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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT 1-THEORY OF VIBRATIONS-SME1306

Chapter 1: Basics of Vibrations for Simple Mechanical Systems

Introduction:

The fundamentals of Sound and Vibrations are part of the broader field of mechanics, with strong connections to classical mechanics, solid mechanics and fluid dynamics. Dynamics is the branch of physics concerned with the motion of bodies under the action of forces. Vibrations or oscillations can be regarded as a subset of dynamics in which a system subjected to restoring forces swings back and forth about an equilibrium position, where a system is defined as an assemblage of parts acting together as a whole. The restoring forces are due to elasticity, or due to gravity.

The subject of Sound and Vibrations encompasses the generation of sound and vibrations, the distribution and damping of vibrations, how sound propagates in a free field, and how it interacts with a closed space, as well as its effect on man and measurement equipment. Technical applications span an even wider field, from applied mathematics and mechanics, to electrical instrumentation and analog and digital signal processing theory, to machinery and building design. Most human activities involve vibration in one form or other. For example, we hear because our eardrums vibrate and see because light waves undergo vibration. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. Human speak due to the oscillatory motion of larynges (tongue).

In most of the engineering applications, vibration is signifying to and fro motion, which is undesirable. Galileo discovered the relationship between the length of a pendulum and its frequency and observed the resonance of two bodies that were connected by some energy transfer medium and tuned to the same natural frequency. Vibration may results in the failure of machines or their critical components. The effect of vibration depends on the magnitude in terms of displacement, velocity or accelerations, exciting frequency and the total duration of the vibration. In this chapter, the vibration of a single-degree-of-freedom (SDOF), Two degree of freedom system with and without damping and introductory multi-degree of freedom system will be discussed in this section.

1. LINEAR SYSTEMS

Often in Vibrations and Acoustics, the calculation of the effect of a certain physical quantity termed as the input signal on another physical quantity, called the output signal; (Figure 1-1). An example is that of calculating vibration velocity *v(t)*, which is obtained in a structure when it is excited by a given force *F(t)*. That problem can be solved by making use of the theory of *linear time- invariant systems*.

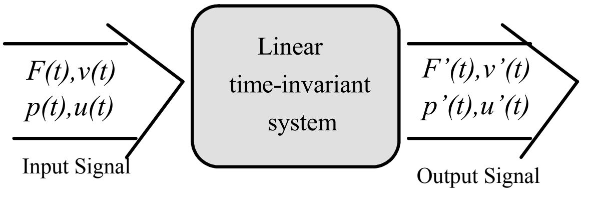
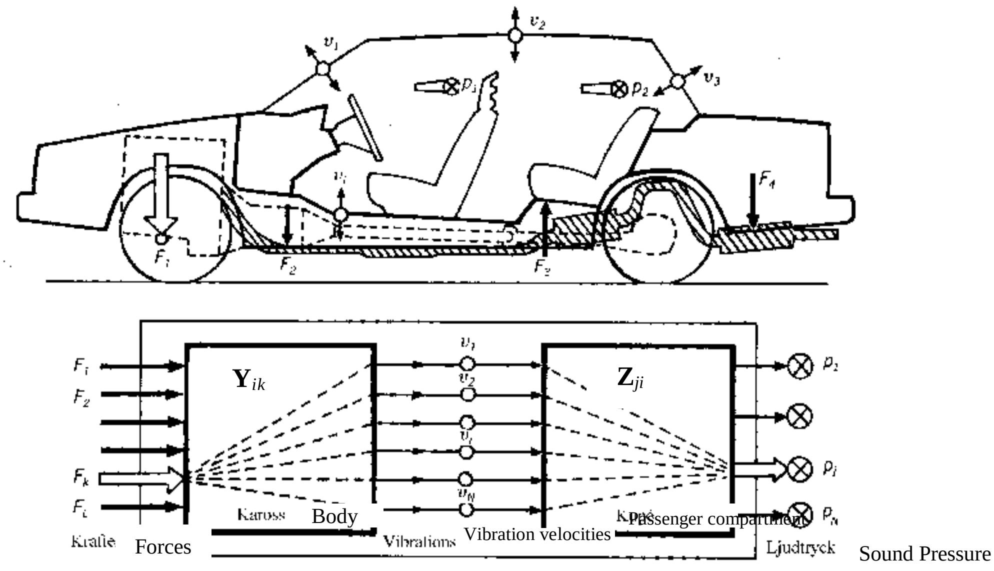


Fig. 0-1 A linear time-invariant system describes the relationship between an input signal and an output signal. For example, the input signal could be a velocity v(t), and the output signal a force F(t), or the input signal an acoustic pressure p(t) and the output signal an acoustic particle velocity u'(t). [Sound and vibration book by KTH[1]]

From a purely mathematical standpoint, a linear system is defined as one in which the relationship between the input and output signals can be described by a linear differential equation. If the coefficients are, moreover, independent of time, i.e., constant, then the system is also time invariant. A linear system has several important features.

Example 0-1 [1]

The figure below, from the introduction, shows an example in which the forces that excite an automobile are inputs to a number of linear systems, the outputs from which are vibration velocities at various points in the structure. The vibration velocities are then, in turn, inputs to a number of linear systems, the outputs from which are sound pressures at various points in the passenger compartment. By adding up the contributions from all of the significant excitation forces, the total sound pressures at points of interest in the passenger compartment can be found. The engine is fixed to the chassis via vibration isolators. If the force F_1 that influences the chassis can be cut in half, then, for a linear system, all vibration velocities $v_1 - v_N$ caused by the force F_1 are also halved. In turn, the sound pressures $p_1 - p_N$, which are brought about by the velocities $v_1 - v_N$, are halved as well. In this chapter, *linear oscillations* in mechanical systems are considered, i.e., oscillations in systems for which there is a linear relation between an exciting force and the resulting motion, as described by displacements, velocities, and accelerations. Linearity is normally applicable whenever the kinematic quantities can be regarded as small variations about an average value, implying that the relation between the input signal and the output signal can be described by linear differential equations with constant coefficients.



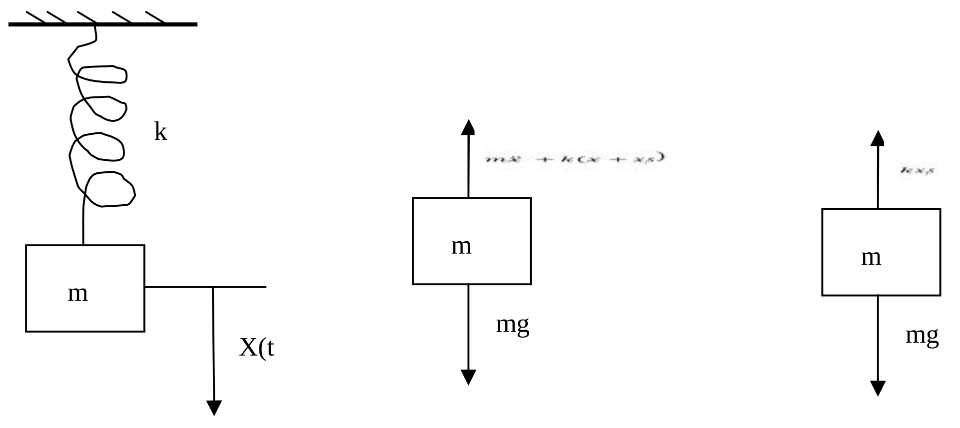
(Picture: Volvo Technology Report, nr 1 1988) [1]

1.1 Single Degree of Freedom Systems

In basic mechanics, one studies *single degree-of-freedom systems* thoroughly. One might wonder why so much attention should be given to such a simple problem. The single degree-of-freedom system is so interesting to study because it gives us information on how a system's characteristics are influenced by different quantities. Moreover, one can model more complex systems, provided that they have isolated resonances, as sums of simple single degree-of-freedom systems.

1.2 Spring Mass System

Most of the system exhibit simple harmonic motion or oscillation. These systems are said to have elastic restoring forces. Such systems can be modeled, in some situations, by a spring-mass schematic, as illustrated in Figure 1.2. This constitutes the most basic vibration model of a machine structure and can be used successfully to describe a surprising number of devices, machines, and structures. This system provides a simple mathematical model that seems to be more sophisticated than the problem requires. This system is very useful to conceptualize the vibration problem in different machine components. The single degree of freedom system is indicating as :



(a) (b) (c)

Fig.1.2 (a) Spring-mass schematic (b) free body diagram, (c) free body diagram in static condition

If $x = x(t)$ denotes the displacement (m) of the mass m (kg) from its equilibrium position as a function of time t (s), the equation of motion for this system becomes,

$$m\ddot{x} + k(x + x_s) - mg = 0 \qquad (1.1)$$

where k =the stiffness of the spring (N/m),

x_s = static deflection

m = the spring under gravity load,

g = the acceleration due to gravity (m/s2),

$\ddot{x}$ = acceleration of the system

Applying static condition as shown in Fig. 1.2 (c) the equation of motion of the system yields

$$m\ddot{x} + kx = 0 \qquad (1.2)$$

This equation of motion of a single-degree-of-freedom system and is a linear, second-order, ordinary differential equation with constant coefficients. A simple experiment for determining the spring stiffness by adding known amounts of mass to a spring and measuring the resulting static deflection x_s is shown in Fig. 1.3. The results of this static experiment can be plotted as force (mass times acceleration) v/s x_s, the slope yielding the value of spring stiffness k for the linear portion of the plot as illustrated in Figure 1.4.

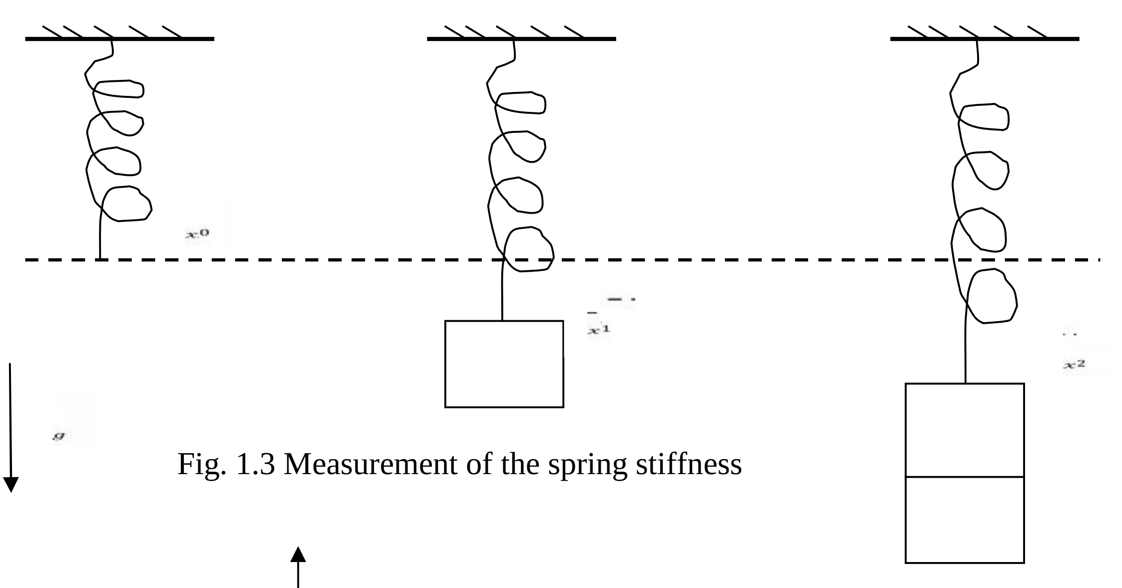


Fig. 1.3 Measurement of the spring stiffness

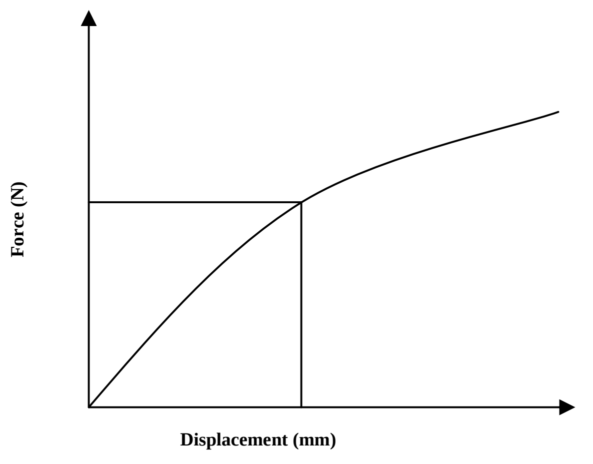


Fig. 1.4 Determination of the spring stiffness

Once m and k are determined from static experiments, Equation (1.2) can be solved to yield the time history of the position of the mass m, given the initial position and velocity of the mass. The form of the solution of previous equation is found from substitution of an assumed periodic motion as,

$$x(t) = A\sin(\omega_n t + \phi) \qquad (1.3)$$

Where, $\omega_n = \sqrt{k/m}$ is the natural frequency (rad/s).

Here, A= the amplitude

Φ= phase shift,

A and Φ are constants of integration determined by the initial conditions.

If x_0 is the specified initial displacement from equilibrium of mass m, and v_0 is its specified initial velocity, simple substitution allows the constants A and Φ to be obtained. The unique displacement may be expressed as,

$$x(t) = \sqrt{\frac{\omega_n^2 x_0^2 + \upsilon_0^2}{\omega_n^2}}\sin\left[\omega_n t + \tan^{-1}\left(\frac{\omega_n x_0}{\upsilon_0}\right)\right] \qquad (1.4)$$

Or,

$$x(t) = \frac{\upsilon_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t$$

Equation 1.2 can also be solved using a pure mathematical approach as described follows.

Substituting $x(t) = C\,e^{\lambda t}$

$$m\lambda^2 e^{\lambda t} + k e^{\lambda t} = 0 \qquad (1.5)$$

Here $C \neq 0$ and $e^{\lambda t} \neq 0$, Hence $m\lambda^2 + k = 0$

Or

$$\lambda = \pm j\left(\frac{k}{m}\right)^{1/2} = \pm\omega_n j$$

where, j is an imaginary number = $\sqrt{-1}$

Hence the generalized solution yields as,

$$x(t) = C_1 e^{\omega_n jt} + C_2 e^{-\omega_n jt} \qquad (1.6)$$

where C_1 and C_2 are arbitrary complex conjugate constants of integration.

The value of the constants C_1 and C_2 can be determined by applying the initial conditions of the system. Note that the equation 1.2 is valid only as long as spring is linear.

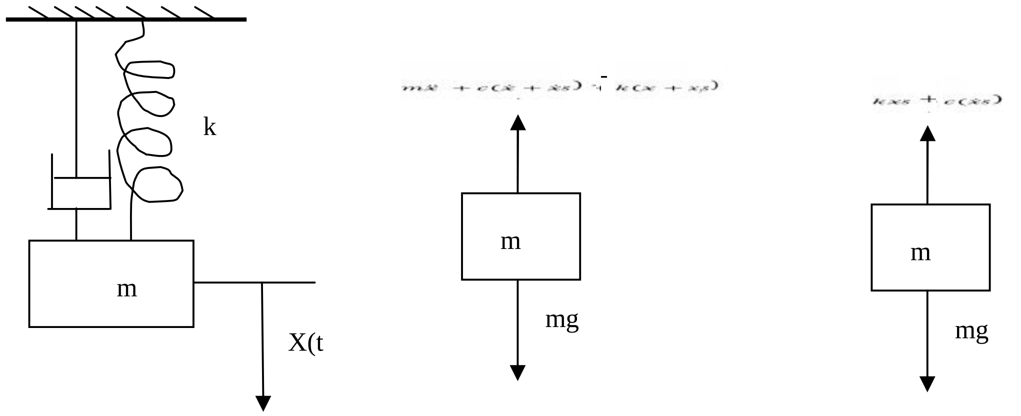
1.3 Spring Mass Damper system

Most systems will not oscillate indefinitely when disturbed, as indicated by the solution in Equation (1.4). Typically, the periodic motion damped out after some time. The easiest way to model this mathematically is to introduce a new term, named as damping force term, into Equation (1.2).

Incorporating the damping term in equation (1.2) yield as,

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (1.7)$$

Physically, the addition of a dashpot or damper results in the dissipation of energy, as illustrated in Figure 1.5 the mass, damper and spring arrangement is as:



(a) **(b)**
(c)

Fig. 1.5 (a) Schematic of the spring–mass–damper system, (b) free body diagram of the system in part (a), (c) free body diagram due to static condition

If the dashpot exerts a dissipative force proportional to velocity on the mass m, the equation (1.7) describes the equation of the motion. Unfortunately, the constant of proportionality, c, cannot be measured by static methods as m and k are measured in spring mass system.

The constant of proportionality c is known as damping coefficient and its unit in MKS is Ns/m. A general mathematical approach can be used to solve the equation 1.7 as described below.

Substituting, $x(t) = a\, e^{\lambda t}$ in equation 1.7, get,

$$a(m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + k e^{\lambda t}) = 0 \qquad (1.8)$$

here $a \neq 0$ and $e^{\lambda t} \neq 0$

hence, $m\lambda^2 + c\lambda + k = 0$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \quad (1.9)$$

The solution of equation 1.8 yields as follows

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}$$

The quantity under the radical is called the discriminant. The value of the discriminant decides that whether the roots are real or complex. Damping ratio: It is relatively convenient to define a non-dimensional quantity named as damping ratio. The damping ratio is generally given by symbol Zeeta (ζ) and mathematically defined as;

$$\zeta = \frac{c}{2\sqrt{km}}$$

Substituting the value of k ,m and c in terms of ζ and ω_n, the equation (1.7) yields as,

$$\ddot{x} + 2\zeta\,\omega_n\dot{x} + \omega_n^2 x = 0 \qquad (1.10)$$

And equation (1.9) yields as

$$\lambda_{1,2} = -\zeta\,\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\xi\,\omega_n \pm \omega\,j \qquad (1.11)$$

where, ω is the damped natural frequency for ($0< \zeta <1$) the damped natural frequency is defined as $\omega = \omega_n\sqrt{1-\zeta^2}$

Clearly, the value of the damping ratio,(ζ), determines the nature of the solution of Equation (1.6).

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UNIT 2-THEORY OF VIBRATIONS-SME1306

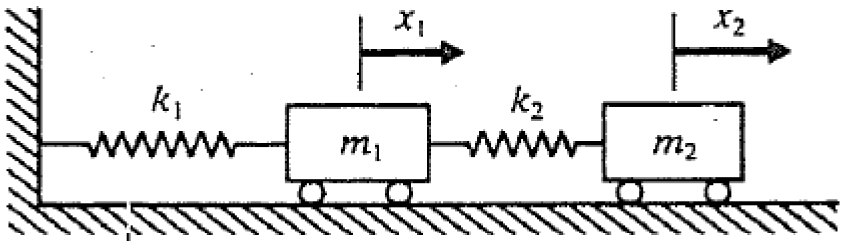
Two degree of freedom systems

- Equations of motion for forced vibration
- Free vibration analysis of an undamped system

Introduction

- Systems that require two independent coordinates to describe their motion are called two degree of freedom systems.

$$\begin{matrix}\text{Number of} \\ \text{degrees of freedom} \\ \text{of the system}\end{matrix} = \begin{matrix}\text{Number of masses} \\ \text{in the system}\end{matrix} \times \begin{matrix}\text{number of possible types} \\ \text{of motion of each mass}\end{matrix}$$



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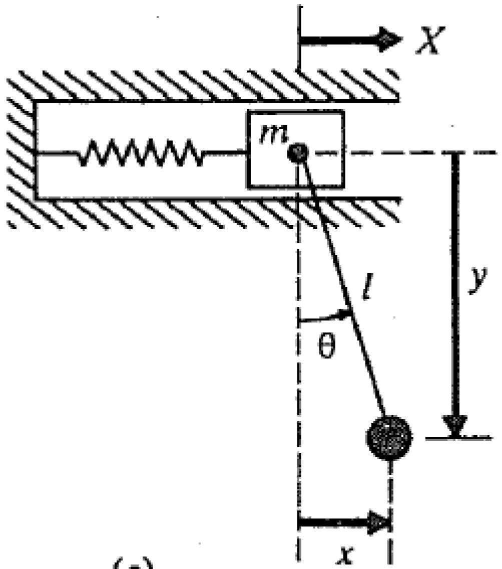
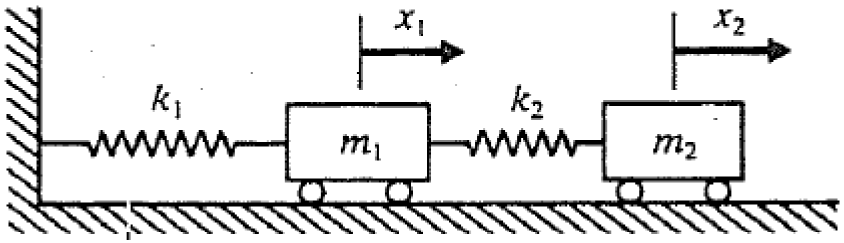


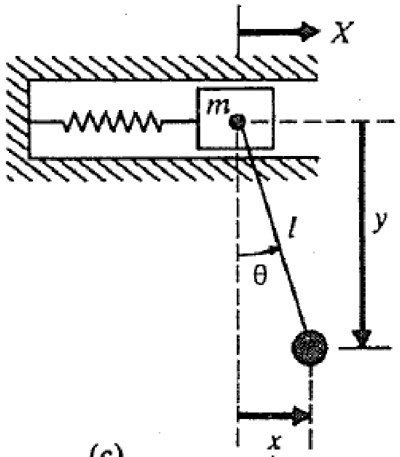
(c)

Introduction

- There are two equations for a two degree of freedom system, one for each mass (precisely one for each degree of freedom).
- They are generally in the form of coupled differential equations-that is, each equation involves all the coordinates.
- If a harmonic solution is assumed for each coordinate,the equations of motion lead to a frequency equation that gives two natural frequencies of the system.



