^{3}\frac{b_i}{2A}.\frac{\partial}{\partial L_i} \end{aligned} \tag{3.1.14}$$

Where, $b_1 = y_2 - y_3$; $b_2 = y_3 - y_1$ and $b_3 = y_1 - y_2$. Similarly, following relation can be obtained.

$$\frac{\partial}{\partial y} = \sum_{i=1}^{3}\frac{c_i}{2A}.\frac{\partial}{\partial L_i} \tag{3.1.15}$$

Where, $c_1 = x_3 - x_2$; $c_2 = x_1 - x_3$ and $c_3 = x_2 - x_1$. The above expressions are looked However, the main advantage is the ease with which polynomial terms can be

cumbersome.

integrated using following area integral expression.

$$\int_A L_1^p L_2^q L_3^r dA = \frac{p!q!r!}{(p+q+r+2)!}2A \tag{3.1.16}$$

Where 0! is defined as unity.

3.1.3 Shape Function using Area Coordinates

The interpolation functions for the triangular element are algebraically complex if expressed in Cartesian coordinates. Moreover, the integration required to obtain the element stiffness matrix is quite cumbersome. This will be discussed in details in next lecture. The interpolation function and subsequently the required integration can be obtained in a simplified manner by the concept of area coordinates. Considering a linear displacement variation of a triangular element as shown in Fig. 3.1.5, the displacement at any point can be written in terms of its area coordinates.

$$u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3$$

Or, $$u = \{\phi\}^T \{\alpha\} \tag{3.1.17}$$

where, $\{\phi\}^T = [L_1 \quad L_2 \quad L_3]$ and $\{\alpha\}^T = \{\alpha_1 \quad \alpha_2 \quad \alpha_3\}$

And $$L_1 = \frac{A_1}{A} \;;\; L_2 = \frac{A_2}{A} \;;\; L_3 = \frac{A_3}{A} \tag{3.1.18}$$

Here, A is the total area of the triangle. It is important to note that the area coordinates are dependent as $L_1 + L_2 + L_3 = 1$. It may be seen from figure that at node 1, $L_1 = 1$ while $L_2 = L_3 = 0$. Similarly for other two nodes: at node 2, $L_2 = 1$ while $L_1 = L_3 = 0$, and$L_3 = 1$ while $L_2 = L_1 = 0$. Now, substituting the area coordinates for node 1, 2 and 3, the displacement components at nodes can be written as

$$\{u_i\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{\alpha\} \tag{3.1.19}$$

Thus, from the above expression, one can obtain the unknown coefficient α :

$$\{\alpha\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{3.1.20}$$

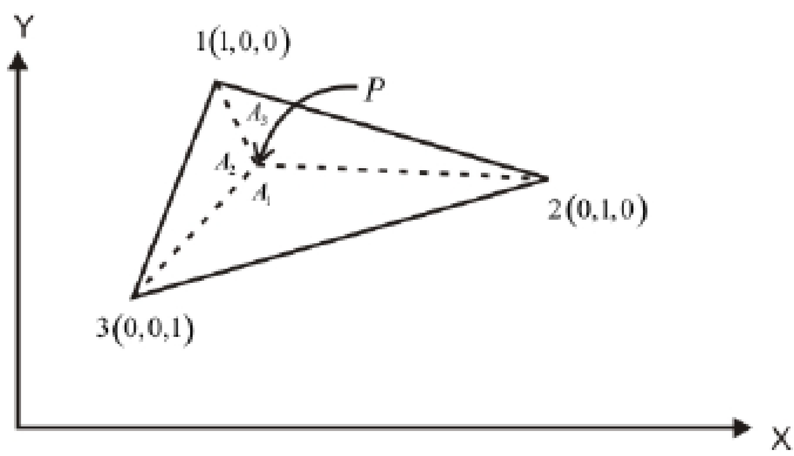


Fig. 3.1.5 Area coordinates for triangular element

Now, eq.(3.1.17) can be written as:

$$\{u\}=\{\phi\}^T\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\{\phi\}^T\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\{u_i\} \tag{3.1.21}$$

The above expression can be written in terms of interpolation function as $u=\{N\}^T\{u_i\}$

Where,

$$\{N\}^T=[L_1 \quad L_2 \quad L_3]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=[L_1 \quad L_2 \quad L_3] \tag{3.1.22}$$

Similarly, the displacement variation *v* in Y direction can be expressed as follows.

$$v=\{N\}^T\{v_i\} \tag{3.1.23}$$

Thus, for two displacement components *u* and *v* of any point inside the element can be written as:

$$\{d\}=\begin{Bmatrix}u\\ v\end{Bmatrix}=\begin{bmatrix}\{N\}^T & \{0\}^T\\ \{0\}^T & \{N\}^T\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\end{Bmatrix} \tag{3.1.24}$$

Thus, the shape function of the element will become

$$[N]=\begin{bmatrix}L_1 & L_2 & L_3 & 0 & 0 & 0\\ 0 & 0 & 0 & L_1 & L_2 & L_3\end{bmatrix} \tag{3.1.25}$$

It is important to note that the shape function N_i become unity at node *i* and zero at other nodes of the element. The displacement at any point of the element can be expressed in terms of its nodal displacement and the interpolation function as given below.

$$\begin{aligned}u&=N_1u_1+N_2u_2+N_3u_3\\ v&=N_1v_1+N_2v_2+N_3v_3\end{aligned} \tag{3.1.26}$$

The triangular element can be used to represent the arbitrary geometry much easily. On the other hand, rectangular elements, in general, are of limited use as they are not well suited for representing curved boundaries. However, an assemblage of rectangular and triangular element with triangular elements near the boundary can be very effective (Fig. 3.2.1). Triangular elements may also be used in 3-dimensionalaxi-symmetric problems, plates and shell structures. The shape function for triangular elements (linear, quadratic and cubic) with various nodes (Fig. 3.2.2) can be formulated. An internal node will exist for cubic element as seen in Fig. 3.2.2(c).

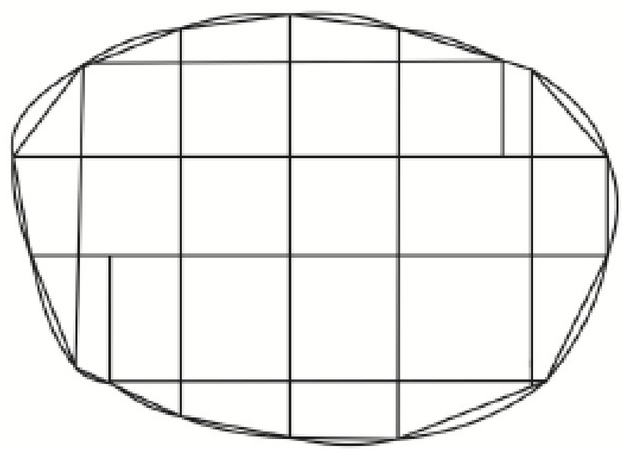


Fig. 3.2.1 Finite element mesh consisting of triangular and rectangular element

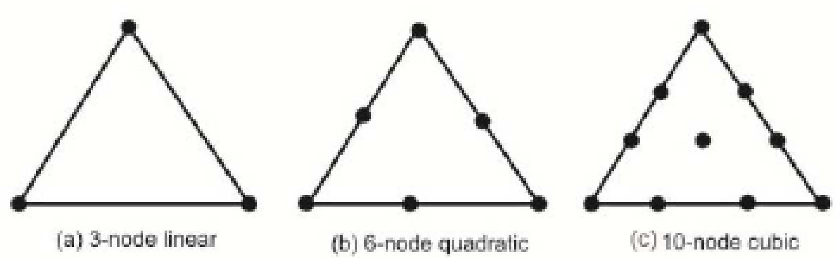


Fig. 3.2.2 Triangular elements

In displacement formulation, it is very important to approximate the variation of displacement in the element by suitable function. The interpolation function can be derived either using the Cartesian coordinate system or by the area coordinates.

3.2.1 Shape function usingCartesiancoordinates

Polynomials are easiest way of mathematical operation for expressing variation of displacement. For example, the displacement variation within the element can be represented by the following function in case of two dimensional plane stress/strain problems.

$$u = \alpha_0 + \alpha_1 x + \alpha_2 y \tag{3.2.1}$$

$$v = \alpha_3 + \alpha_4 x + \alpha_5 y \tag{3.2.2}$$

where α_0, α_1, α_2 ….. are unknown coefficients. Thus the displacement vectors at any point *P*,in the element (Fig.3.2.3) can be expressed with the following relation.

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \tag{3.2.3}$$

$$\text{Or, } \{d\}=[\phi]\{\alpha\} \tag{3.2.4}$$

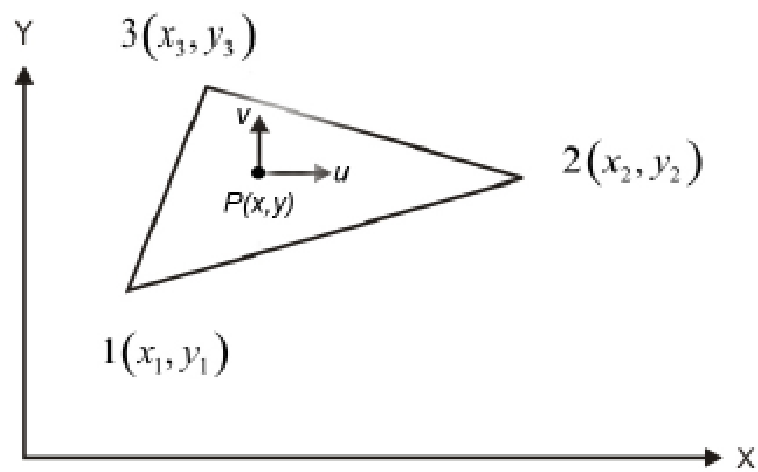


Fig. 3.2.3 Triangular element in Cartesian Coordinates

Similarly, for "*m*" node element having three degrees of freedom at each node, the displacement function can be expressed as

$$\begin{aligned} u &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + \ldots\ldots + \alpha_{m-1} y^n \\ v &= \alpha_m + \alpha_{m+1} x + \alpha_{m+2} y + \alpha_{m+3} x^2 + \alpha_{m+4} xy + \ldots.. + \alpha_{2m-1} y^n \\ w &= \alpha_{2m} + \alpha_{2m+1} x + \alpha_{2m+2} y + \alpha_{2m+3} x^2 + \alpha_{2m+4} xy + \ldots.. + \alpha_{3m-1} y^n \end{aligned} \tag{3.2.5}$$

Hence,in such case,

$$\{d\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} \{\phi\}^T & 0 & 0 \\ 0 & \{\phi\}^T & 0 \\ 0 & 0 & \{\phi\}^T \end{bmatrix} \{\alpha\} \tag{3.2.6}$$

Where, $\{\alpha\}^T = [\alpha_0 \alpha_1 \ldots \alpha_{3m-1}]$ and, $[\phi]^T = [1 \quad x \quad y \quad x^2 \quad xy \ldots. \quad y^n]$

Now, for a linear triangular element with 2 degrees of freedom, eq. (3.2.3) can be written in terms of the nodal displacements as follows.

$$\{d\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \tag{3.2.7}$$

Where, $\{d\}$ is the nodal displacements. To simplify the above expression for finding out the shape function, the displacements in X direction can be separated out which will be as follows:

$$\{u_i\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{Bmatrix} \tag{3.2.8}$$

To obtain the polynomial coefficients, $\{\alpha\}$ the matrix of the above equation are to be inverted. Thus,

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{3.2.9}$$

Where, A is the area of the triangle and can be obtained as follows.

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{3.2.10}$$

Now, eq. (3.2.1) can be written from the above polynomial coefficients.

$$u = \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]u_1$$
$$+ \frac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]u_2 \qquad (3.2.11)$$
$$+ \frac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]u_3$$

Thus, the interpolation function can be obtained from the above as:

$$\{N\}^T = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right] \\ \frac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right] \\ \frac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right] \end{Bmatrix}$$

(3.2.12)

Such three node triangular element is commonly known as constant strain triangle (CST) as its strain is assumed to be constant inside the element. This property may be derived from eq. (3.2.1) and eq.(3.2.2). For example, in case of 2-D plane stress/strain problem, one can express the strain inside the triangle with the help of eq.(3.2.1) and eq.(3.2.2):

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial(\alpha_0 + \alpha_1 x + \alpha_2 y)}{\partial x} = \alpha_1$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial(\alpha_3 + \alpha_4 x + \alpha_5 y)}{\partial y} = \alpha_5 \qquad (3.2.13)$$
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \alpha_2 + \alpha_4$$

CST is the simplest element to develop mathematically. As there is no variation of strain inside the element, the mesh size of the triangular element should be small enough to get correct results. This element produces constant temperature gradients ensuring constant heat flow within the element for heat transfer problems.

3.2.2 Higher Order Triangular Elements

Higher order elements are useful if the boundary of the geometry is curve in nature. For curved case, higher order triangular element can be suited effectively while generating the finite element mesh. Moreover, in case of flexural action in the member, higher order elements can produce more accurate results compare to those using linear elements. Various types of higher order triangular elements are used in practice. However, most commonly used triangular element is the six node element for which development of shape functions are explained below.

3.2.2.1Shape function for six node element

Fig. 3.2.4 shows a triangular element with six nodes. The additional three nodes (4, 5, and 6) are situated at the midpoints of the sides of the element. A complete polynomial representation of the field variable can be expressed with the help of Pascal triangle:

$$\phi(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 \tag{3.2.14}$$

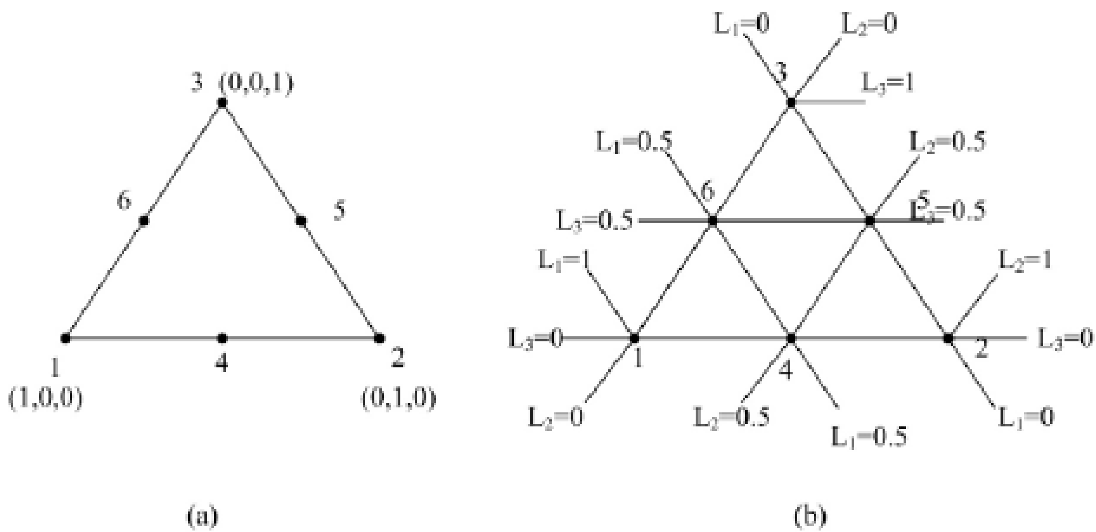


Fig. 3.2.4 (a) Six node triangular element (b) Lines of constant values of the area coordinates

Using the above field variable function, one can reach the following expression using interpolation function and the nodal values.

$$\phi(x, y) = \sum_{i=1}^{6} N_i(x, y)\phi_i \tag{3.2.15}$$

Here, the every shape function must be such that its value will be unity if evaluated at its related node and zero if evaluated at any of the other five nodes. Moreover, as the field variable representation is quadratic, each interpolation function will also become quadratic. Fig. 3.2.4(a) shows the six node element with node numbering convention along with the area coordinates at three corners. The six node element with lines of constant values of the area coordinates passing through the nodes is shown in Fig. 3.2.4(b). Now the interpolation functions can be constructed with the help of area coordinates from the above diagram. For example, the interpolation function N_1 should be unity at node 1 and zero at all other five nodes. According to the above diagram, the value of L_1 is 1 at node 1 and ½ at node 4 and 6. Again, L_1 will be 0 at nodes 2, 3 and 5. To satisfy all these conditions, one can propose following expression:

$$N_1(x,y) = N_1(L_1, L_2, L_3) = L_1\left(L_1 - \frac{1}{2}\right) \tag{3.2.16}$$

Evaluating the above expression, the value of N_1 is becoming ½ at node 1 though it must become unity. Therefore, the above expression is slightly modified satisfying all the conditions and will be as follows:

$$N_1 = 2L_1\left(L_1 - \frac{1}{2}\right) = L_1(2L_1 - 1) \tag{3.2.17}$$

Eq. (3.2.17) assures the required conditions at all the six nodes and is a quadraticfunction, asL_1 is a linear function of x and y. The remaining five interpolationfunctions can also be obtained in similar fashion applying the required nodal conditions. Thus, the shape function for the six node triangle element can be written as given below.

$$\begin{aligned} N_1 &= L_1(2L_1 - 1) \\ N_2 &= L_2(2L_2 - 1) \\ N_3 &= L_3(2L_3 - 1) \\ N_4 &= 4L_1L_2 \\ N_5 &= 4L_2L_3 \\ N_6 &= 4L_3L_1 \end{aligned} \tag{3.2.18}$$

Such six node triangular element is commonly known as linear strain triangle (LST) as its strain is assumed to vary linearly inside the element. In case of 2-D plane stress/strain problem, the element displacement field for such quadratic triangle may be expressed as

$$\begin{aligned} u(x,y) &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 \\ v(x,y) &= \alpha_6 + \alpha_7 x + \alpha_8 y + \alpha_9 x^2 + \alpha_{10} xy + \alpha_{11} y^2 \end{aligned} \tag{3.2.19}$$

So the element strain can be derived from the above displacement field as follows.

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = \alpha_1 + 2\alpha_3 x + \alpha_4 y \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \alpha_8 + \alpha_{10} x + 2\alpha_{11} y \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \alpha_2 + \alpha_4 x + 2\alpha_5 y + \alpha_7 + 2\alpha_9 x + \alpha_{10} y \end{aligned} \tag{3.2.20}$$

The above expression shows that the strain components are linearly varying inside the element. Therefore, this six node element is called linear strain triangle. The main advantage of this element is that it can capture the variation of strains and therefore stresses of the element.

3.2.3Construction of Shape Function by Degrading Technique

Sometimes, the geometry of the structure or its loading and boundary conditions are such that the stresses developed in few locations are quite high. On the other hand, variations of stresses are less in some areas and as a result, refinement of finite element mesh is not necessary. It would be economical in terms of computation if higher order elements are chosen where stress concentration is high and lower order elements at area away from the critical area. Fig.3.2.5 shows graphical representationswhere various order of triangular elements are used for generating a finite element mesh.

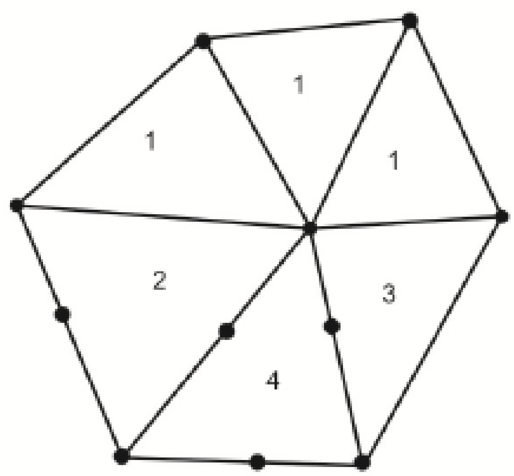


Fig. 3.2.5 Triangular elements with different number of nodes

Fig. 3.2.5contains four types of element. Type 1 has only three nodes, type 2 element has five nodes, type 3 has four nodes and type 4 has six nodes. The shape function for 3-node and 6-node triangular elements has already been derived. The shape functions of 6-node element can suitably be degraded to derive shape functions of other two types of triangular elements.

3.2.3.1 Five node triangular element

Let consider a six node triangular element as shown in Fig. 3.2.6(a) whose shape functions and nodal displacements are (N_1, N_2, N_3, N_4, N_5, N_6) and (u_1, u_2, u_3, u_4, u_5, u_6) respectively. Similarly, for a five node triangular element as shown in Fig. 3.2.6(b), the shape functions and nodal displacements are considered as (N'_1, N'_2, N'_3, N'_4, N'_5) and (u'_1, u'_2, u'_3, u'_4, u'_5) respectively. Thus, the displacementat any point in a six node triangular element will become

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 u_5 + N_6 u_6 \tag{3.2.21}$$

Where, N_1, N_2, …, N_6 are the shape functions and is given in eq.(3.2.18).If there is no node between 1 and 3, the displacement along line 1-3is considered to vary linearly. Thus the displacement at an assumed node6′ may be written as

$$u'_6 = \frac{u_1 + u_3}{2} \tag{3.2.22}$$

Substituting, the value of u'$_5$ for u_5 in eq.(3.2.21) the following expression will be obtained.

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 u_5 + N_6 \frac{u_1 + u_3}{2} \tag{3.2.23}$$

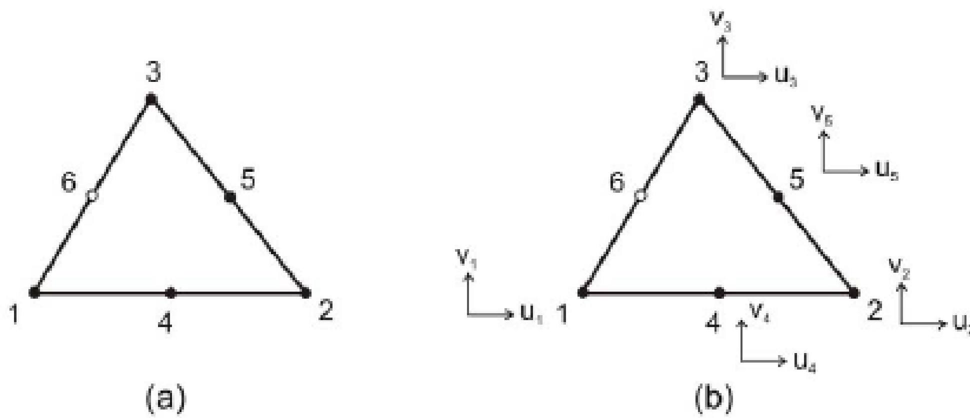


Fig. 3.2.6 Degrading for five node element

Thus, the displacement function can be expressed by five nodal displacements as:

$$u = \left(N_1 + \frac{N_6}{2}\right) u_1 + N_2 u_2 + \left(N_3 + \frac{N_6}{2}\right) u_3 + N_4 u_4 + N_6 u_5 \tag{3.2.24}$$

However, the displacement function for the five node triangular element can be expressed as

$$u = N_1' u_1 + N_2' u_2 + N_3' u_3 + N_4' u_4 + N_5' u_5 \tag{3.2.25}$$

Comparing eq.(3.2.24) and eq.(3.2.25) and observing node 6 of six node triangle corresponds to node 5 of five node triangle, we can write

$$N_1' = N_1 + \frac{N_6}{2},\; N'_2 = N_2,\;\; N_3' = N_3 + \frac{N_6}{2},\;\; N_4' = N_4 \text{ and } N_5' = N_5 \tag{3.2.26}$$

Hence, the shape function of a five node triangular element will be

$$N_1' = N_1 + \frac{N_6}{2} = L_1(2L_1 - 1) + \frac{4L_1L_3}{2} = L_1(1 - 2L_2)$$
$$N_2' = L_2(2L_2 - 1)$$
$$N_3' = N_3 + \frac{N_6}{2} = L_3(2L_3 - 1) + \frac{4L_1L_3}{2} = L_3(1 - 2L_2) \tag{3.2.27}$$
$$N_4' = 4L_1L_2$$
$$N_5' = 4L_2L_3$$

Thus, for a five node triangular element, the above shape function can be used for finite element analysis.

Rectangular elements are suitable for modelling regular geometries. Sometimes, it is used along with triangular elements to represent an arbitrary geometry. The simplest element in the rectangular family is the four node rectangle with sides parallel to x and y axis. Fig. 3.3.1 shows rectangular elements with varying nodes representing linear, quadratic and cubic variation of function.

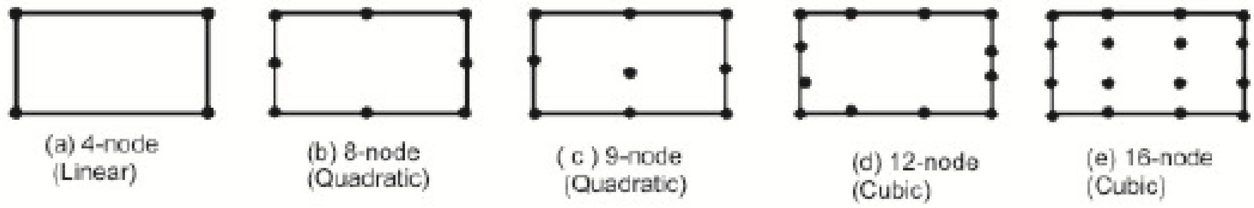


Fig. 3.3.1 Rectangular elements

3.3.1 Shape Function for Four Node Element

Shape functions of a rectangular element can be derived using both Cartesian and natural coordinate systems. A four term polynomial expression for the field variable will be required fora rectangular element with four nodes having four degrees of freedom. Since there is no complete four term polynomial in two dimensions, the incomplete, symmetric expression from the Pascal's triangle may be chosen to ensure geometric isotropy.

3.3.1.1 Shape function using Cartesian coordinates

For the derivation of interpolation function, the sides of the rectangular element (Fig. 3.3.2) are assumed to be parallel to the global Cartesian axes. From the Pascal's triangle, a linear variation may be assumed to define filed variable to ensure inter-element continuity.

$$\phi(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy \tag{3.3.1}$$

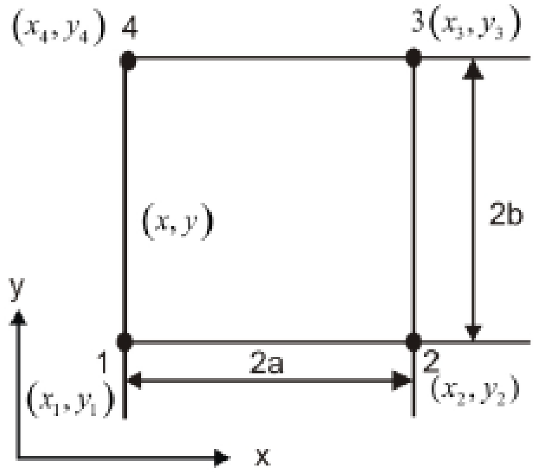


Fig. 3.3.2 Rectangular element in Cartesian coordinate

Applying nodal conditions, the above expression may be written in matrix form as

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{3.3.2}$$

The unknown polynomial coefficients may be obtained from the above equation with the use of nodal field variables.

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{bmatrix}^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{3.3.3}$$

Thus, the field variable at any point inside the element can be described in terms of nodal values as

$$\phi(x,y) = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{bmatrix}^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{3.3.4}$$

From the above expression, the shape function N_i can be derived and will be as follows.

$$N_1 = \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{y - y_4}{y_1 - y_4}\right)$$
$$N_2 = \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{y - y_3}{y_2 - y_3}\right)$$
$$N_3 = \left(\frac{x - x_4}{x_3 - x_4}\right)\left(\frac{y - y_2}{y_3 - y_2}\right) \tag{3.3.5}$$
$$N_4 = \left(\frac{x - x_3}{x_4 - x_3}\right)\left(\frac{y - y_1}{y_4 - y_1}\right)$$

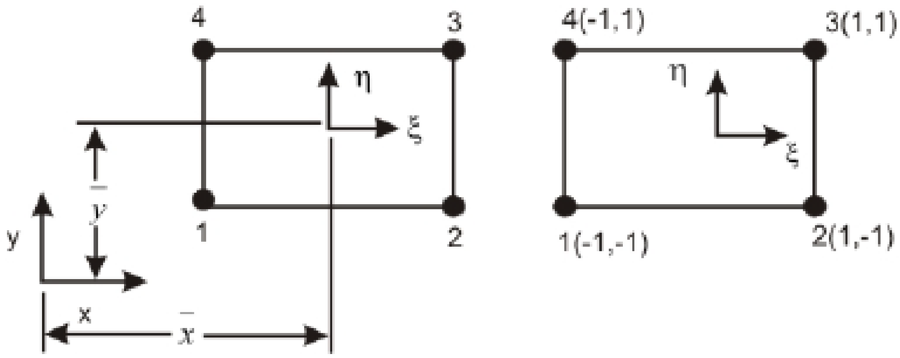
Now, substituting the nodal coordinates in terms of (x_1, y_1) as ($-a$, $-b$) at node 1; (x_2, y_2) as (a, $-b$) at node 2; (x_3, y_3) as (a, b) at node 3 and (x_4, y_4) as ($-a$, b) at node 4 the above expression can be re-written as:

$$N_1 = \frac{1}{4ab}(x-a)(y-b)$$
$$N_2 = \frac{1}{4ab}(x+a)(y-b)$$
$$N_3 = \frac{1}{4ab}(x+a)(y+b) \tag{3.3.6}$$
$$N_4 = \frac{1}{4ab}(x-a)(y+b)$$

Thus, the shape function N can be found from the above expression in Cartesian coordinate system.

3.3.1.2 Shape function using natural coordinates

The derivation of interpolation function in terms of Cartesian coordinate system is algebraically complex as seen from earlier section. However, the complexity can be reduced by the use of natural coordinate system, where the natural coordinates will vary from -1 to +1 in place of –a to +a or –b to +b. The transformation of Cartesian coordinates to Natural coordinates are shown in Fig. 3.3.3.



(a) Transformation of Cartesian to natural coordinate (b) Natural coordinates at nodes

Fig. 3.3.3Four node rectangular element

From the figure, the relation between two coordinate systems can be expressed as

$$\xi = \frac{x - \overline{x}}{a} \quad and \quad \eta = \frac{y - \overline{y}}{b} \tag{3.3.7}$$

Here,2a and 2$b$are the width and height of the rectangle. The coordinate of the center of the rectangle can be written as follows:

$$\overline{x} = \frac{x_1 + x_2}{2} \quad and \quad \overline{y} = \frac{y_1 + y_4}{2} \tag{3.3.8}$$

Thus, from eq. (3.3.7) and eq.(3.3.8), the nodal values in natural coordinate systems can be derived which is shown in Fig. 3.3.4(b). With the above relations variations of $\xi \,\&\, \eta$ will be from -1 to +1. Now the interpolation function can be derived in a similar fashion as done in section 3.3.1.1. The filed variable can be written in natural coordinate system ensuring inter-element continuity as:

$$\phi(\xi, \eta) = \alpha_0 + \alpha_1 \xi + \alpha_2 \eta + \alpha_3 \xi \eta \tag{3.3.9}$$

The coordinates of four nodes of the element in two different systems are shown in Table 3.3.1 for ready reference for the derivation purpose. Applying the nodal values in the above expression one can get

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{3.3.10}$$

Table 3.3.1 Cartesian and natural coordinates for four node element

Node	Cartesian Coordinate		Natural Coordinate	
	x	y	ξ	η
1	x_1	y_1	-1	-1
2	x_2	y_2	1	-1
3	x_3	y_3	1	1
4	x_4	y_4	-1	1

Thus, the unknown polynomial coefficients can be found as

$$\begin{Bmatrix}\alpha_0\\\alpha_1\\\alpha_2\\\alpha_3\end{Bmatrix}=\begin{bmatrix}1 & -1 & -1 & 1\\1 & 1 & -1 & -1\\1 & 1 & 1 & 1\\1 & -1 & 1 & -1\end{bmatrix}^{-1}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}=\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\-1 & 1 & 1 & -1\\-1 & -1 & 1 & 1\\1 & -1 & 1 & -1\end{bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix} \tag{3.3.11}$$

The field variable can be written as follows using eq.(3.3.9) and eq.(3.3.11).

$$\begin{aligned}\phi(\xi,\eta)&=\begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\begin{Bmatrix}\alpha_0\\\alpha_1\\\alpha_2\\\alpha_3\end{Bmatrix}=\begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\-1 & 1 & 1 & -1\\-1 & -1 & 1 & 1\\1 & -1 & 1 & -1\end{bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}\\&=\begin{bmatrix}N_1 & N_2 & N_3 & N_4\end{bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}\end{aligned} \tag{3.3.12}$$

Where, N_i are the interpolation function of the element in natural coordinate system and can be found as:

$$N_i=\begin{Bmatrix}N_1\\N_2\\N_3\\N_4\end{Bmatrix}=\begin{Bmatrix}\dfrac{(1-\xi)(1-\eta)}{4}\\\dfrac{(1+\xi)(1-\eta)}{4}\\\dfrac{(1+\xi)(1+\eta)}{4}\\\dfrac{(1-\xi)(1+\eta)}{4}\end{Bmatrix} \tag{3.3.13}$$

3.3.2 Shape Function for Eight Node Element

The shape function of eight node rectangular element can be derived in similar fashion as done in case of four node element. The only difference will be on choosing of polynomial as this element is of quadratic in nature. The derivation will be algebraically complex in case of using Cartesian coordinate system. However, use of the natural coordinate system will make the process simpler as the natural coordinates vary from -1 to +1 in the element.The variation of filed variable ϕ can be expressed in natural coordinate system by the following polynomial.

$$\phi(\xi,\eta)=\alpha_0+\alpha_1\xi+\alpha_2\eta+\alpha_3\xi^2+\alpha_4\xi\eta+\alpha_5\eta^2+\alpha_6\xi^2\eta+\alpha_7\xi\eta^2 \tag{3.3.14}$$

It may be noted that the cubic terms ξ^3 and $\eta^3$are omitted and geometric invariance is ensured by choosing the above expression. Fig. 3.3.4 shows the natural nodal coordinates of the eight node rectangle element in natural coordinate system.

The nodal field variables can be obtained from the above expression after putting the coordinates at nodes.

$$\{\phi_i\}=\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\\\phi_5\\\phi_6\\\phi_7\\\phi_8\end{Bmatrix}=\begin{bmatrix}1&-1&-1&1&1&1&-1&-1\\1&1&-1&1&-1&1&-1&1\\1&1&1&1&1&1&1&1\\1&-1&1&1&-1&1&1&-1\\1&0&-1&0&0&1&0&0\\1&1&0&1&0&0&0&0\\1&0&1&0&0&1&0&0\\1&-1&0&1&0&0&0&0\end{bmatrix}\begin{Bmatrix}\alpha_0\\\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\\\alpha_5\\\alpha_6\\\alpha_7\end{Bmatrix}=[A]\{\alpha_i\} \tag{3.3.15}$$

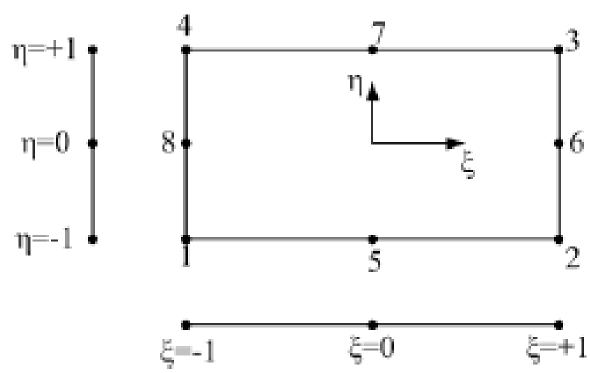


Fig. 3.3.4 Natural coordinates of eight node rectangular element

Replacing the unknown coefficient α_i in eq.(3.3.14) from eq.(3.3.15), the following relations will be obtained.

$$\phi(\xi,\eta)=\left[1\ \xi\ \eta\ \xi^2\ \ \xi\eta\ \ \eta^2\ \ \xi^2\eta\ \ \xi\eta^2\right][A]^{-1}\{\phi_i\}$$

$$=\left[1\ \xi\ \eta\ \xi^2\ \ \xi\eta\ \ \eta^2\ \ \xi^2\eta\ \ \xi\eta^2\right]\frac{1}{4}\begin{bmatrix} -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & -2 & 0 & -2 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & -2 & 0 & -2 \\ -1 & -1 & 1 & 1 & 2 & 0 & -2 & 0 \\ -1 & 1 & 1 & -1 & 0 & -2 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{Bmatrix}$$

$$=\left[N_1\quad N_2\quad N_3\quad N_4\quad N_5\quad N_6\quad N_7\quad N_8\right]\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{Bmatrix} \tag{3.3.16}$$

Thus, the interpolation function will become

$$\begin{aligned} &N_1=\frac{(1-\xi)(1-\eta)(-\xi-\eta-1)}{4};\ \ N_2\frac{(1+\xi)(1-\eta)(\xi-\eta-1)}{4};\\ &N_3=\frac{(1+\xi)(1+\eta)(\xi+\eta-1)}{4};N_4=\frac{(1-\xi)(1+\eta)(-\xi+\eta-1)}{4};\\ &N_5=\frac{(1+\xi)(1-\xi)(1-\eta)}{2};\ N_6=\frac{(1+\xi)(1+\eta)(1-\eta)}{2};\\ &N_7=\frac{(1+\xi)(1-\xi)(1+\eta)}{2};\ N_8=\frac{(1-\xi)(1+\eta)(1-\eta)}{2} \end{aligned} \tag{3.3.17}$$

The shape functions of rectangular elements with higher nodes can be derived in similar manner using appropriate polynomial satisfying all necessary criteria. However, difficulty arises due to the inversion of large size of the matrix because of higher degree of polynomial chosen. In next lecture, the shape functions of rectangular element with higher nodes will be derived in a much simpler way.

In last lecture note, the interpolation functions are derived on the basis of assumed polynomial from Pascal's triangle for the filed variable. As seen, the inverse of the large matrix is quite cumbersome if the element is of higher order.

3.4.1 Lagrange Interpolation Function

An alternate and simpler way to derive shape functions is to use Lagrange interpolation polynomials. This method is suitable to derive shape function for elements having higher order of nodes. The Lagrange interpolation function at node *i* is defined by

$$f_i(\xi)=\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{(\xi-\xi_j)}{(\xi_i-\xi_j)}=\frac{(\xi-\xi_1)(\xi-\xi_2)\dots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\dots(\xi-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)\dots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\dots(\xi_i-\xi_n)} \tag{3.4.1}$$

The function $f_i(\xi)$ produces the Lagrange interpolation function for i^{th} node, and ξ_j denotes ξ coordinate of j^{th} node in the element. In the above equation if we put $\xi = \xi_j$, and $j \neq i$, the value of the function $f_i(\xi)$ will be equal to zero. Similarly, putting $\xi = \xi_i$, the numerator will be equal to denominator and hence $f_i(\xi)$ will have a value of unity.Since, Lagrange interpolation function for i^{th} node includes product of all terms except j^{th}term; for an element with n nodes, $f_i(\xi)$ will have n-1 degrees of freedom. Thus, for one-dimensional elements with n-nodes we can define shape function as $N_i(\xi)=f_i(\xi)$.

3.4.1.1Shape function for two node bar element

Consider the two node bar element discussed as in section 3.1.1. Let us consider the natural coordinate of the center of the element as 0, and the natural coordinate of the nodes 1 and 2 are -1 and +1 respectively. Therefore, the natural coordinate ξ at any point x can be represented by,

$$\xi=\frac{2(x-x_1)}{l}-1 \tag{3.4.2}$$

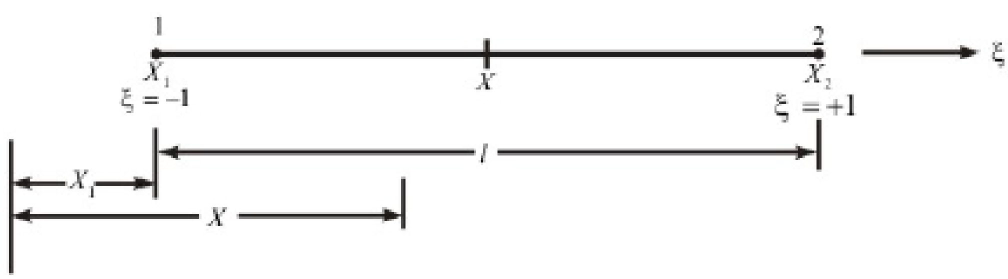


Fig. 3.4.1 Natural coordinates of bar element

The shape function for two node bar element as shown in Fig. 3.4.1 can be derived from eq.(3.4.1) as follows:

$$N_1 = f_1(\xi) = \frac{(\xi - \xi_2)}{(\xi_1 - \xi_2)} = \frac{(\xi - 1)}{-1-(1)} = \frac{1}{2}(1-\xi)$$
$$N_2 = f_1(\xi) = \frac{(\xi - \xi_1)}{(\xi_2 - \xi_1)} = \frac{(\xi + 1)}{1-(1)} = \frac{1}{2}(1+\xi) \qquad (3.4.3)$$

Graphically, these shape functions are represented in Fig.3.4.2.

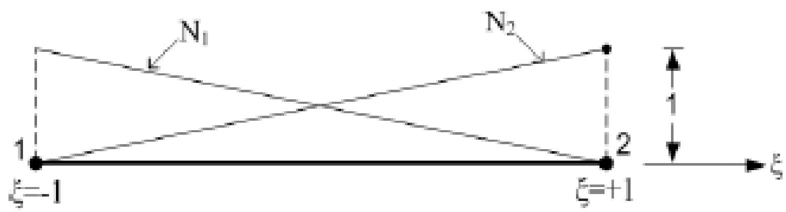


Fig.3.4.2 Shape functions for two node bar element

3.4.1.2 Shape function for three node bar element

For a three node bar element as shown in Fig.3.4.3, the shape function will be quadratic in nature. These can be derived in the similar fashion using eq.(3.4.1) which will be as follows:

$$N_1(\xi) = f_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi)(\xi - 1)}{(-1)(-2)} = \frac{1}{2}\xi(\xi - 1)$$

$$N_2(\xi) = f_2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{(\xi + 1)(\xi - 1)}{(1)(-1)} = (1 - \xi^2) \qquad (3.4.4)$$

$$N_3(\xi) = f_3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = \frac{(\xi + 1)(\xi)}{(2)(1)} = \frac{1}{2}\xi(\xi + 1)$$

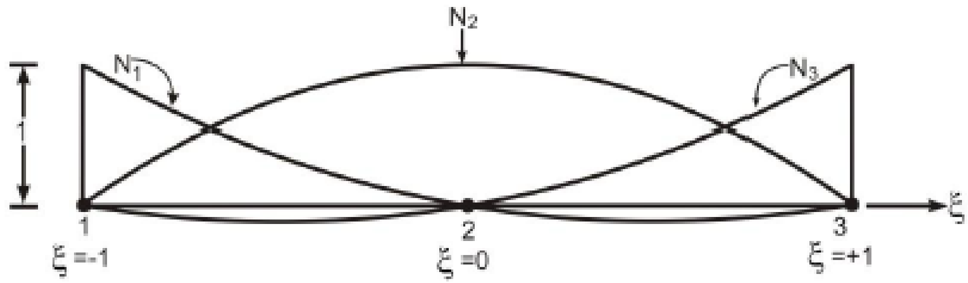


Fig.3.4.3 Quadratic shape functions for three node bar element

3.4.1.3 Shape function for two dimensional elements

We can derive the Lagrange interpolation function for two or three dimensional elements from one dimensional element as discussed above. Those elements whose shape functions are derived from the products of one dimensional Lagrange interpolation functions are called Lagrange elements. The Lagrange interpolation function for a rectangular element can be obtained from the product of appropriate interpolation functions in the ξ direction [$f_i(\xi)$] and η direction [$f_i(\eta)$]. Thus,

$$N_i(\xi,\eta) = f_i(\xi) f_i(\eta) \text{ Where, } i = 1,2,3, \ldots, \text{n-node} \tag{3.4.5}$$

The procedure is described in details in following examples.

Four node rectangular element

The shape functions for the four node rectangular element as shown in the Fig.3.4.4 can be derived by applying eq.(3.4.3) eq.(3.4.5)which will be as follows.

$$\begin{aligned} N_1(\xi,\eta) &= f_1(\xi) f_1(\eta) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)}\frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} \\ &= \frac{(\xi-1)}{-1-(1)} \times \frac{(\eta-1)}{-1-(1)} = \frac{1}{4}(1-\xi)(1-\eta) \end{aligned} \tag{3.4.6}$$

Similarly, other interpolation functions can be derived which are given below.

$$\begin{aligned} N_2(\xi,\eta) &= f_2(\xi) f_1(\eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) &= f_2(\xi) f_2(\eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) &= f_1(\xi) f_2(\eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{3.4.7}$$

These shape functions are exactly same as eq.(3.3.13) which was derivedearlier by choosing polynomials.

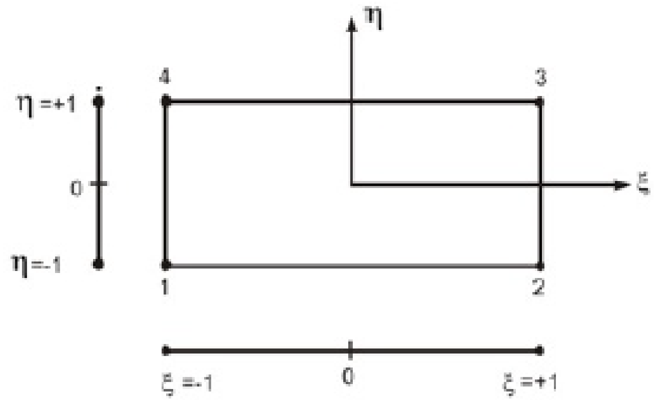


Fig. 3.4.4Four node rectangular element

Nine node rectangular element

In a similar way, to the derivation of four node rectangular element, we can derive the shape functions for a nine node rectangular element. In this case, the shape functions can be derived using eq.(3.4.4) and eq.(3.4.5).

$$N_1(\xi,\eta)=f_1(\xi)f_1(\eta)=\frac{1}{2}\xi(\xi-1)\times\frac{1}{2}\eta(\eta-1)=\frac{1}{4}\xi\eta(\xi-1)(\eta-1) \tag{3.4.8}$$

In a similar way, all the other shape functions of the element can be derived. The shape functions of nine node rectangular element will be:

$$\begin{aligned}
&N_1=\frac{1}{4}\xi\eta(\xi-1)(\eta-1), \qquad N_2=\frac{1}{4}\xi\eta(\xi+1)(\eta-1)\\
&N_3=\frac{1}{4}\xi\eta(\xi+1)(\eta+1), \qquad N_4=\frac{1}{4}\xi\eta(\xi-1)(\eta+1)\\
&N_5=\frac{1}{2}\eta\left(1-\xi^2\right)(\eta-1), \qquad N_6=\frac{1}{2}\xi(\xi+1)\left(1-\eta^2\right)\\
&N_7=\frac{1}{2}\eta\left(1-\xi^2\right)(\eta+1), \qquad N_8=\frac{1}{2}\xi(\xi-1)\left(1-\eta^2\right)\\
&N_9=\left(1-\xi^2\right)\left(1-\eta^2\right)
\end{aligned} \tag{3.4.9}$$

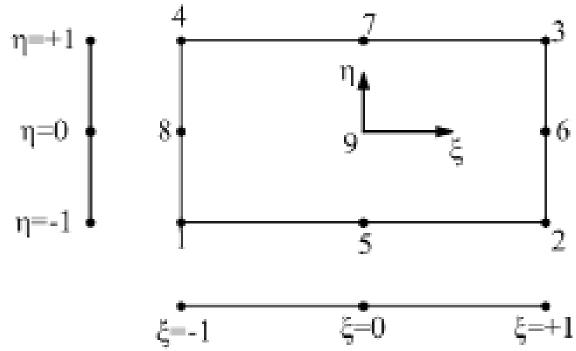


Fig.3.4.5 Nine node rectangular element

Thus, it is observed that the two dimensional Lagrange element contains internal nodes (Fig. 3.4.6) which are not connected to other nodes.

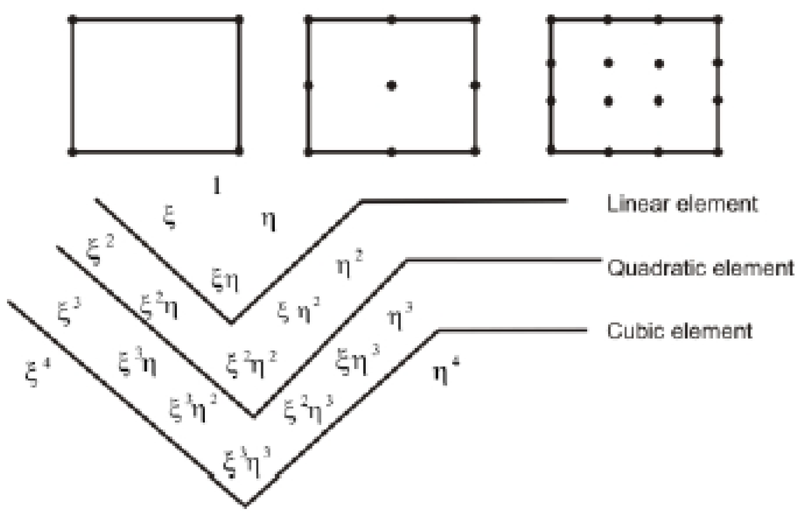


Fig. 3.4.6Two dimensional Lagrange elements and Pascal triangle

3.4.2 Serendipity Elements

Higher order Lagrange elements contains internal nodes, which do not contribute to the inter-element connectivity. However, these can be eliminated by condensation procedure which needs extra computation. The elimination of these internal nodes results in reduction in size of the element matrices.Alternatively, one can develop shape functions of two dimensional elements which contain nodes only on the boundaries. These elements are called serendipity elements (Fig. 3.4.7) and their interpolation functions can be derived by inspection or the procedure described in previous lecture

(Module 3, lecture 3). The interpolation function can be derived by inspection in terms of natural coordinate system as follows:

(a) Linear element

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{3.4.10}$$

(b) Quadratic element

(i) For nodes at $\xi=\pm1,\ \eta=\pm1$

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) \tag{3.4.11a}$$

(ii) For nodes at $\xi=\pm1,\ \eta=0$

$$N_i(\xi,\eta)=\frac{1}{2}(1+\xi\xi_i)(1-\eta^2) \tag{3.4.11b}$$

(iii) For nodes at $\xi=0,\ \eta=\pm1$

$$N_i(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta\eta_i) \tag{3.4.11c}$$

(c) Cubic element

(i) For nodes at $\xi=\pm1,\ \eta=\pm1$

$$N_i(\xi,\eta)=\frac{1}{32}(1+\xi\xi_i)(1+\eta\eta_i)\left[9\left(\xi^2+\eta^2\right)-10\right] \tag{3.4.12a}$$

(ii) For nodes at $\xi=\pm1,\ \eta=\pm\frac{1}{3}$

$$N_i(\xi,\eta)=\frac{9}{32}(1+\xi\xi_i)(1-\eta^2)(1+9\eta\eta_i) \tag{3.4.12b}$$

And so on for other nodes at the boundaries.

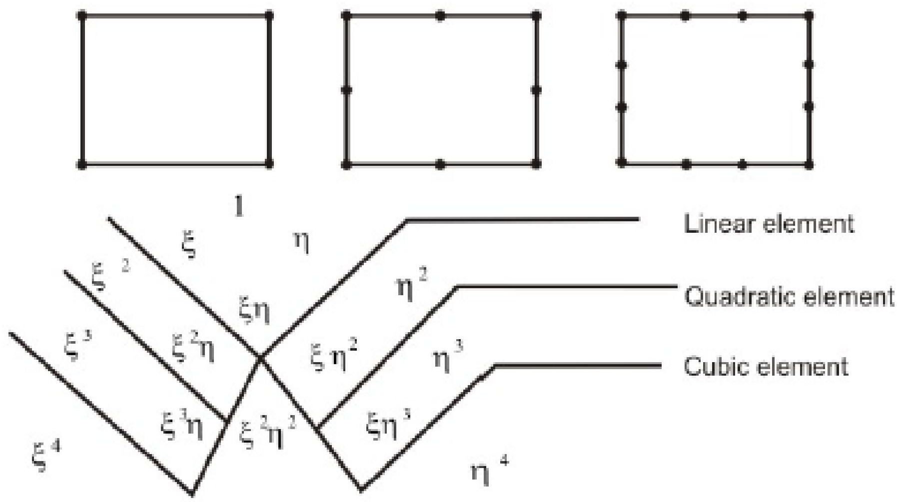


Fig. 3.4.7 Two dimensional serendipity elements and Pascal triangle

Thus, the nodal conditions must be satisfied by each interpolation function to obtain the functions serendipitously. For example, let us consider an eight node element as shown in Fig. 3.4.8 to derive its shape function. The interpolation function N_1 must become zero at all nodes except node 1, where its value must be unity. Similarly, at nodes 2, 3, and 6, $\xi = 1$, so including the term $\xi - 1$ satisfies the zero condition at those nodes. Similarly, at nodes 3, 4 and 7, $\eta = 1$ so the term $\eta - 1$ ensures the zero condition at these nodes.

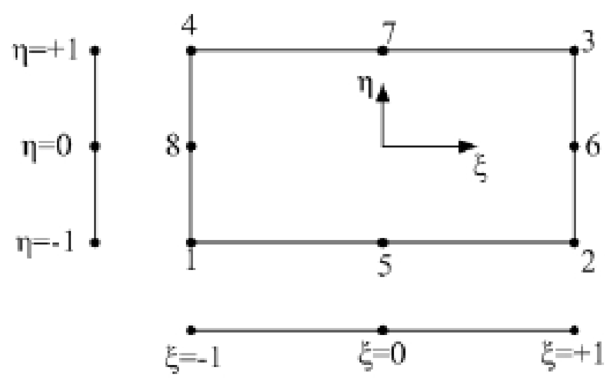


Fig. 3.4.8 Two dimensional eight node rectangular element

Again, at node 5, $(\xi, \eta) = (0, -1)$, and at node 8, $(\xi, \eta) = (-1, 0)$. Hence, at nodes 5 and 8, the term $(\xi + \eta + 1)$ is zero. Using this reasoning, the equation of lines are expressed in Fig. 3.4.9. Thus, theinterpolation function associated with node 1 is to be of the form $N_1 = \psi_1(\eta-1)(\xi-1)(\xi+\eta+1)$ where, ψ_1 is unknown constant. As the value of N_1 is 1 at node 1, the magnitude unknown constant ψ_1 will become -1/4. Therefore, the shape function for node 1 will become $N_1 = -\frac{1}{4}(1-\eta)(1-\xi)(\xi+\eta+1)$.

Similarly, ψ_2 will become -1/4 considering the value of N_2 at node 2 as unity and the shape function for node 2 will be $N_2 = \psi_2(\eta-1)(\xi+1)(\xi-\eta-1) = -\frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$. In a similar fashion one can find out other interpolation functions from Fig. 3.4.9 by putting the respective values at various nodes. Thus, the shape function for 8-node rectangular element is given below.

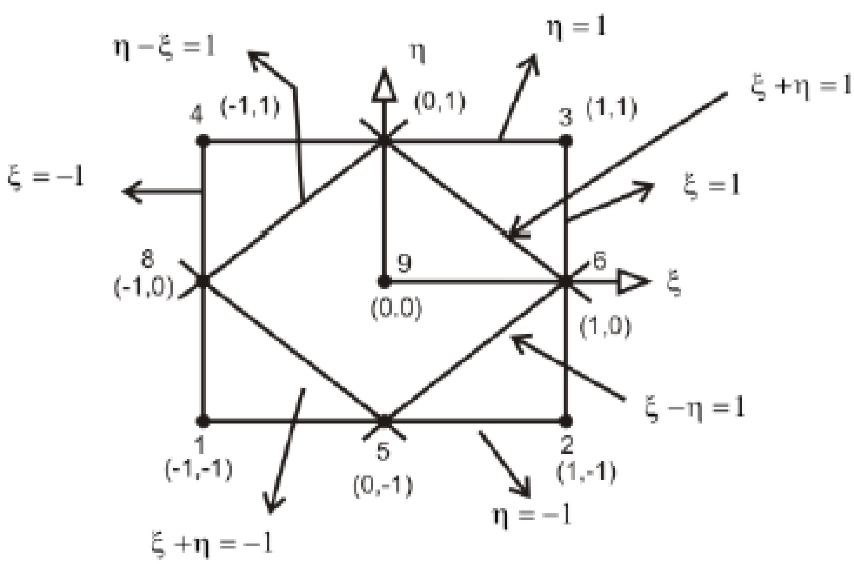
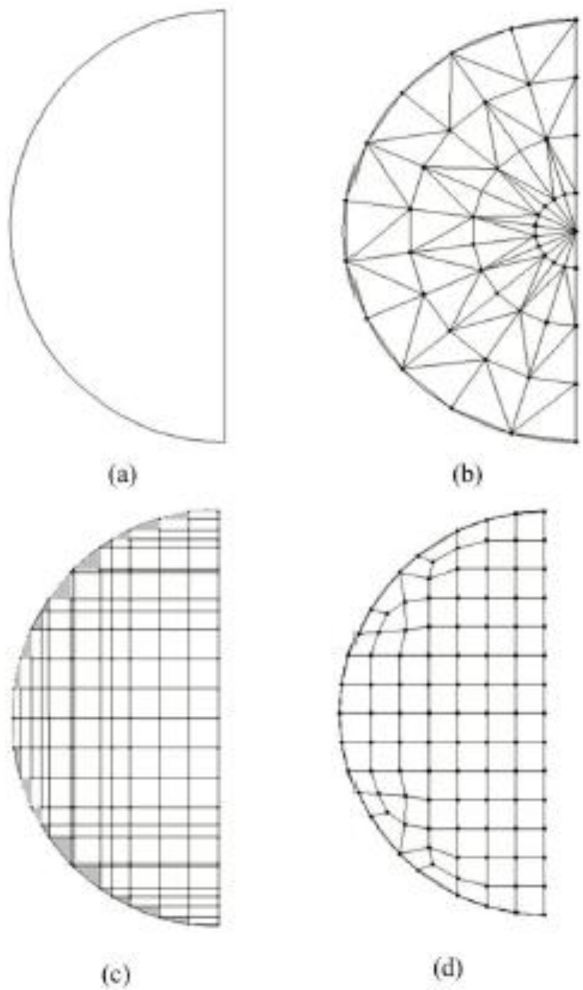


Fig. 3.4.9 Equations of lines for two dimensional eight node element

$$
\begin{aligned}
N_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta), \qquad & N_5 &= \frac{1}{2}(1-\xi^2)(1-\eta),\\
N_2 &= -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta), \qquad & N_6 &= \frac{1}{2}(1+\xi)(1-\eta^2),\\
N_3 &= -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta), \qquad & N_7 &= \frac{1}{2}(1-\xi^2)(1+\eta),\\
N_4 &= -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \text{ and } & N_8 &= \frac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}
\tag{3.4.13}
$$

3.6.1 Necessity of Isoparametric Formulation

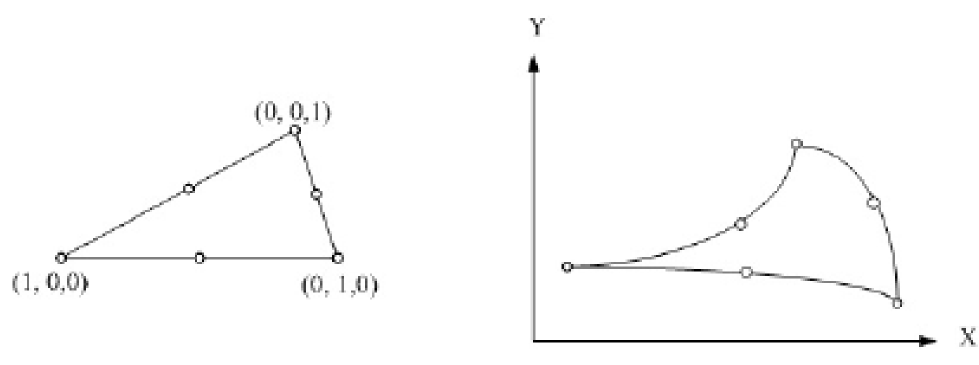
The two or three dimensional elements discussed till now are of regular geometry (e.g. triangular and rectangular element) having straight edge. Hence, for the analysis of any irregular geometry, it is difficult to use such elements directly. For example, the continuum having curve boundary as shown in the Fig. 3.6.1(a) has been discretized into a mesh of finite elements in three ways as shown.

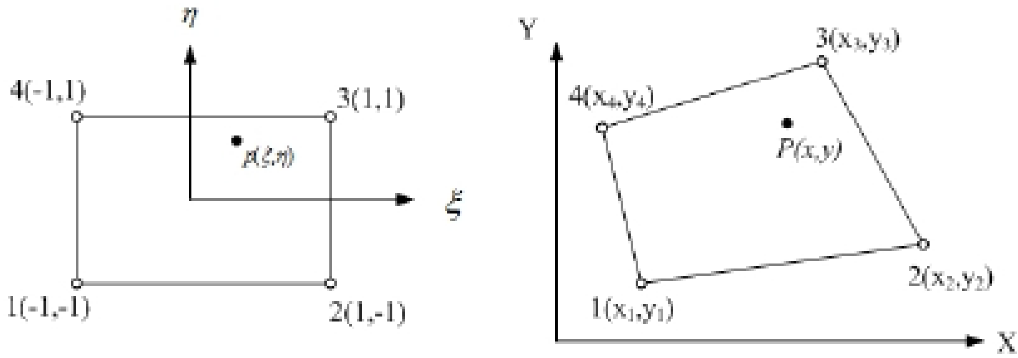


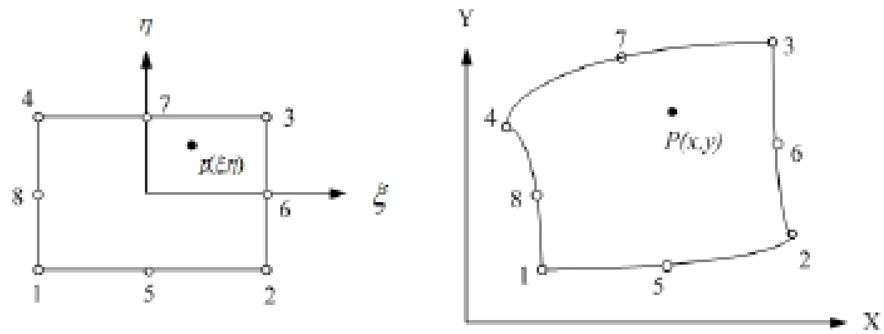
(a) The Continuum to be discritized (b) Discritization using Triangular Elements (c) Discritization using rectangular elements (d) Discritization using a combination of rectangular and quadrilateral elements

Fig 3.6.1 Discretization of a continuum using different elements

Figure 3.6.1(b) presents a possible mesh using triangular elements. Though, triangular elements can suitable approximate the circular boundary of the continuum, but the elements close to the center becomes slender and hence affect the accuracy of finite element solutions. One possible solution to the problem is to reduce the height of each row of elements as we approach to the center. But, unnecessary refining of the continuum generates relatively large number of elements and thus increases computation time. Alternatively, when meshing is done using rectangular elements as shown in Fig 3.6.1(c), the area of continuum excluded from the finite element model is significantlyadequate to provide incorrect results. In order to improve the accuracy of the result one can generate mesh using very small elements. But, this will significantly increase the computation time. Another possible way is to use a combination of both rectangular and triangular elements as discussed in section 3.2. But such types of combination may not provide the best solution in terms of accuracy, since different order polynomials are used to represent the field variables for different types of elements. Also the triangular elements may be slender and thus can affect the accuracy. In Fig.3.6.1(d), the same continuum is discritized with rectangular elements near center and with four-node quadrilateral elements near boundary. This four-node quadrilateral element can be derived from rectangular elements using the concept of mapping. Using the concept of mapping regular triangular, rectangular or solid elements in natural coordinate system (known as parent element) can be transformed into global Cartesian coordinate system having arbitrary shapes (with curved edge or surfaces). Fig.3.6.2 shows the parent elements in natural coordinate system and the mapped elements in global Cartesian system.




η
4(-1,1)
3(1,1)
p(ξ,η)
ξ
1(-1,-1)
2(1,-1)
Y
3(x3,y3)
4(x4,y4)
P(x,y)
2(x2,y2)
1(x1,y1)
X


η
4
7
3
p(ξ,η)
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Y
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P(x,y)
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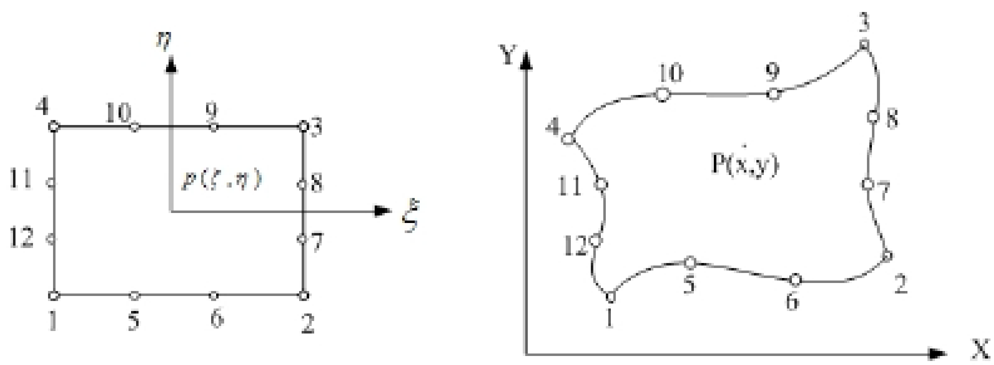
5
X


η
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p(ζ,η)
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12
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P(x,y)
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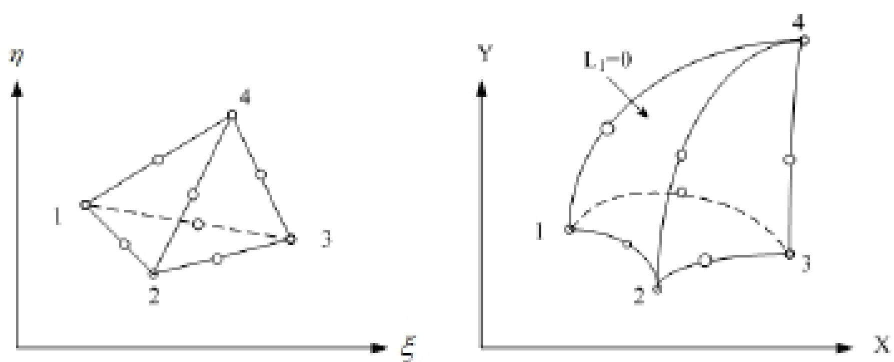
η
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1
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Y
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L1=0
1
3
2
X

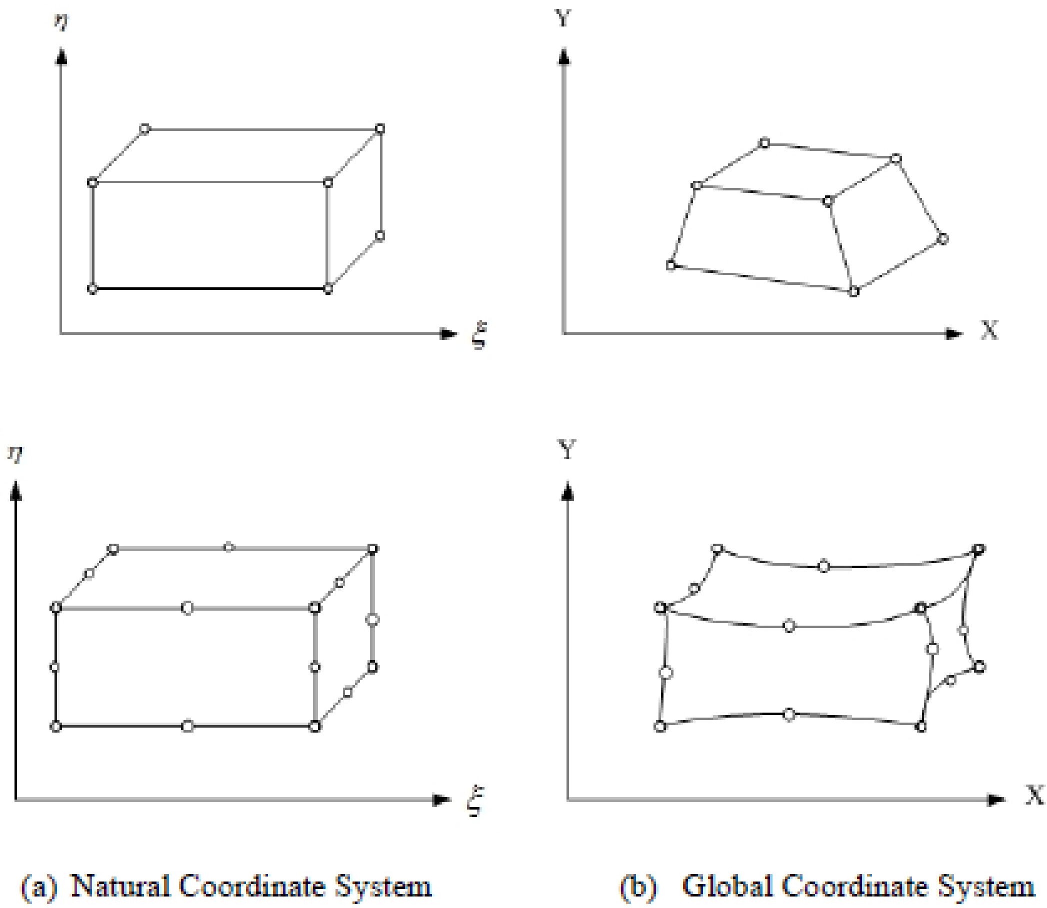

Fig. 3.6.2 Mapping of isoparametric elements in global coordinate system

3.6.2 Coordinate Transformation

The geometry of an element may be expressed in terms of the interpolation functions as follows.

$$
\begin{aligned}
x &= N_1 x_1 + N_2 x_2 + \dots + N_n x_n = \sum_{i=1}^{n} N_i x_i \\
y &= N_1 y_1 + N_2 y_2 + \dots + N_n y_n = \sum_{i=1}^{n} N_i y_i \\
z &= N_1 z_1 + N_2 z_2 + \dots + N_n z_n = \sum_{i=1}^{n} N_i z_i
\end{aligned}
\tag{3.6.1}
$$

Where,

n=No. of Nodes

N_i=Interpolation Functions

x_i, y_i, z_i=Coordinates of Nodal Points of the Element

One can also express the field variable variation in the element as

$$\phi(\xi,\eta,\zeta)=\sum_{i=1}^{n} N_i(\xi,\eta,\zeta)\phi_i \tag{3.6.2}$$

As the same shape functions are used for both the field variableand description of element geometry, the method is known as isoparametricmapping. The element defined by such a method is known as an isoparametric element. This method can be used to transform the natural coordinates of a point to the Cartesian coordinate system and vice versa.

Example 3.6.1

Determine the Cartesian coordinate of the point P (ξ= 0.8, η= 0.9) as shown in Fig. 3.6.3.

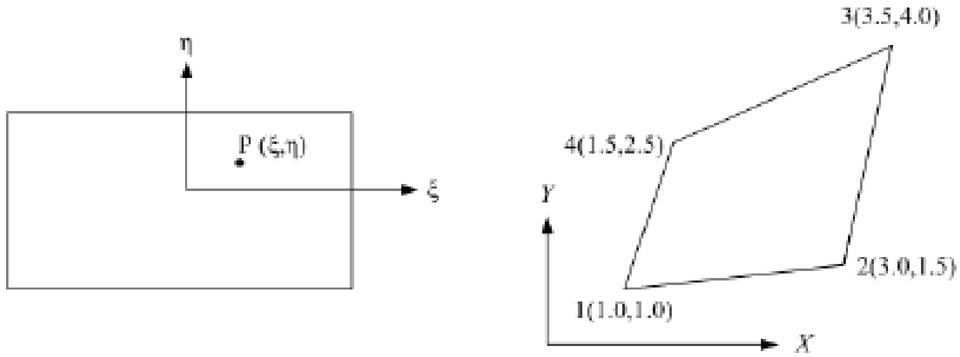

Fig. 3.6.3 Transformation of Coordinates

Solution:

As described above, the relation between two coordinate systems can be represented through their interpolation functions. Therefore, the valuesof the interpolation function at point P will be

$$N_1=\frac{(1-\xi)(1-n)}{4}=\frac{(1-0.8)(1-0.9)}{4}=0.005$$

$$N_2=\frac{(1+\xi)(1-n)}{4}=\frac{(1+0.8)(1-0.9)}{4}=0.045$$

$$N_3=\frac{(1+\xi)(1+n)}{4}=\frac{(1+0.8)(1+0.9)}{4}=0.855$$

$$N_4=\frac{(1-\xi)(1+n)}{4}=\frac{(1-0.8)(1+0.9)}{4}=0.095$$

Thus the coordinate of point *P* in Cartesian coordinate system can be calculated as

$$x=\sum_{i=1}^{4}N_i x_i=0.005\times1+0.045\times3+0.855\times3.5+0.095\times1.5=3.275$$

$$y=\sum_{i=1}^{4}N_i y_i=0.005\times1+0.045\times1.5+0.855\times4.0+0.095\times2.5=3.73$$

Thus the coordinate of point P (ξ= 0.8, η= 0.9) in Cartesian coordinate system will be 3.275, 3.73.

Solid isoparametric elements can easily be formulated by the extension of the procedure followed for 2-D elements. Regardless of the number of nodes or possible curvature of edges, the solid element is just like a plane element which is mapped into the space of natural co-ordinates, i.e, $\xi=\pm1,\eta=\pm1,\zeta=\pm1$.

3.6.3 Concept of Jacobian Matrix

A variety of derivatives of the interpolationfunctions with respect to the global coordinates are necessary to formulate the element stiffness matrices. As the both element geometry andvariation of the shape functions are represented in terms of the naturalcoordinates of the parent element,some additional mathematical obstacle arises. For example, in case of evaluation of the strain vector, the operator matrix is with respect to x and y, but the interpolation function is with ξ *and* η. Therefore, the operator matrix is to be transformed for taking derivative with ξ *and* η. The relationship between two coordinate systems may be computed by using the chain rule of partial differentiation as

$$\frac{\partial}{\partial\xi}=\frac{\partial}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial}{\partial y}\frac{\partial y}{\partial\xi}\quad and\quad \frac{\partial}{\partial\eta}=\frac{\partial}{\partial x}\frac{\partial x}{\partial\eta}+\frac{\partial}{\partial y}\frac{\partial y}{\partial\eta}\tag{3.6.3}$$

The above equations can be expressed in matrix form as well.

$$\begin{Bmatrix}\dfrac{\partial}{\partial\xi}\\ \dfrac{\partial}{\partial\eta}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial x}{\partial\xi} & \dfrac{\partial y}{\partial\xi}\\ \dfrac{\partial x}{\partial\eta} & \dfrac{\partial y}{\partial\eta}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{Bmatrix}=[J]\begin{Bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{Bmatrix}\tag{3.6.4}$$

The matrix [*J*] is denoted as Jacobian matrix which is: $\begin{bmatrix}\dfrac{\partial x}{\partial\xi} & \dfrac{\partial y}{\partial\xi}\\ \dfrac{\partial x}{\partial\eta} & \dfrac{\partial y}{\partial\eta}\end{bmatrix}$.As we know, $x=\sum_{i=1}^{n}N_i x_i$

where, *n*is the number of nodes in an element. Hence, $J_{11}=\dfrac{\partial x}{\partial\xi}=\dfrac{\partial\sum_{i=1}^{n}N_i x_i}{\partial\xi}=\sum_{i=1}^{n}\dfrac{\partial N_i}{\partial\xi}x_i$

Similarly one can calculate the other terms J_{12}, $J_{21}$and J_{22} of the Jacobian matrix. Hence,

$$[J]=\begin{bmatrix}\sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}y_i\\ \sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}y_i\end{bmatrix} \tag{3.6.5}$$

From eq. (3.6.4), one can write

$$\begin{Bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial}{\partial \xi}\\ \frac{\partial}{\partial \eta}\end{Bmatrix} \tag{3.6.6}$$

Considering $\begin{bmatrix}J_{11}^{*} & J_{12}^{*}\\ J_{21}^{*} & J_{22}^{*}\end{bmatrix}$ are the elements of inverted [J] matrix, we may arise into the following relations.

$$\begin{aligned}\frac{\partial}{\partial x}&=J_{11}^{*}\cdot\frac{\partial}{\partial \xi}+J_{12}^{*}\cdot\frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial y}&=J_{21}^{*}\cdot\frac{\partial}{\partial \xi}+J_{22}^{*}\cdot\frac{\partial}{\partial \eta}\end{aligned} \tag{3.6.7}$$

Similarly, for three dimensional case, the following relation exists between the derivative operators in the global and the natural coordinate system.

$$\begin{Bmatrix}\frac{\partial}{\partial \xi}\\ \frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial \zeta}\end{Bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}\\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}\end{bmatrix}\begin{Bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{Bmatrix}=[J]\begin{Bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{Bmatrix} \tag{3.6.8}$$

Where,

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}\\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}\end{bmatrix} \tag{3.6.9}$$

$[J]$ is known as the Jacobian Matrix for three dimensional case. Putting eq. (3.6.1) in eq. (3.6.9) and after simplifying one can get

$$[J]=\sum_{i=1}^{n}\begin{bmatrix}\frac{\partial N_i}{\partial \xi}x_i & \frac{\partial N_i}{\partial \xi}y_i & \frac{\partial N_i}{\partial \xi}z_i\\ \frac{\partial N_i}{\partial \eta}x_i & \frac{\partial N_i}{\partial \eta}y_i & \frac{\partial N_i}{\partial \eta}z_i\\ \frac{\partial N_i}{\partial \zeta}x_i & \frac{\partial N_i}{\partial \zeta}y_i & \frac{\partial N_i}{\partial \zeta}z_i\end{bmatrix} \tag{3.6.10}$$

From eq. (3.6.8), one can find the following expression.

$$\begin{Bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial}{\partial \xi}\\ \frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial \zeta}\end{Bmatrix} \tag{3.6.11}$$

Considering $[J]^{-1}=\begin{bmatrix}J_{11}^{*} & J_{12}^{*} & J_{13}^{*}\\ J_{21}^{*} & J_{22}^{*} & J_{23}^{*}\\ J_{31}^{*} & J_{32}^{*} & J_{33}^{*}\end{bmatrix}$ we can arrived at the following relations.

$$\begin{aligned}\frac{\partial}{\partial x}&=J_{11}^{*}\cdot\frac{\partial}{\partial \xi}+J_{12}^{*}\cdot\frac{\partial}{\partial \eta}+J_{13}^{*}\cdot\frac{\partial}{\partial \zeta}\\ \frac{\partial}{\partial y}&=J_{21}^{*}\cdot\frac{\partial}{\partial \xi}+J_{22}^{*}\cdot\frac{\partial}{\partial \eta}+J_{23}^{*}\cdot\frac{\partial}{\partial \zeta}\\ \frac{\partial}{\partial z}&=J_{31}^{*}\cdot\frac{\partial}{\partial \xi}+J_{32}^{*}\cdot\frac{\partial}{\partial \eta}+J_{33}^{*}\cdot\frac{\partial}{\partial \zeta}\end{aligned} \tag{3.6.12}$$

3.7.1 Evaluation of Stiffness Matrix of 2-D Isoparametric Elements

For two dimensional plane stress/strain formulation, the strain vector can be represented as

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\end{Bmatrix}=\begin{Bmatrix}J_{11}^{*}\cdot\dfrac{\partial u}{\partial \xi}+J_{12}^{*}\cdot\dfrac{\partial u}{\partial \eta}\\ J_{21}^{*}\cdot\dfrac{\partial v}{\partial \xi}+J_{22}^{*}\cdot\dfrac{\partial v}{\partial \eta}\\ J_{11}^{*}\cdot\dfrac{\partial v}{\partial \xi}+J_{12}^{*}\cdot\dfrac{\partial v}{\partial \eta}+J_{21}^{*}\cdot\dfrac{\partial u}{\partial \xi}+J_{22}^{*}\cdot\dfrac{\partial u}{\partial \eta}\end{Bmatrix} \tag{3.7.1}$$

The above expression can be rewritten in matrix form

$$\{\varepsilon\}=\begin{bmatrix}J_{11}^{*} & J_{12}^{*} & 0 & 0\\ 0 & 0 & J_{21}^{*} & J_{22}^{*}\\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u}{\partial \xi}\\ \dfrac{\partial u}{\partial \eta}\\ \dfrac{\partial v}{\partial \xi}\\ \dfrac{\partial v}{\partial \eta}\end{Bmatrix} \tag{3.7.2}$$

For an n node element the displacement$u$ can be represented as, $u=\sum_{i=1}^{n}N_i u_i$ and similarly for v&w.

Thus,

$$\begin{Bmatrix}\dfrac{\partial u}{\partial \xi}\\ \dfrac{\partial u}{\partial \eta}\\ \dfrac{\partial v}{\partial \xi}\\ \dfrac{\partial v}{\partial \eta}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi} & 0 & \cdots & 0\\ \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta} & 0 & \cdots & 0\\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi}\\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta}\end{bmatrix}\begin{Bmatrix}u_1\\ \vdots\\ u_n\\ v_1\\ \vdots\\ v_n\end{Bmatrix} \tag{3.7.3}$$

As a result, eq.(3.7.2) can be written using eq. (3.7.3) which will be as follows.

$$\{\varepsilon\}=\begin{bmatrix} J_{11}^{*} & J_{12}^{*} & 0 & 0 \\ 0 & 0 & J_{21}^{*} & J_{22}^{*} \\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*} \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix}\begin{Bmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ \vdots \\ v_n \end{Bmatrix} \tag{3.7.4}$$

Or,

$$\{\varepsilon\}=[B]\{d\} \tag{3.7.5}$$

Where {d} is the nodal displacement vector and [B] is known as strain displacement relationship matrix and can be obtained as

$$[B]=\begin{bmatrix} J_{11}^{*} & J_{12}^{*} & 0 & 0 \\ 0 & 0 & J_{21}^{*} & J_{22}^{*} \\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*} \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \tag{3.7.6}$$

It is necessary to transform integrals from Cartesian tothe natural coordinates as well for calculation of the elemental stiffness matrix in isoparametric formulation. The differential area relationship can be established from advanced calculus and the elemental area in Cartesian coordinate can be represented in terms of area in natural coordinates as:

$$dA = dx\ dy = |J|\, d\xi\ d\eta \tag{3.7.7}$$

Here $|J|$ is the determinant of the Jacobian matrix. The stiffness matrix for a two dimensional element may be expressed as

$$[k]=\iiint_{\Omega}[B]^{T}[D][B]\,d\Omega = t\iint_{A}[B]^{T}[D][B]\,dxdy \tag{3.7.8}$$

Here, [B] is the strain-displacement relationship matrix and *t* is the thickness of the element. The above expression in Cartesian coordinate system can be changed to the natural coordinate system as follows to obtain the elemental stiffness matrix

$$[k] = t\int_{-1}^{+1}\int_{-1}^{+1}[B]^T[D][B]|J|\,d\xi\,d\eta \tag{3.7.9}$$

Though the isoparametric formulation is mathematically straightforward, the algebraic difficulty is significant.