(a)



(c)

Introduction

- If we give suitable initial excitation, the system vibrates at one of these natural frequencies. During free vibration at one of the natural frequencies, the amplitudes of the two degrees of freedom (coordinates) are related in a specified manner and the configuration is called a **normal mode**, **principle mode**, or **natural mode of vibration**.

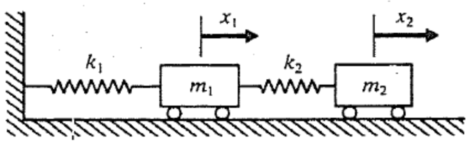
- Thus a **two degree of freedom** system has **two normal modes** of vibration corresponding to **two natural frequencies**.

- If we give an arbitrary initial excitation to the system, the resulting free vibration will be a superposition of the two normal modes of vibration. However, if the system vibrates under the action of an external harmonic force, the resulting forced harmonic vibration takes place at the frequency of the applied force.

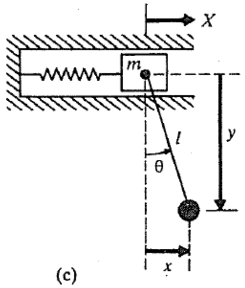
Introduction

- As is evident from the systems shown in the figures, the configuration of a system can be specified by a set of independent coordinates such as length, angle or some other physical parameters. Any such set of coordinates is called **generalized coordinates**.

- Although the equations of motion of a two degree of freedom system are generally coupled so that each equation involves all coordinates, it is always possible to find a particular set of coordinates such that each equation of motion contains only one coordinate. The equations of motion are then **uncoupled** and can be solved independently of each other. Such a set of coordinates, which leads to an uncoupled system of equations, is called **principle copordinates**.



(a)

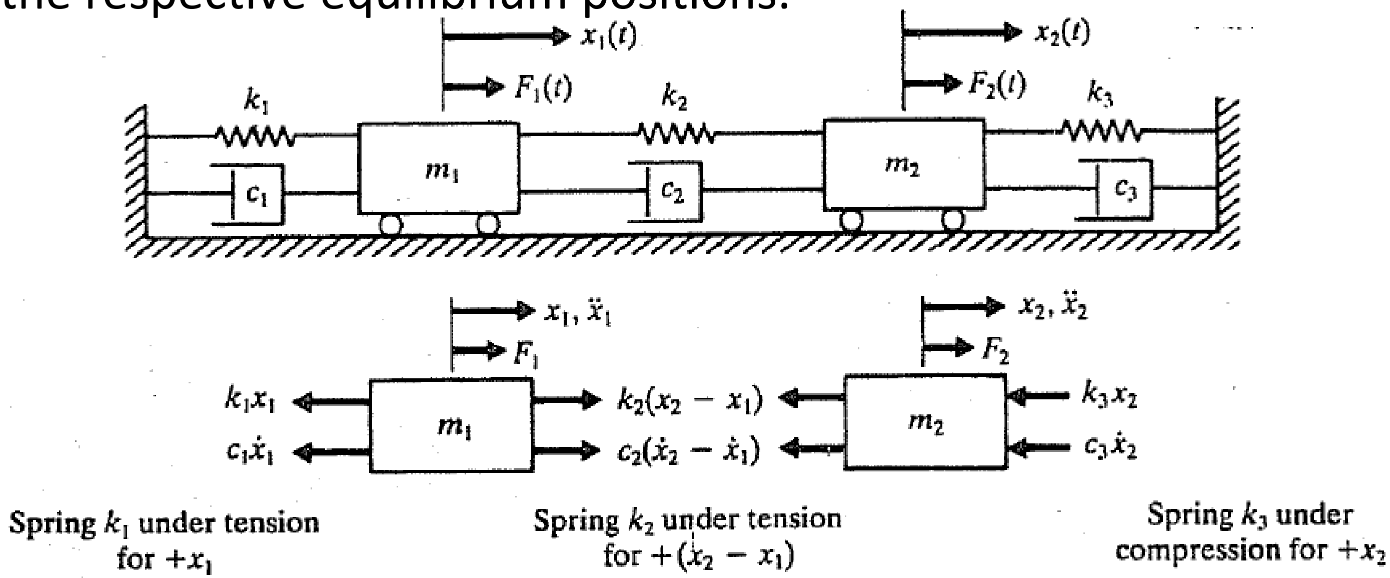


(c)

Equations of motion for forced vibration

- Consider a viscously damped two degree of freedom spring-mass system shown in the figure.

- The motion of the system is completely described by the coordinates $x_1(t)$ and $x_2(t)$, which define the positions of the masses m_1 and m_2 at any time t from the respective equilibrium positions.

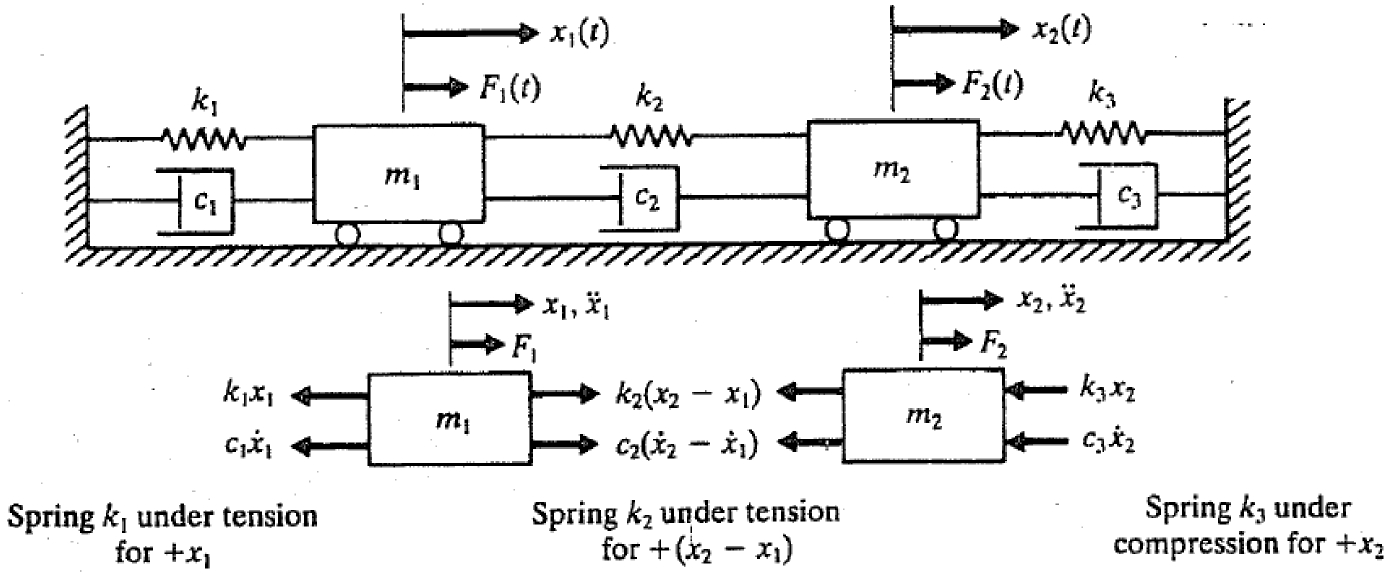


Equations of motion for forced vibration

- The external forces F_1 and F_2 act on the masses m_1 and m_2, respectively. The free body diagrams of the masses are shown in the figure.
- The application of Newton's second law of motion to each of the masses gives the equation of motion:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

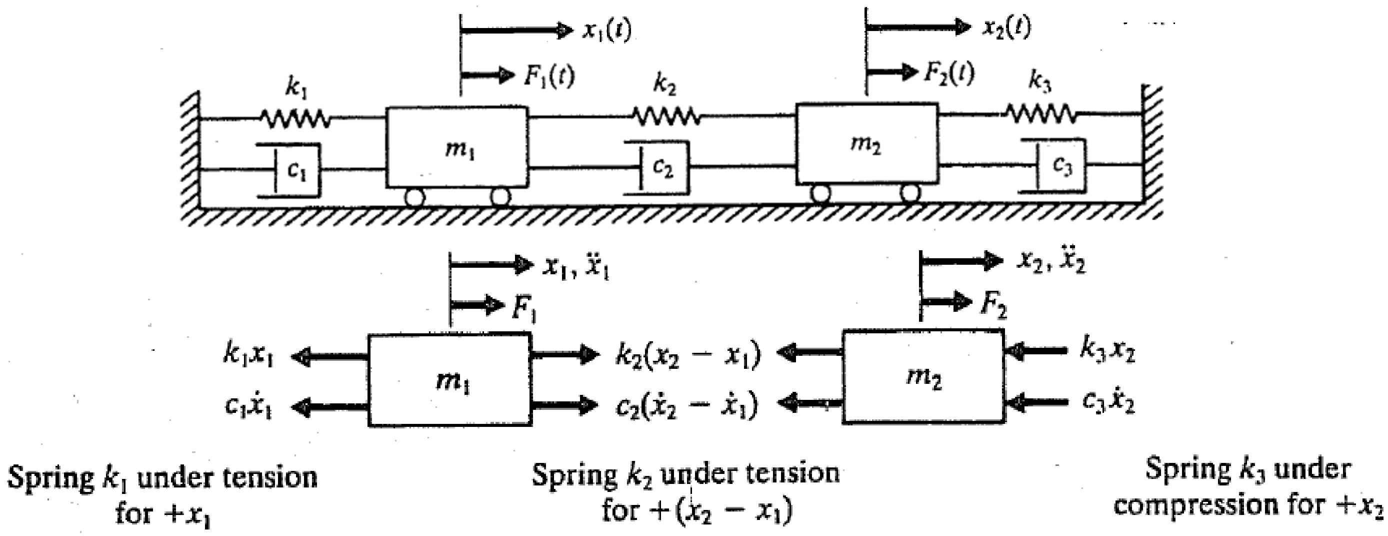


Equations of motion for forced vibration

- It can be seen that the first equation contains terms involving x_2, whereas the second equation contains terms involving x_1. Hence, they represent a system of two coupled second-order differential equations. We can therefore expect that the motion of the m_1 will influence the motion of m_2, and vica versa.

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$



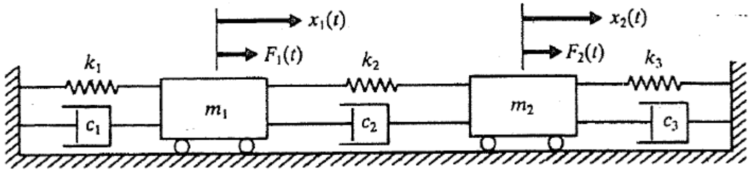
Equations of motion for forced vibration

- The equations can be written in matrix form as:

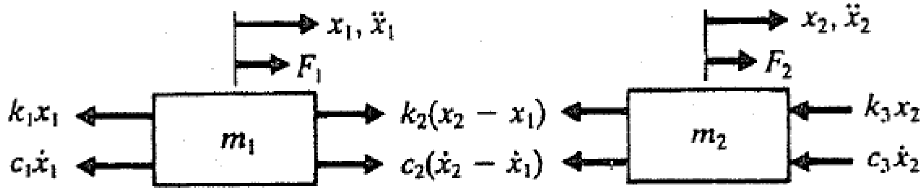
$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{F}(t)$$

where [m], [c] and [k] are mass, damping and stiffness matrices, respectively and x(t) and F(t) are called the displacement and force vectors, respectively.which are given by:

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \qquad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$



$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \qquad \vec{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$



Spring k_1 under tension for $+x_1$

Spring k_2 under tension for $+(x_2 - x_1)$

Spring k_3 under compression for $+x_2$

Equations of motion for forced vibration

- It can be seen that the matrices [m], [c] and [k] are all 2x2 matrices whose elements are the known masses, damping coefficienst, and stiffness of the system, respectively.
- Further, these matrices can be seen to be symmetric, so that:

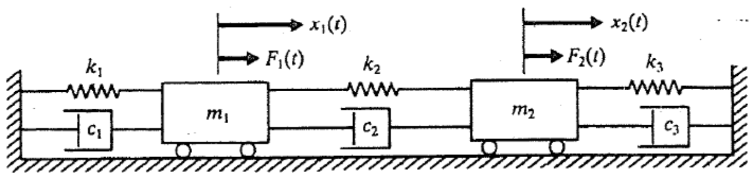
$$[m]^T = [m], \qquad [c]^T = [c], \qquad [k]^T = [k]$$

Free vibration analysis of an undamped system

- For the free vibration analysis of the system shown in the figure, we set $F_1(t)=F_2(t)=0$. Further, if the damping is disregarded, $c_1=c_2=c_3=0$, and the equations of motion reduce to:

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$



(a)

Free vibration analysis of an undamped system

- We are interested in knowing whether m_1 and m_2 can oscillate harmonically with the same frequency and phase angle but with different amplitudes. Assuming that it is possible to have harmonic motion of m_1 and m_2 at the same frequency ω and the same phase angle ϕ, we take the solutions to the equations

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$

as:

$$x_1(t) = X_1 \cos(\omega t + \phi)$$

$$x_2(t) = X_2 \cos(\omega t + \phi)$$

where X_1 and X_2 are constants that denote the maximum amplitudes of $x_1(t)$ and $x_2(t)$ and ϕ is the phase angle.Substituting the above two solutions into the first two equations, we have:

Free vibration analysis of an undamped system

$$[\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2X_2]\cos(\omega t + \phi) = 0$$
$$[-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2]\cos(\omega t + \phi) = 0$$

- Since the above equations must be satisfied for all values of time t, the terms between brackets must be zero. This yields,

$$\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2X_2 = 0$$
$$-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2 = 0$$

  which represents two simultaneous homogeneous algebraic equations in the unknowns X_1 and X_2. It can be seen that the above equation can be satisfied by the trivial soution $X_1=X_2=0$, which implies that there is no vibration. For a nontrivial solution of X_1 and X_2, the determinant of coefficients of X_1 and X_2 must be zero.

Free vibration analysis of an undamped system

$$\det\begin{bmatrix} \{-m_1\omega^2 + (k_1 + k_2)\} & -k_2 \\ -k_2 & \{m_2\omega^2 + (k_2 + k_3)\} \end{bmatrix} = 0$$

$$(m_1 m_2)\omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega^2 + \{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0$$

- The above equation is called the **frequency** or **characteristic equation** because solution of this equation yields the frequencies of the characteristic values of the system. The roots of the above equation are given by:

$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

Free vibration analysis of an undamped system

- This shows that it is possible for the system to have a nontrivial harmonic solution of the form

$$x_1(t) = X_1 \cos(\omega t + \phi)$$

$$x_2(t) = X_2 \cos(\omega t + \phi)$$

when $\omega=\omega_1$ and $\omega=\omega_2$ given by:

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

We shall denote the values of X_1 and X_2 corresponding to ω_1 as $X_1^{(1)}$ and $X_2^{(1)}$ and those corresponding to ω_2 as $X_1^{(2)}$ and $X_2^{(2)}$.

Free vibration analysis of an undamped system

- Further. since

$$\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2 X_2 = 0$$

$$-k_2 X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2 = 0$$

the above equation is homogeneous, only the ratios $r_1 = \{X_2^{(1)}/X_1^{(1)}\}$ and r2: $\{X_2^{(2)}/X_1^{(2)}\}$ can be found. For $\omega^2 = \omega_1^2$ and $\omega^2 = \omega_2^2$, the equations

$$\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2 X_2 = 0$$

$$-k_2 X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2 = 0$$

give:

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

- Notice that the two ratios are identical.

Free vibration analysis of an undamped system

- The normal modes of vibration corresponding to ω_1^2 and ω_2^2 can be expressed, respectively, as:

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix} \qquad \vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix}$$

- The vectors $\vec{X}^{(1)}$ and $\vec{X}^{(2)}$, which denote the normal modes of vibration are known as the **modal vectors of the system**. The free vibration solution or the motion in time can be expressed using

$$x_1(t) = X_1 \cos(\omega t + \phi)$$

$$x_2(t) = X_2 \cos(\omega t + \phi)$$

as:

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode}$$

where the constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 are determined by the initial conditions.

Free vibration analysis of an undamped system

Initial conditions:

Each of the two equations of motion ,

$$m_1\ddot{x}_1 + (c_1 + c_2)\,\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)\,x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\,\dot{x}_2 - k_2x_1 + (k_2 + k_3)\,x_2 = F_2$$

involves second order time derivatives; hence we need to specify two initial conditions for each mass.

The system can be made to vibrate in its ith normal mode (i=1,2) by subjecting it to the specific initial conditions.

$$x_1(t = 0) = X_1^{(i)} = \text{some constant}, \qquad \dot{x}_1(t = 0) = 0,$$

$$x_2(t = 0) = r_1X_1^{(i)}, \qquad \dot{x}_2(t = 0) = 0$$

However, for any other general initial conditions, both modes will be excited. The resulting motion, which is given by the general solution of the equations

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$

can be obtained by a linear superposition of two normal modes.

Free vibration analysis of an undamped system

Initial conditions:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

where c_1 and c_2 are constants.

Since $x_1^{(1)}(t)$ and $x_1^{(2)}(t)$ already involve the unknown constants $X_1^{(1)}$ and $X_1^{(2)}$ we can choose $c_1=c_2=1$ with no loss of generality. Thus, the components of the vector $\vec{x}(t)$ can be expressed as:

$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t)$$

$$= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

where the unknown $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 can be determined from the initial conditions

$$x_1(t = 0) = x_1(0), \qquad \dot{x}_1(t = 0) = \dot{x}_1(0),$$

$$x_2(t = 0) = x_2(0), \qquad \dot{x}_2(t = 0) = \dot{x}_2(0)$$

Free vibration analysis of an undamped system

$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t)$$

$$= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (5.15)$$

Thus if the initial conditions are given by

$$x_1(t = 0) = x_1(0), \qquad \dot{x}_1(t = 0) = \dot{x}_1(0),$$

$$x_2(t = 0) = x_2(0), \qquad \dot{x}_2(t = 0) = \dot{x}_2(0) \qquad (5.16)$$

the constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 can be found by solving the following equations (obtained by substituting Eqs. 5.16 into Eqs. 5.15):

$$x_1(0) = X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2$$

$$\dot{x}_1(0) = -\omega_1 X_1^{(1)} \sin \phi_1 - \omega_2 X_1^{(2)} \sin \phi_2$$

$$x_2(0) = r_1 X_1^{(1)} \cos \phi_1 + r_2 X_1^{(2)} \cos \phi_2$$

$$\dot{x}_2(0) = -\omega_1 r_1 X_1^{(1)} \sin \phi_1 - \omega_2 r_2 X_1^{(2)} \sin \phi_2 \qquad (5.17)$$

Free vibration analysis of an undamped system

Equations (5.17) can be regarded as four algebraic equations in the unknowns $X_1^{(1)} \cos \phi_1$, $X_1^{(2)} \cos \phi_2$, $X_1^{(1)} \sin \phi_1$, and $X_1^{(2)} \sin \phi_2$. The solution of Eqs. (5.17) can be expressed as

$$X_1^{(1)} \cos \phi_1 = \left\{ \frac{r_2 x_1(0) - x_2(0)}{r_2 - r_1} \right\}, \qquad X_1^{(2)} \cos \phi_2 = \left\{ \frac{-r_1 x_1(0) + x_2(0)}{r_2 - r_1} \right\}$$

$$X_1^{(1)} \sin \phi_1 = \left\{ \frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1 (r_2 - r_1)} \right\}, \qquad X_1^{(2)} \sin \phi_2 = \left\{ \frac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2 (r_2 - r_1)} \right\}$$

from which we obtain the desired solution

$$X_1^{(1)} = [\{X_1^{(1)} \cos \phi_1\}^2 + \{X_1^{(1)} \sin \phi_1\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)} \left[\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2 \dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2} \right]^{1/2}$$

$$X_1^{(2)} = [\{X_1^{(2)} \cos \phi_2\}^2 + \{X_1^{(2)} \sin \phi_2\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)} \left[\{-r_1 x_1(0) + x_2(0)\}^2 + \frac{\{r_1 \dot{x}_1(0) - \dot{x}_2(0)\}^2}{\omega_2^2} \right]^{1/2}$$

Free vibration analysis of an undamped system

from which we obtain the desired solution

$$X_1^{(1)} = [\{X_1^{(1)} \cos \phi_1\}^2 + \{X_1^{(1)} \sin \phi_1\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)}\left[\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2 \dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2}\right]^{1/2}$$

$$X_1^{(2)} = [\{X_1^{(2)} \cos \phi_2\}^2 + \{X_1^{(2)} \sin \phi_2\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)}\left[\{-r_1 x_1(0) + x_2(0)\}^2 + \frac{\{r_1 \dot{x}_1(0) - \dot{x}_2(0)\}^2}{\omega_2^2}\right]^{1/2}$$

$$\phi_1 = \tan^{-1}\left\{\frac{X_1^{(1)} \sin \phi_1}{X_1^{(1)} \cos \phi_1}\right\} = \tan^{-1}\left\{\frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1[r_2 x_1(0) - x_2(0)]}\right\}$$

$$\phi_2 = \tan^{-1}\left\{\frac{X_1^{(2)} \sin \phi_2}{X_1^{(2)} \cos \phi_2}\right\} = \tan^{-1}\left\{\frac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2[-r_1 x_1(0) + x_2(0)]}\right\}$$

Frequencies of a mass-spring system

Example: Find the natural frequencies and mode shapes of a spring mass system , which is constrained to move in the vertical direction.