Example 3.7.1:

Calculate the Jacobian matrix and the strain displacement matrix for four nodetwo dimensional quadrilateral elements corresponding to the gauss point (0.57735, 0.57735) as shown in Fig.3.6.4.

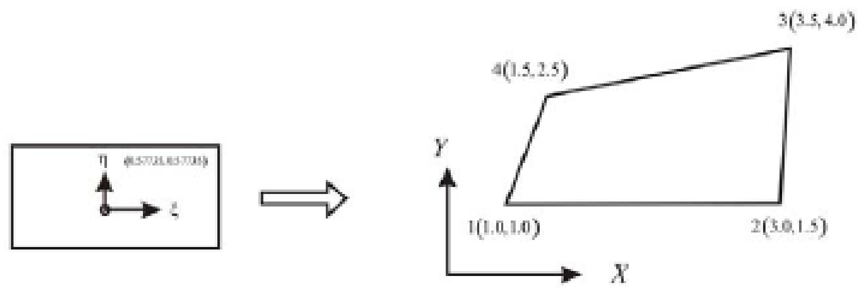

Fig.3.7.1 Two dimensional quadrilateral element

Solution:

The Jacobian matrix for a four node element is given by,

$$[J] = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

For the four node element one can find the following relations.

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}, \quad \frac{\partial N_1}{\partial \xi} = -\frac{1-\eta}{4}, \quad \frac{\partial N_1}{\partial \eta} = -\frac{1-\xi}{4}$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4}, \quad \frac{\partial N_2}{\partial \xi} = \frac{1-\eta}{4}, \quad \frac{\partial N_2}{\partial \eta} = -\frac{1+\xi}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4}, \quad \frac{\partial N_3}{\partial \xi} = \frac{1+\eta}{4}, \quad \frac{\partial N_3}{\partial \eta} = \frac{1+\xi}{4}$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4}, \quad \frac{\partial N_4}{\partial \xi} = -\frac{1+\eta}{4}, \quad \frac{\partial N_4}{\partial \eta} = \frac{1-\xi}{4}$$

Now, for a four node quadrilateral element, the Jacobian matrix will become

$$[J] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1-\eta}{4} & \frac{1-\eta}{4} & \frac{1+\eta}{4} & -\frac{1+\eta}{4} \\ -\frac{1-\xi}{4} & -\frac{1+\xi}{4} & \frac{1+\xi}{4} & \frac{1-\xi}{4} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Putting the values of ξ & η as 0.57735 and 0.57735 respectively, one will obtain the following.

$$\frac{\partial N_1}{\partial \xi} = -0.10566 \qquad \frac{\partial N_1}{\partial \eta} = -0.10566$$

$$\frac{\partial N_2}{\partial \xi} = 0.10566 \qquad \frac{\partial N_2}{\partial \eta} = -0.39434$$

$$\frac{\partial N_3}{\partial \xi} = 0.39434 \qquad \frac{\partial N_3}{\partial \eta} = 0.39434$$

$$\frac{\partial N_4}{\partial \xi} = -0.39434 \qquad \frac{\partial N_4}{\partial \eta} = 0.10566$$

Hence, $J_{11} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i = -0.10566 \times 1 + \ 0.10566 \times 3 + 0.39434 \times 3.5 - 0.39434 \times 1.5 = 1.0$

Similarly, J_{12} =0.64632, J_{21} =0.25462 and J_{22} =1.14962.

Hence,

$$J = \begin{bmatrix} 1.00000 & 0.64632 \\ 0.25462 & 1.14962 \end{bmatrix}$$

Thus, the inverse of the Jacobian matrix will become:

$$[J^*] = \begin{bmatrix} J^*_{11} & J^*_{12} \\ J^*_{21} & J^*_{22} \end{bmatrix} = \begin{bmatrix} 1.1671 & -0.6561 \\ -0.2585 & 1.0152 \end{bmatrix}$$

Hence strain displacement matrix is given by,

$$[B]=\begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix}$$

$$=\begin{bmatrix} 1.1671 & -0.6561 & 0 & 0 \\ 0 & 0 & -0.2585 & 1.0152 \\ -0.2585 & 1.0152 & 1.1671 & -0.6561 \end{bmatrix}\times$$

$$\begin{bmatrix} -0.10566 & 0.10566 & 0.39434 & -0.39434 & 0 & 0 & 0 & 0 \\ -0.10566 & -0.39434 & 0.39434 & 0.10566 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.10566 & 0.10566 & 0.39434 & -0.39434 \\ 0 & 0 & 0 & 0 & -0.10566 & -0.39434 & 0.39434 & 0.10566 \end{bmatrix}$$

$$=\begin{bmatrix} -0.0540 & 0.3820 & 0.2015 & -0.5294 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0800 & -0.4276 & 0.2984 & 0.2092 \\ -0.0800 & -0.4276 & 0.2984 & 0.2092 & -0.0540 & 0.3820 & 0.2015 & -0.5294 \end{bmatrix}$$

3.7.2 Evaluation of Stiffness Matrix of 3-D Isoparametric Elements

Stiffness matrix of 3-D solid isoparametric elements can easily be formulated by the extension of the procedure followed for plane elements. For example, the eight node solid element is analogous to the four node plane element. The strain vector for solid element can be written in the following form.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{Bmatrix} \tag{3.7.10}$$

The above equation can be expressed as

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial x}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{21}^* & J_{22}^* & J_{23}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* \\ J_{21}^* & J_{22}^* & J_{23}^* & J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* & J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* & 0 & 0 & 0 & J_{11}^* & J_{12}^* & J_{13}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} \end{Bmatrix} \tag{3.7.11}$$

For an 8 node brick element u can be represented as, $u = \sum_{i=1}^{8} N_i u_i$ and similarly for v&w.

$$\frac{\partial u}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} u_i \,,\; \frac{\partial u}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} u_i \;\&\; \frac{\partial u}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} u_i$$
$$\frac{\partial v}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} v_i \,,\; \frac{\partial v}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} v_i \;\&\; \frac{\partial v}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} v_i \qquad (3.7.12)$$
$$\frac{\partial w}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} w_i \,,\; \frac{\partial w}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} w_i \;\&\; \frac{\partial w}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} w_i$$

Hence eq. (3.7.11) can be rewritten as

$$\{\varepsilon\} = \begin{bmatrix} J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{21} & J^*_{22} & J^*_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} \\ J^*_{21} & J^*_{22} & J^*_{23} & J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} & J^*_{21} & J^*_{22} & J^*_{23} \\ J^*_{31} & J^*_{32} & J^*_{33} & 0 & 0 & 0 & J^*_{11} & J^*_{12} & J^*_{13} \end{bmatrix} \times \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial \eta} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \xi} & 0 \\ 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} & 0 & \frac{\partial N_i}{\partial \xi} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \qquad (3.7.13)$$

Thu, the strain-displacement relationship matrix [B] for 8 node brick element is

$$[B]=\begin{bmatrix} J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{21} & J^*_{22} & J^*_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} \\ J^*_{21} & J^*_{22} & J^*_{23} & J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} & J^*_{21} & J^*_{22} & J^*_{23} \\ J^*_{31} & J^*_{32} & J^*_{33} & 0 & 0 & 0 & J^*_{11} & J^*_{12} & J^*_{13} \end{bmatrix} \times \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial \eta} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \xi} & 0 \\ 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} & 0 & \frac{\partial N_i}{\partial \xi} \end{bmatrix} \quad (3.7.14)$$

The stiffness matrix may be found by using the following expression in natural coordinate system.

$$[\mathrm{k}]=\iiint_{\Omega}[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}][\mathrm{B}]\,\mathrm{d}\Omega=\iiint_{\mathrm{V}}[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}][\mathrm{B}]\,\mathrm{dxdydz}=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}][\mathrm{B}]\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,|\mathrm{J}| \quad (3.7.15)$$

3.8. NUMERICAL INTEGRATION

The integrations, we generally encounter in finite element methods, are quite complicated and it is not possible to find a closed form solutions to those problems. Exact and explicit evaluation of the integral associated to the element matrices and the loading vector is not always possible because of the algebraic complexity of the coefficient of the different equation (i.e., the stiffness influence coefficients, elasticity matrix, loading functions etc.). In the finite element analysis, we face the problem of evaluating the following types of integrations in one, two and three dimensional cases respectively. These are necessary to compute element stiffness and element load vector.

$$\int \phi(\xi)\,d\xi; \quad \int \phi(\xi,\eta)\,d\xi\,d\eta; \quad \int \phi(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta; \tag{3.8.1}$$

Approximate solutions to such problems are possible using certain numerical techniques. Several numerical techniques are available, in mathematics for solving definite integration problems, including, mid-point rule, trapezoidal-rule, Simpson's 1/3rd rule, Simpson's 3/8th rule and Gauss Quadrature formula. Among these, Gauss Quadrature technique is most useful one for solving problems in finite element method and therefore will be discussed in details here.

3.8.1 Gauss Quadrature for One-Dimensional Integrals

The concept of Gauss Quadrature is first illustrated in one dimension in the context of an integral in the form of $I = \int_{-1}^{+1} \phi(\xi)\,d\xi$ from $\int_{x_1}^{x_2} f(x)dx$. To transform from an arbitrary interval of $x_1 \leq x \leq x_2$ to an interval of $-1 \leq \xi \leq 1$, we need to change the integration function from $f(x)$ to $\phi(\xi)$ accordingly. Thus, for a linear variation in one dimension, one can write the following relations.

$$x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2 = N_1x_1 + N_2x_2$$

$$\text{so for } \xi = -1, x = \frac{1-(-1)}{2}x_1 + \frac{1-1}{2}x_2 = x_1$$

$$\xi = +1, \quad x = x_2$$

$$\therefore I = \int_{x_1}^{x_2} f(x)dx = \int_{-1}^{+1} \phi(\xi)\,d\xi$$

Numerical integration based on Gauss Quadrature assumes that the function $\phi(\xi)$ will be evaluated over an interval $-1 \leq \xi \leq 1$. Considering an one-dimensional integral, Gauss Quadrature represents the integral $\phi(\xi)$ in the form of

$$I = \int_{-1}^{+1} \phi(\xi)\,d\xi \approx \sum_{i=1}^{n} w_i\phi(\xi_i) \approx w_1\phi(\xi_1) + w_2\phi(\xi_2) + \ldots\ldots + w_\eta\phi(\xi_\eta) \tag{3.8.2}$$

Where, the ξ_1, ξ_2, ξ_3, ..., ξ_n represents n numbers of points known as Gauss Points and the corresponding coefficients w_1, w_2, w_3, …, w_n are known as weights. The location and weight coefficients of Gauss points are calculated by Legendre polynomials. Hence this method is also sometimes referred as Gauss-Legendre Quadrature method. The summation of these values at n sampling points gives the exact solution of a polynomial integrand of an order up to $2n$-1. For example, considering sampling at two Gauss points we can get exact solution for a polynomial of an order (2×2-1) or 3. The use of more number of Gauss points has no effect on accuracy of results but takes more computation time.

3.8.2One- Point Formula

Considering $n = 1$, eq.(3.8.2) can be written as

$$\int_{-1}^{1}\phi(\xi)d\xi \approx w_1\phi(\xi_1) \tag{3.8.3}$$

Since there are two parameters w_1 and ξ_1, we need a first order polynomial for $\phi(\xi)$ to evaluate the eq.(3.8.3) exactly. For example, considering, $\phi(\xi) = a_0 + a_1\xi$,

$$\mathrm{Error} = \int_{-1}^{1}(a_0 + a_1\xi)d\xi - w_1\phi(\xi_1) = 0$$

$$\Rightarrow 2a_0 - w_1(a_0 + a_1\xi_1) = 0$$

$$\Rightarrow a_0(2 - w_1) - w_1 a_1 \xi_1 = 0 \tag{3.8.4}$$

Thus, the error will be zero if $w_1 = 2$ and $\xi_1 = 0$. Putting these in eq.(3.8.3), for any general ϕ, we have

$$I = \int_{-1}^{1}\phi(\xi)d\xi = 2\phi(0) \tag{3.8.5}$$

This is exactly similar to the well known midpoint rule.

3.8.3 Two-Point Formula

If we consider n = 2, then the eq.(3.8.2) can be written as

$$\int_{-1}^{1}\phi(\xi)d\xi \approx w_1\phi(\xi_1) + w_2\phi(\xi_2) \tag{3.8.6}$$

This means we have four parameters to evaluate. Hence we need a 3rd order polynomial for $\phi(\xi)$ to exactly evaluate eq.(3.8.6).

Considering, $\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$

$$\mathrm{Error} = \left[\int_{-1}^{1}\left(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3\right)d\xi\right] - \left[w_1\phi(\xi_1) + w_2\phi(\xi_2)\right]$$

$$\Rightarrow 2a_0 + \frac{2}{3}a_2 - w_1\left(a_0 + a_1\xi_1 + a_2\xi_1^{\,2} + a_3\xi_1^{\,3}\right) - w_2\left(a_0 + a_1\xi_2 + a_2\xi_2^{\,2} + a_3\xi_2^{\,3}\right) = 0$$

$$\Rightarrow \left(2 - w_1 - w_2\right)a_0 - \left(w_1\xi_1 + w_2\xi_2\right)a_1 + \left(\frac{2}{3} - w_1\xi_1^{\,2} - w_2\xi_2^{\,2}\right)a_2 - \left(w_1\xi_1^{\,3} + w_2\xi_2^{\,3}\right)a_3 = 0$$

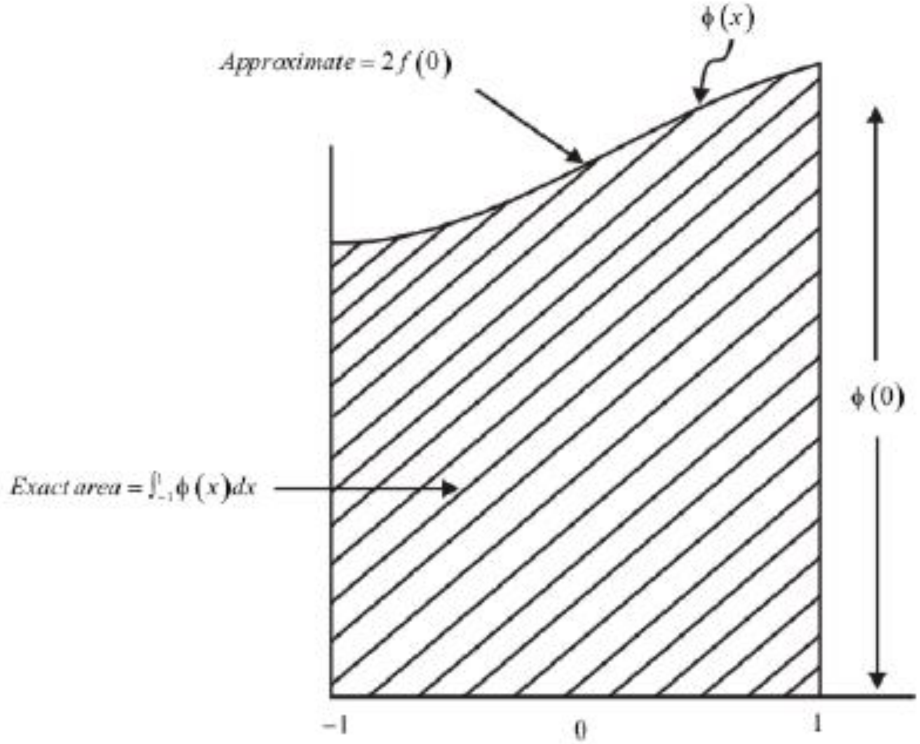

Fig 3.8.1 One-point Gauss Quadrature

Requiring zero error yields

$$\begin{aligned} &w_1 + w_2 = 2 \\ &w_1\xi_1 + w_2\xi_2 = 0 \\ &w_1\xi_1^2 + w_2\xi_2^2 = \frac{2}{3} \\ &w_1\xi_1^3 + w_2\xi_2^3 = 0 \end{aligned} \qquad (3.8.7)$$

These nonlinear equations have the unique solution as

$$w_1 = w_2 = 1 \qquad \xi_1 = -\xi_2 = -1/\sqrt{3} = -0.5773502691 \qquad (3.8.8)$$

From this solution, we can conclude that *n*-point Gaussian Quadrature will provide an exact solution if $\phi(\xi)$ is a polynomial of order $(2n\text{-}1)$ or less. Table 3.8.1 gives the values of w_i and ξ_i for Gauss Quadrature formulas of orders n = 1 through n = 6. From the table it can be observed that the gauss points are symmetrically placed with respect to origin and those symmetrical points have the same weights. For accuracy in the calculation maximum number digits for gauss point and gauss weights should be taken. The Location and weights given in the Table 3.8.1 must be used when the limits of integration ranges from -1 to 1. Integration limits other than [-1, 1], should be appropriately changed to [-1, 1] before applying these values.

Table 3.8.1 Gauss points and corresponding weights

Number of Gauss points, n	**Gauss Point Location, ξ_{ci}**	**Weight, w_i**
1	0.0	2.0
2	±0.5773502692 ($=\pm 1/\sqrt{3}$)	1.0
3	0.0	0.8888888889 (=8/9)
	±0.7745966692 ($=\pm\sqrt{0.6}$)	0.5555555556 (=5/9)
4	±0.3399810436	0.6521451549
	±0.861363116	0.3478548451
5	0.0	0.5688888889
	±0.5384693101	0.4786286705
	±0.9061798459	0.2369268851
6	±0.2386191861	0.4679139346
	±0.6612093865	0.3607615730
	±0.9324695142	0.1713244924

Example 1:

Evaluate $I = \int_0^1 \left(e^x - \frac{2x}{x^2 - 2}\right) dx$ using one, two and three point gauss Quadrature.

Solution:

Before applying the Gauss Quadrature formula, the existing limits of integration should be changed from [0, 1] to [-1, +1].Assuming, $\xi = a + bx$, the upper and lower limit can be changed. i.e., at x = 0,ξ = -1 and at x = 1, ξ = +1. Thus, putting these conditions and solving for a & b, we get a = -1 and b = 2. The relation between two coordinate systems will become $\xi = 2x - 1$ and $d\xi = 2dx$.

Therefore the initial equation can be written as

$$I=\int_{-1}^{1}\left(e^{\left(\frac{\xi+1}{2}\right)}-\frac{2\left(\frac{\xi+1}{2}\right)}{\left(\frac{\xi+1}{2}\right)^{2}-2}\right)dx$$

Or, $I=\frac{1}{2}\int_{-1}^{1}\left(e^{\frac{(\xi+1)}{2}}-\frac{4(\xi+1)}{(\xi+1)^{2}-8}\right)d\xi$

Using one point gauss Quadrature:

$w_1=2,\ \xi_1=0$ and

$I\approx 2\phi(0)$

Or $I\approx 2\left(\frac{1}{2}\left(e^{0.5}+\frac{4}{7}\right)\right)=2.22015$

Using two point gauss Quadrature:

$w_1=w_2=1$

$\xi_1=-0.5773502692$

$\xi_2=0.5773502692$

Putting these values and calculating, $I=2.39831$

Using three point gauss Quadrature:

$w_1=\ \ 0.555555556$

$\xi_1=-0.774596669$

$w_2=\ \ 0.888888889$

$\xi_2=\ \ 0.000000000$

$w_3=\ \ 0.555555556$

$\xi_3=\ \ 0.774596669$

and $I=2.41024$

This may be compared with the exact solution as $I_{exact}=2.41193$

Numerical integrations using Gauss Quadrature method can be extended to two and three dimensional cases in a similar fashion. Such integrations are necessary to perform for the analysis of plane stress/strain problem, plate and shell structures and for the three dimensional stress analysis.

3.9.1 Gauss Quadrature for Two-Dimensional Integrals

For two dimensional integration problems the above mentioned method can be extended by first evaluating the inner integral, keeping η constant, and then evaluating the outer integral. Thus,

$$I=\int_{-1}^{1}\int_{-1}^{1}\phi(\xi,\eta)d\xi d\eta \approx \int_{-1}^{1}\left[\sum_{i=1}^{n}w_i\phi(\xi_i,\eta)\right]d\eta \approx \sum_{i=1}^{n}w_j\left[\sum_{i=1}^{n}w_i\phi(\xi_i,\eta_j)\right]$$

Or,

$$I\approx\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\phi(\xi_i,\eta_j) \tag{3.9.1}$$

In a matrix form we can rewrite the above expression as

$$I\approx\begin{bmatrix}w_1 & w_2 & \dots & w_n\end{bmatrix}\begin{bmatrix}\phi(\xi_1,\eta_1) & \phi(\xi_1,\eta_2) & & \phi(\xi_1,\eta_n)\\ \phi(\xi_2,\eta_1) & \phi(\xi_2,\eta_2) & & \phi(\xi_2,\eta_n)\\ & & \ddots & \\ \phi(\xi_n,\eta_1) & \phi(\xi_n,\eta_2) & & \phi(\xi_n,\eta_n)\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ \vdots\\ w_n\end{Bmatrix} \tag{3.9.2}$$

Example 1:

Evaluate the integral: $I=\int\limits_{y=c=-4}^{y=d=4}\int\limits_{x=a=2}^{x=b=3}(1-x)^2(2-y)^2dxdy$

Solution:

Before applying the Gauss Quadrature formula, the above integral should be converted in terms of ξ and η and the existing limits of y should be changed from [-4,4] to [-1, 1] and that of x is from [2,3] to [-1,1].

$$x = \frac{(b-a)}{2}\xi + \frac{(b+a)}{2} = \frac{(\xi+5)}{2}; \quad dx = \frac{d\xi}{2}$$

$$y = \frac{(d-c)}{2}\eta + \frac{(d+c)}{2} = 4\eta; \quad dy = 4d\eta$$

$$I = 2\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1}\left(\frac{3+\xi}{2}\right)^2 (2-4\eta)^2\, d\xi\, d\eta = \int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1}\phi(\xi,\eta)\, d\xi\, d\eta$$

where $\quad \phi(\xi,\eta) = 2\left(\frac{3+\xi}{2}\right)^2 (2-4\eta)^2 = 2(3+\xi)^2(1-2\eta)^2$

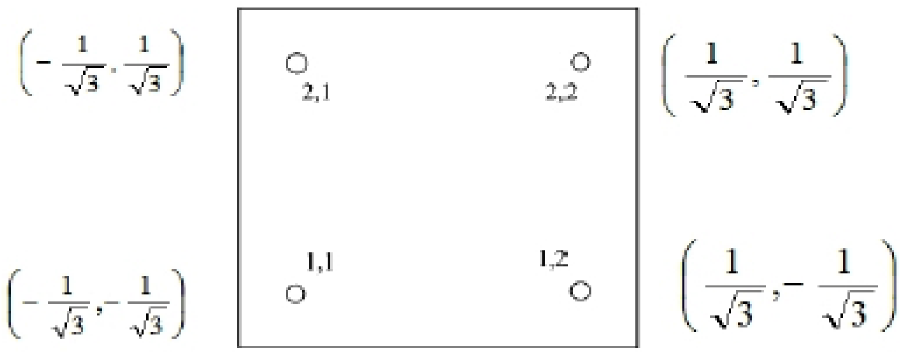

Fig. 3.9.1 Gauss points for two-dimensional integral

$$\xi_1 = -\frac{1}{\sqrt{3}}; \eta_1 = -\frac{1}{\sqrt{3}}; \xi_2 = \frac{1}{\sqrt{3}}; \eta_2 = \frac{1}{\sqrt{3}}$$

$$\phi(\xi_1, \eta_1) = 2\left(3 - \frac{1}{\sqrt{3}}\right)^2 \left(1 + \frac{2}{\sqrt{3}}\right)^2 = 54.49857$$

$$\phi(\xi_2, \eta_1) = \left(\frac{3 + \frac{1}{\sqrt{3}}}{2}\right)^2 \left(2 + \frac{4}{\sqrt{3}}\right)^2 = 118.83018$$

$$\phi(\xi_2, \eta_2) = \left(\frac{3 + \frac{1}{\sqrt{3}}}{2}\right)^2 \left(2 - \frac{4}{\sqrt{3}}\right)^2 = 0.61254$$

$$\phi(\xi_1, \eta_2) = \left(\frac{3 - \frac{1}{\sqrt{3}}}{2}\right)^2 \left(2 - \frac{4}{\sqrt{3}}\right)^2 = 0.28093$$

$$I = \{w_1 \quad w_2\} \begin{bmatrix} \phi(\xi_1, \eta_1) & \phi(\xi_1, \eta_2) \\ \phi(\xi_2, \eta_1) & \phi(\xi_2, \eta_2) \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}$$

$$= \{1 \quad 1\} \begin{bmatrix} 54.49857 & 0.28093 \\ 118.83018 & 0.61254 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

= 174.22222 agrees with the exact value 174.22222

3.9.3 Numerical Integration of Element Stiffness Matrix

As discussed earlier notes, the element stiffness matrix for three dimensional analyses in natural coordinate system can be written as

$$[k] = \iiint_{\Omega} [B]^T [D][B] d\Omega = \iiint_{V} [B]^T [D][B] dxdydz = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [B]^T[D][B] d\xi d\eta d\zeta |J| \quad (3.9.4)$$

Here, $[B]$ and $[D]$ are the strain displacement relationship matrix and constitutive matrix respectively and integration is performed over the domain. As the element stiffness matrix will be calculated in natural coordinate system, the strain displacement matrix [B] and Jacobian matrix [J] are functions of $\xi, \eta \text{ and } \zeta$. In case of two dimensional isoparametric element, the stiffness matrix will be simplified to

$$[k] = t\int_{-1}^{+1}\int_{-1}^{+1} [B]^T[D][B] d\xi d\eta |J| \quad (3.9.5)$$

This is actually an 8×8 matrix containing the integrals of each element. We do not need to integrate elements below the main diagonal of the stiffness matrix as it is symmetric. Considering, $\phi(\xi,\eta) = t[B]^T[D][B]|J|$, the element stiffness matrix will become after numerical integration as

$$[k] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \phi(\xi_i, \eta_j) \quad (3.9.6)$$

Using a 2×2 rule, we get

$$[k] = w_1^2\phi(\xi_1, \eta_1) + w_1 w_2 \phi(\xi_1, \eta_2) + w_2 w_1 \phi(\xi_2, \eta_1) + w_2^2 \phi(\eta_2, \eta_2) \quad (3.9.7)$$

Where $w_1 = w_2 = 1.0, \xi_1 = \eta_1 = -0.57735....,\text{and } \xi_2 = \eta_2 = +0.57735....$ Here, w_n is the weight factor at integration point n. A suitable computer program can be written to calculate the element stiffness matrix through the numerical integration. The process of obtaining stiffness matrix using Gauss Quadrature integration will be demonstrated through a numerical example in module 5.

3.10.4 Gauss Quadrature for Triangular Elements

The procedure described for the rectangular element will not be applicable directly. The Gauss Quadrature is extended to include triangular elements in terms of triangular area coordinates.

$$I = \iint_{A} \phi(L_1, L_2, L_3) dA \approx \sum_{i=1}^{n} w_i \phi(L_1^i, L_2^i, L_3^i) \quad (3.9.8)$$

Where, L terms are the triangular area coordinates and the w_i terms are the weights associated with those coordinates. The locations of integration points are shown in Fig. 3.9.2.

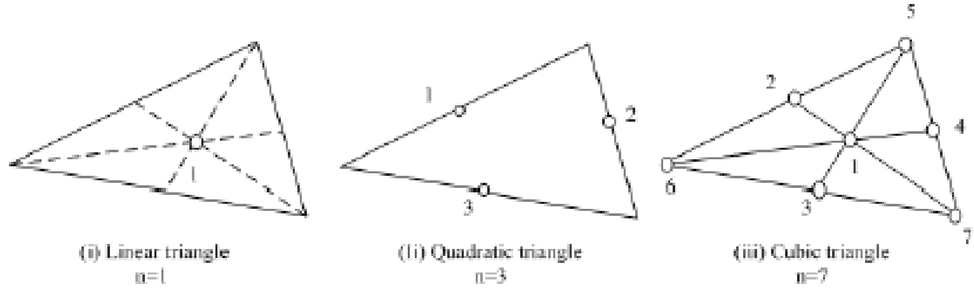

Fig. 3.9.2 Gauss points for triangles

The sampling points and their associated weights are described below:

For sampling point =1 (Linear triangle)

$$w_1 = 1 \qquad L_1^1 = L_2^1 = L_3^1 = \frac{1}{3} \tag{3.9.9}$$

For sampling points =3 (Quadratic triangle)

$$\begin{aligned} w_1 &= \frac{1}{3} && L_1^1 = L_2^1 = \frac{1}{2},\ L_3^1 = 0 \\ w_2 &= \frac{1}{3} && L_1^2 = 0,\ L_2^2 = L_3^2 = \frac{1}{2} \\ w_3 &= \frac{1}{3} && L_1^3 = \frac{1}{2},\ L_2^3 = 0,\ L_3^3 = \frac{1}{2} \end{aligned} \tag{3.9.10}$$

For sampling point = 7 (Cubic triangle)

$$\begin{aligned} w_1 &= \frac{27}{60} && L_1^1 = L_2^1 = L_3^1 = \frac{1}{3} \\ w_2 &= \frac{8}{60} && L_1^2 = L_2^2 = \frac{1}{2}, L_3^2 = 0 \\ w_3 &= \frac{8}{60} && L_1^3 = 0, L_2^3 = L_3^3 = \frac{1}{2} \\ w_4 &= \frac{8}{60} && L_1^4 = L_3^4 = \frac{1}{2}, L_2^4 = 0 \\ w_5 &= \frac{3}{60} && L_1^5 = 1, L_2^5 = L_3^5 = 0 \\ w_6 &= \frac{3}{60} && L_1^6 = L_3^6 = 0, L_2^6 = 1 \\ w_7 &= \frac{3}{60} && L_1^7 = L_2^7 = 0, L_3^7 = 1 \end{aligned} \tag{3.9.11}$$

Worked out Examples

Example 3.1 Calculation of displacement using area coordinates

The coordinates of a three node triangular element is given below. Calculate the displacement at point P if the displacements of nodes 1, 2 and 3 are 11 mm, 14mm and 17mm respectively using the concepts of area coordinates.

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 4 \\ 1 & 3 & 6 \end{vmatrix} = \frac{1}{2}[(30-12) - (12-9) + (8-15)] = \frac{8}{2} = 4$$

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 5 & 4 \\ 1 & 3 & 6 \end{vmatrix} = \frac{1}{2}[(30-12) - (18-12) + (12-20)] = \frac{4}{2} = 2$$

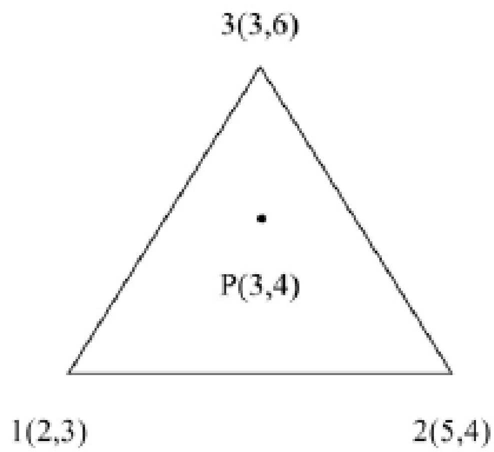

Fig.Ex.3.1 Nodal coordinates of a triangular element

$$A_2 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 6 \\ 1 & 2 & 3 \end{vmatrix} = \frac{1}{2}[(9-12) - (9-8) + (18-12)] = \frac{2}{2} = 1$$

$$A_3 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 5 & 4 \end{vmatrix} = \frac{1}{2}\,[(8\text{-}15) - (12\text{-}20) + (9\text{-}8)] = \frac{2}{2} = 1$$

$$N_1 = \frac{A_1}{A} = \frac{2}{4} = 0.5$$

$$N_2 = \frac{A_2}{A} = \frac{1}{4} = 0.25$$

$$N_3 = \frac{A_3}{A} = \frac{1}{4} = 0.25$$

u = $N_1 u_1 + N_2 u_2 + N_3 u_3$

= 0.5 x 11 + 0.25 x 14 + 0.25 x 17 = 13.25 mm

Example 3.2 Derivation of shape function of four node triangular element

Derive the shape function of a four node triangular element.

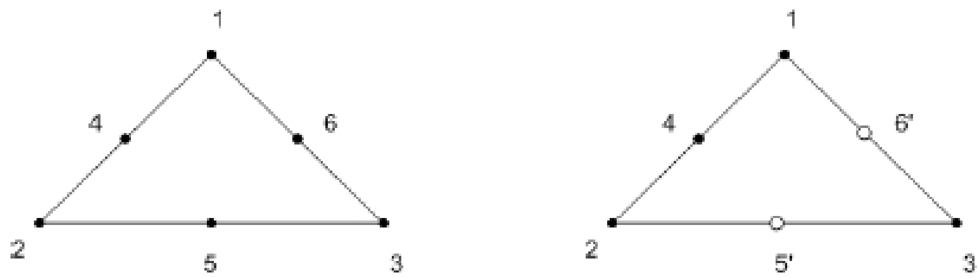

Fig.Ex.3.2Degrading for four node element

The procedure for four node triangular element is the same as five node triangular element to derive its interpolation functions. Here, node 5 and 6 are omitted and therefore displacements in these nodes can be expressed in terms of the displacements at their corner nodes. Hence,

$$u'_5 = \frac{u_2 + u_3}{2} \quad \text{and} \quad u'_6 = \frac{u_1 + u_3}{2} \tag{3.11.1}$$

Substituting the values of u'_5 and u'_6 in eq.(3.3.8), the following relations can be obtained.

$$\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 \frac{(u_2 + u_3)}{2} + N_6 \frac{(u_3 + u_1)}{2} \\ &= \left(N_1 + \frac{N_6}{2}\right) u_1 + \left(N_2 + \frac{N_5}{2}\right) u_2 + \left(N_3 + \frac{N_5 + N_6}{2}\right) u_3 + N_4 u_4 \end{aligned} \tag{3.11.2}$$

Now, the displacement at any point inside the four node element can be expressed by its nodal displacement with help of shape function.

$$u = N'_1 u_1 + N'_2 u_2 + N'_3 u_3 + N'_4 u_4 \tag{3.11.3}$$

Comparing eq. (3.11.2) and eq. (3.11.3), one can find the following relations.

$$\begin{aligned} N'_1 &= N_1 + \frac{N_6}{2} = L_1(2L_1 - 1) + \frac{4L_3 L_1}{2} = L_1(1 - 2L_2) \\ N'_2 &= N_2 + \frac{N_5}{2} = L_2(2L_2 - 1) + \frac{4L_2 L_3}{2} = L_2(1 - 2L_1) \\ N'_3 &= N_3 + \frac{N_5 + N_6}{2} = L_3(2L_3 - 1) + \frac{4L_2 L_3 + 4L_3 L_1}{2} = L_3 \\ N'_4 &= N_4 = 4L_1 L_2 \end{aligned} \tag{3.11.4}$$

Thus, the shape functions for the four node triangular element are

$$
\begin{aligned}
N_1' &= L_1(1-2L_2)\\
N_2' &= L_2(1-2L_1)\\
N_3' &= L_3\\
N_4' &= 4L_1L_2
\end{aligned}
\tag{3.11.5}
$$

Example 3.3 Numerical integration for two dimensional problems

Evaluate the integral: $I=\int_{-2}^{3}\left(x^2+11x-32\right)dx$ using one, two and three point gauss Quadrature. Also, find the exact solution for comparison of accuracy.

Solution:

The existing limits of integration should be changed from [-2, +3] to [-1, +1]. Assuming, $\xi = a+bx$, the upper and lower limit can be changed. i.e., at $x_1=-2, \xi_1=-1$ and at $x_2=3, \xi_2=+1$. Thus, putting these limits and solving for $a\&b$, we get a = -0.2 and b = 0.4. The relation between two coordinate systems will become:

$$\xi=-0.2+0.4\text{x} \text{ or } \text{x}=\frac{5\xi+1}{2} \text{ and } \text{dx}=2.5\text{d}\xi$$

Thus, the initial equation can be written as

$$I=\int_{-2}^{3}\left(x^2+11x-32\right)dx = 2.5\int_{-1}^{+1}\left[\left(\frac{5\xi+1}{2}\right)^2+11\left(\frac{5\xi+1}{2}\right)-32\right]d\xi$$

(i) <u>Exact Solution:</u>

$$I=\int_{-2}^{3}\left(x^2+11x-32\right)dx$$

$$=\left[\frac{x^3}{3}+\frac{11x^2}{2}-32x\right]_{-2}^{3}$$

$$=\left[9+\frac{99}{2}-96\right]-\left[-\frac{8}{3}+22+64\right]$$

$$=-37.5-83.33333=-120.83333$$

Thus, I_{exact} = -120.83333

(ii) <u>One Point Formula:</u>

$$I=\int_{-1}^{+1}\phi(\xi)d\xi=w_1\phi(\xi_1)$$

For one point formula in Gauss Quadrature integration, $w_1=2,\ \xi_1=0$. Thus,

$$I_1=2\times 2.5\left[\left(\frac{5\times 0+1}{2}\right)^2+11\left(\frac{5\times 0+1}{2}\right)-32\right]$$

$$=5\left[\frac{1}{4}+\frac{11}{2}-32\right]=-131.25$$

Thus, % of error = (120.83333-131.25)×100/120.83333 = 8.62%

(iii) <u>Two Point Formula:</u>

Here, for two point formula in Gauss Quadrature integration,

$\mathrm{w}_1=\mathrm{w}_2=1.0$ and $\xi_1=-\xi_2=-\dfrac{1}{\sqrt{3}}$. Thus,

$$I_2=w_1\phi(\xi_1)+w_2\phi(\xi_2)$$

$$1.0\times 2.5\times\left[\left(\frac{\frac{-5}{\sqrt{3}}+1}{2}\right)^2+11\left(\frac{\frac{-5}{\sqrt{3}}+1}{2}\right)-32\right]+1.0\times 2.5\times\left[\left(\frac{\frac{5}{\sqrt{3}}+1}{2}\right)^2+11\left(\frac{\frac{5}{\sqrt{3}}+1}{2}\right)-32\right]$$

$$=(0.88996-10.37713-32)\times 2.5+(3.77671+21.3771-32)\times 2.5$$

$$=-48.3333\times 2.5$$

$$=-120.83325$$

Thus, % of error = (120.83333-120.83325)×100/120.83333 = 6.62×10^{-05}

(iv) <u>Three Point Formula:</u>

Here, for three point formula in Gauss Quadrature integration,

$\mathrm{w}_1=0.8889,\qquad \xi_1=0.0$

$w_2=0.5556,\qquad \xi_2=+0.7746$

$w_3=0.5556,\qquad \xi_3=-0.7746$

Thus,

$$I_3=w_1\phi(\xi_1)+w_2\phi(\xi_2)+w_3\phi(\xi_3)$$

$$I_3 = 0.8889\times 2.5\times\left[\left(\frac{5\times 0+1}{2}\right)^2 + 11\times\frac{5\times 0+1}{2} - 32\right]$$
$$+0.5556\times 2.5\times\left[\left(\frac{5\times 0.7746+1}{2}\right)^2 + 11\times\frac{5\times 0.7746+1}{2} - 32\right]$$
$$+0.5556\times 2.5\times\left[\left(\frac{-5\times 0.7746+1}{2}\right)^2 + 11\times\frac{-5\times 0.7746+1}{2} - 32\right]$$

$$I_3 = 0.8889\times 2.5\times[0.25+5.5-32]$$
$$+0.5556\times 2.5\times[5.9365+26.8015-32]$$
$$+0.5556\times 2.5\times[2.0635-15.8015-32]$$
$$=2.5\times(-23.3336+0.4100-25.4120)$$
$$=-2.5\times 48.3356 = -120.839$$

Thus, % of error = (120.83333-120.839)×100/120.83333 = 4.69×10^{-03}. However, difference of results will approach to zero, if few more digits after decimal points are taken in calculation.

Example 3.4 Numerical integration for three dimensional problems

Evaluate the integral: $I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(1-2\xi)^2(1-\eta)^2(3\zeta-2)^2\, d\xi d\eta d\zeta$

Solution:

Using two point gauss Quadrature formula for the evaluation of three dimensional integration, we have the following sampling points and weights.

$w_1 = w_2 = 1$

$\xi_1 = -0.5773502692$

$\xi_2 = 0.5773502692$

$\eta_1 = -0.5773502692$

$\eta_2 = 0.5773502692$

$\zeta_1 = -0.5773502692$

$\zeta_2 = 0.5773502692$

Putting the above values, in $\phi(\xi,\eta,\zeta) = (1-2\xi)^2(1-\eta)^2(3\zeta-2)^2$ one can find the following values in 8 (i.e., 2 × 2 × 2) sampling points.