Solution: The equations of motion are given by:

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0$$

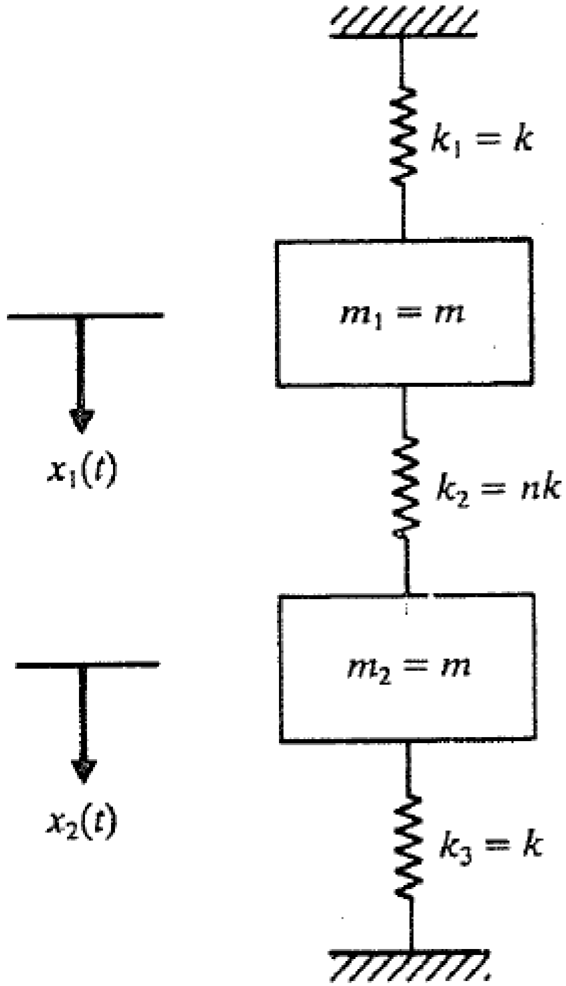
By assuming harmonic solution as:

$$x_i(t) = X_i \cos(\omega t + \phi);\ i = 1, 2$$

the frequency equation can be obtained by:

$$\begin{vmatrix} (-m\omega^2 + 2k) & (-k) \\ (-k) & (-m\omega^2 + 2k) \end{vmatrix} = 0$$

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$



Frequencies of a mass-spring system

$$\begin{vmatrix} (-m\omega^2 + 2k) & (-k) \\ (-k) & (-m\omega^2 + 2k) \end{vmatrix} = 0$$

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

- The solution to the above equation gives the natural frequencies:

$$\omega_1 = \left\{ \frac{4km - [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \left\{ \frac{4km + [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{3k}{m}}$$

Frequencies of a mass-spring system

- From

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

the amplitude ratios are given by:

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\omega_1^2 + 2k}{k} = \frac{k}{-m\omega_1^2 + 2k} = 1$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + 2k}{k} = \frac{k}{-m\omega_2^2 + 2k} = -1$$

Frequencies of a mass-spring system

- From $$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode}$$
$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode}$$
- The natural modes are given by

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \end{Bmatrix}$$

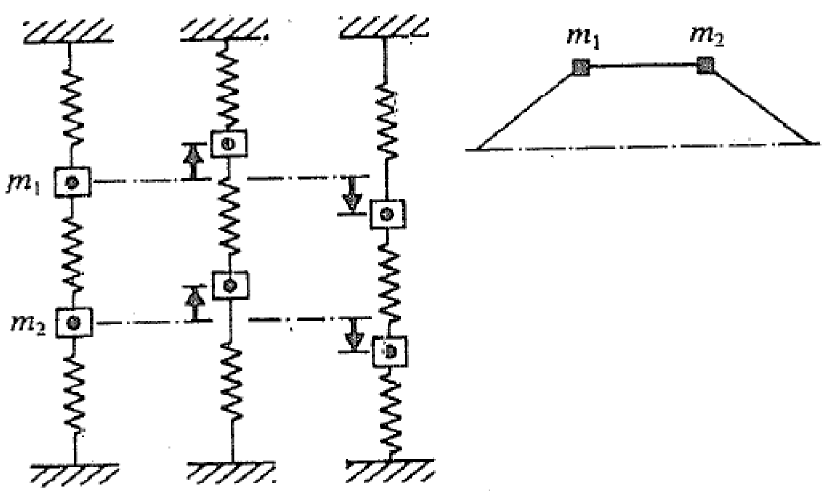
$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

Frequencies of a mass-spring system

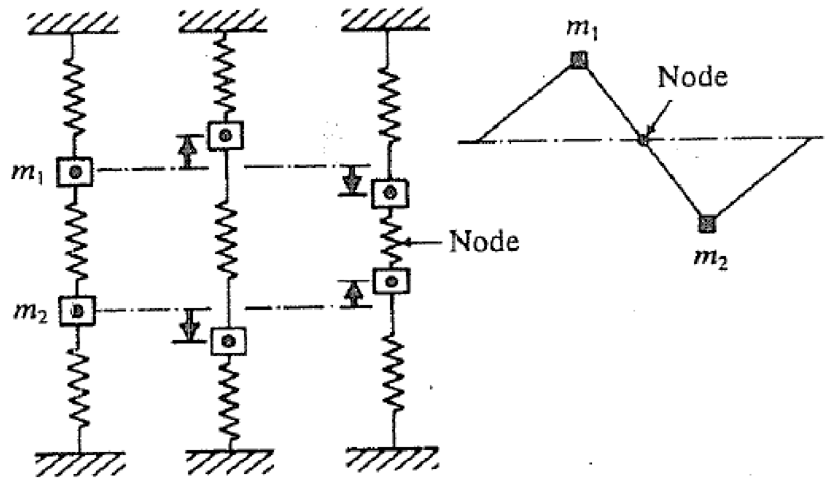
- The natural modes are given by:

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\,t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\,t + \phi_1\right) \end{Bmatrix}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\,t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\,t + \phi_2\right) \end{Bmatrix}$$



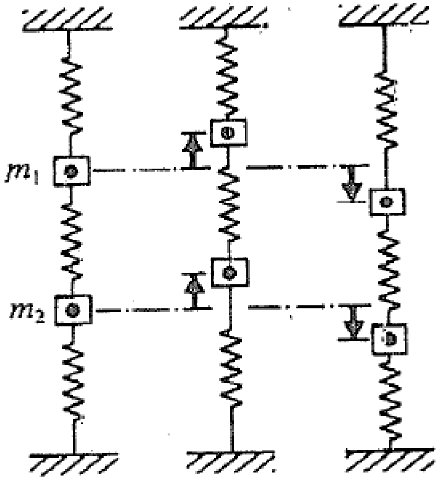
(a) First mode



(b) Second mode

Frequencies of a mass-spring system

- It can be seen that when the system vibrates in its first mode, the amplitudes of the two masses remain the same. This implies that the length of the middle spring remains constant. Thus the motions of the mass 1 and mass 2 are in phase.



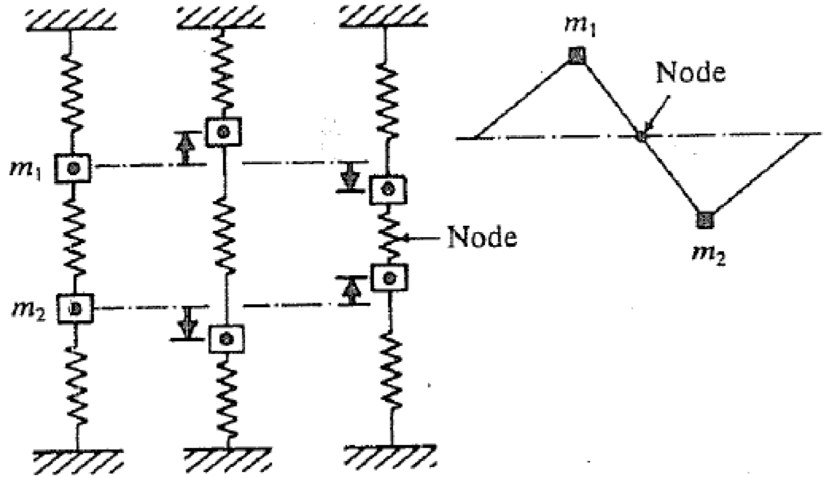
(a) First mode

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \end{Bmatrix}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

Frequencies of a mass-spring system

- When the system vibrates in its second mode, the equations below show that the displacements of the two masses have the same magnitude with opposite signs. Thus the motions of the mass 1 and mass 2 are out of phase. In this case, the midpoint of the middle spring remains stationary for all time. Such a point is called a **node**.



(b) Second mode

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \end{Bmatrix}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

Frequencies of a mass-spring system

- Using equations

$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \end{Bmatrix}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

the motion (general solution) of the system can be expressed as:

$$x_1(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right)$$

$$x_2(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right)$$

Forced vibration analysis

- The equation of motion of a general two degree of freedom system under external forces can be written as:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

- We shall consider the external forces to be harmonic:

$$F_j(t) = F_{j0} e^{i\omega t}, \qquad j = 1, 2$$

where ω is the forcing frequency. We can write the steady state solution as:

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1, 2$$

where X_1 and X_2 are, in general, complex quantities that depend on ω and the system parameters. Substituting the above two equations into the first one:

Forced vibration analysis

- We obtain:

$$\begin{bmatrix} (-\omega^2 m_{11} + i\omega c_{11} + k_{11}) & (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) & (-\omega_2 m_{22} + i\omega c_{22} + k_{22}) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

(

- If we define a term called 'mechanical impedance' $Z_{rs}(i\omega)$ as:

$$Z_{rs}(i\omega) = -\omega^2 m_{rs} + i\omega c_{rs} + k_{rs}, \qquad r, s = 1, 2$$

and write the first equation as: $[Z(i\omega)]\vec{X} = \vec{F}_0$

where

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{12}(i\omega) & Z_{22}(i\omega) \end{bmatrix} = \text{Impedance matrix}$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \qquad \vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Forced vibration analysis

- The equation $[Z(i\omega)]\vec{X} = \vec{F}_0$

  can be solved to obtain: $X = [Z(i\omega)]^{-1}\vec{F}_0$
- Where the inverse of the impedance matrix is given by:

$$[Z(i\omega)]^{-1} = \frac{1}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}\begin{bmatrix} Z_{22}(i\omega) & -Z_{12}(i\omega) \\ -Z_{12}(i\omega) & Z_{11}(i\omega) \end{bmatrix}$$

- Therefore, the solutions are:

$$X_1(i\omega) = \frac{Z_{22}(i\omega)F_{10} - Z_{12}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)F_{10} + Z_{11}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

- By substituting these into the below equation, the solutions can be obtained.

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1, 2$$

Multi-degree of freedom systems

- Modeling of continuous systems as multidegree of freedom systems
- Eigenvalue problem

Multidegree of freedom systems

- As stated before, most engineering systems are **continuous** and have an **infinite number of degrees of freedom**. The vibration analysis of continuous systems requires the solution of partial differential equations, which is quite difficult.

- In fact, analytical solutions do not exist for many partial differential equations. The analysis of a multidegree of freedom system on the other hand, requires the solution of a set of ordinary differential equations, which is relatively simple. Hence, for simplicity of analysis, **continuous systems** are often approximated as **multidegree of freedom systems**.

- For a system having n degrees of freedom, there are n associated natural frequencies, each associated with its own mode shape.

Multidegree of freedom systems

- Different methods can be used to **approximate a continuous system** as a **multidegree of freedom system**. A simple method involves **replacing the distributed mass** or inertia of the system **by a finite number of lumped masses** or rigid bodies.

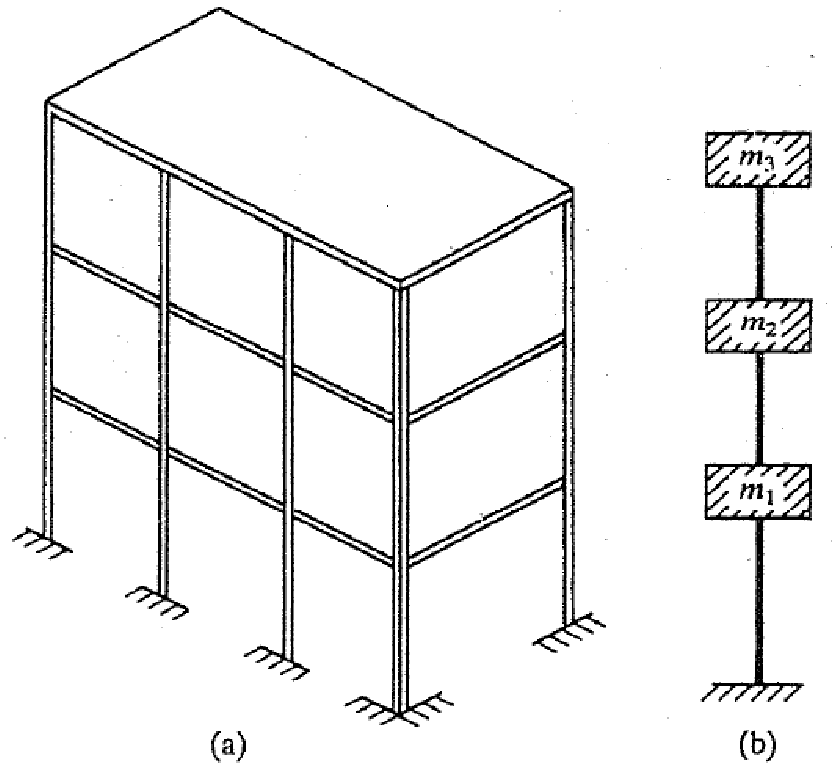
- The lumped masses are assumed to be connected by massless elastic and damping members.

- Linear coordinates are used to describe the motion of the lumped masses. Such models are called **lumped parameter of lumped mass or discrete mass systems**.

- The **minimum number of coordinates** necessary to describe the motion of the lumped masses and rigid bodies defines **the number of degrees of freedom** of the system. Naturally, the larger the number of lumped masses used in the model, the higher the accuracy of the resulting analysis.

Multidegree of freedom systems

- Some problems automatically indicate the type of lumped parameter model to be used.
- For example, the three storey building shown in the figure automatically suggests using a three lumped mass model as indicated in the figure.
- In this model, the inertia of the system is assumed to be concentrated as three point masses located at the floor levels, and the elasticities of the columns are replaced by the springs.



Multidegree of freedom systems

- Another popular method of approximating a continuous system as a multidegree of freedom system involves replacing the geometry of the system by a large number of small elements.
- By assuming a simple solution within each element, the principles of **compatibility** and **equilibrium** are used to find an approximate solution to the original system. This method is known as the **finite element method**.

Using Newton’s second law to derive equations of motion

The following procedure can be adopted to derive the equations of motion of a multidegree of freedom system using Newton’s second law of motion.

1. Set up suitable coordinates to describe the positions of the various point masses and rigid bodies in the system. Assume suitable positive directions for the displacements, velocities and accelerations of the masses and rigid bodies.

2. Determine the static equilibrium configuration of the system and measure the displacements of the masses and rigid bodies from their respective static equilibrium positions.

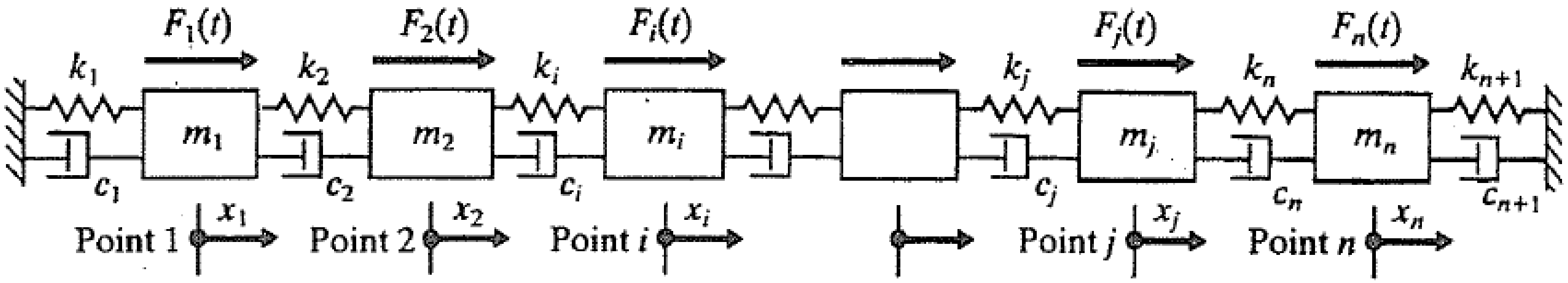
3. Draw the free body diagram of each mass or rigid body in the system. Indicate the spring, damping and external forces acting on each mass or rigid body when positive displacement or velocity are given to that mass or rigid body.

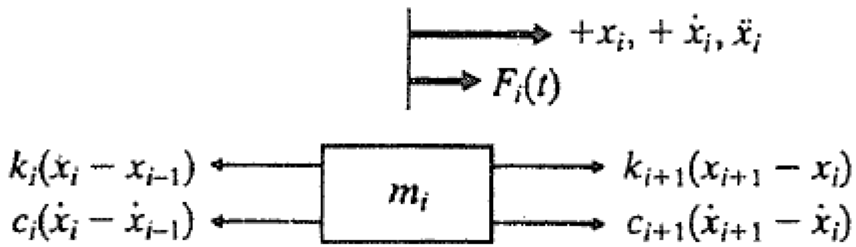
Using Newton's second law to derive equations of motion

4. Apply Newton's second law of motion to each mass or rigid body shown by the free body diagram as:

$$m_i \ddot{x}_i = \sum_j F_{ij} \text{ (for mass } m_i\text{)}$$

Example: Derive the equations of motion of the spring-mass-damper system shown in the figure.





Using Newton's second law to derive equations of motion

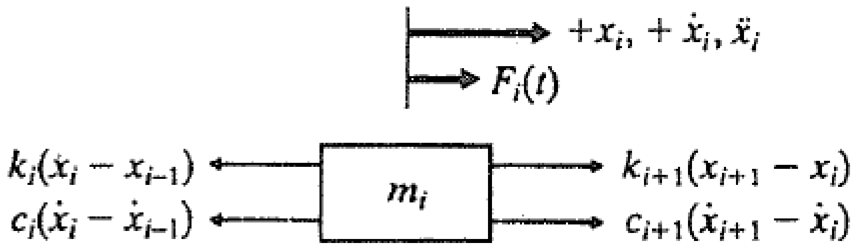
- Draw free-body diagrams of masses and apply Newton's second law of motion. The coordinates describing the positions of the masses, $x_i(t)$, are measured from their respective static equilibrium positions, as indicated in the figure. The application of the Newton's second law of motion to mass m_i gives:

$$m_i\ddot{x}_i = -k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) - c_i(\dot{x}_i - \dot{x}_{i-1})$$
$$+ c_{i+1}(\dot{x}_{i+1} - \dot{x}_i) + F_i\,;\; i = 2, 3, \ldots, n-1$$

- or

$$m_i\ddot{x}_i - c_i\dot{x}_{i-1} + (c_i + c_{i+1})\dot{x}_i - c_{i+1}\dot{x}_{i+1} - k_i x_{i-1}$$
$$+ (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = F_i\,;\; i = 2, 3, \ldots, n-1$$

- The equations of motion of the masses m_1 and m_2 can be derived from the above equations by setting i=1 along with x_o=0 and i=n along with x_{n+1}=0, respectively.



Equations of motion in matrix form

- The equations of motion in matrix form in the above example can be expressed as:

$$[m]\,\ddot{\vec{x}} + [c]\,\dot{\vec{x}} + [k]\,\vec{x} = \vec{F}$$

where $[m]$, $[c]$, and $[k]$ are called the mass, damping, and stiffness matrices, respectively, and are given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & m_3 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & 0 & m_n \end{bmatrix}$$

$$[c] = \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & (c_3 + c_4) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -c_n & (c_n + c_{n+1}) \end{bmatrix}$$

Equations of motion in matrix form

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & (k_3 + k_4) & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & -k_n & (k_n + k_{n+1}) \end{bmatrix}$$

and $\vec{x}$, $\dot{\vec{x}}$, $\ddot{\vec{x}}$, and $\vec{F}$ are the displacement, velocity, acceleration, and force vectors, given by

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{Bmatrix}, \qquad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(t) \end{Bmatrix},$$

$$\ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n(t) \end{Bmatrix}, \qquad \vec{F} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ \cdot \\ \cdot \\ \cdot \\ F_n(t) \end{Bmatrix}$$

Equations of motion in matrix form

- For an undamped system, the equations of motion reduce to:

$$[m]\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{F}$$

- The differential equations of the spring-mass system considered in the example, can be seen to be coupled. Each equation involves more than one coordinate. This means that the equations can not be solved individually one at a time; they can only be solved simultaneously.

- In addition, the system can be seen to be statically coupled since stiffnesses are coupled- that is the stiffness matrix has at least one nonzero off-diagonal term. On the other hand, if the mass matrix has at least one off-diagonal term nonzero, the system is said to be dynamically coupled. Further, if both the stiffness and the mass matrices have nonzero off-diagonal terms, the system is said to be coupled both statically and dynamically.

Undamped free vibrations

- The equations of motion for a freely vibrating undamped system can be obtained by omitting the damping matrix and applied load vector from:

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{0}$$

  in which **0** is a zero vector. The problem of vibration analysis consists of determining the conditions under which the equilibrium condition expressed by the above equation will be satisfied.

- By analogy with the behaviour of SDOF systems, it will be assumed that the free-vibration motion is simple harmonic (the first equation below), which may be expressed for a multi degree of freedom system as:

$$\mathbf{x}(t) = \hat{\mathbf{x}}\sin(\omega t + \theta)$$

$$\ddot{\mathbf{x}} = -\omega^2\hat{\mathbf{x}}\sin(\omega t + \theta) = -\omega^2\mathbf{x}$$

- In the above expressions, $\hat{\mathbf{x}}$ represents the shape of the system (which does not change with time; only the amplitude varies) and θ is a phase angle. The third equation above represents the accelerations in the free vibration.

Undamped free vibrations

- Substituting

$$\mathbf{x}(t) = \hat{\mathbf{x}} \sin(\omega t + \theta)$$
$$\ddot{\mathbf{x}} = -\omega^2 \hat{\mathbf{x}} \sin(\omega t + \theta) = -\omega^2 \mathbf{x}$$

in the equation

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{0}$$

we obtain:

$$-\omega^2 \mathbf{m}\hat{\mathbf{x}} \sin(\omega t + \theta) + \mathbf{k}\hat{\mathbf{x}} \sin(\omega t + \theta) = \mathbf{0}$$

which (since the sine term is arbitrary and may be omitted) may be written:

$$\left[\mathbf{k} - \omega^2 \mathbf{m}\right]\hat{\mathbf{x}} = \mathbf{0}$$

- The above equation is one way of expressing what is called an **eigenvalue or characteristic value problem**. The quantities ω^2 are the **eigenvalues or characteristic values indicating the square of the free-vibration frequencies**, while the corresponding displacement vectors $\hat{\mathbf{x}}$ express the corresponding shapes of the vibrating system- known as the **eigenvectors or mode shapes**.

Undamped free vibrations

- It can be shown by Cramer's rule that the solution of this set of simultaneous equations is of the form:

$$\hat{\mathbf{x}} = \frac{\mathbf{0}}{\left\| \mathbf{k} - \omega^2 \mathbf{m} \right\|}$$

- Hence a nontrivial solution is possible only when the denominator determinant vanishes. In other words, finite amplitude free vibrations are possible only when

$$\left\| \mathbf{k} - \omega^2 \mathbf{m} \right\| = 0$$

- The above equation is called the frequency equation of the system. Expanding the determinant will give an algebraic equation of the Nth degree in the frequency parameter ω^2 for a system having N degrees of freedom.
- The N roots of this equation $\left(\omega_1^2, \omega_2^2, \omega_3^2, \ldots, \omega_N^2\right)$ represent the frequencies of the N modes of vibration which are possible in the system.

Undamped free vibrations

- The mode having the lowest frequency is called the first mode, the next higher frequency is the second mode, etc.
- The vector made up of the entire set of modal frequencies, arranged in sequence, will be called the frequency vector ω.

$$\omega = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_N \end{Bmatrix}$$

Normalization:

It was noted earlier that the vibration mode amplitudes obtained from the eigenproblem solution are arbitrary; any amplitude will satisfy the basic frequency equation

$$\left\| \mathbf{k} - \omega^2 \mathbf{m} \right\| = 0$$

and only the resulting shapes are uniquely defined.

Normalization of modes

- In the analysis process described above, the amplitude of one degree of freedom (the first actually) has been set to unity, and the other displacements have been determined relative to this reference value. This is called **normalizing the mode shapes** with respect to the specified reference coordinate.

- Other normalizing procedures also are frequently used; e.g., in many computer programs, the shapes are normalized relative to the maximum displacement value in each mode rather than with respect to any particular coordinate. Thus, the maximum value in each modal vector is unity, which provides convenient numbers for use in subsequent calculations.

Normalization of modes

- The normalizing procedure most often used in computer programs for structural vibration analysis, however, involves adjusting each modal amplitude to the amplitude $\hat{\phi}_n$ which satisfies the condition

$$\hat{\phi}_n^T \mathbf{m} \hat{\phi}_n = 1$$

- This can be accomplished by computing the scalar factor

$$\hat{\mathbf{v}}_{\mathbf{n}}^{\mathbf{T}} \mathbf{m} \hat{\mathbf{v}}_{\mathbf{m}} = \hat{M}_n$$

where $\hat{\mathbf{v}}_{\mathbf{n}}$ represents an arbitrarily determined modal amplitude, and then computing the normalized mode shapes as follows:

$$\hat{\phi}_n = \hat{\mathbf{v}}_{\mathbf{n}} \hat{M}_n^{-1/2}$$

By simple substitution, it is easy to show that this givs the desired result. A consequence of this type of normalizing together with the modal orthogonality relationships relative to the mass matrix is that

$$\hat{\boldsymbol{\varphi}}_{\mathbf{n}}^{\mathbf{T}} \mathbf{m} \hat{\boldsymbol{\varphi}}_{\mathbf{n}} = \mathbf{I}$$

where ϕ is the complete set of N normalized mode shapes and I is an NxN identity matrix. The mode shapes normalized in this fashion are said to be orthonormal relative to the mass matrix.

Mode shapes of a four storey 2D frame

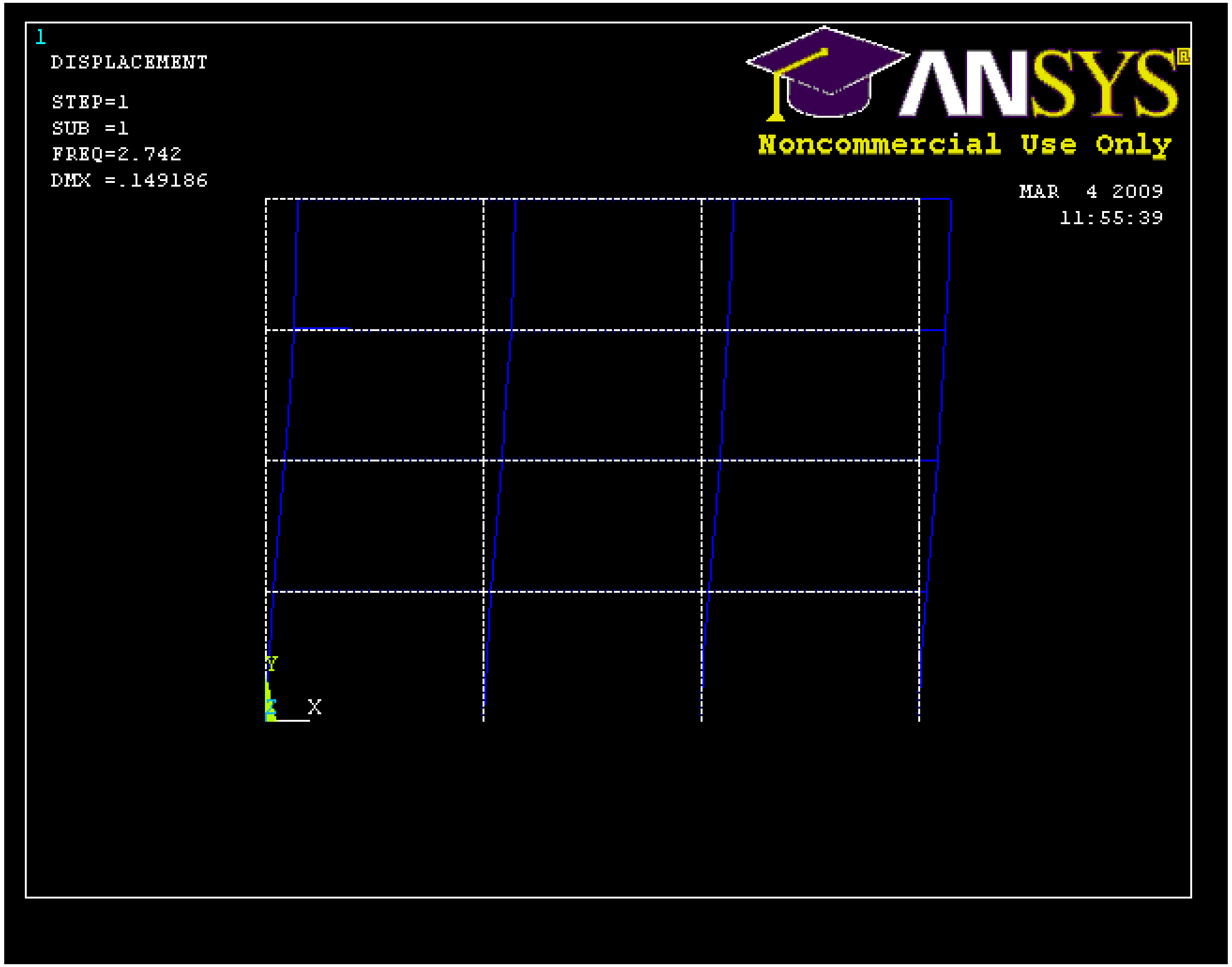
- A model of a four-story three-bay frame can be evaluated to determine the mode shapes. This 2 D model is from a typical building from the Marmara region in Turkey.

- Generally, the first mode of vibration is the one of primary interest. The first mode usually has the largest contribution to the structure's motion. The period of this mode is the longest and the natural frequency is the lowest.

- Please click on the movie to start!

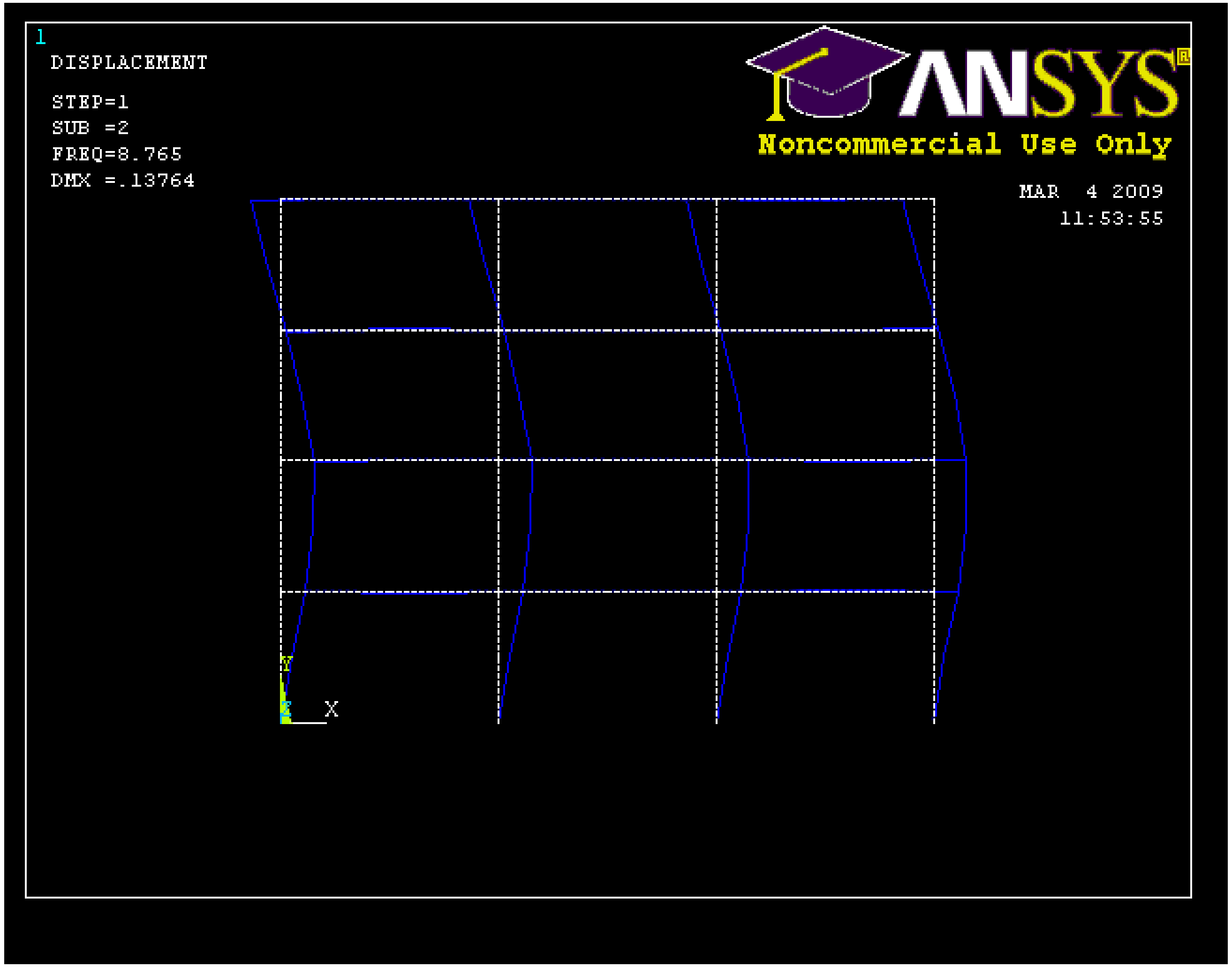
Mode shapes of a four storey 2D frame

- First mode shape



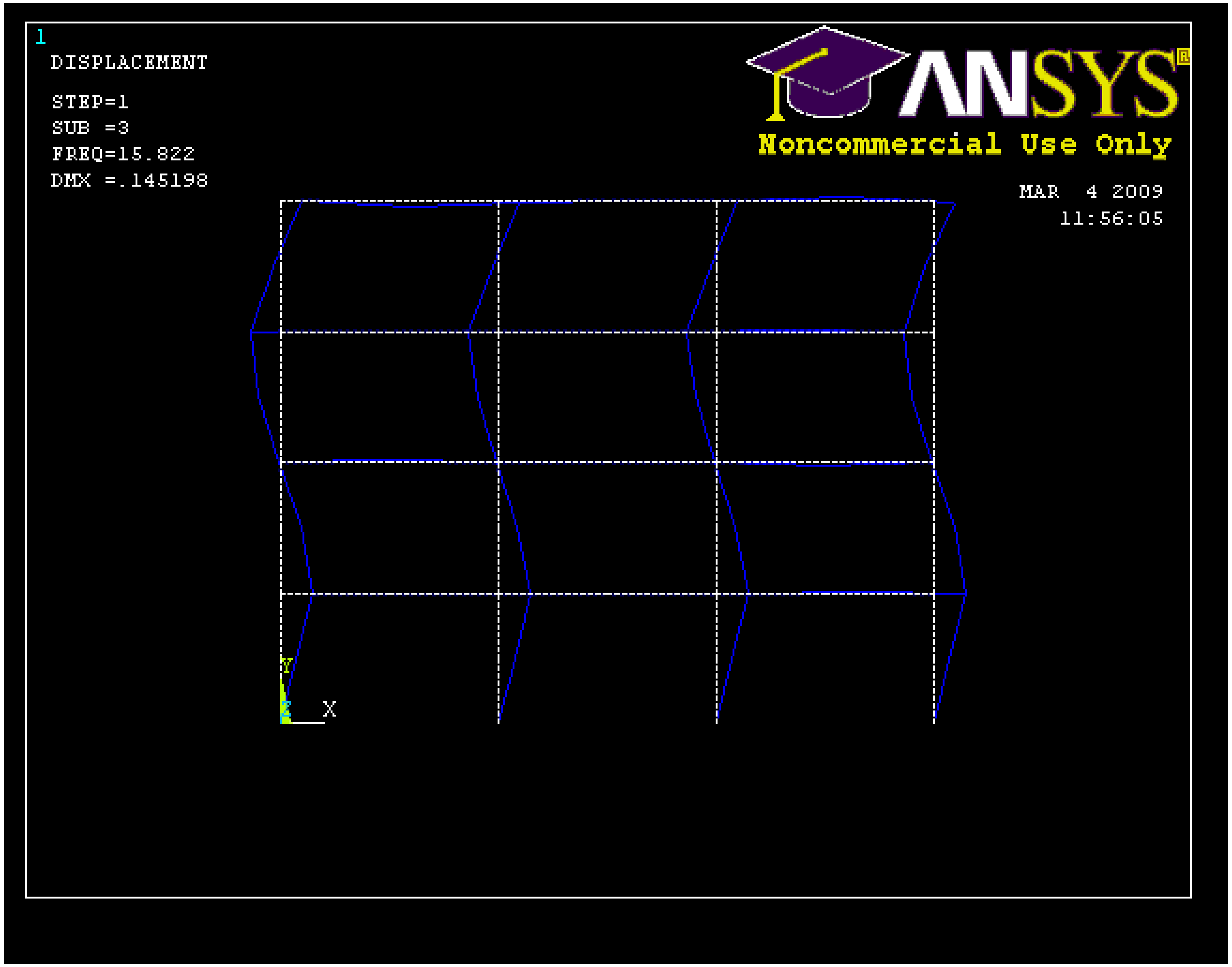
Mode shapes of a four storey 2D frame

- Second mode shape



Mode shapes of a four storey 2D frame

- Third mode shape



Undamped free vibrations

Example:

Determine the eigenvalues and eigenvectors of a vibrating system for which

$$[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [k] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Given: Mass and stiffness matrices.

Find: Eigenvalues and eigenvectors.

Example

Solution: The eigenvalue equation $[[k] - \lambda[m]]\vec{X} = \vec{0}$ can be written in the form

$$\begin{bmatrix} (1-\lambda) & -2 & 1 \\ -2 & 2(2-\lambda) & -2 \\ 1 & -2 & (1-\lambda) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

where $\lambda = \omega^2$. The characteristic equation gives

$$|[k] - \lambda[m]| = \lambda^2(\lambda - 4) = 0$$

so

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 4$$

Eigenvector for $\lambda_3 = 4$: Using $\lambda_3 = 4$, Eq. (E.1) gives

$$-3X_1^{(3)} - 2X_2^{(3)} + X_3^{(3)} = 0$$

$$-2X_1^{(3)} - 4X_2^{(3)} - 2X_3^{(3)} = 0$$

$$X_1^{(3)} - 2X_2^{(3)} - 3X_3^{(3)} = 0$$

If $X_1^{(3)}$ is set equal to 1, Eqs. (E.3) give the eigenvector $\vec{X}^{(3)}$:

$$\vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

Example

Solution:

When the characteristic equation possesses repeated roots, the corresponding mode shapes are not unique.

Eigenvector for $\lambda_1 = \lambda_2 = 0$: The value $\lambda_1 = 0$ or $\lambda_2 = 0$ indicates that the system is degenerate (see Section 6.12). Using $\lambda_1 = 0$ in Eq. (E.1), we obtain

$$X_1^{(1)} - 2X_2^{(1)} + X_3^{(1)} = 0$$

$$-2X_1^{(1)} + 4X_2^{(1)} - 2X_3^{(1)} = 0$$

$$X_1^{(1)} - 2X_2^{(1)} + X_3^{(1)} = 0 \qquad \text{(E.5)}$$

Example

Solution:

All these equations are of the form

$$X_1^{(1)} = 2\,X_2^{(1)} - X_3^{(1)}$$

Thus the eigenvector corresponding to $\lambda_1 = \lambda_2 = 0$ can be written

$$\vec{X}^{(1)} = \begin{Bmatrix} 2\,X_2^{(1)} - X_3^{(1)} \\ X_2^{(1)} \\ X_3^{(1)} \end{Bmatrix}$$

If we choose $X_2^{(1)} = 1$ and $X_3^{(1)} = 1$, we obtain

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

If we select $X_2^{(1)} = 1$ and $X_3^{(1)} = -1$, Eq. (E.6) gives

$$\vec{X}^{(1)} = \begin{Bmatrix} 3 \\ 1 \\ -1 \end{Bmatrix}$$

Rigid body motion

- An unrestrained system is one that has no restraints or supports and that can move as a rigid body. It is not uncommon to see in practice systems that are not attached to any stationary frame.
- Such systems are capable of moving as rigid bodies, which can be considered as modes of oscillation with zero frequency.
- A semidefinite system such as this, has a singular stiffness matrix. In systems that are not properly restrained, rigid-body displacements can take place without the application of any force. Thus, denoting a possible rigid-body displacement by $\mathbf{u}_r$, we have

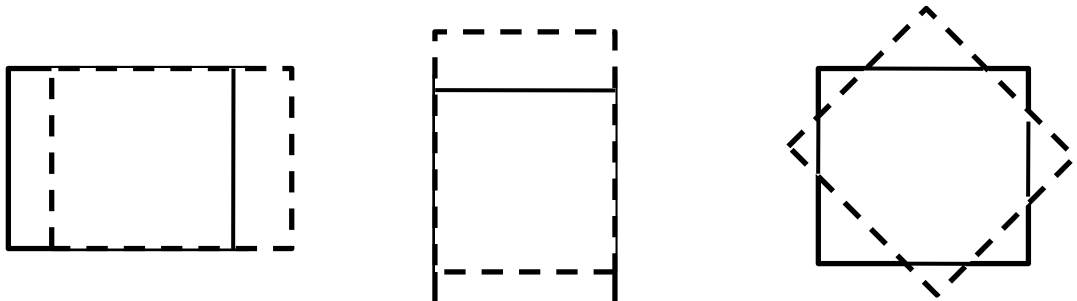
$$\mathbf{f}_\mathbf{r} = \mathbf{K}\mathbf{u}_\mathbf{r} = \mathbf{0}$$

- For a nonzero $\mathbf{u}_r$, the above equation can be satisfied provided only that **K** is singular. In this case, the below equation can only be satisfied when ω=0.