$\phi(\xi_1, \eta_1, \zeta_1) = 160.8886$

$\phi(\xi_1, \eta_1, \zeta_2) = 0.8293$

$\phi(\xi_1, \eta_2, \zeta_1) = 11.5513$

$\phi(\xi_1, \eta_2, \zeta_2) = 0.0595$

$\phi(\xi_2, \eta_1, \zeta_1) = 0.8293$

$\phi(\xi_2, \eta_1, \zeta_2) = 0.0043$

$\phi(\xi_2, \eta_2, \zeta_1) = 0.0595$

$\phi(\xi_2, \eta_2, \zeta_2) = 0.0003$

Now, $I = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_i w_j w_k \phi(\xi_i, \eta_j, \zeta_k)$

Thus, $I = w_1 w_1 w_1 \phi(\xi_1, \eta_1, \zeta_1) + w_1 w_1 w_2 \phi(\xi_1, \eta_1, \zeta_2) + \ldots + w_2 w_2 w_2 \phi(\xi_2, \eta_2, \zeta_2) =$ 174.222,where asI_{exact} = 174.222.

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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT – III –FINITE ELEMENT ANALYSIS – SME1308

UNIT – 3

SOLUTIONS TO PLANE ELASTICITY PROBLEMS

Lecture 3: Introduction to Elasticity

1.3.1 Stresses and Equilibrium

Let consider an infinitesimal element of sides dx, dy and dz as shown in Fig. 1.3.1. The stresses are acting on the elemental volume dV because of external and or body forces. These stresses can be represented by six independent components as given below.

$$\{\sigma\} = \left[\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{yz}, \quad \tau_{zx}\right]^T \tag{1.3.1}$$

Here, σ_x, σ_y *and* σ_z are normal stresses and τ_{xy}, τ_{yz} *and* τ_{zx} are shear stresses. Applying the conditions of static equilibrium for forces along the direction of X axis (i.e., $\sum F_x = 0$), following expression will be obtained.

$$\frac{\partial \sigma_x}{\partial x} dxdydz + \frac{\partial \tau_{yx}}{\partial y} dxdydz + \frac{\partial \tau_{zx}}{\partial z} dxdydz + F_{\Omega x} dxdydz = 0 \tag{1.3.2}$$

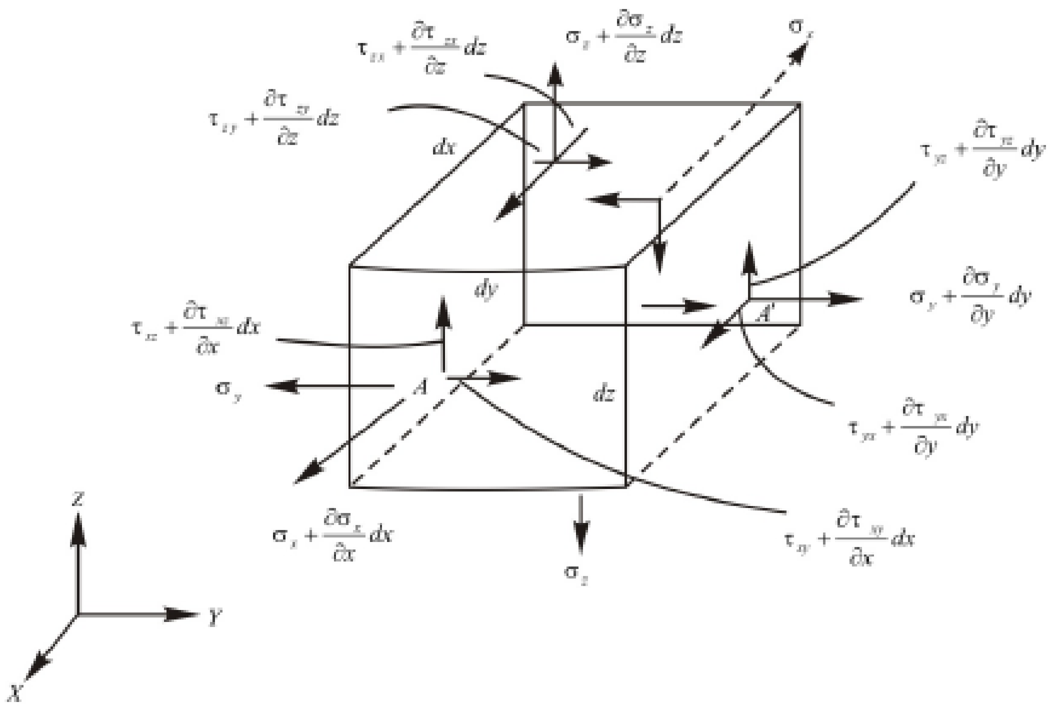


Fig. 1.3.1 Stresses on an infinitesimal element

Where, $F_{\Omega x}$ is the component of body force along x direction. Now, dividing dxdydz on the above expression, following equilibrium condition is obtained.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = -F_{\Omega x} \quad (1.3.3)$$

Similarly, applying equilibrium condition along Y and Z directions, one can find the following relations.

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = -F_{\Omega y} \quad (1.3.4)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = -F_{\Omega z} \quad (1.3.5)$$

Here, $F_{\Omega y}$ and $F_{\Omega z}$ are the component of body forces along Y and Zdirections respectively. Satisfying moment equations (i.e., $\sum M_x = 0; \sum M_y = 0 \; and \; \sum M_z = 0;$), one can obtain the following relations.

$$\tau_{xy} = \tau_{yx}; \quad \tau_{yz} = \tau_{zy} \text{ and } \tau_{xz} = \tau_{zx} \quad (1.3.6)$$

Using eq. (1.3.6), the equilibrium equations (1.3.3 to 1.3.5), can be rewritten in the following form.

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = -F_{\Omega x} \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = -F_{\Omega y} \\ &\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = -F_{\Omega z} \end{aligned} \quad (1.3.7)$$

Eq. (1.3.7) is known as equation of equilibrium.

Let assume an element of area $\Delta\Gamma$ on the surface of the solid in equilibrium (Fig.1.3.2) and $F_{\Gamma x}$, $F_{\Gamma y}$ and $F_{\Gamma z}$ are the components of external forces per unit area and are acting on the surface. Consideration of equilibrium along the three axes directions gives the following relations.

$$\begin{aligned} &\sigma_x l + \tau_{xy} m + \tau_{zx} n = F_{\Gamma x} \\ &\tau_{xy} l + \sigma_y m + \tau_{yz} n = F_{\Gamma y} \\ &\tau_{zx} l + \tau_{yz} m + \sigma_z n = F_{\Gamma z} \end{aligned} \quad (1.3.8)$$

Here, l, m and n are the direction cosines of the normal to the boundary surface. Eq. (1.3.8) is known as **static boundary condition**.

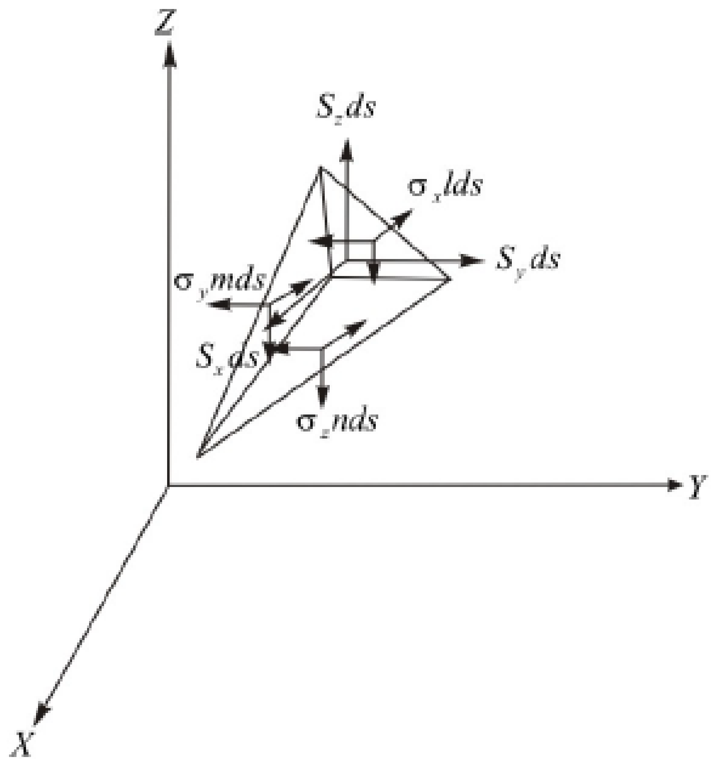


Fig. 1.3.2 Forces acting on an element on the boundary

1.3.2 Strain-Displacement Relations

The displacement at any point of a deformable body may be expressed by the components of *u*, *v* and *w* parallel to the Cartesian coordinate's axes. The components of the displacements can be described as functions of *x*, *y* and *z*. Displacementsbasically the change of position during deformation. If point P (x,y,z) is displaced to P' (x',y',z'), then the displacement along X, Y and Z direction (Fig. 1.3.3) will become

$$x' = x + u \text{ or } u = x' - x$$
$$y' = y + v \text{ or } v = y' - y$$
$$z' = z + w \text{ or } w = z' - z$$

Therefore, the normal strain can be written as:

$$\varepsilon_x = \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \text{(As } \varepsilon = \frac{\Delta L}{L} \text{ for uniform strain in axial member)}$$

Similarly, $\varepsilon_y = \frac{\partial v}{\partial y}$ and $\varepsilon_z = \frac{\partial w}{\partial z}$

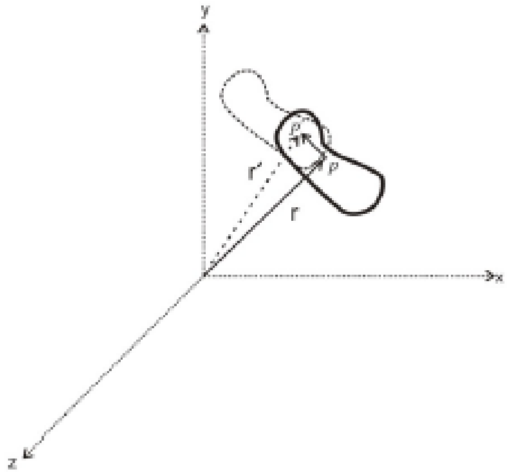


Fig. 1.3.3 Deformation of an elastic body

Let consider points P,Q and R are before deformation and points P',Q' and R' are after deformation as shown in Fig. 1.3.4 below. Now for small deformation, rotation of PQ will become

$$\theta_1 = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x}$$

Similarly, rotation of PR due to deformation will be: $\theta_2 = \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta y} = \frac{\partial u}{\partial y}$

Thus, the total change of angle between PQ and PR after deformation is as follows which is defined as shear strain in X-Y plane.

$$\gamma_{xy} = \gamma_{yx} = \theta_1 + \theta_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

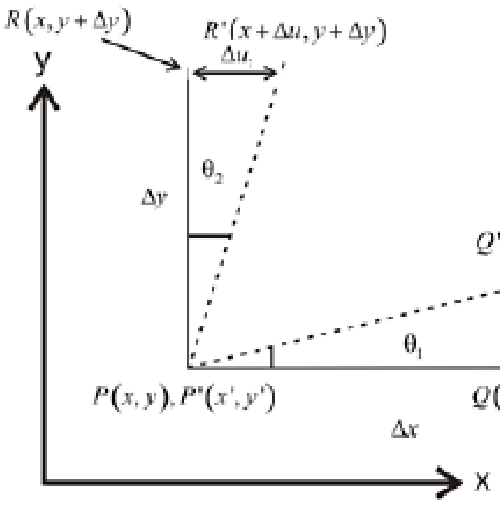


Fig. 1.3.4 Derivation of shear strain

Similarly, shear strains in Y-Z and X-Z plane will become

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}; \ \gamma_{xz} = \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

The strain can be expressed as partial derivatives of the displacements *u*, *v* and *w*. The above expressions for strain-displacement relationship are true only for small amplitude of deformation. However, the strain-displacement relations are expressed by the following equations for large magnitude of deformation.

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{1.3.9}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{1.3.10}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \tag{1.3.11}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \tag{1.3.12}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right] \tag{1.3.13}$$

$$\gamma_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}+\left[\frac{\partial u}{\partial z}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right] \tag{1.3.14}$$

The eqs.(1.3.9 to 1.3.14) are known as Green-Lagrange strain displacement equation. The components of the strain ε_x, ε_y, ε_z, γ_{xy}, γ_{yz} and γ_{zx} define the state of strains in the deformed body, and can be written in a matrix form as

$$\{\varepsilon\}^T=\begin{bmatrix}\varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx}\end{bmatrix} \tag{1.3.15}$$

The relations given in eqs.(1.3.9 to 1.3.14) are non-linear partial differential equations in the unknown component of the displacements. In case of small deformations, the products and squares of the first derivatives are assumed to be negligible compared with the derivatives themselves in many problems of stress analysis. Thus the strain-displacement relations in eqs. (1.3.9 to 1.3.14) reduce to linear relations as follows.

$$\begin{aligned}
\varepsilon_x&=\frac{\partial u}{\partial x}\\
\varepsilon_y&=\frac{\partial v}{\partial y}\\
\varepsilon_z&=\frac{\partial w}{\partial z}\\
\gamma_{xy}&=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\\
\gamma_{yz}&=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\\
\gamma_{zx}&=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}
\end{aligned} \tag{1.3.16}$$

Eq.(1.3.16)is known as Von-Karman strain displacement equation.The above equation can be expressed in a matrix form as given below.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{1.3.17}$$

The above assumption will be incorrect in case of large deformation problems. In these cases, geometric nonlinearity has to be considered.

1.3.3 Linear Constitutive Relations

Hooke's law states that the six component of stress may be described as linear function of six components of strain. The relation for a linear elastic, anisotropic and homogeneous material are expressed as follows.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & . & . & . & C_{16} \\ C_{21} & C_{22} & . & . & . & C_{26} \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ C_{61} & C_{62} & . & . & . & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{1.3.18}$$

or $$\{\sigma\} = [C]\{\varepsilon\} \tag{1.3.19}$$

Where [C] is constitutive matrix. If the material has three orthogonal planes of symmetry, it is said to be orthotropic. In this case only nine constants are required for describing constitutive relations as given below.

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Symmetry & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{1.3.20}
$$

The inverse relation for strains and stresses may be expressed as

$$
[\varepsilon] = [C^{-1}]\{\sigma\} = [D]\{\sigma\} \tag{1.3.21}
$$

An isotropic is one for which every plane is a plane of symmetry of material behavior and only two constants (Young Modulus, E and Poisson ratioμ) are required to describe the constitutive relation. The following equation includes the effect due to temperature changes as may be necessary in certain cases of stress analysis.

or

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ & 1 & -\mu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\mu) & 0 & 0 \\ & & Symmetry & & 2(1+\mu) & 0 \\ & & & & & 2(1+\mu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} + \alpha T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.3.22}
$$

T and α in eq.(1.3.22) denote the difference of temperature and coefficient of thermal expansion respectively.

The inverse relation of stresses in terms of strain components can be expressed as

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \overline{E} \begin{bmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ & (1-\mu) & \mu & 0 & 0 & 0 \\ & & (1-\mu) & 0 & 0 & 0 \\ & & & \frac{1-2\mu}{2} & 0 & 0 \\ & Symmetry & & & \frac{1-2\mu}{2} & 0 \\ & & & & & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} - \frac{E\alpha T}{1-2\mu} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.3.23}
$$

where $\overline{E} = \dfrac{E}{(1+\mu)(1-2\mu)}$

1.3.4 Two-Dimensional Stress Distribution

The problems of solid mechanics may be formulated as three-dimensional problems and finite element technique may be used to solve them. In many practical situations, the geometry and loading will be such that the problems may be formulated to two-dimensional or one-dimensional problems without much loss of accuracy. The relation between strain and displacement for two dimensional problems can be simplified from eq. (1.3.16) and can be written as follows.

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned} \tag{1.3.24}$$

The above expression can be written in a combined form:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{1.3.25}$$

Eq. (1.3.25) is the compatibility equation since it states the geometric requirements. This condition will ensure adjacent elements to remain free from discontinuities such as gaps and overlaps.

1.3.4.1 Plane stress problem

The plane stress problem is characterized by very small dimensions in one of the normal directions. Some typical examples are shown in Fig. 1.3.5. In these cases, it is assumed that no stress component varies across the thickness and the stress components σ_z, τ_{xz} and τ_{yz} are zero. The state of stress is specified by σ_x, σ_y and τ_{xy} only and is called plane stress.

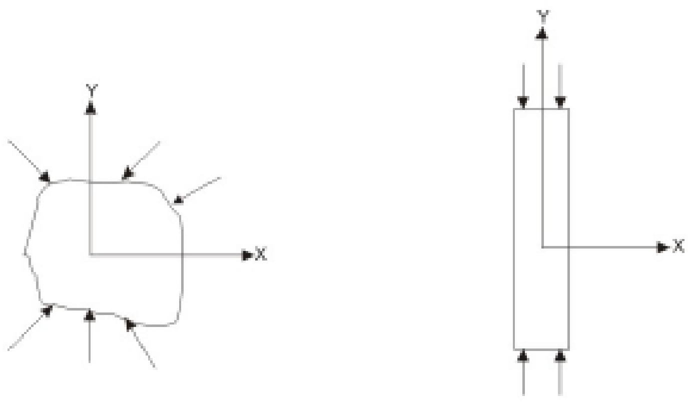


Fig. 1.3.5 Plane stress example: Thin plate with in-plane loading

The stress components may be expressed in terms of strain, which is as follows.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \frac{E\alpha T}{1-\mu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{1.3.26}$$

The strain components can also be expressed in terms of the stress, which is given below.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \alpha T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{1.3.27}$$

It can also be shown that

$$\varepsilon_z = \frac{-\mu}{1-\mu}\left(\varepsilon_x + \varepsilon_y\right) + \frac{1+\mu}{1-\mu}\alpha T \text{ and } \quad \gamma_{yz} = \gamma_{zx} = 0 \tag{1.3.28}$$

1.3.4.2 Plane strain problem

Problems involving long bodies whose geometry and loading do not vary significantly in the longitudinal direction are referred to as plane strain problems. Some typical examples are given in Fig. 1.3.6. In these cases, a constant longitudinal displacement corresponding to a rigid body translation and displacements linear in z corresponding to rigid body rotation do not result in strain. As a result, the following relations arise.

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \tag{1.3.29}$$

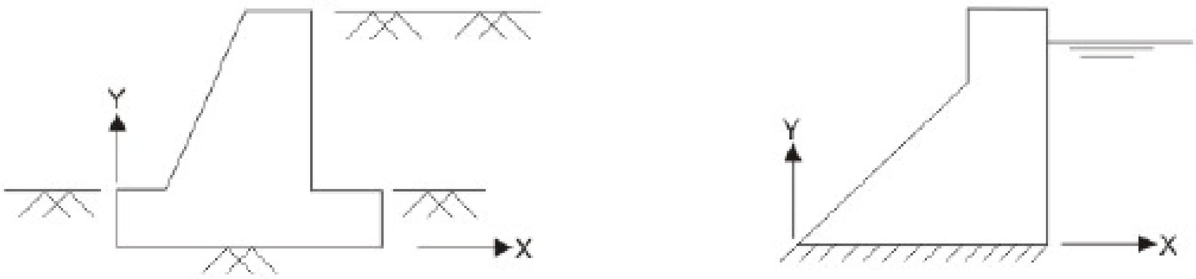
The constitutive relation for elastic isotropic material for this case may be given by,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \frac{E\alpha T}{1-2\mu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{1.3.30}$$

$$\text{Also} \quad \sigma_z = \mu(\sigma_x + \sigma_y) - E\alpha T \text{ and } \tau_{yz} = \tau_{zx} = 0 \tag{1.3.31}$$

The strain components can be expressed in terms of the stress as follows.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{(1+\mu)}{E} \begin{bmatrix} (1-\mu) & -\mu & 0 \\ -\mu & (1-\mu) & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + (1+\mu)\alpha T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{1.3.32}$$



(a) Retaining wall **(b) Dam**

Fig. 1.3.6 Plane strain examples

1.3.4.3Axisymmetric Problem

Many problems in stress analysis which are of practical interest involve solids of revolution subject to axially symmetric loading. A circular cylinder loaded by a uniform internal or external pressure, circular footing resting on soil mass, pressure vessels, rotating wheels, flywheels etc. The strain-displacement relations in these type of problems are given by

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_\theta &= \frac{u}{x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \tag{1.3.33}$$

The two components of displacements in any plane section of the body along its axis of symmetry define completely the state of strain and therefore the state of stress. The constitutive relations are given below for such types of problems.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_\theta \\ \gamma_{xy} \end{Bmatrix} \tag{1.3.34}$$

3.5. CST:

The triangular elements with different numbers of nodes are used for solving two dimensional solid members. The linear triangular element was the first type of element developed for the finite element analysis of 2D solids. However, it is observed that the linear triangular element is less accurate compared to linear quadrilateral elements. But the triangular element is still a very useful element for its adaptivity to complex geometry. These are used if the geometry of the 2D model is complex in nature. Constant strain triangle (CST) is the simplest element to develop mathematically. In CST, strain inside the element has no variation (Ref. module 3, lecture 2) and hence element size should be small enough to obtain accurate results. As indicated earlier, the displacement is expressed in two orthogonal directions in case of 2D solid elements. Thus the displacement field can be written as

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{5.1.1}$$

Here, u and v are the displacements parallel to x and y directions respectively.

5.1.1 Element Stiffness Matrix for CST

A typical triangular element assumed to represent a subdomain of a plane body under plane stress/strain condition is represented in Fig. 5.1.1. The displacement (u, v) of any point P is represented in terms of nodal displacements

$$\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 \\ v &= N_1 v_1 + N_2 v_2 + N_3 v_3 \end{aligned} \tag{5.1.2}$$

Where, N_1, N_2, N_3 are the shape functions as described in module 3, lecture 2.

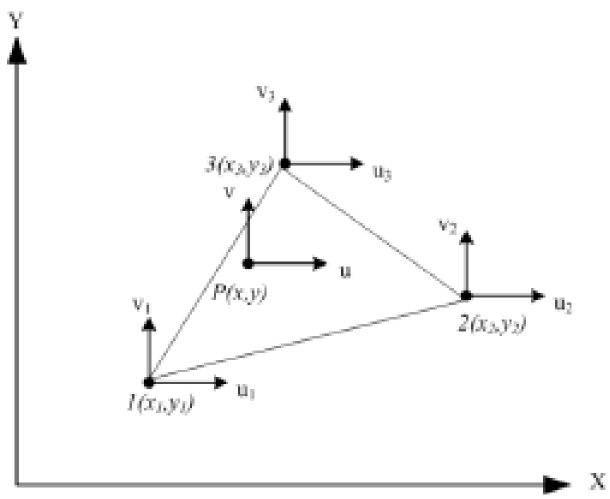


Fig. 5.1.1 Linear triangular element for plane stress/strain

The strain-displacement relationship for two dimensional plane stress/strain problem can be simplified in the following form from three dimensional cases (eq.1.3.9 to1.3.14).

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]
\end{aligned} \tag{5.1.3}$$

In case of small amplitude of displacement, one can ignore the nonlinear term of the above equation and will reach the following expression.

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}
\end{aligned} \tag{5.1.4}$$

Hence the element strain components can be represented as,

$$\varepsilon = \begin{cases}
\varepsilon_x = \dfrac{\partial u}{\partial x} = \dfrac{\partial N_1}{\partial x}u_1 + \dfrac{\partial N_2}{\partial x}u_2 + \dfrac{\partial N_3}{\partial x}u_3 \\
\varepsilon_y = \dfrac{\partial v}{\partial y} = \dfrac{\partial N_1}{\partial y}v_1 + \dfrac{\partial N_2}{\partial y}v_2 + \dfrac{\partial N_3}{\partial y}v_3 \\
\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = \dfrac{\partial N_1}{\partial y}u_1 + \dfrac{\partial N_2}{\partial y}u_2 + \dfrac{\partial N_3}{\partial y}u_3 + \dfrac{\partial N_1}{\partial x}v_1 + \dfrac{\partial N_2}{\partial x}v_2 + \dfrac{\partial N_3}{\partial x}v_3
\end{cases}$$

Or,

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix}
\dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \\
\dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x}
\end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{5.1.5}$$

Or, $\varepsilon = [B]\{d\}$ (5.1.6)

In the above equation [B] is called as strain displacement relationship matrix. The shape functions for the 3 node triangular element in Cartesian coordinate is represented as,

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right] \\ \dfrac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right] \\ \dfrac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right] \end{Bmatrix}$$

Or,

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{2A}\left[\alpha + \beta_1 x + \gamma_1 y\right] \\ \dfrac{1}{2A}\left[\alpha_2 + \beta_2 x + \gamma_2 y\right] \\ \dfrac{1}{2A}\left[\alpha_3 + \beta_3 x + \gamma_3 y\right] \end{Bmatrix} \tag{5.1.7}$$

Where,

$$\begin{matrix} \alpha_1 = (x_2y_3 - x_3y_2), & \alpha_2 = (x_3y_1 - x_1y_3), & \alpha_3 = (x_1y_2 - x_2y_1), \\ \beta_1 = (y_2 - y_3), & \beta_2 = (y_3 - y_1), & \beta_3 = (y_1 - y_2), \\ \gamma_1 = (x_3 - x_2), & \gamma_2 = (x_2 - x_1), & \gamma_3 = (x_2 - x_1), \end{matrix} \tag{5.1.8}$$

Hence the required partial derivatives of shape functions are,

$$\begin{matrix} \dfrac{\partial N_1}{\partial x} = \dfrac{\beta_1}{2A}, & \dfrac{\partial N_2}{\partial x} = \dfrac{\beta_2}{2A}, & \dfrac{\partial N_3}{\partial x} = \dfrac{\beta_3}{2A}, \\ \dfrac{\partial N_1}{\partial y} = \dfrac{\gamma_1}{2A}, & \dfrac{\partial N_2}{\partial x} = \dfrac{\gamma_2}{2A}, & \dfrac{\partial N_3}{\partial x} = \dfrac{\gamma_3}{2A}, \end{matrix}$$

Hence the value of [B] becomes:

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \end{bmatrix}$$

Or, $$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \tag{5.1.9}$$

According to Variational principle described in module 2, lecture 1, the stiffness matrix is represented as,

$$[k]=\iiint_{\Omega}[B]^{T}[D][B]\,d\Omega \tag{5.1.10}$$

Since, [B] and [D] are constant matrices; the above expression can be expressed as

$$[k]=[B]^{T}[D][B]\iiint_{V}dV=[B]^{T}[D][B]V \tag{5.1.11}$$

For a constant thickness (t), the volume of the element will become $A.t$. Hence the above equation becomes,

$$[k]=[B]^{T}[D][B]At \tag{5.1.12}$$

For plane stress condition, [D] matrix will become:

$$[D]=\frac{E}{1-\mu^{2}}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \frac{1-\mu}{2}\end{bmatrix} \tag{5.1.13}$$

Therefore, for a plane stress problem, the element stiffness matrix becomes,

$$[k]=\frac{Et}{4A\left(1-\mu^{2}\right)}\begin{bmatrix}\beta_1 & 0 & \gamma_1\\ \beta_2 & 0 & \gamma_2\\ \beta_3 & 0 & \gamma_3\\ 0 & \gamma_1 & \beta_1\\ 0 & \gamma_2 & \beta_2\\ 0 & \gamma_3 & \beta_3\end{bmatrix}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \frac{1-\mu}{2}\end{bmatrix}\begin{bmatrix}\beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3\\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3\end{bmatrix} \tag{5.1.14}$$

Or,

$$[k]=\frac{Et}{4A\left(1-\mu^{2}\right)}\begin{bmatrix}\beta_1^2+C\gamma_1^2 & \beta_1\beta_2+C\gamma_1\gamma_2 & \beta_1\beta_3+C\gamma_1\gamma_3 & \frac{(1+\mu)}{2}\beta_1\gamma_1 & \mu\beta_1\gamma_2+C\beta_2\gamma_1 & \mu\beta_1\gamma_3+C\beta_3\gamma_1\\ & \beta_2^2+C\gamma_2^2 & \beta_2\beta_3+C\gamma_2\gamma_3 & \mu\beta_2\gamma_1+C\beta_1\gamma_2 & \frac{(1+\mu)}{2}\beta_2\gamma_2 & \mu\beta_2\gamma_3+C\beta_3\gamma_2\\ & & \beta_3^2+C\gamma_3^2 & \mu\beta_3\gamma_1+C\beta_1\gamma_3 & \mu\beta_3\gamma_2+C\beta_2\gamma_3 & \frac{(1+\mu)}{2}\beta_3\gamma_3\\ & & & \gamma_1^2+C\beta_1^2 & \gamma_1\gamma_2+C\beta_1\beta_2 & \gamma_1\gamma_3+C\beta_1\beta_3\\ & Sym. & & & \gamma_2^2+C\beta_2^2 & \gamma_2\gamma_3+C\beta_2\beta_3\\ & & & & & \gamma_3^2+C\beta_3^2\end{bmatrix} \tag{5.1.15}$$

Where, $C=\dfrac{(1-\mu)}{2}$

Similarly for plane strain condition, [D] matrix is equal to,

$$[D]=\frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix}(1-\mu) & \mu & 0\\ \mu & (1-\mu) & 0\\ 0 & 0 & \frac{1-2\mu}{2}\end{bmatrix} \tag{5.1.16}$$

Hence the element stiffness matrix will become:

$$[k]=\frac{Et}{2A(1+\mu)}\begin{bmatrix} M\beta_1^2+\gamma_1^2 & M\beta_1\beta_2+\gamma_1\gamma_2 & M\beta_1\beta_3+\gamma_1\gamma_3 & (\mu+1)\beta_1\gamma_1 & \mu\beta_1\gamma_2+\beta_2\gamma_1 & \mu\beta_1\gamma_3+\beta_3\gamma_1\\ & M\beta_2^2+\gamma_2^2 & M\beta_2\beta_3+\gamma_2\gamma_3 & \mu\beta_2\gamma_1+\beta_1\gamma_2 & (\mu+1)\beta_2\gamma_2 & \mu\beta_2\gamma_3+\beta_3\gamma_2\\ & & M\beta_3^2+\gamma_3^2 & \mu\beta_3\gamma_1+\beta_1\gamma_3 & \mu\beta_3\gamma_2+C\beta_2\gamma_3 & (\mu+1)\beta_3\gamma_3\\ & & & M\gamma_1^2+\beta_1^2 & M\gamma_1\gamma_2+\beta_1\beta_2 & M\gamma_1\gamma_3+\beta_1\beta_3\\ & \mathit{Sym.} & & & M\gamma_2^2+\beta_2^2 & M\gamma_2\gamma_3+\beta_2\beta_3\\ & & & & & M\gamma_3^2+\beta_3^2\end{bmatrix} \tag{5.1.17}$$

Where $M=(1-\mu)$

5.1.2 Nodal Load Vector for CST

From the principle of virtual work,

$$\int_\Omega \delta\{\varepsilon\}^T\{\sigma\}d\Omega=\int_\Gamma \delta\{u\}^T\{F_\Gamma\}d\Gamma+\int_\Omega \delta\{u\}^T\{F_\Omega\}d\Omega \tag{5.1.18}$$

Where, F_Γ, and F_Ω are the surface and body forces respectively. Using the relationship between stress-stain and strain displacement, one can derive the following expressions:

$$\{\sigma\}=[D][B]\{d\},\quad \delta\{\varepsilon\}=[B]\delta\{d\}\quad \text{and}\quad \delta\{u\}=[N]\delta\{d\} \tag{5.1.19}$$

Hence eq. (5.1.18) can be rewritten as,

$$\int_\Omega \delta\{d\}^T[B]^T[D][B]\{d\}d\Omega=\int_\Gamma \delta\{d\}^T\left[N^S\right]^T\{F_\Gamma\}d\Gamma+\int_\Omega \delta\{d\}^T[N]^T\{F_\Omega\}d\Omega \tag{5.1.20}$$

Or,

$$\int_\Omega [B]^T[D][B]\{d\}d\Omega=\int_\Gamma \left[N^S\right]^T\{F_\Gamma\}d\Gamma+\int_\Omega [N]^T\{F_\Omega\}d\Omega \tag{5.1.21}$$

Here, $[N^s]$ is the shape function along the boundary where forces are prescribed. Eq.(5.1.21) is equivalent to $[k]\{d\}=\{F\}$, and thus, the nodal load vector becomes

$$\{F\}=\int_\Gamma \left[N^s\right]^T\{F_\Gamma\}d\Gamma+\int_\Omega [N]^T\{F_\Omega\}d\Omega \tag{5.1.22}$$

For a constant thickness of the triangular element eq.(5.1.22) can be rewritten as

$$\{F\}=t\int_S \left[N^s\right]^T\{F_\Gamma\}ds+t\int_A [N]^T\{F_\Omega\}dA \tag{5.1.23}$$

For the a three node triangular two dimensional element, one can represent F_Ω and F_Γ as,

$$\{F_\Omega\} = \begin{Bmatrix} F_{\Omega x} \\ F_{\Omega y} \end{Bmatrix} \text{ and } \{F_\Gamma\} = \begin{Bmatrix} F_{\Gamma x} \\ F_{\Gamma y} \end{Bmatrix}$$

For example, in case of gravity load on CST element, $\{F_\Omega\} = \begin{Bmatrix} F_{\Omega x} \\ F_{\Omega y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix}$

For this case, the shape functions in terms of area coordinates are:

$$[N] = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix} \tag{5.1.24}$$

As a result, the force vector on the element considering only gravity load, will become,

$$\{F\} = t\int_A \begin{bmatrix} L_1 & 0 \\ L_2 & 0 \\ L_3 & 0 \\ 0 & L_1 \\ 0 & L_2 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix} dA = t\int_A \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -L_1\rho g \\ -L_2\rho g \\ -L_3\rho g \end{Bmatrix} dA = -\rho g t\int_A \begin{Bmatrix} 0 \\ 0 \\ 0 \\ L_1 \\ L_2 \\ L_3 \end{Bmatrix} dA \tag{5.1.25}$$

The integration in terms of area coordinate is given by,

$$\int_A L_1^p L_2^q L_3^r dA = \frac{p!q!r!}{(p+q+r+2)!} 2A \tag{5.1.26}$$

Thus, the nodal load vector will finally become

$$\{F\} = -\rho g t \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{1!0!0!}{(1+0+0+2)!} 2A \\ \frac{0!1!0!}{(0+1+0+2)!} 2A \\ \frac{0!0!1!}{(0+0+1+2)!} 2A \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\frac{\rho g A t}{3} \\ -\frac{\rho g A t}{3} \\ -\frac{\rho g A t}{3} \end{Bmatrix} = -\frac{\rho g A t}{3} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{5.1.27}$$

5.2.1 Element Stiffness Matrix for LST

In case of CST, it is observed that the strain within the element remains constant. Though, these elements are able to provide enough information about displacement pattern of the element, but it is unable to provide adequate information about stress inside an element. This limitation will be significant enough in regions of high strain gradients. The use of a higher order triangular element called Linear Strain Triangle (LST) significantly improves the results at these areas as the strin inside the element is varying. The LST element has six nodes (Fig. 5.2.1) and hence, twelve degrees of freedom. Thus the displacement function can be chosen as follows.

$$\begin{aligned} u &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 \\ v &= a_6 + a_7 x + a_8 y + a_9 x^2 + a_{10} xy + a_{11} y^2 \end{aligned} \qquad (5.2.1)$$

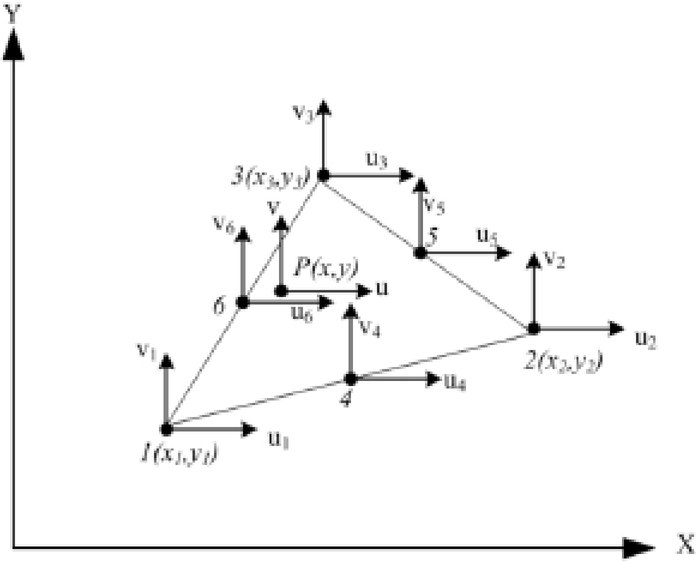


Fig. 5.2.1 Linear strain triangle element

Therefore, the element strain matrix is obtained as

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = \alpha_1 + 2\alpha_3 x + \alpha_4 y \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \alpha_8 + \alpha_{10} x + 2\alpha_{11} y \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = (\alpha_2 + \alpha_7) + (\alpha_4 + 2\alpha_9)x + (2\alpha_5 + \alpha_{10})y \end{aligned} \qquad (5.2.2)$$

In the area coordinate system as discussed in module 3, lecture 3 we can write the shape function for the six node triangular element as

$$\begin{array}{lll} N_1 = L_1(2L_1 - 1) & N_2 = L_2(2L_2 - 1) & N_3 = L_3(2L_3 - 1) \\ N_4 = 4L_1L_2 & N_5 = 4L_2L_3 & N_6 = 4L_3L_1 \end{array} \tag{5.2.3}$$

The displacement *(u,v)* of any point within the element can be represented in terms of their nodal displacements with the use of interpolation function.

$$\begin{aligned} u &= \sum_{i=1}^{6} N_i u_i \\ v &= \sum_{i=1}^{6} N_i v_i \end{aligned} \tag{5.2.4}$$

Using eq.(5.2.4) we can rewrite eq.(5.2.2) as,

$$\varepsilon = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

Or,

$$\varepsilon = [B]\{d\} \tag{5.2.5}$$

Where,

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \end{bmatrix} \tag{5.2.6}$$

Using Chain rule,

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial L_1}\cdot\frac{\partial L_1}{\partial x} + \frac{\partial N_1}{\partial L_2}\cdot\frac{\partial L_2}{\partial x} + \frac{\partial N_1}{\partial L_3}\cdot\frac{\partial L_3}{\partial x}$$

As discussed in module 3, lecture 1, we can write the above expression as,

$$\frac{\partial N_1}{\partial x} = \frac{b_1}{2A}.\frac{\partial N_1}{\partial L_1} + \frac{b_2}{2A}.\frac{\partial N_1}{\partial L_2} + \frac{b_3}{2A}.\frac{\partial N_1}{\partial L_3}$$

$$\frac{\partial N_1}{\partial x} = \frac{b_1}{2A}.(4L_1 - 1)$$

Similarly we can evaluate expressions for other terms and can be written as,

$$\frac{\partial N_1}{\partial x} = \frac{b_1}{2A}.(4L_1 - 1) \qquad \frac{\partial N_2}{\partial x} = \frac{b_2}{2A}.(4L_2 - 1) \qquad \frac{\partial N_3}{\partial x} = \frac{b_3}{2A}.(4L_3 - 1)$$

$$\frac{\partial N_4}{\partial x} = 4\left(L_2 b_1 + L_1 b_2\right) \qquad \frac{\partial N_5}{\partial x} = 4\left(L_3 b_2 + L_2 b_3\right) \qquad \frac{\partial N_6}{\partial x} = 4\left(L_1 b_3 + L_3 b_1\right)$$

And,

$$\frac{\partial N_1}{\partial y} = \frac{a_1}{2A}.(4L_1 - 1) \qquad \frac{\partial N_2}{\partial y} = \frac{a_2}{2A}.(4L_2 - 1) \qquad \frac{\partial N_3}{\partial y} = \frac{a_3}{2A}.(4L_3 - 1)$$

$$\frac{\partial N_4}{\partial y} = 4\left(L_2 a_1 + L_1 a_2\right) \qquad \frac{\partial N_5}{\partial y} = 4\left(L_3 a_2 + L_2 a_3\right) \qquad \frac{\partial N_6}{\partial y} = 4\left(L_1 a_3 + L_3 a_1\right)$$

Where,

$$a_1 = x_2 - x_3 \qquad a_2 = x_3 - x_1 \qquad a_3 = x_1 - x_2$$

$$b_1 = y_2 - y_3 \qquad b_2 = y_3 - y_1 \qquad b_3 = y_1 - y_2$$

The stiffness matrix of the element is represented by,

$$[k] = \iiint_{\Omega} [B]^T [D][B] \, d\Omega \tag{5.2.7}$$

The, [D] matrix is the constitutive matrix which will be taken according to plane stress or plane strain condition. The nodal strain and stress vectors are given by,

$$\{\varepsilon_n\} = \{\varepsilon_{x1} \quad \varepsilon_{x2} \quad \varepsilon_{x3} \quad \varepsilon_{y1} \quad \varepsilon_{y2} \quad \varepsilon_{y3} \quad \gamma_{xy1} \quad \gamma_{xy2} \quad \gamma_{xy3}\}^T \tag{5.2.8}$$

$$\{\sigma_n\} = \{\sigma_{x1} \quad \sigma_{x2} \quad \sigma_{x3} \quad \sigma_{y1} \quad \sigma_{y2} \quad \sigma_{y3} \quad \tau_{xy1} \quad \tau_{xy2} \quad \tau_{xy3}\}^T \tag{5.2.9}$$

$$\{\varepsilon_n\} = \begin{bmatrix} [B_{n1}] & [0] \\ [0] & [B_{n2}] \\ [B_{n2}] & [B_{n1}] \end{bmatrix} \{d\} \tag{5.2.10}$$

Referring to section 3.3.1, using proper values of area coordinates in [B] matrix, one can find

$$[B_{n1}] = \frac{1}{2A} \begin{bmatrix} 3b_1 & -b_2 & -b_3 & 4b_2 & 0 & 4b_3 \\ -b_1 & 3b_2 & -b_3 & 4b_1 & 4b_3 & 0 \\ -b_1 & -b & 3b_3 & 0 & 4b_2 & 4b_1 \end{bmatrix} \tag{5.2.11a}$$

And,

$$[B_{n2}]=\frac{1}{2A}\begin{bmatrix} 3a_1 & -a_2 & -a_3 & 4a_2 & 0 & 4a_3 \\ -a_1 & 3a_2 & -a_3 & 4a_1 & 4a_3 & 0 \\ -a_1 & -a & 3a_3 & 0 & 4a_2 & 4a_1 \end{bmatrix} \tag{5.2.11b}$$

Thus, the element stiffness can be evaluated by putting the values from eq. (5.2.11) in eq. (5.2.7).