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{u}_\mathbf{r} = \mathbf{0}$$

Rigid body motion

- The rigid body displacements are those displacement modes that the element must be able to undergo as a rigid body without stresses being developed in it.
- Rigid body displacement shapes are also referred to as rigid body modes.
- A system can, of course, have more than one rigid body mode. In the most general case, up to six rigid body modes are possible. For example, a spacecraft or an aeroplane in flight has all six possible rigid-body modes, three translations and three rotations, one along each of the three axis.



Rigid body modes of a plane stress element

Orthogonality of modes

- The natural modes corresponding to different natural frequencies can be shown to satisfy the following orthogonality conditions. When $\omega_n \neq \omega_r$:

$$\phi_n^T \mathbf{k} \phi_r = 0 \qquad \phi_n^T \mathbf{m} \phi_r = 0$$

- **Proof:** The nth natural frequency and mode satisfy

$$\mathbf{k}\phi_n = \omega_n^2 \mathbf{m} \phi_n$$

Premultiplying the above equation by ϕ_r^T

$$\phi_r^T \mathbf{k} \phi_n = \omega_n^2 \phi_r^T \mathbf{m} \phi_n$$

Similarly the rth natural frequency and mode shape satisfy

$$\mathbf{k}\phi_r = \omega_r^2 \mathbf{m} \phi_r$$

Orthogonality of modes

Premultiplying $\mathbf{k}\phi_r = \omega_r^2 \mathbf{m}\phi_r$ by ϕ_n^T gives:

$$\phi_n^T \mathbf{k}\phi_r = \omega_r^2 \phi_n^T \mathbf{m}\phi_r$$

The transpose of the matrix on the left side of $\phi_r^T \mathbf{k}\phi_n = \omega_n^2 \phi_r^T \mathbf{m}\phi_n$ will equal the transpose of the matrix on the right side of the equation:

$$\phi_n^T \mathbf{k}\phi_r = \omega_n^2 \phi_n^T \mathbf{m}\phi_r$$

Subtracting the first equation from the second equation:

$$\left(\omega_n^2 - \omega_r^2\right)\phi_n^T \mathbf{m}\phi_r = 0$$

The equation $\phi_n^T \mathbf{m}\phi_r = 0$ is true when $\omega_n \neq \omega_r$ which for systems with positive natural frequencies implies that $\omega_n \neq \omega_r$

Modal equations for undamped systems

- The equations of motion for a linear MDOF system without damping is:

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{p}(t)$$

- The simultaneous solution of these coupled equations of motion that we have illustrated before for a 2 dof system subjected to harmonic excitation is not efficient for systems with more DOF, nor is it feasible for systems excited by other types of forces. Consequently, it is advantegous to transform these equations to modal coordinates.
- The displacement vector **x** of a MDOF system can be expanded in terms of modal contributions. Thus, the dynamic response of a system can be expressed as:

$$\mathbf{x}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \boldsymbol{\varphi}\mathbf{q}(t)$$

Modal equations for undamped systems

- Using the equation $\mathbf{x}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \boldsymbol{\varphi}\mathbf{q}(t)$, the coupled equations in $x_j(t)$ given below

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{p}(t)$$

can be transformed to a set of uncoupled equations with modal coordinates $q_n(t)$ as the unknowns. Substituting the first equation into the second:

$$\sum_{r=1}^{N} \mathbf{m}\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \mathbf{k}\phi_r q_r(t) = \mathbf{p}(t)$$

Premultiplying each term in this equation by ϕ_n^T gives:

$$\sum_{r=1}^{N} \phi_n^T \mathbf{m}\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{k}\phi_r q_r(t) = \phi_n^T \mathbf{p}(t)$$

Modal equations for undamped systems

- Because of the orthogonality relations $\phi_n^T \mathbf{k} \phi_r = 0 \qquad \phi_n^T \mathbf{m} \phi_r = 0$, all terms in each of the summations vanish except the r=n term, reducing the equation to:

$$\left(\phi_n^T \mathbf{m} \phi_n\right) \ddot{q}_n(t) + \left(\phi_n^T \mathbf{k} \phi_n\right) q_n(t) = \phi_n^T \mathbf{p}(t)$$

or

$$M_n \ddot{q}_n(t) + K_n q_n(t) = P_n(t)$$

where

$$M_n = \phi_n^T \mathbf{m} \phi_n \qquad K_n = \phi_n^T \mathbf{k} \phi_n \qquad P_n(t) = \phi_n^T \mathbf{p}(t)$$

- The above equation may be interpreted as the equation governing the response $q_n(t)$ of the SDOF system with mass M_n, stiffness K_n, and exciting force $P_n(t)$.
- Therefore M_n is called the generalized mass for the nth natural mode, K_n the generalized stiffness for the nth mode, and $P_n(t)$ the generalized force for the nth mode. These parameters only depend on the nth mode.

Modal equations for damped systems

- When damping is included, the equations of motion for a MDOF system are:

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{p}(t)$$

- Using the transformation

$$\mathbf{x}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \boldsymbol{\varphi}\mathbf{q}(t)$$

where ϕ_r are the natural modes of the system without damping, these equations can be written in terms of the modal coordinates. Unlike the case of undamped systems, these modal equations may be coupled through the damping terms. However, for certain forms of damping that are reasonable idealizations for many structures, the equations become uncoupled, just as for undamped systems. Substituting the second equation into the first, we obtain:

$$\sum_{r=1}^{N} \mathbf{m}\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \mathbf{c}\phi_r \dot{q}_r(t) + \sum_{r=1}^{N} \mathbf{k}\phi_r q_r(t) = \mathbf{p}(t)$$

Modal equations for damped systems

- Premultiplying each term in this equation by ϕ_n^T gives:

$$\sum_{r=1}^{N} \phi_n^T \mathbf{m} \phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{c} \phi_r \dot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{k} \phi_r q_r(t) = \phi_n^T \mathbf{p}(t)$$

which can be rewritten as:

$$M_n \ddot{q}_n(t) + \sum_{r=1}^{N} C_{nr} \dot{q}_r(t) + K_n q_n(t) = P_n(t)$$

where

$$C_{nr} = \phi_n^T \mathbf{c} \phi_r$$

The above N equations can be written in matrix form as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$$

Here C is a nondiagonal matrix of coefficients C_{nr}.

Modal equations for damped systems

- The modal equations will be uncoupled if the system has classical damping. For such systems $C_{nr}=0$ if $n \neq r$ and C_n can be expressed as:

$$C_n = 2\zeta_n M_n \omega_n$$

- For such systems:

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_n(t)$$

- Dividing by M_n:

$$\ddot{q}_n + 2\zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n}$$

where ζ_n is the damping ratio for the nth mode.

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DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT 3-THEORY OF VIBRATIONS-SME1306

CHAPTER 9
MULTI-DEGREE-OF-FREEDOM SYSTEMS
Equations of Motion, Problem Statement, and Solution Methods

Two-story shear building

A shear building is the building whose floor systems are rigid in flexure and several factors are neglected, for example, axial deformation of beams and columns.

We will formulate the equations of motion of a simple 2-story shear building whose mass are lumped at the floor.

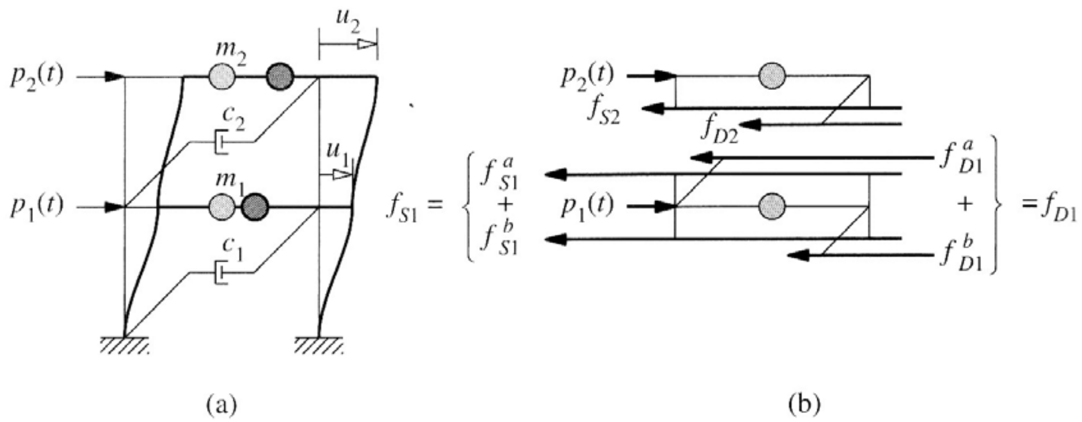


Figure 9.1.1 (a) Two-story shear frame; (b) forces acting on the two masses.

The equations of motion are formulated by considering equilibrium of forces acting on each mass. Any of the two approaches can be used

(1) Newton's second law of motion

(2) D'Alembert's principle of dynamic equilibrium

Newton's second law of motion

$$\sum F = m\ddot{u}$$

For each floor mass (j=1 and 2)

$$p_j - f_{Sj} - f_{Dj} = m_j\ddot{u}_j \quad \text{or} \quad m_j\ddot{u}_j + f_{Sj} + f_{Dj} = p_j(t)$$

Two equations can be written in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{Bmatrix} f_{D1} \\ f_{D2} \end{Bmatrix} + \begin{Bmatrix} f_{S1} \\ f_{S2} \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

$$\mathbf{m\ddot{u}} + \mathbf{f}_D + \mathbf{f}_S = \mathbf{p}(t)$$

where

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \mathbf{m} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \mathbf{f}_D = \begin{Bmatrix} f_{D1} \\ f_{D2} \end{Bmatrix} \quad \mathbf{f}_S = \begin{Bmatrix} f_{S1} \\ f_{S2} \end{Bmatrix} \quad \mathbf{p} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

Because all beams are assumed rigid, the story shear force can be directly related to the relative displacement between stories.

$$V_j = k_j \Delta_j$$

where $\Delta_j = u_{j+1} - u_j$ and $k_j = \sum_{\text{columns}} \frac{12EI_c}{h^3}$

The elastic force acting on the first story mass comes from columns below (f_{S1}^b) and above (f_{S1}^a) the floor.

$$f_{S1} = f_{S1}^b + f_{S1}^a$$

$$f_{S1} = k_1 u_1 + k_2(u_1 - u_2)$$

$$f_{S2} = k_2(u_2 - u_1)$$

$$\begin{Bmatrix} f_{S1} \\ f_{S2} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{or} \qquad \mathbf{f}_S = \mathbf{k}\mathbf{u}$$

The elastic resisting force vector $\mathbf{f}_S$ is related to displacement vector $\mathbf{u}$ through the stiffness matrix $\mathbf{k}$.

The damping forces f_{D1} and f_{D2} are related to floor velocities $\dot{u}_1$ and $\dot{u}_2$. The j^{th} story damping coefficient c_j relates the story shear V_j due to the damping effects to the velocity $\dot{\Delta}_j$ associated with the story deformation by

$$V_j = c_j \dot{\Delta}_j$$

We can derive

$$f_{D1} = c_1 \dot{u}_1 + c_2 \left(\dot{u}_1 - \dot{u}_2 \right)$$

$$f_{D2} = c_2 \left(\dot{u}_2 - \dot{u}_1 \right)$$

$$\begin{Bmatrix} f_{D1} \\ f_{D2} \end{Bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \qquad \text{or} \qquad \mathbf{f}_D = \mathbf{c}\dot{\mathbf{u}}$$

The damping force vector $\mathbf{f}_D$ and velocity vector $\dot{\mathbf{u}}$ are related through the damping matrix $\mathbf{c}$.

Therefore, the equations of motion are

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$

This matrix equation represents two ordinary differential equations governing the displacements $u_1(t)$ and $u_2(t)$ of the two-story frame subjected external dynamic forces $p_1(t)$ and $p_2(t)$.

Each equation contains both unknowns u_1 and u_2, so two equations are coupled and must be solved simultaneously.

Dynamic Equilibrium (D'Alembert's principle)

For each of the mass in the system, the external force must be in balance with

(1) inertia force (resisting acceleration) acting in the opposite direction to acceleration

(2) damping force (resisting velocity) acting in the opposite direction to velocity and

(3) elastic force resisting deformation

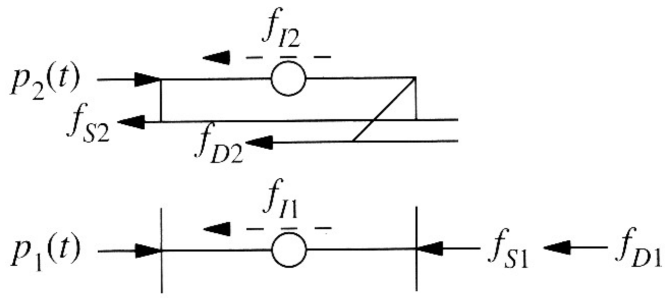


Figure 9.1.2 Free-body diagrams.

Example 9.1a

Formulate the equations of motion for the two-story shear frame shown in Fig. E9.1a.

Solution Equation (9.1.11) is specialized for this system to obtain its equation of motion. To do so, we note that

$$m_1 = 2m \qquad m_2 = m$$

$$k_1 = 2\frac{12(2EI_c)}{h^3} = \frac{48EI_c}{h^3} \qquad k_2 = 2\frac{12(EI_c)}{h^3} = \frac{24EI_c}{h^3}$$

Substituting these data in Eqs. (9.1.2) and (9.1.7) gives the mass and stiffness matrices:

$$\mathbf{m} = m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{k} = \frac{24EI_c}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

Substituting these **m** and **k** in Eq. (9.1.11) gives the governing equations for this system without damping:

$$m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + 24\frac{EI_c}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

Observe that the stiffness matrix is nondiagonal, implying that the two equations are coupled, and in their present form must be solved simultaneously.

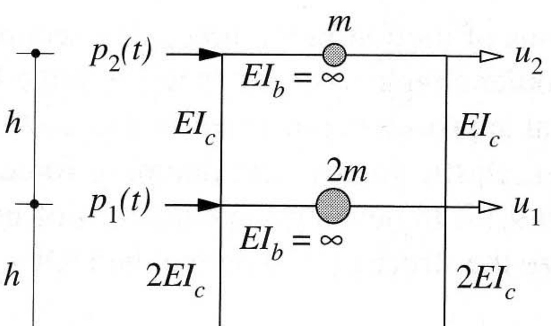


Figure E9.1a

Example 9.1b

Formulate the equations of motion for the two-story shear frame in Fig. E9.1a using influence coefficients.

Solution

The two DOFs of this system are shown in Fig. E9.1a; thus, $\mathbf{u} = \langle u_1 \quad u_2 \rangle^T$.

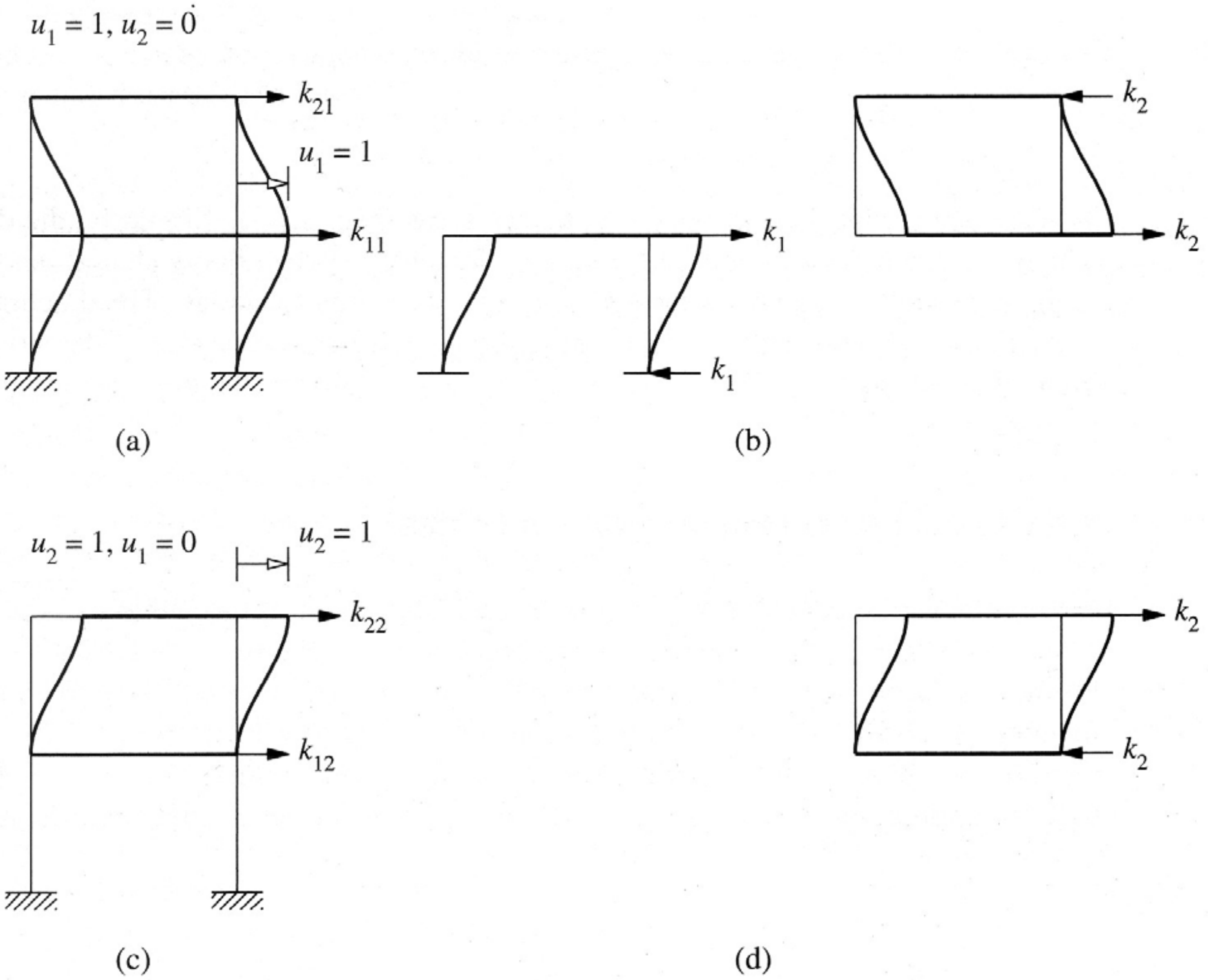


Figure E9.1b

1. *Determine the stiffness matrix.* To obtain the first column of the stiffness matrix, we impose $u_1 = 1$ and $u_2 = 0$. The stiffness influence coefficients are k_{ij} (Fig. E9.1b). The forces necessary at the top and bottom of each story to maintain the deflected shape are expressed in terms of story stiffnesses k_1 and k_2 [part (b) of the figure], defined in Section 9.1.1 and determined in Example 9.1a:

$$k_1 = \frac{48EI_c}{h_3} \qquad k_2 = \frac{24EI_c}{h^3} \tag{a}$$

The two sets of forces in parts (a) and (b) of the figure are one and the same. Thus,

$$k_{11} = k_1 + k_2 = \frac{72EI_c}{h^3} \qquad k_{21} = -k_2 = -\frac{24EI_c}{h^3} \tag{b}$$

The second column of the stiffness matrix is obtained in a similar manner by imposing $u_2 = 1$ with $u_1 = 0$. The stiffness influence coefficients are k_{i2} [part (c) of the figure] and the forces necessary to maintain the deflected shape are shown in part (d) of the figure. The two sets of forces in parts (c) and (d) of the figure are one and the same. Thus,

$$k_{12} = -k_2 = -\frac{24EI_c}{h^3} \qquad k_{22} = k_2 = \frac{24EI_c}{h^3} \tag{c}$$

With the stiffness influence cofficients determined, the stiffness matrix is

$$\mathbf{k} = \frac{24EI_c}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \tag{d}$$

2. *Determine the mass matrix.* With the DOFs defined at the locations of the lumped masses, the diagonal mass matrix is given by Eq. (9.2.10):

$$\mathbf{m} = m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{e}$$

3. *Determine the equations of motion.* The governing equations are

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \tag{f}$$

where $\mathbf{m}$ and $\mathbf{k}$ are given by Eqs. (e) and (d), and $\mathbf{p}(t) = \langle\, p_1(t) \quad p_2(t)\,\rangle^T$.