5.2.2 Nodal Load Vector for LST

Similar to 3-node triangular element, the load will be lumped at each node which can be computed using the earlier expression,

$$\{F\}=\int_{\Gamma}[N^s]^T\{F_r\}d\Gamma+\int_{\Omega}[N]^T\{F_\Omega\}d\Omega \tag{5.2.12}$$

And for element with constant thickness,

$$\{F\}=t\int_{s}[N^s]^T\{F_r\}ds+t\int_{A}[N]^T\{F_\Omega\}dA \tag{5.2.13}$$

5.2.3 Numerical Example using CST

Determine the displacements at the nodes for the following 2D solid continuum considering a constant thickness of 25 mm, Poisson's ratio, μ as 0.25 and modulus of elasticity E as 2 x 10^5 N/mm^2. The continuum is discritized with two CST plane stress elements.

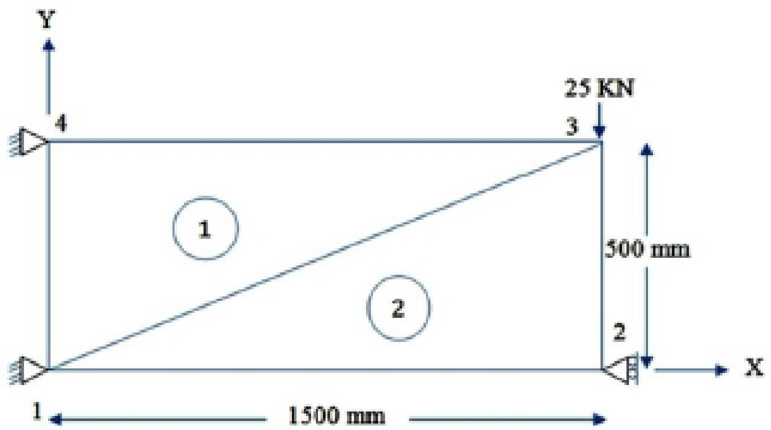


Fig. 5.2.2 Geometry and discretization of the continuum

The element 1 is connected with node 1, 3 and 4 and let assume its Cartesian coordinates are (x_1, y_1), (x_3, y_3) and (x_4, y_4) respectively. If we consider nodes 1, 3 and 4 are similar to node 1, 2 and 3 in eq.(5.1.9) then the $[B]$ can be written as

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

By introducing values of β & γ discussed in previous lecture note, we can get value of $[B]$ as

$$[B] = \frac{1}{1500}\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ -3 & 0 & 3 & 0 & 1 & -1 \end{bmatrix}$$

For plain stress problem, putting the values of E and μ one can find the following values.

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = \frac{4\times10^4}{3}\begin{bmatrix} 16 & 4 & 0 \\ 4 & 16 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Therefore the stiffness matrix for the element 1 will be

$$[k]_1 = tA[B]^T[D][B]$$

Putting values of *t, A, [B] & [D]* we will get,

$$[k]_1 = 4\times10^3 \times \begin{bmatrix} 750 & 0 & -750 & 0 & -250 & 250 \\ 0 & 222.2222 & -222.2222 & -166.6667 & 0 & 166.6667 \\ -750 & -222.2222 & 972.2222 & 166.6667 & 250 & -416.6667 \\ 0 & -166.6667 & 166.6667 & 2000 & 0 & -2000 \\ -250 & 0 & 250 & 0 & 83.3333 & -83.3333 \\ 250 & 166.6667 & -416.6667 & -2000 & -83.3333 & 2083.3333 \end{bmatrix}$$

Similarly element 2 is connected with nodes 1, 2 and 3 and global coordinates of these nodes are (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. For this element, by proceeding in a similar manner to element 1 we can calculate $[B]$ matrix as,

$$[B] = \frac{1}{1500}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & -3 & 3 & -1 & 1 & 0 \end{bmatrix}$$

Hence, the elemental stiffness matrix becomes,

$$[k]_2 = 4\times10^3 \times \begin{bmatrix} 222.2222 & -222.2222 & 0 & 0 & 166.6667 & -166.6667 \\ -222.2222 & 972.2222 & -750 & 250 & -416.6667 & 166.6667 \\ 0 & -750 & 750 & -250 & 250 & 0 \\ 0 & 250 & -250 & 83.3333 & -83.3333 & 0 \\ 166.6667 & -416.6667 & 250 & -83.3333 & 2083.3333 & -2000 \\ -166.6667 & 166.6667 & 0 & 0 & -2000 & 2000 \end{bmatrix}$$

By assembling the stiffness matrices into global stiffness matrix [K],

$$[K] = 4\times10^3 \times \begin{bmatrix} 972.2222 & -222.2222 & 0 & -750 & 0 & 166.6667 & -416.6667 & 250 \\ -222.2222 & 972.2222 & -750 & 0 & 250 & -416.6667 & 166.6667 & 0 \\ 0 & -750 & 972.2222 & -222.2222 & -416.6667 & 250 & 0 & 166.6667 \\ -750 & 0 & -222.2222 & 972.2222 & 166.6667 & 0 & 250 & -416.6667 \\ 0 & 250 & -416.6667 & 166.6667 & 2083.3333 & -83.3333 & 0 & -2000 \\ 166.6667 & -416.6667 & 250 & 0 & -83.3333 & 2083.3333 & -2000 & 0 \\ -416.6667 & 166.6667 & 0 & 250 & 0 & -2000 & 2083.3333 & -83.3333 \\ 250 & 0 & 166.6667 & -416.6667 & -2000 & 0 & -83.3333 & 2083.3333 \end{bmatrix}$$

Now, applying equation $[F]=[K]\{d\}$, the following expression can be written.

$$\begin{Bmatrix} F_{u1} \\ F_{u2} \\ F_{u3} \\ F_{u4} \\ F_{v1} \\ F_{v2} \\ F_{v3} \\ F_{v4} \end{Bmatrix} = 4\times10^3 \times \begin{bmatrix} 972.2222 & -222.2222 & 0 & -750 & 0 & 166.6667 & -416.6667 & 250 \\ -222.2222 & 972.2222 & -750 & 0 & 250 & -416.6667 & 166.6667 & 0 \\ 0 & -750 & 972.2222 & -222.2222 & -416.6667 & 250 & 0 & 166.6667 \\ -750 & 0 & -222.2222 & 972.2222 & 166.6667 & 0 & 250 & -416.6667 \\ 0 & 250 & -416.6667 & 166.6667 & 2083.3333 & -83.3333 & 0 & -2000 \\ 166.6667 & -416.6667 & 250 & 0 & -83.3333 & 2083.3333 & -2000 & 0 \\ -416.6667 & 166.6667 & 0 & 250 & 0 & -2000 & 2083.3333 & -83.3333 \\ 250 & 0 & 166.6667 & -416.6667 & -2000 & 0 & -83.3333 & 2083.3333 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

Putting boundary conditions $u_1 = v_1 = u_2 = u_4 = v_4 = 0$ and adopting elimination technique for applying boundary condition we get expression,

$$\begin{Bmatrix} 0 \\ 0 \\ -25000 \end{Bmatrix} = 4\times10^3 \times \begin{bmatrix} 972.2222 & 250 & 0 \\ 250 & 2083.3333 & -2000 \\ 0 & -2000 & 2083.3333 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_2 \\ v_3 \end{Bmatrix}$$

Solving the above expression, the unknown nodal displacements may be obtained as follows.

$v_2 = -0.0606$ mm, $u_3 = 0.0156$ mm and $v_3 = -0.0612$ mm.

Derivation of element stiffness for a four node rectangle element has been demonstrated in last lecture. The stiffness matrix of each element can be calculated easily by developing a suitable computer algorithm. To help students for developing their own computer code, a numerical example has been solved and demonstrated here.

5.4.1 Numerical Example

Calculate the stiffness matrix for the given four node rectangular element by the Gauss Quadrature integration rule using one point and two point formula assuming plane stress formulation. Consider, the thickness of element = 20 cm, E=2 $\times$ 10^3 kN/cm^2 and μ =0.

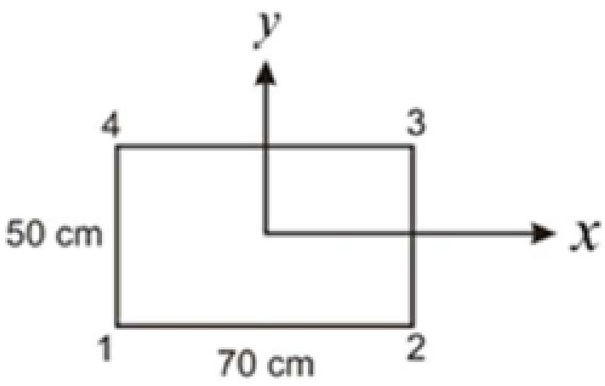


Fig. 5.4.1 Element Dimension

5.4.2 Evaluation of Stiffness using One Point Gauss Quadrature

For the calculation of stiffness matrix, first, 1×1 Gauss Quadrature integration procedure has been carried out. Thus, the natural coordinate of the sampling point will become 0,0 and weight will become 2.0 which is shown in the figure below.

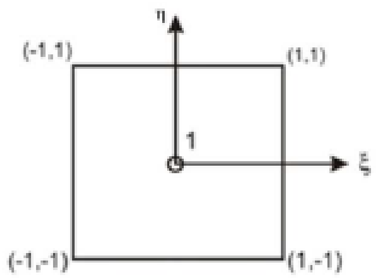


Fig. 5.4.2 Natural coordinates for one point Gauss Quadrature

For a four node quadrilateral element, the shape functions and their derivatives are as follows.

$$N_1 = \frac{(1-\xi)(1-\eta)}{4},\ N_2 = \frac{(1+\xi)(1-\eta)}{4} \text{ and } N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \xi} = \frac{-(1-\eta)}{4};\quad \frac{\partial N_2}{\partial \xi} = \frac{(1-\eta)}{4};\quad \frac{\partial N_3}{\partial \xi} = \frac{(1+\eta)}{4};\quad \frac{\partial N_4}{\partial \xi} = \frac{-(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \eta} = \frac{-(1-\xi)}{4};\quad \frac{\partial N_2}{\partial \eta} = \frac{-(1+\xi)}{4};\quad \frac{\partial N_3}{\partial \eta} = \frac{(1+\xi)}{4};\quad \frac{\partial N_4}{\partial \eta} = \frac{(1-\xi)}{4}$$

The Jacobian matrix can be found from the following relations.

$$|J| = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} \frac{-(1-\eta)}{4} & \frac{+(1-\eta)}{4} & \frac{+(1+\eta)}{4} & \frac{-(1+\eta)}{4} \\ \frac{-(1-\xi)}{4} & \frac{-(1+\xi)}{4} & \frac{+(1+\xi)}{4} & \frac{+(1-\xi)}{4} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Considering the sampling point, (ξ=0 and η=0), the value of the Jacobian, $[J]$ is

$$[J] = \begin{bmatrix} \frac{-1}{4} & \frac{+1}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 70 & 0 \\ 70 & 50 \\ 0 & 50 \end{bmatrix}$$

Thus, $[J] = \begin{bmatrix} 35 & 0 \\ 0 & 25 \end{bmatrix}$ and $[J]^{-1} = \begin{bmatrix} \frac{1}{35} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix}$; and $|J| = 875$

Now, the strain vector for the element will become

$$[\varepsilon] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

$$[\varepsilon]=\begin{bmatrix} J_{11}^{*} & J_{12}^{*} & 0 & 0 \\ 0 & 0 & J_{21}^{*} & J_{22}^{*} \\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*} \end{bmatrix}\times\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$[\varepsilon]=\begin{bmatrix} J_{11}^{*} & J_{12}^{*} & 0 & 0 \\ 0 & 0 & J_{21}^{*} & J_{22}^{*} \\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*} \end{bmatrix}\times[B']\{d\}=[B]\{d\}$$

$$[B']=\begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \text{ and } [B]=\begin{bmatrix} \frac{1}{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{25} \\ 0 & \frac{1}{25} & \frac{1}{35} & 0 \end{bmatrix}[B']$$

$$[\mathrm{B}]=\begin{bmatrix} -7.143 & 0 & 7.143 & 0 & 7.143 & 0 & -7.143 & 0 \\ 0 & -10 & 0 & -10 & 0 & 10 & 0 & 10 \\ -10 & -7.143 & -10 & 7.143 & 10 & 7.143 & 10 & -7.143 \end{bmatrix}\times10^{-3}$$

<u>For plane stress condition</u>

$$[C]=\frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}=2\times10^{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$[C][B] = 2\times10^3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}\times$$

$$\begin{bmatrix}-7.143 & 0 & 7.143 & 0 & 7.143 & 0 & -7.143 & 0\\ 0 & -10 & 0 & -10 & 0 & 10 & 0 & 10\\ -10 & -7.143 & -10 & 7.143 & 10 & 7.143 & 10 & -7.143\end{bmatrix}\times10^{-3}$$

Assume the values of gauss weight, $w = 2$, the stiffness matrix $[k]$ at this sampling point is $[k] = tww[B]^T[C]\,[B]\,\,|J|$, Where t is thickness of the element. Thus,

$$[k] = 20\times2\times2\times875\times2\times10^3\times\begin{bmatrix}-7.143 & 0 & -10\\ 0 & -10 & -7.143\\ 7.143 & 0 & -10\\ 0 & -10 & 7.143\\ 7.143 & 0 & 10\\ 0 & 10 & 7.143\\ -7.143 & 0 & 4\\ 0 & 10 & -7.143\end{bmatrix}10\times\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}$$

$$\times\begin{bmatrix}-7.143 & 0 & 7.143 & 0 & 7.143 & 0 & -7.143 & 0\\ 0 & -10 & 0 & -10 & 0 & 10 & 0 & 10\\ -10 & -7.143 & -10 & 7.143 & 10 & 7.143 & 10 & -7.143\end{bmatrix}$$

$$= 2\times10^3\times\begin{bmatrix}7.071 & 2.5 & -0.0714 & -2.5 & -7.071 & -2.5 & 0.0714 & 2.5\\ 2.5 & 8.785 & 2.5 & 5.214 & -2.5 & -8.785 & 2.5 & -5.214\\ -0.0714 & 2.5 & 7.071 & -2.5 & 0.0714 & -2.5 & -7.071 & 2.5\\ -2.5 & 5.214 & -2.5 & 8.785 & 2.5 & -5.214 & 2.5 & -8.785\\ -7.071 & -2.5 & 0.0714 & 2.5 & 7.071 & 2.5 & -0.0714 & -2.5\\ -2.5 & -8.785 & -2.5 & -5.214 & 2.5 & 8.785 & 2.5 & 5.214\\ 0.0714 & -2.5 & -7.071 & 2.5 & -0.0714 & 2.5 & 7.071 & -2.5\\ 2.5 & -5.214 & 2.5 & -8.785 & -2.5 & 5.214 & -2.5 & 8.785\end{bmatrix}$$

5.4.3 Evaluation of Stiffness using Two Point Gauss Quadrature

In this case, 2×2 Gauss Quadrature integration procedure has been carried out to the calculate the stiffness matrix of the same element for a comparison. The natural coordinate of the sampling point is shown in the figure below.

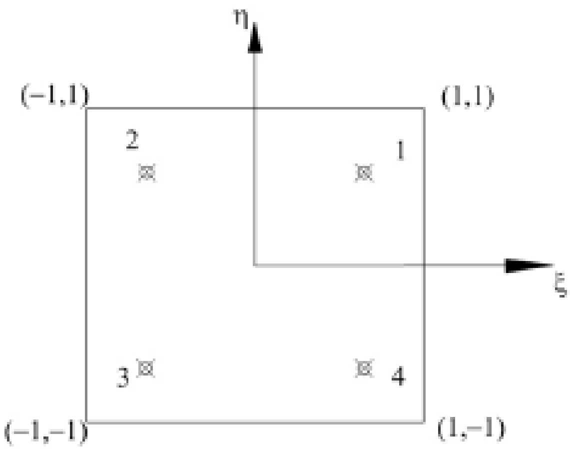


Fig. 5.4.3 Natural Coordinates for Two Points Gauss Quadrature

The natural co-ordinates of the sampling points for 2×2 Gauss Quadrature integration are

1	+0.57735	+0.57735
2	-0.57735	+0.57735
3	-0.57735	-0.57735
4	+0.57735	-0.57735

For a four node quadrilateral element, the shape functions and their derivatives are as follows.

$$\mathrm{N}_1 = \frac{(1-\xi)(1-\eta)}{4},\ \mathrm{N}_2 = \frac{(1+\xi)(1-\eta)}{4},\ \mathrm{N}_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } \mathrm{N}_4 = \frac{(1-\xi)(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \xi} = \frac{-(1-\eta)}{4};\qquad \frac{\partial N_2}{\partial \xi} = \frac{(1-\eta)}{4};\qquad \frac{\partial N_3}{\partial \xi} = \frac{(1+\eta)}{4};\qquad \frac{\partial N_4}{\partial \xi} = \frac{-(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \eta} = \frac{-(1-\xi)}{4};\qquad \frac{\partial N_2}{\partial \eta} = \frac{-(1+\xi)}{4};\qquad \frac{\partial N_3}{\partial \eta} = \frac{(1+\xi)}{4};\qquad \frac{\partial N_4}{\partial \eta} = \frac{(1-\xi)}{4}$$

The Jacobian matrix will be

$$|J| = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} \frac{-(1-\eta)}{4} & \frac{+(1-\eta)}{4} & \frac{+(1+\eta)}{4} & \frac{-(1+\eta)}{4} \\ \frac{-(1-\xi)}{4} & \frac{-(1+\xi)}{4} & \frac{+(1+\xi)}{4} & \frac{+(1-\xi)}{4} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

(a) At sampling point 1, (ξ=0.57735, η=0.57735)

The value of the Jacobian, [J] at sampling point 1 will become

$$[J]=\begin{bmatrix}\frac{-(1-0.57735)}{4} & \frac{+(1-0.57735)}{4} & \frac{+(1+0.57735)}{4} & \frac{-(1+0.57735)}{4}\\ \frac{-(1-0.57735)}{4} & \frac{-(1+0.57735)}{4} & \frac{+(1+0.57735)}{4} & \frac{+(1-0.57735)}{4}\end{bmatrix}\begin{bmatrix}0 & 0\\ 70 & 0\\ 70 & 50\\ 0 & 50\end{bmatrix}$$

Thus, $[J]=\begin{bmatrix}35 & 0\\ 0 & 25\end{bmatrix}$ and $[J]^{-1}=\begin{bmatrix}\frac{1}{35} & 0\\ 0 & \frac{1}{25}\end{bmatrix}=\begin{bmatrix}J_{11}^{*} & J_{12}^{*}\\ J_{21}^{*} & J_{22}^{*}\end{bmatrix}$; Thus $|J|=875$

$$[\varepsilon]=\begin{bmatrix}J_{11}^{*} & J_{12}^{*} & 0 & 0\\ 0 & 0 & J_{21}^{*} & J_{22}^{*}\\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*}\end{bmatrix}\times\begin{bmatrix}\frac{\partial u}{\partial\xi}\\ \frac{\partial u}{\partial\eta}\\ \frac{\partial v}{\partial\xi}\\ \frac{\partial v}{\partial\eta}\end{bmatrix}$$

$$[\varepsilon]=\begin{bmatrix}J_{11}^{*} & J_{12}^{*} & 0 & 0\\ 0 & 0 & J_{21}^{*} & J_{22}^{*}\\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*}\end{bmatrix}\times\begin{bmatrix}\frac{\partial N_1}{\partial\xi} & 0 & \frac{\partial N_2}{\partial\xi} & 0 & \frac{\partial N_3}{\partial\xi} & 0 & \frac{\partial N_4}{\partial\xi} & 0\\ \frac{\partial N_1}{\partial\eta} & 0 & \frac{\partial N_2}{\partial\eta} & 0 & \frac{\partial N_3}{\partial\eta} & 0 & \frac{\partial N_4}{\partial\eta} & 0\\ 0 & \frac{\partial N_1}{\partial\xi} & 0 & \frac{\partial N_2}{\partial\xi} & 0 & \frac{\partial N_3}{\partial\xi} & 0 & \frac{\partial N_4}{\partial\xi}\\ 0 & \frac{\partial N_1}{\partial\eta} & 0 & \frac{\partial N_2}{\partial\eta} & 0 & \frac{\partial N_3}{\partial\eta} & 0 & \frac{\partial N_4}{\partial\eta}\end{bmatrix}\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{bmatrix}$$

$$[\varepsilon]=\begin{bmatrix}J_{11}^{*} & J_{12}^{*} & 0 & 0\\ 0 & 0 & J_{21}^{*} & J_{22}^{*}\\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*}\end{bmatrix}\times[B']\{d\}=[B]\{d\}$$

$$[B']=\begin{bmatrix} -\frac{1-0.57735}{4} & 0 & \frac{1-0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & -\frac{1+0.57735}{4} & 0 \\ -\frac{1-0.57735}{4} & 0 & -\frac{1+0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & \frac{1-0.57735}{4} & 0 \\ 0 & -\frac{1-0.57735}{4} & 0 & \frac{1-0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & -\frac{1+0.57735}{4} \\ 0 & -\frac{1-0.57735}{4} & 0 & -\frac{1+0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & \frac{1-0.57735}{4} \end{bmatrix}$$

$$[B']=\begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 & 0 \\ -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 & 0 \\ 0 & -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 \\ 0 & -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 \end{bmatrix}$$

Thus,

$$[B]=\begin{bmatrix} \frac{1}{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{25} \\ 0 & \frac{1}{25} & \frac{1}{35} & 0 \end{bmatrix}\times\begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 & 0 \\ -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 & 0 \\ 0 & -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 \\ 0 & -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 \end{bmatrix}$$

$$[B]=\begin{bmatrix} -0.003 & 0 & 0.003 & 0 & 0.0113 & 0 & -0.0113 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0042 & -0.003 & -0.0158 & 0.003 & 0.0158 & 0.0113 & 0.0042 & -0.0113 \end{bmatrix}$$

For plane stress condition

$$[C]=\frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}=2\times10^3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$[C][B] = 2\times10^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.003 & 0 & 0.003 & 0 & 0.0113 & 0 & -0.0113 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0042 & -0.003 & -0.0158 & 0.003 & 0.0158 & 0.0113 & 0.0042 & -0.0113 \end{bmatrix}$$

The values of gauss weights are w_i=w_j=1.0. Therefore, the stiffness matrix $[k]$ at this sampling point is $[k] = tw_iw_j[B]_{ij}^T[C]_{ij}[B]_{ij}\,|J|$, where t is thickness of the element. Thus at sampling point 1,

$$[k_1] = 20\times1\times1\times875\times2\times10^3\times \begin{bmatrix} -0.003 & 0 & -0.0042 \\ 0 & -0.0042 & -0.003 \\ 0.003 & 0 & -0.0158 \\ 0 & -0.0158 & 0.003 \\ 0.0113 & 0 & 0.0158 \\ 0 & 0.0158 & 0.0113 \\ -0.0113 & 0 & 0.0042 \\ 0 & 0.0042 & -0.0113 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.003 & 0 & 0.003 & 0 & 0.0113 & 0 & -0.0113 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0042 & -0.003 & -0.0158 & 0.003 & 0.0158 & 0.0113 & 0.0042 & -0.0113 \end{bmatrix}$$

$$[k_1]=10^4\times\begin{bmatrix} 0.0632 & 0.0223 & 0.0848 & -0.0223 & -0.2358 & -0.0834 & 0.0878 & 0.0834 \\ & 0.0785 & 0.0834 & 0.2174 & -0.0834 & -0.2929 & -0.0223 & -0.0030 \\ & & 0.4672 & -0.0834 & -0.3162 & -0.3109 & -0.2358 & 0.3109 \\ & & & 0.8866 & 0.0834 & -0.8111 & 0.0223 & -0.2929 \\ & & & & 0.8795 & 0.3109 & -0.3275 & -0.3109 \\ & & sym & & & 1.0927 & 0.0834 & 0.0113 \\ & & & & & & 0.4755 & -0.0834 \\ & & & & & & & 0.2847 \end{bmatrix}$$

(b) At sampling point 2, (ξ=-0.57735, η=0.57735)

The value of the Jacobian, [*J*] at sampling point 2 can be calculated in a similar way and finally the strain-displacement relationship matrix and then the stiffness matrix [k_2] can be evaluated and is shown below.

$$[k_2]=20\times1\times1\times875\times2\times10^3\begin{bmatrix} -0.0113 & 0 & -0.0042 \\ 0 & -0.0042 & -0.0113 \\ 0.0113 & 0 & -0.0158 \\ 0 & -0.0158 & 0.0113 \\ 0.003 & 0 & 0.0158 \\ 0 & 0.0158 & 0.003 \\ -0.003 & 0 & 0.0042 \\ 0 & 0.0042 & -0.0030 \end{bmatrix}\times$$
$$\begin{bmatrix} -0.0113 & 0 & 0.0113 & 0 & 0.0030 & 0 & -0.0030 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0021 & -0.0056 & -0.0079 & 0.0056 & 0.0079 & 0.0015 & 0.0021 & -0.0015 \end{bmatrix}$$

$$[k_2]=10^4\times\begin{bmatrix} 0.4756 & 0.0833 & -0.3276 & -0.0833 & -0.2357 & -0.0223 & 0.0878 & 0.0223 \\ & 0.2847 & 0.3110 & 0.0112 & -0.3110 & -0.2929 & -0.0833 & -0.0030 \\ & & 0.8797 & -0.3110 & -0.3164 & -0.0833 & -0.2357 & 0.0833 \\ & & & 1.0930 & 0.3110 & -0.8113 & 0.0833 & -0.2929 \\ & & & & 0.4673 & 0.0833 & 0.0848 & -0.0833 \\ & & sym & & & 0.8868 & 0.0223 & 0.2174 \\ & & & & & & 0.0632 & -0.0223 \\ & & & & & & & 0.0785 \end{bmatrix}$$

(c) At sampling point 3, (ξ=-0.57735, η=-0.57735)

The value of the strain-displacement relationship matrix and then the stiffness matrix $[k_3]$ can be evaluated and is shown below.

$$[k_3] = 20\times1\times1\times875\times2\times10^3 \begin{bmatrix} -0.0113 & 0 & -0.0158 \\ 0 & -0.0158 & -0.0113 \\ 0.0113 & 0 & -0.0042 \\ 0 & -0.0042 & 0.0113 \\ 0.0030 & 0 & 0.0042 \\ 0 & 0.0042 & 0.0030 \\ -0.0030 & 0 & 0.0158 \\ 0 & 0.0158 & -0.0030 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.0030 & 0 & 0.0113 & 0 & 0.0030 & 0 & -0.0030 & 0 \\ 0 & -0.0158 & 0 & -0.0042 & 0 & 0.0042 & 0 & 0.0158 \\ -0.0079 & -0.0056 & -0.0021 & 0.0056 & 0.0021 & 0.0015 & 0.0079 & -0.0015 \end{bmatrix}$$

$$[k_3] = 10^4 \times \begin{bmatrix} 0.8797 & 0.3110 & -0.3276 & -0.3110 & -0.2357 & -0.0833 & -0.3164 & 0.0833 \\ & 1.0930 & 0.0833 & 0.0112 & -0.0833 & -0.2929 & -0.3110 & -0.8113 \\ & & 0.4756 & -0.0833 & 0.0878 & -0.0223 & -0.2357 & 0.0223 \\ & & & 0.2847 & 0.0833 & -0.0030 & 0.3110 & -0.2929 \\ & & & & 0.0632 & 0.0223 & 0.0848 & -0.0223 \\ & & sym & & & 0.0785 & 0.0833 & 0.2174 \\ & & & & & & 0.4673 & -0.0833 \\ & & & & & & & 0.8868 \end{bmatrix}$$

(d) At sampling point 4, (ξ=0.57735, η=-0.57735)

The value of the strain-displacement relationship matrix and then the stiffness matrix $[k_3]$ can be evaluated and is shown below.

$$[k_4]=20\times1\times1\times875\times2\times10^3\begin{bmatrix}-0.0030 & 0 & -0.0158\\ 0 & -0.0158 & -0.0030\\ 0.0030 & 0 & -0.0042\\ 0 & -0.0042 & 0.0030\\ 0.0113 & 0 & 0.0042\\ 0 & 0.0042 & 0.0113\\ -0.0113 & 0 & 0.0158\\ 0 & 0.0158 & -0.0113\end{bmatrix}\times$$

$$\begin{bmatrix}-0.0030 & 0 & 0.0030 & 0 & 0.0113 & 0 & -0.0113 & 0\\ 0 & -0.0158 & 0 & -0.0042 & 0 & 0.0042 & 0 & 0.0158\\ -0.0079 & -0.0015 & -0.0021 & 0.0015 & 0.0021 & 0.0056 & 0.0079 & -0.0056\end{bmatrix}$$

$$[k_4]=10^4\times\begin{bmatrix}0.4673 & 0.0833 & 0.0848 & -0.0833 & -0.2357 & -0.3110 & -0.3164 & 0.3110\\ & 0.8868 & 0.0223 & 0.2174 & -0.0223 & -0.2929 & -0.0833 & -0.8113\\ & & 0.0632 & -0.0223 & 0.0878 & -0.0833 & -0.2357 & 0.0833\\ & & & 0.0785 & 0.0223 & -0.0030 & 0.0833 & -0.2929\\ & & & & 0.4756 & 0.0833 & -0.3276 & -0.0833\\ & & sym & & & 0.2847 & 0.3110 & 0.0112\\ & & & & & & 0.8797 & -0.3110\\ & & & & & & & 1.0930\end{bmatrix}$$

The stiffness matrix of the element can be computed as the sum of the values at the four sampling points: $[k]=[k_1]+[k_2]+[k_3]+[k_4]$. Thus, the final value of the stiffness matrix will become

$$[k]=10^4\times\begin{bmatrix}1.8857 & 0.5000 & -0.4857 & -0.5000 & -0.9429 & -0.5000 & -0.4571 & 0.3110\\ & 2.3429 & 0.5000 & 0.4571 & -0.5000 & -1.1714 & -0.5000 & -1.6286\\ & & 1.8857 & -0.5000 & -0.4571 & -0.5000 & -0.9429 & 0.5000\\ & & & 2.3429 & -0.5000 & -1.6286 & 0.5000 & -1.1714\\ & & & & 1.8857 & 0.5000 & -0.4857 & -0.5000\\ & & sym & & & 2.3429 & 0.5000 & 0.4571\\ & & & & & & 1.8857 & -0.5000\\ & & & & & & & 2.3429\end{bmatrix}$$

5.6. AXISYMMETRIC ELEMENT

5.6.1 Introduction

Many three-dimensional problems show symmetry about an axis of rotation. If the problem geometry is symmetric about an axis and the loading and boundary conditions are symmetric about the same axis, the problem is said to be axisymmetric. Such three-dimensional problems can be solved using two-dimensional finite elements. The axisymmetric problem are most conveniently defined by polar coordinate system with coordinates (r, θ, z) as shown in Fig. 5.6.1. Thus, for axisymmetric analysis, following conditions are to be satisfied.

1. The domain should have an axis of symmetry and is considered as *z* axis.
2. The loadings on the domain has to be symmetric about the axis of revolution, thus they are independent of circumferential coordinate θ.
3. The boundary condition and material properties are symmetric about the same axis and will be independent of circumferential coordinate.

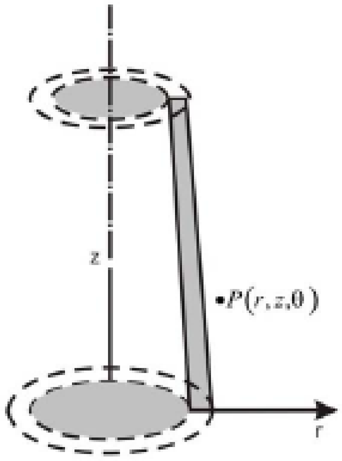


Fig. 5.6.1 Cylindrical coordinates

Axisymmetric solids are of total symmetry about the axis of revolution (i.e., *z*-axis), the field variables, such as the stress and deformation is independent of rotational angle θ. Therefore, the field variables can be defined as a function of (r,z) and hence the problem becomes a two dimensional problem similar to those of plane stress/strain problems. Axisymmetric problems includes, circular cylinder loaded with uniform external or internal pressure, circular water tank, pressure vessels, chimney, boiler, circular footing resting on soil mass, etc.

5.6.2 Relation between Strain and Displacement

An axisymmetric problem is readily described in cylindrical polar coordinate system: r, z and θ. Here, θ measures the angle between the plane containing the point and the axis of the coordinate system. At θ = 0, the radial and axial coordinates coincide with the global Cartesian X and Y coordinates. Fig. 5.6.2 shows a cylindrical coordinate system and the definition of the position vectors. Let $\hat{\mathbf{r}}, \hat{\mathbf{z}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the radial, axial, and circumferential directions at a point in the cylindrical coordinate system.

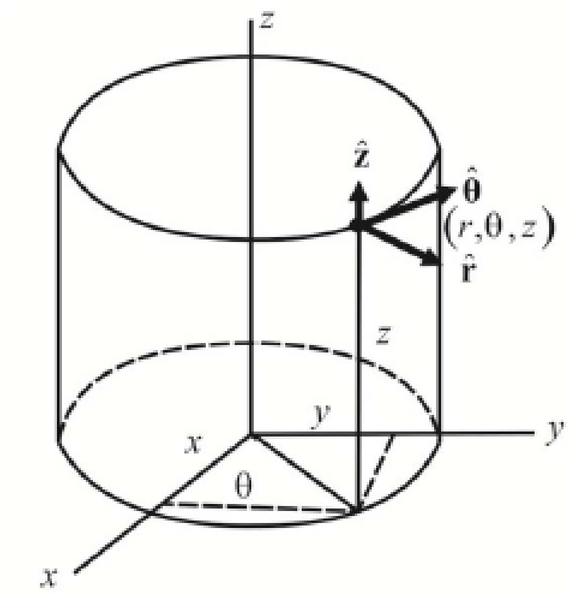
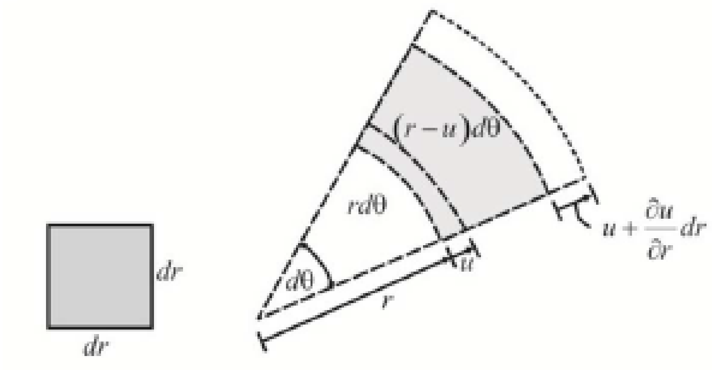


Fig. 5.6.2 Cylindrical Coordinate System

If the loading consists of radial and axial components that are independent of θ and the material is either isotropic or orthotropic and the material properties are independent of θ, the displacement at any point will only have radial (u_r) and axial (u_z) components. The only stress components that will be nonzero are σ_{rr}, σ_{zz}, $\sigma_{\theta\theta}$ *and* τ_{rz} .



(a) Element in r-z plane **(b) Element in r-θ plane**

Fig. 5.6.3 Deformation of the axisymmetric element

A differential element of the body in the *r-z* plane is shown in Fig. 5.6.3(a). The element undergoes deformation in the radial direction. Therefore, it initiates increase in circumference and associated circumferential strain. Let denote the radial displacement as *u*, the circumferential displacement as *v*, and the axial displacement as *w*. Dashed line represents the deformed positions of the body in Fig. 5.6.3(b). The radial strain can be calculated from the above diagram as

$$\varepsilon_r = \frac{1}{dr}\left(u + \frac{\partial u}{\partial r}\times dr - u\right) = \frac{\partial u}{\partial r} \tag{5.6.1}$$

Since the *rz* plane is effectively the same as a rectangular coordinate system, the axial strain will become

$$\varepsilon_z = \frac{1}{dz}\left(w + \frac{\partial w}{\partial z}\times dz - w\right) = \frac{\partial w}{\partial z} \tag{5.6.2}$$

Considering the original arc length versus the deformed arc length, the differential element undergoes an expansion in the circumferential direction. Before deformation, let the arc length is assumed as d*s* = *r*dθ. After deformation, the arc length will become d*s* = (*r*+*u*) dθ. Thus, the tangential strain will be

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r} \tag{5.6.3}$$

Similarly, the shear strain will be

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{5.6.4}$$

$$\gamma_{r\theta} = 0 \text{ and } \gamma_{z\theta} = 0$$

Thus, there are four strain components present in this case and is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix} \tag{5.6.5}$$

5.6.3 Relation between Stress and Strain

The stress strain relation for axisymmetric case can be derived from the three dimensional constitutive relations. We know the stress-strain relation for a three-dimensional solid is

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix}1-\mu & \mu & \mu & 0 & 0 & 0\\ \mu & 1-\mu & \mu & 0 & 0 & 0\\ \mu & \mu & 1-\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2}\end{bmatrix}\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \nu_{xy}\\ \nu_{yz}\\ \nu_{zx}\end{Bmatrix}$$

(5.6.6)

The stresses acting on a differential volume of an axisymmetric solid under axisymmetric loading is shown in Fig. 5.6.4.

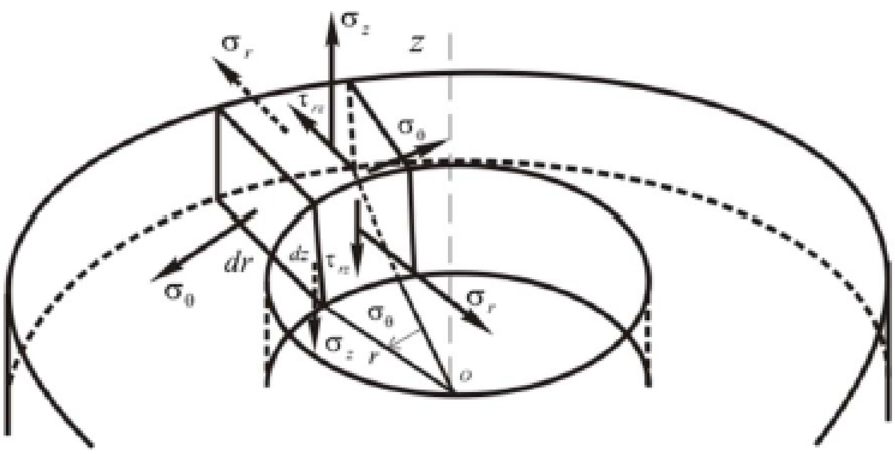


Fig. 5.6.4 Stresses acting on a differential volume

Now, comparing the stress-strain components present in the axisymmetric case, the stress-strain relation can be expressed from the above expression as follows

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix}1-\mu & \mu & \mu & 0\\ \mu & 1-\mu & \mu & 0\\ \mu & \mu & 1-\mu & 0\\ 0 & 0 & 0 & \frac{1-2\mu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \nu_{rz}\end{Bmatrix} \qquad (5.6.7)$$

Thus, the constitutive matrix [D] for the axisymmetric elastic solid will be

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \tag{5.6.8}$$

5.6.4 Axisymmetric Shell Element

A cylindrical liquid storage container like structures (Fig. 5.6.5) may be idealized using axisymmetric shell element for the finite element analysis. It may be noted that the liquid in the container may be idealized with two dimensional axisymmetric elements. Let us consider the radius, height and, thickness of the circular tank are R, H and h respectively.