General Approach for Linear Systems

Discretization

A frame structure can be idealized by an assemblage of elements—beams, columns, walls—interconnected at nodal points or nodes. Displacements of nodes are degrees of freedom. A node in a planar two-dimension frame has 3 DOFs—two translations and one rotation.

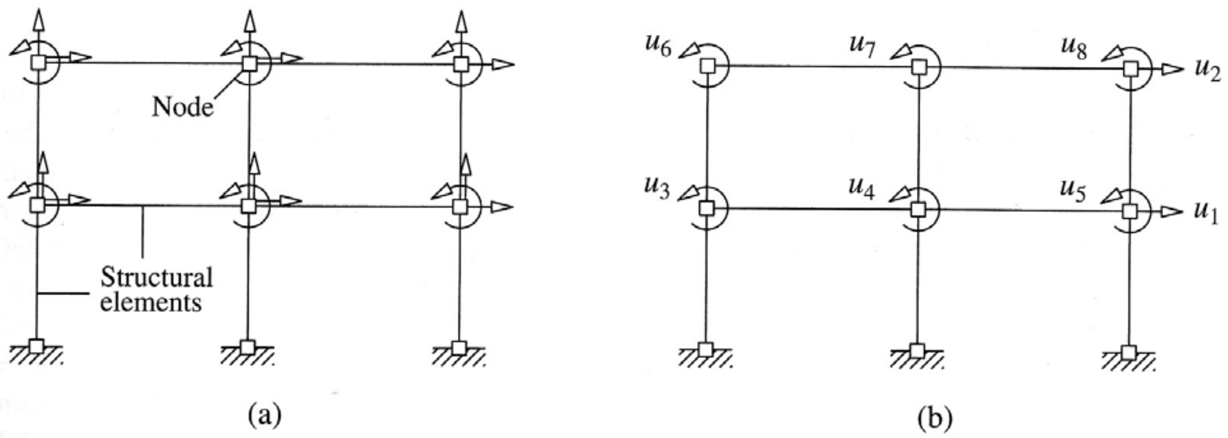


Figure 9.2.1 Degrees of freedom: (a) axial deformation included, 18 DOFs; (b) axial deformation neglected, 8 DOFs.

If axial deformations are neglected, the number of DOFs can be reduced because some translational DOF are equal.

The external forces are applied at the nodes which correspond to the DOFs.

Elastic forces

The elastic forces are related to displacement through stiffness matrix. The stiffness matrix can be obtained from stiffness influence coefficient k_{ij}, which is the force required along DOF i due to a unit displacement at DOF j and zero displacement at all other DOFs.

For example, the force $k_{i1}(i=1,2,...,8)$ are required to maintain the deflected shape associated with $u_1 = 1$ and all other $u_j = 0$.

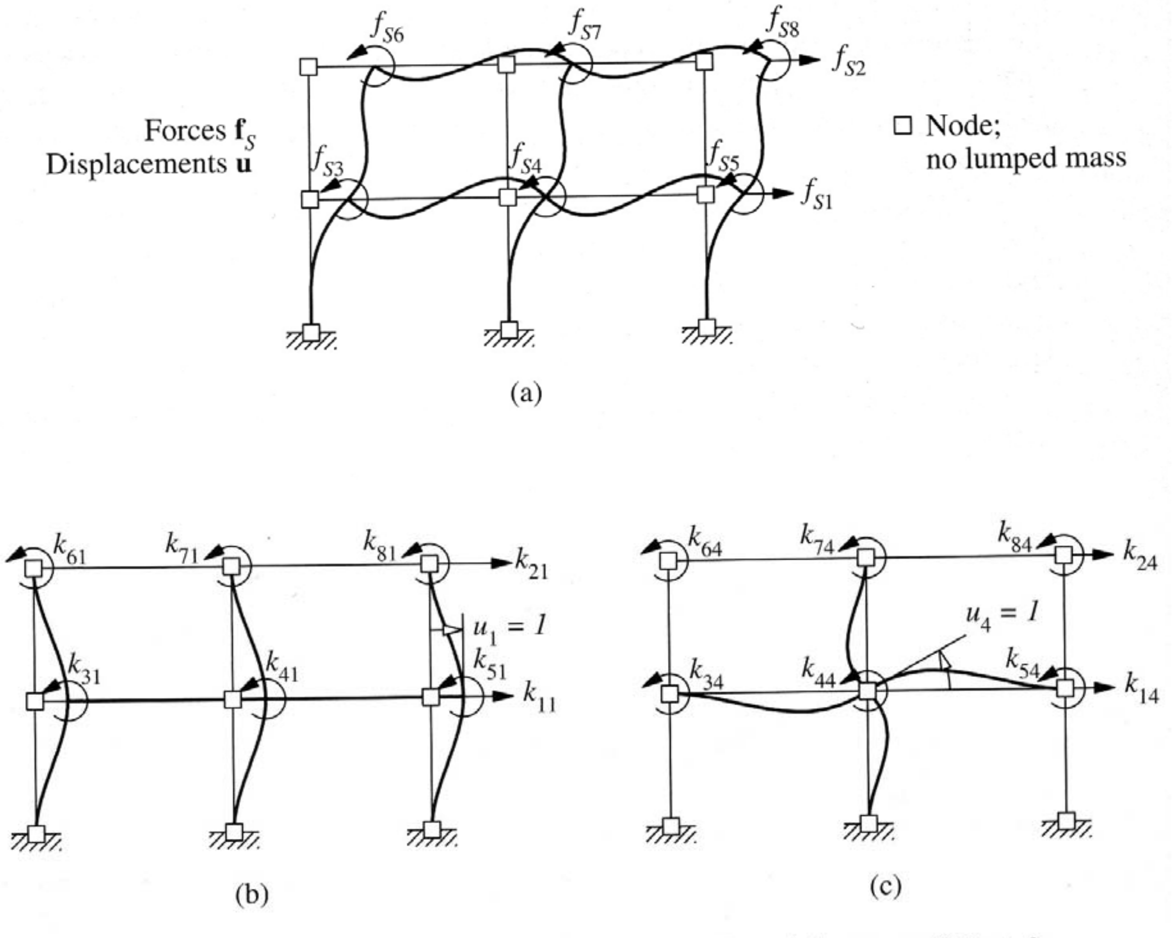


Figure 9.2.3 (a) Stiffness component of frame; (b) stiffness influence coefficients for $u_1 = 1$; (c) stiffness influence coefficients for $u_4 = 1$.

The force f_{Si} at DOF i associated with displacement u_j ($j = 1$ to N) is obtained by superposition:

$$f_{Si} = k_{i1}u_1 + k_{i2}u_2 + ... + k_{iN}u_N$$

Such equation applies to each of f_{Si} where $i = 1$ to N, so

$$\begin{Bmatrix} f_{S1} \\ f_{S2} \\ : \\ f_{SN} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & .. & k_{1N} \\ k_{21} & k_{22} & .. & k_{2N} \\ : & : & .. & : \\ k_{N1} & k_{N2} & .. & k_{NN} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ : \\ u_N \end{Bmatrix} \quad \text{or} \quad \mathbf{f}_S = \mathbf{ku}$$

where $\mathbf{k}$ is the stiffness matrix of the structure.

This approach can be cumbersome for complex structures in order to visualize a deflected shape with a unit displacement at DOF j and zero displacement at all other DOFs.

The direct stiffness method must be used instead. It involves assembling of stiffness matrices of structural members into the stiffness matrix of the whole system. The appropriate method should be used for a given problem.

Damping forces

Damping forces are related to velocities of nodes through damping matrix. The method of damping influence coefficient c_{ij} can be used to derive the damping matrix in a similar manner as stiffness matrix relating elastic forces to displacements.

However, it is impractical to compute the coefficient c_{ij} of damping matrix directly from the size of the structural elements. Instead, damping of a MDF system is usually specified in term of damping ratio and the corresponding damping matrix can be constructed accordingly.

Inertia forces

Inertia forces are forces related to acceleration of the mass. An approach to consider inertia forces acting at nodes is to lump the mass of structural components to nodes.

Inertial forces are related to acceleration at nodes through the mass matrix $\mathbf{m}$. Mass matrix can be derived using mass influence coefficient m_{ij} which is the external force in DOF i due to unit acceleration along DOF j. For example, the force m_{i1} $(i = 1, 2, ... 8)$ are required in various DOF to equilibrate the inertia forces associated with $\ddot{u}_1 = 1$ and all other $\ddot{u}_j = 0$.

The force at DOF i due to acceleration at various nodes can be obtained by superposition

$$f_{Ii} = m_{i1}\ddot{u}_1 + m_{i2}\ddot{u}_2 + ... + m_{iN}\ddot{u}_N$$

Such inertia forces at all DOFs are written together in the inertia force vector $\mathbf{f}_I$, which is equal to

$$\begin{Bmatrix} f_{I1} \\ f_{I2} \\ \vdots \\ f_{IN} \end{Bmatrix} = \begin{bmatrix} m_{11} & m_{12} & .. & m_{1N} \\ m_{21} & m_{22} & .. & m_{2N} \\ \vdots & \vdots & .. & \vdots \\ m_{N1} & m_{N2} & .. & m_{NN} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_N \end{Bmatrix} \quad \text{or} \quad \mathbf{f}_I = \mathbf{m}\ddot{\mathbf{u}}$$

When lumped-mass model is used, the mass matrix will be diagonal. Rotational inertia forces at the nodes are neglected, so the mass associated with rotational DOFs are zero.

$$m_{ij} = 0 \text{ if } i \neq j \qquad m_{jj} = m_j \text{ or } 0$$

Equations of motion

$$\mathbf{f}_I + \mathbf{f}_D + \mathbf{f}_S = \mathbf{p}(t)$$

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$

The off-diagonal terms in the coefficient matrices **m**, **c**, and **k** are known as coupling terms. The coupling in a system also depends on the choice of DOFs.

Example 9.2

A uniform rigid bar of total mass m is supported on two springs k_1 and k_2 at the two ends and subjected to dynamic forces shown in Fig. E9.2a. The bar is constrained so that it can move only vertically in the plane of the paper; with this constraint the system has two DOFs.

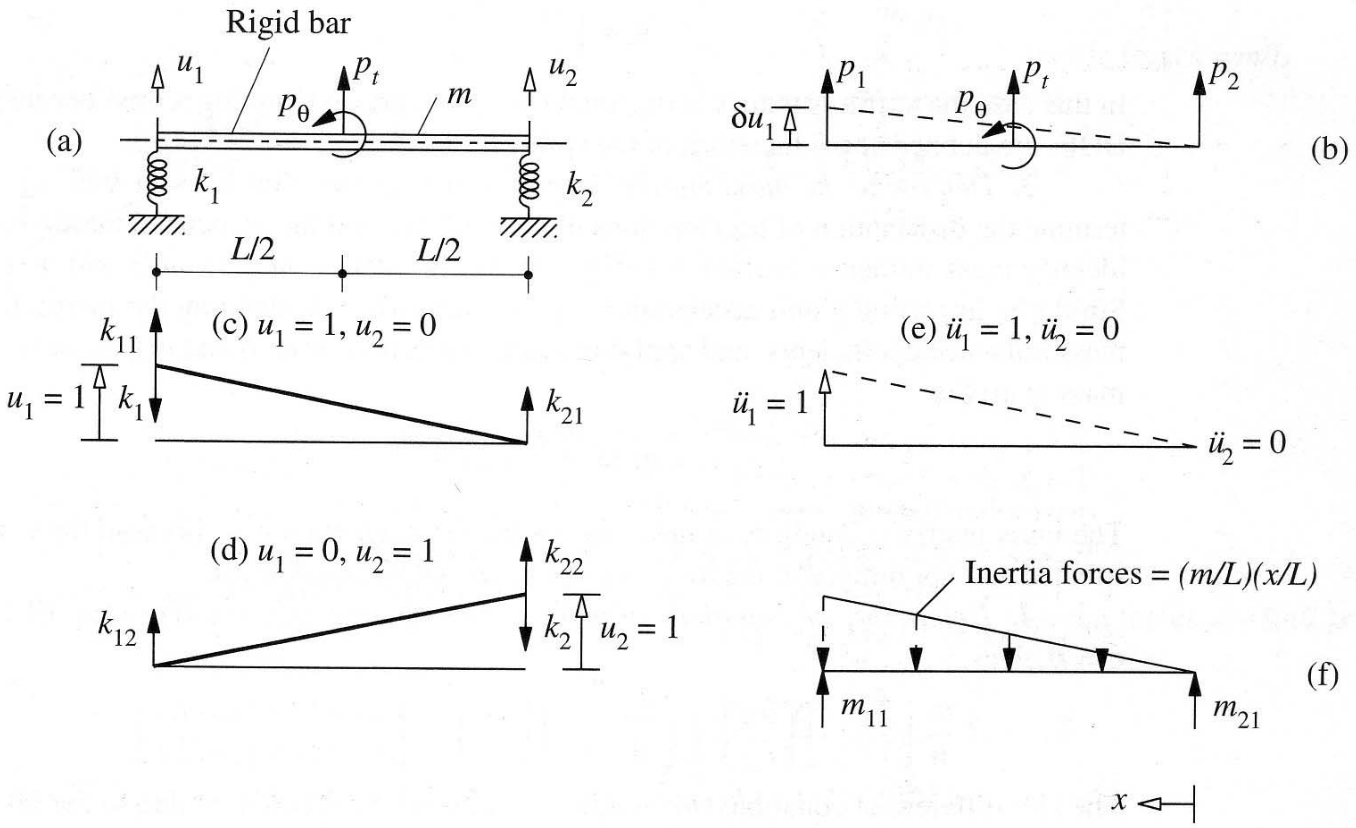


Figure E9.2

Formulate the equations of motion with respect to displacements u_1 and u_2 of the two ends as the two DOFs.

Solution

1. *Determine the applied forces.* The external forces do not act along the DOFs and should therefore be converted to equivalent forces p_1 and p_2 along the DOFs (Fig. E9.2b) using equilibrium equations. This can also be achieved by the principle of virtual displacements. Thus if we introduce a virtual displacement δu_1 along DOF 1, the work done by the applied forces is

$$\delta W = p_t \frac{\delta u_1}{2} - p_\theta \frac{\delta u_1}{L} \tag{a}$$

Similarly, the work done by the equivalent forces is

$$\delta W = p_1 \delta u_1 + p_2(0) \tag{b}$$

Because the work done by the two sets of forces should be the same, we equate Eqs. (a) and (b) and obtain

$$p_1 = \frac{p_t}{2} - \frac{p_\theta}{L} \tag{c}$$

In a similar manner, by introducing a virtual displacement δu_2, we obtain

$$p_2 = \frac{p_t}{2} + \frac{p_\theta}{L} \tag{d}$$

2. *Determine the stiffness matrix.* Apply a unit displacement $u_1 = 1$ with $u_2 = 0$ and identify the resulting elastic forces and the stiffness influence coefficients k_{11} and k_{21} (Fig. E9.2c). By statics, $k_{11} = k_1$ and $k_{21} = 0$. Now apply a unit displacement $u_2 = 1$ with $u_1 = 0$ and identify the resulting elastic forces and the stiffness influence coefficients (Fig. E9.2d). By statics, $k_{12} = 0$ and $k_{22} = k_2$. Thus the stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \tag{e}$$

In this case the stiffness matrix is diagonal (i.e., there are no coupling terms) because the two DOFs are defined at the locations of the springs.

3. *Determine the mass matrix.* Impart a unit acceleration $\ddot{u}_1 = 1$ with $\ddot{u}_2 = 0$, determine the distribution of accelerations of (Fig. E9.2e) and the associated inertia forces, and identify mass influence coefficients (Fig. E9.2f). By statics, $m_{11} = m/3$ and $m_{21} = m/6$. Similarly, imparting a unit acceleration $\ddot{u}_2 = 1$ with $\ddot{u}_1 = 0$, defining the inertia forces and mass influence coefficients, and applying statics gives $m_{12} = m/6$ and $m_{22} = m/3$. Thus the mass matrix is

$$\mathbf{m} = \frac{m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{f}$$

The mass matrix is coupled, as indicated by the off-diagonal terms, because the mass is distributed and not lumped at the locations where the DOFs are defined.

4. *Determine the equations of motion.* Substituting Eqs. (c)–(f) in Eq. (9.2.12) with $\mathbf{c} = \mathbf{0}$ gives

$$\frac{m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (p_t/2) - (p_\theta/L) \\ (p_t/2) + (p_\theta/L) \end{bmatrix} \tag{g}$$

The two differential equations are coupled because of mass coupling due to the off-diagonal terms in the mass matrix.

Example 9.3

Formulate the equations of motion of the system of Fig. E9.2a with the two DOFs defined at the center of mass O of the rigid bar: translation u_t and rotation u_θ (Fig. E9.3a).

Solution

1. *Determine the stiffness matrix.* Apply a unit displacement $u_t = 1$ with $u_\theta = 0$ and identify the resulting elastic forces and k_{tt} and $k_{\theta t}$ (Fig. E9.3b). By statics, $k_{tt} = k_1 + k_2$ and $k_{\theta t} = (k_2 - k_1)L/2$. Now, apply a unit rotation $u_\theta = 1$ with $u_t = 0$ and identify the resulting elastic forces and $k_{t\theta}$ and $k_{\theta\theta}$ (Fig. E9.3c). By statics, $k_{t\theta} = (k_2 - k_1)L/2$ and $k_{\theta\theta} = (k_1 + k_2)L^2/4$. Thus the stiffness matrix is

$$\bar{\mathbf{k}} = \begin{bmatrix} k_1 + k_2 & (k_2 - k_1)L/2 \\ (k_2 - k_1)L/2 & (k_1 + k_2)L^2/4 \end{bmatrix} \tag{a}$$

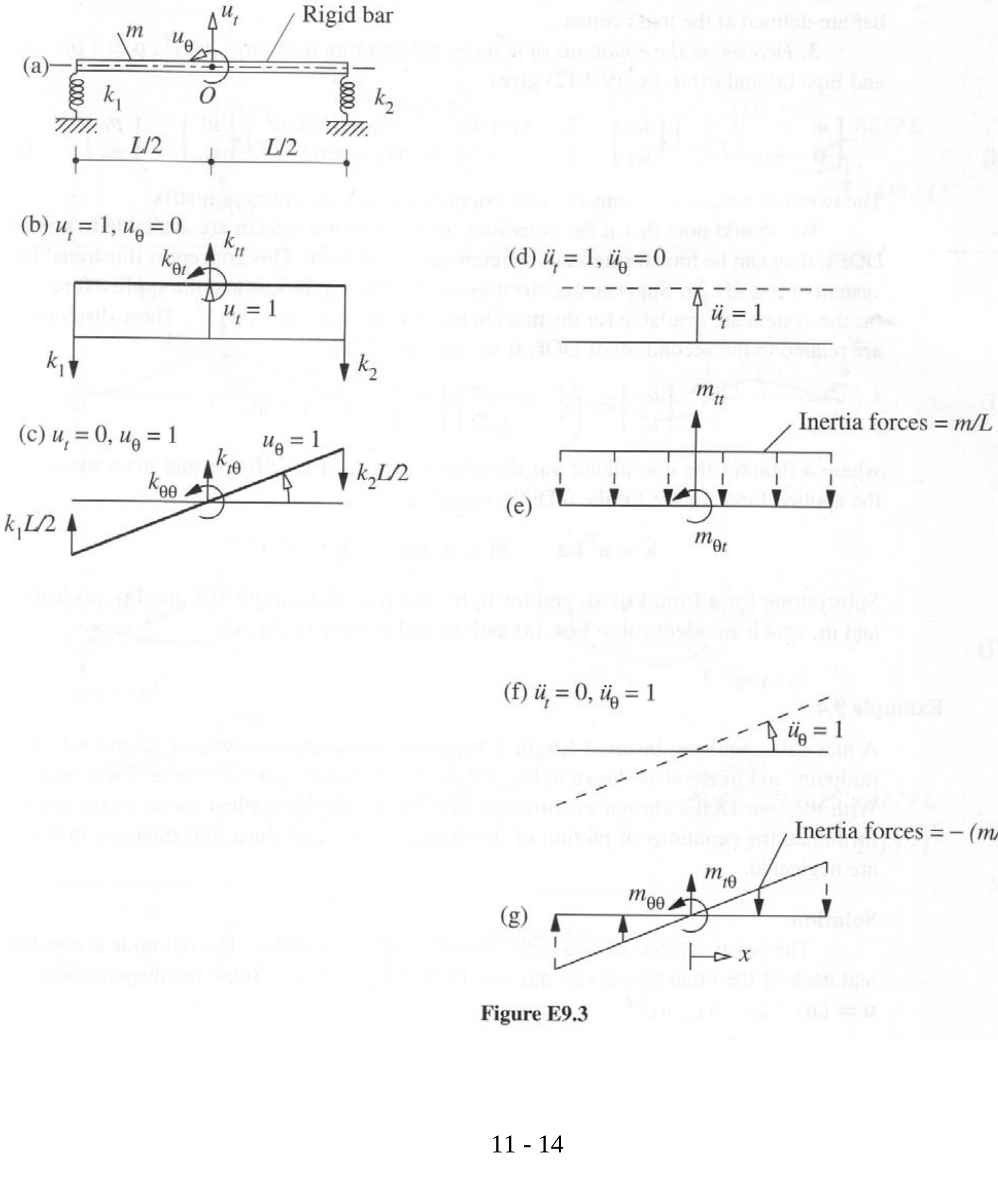


Figure E9.3

Observe that now the stiffness matrix has coupling terms because the chosen DOFs are not the displacements at the locations of the springs.

2. *Determine the mass matrix.* Impart a unit acceleration $\ddot{u}_t = 1$ with $\ddot{u}_\theta = 0$, determine the acceleration distribution (Fig. E9.3d) and the associated inertia forces, and identify m_{tt} and $m_{\theta t}$ (Fig. E9.3e). By statics, $m_{tt} = m$ and $m_{\theta t} = 0$. Now impart a unit rotational acceleration $\ddot{u}_\theta = 1$ with $\ddot{u}_t = 0$, determine the resulting accelerations (Fig. E9.3f) and the associated inertia forces, and identify $m_{t\theta}$ and $m_{\theta\theta}$ (Fig. E9.3g). By statics, $m_{t\theta} = 0$ and $m_{\theta\theta} = mL^2/12$. Note that $m_{\theta\theta} = I_O$, the moment of inertia of the bar about an axis that passes through O and is perpendicular to the plane of rotation. Thus the mass matrix is

$$\bar{\mathbf{m}} = \begin{bmatrix} m & 0 \\ 0 & mL^2/12 \end{bmatrix} \tag{b}$$

Now the mass matrix is diagonal (i.e., it has no coupling terms) because the DOFs of this rigid bar are defined at the mass center.

3. *Determine the equations of motion.* Substituting $\mathbf{u} = \langle u_t \quad u_\theta \rangle^T$, $\mathbf{p} = \langle p_t \quad p_\theta \rangle^T$, and Eqs. (a) and (b) in Eq. (9.2.12) gives

$$\begin{bmatrix} m & 0 \\ 0 & mL^2/12 \end{bmatrix} \begin{Bmatrix} \ddot{u}_t \\ \ddot{u}_\theta \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & (k_2 - k_1)L/2 \\ (k_2 - k_1)L/2 & (k_1 + k_2)L^2/4 \end{bmatrix} \begin{Bmatrix} u_t \\ u_\theta \end{Bmatrix} = \begin{Bmatrix} p_t \\ p_\theta \end{Bmatrix} \tag{c}$$

The two differential equations are now coupled through the stiffness matrix.

We should note that if the equations of motion for a system are available in one set of DOFs, they can be transformed to a different choice of DOF. This concept is illustrated for the system of Fig. E9.2a. Suppose that the mass and stiffness matrices and the applied force vector for the system are available for the first choice of DOF, $\mathbf{u} = \langle u_1 \quad u_2 \rangle^T$. These displacements are related to the second set of DOF, $\bar{\mathbf{u}} = \langle u_t \quad u_\theta \rangle^T$, by

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & -L/2 \\ 1 & L/2 \end{bmatrix} \begin{Bmatrix} u_t \\ u_\theta \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{a}\bar{\mathbf{u}} \tag{d}$$

where $\mathbf{a}$ denotes the coordinate transformation matrix. The stiffness and mass matrices and the applied force vector for the $\bar{\mathbf{u}}$ DOFs are given by

$$\bar{\mathbf{k}} = \mathbf{a}^T\mathbf{k}\mathbf{a} \qquad \bar{\mathbf{m}} = \mathbf{a}^T\mathbf{m}\mathbf{a} \qquad \bar{\mathbf{p}} = \mathbf{a}^T\mathbf{p} \tag{e}$$

Substituting for $\mathbf{a}$ from Eq. (d) and for $\mathbf{k}$, $\mathbf{m}$, and $\mathbf{p}$ from Example 9.2 into Eq. (e) leads to $\bar{\mathbf{k}}$ and $\bar{\mathbf{m}}$, which are identical to Eqs. (a) and (b) and to the $\bar{\mathbf{p}}$ in Eq. (c).

Example 9.4

A massless cantilever beam of length L supports two lumped masses $mL/2$ and $mL/4$ at the midpoint and free end as shown in Fig. E9.4a. The flexural rigidity of the uniform beam is EI. With the four DOFs chosen as shown in Fig. E9.4b and the applied forces $p_1(t)$ and $p_2(t)$, formulate the equations of motion of the system. Axial and shear deformations in the beam are neglected.

Solution

The beam consists of two beam elements and three nodes. The left node is constrained and each of the other two nodes has two DOFs (Fig. E9.4b). Thus, the displacement vector $\mathbf{u} = \langle u_1 \quad u_2 \quad u_3 \quad u_4 \rangle^T$.

1. *Determine the mass matrix.* With the DOFs defined at the locations of the lumped masses, the diagonal mass matrix is given by Eq. (9.2.10):

$$\mathbf{m} = \begin{bmatrix} mL/4 & & & \\ & mL/2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad \text{(a)}$$

2. *Determine the stiffness matrix.* Several methods are available to determine the stiffness matrix. Here we use the direct equilibrium method based on the definition of stiffness influence coefficients (Appendix 1).

To obtain the first column of the stiffness matrix, we impose $u_1 = 1$ and $u_2 = u_3 = u_4 = 0$. The stiffness influence coefficients are k_{i1} (Fig. E9.4c). The forces necessary at the nodes of each beam element to maintain the deflected shape are determined from the beam stiffness coefficients (Fig. E9.4d). The two sets of forces in figures (c) and (d) are one and the same. Thus $k_{11} = 96EI/L^3$, $k_{21} = -96EI/L^3$, $k_{31} = -24EI/L^2$, and $k_{41} = -24EI/L^2$.

The second column of the stiffness matrix is obtained in a similar manner by imposing $u_2 = 1$ with $u_1 = u_3 = u_4 = 0$. The stiffness influence coefficients are k_{i2} (Fig. E9.4e) and the forces on each beam element necessary to maintain the imposed displacements are shown in Fig. E9.4f. The two sets of forces in figures (e) and (f) are one and the same. Thus $k_{12} = -96EI/L^3$, $k_{32} = 24EI/L^2$, $k_{22} = 96EI/L^3 + 96EI/L^3 = 192EI/L^3$, and $k_{42} = -24EI/L^2 + 24EI/L^2 = 0$.

The third column of the stiffness matrix is obtained in a similar manner by imposing $u_3 = 1$ with $u_1 = u_2 = u_4 = 0$. The stiffness influence coefficients k_{i3} are shown in Fig. E9.4g and the nodal forces in Fig. E9.4h. Thus $k_{13} = -24EI/L^2$, $k_{23} = 24EI/L^2$, $k_{33} = 8EI/L$, and $k_{43} = 4EI/L$.

The fourth column of the stiffness matrix is obtained in a similar manner by imposing $u_4 = 1$ with $u_1 = u_2 = u_3 = 0$. The stiffness influence coefficients k_{i4} are shown in Fig. E9.4i, and the nodal forces in Fig. E9.4j. Thus $k_{14} = -24EI/L^2$, $k_{34} = 4EI/L$, $k_{24} = -24EI/L^2 + 24EI/L^2 = 0$, and $k_{44} = 8EI/L + 8EI/L = 16EI/L$.

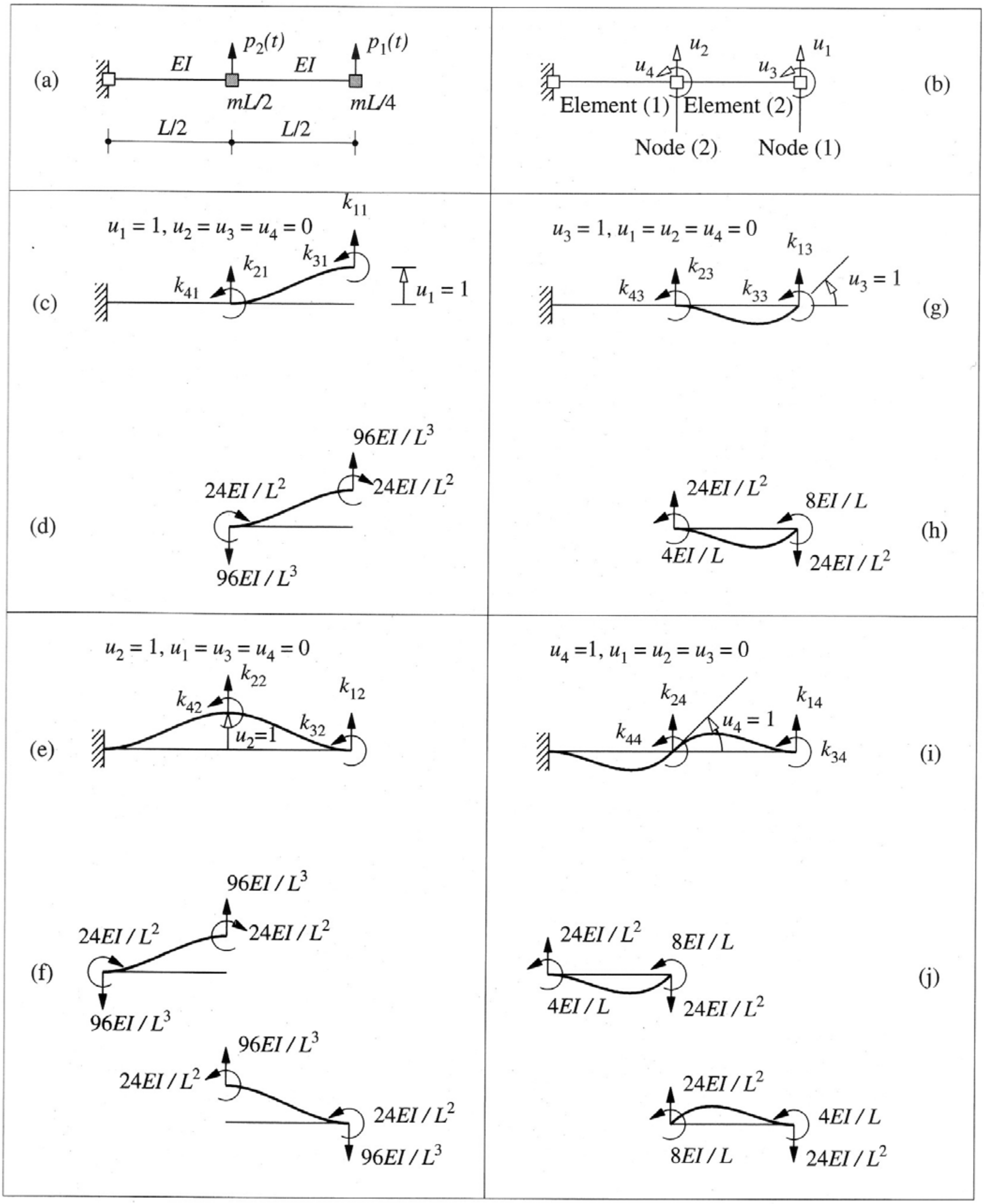
With all the stiffness influence coefficients determined, the stiffness matrix is

$$\mathbf{k} = \frac{8EI}{L^3} \begin{bmatrix} 12 & -12 & -3L & -3L \\ -12 & 24 & 3L & 0 \\ -3L & 3L & L^2 & L^2/2 \\ -3L & 0 & L^2/2 & 2L^2 \end{bmatrix} \qquad \text{(b)}$$

3. *Determine the equations of motion.* The governing equations are

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \qquad \text{(c)}$$

where $\mathbf{m}$ and $\mathbf{k}$ are given by Eqs. (a) and (b), and $\mathbf{p}(t) = \langle p_1(t) \quad p_2(t) \quad 0 \quad 0 \rangle^T$.



(a)
$p_2(t)$
$p_1(t)$
EI
EI
$mL/2$
$mL/4$
$L/2$
$L/2$
(b)
u_2
u_1
u_4
u_3
Element (1)
Element (2)
Node (2)
Node (1)
(c)
$u_1 = 1, u_2 = u_3 = u_4 = 0$
k_{11}
k_{21}
k_{31}
k_{41}
$u_1 = 1$
(d)
$96EI/L^3$
$24EI/L^2$
$24EI/L^2$
$96EI/L^3$
(e)
$u_2 = 1, u_1 = u_3 = u_4 = 0$
k_{22}
k_{42}
k_{12}
k_{32}
$u_2=1$
(f)
$96EI/L^3$
$24EI/L^2$
$24EI/L^2$
$96EI/L^3$
$96EI/L^3$
$24EI/L^2$
$24EI/L^2$
$96EI/L^3$
(g)
$u_3 = 1, u_1 = u_2 = u_4 = 0$
k_{13}
k_{23}
k_{43}
k_{33}
$u_3 = 1$
(h)
$24EI/L^2$
$8EI/L$
$4EI/L$
$24EI/L^2$
(i)
$u_4 =1, u_1 = u_2 = u_3 = 0$
k_{24}
k_{14}
k_{44}
$u_4 = 1$
k_{34}
(j)
$24EI/L^2$
$8EI/L$
$4EI/L$
$24EI/L^2$
$24EI/L^2$
$4EI/L$
$8EI/L$
$24EI/L^2$

Figure E9.4

Example 9.5

Derive the equations of motion of the beam of Example 9.4 (also shown in Fig. E9.5a) expressed in terms of the displacements u_1 and u_2 of the masses (Fig. E9.5b).

Solution This system is the same as that in Example 9.4, but its equations of motion will be formulated considering only the translational DOFs u_1 and u_2 (i.e., the rotational DOFs u_3 and u_4 will be excluded).

1. *Determine the stiffness matrix.* In a statically determinate structure such as the one in Fig. E9.5a, it is usually easier to calculate first the flexibility matrix and invert it to obtain

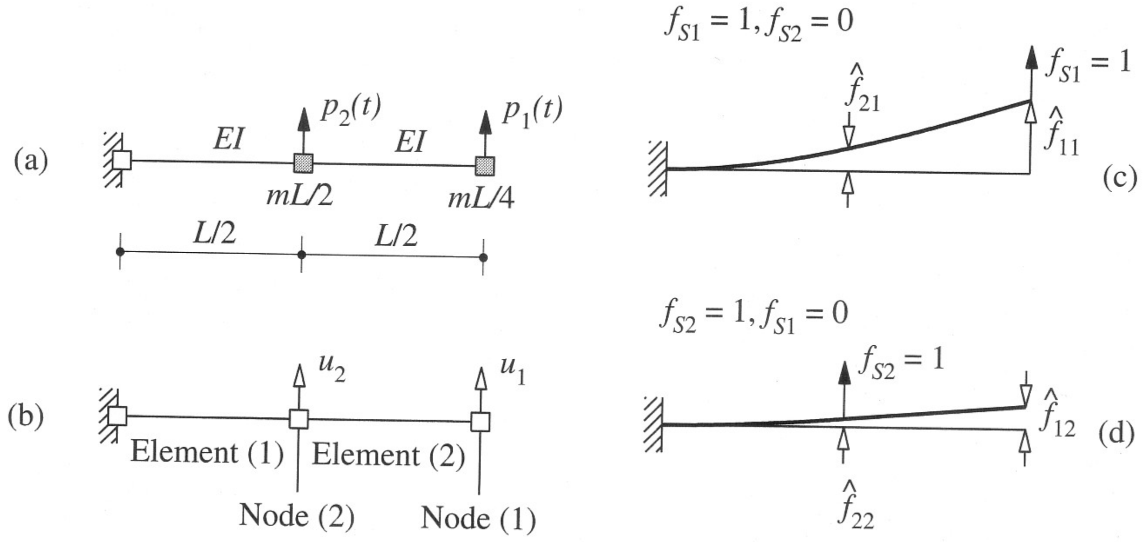


Figure E9.5

the stiffness matrix. The flexibility influence coefficient $\hat{f}_{ij}$ is the displacement in DOF i due to unit force applied in DOF j (Fig. E9.4c and d). The deflections are computed by standard procedures of structural analysis to obtain the flexibility matrix:

$$\hat{\mathbf{f}} = \frac{L^3}{48EI}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$$

The off-diagonal elements $\hat{f}_{12}$ and $\hat{f}_{21}$ are equal, as expected, because of Maxwell's theorem of reciprocal deflections. By inverting $\hat{\mathbf{f}}$, the stiffness matrix is obtained:

$$\mathbf{k} = \frac{48EI}{7L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix} \tag{a}$$

2. *Determine the mass matrix.* This is a diagonal matrix because the lumped masses are located where the DOFs are defined:

$$\mathbf{m} = \begin{bmatrix} mL/4 & \\ & mL/2 \end{bmatrix} \tag{b}$$

3. *Determine the equations of motion.* Substituting **m**, **k**, and $\mathbf{p}(t) = \langle\, p_1(t) \quad p_2(t)\,\rangle^T$ in Eq. (9.2.12) with $\mathbf{c} = \mathbf{0}$ gives

$$\begin{bmatrix} mL/4 & \\ & mL/2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{48EI}{7L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{c}$$

Example 9.6

Formulate the free vibration equations for the two-element frame of Fig. E9.6a. For both elements the flexural stiffness is EI, and axial deformations are to be neglected. The frame is massless with lumped masses at the two nodes as shown.

Solution The two degrees of freedom of the frame are shown. The mass matrix is

$$\mathbf{m} = \begin{bmatrix} 3m & \\ & m \end{bmatrix} \quad \text{(a)}$$

Note that the mass corresponding to $\ddot{u}_1 = 1$ is $2m+m = 3m$ because both masses will undergo the same acceleration since the beam connecting the two masses is axially inextensible.

The stiffness matrix is formulated by first evaluating the flexibility matrix and then inverting it. The flexibility influence coefficients are identified in Fig. E9.6b and c, and the

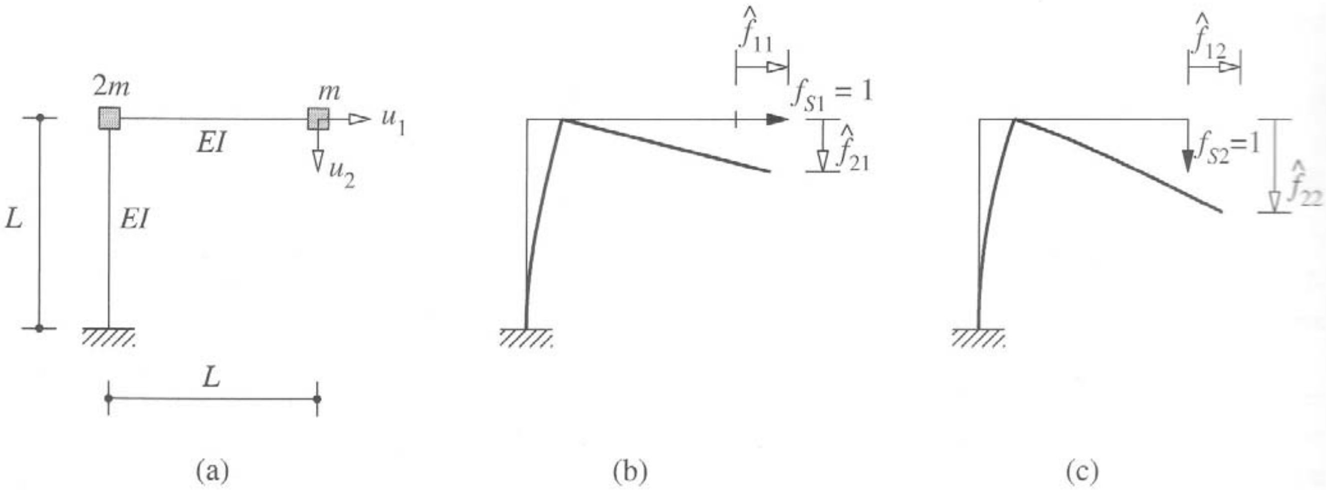


Figure E9.6

deflections are computed by standard procedures of structural analysis to obtain the flexibility matrix:

$$\hat{\mathbf{f}} = \frac{L^3}{6EI}\begin{bmatrix} 2 & 3 \\ 3 & 8 \end{bmatrix}$$

This matrix is inverted to determine the stiffness matrix:

$$\mathbf{k} = \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$$

Thus the equations in free vibration of the system (without damping) are

$$\begin{bmatrix} 3m & \\ & m \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Example 9.7

Formulate the equations of motion for the two-story frame in Fig. E9.7a. The flexural rigidity of the beams and columns and the lumped masses at the floor levels are as noted. The dynamic excitation consists of lateral forces $p_1(t)$ and $p_2(t)$ at the two floor levels. The story height is h and the bay width $2h$. Neglect axial deformations in the beams and the columns.