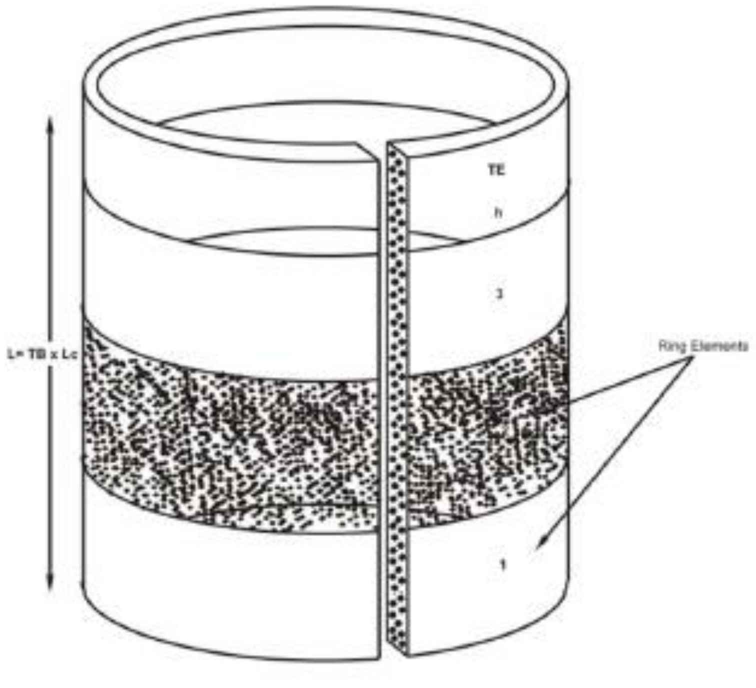


Fig. 5.6.5 Thin wall cylindrical container

The strain energy of the axisymmetric shell element (Fig. 5.6.6) including the effect of both stretching and bending are expressed as

$$U = \frac{1}{2}\int_0^H \left(N_y\varepsilon_y + N_\theta\varepsilon_\theta + M_y\chi_y\right)2\pi R dy \tag{5.6.9}$$

Here, N_y and N_θ are the membrane force resultants and M_y is the bending moment resultant. The shell is assumed to be linearly elastic, homogeneous and isotropic. Thus the force and moment resultants can be expressed in terms of the mid-surface change in curvature χ_y as follows.

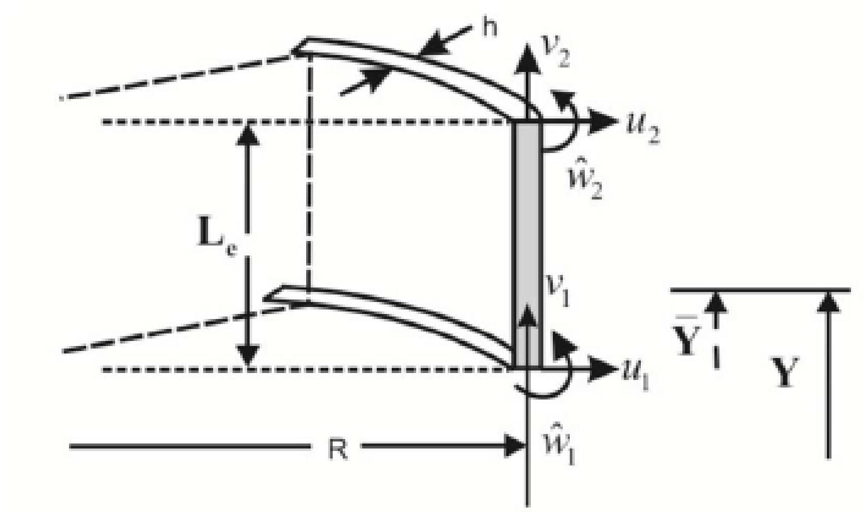


Fig 5.6.6 Axisymmetric plate element

Here, the strain-displacement relation is given by

$$\{\sigma\} = [D]\{\varepsilon\} \tag{5.6.10}$$

In which,

$$\{\sigma\} = \begin{Bmatrix} N_y \\ N_\theta \\ M_y \end{Bmatrix}, \ \{\varepsilon\} = \begin{Bmatrix} \varepsilon_y \\ \varepsilon_\theta \\ \chi_y \end{Bmatrix} \text{ and } [D] = \frac{Eh}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{h^2}{12} \end{bmatrix} \tag{5.6.11}$$

The generalized strain vector can be expressed in terms of the displacement vectors as follows.

$$\{\varepsilon\} = [B]\{d\} \tag{5.6.12}$$

Where,

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} \text{and } [B] = \begin{bmatrix} 0 & \frac{\partial}{\partial y} \\ \frac{1}{R} & 0 \\ -\frac{\partial^2}{\partial y^2} & 0 \end{bmatrix} \tag{5.6.13}$$

Here, u and v are the displacement components in two perpendicular directions. With the use of stress and strain vectors, the potential energy expression are written in terms of displacement vectors as

$$U = \frac{1}{2} \times 2\pi R \int_0^H \left(\{d\}^T [B]^T [D][B]\{d\} \right) dy \tag{5.6.14}$$

Thus, the element stiffness are derived as

$$[k] = 2\pi R \int_0^H [B]^T [D][B] dy \tag{5.6.15}$$

Similarly, neglecting the rotary inertia, the kinetic energy can be expressed as

$$T = \frac{1}{2} \times 2\pi R \int_0^H \left(\{\dot{d}\}^T [N]^T m[N]\{\dot{d}\} \right) dy \tag{5.6.16}$$

Where, m denotes the mass of the shell element per unit area and $\{\dot{d}\}$ represents the velocity vector. Thus, the element mass matrix is given by

$$[M] = 2\pi Rm \int_0^{L_e} [N]^T [N] dy \tag{5.6.17}$$

Lecture 7: Finite Element Formulation of Axisymmetric Element

Finite element formulation for the axisymmetric problem will be similar to that of the two dimensional solid elements. As the field variables, such as the stress and strain is independent of rotational angle θ, circumferential displacement will not appear. Thus, the displacement field variables are expressed as

$$\begin{aligned} u(r,z) &= \sum_{i=1}^{n} N_i(r,z) u_i \\ w(r,z) &= \sum_{i=1}^{n} N_i(r,z) w_i \end{aligned} \tag{5.7.1}$$

Here, u_i and w_i represent radial and axial displacements respectively at nodes. $N_i(r, z)$ are the shape functions. As the geometry and field variables are independent of rotational angle θ, the interpolation function $N_i(r, z)$ can be expressed similar to 2-dimensional problems by replacing the x and y terms with r and z terms respectively.

5.7.1 Stiffness Matrix of a Triangular Element

Fig. 5.7.1 shows the cylindrical coordinates of a three node triangular element. Hence the analysis of the axisymmetric element can be approached in a similar way as the CST element. Thus the field variables of such an element can be expressed as

$$\begin{aligned} u &= \alpha_0 + \alpha_1 r + \alpha_2 z \\ w &= \alpha_3 + \alpha_4 r + \alpha_5 z \end{aligned} \tag{5.7.2}$$

Or,

$$\{d\} = [\phi]\{\alpha\} \tag{5.7.3}$$

Where,

$$\{d\} = \begin{Bmatrix} u \\ w \end{Bmatrix},\ [\phi] = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \text{ and } \{\alpha\}^T = \{\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5\}$$

Using end conditions,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i & 0 & 0 & 0 \\ 1 & r_j & z_j & 0 & 0 & 0 \\ 1 & r_k & z_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_i & z_i \\ 0 & 0 & 0 & 1 & r_j & z_j \\ 0 & 0 & 0 & 1 & r_k & z_k \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \tag{5.7.4}$$

Or,

$$\begin{aligned} \{\bar{d}\} &= [A]\{\alpha\} \\ \Rightarrow \{\alpha\} &= [A]^{-1}\{\bar{d}\} \end{aligned} \tag{5.7.5}$$

Here $\{\bar{d}\}$ are the nodal displacement vectors.

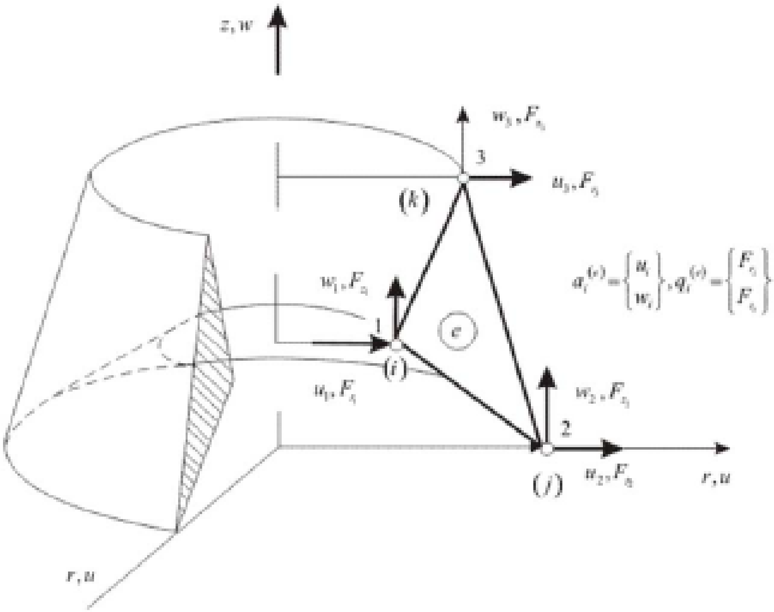


Fig. 5.7.1 Axisymmetric three node triangle in cylindrical coordinates

Putting above values in eq.(5.7.3), the following relations will be obtained.

$$\{d\} = [\phi][A]^{-1}\{\bar{d}\} = [N]\{\bar{d}\} \tag{5.7.6}$$

Or,

$$\{d\} = \begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{bmatrix} N_i & N_j & N_k & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & N_j & N_k \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ z_1 \\ z_2 \\ z_3 \end{Bmatrix} \tag{5.7.7}$$

Using a similar approach as in case of CST elements, the three shape functions $[N_1, N_2, N_3]$ can be assumed as,

$$N_1(r,z) = \frac{1}{2A}\left[(r_2 z_3 - r_3 z_2) + (z_2 - z_3)r + (r_3 - r_2)z\right]$$

$$N_2(r,z) = \frac{1}{2A}\left[(r_3 z_1 - r_1 z_3) + (z_3 - z_1)r + (r_1 - r_3)z\right]$$

$$N_3(r,z) = \frac{1}{2A}\left[(r_1 z_2 - r_2 z_1) + (z_1 - z_2)r + (r_2 - r_1)z\right]$$

Or,

$$N_i(r,z)=\frac{1}{2A}(\alpha_i+r\beta_i+z\gamma_i)$$
$$N_j(r,z)=\frac{1}{2A}(\alpha_j+r\beta_j+z\gamma_j) \qquad (5.7.8)$$
$$N_k(r,z)=\frac{1}{2A}(\alpha_k+r\beta_k+z\gamma_k)$$

Where,

$$\alpha_i=r_jz_k-r_kz_j \qquad \alpha_j=r_kz_i-r_iz_k \qquad \alpha_k=r_iz_j-r_jz_i$$
$$\beta_i=z_j-z_k \qquad \beta_j=z_k-z_i \qquad \beta_k=z_i-z_j$$
$$\gamma_i=r_k-r_j \qquad \gamma_i=r_i-r_k \qquad \gamma_i=r_j-r_i \qquad (5.7.9)$$
$$2A=\frac{1}{2}(r_iz_j+r_jz_k+r_kz_i-r_iz_k-r_jz_i-r_kz_j)$$

Putting the value of {u,w} in eq. (5.7.7) from eq. (5.6.5),

$$\{\varepsilon\}=\begin{bmatrix} \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial r} & 0 & 0 & 0 \\ \frac{N_i}{r} & \frac{N_j}{r} & \frac{N_k}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial r} \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ z_1 \\ z_2 \\ z_3 \end{Bmatrix}=[B]\{\bar{d}\} \qquad (5.7.10)$$

Thus, the strain displacement matrix can be expressed as,

$$[B]=\frac{1}{2A}\begin{bmatrix} \beta_i & \beta_j & \beta_k & 0 & 0 & 0 \\ \frac{N_i}{r} & \frac{N_j}{r} & \frac{N_k}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_i & \gamma_j & \gamma_k \\ \gamma_i & \gamma_j & \gamma_k & \beta_i & \beta_j & \beta_k \end{bmatrix} \qquad (5.7.11)$$

Where, $r=\frac{r_i+r_j+r_k}{3}$. Thus the stiffness matrix will become

$$[k]=\iiint [B]^T[D][B]\,d\Omega$$

Or, $$[k]=\iint\int_0^{2\pi}[B]^T[D][B]\,r\,d\theta\,dA=2\pi\iint [B]^T[D][B]\,r\,dr\,dz \qquad (5.7.12)$$

Since, the term [B] is dependent of 'r' terms; the term $[B]^T[D][B]$ cannot be taken out of integration. Yet, a reasonably accurate solution can be obtained by evaluating the [B] (denoted as [$\underline{B}$]) matrix at the centroid.

Hence, $[k] = 2\pi\underline{r}[\underline{B}]^T[D][\underline{B}]\iint dr\,dz$

Or,

$$[k] = [\underline{B}]^T[D][\underline{B}]2\pi\underline{r}A \tag{5.7.13}$$

5.7.2 Stiffness Matrix of a Quadrilateral Element

The strain-displacement relation for axisymmetric problem derived earlier (eq.(5.6.5)) can be rewritten as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{r} \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \\ \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ u \end{Bmatrix} \tag{5.7.14}$$

Applying chain rule of differentiation equation we get,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \\ \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ u \end{Bmatrix} = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 & 0 \\ J^*_{21} & J^*_{22} & 0 & 0 & 0 \\ 0 & 0 & J^*_{11} & J^*_{12} & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \\ u \end{Bmatrix} \tag{5.7.15}$$

Hence, the strain components are calculated as

$$\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \gamma_{rz}\end{Bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&\frac{1}{r}\\0&1&1&0&0\end{bmatrix}\begin{bmatrix}J^*_{11}&J^*_{12}&0&0&0\\J^*_{21}&J^*_{22}&0&0&0\\0&0&J^*_{11}&J^*_{12}&0\\0&0&J^*_{21}&J^*_{22}&0\\0&0&0&0&1\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial w}{\partial \xi}\\ \frac{\partial w}{\partial \eta}\\ u\end{Bmatrix}$$

Or,

$$\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \gamma_{rz}\end{Bmatrix}=\begin{bmatrix}J^*_{11}&J^*_{12}&0&0&0\\0&0&J^*_{21}&J^*_{22}&0\\0&0&0&0&\frac{1}{r}\\J^*_{21}&J^*_{22}&J^*_{11}&J^*_{12}&0\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial w}{\partial \xi}\\ \frac{\partial w}{\partial \eta}\\ u\end{Bmatrix} \tag{5.7.16}$$

With the use of interpolation function and nodal displacements, $\left(\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta},\frac{\partial w}{\partial \xi},\frac{\partial w}{\partial \eta}\right)$ can be expressed for a four node quadrilateral element as

$$\begin{Bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial w}{\partial \xi}\\ \frac{\partial w}{\partial \eta}\end{Bmatrix}=\begin{bmatrix}\frac{\partial N_1}{\partial \xi}&\frac{\partial N_2}{\partial \xi}&\frac{\partial N_3}{\partial \xi}&\frac{\partial N_4}{\partial \xi}&0&0&0&0\\ \frac{\partial N_1}{\partial \eta}&\frac{\partial N_2}{\partial \eta}&\frac{\partial N_3}{\partial \eta}&\frac{\partial N_4}{\partial \eta}&0&0&0&0\\ 0&0&0&0&\frac{\partial N_1}{\partial \xi}&\frac{\partial N_2}{\partial \xi}&\frac{\partial N_3}{\partial \xi}&\frac{\partial N_4}{\partial \xi}\\ 0&0&0&0&\frac{\partial N_1}{\partial \eta}&\frac{\partial N_2}{\partial \eta}&\frac{\partial N_3}{\partial \eta}&\frac{\partial N_4}{\partial \eta}\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\\w_1\\w_2\\w_3\\w_4\end{Bmatrix} \tag{5.7.17}$$

Putting eq. (5.7.17) in eq. (5.7.16) we get,

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \\ N_1 & N_2 & N_3 & N_4 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} \tag{5.7.18}$$

Thus, the strain displacement relationship matrix [*B*] becomes

$$[B] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \\ N_1 & N_2 & N_3 & N_4 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \tag{5.7.19}$$

For a four node quadrilateral element,

$$\begin{aligned} N_1 &= \frac{(1-\xi)(1-\eta)}{4} \Rightarrow \frac{\partial N_1}{\partial \xi} = -\frac{(1-\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = -\frac{(1-\xi)}{4} \\ N_2 &= \frac{(1+\xi)(1-\eta)}{4} \Rightarrow \frac{\partial N_2}{\partial \xi} = \frac{(1-\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = -\frac{(1+\xi)}{4} \\ N_3 &= \frac{(1+\xi)(1+\eta)}{4} \Rightarrow \frac{\partial N_2}{\partial \xi} = \frac{(1+\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = \frac{(1+\xi)}{4} \\ N_4 &= \frac{(1-\xi)(1+\eta)}{4} \Rightarrow \frac{\partial N_2}{\partial \xi} = -\frac{(1+\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = \frac{(1-\xi)}{4} \end{aligned} \tag{5.7.20}$$

Thus, the [B] matrix will become

$$[B]=\begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} & 0 \end{bmatrix}\times$$

$$\begin{bmatrix} -\frac{(1-\eta)}{4} & \frac{(1-\eta)}{4} & \frac{(1+\eta)}{4} & -\frac{(1+\eta)}{4} & 0 & 0 & 0 & 0 \\ -\frac{(1-\xi)}{4} & -\frac{(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(1-\eta)}{4} & \frac{(1-\eta)}{4} & \frac{(1+\eta)}{4} & -\frac{(1+\eta)}{4} \\ 0 & 0 & 0 & 0 & -\frac{(1-\xi)}{4} & -\frac{(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} \\ \frac{(1-\xi)(1-\eta)}{4} & \frac{(1+\xi)(1-\eta)}{4} & \frac{(1+\xi)(1+\eta)}{4} & \frac{(1-\xi)(1+\eta)}{4} & \frac{(1-\xi)(1-\eta)}{4} & \frac{(1+\xi)(1-\eta)}{4} & \frac{(1+\xi)(1+\eta)}{4} & \frac{(1-\xi)(1+\eta)}{4} \end{bmatrix}$$

(5.7.21)

The stiffness matrix for the axisymmetric element finally can be found from the following expression after numerical integration.

$$[k]=\int_{\Omega}[B]^T[D][B]\,d\Omega=\int_{-1}^{+1}\int_{-1}^{+1}[\mathrm{B}]^T[\mathrm{D}][\mathrm{B}].2\pi\mathrm{r}.|J|.\mathrm{d}\xi\mathrm{d}\eta \tag{5.7.22}$$

Finite element formulation for the axisymmetric problem will be similar to that of the two dimensional solid elements. As the field variables, such as the stress and strain is independent of rotational angle θ, circumferential displacement will not appear. Thus, the displacement field variables are expressed as

$$\begin{aligned} u(r,z) &= \sum_{i=1}^{n} N_i(r,z)\,u_i \\ w(r,z) &= \sum_{i=1}^{n} N_i(r,z)\,w_i \end{aligned} \qquad (5.7.1)$$

Here, u_i and w_i represent radial and axial displacements respectively at nodes. $N_i(r, z)$ are the shape functions. As the geometry and field variables are independent of rotational angle θ, the interpolation function $N_i(r, z)$ can be expressed similar to 2-dimensional problems by replacing the x and y terms with r and z terms respectively.

5.7.1 Stiffness Matrix of a Triangular Element

Fig. 5.7.1 shows the cylindrical coordinates of a three node triangular element. Hence the analysis of the axisymmetric element can be approached in a similar way as the CST element. Thus the field variables of such an element can be expressed as

$$\begin{aligned} u &= \alpha_0 + \alpha_1 r + \alpha_2 z \\ w &= \alpha_3 + \alpha_4 r + \alpha_5 z \end{aligned} \qquad (5.7.2)$$

Or,

$$\{d\} = [\phi]\{\alpha\} \qquad (5.7.3)$$

Where,

$$\{d\} = \begin{Bmatrix} u \\ w \end{Bmatrix},\ [\phi] = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \text{ and } \{\alpha\}^T = \{\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5\}$$

Using end conditions,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i & 0 & 0 & 0 \\ 1 & r_j & z_j & 0 & 0 & 0 \\ 1 & r_k & z_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_i & z_i \\ 0 & 0 & 0 & 1 & r_j & z_j \\ 0 & 0 & 0 & 1 & r_k & z_k \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \qquad (5.7.4)$$

Or,

$$\{\overline{d}\} = [A]\{\alpha\}$$
$$\Rightarrow \{\alpha\} = [A]^{-1}\{\overline{d}\} \tag{5.7.5}$$

Here $\{\overline{d}\}$ are the nodal displacement vectors.

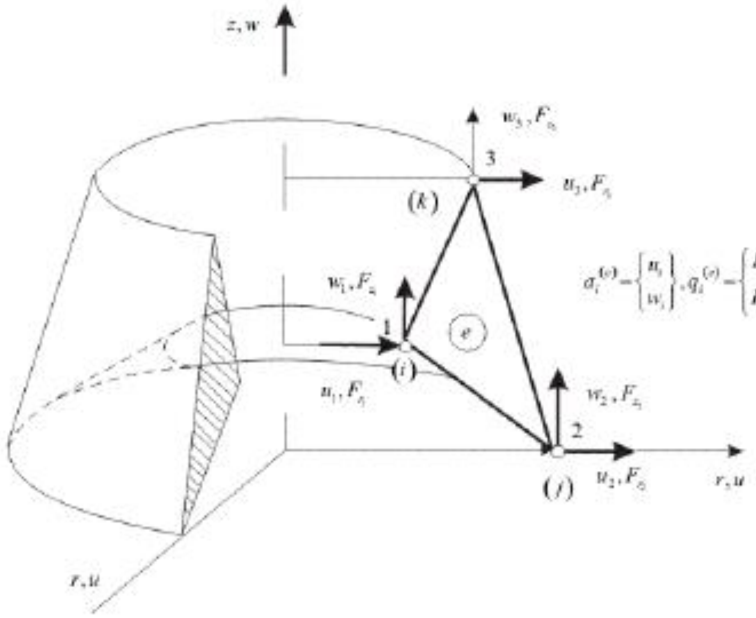


Fig. 5.7.1Axisymmetric three node triangle in cylindrical coordinates

Putting above values in eq.(5.7.3), the following relations will be obtained.

$$\{d\} = [\phi][A]^{-1}\{\overline{d}\} = [N]\{\overline{d}\} \tag{5.7.6}$$

Or,

$$\{d\} = \begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{bmatrix} N_i & N_j & N_k & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & N_j & N_k \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ z_1 \\ z_2 \\ z_3 \end{Bmatrix} \tag{5.7.7}$$

Using a similar approach as in case of CST elements, the three shape functions $[N_1, N_2, N_3]$ can be assumed as,

$$N_1(r,z)=\frac{1}{2A}\left[(r_2z_3-r_3z_2)+(z_2-z_3)r+(r_3-r_2)z\right]$$
$$N_2(r,z)=\frac{1}{2A}\left[(r_3z_1-r_1z_3)+(z_3-z_1)r+(r_1-r_3)z\right]$$
$$N_3(r,z)=\frac{1}{2A}\left[(r_1z_2-r_2z_1)+(z_1-z_2)r+(r_2-r_1)z\right]$$

Or,

$$\begin{aligned}N_i(r,z)&=\frac{1}{2A}(\alpha_i+r\beta_i+z\gamma_i)\\ N_j(r,z)&=\frac{1}{2A}(\alpha_j+r\beta_j+z\gamma_j)\\ N_k(r,z)&=\frac{1}{2A}(\alpha_k+r\beta_k+z\gamma_k)\end{aligned}\qquad(5.7.8)$$

Where,

$$\begin{array}{lll}\alpha_i=r_jz_k-r_kz_j & \alpha_j=r_kz_i-r_iz_k & \alpha_k=r_iz_j-r_jz_i\\ \beta_i=z_j-z_k & \beta_j=z_k-z_i & \beta_k=z_i-z_j\\ \gamma_i=r_k-r_j & \gamma_i=r_i-r_k & \gamma_i=r_j-r_i\end{array}\qquad(5.7.9)$$

$$2A=\frac{1}{2}(r_iz_j+r_jz_k+r_kz_i-r_iz_k-r_jz_i-r_kz_j)$$

Putting the value of {u,w} in eq. (5.7.7) from eq. (5.6.5),

$$\{\varepsilon\}=\begin{bmatrix}\frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial r} & 0 & 0 & 0\\ \frac{N_i}{r} & \frac{N_j}{r} & \frac{N_k}{r} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z}\\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial r}\end{bmatrix}\begin{Bmatrix}r_1\\ r_2\\ r_3\\ z_1\\ z_2\\ z_3\end{Bmatrix}=[B]\{\bar{d}\}\qquad(5.7.10)$$

Thus, the strain displacement matrix can be expressed as,

$$[B]=\frac{1}{2A}\begin{bmatrix}\beta_i & \beta_j & \beta_k & 0 & 0 & 0\\ \frac{N_i}{r} & \frac{N_j}{r} & \frac{N_k}{r} & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma_i & \gamma_j & \gamma_k\\ \gamma_i & \gamma_j & \gamma_k & \beta_i & \beta_j & \beta_k\end{bmatrix}\qquad(5.7.11)$$

Where, $r = \dfrac{r_i + r_j + r_k}{3}$. Thus the stiffness matrix will become

$$[k] = \iiint [B]^T [D][B]\, d\Omega$$

Or, $$[k] = \iint \int_0^{2\pi} [B]^T [D][B]\; r\, d\theta\, dA \; = 2\pi \iint [B]^T [D][B]\, r\, dr\, dz \qquad (5.7.12)$$

Since, the term [B] is dependent of 'r' terms; the term $[B]^T [D][B]$ cannot be taken out of integration. Yet, a reasonably accurate solution can be obtained by evaluating the [B] (denoted as [B̲]) matrix at the centroid.

Hence, $[k] = 2\pi \underline{r} [\underline{B}]^T [D][\underline{B}] \iint \; dr\, dz$

Or,

$$[k] = [\underline{B}]^T [D][\underline{B}]\, 2\pi \underline{r} A \qquad (5.7.13)$$

5.7.2 Stiffness Matrix of a Quadrilateral Element

The strain-displacement relation for axisymmetric problem derived earlier (eq.(5.6.5)) can be rewritten as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{r} \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \\ \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ u \end{Bmatrix} \qquad (5.7.14)$$

Applying chain rule of differentiation equation we get,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \\ \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ u \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \\ u \end{Bmatrix} \tag{5.7.15}$$

Hence, the strain components are calculated as

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{r} \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \\ u \end{Bmatrix}$$

Or,

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \\ u \end{Bmatrix} \tag{5.7.16}$$

With the use of interpolation function and nodal displacements, $\left(\dfrac{\partial u}{\partial \xi}, \dfrac{\partial u}{\partial \eta}, \dfrac{\partial w}{\partial \xi}, \dfrac{\partial w}{\partial \eta}\right)$ can be expressed for a four node quadrilateral element as

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} \tag{5.7.17}$$

Putting eq. (5.7.17) in eq. (5.7.16) we get,

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \\ N_1 & N_2 & N_3 & N_4 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} \tag{5.7.18}$$

Thus, the strain displacement relationship matrix $[B]$ becomes

$$[B] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \\ N_1 & N_2 & N_3 & N_4 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \tag{5.7.19}$$

For a four node quadrilateral element,

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} \Rightarrow \frac{\partial N_1}{\partial \xi} = -\frac{(1-\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = -\frac{(1-\xi)}{4}$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4} \Rightarrow \frac{\partial N_2}{\partial \xi} = \frac{(1-\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = -\frac{(1+\xi)}{4} \tag{5.7.20}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \Rightarrow \frac{\partial N_2}{\partial \xi} = \frac{(1+\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = \frac{(1+\xi)}{4}$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4} \Rightarrow \frac{\partial N_2}{\partial \xi} = -\frac{(1+\eta)}{4} \text{ and } \frac{\partial N_1}{\partial \eta} = \frac{(1-\xi)}{4}$$

Thus, the [B] matrix will become

$$[B] = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} -\frac{(1-\eta)}{4} & \frac{(1-\eta)}{4} & \frac{(1+\eta)}{4} & -\frac{(1+\eta)}{4} & 0 & 0 & 0 & 0 \\ -\frac{(1-\xi)}{4} & -\frac{(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(1-\eta)}{4} & \frac{(1-\eta)}{4} & \frac{(1+\eta)}{4} & -\frac{(1+\eta)}{4} \\ 0 & 0 & 0 & 0 & -\frac{(1-\xi)}{4} & -\frac{(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} \\ \frac{(1-\xi)(1-\eta)}{4} & \frac{(1+\xi)(1-\eta)}{4} & \frac{(1+\xi)(1+\eta)}{4} & \frac{(1-\xi)(1+\eta)}{4} & \frac{(1-\xi)(1-\eta)}{4} & \frac{(1+\xi)(1-\eta)}{4} & \frac{(1+\xi)(1+\eta)}{4} & \frac{(1-\xi)(1+\eta)}{4} \end{bmatrix}$$

(5.7.21)

The stiffness matrix for the axisymmetric element finally can be found from the following expression after numerical integration.

$$[k] = \int_{\Omega} [B]^T [D][B] \, d\Omega = \int_{-1}^{+1}\int_{-1}^{+1} [B]^T [D][B].2\pi r.|J|.d\xi d\eta \tag{5.7.22}$$

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UNIT – IV –FINITE ELEMENT ANALYSIS – SME1308

UNIT – IV

APPLICATIONS IN HEAT TRANSFER & FLUID MECHANICS

7.2.1 Governing Fluid Equations

The fluid mechanics topic covers a wide range of problems ofinterest in engineering applications. Basically fluid is a material that conforms to the shape of its container. Thus, both the liquids and gases are considered as fluid. However, the physical behaviour of liquids and gases is very different. The differences inbehaviour lead to a variety of subfields in fluid mechanics. In case of constant density of liquid, the flow is generally referredto as incompressible flow. The density of gases not constant and therefore, their flow is compressible flow.The Navier–Stokes equations are the fundamental basis of almost all fluid dynamics related problems. Any single-phase fluid flow can be defined by this expression. The general form of motion of a two dimensional viscous Newtonian fluid may be expressed as

$$\frac{1}{\rho} p_{,i} + \dot{v}_i + v_j v_{i,j} - \nu v_{i,ji} = f_i \qquad (7.2.1)$$

Here,

ν = kinematic viscosity

ρ = mass density of fluid

v_i = Velocity components

f_i = Body forces

p = fluid pressure

The suffix ,j and ,ji are the derivatives along j and j & i direction respectively. The dot represents the derivative with respect to time.Neglecting non linear convective terms, viscosity and body forces, eq. (7.2.1) can be simplified as:

$$p_{,i} + \rho \dot{v}_i = 0 \qquad (7.2.2)$$

Now, the continuity equation of the fluid is expressed by

$$\dot{p} + \rho c^2 v_{k,k} = 0 \qquad (7.2.3)$$

Here, *c* is the acoustic wave speed in fluid. In the above expression, two sets of variables, the velocity and the pressure are used to describe the behaviour of fluid. Now it is possible to combine equation (7.2.2) and (7.2.3) to obtain a single variable formulation.For the small amplitude of fluid motion, one can assume

$$v_i = \dot{u}_i \qquad (7.2.4)$$

Where u_i is the displacement component of fluid.To obtain single variable formulation, eq. (7.2.4) may be substituted into eq. (7.2.3) and one can get

$$\dot{p} + \rho c^2 \dot{u}_{k,k} = 0 \qquad (7.2.5)$$

Integrating eq. (5) w.r.t. time we have

$$p = -\rho c^2 u_{k,k} \tag{7.2.6}$$

Now differentiating the above expression w.r.t. x_i following expression will be arrived:

$$p_{,i} = -\rho c^2 u_{k,ki} \tag{7.2.7}$$

Substituting the above in to eq.(7.2.2) one can have

$$\rho \dot{v}_i - \rho c^2 u_{k,ki} = 0 \tag{7.2.8}$$

Thus, eq. (7.2.8) is expressed in terms of displacement variables only and known as displacement based equation.

Similarly, it is possible to obtain the fluid equation in terms of pressure variable only. Differentiating eq. (7.2.3) w.r.t.Time, the following expression can be obtained.

$$\ddot{p} + \rho c^2 \dot{v}_{k,k} = 0 \tag{7.2.9}$$

Again, differentiating eq. (7.2.2) w.r.t. x_i, we have

$$p_{,ii} + \rho \dot{v}_{i,i} = 0 \tag{7.2.10}$$

From eqs. (7.2.9) and (7.2.10), the pressure based single variable expression can be arrived as given below.

$$\ddot{p} - c^2 p_{,ii} = 0 \tag{7.2.11}$$

The above expression is basically the Helmholtz wave equation for a compressible fluid having acoustic speed *c*.

$$\nabla^2 p - \frac{1}{c^2}\ddot{p} = 0 \tag{7.2.12}$$

Thus, the general form of fluid equations of 2D linear steady state problems can be expressed by the Helmholtz equation.For incompressible fluid *c* becomes infinitely large. Hence for incompressible fluid, eq. (7.2.12)can be written as

$$\nabla^2 p = 0 \tag{7.2.13}$$

For the ideal, irrotational fluid flow problems, the field variables are the streamline, ϕ and potential ϕ functions which are governed by Laplace's equations

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{7.2.14}$$

Derivation of the above expressionand many other related equations can be found in details in fluid mechanics related text books.

7.2.2 Finite Element Formulation

The equation of motion of fluid can be expressed in various ways and some of those are shown in previous section. Finite element form of those expressions can be derived using various methods considering pressure, displacement, velocity, velocity potential, stream functions and their combinations. Here, displacement and pressure based formulations will be derived using finite element method.

7.2.2.1 Displacement based finite element formulation

Consider the equation (7.2.8) which can be expressed only in terms of displacement variables.

$$\rho\ddot{u}_i - \rho c^2 u_{k,ki} = 0 \tag{7.2.15}$$

Here, u is the displacement vector. Now, the weak form of the above equation will become

$$\int_\Omega w_i\left(\rho\ddot{u}_i - \rho c^2 u_{k,ki}\right) d\Omega = 0 \tag{7.2.16}$$

Performing integration by parts on the second terms, one can arrive at the following expression:

$$\begin{aligned} &\int_\Omega w_i\ \rho\ddot{u}_i\ d\Omega - \int w_i\ \rho c^2 u_{k,k} d\Gamma + \int w_{i,i}\rho c^2 u_{k,k} d\Omega = 0 \\ \text{or,} &\int_\Omega w_i\ \rho\ddot{u}_i\ d\Omega + \int_\Omega w_{i,i}\ \rho c^2 u_{k,ki} d\Omega = \int_\Gamma w_i \rho c^2 u_{k,k} d\Gamma \end{aligned} \tag{7.2.17}$$

Now from earlier relation (eq.7.2.6) we have, $p = -\rho c^2 u_{k,k}$. Thus, the above equation may be written as:

$$\int_\Omega w_i\ \rho\ddot{u}_i\ d\Omega + \int_\Omega w_{i,i}\ \rho c^2 u_{k,ki} d\Omega = -\int_\Gamma w_i p d\Gamma \tag{7.2.18}$$

In case of fluid filled rigid tank, the weighting function *wi*must satisfy the condition $w_i n_i = 0$ on its rigid boundary. Therefore, the above eq. will become

$$\int_\Omega \left(w_i \rho\ddot{u}_i + w_{i,i}\rho c^2 u_{k,k}\right) d\Omega = -\int_{\Gamma_P} w_i\ n_i\ p\Gamma \tag{7.2.19}$$

For finite element implementation of the above expression, let consider the interpolation function as *N* and $\overline{u}$ as the nodal displacement vector. Thus,

$$u = N\bar{u} \text{ and } w = N\bar{w} \tag{7.2.20}$$

Now the divergence of the displacement vector can be expressed as:

$$u_{i,i} = Lu = LN\bar{u} = B\bar{u} \tag{7.2.21}$$

Where $L = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$ is the differential operator.Thus eq. (7.2.19) may be written as

$$w^T \int_\Omega \left[N^T \rho N\ddot{\bar{u}} + B^T \rho c^2 B\bar{u}\right] d\Omega = -w^T \int_{\Gamma_p} N^T n\bar{p}\, d\Gamma \tag{7.2.22}$$

Or,

$$[M]\{\ddot{\bar{u}}\} + [K]\{\bar{u}\} = \{F\} \tag{7.2.23}$$

Where,

$$\begin{aligned} [M] &= \int_\Omega \rho [N]^T [N] d\Omega \\ [K] &= \int_\Omega \rho c^2 [B]^T [B] d\Omega \\ \{F\} &= -\int_{\Gamma_p} [N]^T n\ \{\bar{p}\}\ d\Gamma \end{aligned} \tag{7.2.24}$$

Using eq. (7.2.23), the displacements in fluid domain can be determined under external forces applying proper boundary conditions.

7.2.2.2 Pressure based finite element formulation

The Helmholtz equation (7.2.12) for a compressible fluid in two dimension can be used to determine the pressure distribution in the fluid domain using finite element technique.

$$p,ii - \frac{1}{c^2}\ddot{p} = 0 \tag{7.2.25}$$

The weak form of the above expression can be written as

$$\int_\Omega w_i \left(p,ii - \frac{1}{c^2}\ddot{p}\right) d\Omega = 0 \tag{7.2.26}$$

Now, performing integration by parts on the first term, the following expression can be obtained.

$$\int_\Gamma w_i\ p,i\ d\Gamma - \int_\Omega w_{i,i}\ p,i\ d\Omega - \int_\Omega \frac{1}{c^2} w_i\ \ddot{p} d\Omega = 0 \tag{7.2.27}$$

Thus,

$$\frac{1}{c^2}\int w_i \ddot{p} d\Omega + \int_\Omega w_{i,i} p_{,i} d\Omega = \int_\Gamma w_i\ p_{,i} d\,\Gamma \tag{7.2.28}$$

Assuming interpolation function as N and $\overline{p}$ as the nodal pressure vector, the pressure (p) at any point can be written as: $p = N\overline{p}$ and similarly, $w = N\overline{w}$. The divergence of the pressure can be expressed as: $p,i = Lp = LN\overline{p} = B\overline{p}$, where, $L = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$. Again,

$$
\begin{aligned}
& w_{i,i} = Lw = LN\overline{w} = B\overline{w} \\
& w_i\ddot{p} = [N\overline{w}]^T [N\ddot{\overline{p}}] = \overline{w}^T N^T N\ddot{\overline{p}}
\end{aligned}
\tag{7.2.29}
$$

Thus, eq. (7.2.28) will become:

$$
\frac{1}{c^2}\int_{\Omega} \overline{w}^T N^T N\ddot{\overline{p}} d\Omega + \int_{\Omega} \overline{w}^T B^T B\overline{p} d\Omega = \int_{\Gamma} w^T N^T \frac{\partial p}{\partial n} d\Gamma \tag{7.2.30}
$$

Or,

$$
[E]\{\ddot{\overline{p}}\} + [G]\{\overline{p}\} = \{B\} \tag{7.2.31}
$$

Where,

$$
\begin{aligned}
& [E] = \frac{1}{c^2}\int_{\Omega} [N]^T [N] d\Omega \\
& [G] = \int_{\Omega} [B]^T [B] d\Omega \\
& \{B\} = \int_{\Gamma} [N]^T \frac{\partial p}{\partial n} d\Gamma
\end{aligned}
\tag{7.2.32}
$$

Applying boundary conditions, eq. (7.2.31) can be solved to calculate the dynamic pressure developed in the fluid under applied accelerations on the domain.

7.2.3 Finite Element Formulation of Infinite Reservoir

Let consider an infinite reservoir adjacent to a dam like structure. In such case, if the dam is vibrated, the hydrodynamic pressure will be developed in the reservoir which can be calculated using above method. For finite element analysis,itis necessary to truncate such infinite domain at a certain distance away from structure to have a manageable computational domain. The reservoir has four sides (Fig. 7.2.1) and as a result four types of boundary conditions need to be specified.

$$
\{B\} = \{B\}_1 + \{B\}_2 + \{B\}_3 + \{B\}_4 \tag{7.2.33}
$$

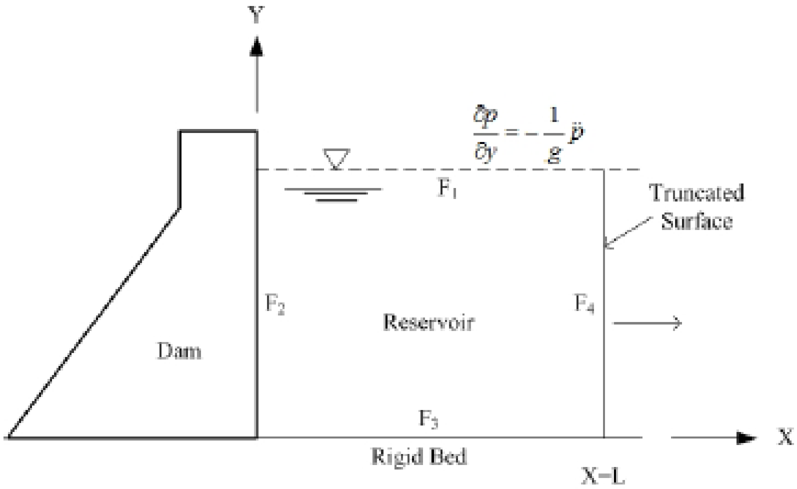

Fig. 7.2.1 Reservoir and its boundary conditions

(i) At the free surface (Γ_1)

Neglecting the effects of surface waves of the water, the boundary condition of the free surface may be expressed as

$$p(x, H, t) = 0 \qquad (7.2.34)$$

Here, *H* is the depth of the reservoir. However, sometimes, the effect of surface waves of the water needs to be considered at the free surface. This can be approximated by assuming the actual surface to be at an elevation relative to the mean surface and the following linearised free surface condition may be adopted.

$$\frac{1}{g}\ddot{p} + \frac{\partial p}{\partial y} = 0 \qquad (7.2.35)$$

Thus, the above expression may be written in finite element form as

$$\{B_1\} = \frac{\partial p}{\partial n} = -\frac{1}{g}[R_1]\{\ddot{p}\} \qquad (7.2.36)$$

In which,

$$[R_1] = \int_{\Gamma_1}[N]^T[N]d\Gamma \qquad (7.2.37)$$

(ii) At the dam-reservoir interface (Γ_2)

At the dam-reservoir interface, the pressure should satisfy

$$\frac{\partial p}{\partial n}(0, y, t) = \rho a e^{i\omega t} \qquad (7.2.38)$$

where $ae^{i\omega t}$ is the horizontal component of the ground acceleration in which, ω is the circular frequency of vibration and $i=\sqrt{-1}$, n is the outwardly directed normal to the elemental surface along the interface. In case of vertical dam-reservoir interface $\partial p/\partial n$ may be written as $\partial p/\partial x$ as both will represent normal to the element surface. For an inclined dam-reservoir interface $\partial p/\partial x$ cannot represent the normal to the element surface. Therefore, to generalize the expressions $\partial p/\partial n$ is used in eq. (7.2.38).If $\{a\}$ is the vector of nodal accelerations of generalized coordinates, $\{B_2\}$ may be expressed as

$$\{B_2\} = -\rho[R_2]\{a\} \tag{7.2.39}$$

where,

$$[R_2] = \sum \int_{\Gamma_2} [N_r]^T [T][N_d] d\Gamma \tag{7.2.40}$$

Here, $[T]$ is the transformation matrix for generalized accelerations of a point on the dam reservoir interface and $[N_d]$ is the matrix of shape functions of the dam used to interpolate the generalized acceleration at any point on their interface in terms of generalized nodal accelerations of an element.

(iii) At the reservoir bed interface (Γ_3)

At the interface between the reservoir and the elastic foundation below the reservoir, the accelerations should not be specified as rigid foundation because they depend on the interaction between the reservoir and the foundation. However, for the sake of simplicity, the reservoir bed can be assumed as rigid and following boundary condition may be adopted.

$$\{B_3\} = \frac{\partial p}{\partial \eta}(x,0,t) = 0 \tag{7.2.41}$$

(iv) At the truncation boundary (Γ_4)

The specification of the far boundary condition is one of the most important features in the finite element analysis of a semi-infinite or infinite reservoir. This is due to the fact that the developed hydrodynamic pressure, which affects the response of the structure, is dependent on the truncation boundary condition. The infinite fluid domain may truncated at a finite distance away from the structure for finite element analysis satisfying Sommerfeld radiation boundary condition. Application of Sommerfeld radiation condition at the truncation boundary leads to

$$\{B_4\} = \frac{\partial p}{\partial x}(L, y, t) = 0 \tag{7.2.42}$$

Here, L represents the distance between the structure and the truncation boundary. Thus, the hydrodynamic pressure developed on the dam-reservoir interface can be calculated under external excitation by the use of finite element technique.

2.25. HEAT TRANSFER

Heat transfer can be defined as the transmission of energy from one region to another region due to temperature difference. A knowledge of the temperature distribution within a body is important in many engineering problems. There are three modes of heat transfer.

They are: (*i*) Conduction

(*ii*) Convection

(*iii*) Radiation

(i) Conduction

Heat conduction is a mechanism of heat transfer from a region of high temperature to a region of low temperature within a medium (solid, liquid or gases) or between different medium in direct physical contact.

In conduction, energy exchange takes place by the kinematic motion or direct impact of molecules. Pure conduction is found only in solids.

(ii) Convection

Convection is a process of heat transfer that will occur between a solid surface and a fluid medium when they are at different temperatures.

Convection is possible only in the presence of fluid medium.

(iii) Radiation

The heat transfer from one body to another without any transmitting medium is known as radiation. It is an electromagnetic wave phenomenon.

2.26. DERIVATION OF TEMPERATURE FUNCTION (T) AND SHAPE FUNCTION (N) FOR ONE DIMENSIONAL HEAT CONDUCTION ELEMENT

Consider a bar element with nodes 1 and 2 as shown in Fig.2.34. T_1 and T_2 are the temperatures at the respective nodes. So, T_1 and T_2 are considered as degrees of freedom of this bar element.

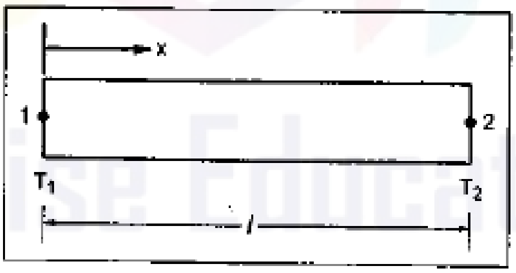

Fig. 2.34.

Since the element has got two degrees of freedom, it will have two generalized co-ordinates.

$$\Rightarrow \qquad T = a_0 + a_1 x \qquad \dots (2.121)$$

where, a_0 and a_1 are global or generalized co-ordinates.

Writing the equation (2.121) in matrix form,

$$T = [1 \;\; x]\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \qquad \dots (2.122)$$

At node 1, $\quad T = T_1, \quad x = 0$

At node 2, $\quad T = T_2, \quad x = l$

Substitute the above values in equation (2.121),

$$\Rightarrow \qquad T_1 = a_0 \qquad \dots (2.123)$$

$$\Rightarrow \qquad T_2 = a_0 + a_1 l \qquad \dots (2.124)$$

Assembling the equations (2.123) and (2.124) in matrix form,

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l-0}\begin{bmatrix} l & -0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\left[\text{Note: } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\right]$$

$$\Rightarrow \qquad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ values in equation (2.122)

$$\Rightarrow \qquad T = [1 \;\; x]\frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l}[1 \;\; x]\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l}[l - x \quad 0 + x]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$[\because$ Matrix multiplication $(1 \times 2) \times (2 \times 2) = (1 \times 2)]$

$$T = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \quad \dots (2.125)$$

$$T = [N_1 \quad N_2] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\text{Temperature function, } T = N_1 T_1 + N_2 T_2 \quad \dots (2.126)$$

$$\text{where, Shape functions, } N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

2.27. DERIVATION OF STIFFNESS MATRIX FOR ONE DIMENSIONAL HEAT CONDUCTION ELEMENT

Consider a one dimensional bar element with nodes 1 and 2 as shown in Fig.2.35. Let T_1 and T_2 be the temperatures at the respective nodes and k be the thermal conductivity of the material.

Fig. 2.35.

We know that,

$$\text{Stiffness matrix } [K] = \int_v [B]^T [D] [B] \, dv$$

In one dimensional element,

$$\text{Temperature function, } T = N_1 T_1 + N_2 T_2$$

$$\text{where, } N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

We know that,

$$\text{Strain–Displacement matrix, } [B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$\Rightarrow \qquad [B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \qquad \ldots (2.127)$$

$$\Rightarrow \qquad [B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \qquad \ldots (2.128)$$

In one dimensional heat conduction problems,

$$[D] = [K] = k = \text{Thermal conductivity of the material}$$

Substitute $[B]$, $[B]^T$ and $[D]$ values in stiffness matrix equation

$$\Rightarrow \quad \left.\begin{array}{r}\text{Stiffness matrix for} \\ \text{heat conduction}\end{array}\right\} [K_C] = \int_0^l \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \times k \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} k \ dv$$

[Matrix multiplication $(2 \times \underbrace{1) \times (1}_{} \times 2) = (2 \times 2)$]

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} k \, A \ dx \qquad [\because dv = A \times dx]$$

$$= A\,k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^l dx$$

$$= A\,k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$= A\,k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} (l - 0)$$

$$= A\,k\,l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix}$$

$$= \frac{A\,k\,l}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_C] = \frac{A\,k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \dots (2.129)$$

where, A = Area of the element, m^2

k = Thermal conductivity of the element, W/mK

l = Length of the element, m

2.28. FINITE ELEMENT EQUATIONS FOR ONE DIMENSIONAL HEAT CONDUCTION PROBLEMS

We know that,

General force equation is, $\{F\} = [K_C]\{T\}$ ... (2.130)

where, $\{F\}$ is a element force vector [Column matrix]

$[K_C]$ is a stiffness matrix [Row matrix]

$\{T\}$ is a nodal temperature [Column matrix]

For one dimensional heat conduction problems, stiffness matrix [K] is given by

$$[K_C] = \frac{A\,k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Consider a two noded element as shown in Fig.2.36.

Force vector $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$

Nodal temperature $\{T\} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$

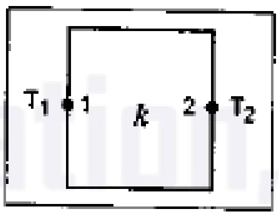

Fig. 2.36.

Substitute $[K_C]$ $\{F\}$ and $\{T\}$ values in equation (2.130).

$$\Rightarrow \quad \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A\,k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \qquad \dots (2.131)$$

Case (i): One dimensional heat conduction with free end convection

Consider a one dimensional element with nodes 1 and 2 as shown in Fig.2.37. T_1 and T_2 are the temperatures at the respective nodes. Assume convection occurs only from the right end of the element as shown in Fig.2.37.

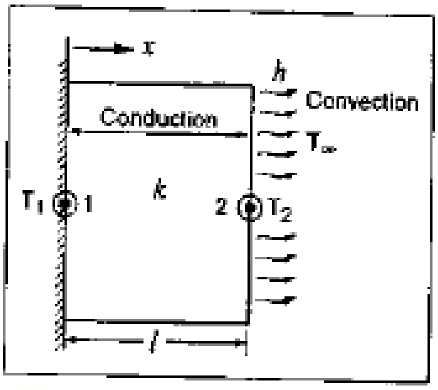

Fig. 2.37.

Stiffness matrix $[K_C]$ for one dimensional heat conduction element is given by

$$[K_C] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{[From equation no.(2.129)]}$$

The convection term contribution to the stiffness matrix is given by

$$[K_h]_{end} = \iint_A h\,[N]^T\,[N]\,dA \qquad \dots (2.132)$$

where, h = Heat transfer coefficient, W/m²K

N = Shape factor

We know that,

$$\text{Shape factor, } [N] = [N_1 \;\; N_2] = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

[From equation no.(2.125)]

At node 2, $x = l$

$$\Rightarrow \quad [N] = [N_1 \;\; N_2] = [0 \;\; 1]$$

$$\Rightarrow \quad [N]^T = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Substitute $[N]$ and $[N]^T$ values in equation (2.132).

$$\Rightarrow \quad [K_h]_{end} = \iint_A h \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} [0 \;\; 1]\,dA$$

$$[(2 \times \underbrace{1) \times (1}_{} \times 2) = (2 \times 2)]$$

$$[K_h]_{end} = h\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\int dA$$

$$[K_h]_{end} = hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \dots (2.133)$$

$$\text{Stiffness matrix } [K] = [K_C] + [K_h]$$

$$\Rightarrow \qquad [K] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{... (2.134)}$$

The convection force from the free end of the element is obtained from the following relation,

$$\{F_h\}_{end} = h T_\infty A \begin{Bmatrix} N_1 \ (x = l) \\ N_2 \ (x = l) \end{Bmatrix}$$

$$\{F_h\}_{end} = h T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad \text{... (2.135)}$$

We know that, General force equation is

$$\{F\} = [K]\{T\}$$

Substitute $\{F\}$ and $[K]$ values,

$$\Rightarrow \qquad h T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left[\frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\Rightarrow \left[\frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = h T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad \text{... (2.136)}$$

where, A = Area of the element, m^2
k = Thermal conductivity of the element, W/mK
l = Length of the element
h = Heat transfer coefficient, W/m^2K
T_∞ = Fluid temperature, K
T = Temperature, K

This is a finite element equation for one dimensional heat conduction element with free end convection.

Case (ii): One dimensional element with conduction, convection and internal heat generation:

Consider a rod with nodes 1 and 2 as shown in Fig.2.38. This rod is subjected to conduction, convection and internal heat generation.

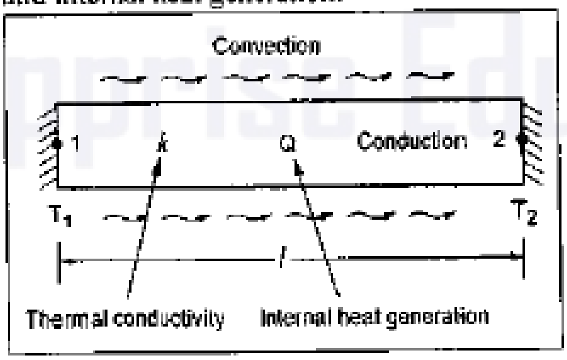

Fig. 2.38.

We know that, heat conduction part of the stiffness matrix [K] for the one dimensional element is

$$[\mathrm{K_C}] = \frac{\mathrm{A}k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{[From equation no.(2.129)]}$$

Heat convection part of the stiffness matrix [K] for the one dimensional element is given by

$$[\mathrm{K}_h] = \iint\limits_{S} h\,[\mathrm{N}]^{\mathrm{T}}\,[\mathrm{N}]\,d\mathrm{S}$$

$$= h\mathrm{P}\int_0^l [\mathrm{N}]^{\mathrm{T}}\,[\mathrm{N}]\,dx$$

$$[\because d\mathrm{S} = \mathrm{P}\times dx \quad \text{where } \mathrm{P} = \text{Perimeter of the element}]$$

$$= h\mathrm{P}\int_0^l \begin{Bmatrix} \dfrac{l-x}{l} \\[2mm] \dfrac{x}{l} \end{Bmatrix}\begin{bmatrix} \dfrac{l-x}{l} & \dfrac{x}{l} \end{bmatrix} dx \qquad \text{[From equation no.(2.125)]}$$

$$= h\mathrm{P}\int_0^l \begin{bmatrix} \left(\dfrac{l-x}{l}\right)^2 & \left(\dfrac{l-x}{l}\right)\times\dfrac{x}{l} \\[3mm] \dfrac{x}{l}\times\left(\dfrac{l-x}{l}\right) & \dfrac{x}{l}\times\dfrac{x}{l} \end{bmatrix} dx$$

$$= h\mathrm{P}\int_0^l \begin{bmatrix} \left(1-\dfrac{x}{l}\right)^2 & \dfrac{x}{l}-\dfrac{x^2}{l^2} \\[3mm] \dfrac{x}{l}-\dfrac{x^2}{l^2} & \dfrac{x^2}{l^2} \end{bmatrix} dx$$

$$= h\mathrm{P}\begin{bmatrix} \dfrac{\left(1-\dfrac{x}{l}\right)^3}{3\times\left(\dfrac{-1}{l}\right)} & \dfrac{x^2}{2l}-\dfrac{x^3}{3l^2} \\[5mm] \dfrac{x^2}{2l}-\dfrac{x^3}{3l^2} & \dfrac{x^3}{3l^2} \end{bmatrix}_0^l$$

$$= h\,P \begin{bmatrix} \dfrac{\left(1-\frac{l}{l}\right)^3}{\frac{-3}{l}} - \dfrac{l^3}{\frac{-3}{l}} & \dfrac{l^2}{2\,l} - \dfrac{l^3}{3\,l^2} - 0 \\ \dfrac{l^2}{2\,l} - \dfrac{l^3}{3\,l^2} - 0 & \dfrac{l^3}{3\,l^2} - 0 \end{bmatrix}$$

$$= h\,P \begin{bmatrix} \dfrac{l}{3} & \dfrac{l}{2} - \dfrac{l}{3} \\ \dfrac{l}{2} - \dfrac{l}{3} & \dfrac{l}{3} \end{bmatrix}$$

$$= h\,P \begin{bmatrix} \dfrac{l}{3} & \dfrac{l}{6} \\ \dfrac{l}{6} & \dfrac{l}{3} \end{bmatrix}$$

$$[\,K_h\,] = \frac{h\,P\,l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \ldots (2.137)$$

Stiffness matrix $[\,K\,] = [\,K_C\,] + [\,K_h\,]$

$$\Rightarrow \qquad [\,K\,] = \frac{A\,k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \ldots (2.138)$$

Force matrix due to heat generation is given by,

$$\{\,F_Q\,\} = \iiint_V [\,N\,]^T\; Q\; dV$$

$$= \int_0^l [\,N\,]^T \times Q \times A \times dx \qquad [\because dV = A \times dx]$$

$$= Q \times A \int_0^l [\,N\,]^T\; dx$$

$$= Q \times A \int_0^l \begin{Bmatrix} \dfrac{l-x}{l} \\ \dfrac{x}{l} \end{Bmatrix} dx$$

$$= Q \times A \int_0^l \begin{Bmatrix} 1 - \frac{x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= Q \times A \begin{Bmatrix} x - \frac{x^2}{2l} \\ \frac{x^2}{2l} \end{Bmatrix}_0^l$$

$$= Q \times A \begin{Bmatrix} l - \frac{l^2}{2l} - 0 \\ \frac{l^2}{2l} - 0 \end{Bmatrix}$$

$$= Q \times A \begin{Bmatrix} \frac{l^2}{2l} \\ \frac{l^2}{2l} \end{Bmatrix} = Q \times A \begin{Bmatrix} \frac{l}{2} \\ \frac{l}{2} \end{Bmatrix}$$

$$\{F_Q\} = Q \times A \times \frac{l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \ldots (2.139)$$

Force matrix due to convection is given by

$$\{F_h\} = \iint_S h\, T_\infty\, [N]^T\, dS$$

$$= \iint_S h\, T_\infty\, [N]^T\, P \times dx \qquad [\because dS = P \times dx]$$

$$= P\, h\, T_\infty \int_0^l [N]^T\, dx$$

$$= P\, h\, T_\infty \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= P\, h\, T_\infty \begin{Bmatrix} x - \frac{x^2}{2l} \\ \frac{x^2}{2l} \end{Bmatrix}_0^l$$

$$= \mathrm{P}\,h\,\mathrm{T}_\infty \begin{Bmatrix} l - \dfrac{l^2}{2l} - 0 \\ \dfrac{l^2}{2l} - 0 \end{Bmatrix}$$

$$= \mathrm{P}\,h\,\mathrm{T}_\infty \begin{Bmatrix} \dfrac{l}{2} \\ \dfrac{l}{2} \end{Bmatrix}$$

$$\{\mathrm{F}_h\} = \frac{\mathrm{P}\,h\,\mathrm{T}_\infty\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \ldots (2.140)$$

Adding equations (2.139) and (2.140),

$$\text{Force matrix, } \{\mathrm{F}\} = \{\mathrm{F_Q}\} + \{\mathrm{F}_h\}$$

$$= \frac{\mathrm{Q\,A}\,l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{\mathrm{P}\,h\,\mathrm{T}_\infty\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{\mathrm{F}\} = \frac{\mathrm{Q\,A}\,l + \mathrm{P}\,h\,\mathrm{T}_\infty\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \ldots (2.141)$$

We know that, General force equation is

$$\{\mathrm{F}\} = [\mathrm{K}]\{\mathrm{T}\}$$

Substitute { F } and [K] values,

$$\Rightarrow \quad \frac{\mathrm{Q\,A}\,l + \mathrm{P}\,h\,\mathrm{T}_\infty\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \left[\frac{\mathrm{A}\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,\mathrm{P}\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right] \begin{Bmatrix} \mathrm{T}_1 \\ \mathrm{T}_2 \end{Bmatrix}$$

$$\Rightarrow \quad \left[\frac{\mathrm{A}\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,\mathrm{P}\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right] \begin{Bmatrix} \mathrm{T}_1 \\ \mathrm{T}_2 \end{Bmatrix} = \frac{\mathrm{Q\,A}\,l + \mathrm{P}\,h\,\mathrm{T}_\infty\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \ldots (2.142)$$

where,
- A = Area of the element, m^2
- k = Thermal conductivity of the element, W/mK
- l = Length of the element, m
- h = Heat transfer coefficient, $\mathrm{W/m^2K}$
- P = Perimeter, m
- T = Temperature, K
- Q = Heat generation, W
- T_∞ = Fluid temperature, K

This is a finite element equation for one dimensional element which is subjected to conduction, convection and internal heat generation.

3.15. HEAT TRANSFER IN 2-DIMENSION – (THERMAL PROBLEMS)

3.15.1. Shape Function Derivation for Heat Transfer in 2D Element

We begin this section with the development of the shape function for a basic two dimensional triangular element.

We consider this triangular element because its derivation is the simplest among the available two dimensional elements.

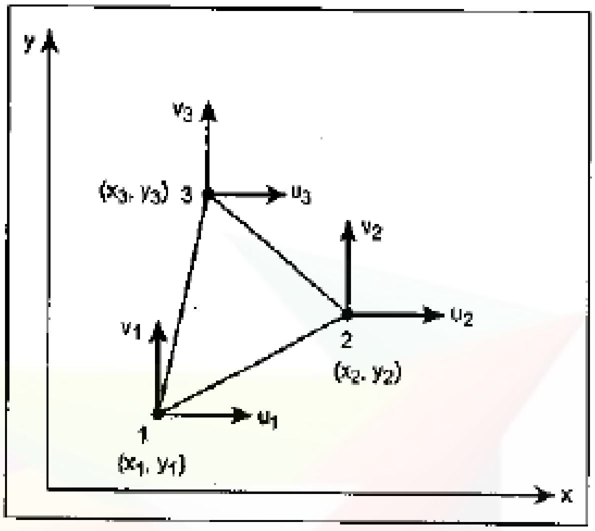

Fig. 3.16. Three noded triangular element

Consider a typical triangular element with nodes 1, 2 and 3 as shown in Fig.3.16. Let the nodal displacements be u_1, u_2, u_3, v_1, v_2 and v_3.

$$\text{Displacement } \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Since the triangular element has got two degrees of freedom at each node (u, v), the total degrees of freedom is 6. Hence it has 6 generalized coordinates.

Let,
$$u = \alpha_1 + \alpha_2 x + \alpha_3 y \qquad \text{... (3.51)}$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \qquad \text{... (3.52)}$$

where, α_1, α_2, α_3, α_4, α_5 and α_6 are global or generalized co-ordinates.

$\Rightarrow$
$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

Write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{Let } D = \begin{bmatrix} \overset{+}{1} & \overset{-}{x_1} & \overset{+}{y_1} \\ \overset{-}{1} & \overset{+}{x_2} & \overset{-}{y_2} \\ \overset{+}{1} & \overset{-}{x_3} & \overset{+}{y_3} \end{bmatrix}$$

We know, $\quad D^{-1} = \dfrac{C^T}{|D|}$

Find the co-factors of matrix D.

$$c_{11} = + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2)$$

$$c_{12} = - \begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} = -(y_3 - y_2) = y_2 - y_3$$

$$c_{13} = + \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = (x_3 - x_2)$$

$$c_{21} = - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -(x_1 y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3$$

$$c_{22} = + \begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$c_{23} = - \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} = -(x_3 - x_1) = x_1 - x_3$$

$$c_{31} = + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$c_{32} = - \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} = -(y_2 - y_1) = y_1 - y_2$$

$$c_{33} = + \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$\Rightarrow \qquad C = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{vmatrix}$$

$$\Rightarrow \qquad C^T = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{vmatrix} \qquad \text{... (3.55)}$$

We know that, $D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$|D| = 1\,(x_2 y_3 - x_3 y_2) - x_1\,(y_3 - y_2) + y_1\,(x_3 - x_2) \qquad \text{... (3.56)}$$

Substitute C^T and D values in equation (3.54),

$$\Rightarrow \qquad D^{-1} = \frac{1}{(x_2 y_3 - x_3 y_2) - x_1\,(y_3 - y_2) + y_1\,(x_3 - x_2)} \times \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

Substitute D^{-1} value in equation (3.53),

$$\Rightarrow \qquad \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{(x_2 y_3 - x_3 y_2) - x_1\,(y_3 - y_2) + y_1\,(x_3 - x_2)} \times \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{... (3.57)}$$

The area of the triangle can be expressed as a function of the x, y co-ordinates of the nodes 1, 2 and 3.

$$\Rightarrow \qquad A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$|A| = \frac{1}{2}\left[1\,(x_2 y_3 - x_3 y_2) - x_1\,(y_3 - y_2) + y_1\,(x_3 - x_2)\right]$$

$$\Rightarrow \qquad 2A = (x_2 y_3 - x_3 y_2) - x_1\,(y_3 - y_2) + y_1\,(x_3 - x_2) \qquad \ldots (3.58)$$

Substitute 2A values in equation (3.57),

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots (3.59)$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots (3.60)$$

where, $a_1 = x_2 y_3 - x_3 y_2$; $\quad a_2 = x_3 y_1 - x_1 y_3$; $\quad a_3 = x_1 y_2 - x_2 y_1$

$b_1 = y_2 - y_3$; $\quad b_2 = y_3 - y_1$; $\quad b_3 = y_1 - y_2$

$c_1 = x_3 - x_2$; $\quad c_2 = x_1 - x_3$; $\quad c_3 = x_2 - x_1$

From equation (3.51), we know that,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

We can write this equation in matrix form,

$$u = [1 \;\; x \;\; y]\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$ value, from equation no.(3.60)

$$\Rightarrow u = [1 \;\; x \;\; y] \times \frac{1}{2A}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A}[1 \;\; x \;\; y]\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A}[a_1 + b_1 x + c_1 y \quad a_2 + b_2 x + c_2 y \quad a_3 + b_3 x + c_3 y] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[\because (1 \times 3) \times (3 \times 3) = 1 \times 3]$$

$$u = \left[\frac{a_1 + b_1 x + c_1 y}{2A} \quad \frac{a_2 + b_2 x + c_2 y}{2A} \quad \frac{a_3 + b_3 x + c_3 y}{2A}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The above equation is in the form of

$$u = [N_1 \;\; N_2 \;\; N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \dots (3.61)$$

$$\text{Similarly, } v = [N_1 \;\; N_2 \;\; N_3]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \qquad \dots (3.62)$$

where, Shape function, $N_1 = \dfrac{a_1 + b_1 x + c_1 y}{2A}$

$$N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$$

$$N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$$

Assembling the equations (3.61) and (3.62) in matrix form,

$$\text{Displacement function, } u = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad \dots (3.63)$$

3.15.2. Stiffness Matrix and Load Vector for Heat Transfer in Two-dimensional Element

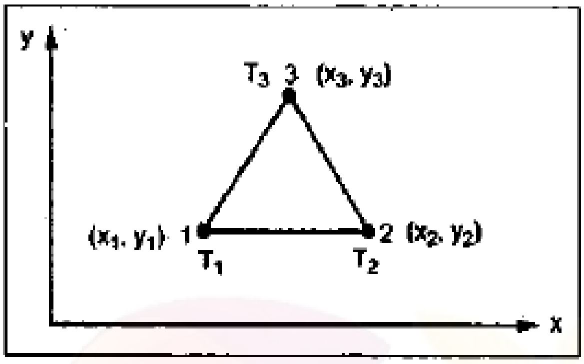

Fig. 3.17. Triangular element with nodal temperature

Triangular element is the basic element for solution of two-dimensional heat transfer problems. Consider the three-noded triangular element with nodal temperatures T_1, T_2 and T_3 as shown in Fig.3.17.

The temperature function is given by,

$$T(x, y) = N_1 T_1 + N_2 T_2 + N_3 T_3 \quad \ldots (3.64)$$

We know that,

$$\left.\begin{aligned} \text{Shape functions, } N_1 &= \frac{a_1 + b_1 x + c_1 y}{2A} \\ N_2 &= \frac{a_2 + b_2 x + c_2 y}{2A} \\ \text{and } N_3 &= \frac{a_3 + b_3 x + c_3 y}{2A} \end{aligned}\right\} \quad \ldots (3.65)$$

We know that,

$$\text{Stiffness matrix } [K_C] = \int [B]^T [D] [B] \, dv \quad \ldots (3.66)$$

We know that

$$\text{Strain – Displacement matrix, } [B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix} \quad \ldots (3.67)$$

By partial differentiation,

$$\Rightarrow \left.\begin{aligned} \frac{\partial N_1}{\partial x} &= \frac{b_1}{2A}, & \frac{\partial N_1}{\partial y} &= \frac{c_1}{2A}, \\ \frac{\partial N_2}{\partial x} &= \frac{b_2}{2A}, & \frac{\partial N_2}{\partial y} &= \frac{c_2}{2A} \\ \frac{\partial N_3}{\partial x} &= \frac{b_3}{2A}, & \frac{\partial N_3}{\partial y} &= \frac{c_3}{2A} \end{aligned}\right\} \quad \ldots (3.68)$$

Substitute the equation (3.68) in equation (3.67)

$$[B] = \begin{bmatrix} \frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} \\ \frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} \end{bmatrix} \quad \ldots\ldots (3.69)$$

$$[B] = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \ldots\ldots (3.70)$$

$$[B]^T = \frac{1}{2A}\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} \quad \ldots (3.71)$$

We know that,

$$\text{Stress-strain matrix, } [D] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \quad \ldots (3.72)$$

Assuming a unit thickness, the elemental volume can be expressed as $dv = dA$

...(3.73)

Substitute the (3.70), (3.71), (3.72), (3.73) in equation (3.66)

$$\Rightarrow [K_C] = \int \frac{1}{2A}\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \times \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} dA$$

$$= \frac{1}{4A^2} \int \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} dA$$

$$= \frac{1}{4A^2} \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \int dA$$

$$= \frac{1}{4A} \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$= \frac{1}{4A} \begin{bmatrix} b_1 k_x + 0 & 0 + c_1 k_y \\ b_2 k_x + 0 & 0 + c_2 k_y \\ b_3 k_x + 0 & 0 + c_3 k_y \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$[K_C] = \frac{1}{4A} \begin{bmatrix} (b_1^2\ k_x + c_1^2\ k_y) & (b_1 b_2 k_x + c_1 c_2 k_y) & (b_1 b_3 k_x + c_1 c_3 k_y) \\ (b_1 b_2 k_x + c_1 c_2 k_y) & (b_2^2\ k_x + c_2^2\ k_y) & (b_2 b_3 k_x + c_2 c_3 k_y) \\ (b_1 b_3 k_x + c_1 c_3 k_y) & (b_2 b_3 k_x + c_2 c_3 k_y) & (b_3^2\ k_x + c_3^2\ k_y) \end{bmatrix}$$

For an isotropic material with $k_x = k_y = k$,

Stiffness matrix for conduction,

$$\Rightarrow \quad [K_C] = \frac{k}{4A} \begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix} \quad \ldots (3.74)$$

To determine the stiffness matrix for convection,

$$[K_h] = \int_S h\,[N]^T [N]\, ds \quad \ldots (3.75)$$

$$= h \int \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} [N_1 \ \ N_2 \ \ N_3]\, ds$$

$$= h \int \begin{bmatrix} N_1^2 & N_1 N_2 & N_1 N_3 \\ N_1 N_2 & N_2^2 & N_2 N_3 \\ N_1 N_3 & N_2 N_3 & N_3^2 \end{bmatrix} ds \quad \ldots (3.76)$$

Let the edge 1-2 of element lies on the boundary as shown in Fig.3.18. So that $N_3 = 0$ along this edge.

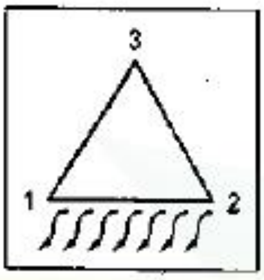

Fig. 3.18. Heat loss by convection from sides 1-2

Substitute $N_3 = 0$ in equation (3.76),

$$[K_h] = h \int \begin{bmatrix} N_1^2 & N_1 N_2 & 0 \\ N_1 N_2 & N_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} ds$$

Substitute $N_1 = L_1$, $N_2 = L_2$ and $N_3 = L_3$, along the edge 1–2, $N_3 = L_3 = 0$.

Hence, $$\Rightarrow [K_h] = h \int_{s_1}^{s_2} \begin{bmatrix} L_1^2 & L_1 L_2 & 0 \\ L_1 L_2 & L_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} ds \qquad \dots (3.77)$$

Where, s-denotes the direction along the edge 1–2.

We know that $$\int L_1^{\alpha} L_2^{\beta}\, ds = \frac{\alpha!\,\beta!}{(\alpha + \beta + 1)!}\, s$$

Therefore, $$\int L_1^2\, ds = \frac{2!}{(2+1)!}\, s$$

$$= \frac{1 \times 2}{1 \times 2 \times 3}\, s$$

$$\int L_1^2\, ds = \frac{s}{3} \qquad \dots (3.78)$$

Similarly, $$\int L_1 L_2\, ds = \frac{1!\,1!}{(1+1+1)!}\, s$$

$$= \frac{1}{1 \times 2 \times 3}\, s$$

$$\int L_1 L_2\, ds = \frac{s}{6} \qquad \dots (3.79)$$

Similarly, $$\int L_2^2 \, ds = \frac{2\,!}{(2+1)!} \, s$$

$$= \frac{1 \times 2}{1 \times 2 \times 3} \, s$$

$$\int L_2^2 \, ds = \frac{s}{3} \qquad \ldots (3.80)$$

Substitute the equation (3.78), (3.79) and (3.80) in equation (3.77),

$$\Rightarrow [K_h]_{1-2} = h_{1-2} \begin{bmatrix} \frac{s_{1-2}}{3} & \frac{s_{1-2}}{6} & 0 \\ \frac{s_{1-2}}{6} & \frac{s_{1-2}}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[Direction along the edge (1 – 2)]

$$[K_h]_{1-2} = \frac{h_{1-2}\, s_{1-2}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \ldots (3.81)$$

Now, consider the edge 2 – 3 of element lies on the boundary.

Hence, $N_1 = L_1 = 0$, $N_2 = L_2$, $N_3 = L_3$.

Substitute the N_1, N_2 and N_3 values in equation (3.76), we get

$$[K_h]_{2-3} = h \int_{s_2}^{s_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & L_2^2 & L_2 L_3 \\ 0 & L_2 L_3 & L_3^2 \end{bmatrix} ds \qquad \ldots (3.82)$$

Substitute the equation (3.78), (3.79) and (3.80) in equation (3.82), we get

$$[K_h]_{2-3} = h_{2-3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{s_{2-3}}{3} & \frac{s_{2-3}}{6} \\ 0 & \frac{s_{2-3}}{6} & \frac{s_{2-3}}{3} \end{bmatrix}$$

[Direction along the edge (2 – 3)]

$$= \frac{h_{2-3}\, s_{2-3}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \ldots (3.83)$$

Similarly, let the edge 3 – 1 of elements lies on the boundary.

Hence, $N_1 = L_1, \quad N_2 = L_2 = 0, \quad N_3 = L_3$

Substitute the N_1, N_2 and N_3 values in equation (3.76), we get

$$[K_h]_{3-1} = h \int_{s_3}^{s_1} \begin{bmatrix} L_1^2 & 0 & L_1 L_3 \\ 0 & 0 & 0 \\ L_1 L_3 & 0 & L_3^2 \end{bmatrix} ds \qquad \ldots (3.84)$$

Substitute the equation (3.78), (3.79) and (3.80) in equation (3.84), we get

$$[K_h]_{3-1} = h_{3-1} \begin{bmatrix} \frac{s_{3-1}}{3} & 0 & \frac{s_{3-1}}{6} \\ 0 & 0 & 0 \\ \frac{s_{3-1}}{6} & 0 & \frac{s_{3-1}}{3} \end{bmatrix}$$

$$[K_h]_{3-1} = \frac{h_{3-1}\, s_{3-1}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad \ldots (3.85)$$

Stiffness matrix for convection,

$$[K_h] = [K_h]_{1-2} + [K_h]_{2-3} + [K_h]_{3-1}$$

$$[K_h] = \frac{h_{1-2}\, s_{1-2}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{h_{2-3}\, s_{2-3}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \frac{h_{3-1}\, s_{3-1}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad \ldots (3.86)$$

Stiffness matrix for 2-dimensional heat transfer element is given by,

$$[K] = [K_C] + [K_h]$$

$$[K] = \frac{k}{4A} \begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix} +$$

$$\frac{h_{1-2}\, s_{1-2}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{h_{2-3}\, s_{2-3}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} +$$

$$\frac{h_{3-1}\, s_{3-1}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad \ldots (3.87)$$

3.15.2.1. Force Vector or Load Vector, (F)

The force vector for 2-dimensional heat transfer element is given by

$$[F_1] = \int q_0 [N]^T dv = q_0 \int \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} dA \qquad \text{... (3.88)}$$

$$[F_1] = q_0 \int \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} dA$$

$$= q_0 \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} dA \qquad [\because \ dv = dA \text{ (unit thickness)}] \qquad \text{... (3.89)}$$

By using area co-ordinates system,

$$\int L_1^{\alpha} L_2^{\beta} L_3^{\gamma} \, dA = \frac{\alpha! \, \beta! \, \gamma!}{(\alpha+\beta+\gamma+2)!} \times 2A$$

We know that, $\int L_1 \, dA = \frac{1!}{(1+2)!} \times 2A = \frac{1}{1 \times 2 \times 3} \times 2A$

$$\int L_1 \, dA = \frac{A}{3} \qquad \text{... (3.90)}$$

Similarly, $\int L_2 \, dA = \frac{1!}{(1+2)!} \times 2A = \frac{1}{1 \times 2 \times 3} \times 2A$

$$\int L_2 \, dA = \frac{A}{3} \qquad \text{... (3.91)}$$

Similarly, $\int L_3 \, dA = \frac{1!}{(1+2)!} \times 2A = \frac{1}{1 \times 2 \times 3} \times 2A$

$$\int L_3 \, dA = \frac{A}{3} \qquad \text{... (3.92)}$$

Substitute the equations (3.90), (3.91) and (3.92) values in equation (3.89),

$$\{F_1\} = \frac{q_0 A}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \text{... (3.93)}$$

Similarly, $\{F_2\} = \int_{s_2} q \, [N]^T \, ds$ ... (3.94)

$$= \int q \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} ds \quad \text{... (3.95)}$$

If the edge 1 – 2 lies on s_2, substitute $N_1 = L_1$, $N_2 = L_2$ and $N_3 = L_3$ along the edge 1 – 2, $N_3 = L_3 = 0$.

$$\{ F_2 \} = q \int_{s_1}^{s_2} \begin{Bmatrix} L_1 \\ L_2 \\ 0 \end{Bmatrix} ds \quad \text{... (3.96)}$$

By using surface edges,

$$\int L_1^{\alpha} L_2^{\beta} \, ds = \frac{\alpha! \, \beta!}{(\alpha + \beta + 1)!} s$$

We know that,

$$\int L_1 \, ds = \frac{1!}{(1+1)!} s = \frac{1}{2!} s = \frac{s}{2} \quad \text{... (3.97)}$$

Similarly,

$$\int L_2 \, ds = \frac{1!}{(1+1)!} s = \frac{s}{2} \quad \text{... (3.98)}$$

Substitute the equation (3.97) and (3.98) in equation (3.96),

$$\{ F_2 \} = \frac{q_{1-2} \, s_{1-2}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad \text{... (3.99)}$$

Similarly, the vector $\{ F_3 \}$ can be obtained as,

$$\{ F_3 \} = \int h \, T_{\infty} [N]^T \, ds \quad \text{... (3.100)}$$

If the edges 1 – 2 lies on s_3, substitute $N_1 = L_1$, $N_2 = L_2$, $N_3 = L_3$ along the edge 1 – 2,

$$N_3 = L_3 = 0$$

$$\{ F_3 \} = h \, T_{\infty} \int \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} ds \quad \text{... (3.101)}$$

$$= h \, T_{\infty} \int \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} ds$$

$$= h \, T_{\infty} \int \begin{Bmatrix} L_1 \\ L_2 \\ 0 \end{Bmatrix} ds \quad \text{... (3.102)}$$

Substitute the equation (3.97) and (3.98) in equation (3.102),

$$\{F_3\} = \frac{h_{1-2}\, T_\infty\, s_{1-2}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT – V –FINITE ELEMENT ANALYSIS – SME1308

UNIT – V

SPECIAL TOPICS

2.30.2. Fundamentals of Vibration

- ✓ *Any motion which repeats itself after an interval of time is called* ***vibration*** *or* ***oscillation*** *or* ***periodic motion.***
- ✓ All bodies possessing mass and elasticity are capable of producing vibrations.
- ✓ Vibration problems, in practice, occur wherever there are rotating or moving parts in a machinery. The study of vibration is concerned with oscillatory motions of the bodies and the forces associated with them.
- ✓ ***Illustration:*** Consider a spring-mass system constrained to move in a rectilinear manner along the axis of the spring, as shown in Fig.2.39.
  - When the mass is displaced from its equilibrium position A, the internal forces in the form of elastic or strain energy are present in the body; and hence the mass reaches position B.
  - At release, these forces bring the mass to its original position. At the equilibrium position A, the whole of the elastic or strain energy is converted into kinetic energy due to which the mass continues to move in the opposite direction to position C.

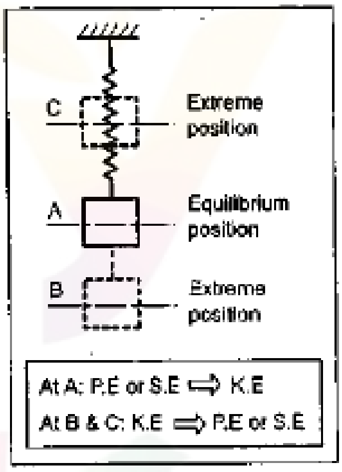


Fig. 2.39. Vibration of a spring-mass system

  - At C, the whole of the kinetic energy is again converted into elastic or strain energy due to which the body again returns to the equilibrium position A.
  - In this way, vibratory motion is repeated indefinitely and exchange of energy takes place.
- ✓ Similarly, the swinging of simple pendulum is an another example of vibration as the motion of ball is to and fro from its mean position repeatedly.

2.30.3. Causes of Vibrations

The main causes of vibration are as follows:

1. ***Unbalanced forces*** in the machine. These forces are produced from within the machine itself.
2. ***Elastic nature*** of the system.
3. ***Self excitations*** produced by the dry friction between the two mating surfaces.

4. ***External excitations*** applied on the system.
5. ***Winds*** may cause the vibrations in certain systems such as transmission and telephone lines under certain conditions.
6. ***Earthquakes*** also cause vibrations and are greatly responsible for the failure of dams, many buildings, *etc.*

2.30.4. Effects of Vibrations

(i) Negative effects: The existence of vibrating elements in any mechanical system produces unwanted noise, high stresses, wear, poor reliability and premature failure of one or more of the parts. In addition to this, vibrations are a great source of human discomfort in the form of physical and mental strains.

(ii) Positive effects: Inspite of the harmful effects, the vibratory systems are built into the machines. Examples are almost all musical instruments, vibrating conveyors, vibrating screens, shakers, stress relievers, *etc.*

2.30.5. Methods of Elimination/Reduction of the Undesirable Vibrations

The undesirable vibrations can be eliminated or reduced by one or more of the following methods.

- By removing the causes of vibration.
- By resting the machinery on proper type of isolators.
- By using shock absorbers.
- By using dynamic vibration absorbers.
- By using the screens (if noise is the objection).

2.30.6. Terminology Used in Vibratory Motion

The terms commonly used in the study of vibrations are presented in Table 2.2.

Table 2.2. Terms used in vibratory motion

1.	***Periodic motion:*** A motion which repeats itself after equal interval of time.
2.	***Time period (t_p):*** It is the time taken by a motion to repeat itself. It is also called as ***period of vibration***, and is measured in seconds.
3.	***Cycle:*** It is the motion completed during one time period.
4.	***Frequency (f):*** It is the number of cycles completed in one second. It is expressed in hertz (Hz). It is a reciprocal of time period. Mathematically, $f = \frac{1}{t_p}$ Hz.
5.	***Natural frequency:*** Frequency of free vibration of the system.
6.	***Amplitude (X):*** The maximum displacement of a vibrating body from the mean position.
7.	***Resonance:*** When the frequency of the external force is equal to the natural frequency of a vibrating body, the amplitude of vibration becomes excessively large. This phenomenon is known as resonance.
8.	***Damping:*** It is the resistance to the motion of a vibrating body.

2.30.7. A Note on Simple Harmonic Motion

- ✓ Since most of the vibrating systems follow simple harmonic motion (SHM), therefore it is essential to have proper understanding of SHM related basic concepts.
- ✓ *A body is said to have simple harmonic motion (SHM), if it moves or vibrates about a mean position such that its acceleration is always proportional to its distance from the mean position and is directed towards the mean position or equilibrium position.*

Differential Equation of SHM

Consider a particle 'P' moving around a circle with a uniform angular velocity ω rad/s as shown in Fig.2.40.

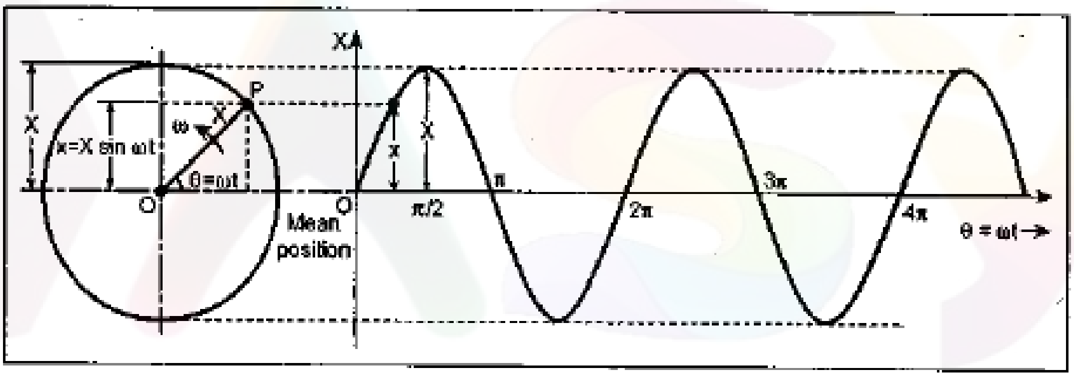


Fig. 2.40. Simple harmonic motion of a particle moving around a circle

Displacement of particle 'P' from mean position after time 't', as shown in Fig.2.40, is given by

$$x = X \sin \omega t$$

where X = Maximum displacement (or amplitude) of particle from mean position.

Velocity of particle after time 't' is given by

$$v = \frac{dx}{dt} = \omega X \cos \omega t$$

Acceleration of particle after time 't' is

$$a = \frac{d^2x}{dt^2} = -\omega^2 X \sin \omega t = -\omega^2 x \qquad [\because x = X \sin \omega t]$$

or

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \qquad \text{... (2.143)}$$

The above equation is known as ***differential equation*** or ***fundamental equation of S.H.M.***

Time period and frequency:

$$\boxed{\text{Time period, } t_p = \frac{2\pi}{\omega}} \quad \text{... (2.144)}$$

and

$$\boxed{\text{Frequency, } f = \frac{1}{t_p} = \frac{\omega}{2\pi}} \quad \text{... (2.145)}$$

2.30.8. Types of Vibrations

Vibrations may be classified according to:

(*a*) the actuating force on the body, and

(*b*) the stresses in the supporting medium, as shown in Fig.2.41.

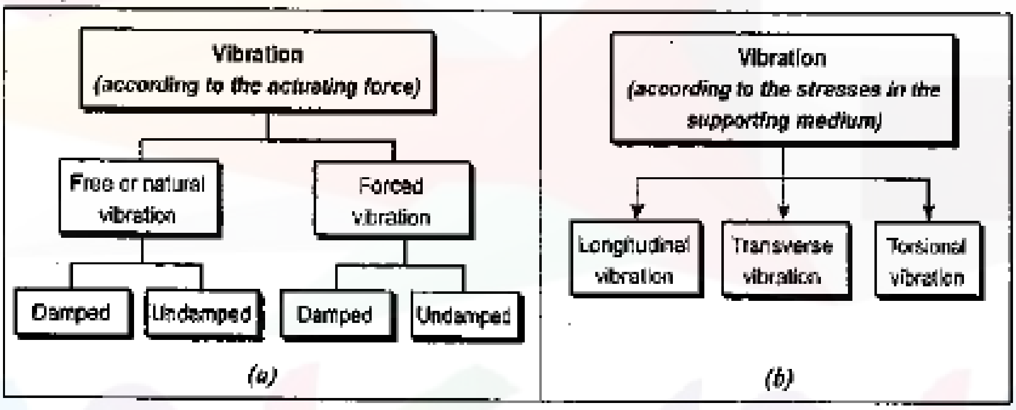


Fig. 2.41. Types of vibrations

I. According to the Actuating Force

1. Free or Natural Vibrations

✓ If the periodic motion continues after the cause of original disturbance (*i.e.*, initial displacement) is removed, then the body is said to be under ***free or natural vibrations.***

✓ The frequency of the free vibrations is called ***free or natural frequency.***

✓ *Example:* Oscillation of a simple pendulum

2. Forced Vibrations

✓ When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations.***

✓ The vibrations have the same frequency as the applied force.

✓ *Examples:* Vibrations in machine tools, electric bells, vibratory conveyors, *etc.*

3. Damped Vibrations

✓ When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration.***

- ✓ That is, if the vibratory system has a damper, the motion of the system will be opposed by it and the energy of the system will be dissipated in friction.
- ✓ On the contrary, the system having no damper is known as ***undamped vibration.***
    - ■ If the damper is connected with free vibrating body to control vibrations, then it is called ***free damped vibrations.***
    - ■ If the damper is connected with forced vibrating body to control vibrations, then it is called ***forced damped vibrations.***
- ✓ *Examples:* Vibrations in all machinery in actual use are damped in nature.

4. Undamped vibrations

- ✓ If no energy is lost or dissipated in friction or other resisting force during vibration, then such vibration is known as ***undamped vibration.***
- ✓ In other words, the system having no damper produces undamped vibrations.
- ✓ In the vibratory system, if the amount of external excitation is known in magnitude, it causes ***deterministic vibration.***

II. According to Motion of System with Respect to Axis

Consider a vibrating body, *e.g.*, a rod, shaft or spring. Fig.2.42 shows a heavy disc carried on one end of a weightless shaft, the other end being fixed. This system can execute any one of the following types of vibrations.

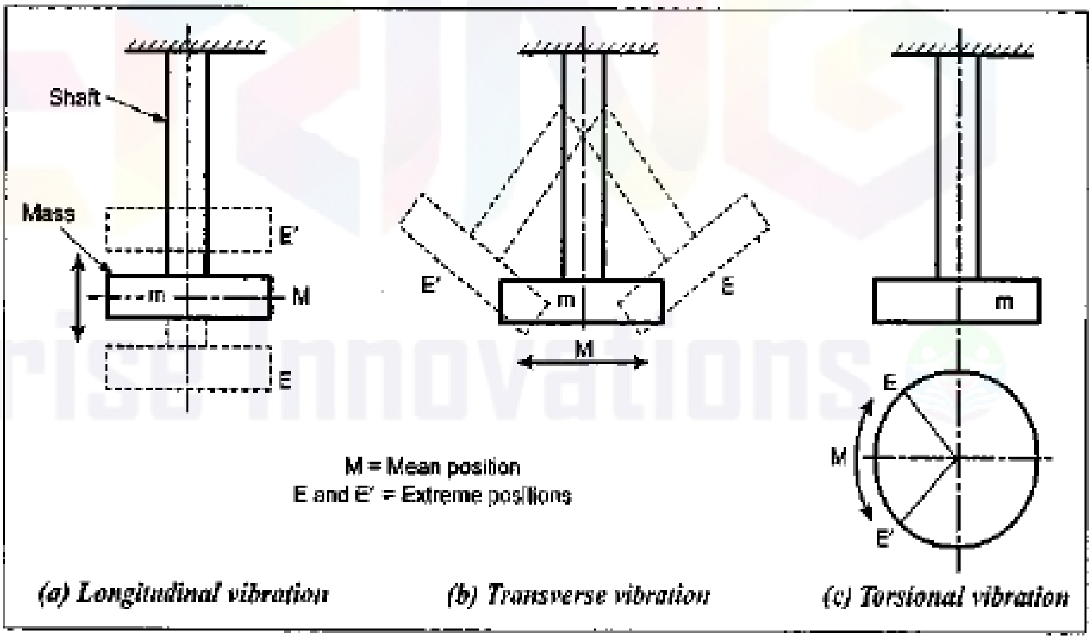


Fig. 2.42. Types of vibration

1. Longitudinal Vibrations

When the particles of the shaft or disc moves parallel to the axis of the shaft, then the vibrations are known as ***longitudinal vibrations,*** as shown in Fig.2.42(a).

2. Transverse Vibrations

When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, then the vibrations are known as ***transverse vibrations,*** as shown in Fig.2.42(b).

3. Torsional Vibrations

When the particles of the shaft or disc move in a circle about the axis of the shaft, then the vibrations are known as ***torsional vibrations,*** as shown in Fig.2.42(c).

2.31. EQUATIONS OF MOTION BASED ON WEAK FORM

2.31.1. Longitudinal Vibration of Bars or Axial Vibration of a Rod

Consider a free body diagram of a differential element of length dx as shown in Fig.2.43.

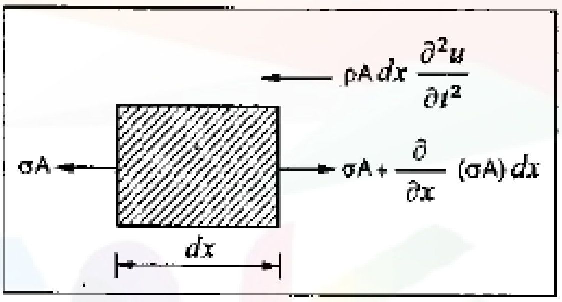


Fig. 2.43. Free body diagram of a differential element

Let, σ = Stress induced in N/m²

m = Mass of body in N

a = Acceleration due to gravity in m²/s

dx = Elemental length in m

Applying Newton's Second Law,

$$\left(\sigma A + \frac{\partial}{\partial x}(\sigma A)\, dx\right) - \sigma A - (\rho\, A\, dx)\frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{... (2.146)}$$

$$\frac{\partial}{\partial x}(\sigma A)\, dx - \rho\, A\, dx \frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial}{\partial x}\sigma A - \rho\, A \frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{... (2.147)}$$

We know that, $$\sigma = E\frac{\partial u}{\partial x} \qquad \text{... (2.148)}$$

Substitute the equation (2.148) in (2.147),

$$\frac{\partial}{\partial x}\left[\mathrm{E\,A}\frac{\partial u}{\partial x}\right]-\rho\,\mathrm{A}\frac{\partial^2 u}{\partial t^2} = 0$$

$$\mathrm{A\,E}\frac{\partial^2 u}{\partial x^2} = \rho\,\mathrm{A}\frac{\partial^2 u}{\partial t^2} \qquad \ldots (2.149)$$

Using the technique of separation of variables and assuming harmonic vibration, we have,

$$u(x, t) = \bar{u}(x)\, e^{-i\omega t} \qquad \ldots (2.150)$$

Equation (2.150), by using partial differentiation,

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = u(x)\cdot e^{-i\omega t}\cdot(-i\omega)\times(-i\omega)$$

$$\frac{\partial^2 u}{\partial t^2} = u\cdot(-\omega^2) \qquad \ldots (2.151)$$

$$[\because -i\times -i = i^2 \Rightarrow i^2 = -1]$$

Equation (2.150) $\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{du}{dx}\cdot e^{-i\omega t}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{dx^2}\cdot e^{-i\omega t} \qquad [\because t \text{ is very small, } i.e.,\ t = 0]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{dx^2} \qquad \ldots (2.152)$$

Substituting equations (2.151) and (2.152) in equation (2.149), we obtain,

$$\mathrm{A\,E}\frac{d^2 u}{dx^2} = -\rho\,\mathrm{A}\,\omega^2 u$$

$$\mathrm{A\,E}\frac{d^2 u}{dx^2}+\rho\,\mathrm{A}\,\omega^2 u = 0 \qquad \ldots (2.153)$$

Weighted-Residual statement for the above equation is

$$\int_0^l w(x)\left(\mathrm{A\,E}\frac{d^2 u}{dx^2}+\rho\,\mathrm{A}\,\omega^2 u\right)dx = 0 \qquad \ldots (2.154)$$

Integrating the above equation, the weak form of the Weighted-Residual statement can be rewritten as,

$$\left[w(x)\,\mathrm{A\,E}\frac{du}{dx}\right]_0^l - \int_0^l \mathrm{A\,E}\frac{du}{dx}\frac{dw}{dx}\,dx + \int_0^l w(x)\,\rho\,\mathrm{A}\,\omega^2 u(x)\,dx = 0 \qquad \ldots (2.155)$$

Bar element: Consider a one dimensional bar element with nodes 1 and 2 as shown in Fig.2.44. Let u_1 and u_2 be the nodal displacement parameters or otherwise known as degrees of freedom.

Fig. 2.44. Bar element

The shape functions are given by equation,

$$u(x) = N_1 u_1 + N_2 u_2$$

$$= \left(1 - \frac{x}{l}\right) u_1 + \left(\frac{x}{l}\right) u_2 \qquad \dots (2.156)$$

In the Galerkin formation, the weight functions are the same as the shape functions. So we have,

$$\left.\begin{aligned} w_1(x) &= 1 - \frac{x}{l}, \\ w_2(x) &= \frac{x}{l} \end{aligned}\right\} \qquad \dots (2.157)$$

Now differentiating with respect to x,

$$\Rightarrow \frac{dw_1}{dx} = \left(\frac{-1}{l}\right) \qquad \dots (2.158)$$

Substitute the equation (2.156) , (4.15) and (2.158) in equation (2.155).

$$\Rightarrow \left[\left(1 - \frac{x}{l}\right) \mathrm{A\,E} \frac{du}{dx}\right]_0^l - \int_0^l \mathrm{A\,E} \frac{du}{dx} \left(\frac{-1}{l}\right) dx + \int_0^l \left(1 - \frac{x}{l}\right) \rho\,\mathrm{A}\,\omega^2 \left[\left(1 - \frac{x}{l}\right) u_1 + \left(\frac{x}{l}\right) u_2\right] dx = 0$$

$$\Rightarrow \left[\left[\left(1 - \frac{l}{l}\right) \mathrm{A\,E} \left(\frac{u_2 - u_1}{l}\right)\right] - \left[\left(1 - \frac{0}{l}\right) \mathrm{A\,E} \left(\frac{u_2 - u_1}{l}\right)\right]\right]$$

$$- \int_0^l \mathrm{AE} \left(\frac{u_2 - u_1}{l}\right) \left(\frac{-1}{l}\right) dx + \int_0^l \left(1 - \frac{x}{l}\right) \rho \mathrm{A} \omega^2 \left[\left(1 - \frac{x}{l}\right) u_1 + \left(\frac{x}{l}\right) u_2\right] dx = 0$$

$$\left[0 - A\,E\left(\frac{u_2-u_1}{l}\right)\right] - \int_0^l A\,E\left(\frac{u_2-u_1}{l}\right)\left(\frac{-1}{l}\right)dx$$

$$+ \int_0^l \left(1-\frac{x}{l}\right)\rho\,A\,\omega^2\left[\left(1-\frac{x}{l}\right)u_1+\left(\frac{x}{l}\right)u_2\right]dx = 0$$

$$\left[\text{We know that, } E = \frac{\sigma}{e} = \frac{P/A}{\frac{du}{dx}} = \frac{P/A}{\left(\frac{u_2-u_1}{l}\right)} \Rightarrow P_0 = A\,E\left(\frac{u_2-u_1}{l}\right)\right]$$

$$-P_0 - \left[A\,E\left(\frac{u_2-u_1}{l}\right)\left(\frac{-1}{l}\right)(x)_0^l\right] +$$

$$\int_0^l \left(1-\frac{x}{l}\right)\rho\,A\,\omega^2\left[\left(1-\frac{x}{l}\right)u_1+\left(\frac{x}{l}\right)u_2\right]dx = 0$$

$$-P_0 - \left[A\,E\left(\frac{u_2-u_1}{l}\right)\left(\frac{-1}{l}\right)(l)\right] +$$

$$\int_0^l \left(1-\frac{x}{l}\right)\rho\,A\,\omega^2\left[\left(1-\frac{x}{l}\right)u_1+\left(\frac{x}{l}\right)u_2\right]dx = 0$$

$$-P_0 - \left[A\,E\left(\frac{u_1-u_2}{l}\right)\right] + \int_0^l \left(1-\frac{x}{l}\right)\rho\,A\,\omega^2\left[\left(1-\frac{x}{l}\right)u_1+\left(\frac{x}{l}\right)u_2\right]dx = 0 \qquad \text{... (2.159)}$$

Similarly, $\qquad w_2(x) = \frac{x}{l} \Rightarrow \frac{dw_2}{dx} = \frac{1}{l} \qquad$ ... (2.160)

Substitute the equation (2.156), (2.157) and (2.160) in equation (2.155),

$$\Rightarrow \left[\frac{x}{l}A\,E\frac{du}{dx}\right]_0^l - \int_0^l A\,E\frac{du}{dx}\left(\frac{1}{l}\right)dx +$$

$$\int_0^l \left(\frac{x}{l}\right)\rho\,A\,\omega^2\left[\left(1-\frac{x}{l}\right)u_1+\left(\frac{x}{l}\right)u_2\right]dx = 0$$

$$\Rightarrow \left[\left[\frac{l}{l} A E \left(\frac{u_2 - u_1}{l} \right) \right] - \left[\frac{0}{l} A E \left(\frac{u_2 - u_1}{l} \right) \right] \right] - \left[A E \left(\frac{u_2 - u_1}{l} \right) \left(\frac{1}{l} \right) (x)_0^l \right]$$

$$+ \int_0^l \left(\frac{x}{l} \right) \rho A \omega^2 \left[\left(1 - \frac{x}{l} \right) u_1 + \left(\frac{x}{l} \right) u_2 \right] dx = 0$$

$$\left[A E \left(\frac{u_2 - u_1}{l} \right) \right] - \left[A E \left(\frac{u_2 - u_1}{l} \right) \right] + \int_0^l \left(\frac{x}{l} \right) \rho A \omega^2 \left[\left(1 - \frac{x}{l} \right) u_1 + \left(\frac{x}{l} \right) u_2 \right] dx = 0$$

$$\left[P_l - A E \left(\frac{u_2 - u_1}{l} \right) + \int_0^l \left(\frac{x}{l} \right) \rho A \omega^2 \left[\left(1 - \frac{x}{l} \right) u_1 + \left(\frac{x}{l} \right) u_2 \right] dx \right] = 0 \quad \ldots (2.161)$$

$$\left[\because P_l = A E \left(\frac{u_2 - u_1}{l} \right) \right]$$

Writing equations (2.159) and (2.161) in matrix form,

$$\begin{Bmatrix} -P_0 \\ P_l \end{Bmatrix} - \frac{A E}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A \omega^2$$

$$\begin{bmatrix} \int_0^l \left(1 - \frac{x}{l} \right) \left(1 - \frac{x}{l} \right) dx & \int_0^l \left(1 - \frac{x}{l} \right) \left(\frac{x}{l} \right) dx \\ \int_0^l \left(\frac{x}{l} \right) \left(1 - \frac{x}{l} \right) dx & \int_0^l \left(\frac{x}{l} \right) \left(\frac{x}{l} \right) dx \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$\Rightarrow \quad \frac{A E}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -P_0 \\ P_l \end{Bmatrix}$$

$$+ \rho A \omega^2 \int_0^l \begin{bmatrix} \left(1 - \frac{x}{l} \right)^2 & \left(\frac{x}{l} - \frac{x^2}{l^2} \right) \\ \left(\frac{x}{l} - \frac{x^2}{l^2} \right) & \frac{x^2}{l^2} \end{bmatrix} dx \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\Rightarrow \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}-P_0\\ P_l\end{Bmatrix} + \rho A \omega^2$$

$$\left[\begin{matrix}\dfrac{\left(1-\dfrac{x}{l}\right)^3}{3\left(\dfrac{-1}{l}\right)} & \dfrac{x^2}{2l}-\dfrac{x^3}{3l^2}\\ \dfrac{x^2}{2l}-\dfrac{x^3}{3l^2} & \dfrac{x^3}{3l^2}\end{matrix}\right]_0^l \begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

$$\Rightarrow \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}-P_0\\ P_l\end{Bmatrix} + \rho A \omega^2$$

$$\begin{bmatrix}\left(\dfrac{\left(1-\dfrac{l}{l}\right)^3}{\dfrac{-3}{l}} - \dfrac{l^3}{\dfrac{-3}{l}}\right) & \left(\dfrac{l^2}{2l}-\dfrac{l^3}{3l^2}-0\right)\\ \left(\dfrac{l^2}{2l}-\dfrac{l^3}{3l^2}-0\right) & \left(\dfrac{l^3}{3l^2}-0\right)\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$\Rightarrow \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}-P_0\\ P_l\end{Bmatrix} + \rho A \omega^2 \begin{bmatrix}\dfrac{l}{3} & \dfrac{l}{6}\\ \dfrac{l}{6} & \dfrac{l}{3}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}-P_0\\ P_l\end{Bmatrix} + \frac{\rho A l \omega^2}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} \quad \ldots (2.162)$$

where, Element stiffness matrix, $[K] = \dfrac{A.E}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

Element mass matrix, $[m] = \dfrac{\rho A l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

$$\therefore \boxed{[K]\{u\} = \begin{Bmatrix}-P_0\\ P_l\end{Bmatrix} + \omega^2 [m]\{u\}}$$

$$\boxed{[[K]-[m]\omega^2]\{u\} = \begin{Bmatrix}-P_0\\ P_l\end{Bmatrix}} \quad \ldots (2.163)$$

In free vibration problems, external forces are zero. So, the above equation reduces to

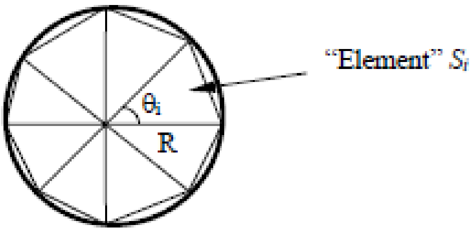
$$\boxed{[[K]-[m]\omega^2]\{u\} = 0}$$

I. Basic Concepts

The *finite element method* (FEM), or *finite element analysis* (FEA), is based on the idea of building a complicated object with simple blocks, or, dividing a complicated object into small and manageable pieces. Application of this simple idea can be found everywhere in everyday life, as well as in engineering.

Examples:

- Lego (kids' play)
- Buildings
- Approximation of the area of a circle:



Area of one triangle: $S_i = \frac{1}{2} R^2 \sin\theta_i$

Area of the circle: $S_N = \sum_{i=1}^{N} S_i = \frac{1}{2} R^2 N \sin\left(\frac{2\pi}{N}\right) \to \pi R^2 \ as\ N \to \infty$

where N = total number of triangles (elements).

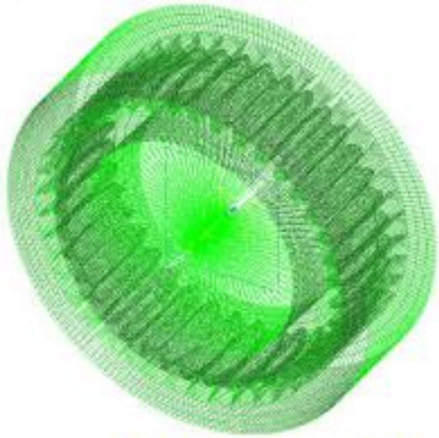
Observation: Complicated or smooth objects can be represented by geometrically simple pieces (elements).

Why Finite Element Method?

- *Design analysis*: hand calculations, experiments, and computer simulations
- FEM/FEA is the most widely applied computer simulation method in engineering
- Closely integrated with CAD/CAM applications
- ...

Applications of FEM in Engineering

- Mechanical/Aerospace/Civil/Automobile Engineering
- Structure analysis (static/dynamic, linear/nonlinear)
- Thermal/fluid flows
- Electromagnetics
- Geomechanics
- Biomechanics
- ...



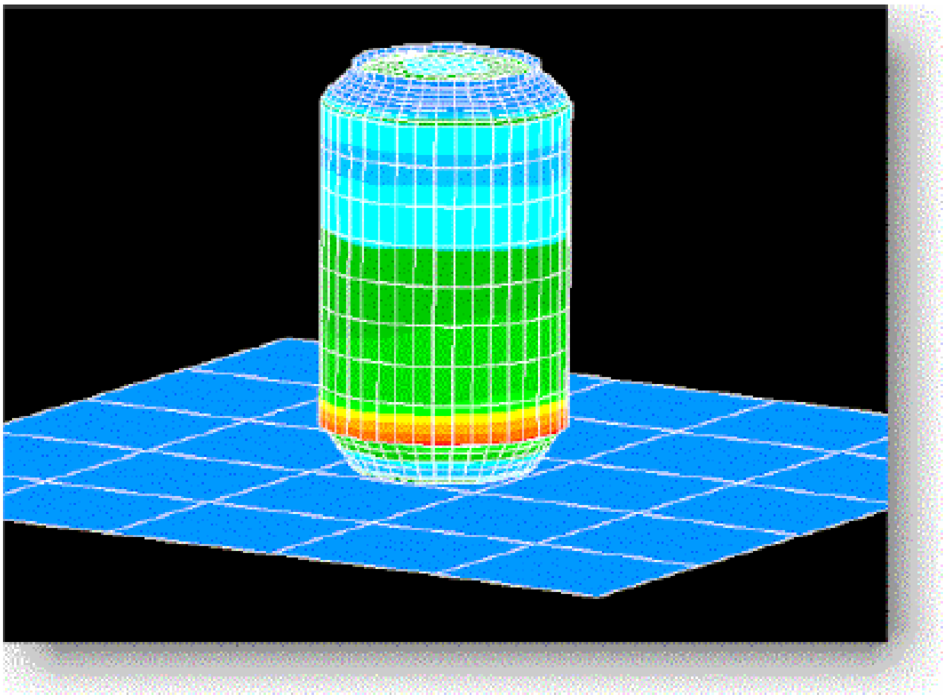
Modeling of gear coupling

Examples:

A Brief History of the FEM

- 1943 ----- Courant (Variational methods)
- 1956 ----- Turner, Clough, Martin and Topp (Stiffness)
- 1960 ----- Clough ("Finite Element", plane problems)

- 1970s ----- Applications on mainframe computers
- 1980s ----- Microcomputers, pre- and postprocessors
- 1990s ----- Analysis of large structural systems

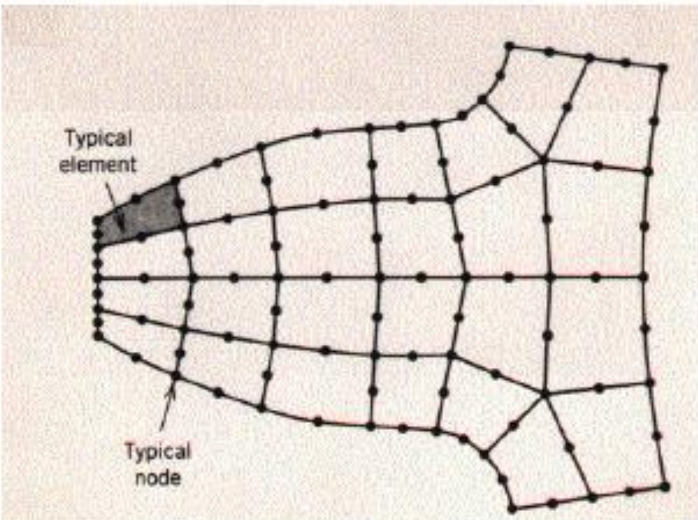


Can Drop Test (Click for more information and an animation)

FEM in Structural Analysis (The Procedure)

- Divide structure into pieces (elements with nodes)
- Describe the behavior of the physical quantities on each element
- Connect (assemble) the elements at the nodes to form an approximate system of equations for the whole structure
- Solve the system of equations involving unknown quantities at the nodes (e.g., displacements)
- Calculate desired quantities (e.g., strains and stresses) at selected elements

Example:



FEM model for a gear tooth (From Cook's book, p.2).

Computer Implementations

- Preprocessing (build FE model, loads and constraints)
- FEA solver (assemble and solve the system of equations)
- Postprocessing (sort and display the results)

Available Commercial FEM Software Packages

- *ANSYS* (General purpose, PC and workstations)
- *SDRC/I-DEAS* (Complete CAD/CAM/CAE package)
- *NASTRAN* (General purpose FEA on mainframes)
- *ABAQUS* (Nonlinear and dynamic analyses)
- *COSMOS* (General purpose FEA)
- *ALGOR* (PC and workstations)
- *PATRAN* (Pre/Post Processor)
- *HyperMesh* (Pre/Post Processor)
- *Dyna-3D* (Crash/impact analysis)
- ...

A Link to CAE Software and Companies

II. Substructures (Superelements)

Substructuring is a process of analyzing a large structure as a collection of (natural) components. The FE models for these components are called *substructures* or *superelements* (SE).

Physical Meaning:

A finite element model of a portion of structure.

Mathematical Meaning:

Boundary matrices which are load and stiffness matrices reduced (condensed) from the *interior* points to the *exterior* or boundary points.

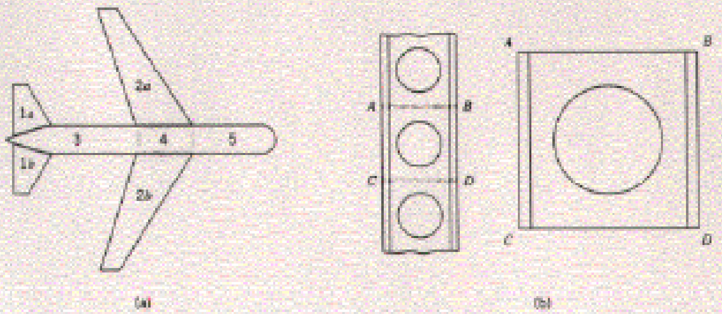


Fig. 4.11-1. (a) Possible substructures 1*a*, 1*b*, ..., 5 of a hypothetical aircraft. (b) Castellated beam, with typical repeating substructure *ABCD*. Elements of the substructures are not shown.

Advantages of Using Substructures/Superelements:

- Large problems (which will otherwise exceed your computer capabilities)
- Less CPU time per run once the superelements have been processed (i.e., matrices have been saved)
- Components may be modeled by different groups
- Partial redesign requires only partial reanalysis (reduced cost)
- Efficient for problems with local nonlinearities (such as confined plastic deformations) which can be placed in one superelement (residual structure)
- Exact for static stress analysis

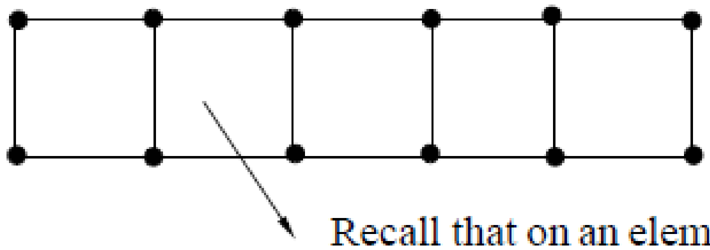
Disadvantages:

- Increased overhead for file management
- Matrix condensation for dynamic problems introduce new approximations
- ...

IV. Nature of Finite Element Solutions

- FE Model – A mathematical model of the real structure, based on many approximations.
- Real Structure -- Infinite number of nodes (physical points or particles), thus infinite number of DOF's.
- FE Model – finite number of nodes, thus finite number of DOF's.

⇨ Displacement field is controlled (or constrained) by the values at a limited number of nodes.



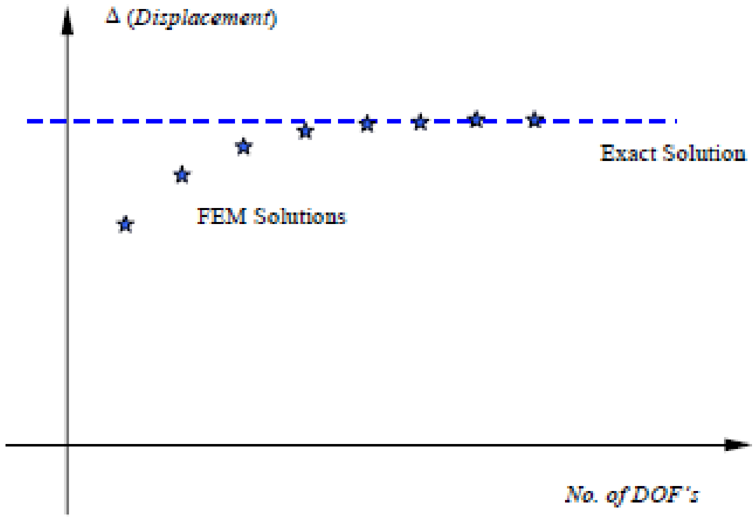
Recall that on an element :

$$u = \sum_{\alpha=1}^{4} N_\alpha u_\alpha$$

Stiffening Effect:

- FE Model is stiffer than the real structure.
- In general, displacement results are smaller in magnitudes than the exact values.

Hence, FEM solution of displacement provides a *lower bound* of the exact solution.



The FEM solution approaches the exact solution from below.

This is true for displacement based FEA!

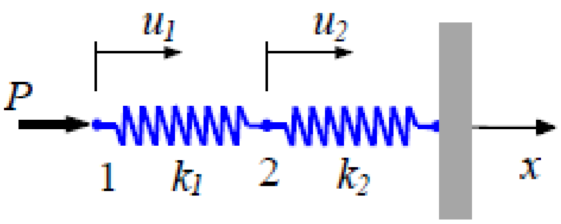
V. Numerical Error

Error ≠ Mistakes in FEM (modeling or solution).

Type of Errors:

- Modeling Error (beam, plate … theories)
- Discretization Error (finite, piecewise …)
- Numerical Error (in solving FE equations)

Example (numerical error):

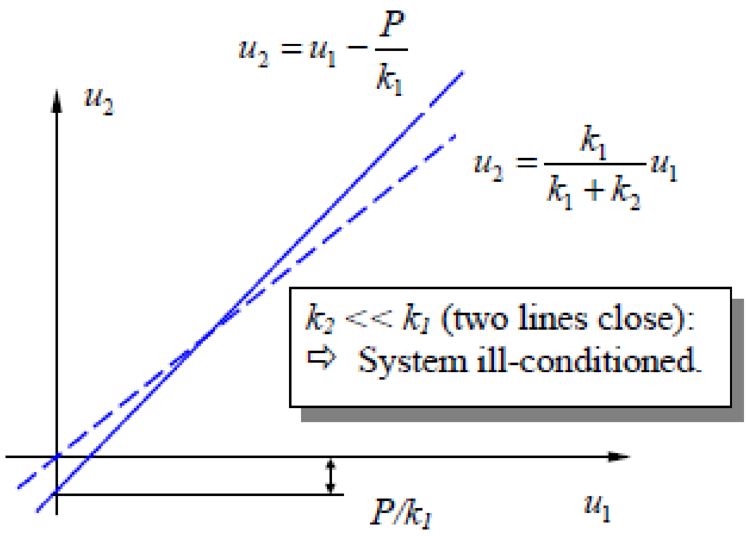


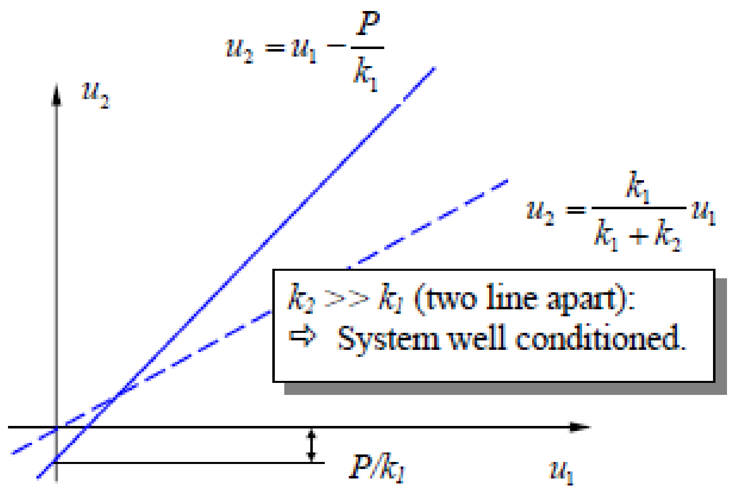
FE Equations:

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

and $\qquad Det\,\mathbf{K} = k_1 k_2 .$

The system will be *singular* if k_2 is small compared with k_1.





- Large difference in stiffness of different parts in FE model may cause ill-conditioning in FE equations. Hence giving results with large errors.
- Ill-conditioned system of equations can lead to large changes in solution with small changes in input (right hand side vector).

VI. Convergence of FE Solutions

As the mesh in an FE model is "refined" repeatedly, the FE solution will converge to the exact solution of the mathematical model of the problem (the model based on bar, beam, plane stress/strain, plate, shell, or 3-D elasticity theories or assumptions).

Type of Refinements:

h-refinement: reduce the size of the element ("*h*" refers to the typical size of the elements);

p-refinement: Increase the order of the polynomials on an element (linear to quadratic, etc.; "*h*" refers to the highest order in a polynomial);

r-refinement: re-arrange the nodes in the mesh;

hp-refinement: Combination of the h- and p-refinements (better results!).

Examples:

…

VII. Adaptivity (h-, p-, and hp-Methods)

- Future of FE applications
- Automatic refinement of FE meshes until converged results are obtained
- User's responsibility reduced: only need to generate a good initial mesh

Error Indicators:

Define,

σ --- element by element stress field (discontinuous),

σ^*--- averaged or smooth stress (continuous),

$\sigma_E = \sigma - \sigma^*$ --- the error stress field.

Compute strain energy,

$$U = \sum_{i=1}^{M} U_i, \qquad U_i = \int_{V_i} \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{E}^{-1} \boldsymbol{\sigma} dV;$$

$$U^* = \sum_{i=1}^{M} U_i^*, \qquad U_i^* = \int_{V_i} \frac{1}{2} \boldsymbol{\sigma}^{*T} \mathbf{E}^{-1} \boldsymbol{\sigma}^* dV;$$

$$U_E = \sum_{i=1}^{M} U_{Ei}, \qquad U_{Ei} = \int_{V_i} \frac{1}{2} \boldsymbol{\sigma}_E^T \mathbf{E}^{-1} \boldsymbol{\sigma}_E dV;$$

where M is the total number of elements, V_i is the volume of the element i.

One error indicator --- the relative energy error:

$$\eta = \left[\frac{U_E}{U + U_E} \right]^{1/2}. \qquad (0 \le \eta \le 1)$$

The indicator η is computed after each FE solution. Refinement of the FE model continues until, say

$$\eta \le 0.05.$$

=> converged FE solution.