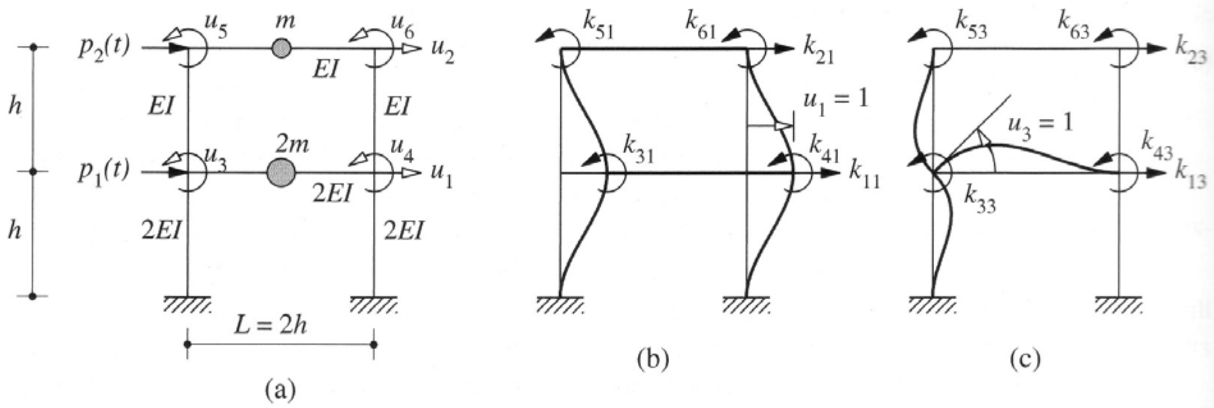


Figure E9.7

Solution The system has six degrees of freedom shown in Fig. E9.7a: lateral displacements u_1 and u_2 of the floors and joint rotations u_3, u_4, u_5, and u_6. The displacement vector is

$$\mathbf{u} = \langle u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \rangle^T \tag{a}$$

The mass matrix is given by Eq. (9.2.10):

$$\mathbf{m} = m \begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} \tag{b}$$

The stiffness influence coefficients are evaluated following the procedure of Example 9.4. A unit displacement is imposed, one at a time, in each DOF while constraining the other five DOFs, and the stiffness influence coefficients (e.g., shown in Fig. E9.7b and c for $u_1 = 1$ and $u_3 = 1$, respectively) are calculated by statics from the nodal forces for individual structural elements associated with the imposed displacements. These nodal forces are determined from the beam stiffness coefficients (Appendix 1). The result is

$$\mathbf{k} = \frac{EI}{h^3} \begin{bmatrix} 72 & -24 & 6h & 6h & -6h & -6h \\ -24 & 24 & 6h & 6h & 6h & 6h \\ 6h & 6h & 16h^2 & 2h^2 & 2h^2 & 0 \\ 6h & 6h & 2h^2 & 16h^2 & 0 & 2h^2 \\ -6h & 6h & 2h^2 & 0 & 6h^2 & h^2 \\ -6h & 6h & 0 & 2h^2 & h^2 & 6h^2 \end{bmatrix} \tag{c}$$

The dynamic forces applied are lateral forces $p_1(t)$ and $p_2(t)$ at the two floors without any moments at the nodes. Thus the applied force vector is

$$\mathbf{p}(t) = \langle p_1(t) \quad p_2(t) \quad 0 \quad 0 \quad 0 \quad 0 \rangle^T \tag{d}$$

The equations of motion are

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \tag{e}$$

where $\mathbf{u}$, $\mathbf{m}$, $\mathbf{k}$, and $\mathbf{p}(t)$ are given by Eqs. (a), (b), (c), and (d), respectively.

Static Condensation

Static condensation is a method to exclude the DOFs with no force from dynamic analysis. Typically the formulation of stiffness matrix in static analysis considers all unrestrained DOFs at joints between structural members. Some of DOFs may not be associated with any mass in dynamic analysis, for example, rotation DOFs in a lumped-mass model, so they should be excluded to simplify the dynamic analysis.

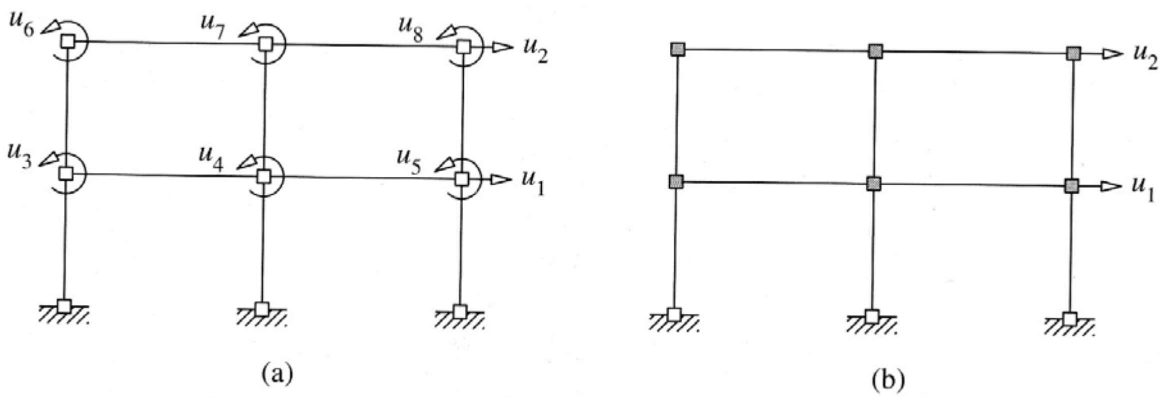


Figure 9.3.1 (a) Degrees of freedom (DOFs) for elastic forces—axial deformations neglected; (b) DOFs for inertia forces.

The equations of motion for a building shown above is

$$\begin{bmatrix} \mathbf{m}_{tt} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_t \\ \ddot{\mathbf{u}}_o \end{Bmatrix} + \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{to} \\ \mathbf{k}_{ot} & \mathbf{k}_{oo} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_o \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}_t(t) \\ 0 \end{Bmatrix}$$

It is partitioned into translation ($\mathbf{u}_t$) and rotation ($\mathbf{u}_o$) DOFs. Each part involves vectors and sub-matrices.

Each group of partitioned equations are

$$\mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \mathbf{k}_{tt}\mathbf{u}_t + \mathbf{k}_{to}\mathbf{u}_o = \mathbf{p}_t(t) \quad \text{and} \quad \mathbf{k}_{ot}\mathbf{u}_t + \mathbf{k}_{oo}\mathbf{u}_o = 0$$

Because no inertia terms and external forces are associated with the rotations, $\mathbf{u}_o$ can be solved.

$$\mathbf{u}_o = -\mathbf{k}_{oo}{}^{-1}\mathbf{k}_{ot}\mathbf{u}_t$$

Then, we can substitute $\mathbf{u}_o$ into the equation for translational DOFs and obtain equations of motion which are simpler as they involve only translation DOFs.

$$\mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \hat{\mathbf{k}}_{tt}\mathbf{u}_t = \mathbf{p}_t\left(t\right)$$

where the condensed stiffness matrix is

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{to}{}^{T}\mathbf{k}_{oo}^{-1}\mathbf{k}_{ot}$$

Note that $\mathbf{k}_{to} = \mathbf{k}_{ot}^{T}$ because **k** is a symmetric matrix.

Example 9.8

Examples 9.4 and 9.5 were concerned with formulating the equations of motion for a cantilever beam with two lumped masses. The degrees of freedom chosen in Example 9.5 were the translational displacements u_1 and u_2 at the lumped masses; in Example 9.4 the four DOFs were u_1, u_2, and node rotations u_3 and u_4. Starting with the equations governing these four DOFs, derive the equations of motion in the two translational DOFs.

Solution The vector of four DOFs is partitioned in two parts: $\mathbf{u}_t = \langle u_1 \quad u_2 \rangle^T$ and $\mathbf{u}_0 = \langle u_3 \quad u_4 \rangle^T$. The equations of motion governing $\mathbf{u}_t$ are given by Eq. (9.3.4), where

$$\mathbf{m}_{tt} = \begin{bmatrix} mL/4 & \\ & mL/2 \end{bmatrix} \qquad \mathbf{p}_t(t) = \langle\, p_1(t) \quad p_2(t) \,\rangle^T \tag{a}$$

To determine $\hat{\mathbf{k}}_{tt}$, the 4×4 stiffness matrix determined in Example 9.4 is partitioned:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix} = \frac{8EI}{L^3} \left[\begin{array}{cc:cc} 12 & -12 & -3L & -3L \\ -12 & 24 & 3L & 0 \\ \hdashline -3L & 3L & L^2 & L^2/2 \\ -3L & 0 & L^2/2 & 2L^2 \end{array}\right] \tag{b}$$

Substituting these submatrices in Eq. (9.3.5) gives the condensed stiffness matrix:

$$\hat{\mathbf{k}}_{tt} = \frac{48EI}{7L^3} \begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix} \tag{c}$$

This stiffness matrix of Eq. (c) is the same as that obtained in Example 9.5 by inverting the flexibility matrix corresponding to the two translational DOFs.

Substituting the stiffness submatrices in Eq. (9.3.3) gives the relation between the condensed DOF $\mathbf{u}_0$ and the dynamic DOF $\mathbf{u}_t$:

$$\mathbf{u}_0 = \mathbf{T}\mathbf{u}_t \qquad \mathbf{T} = \frac{1}{L} \begin{bmatrix} 2.57 & -3.43 \\ 0.857 & 0.857 \end{bmatrix} \tag{d}$$

The equations of motion are given by Eq. (9.3.4), where $\mathbf{m}_{tt}$ and $\mathbf{p}_t(t)$ are defined in Eq. (a) and $\hat{\mathbf{k}}_{tt}$ in Eq. (c). These are the same as Eq. (c) of Example 9.5.

Equation of motion:
Planar systems subjected to translational ground motion

At each instant of time, displacement of each mass is

$$u_j^t(t) = u_j(t) + u_g(t)$$

For N masses, the displacements can be written in compact form as a vector.

$$\mathbf{u}^t(t) = \mathbf{u}(t) + u_g(t)\mathbf{1}$$

where **1** is a vector of order N with each element equal to unity.

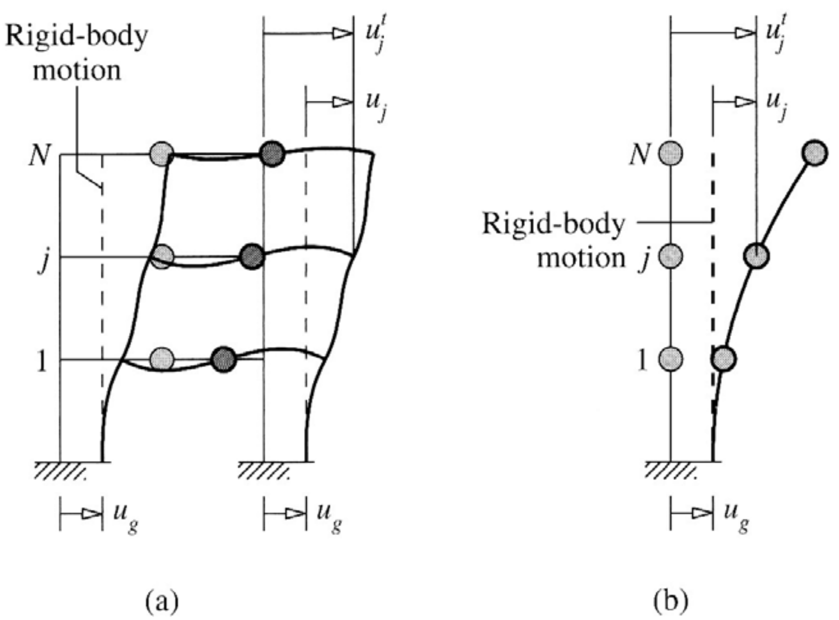


Figure 9.4.1 (a) Building frame; (b) tower.

The equations of motion previously derived for a MDF system subjected to external for $\mathbf{p}(t)$ is still valid except that the external force for this case (ground excitation) is zero.

$$\mathbf{f}_I + \mathbf{f}_D + \mathbf{f}_S = \mathbf{0}$$

Only relative displacements $\mathbf{u}$ between masses and the base produce deformation and elastic and damping forces.

The inertia forces $\mathbf{f}_I$ are related to the total acceleration $\ddot{\mathbf{u}}^t$.

$$\mathbf{f}_I = \mathbf{m}\ddot{\mathbf{u}}^t$$

Substituting $\mathbf{f}_I = \mathbf{m}\ddot{\mathbf{u}}^t$ in the equilibrium equation, we get

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\mathbf{1}\ddot{u}_g(t)$$

The right hand side is the effective earthquake forces due to ground motion excitation.

$$\mathbf{p}_{eff}(t) = -\mathbf{m}\mathbf{1}\ddot{u}_g(t)$$

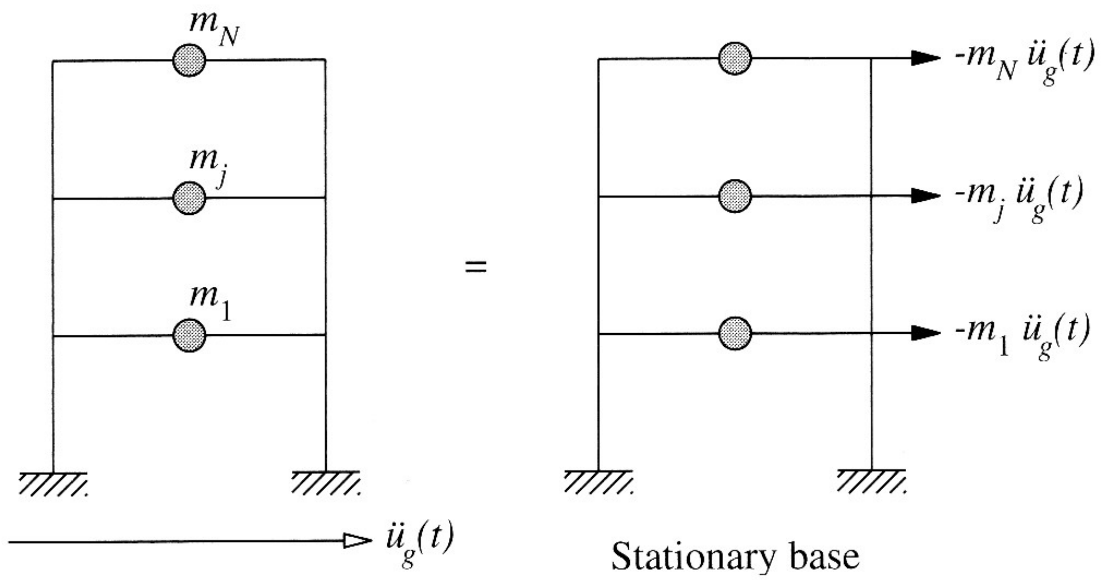


Figure 9.4.2 Effective earthquake forces.

This is valid when a unit ground displacement results in a unit total displacement of all DOFs. In general, this is not always the case. We introduce the influence vector $\boldsymbol{\iota}$ to represent the influence of ground displacement on total displacement at DOFs.

$$\mathbf{u}^t(t) = \boldsymbol{\iota} u_g(t) + \mathbf{u}(t)$$

The equations of motion are

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\boldsymbol{\iota}\ddot{u}_g(t)$$

For example, vertical DOF u_3 is not displaced when the ground moves horizontally. The influence vector is

$$\boldsymbol{\iota} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

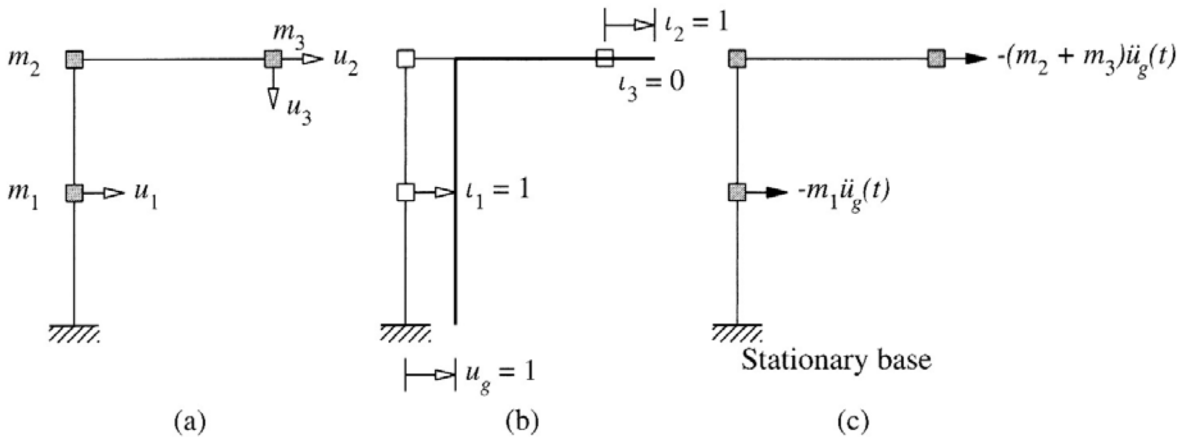


Figure 9.4.4 (a) L-shaped frame; (b) influence vector ι: static displacements due to $u_g = 1$; (c) effective earthquake forces.

The effective earthquake force is

$$\mathbf{p}_{eff}\left(t\right) = -\mathbf{m}\boldsymbol{\iota}\ddot{u}_g\left(t\right) = -\ddot{u}_g\left(t\right)\begin{bmatrix} m_1 & & \\ & m_2 + m_3 & \\ & & m_3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = -\ddot{u}_g\left(t\right)\begin{Bmatrix} m_1 \\ m_2 + m_3 \\ 0 \end{Bmatrix}$$

Note that the mass corresponding to $\ddot{u}_2 = 1$ is $m_2 + m_3$ because both masses will undergo the same acceleration since the connecting beam is axially rigid.

The effective earthquake force is zero in the vertical DOF because the ground motion is horizontal.

Inelastic systems

For inelastic systems, the force resisting deformation is no longer linear relationship and is described by a nonlinear function

$$\mathbf{f}_s = \mathbf{f}_s\left(\mathbf{u}, \dot{\mathbf{u}}\right)$$

The equation of motion becomes

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{f}_s\left(\mathbf{u}, \dot{\mathbf{u}}\right) = -\mathbf{m}\boldsymbol{\iota}\ddot{u}_g\left(t\right)$$

Such equation has to be solved by numerical methods as presented in Chapter 5.

Problem statement

Given a system with known, mass matrix **m**, damping matrix **c**, stiffness matrix **k**, and excitation **p**(t) or $\ddot{u}_g\left(t\right)$, we want to determine the response of the system.

Response can be any response quantity such as displacement $\mathbf{u}(t)$, velocity, acceleration of masses or internal forces, which is closely related to the relative displacement.

By the concept of equivalent static force, internal forces can be obtained by static analysis of structure subjected to a set of equivalent static forces

$$\mathbf{f}_S\left(t\right) = \mathbf{k}\mathbf{u}\left(t\right)$$

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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT 4-THEORY OF VIBRATIONS-SME1306

TORSIONAL VIBRATION

Single Rotor System

If a rigid body oscillates about a specific reference axis, the resulting motion is called torsional vibration. In this case, the displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple. Figure 1 shows a disc, which has a polar mass moment of inertia J_0 mounted at one end of a solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be θ, θ also represents the shaft's angle of twist.

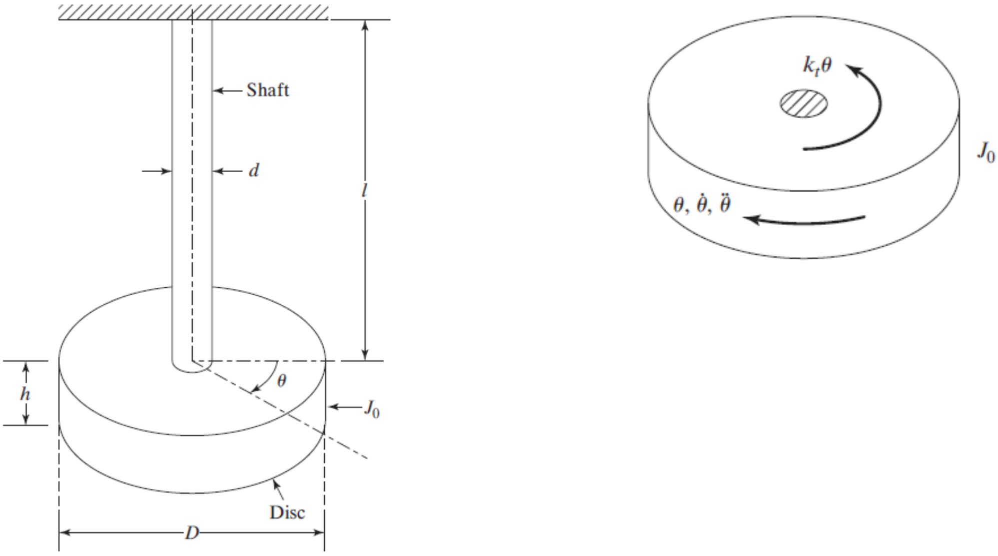

Figure 1 Torsional vibration of a disc

Let

θ = angular twist of the disc from its equilibrium position

T = torque required to produce the twist = $\frac{GJ}{l}\theta$

J is the polar moment inertia of the rod = $\pi d^4/32$

d = rod dia.

l = rod length

Then the torsional spring constant can be defined as,

$$\boxed{k_t = \frac{T}{\theta} = \frac{GJ}{l}}$$

Applying D'Alembert's principle the equation of motion may be written as

$$I\ddot{\theta} + k_t\theta = 0$$

$$\ddot{\theta} + \frac{k_t}{I}\theta = 0$$

So the natural frequency ω_n may written as

$$\omega_n = \sqrt{\frac{k_t}{I}}$$

And $\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_t}{I}}}$ Hz

Double Rotor System

Consider a torsional system consisting of two discs mounted on a shaft, as shown in Fig. 2. The three segments of the shaft have rotational spring constants k_{t1}, k_{t2}, k_{t3} and as indicated in the figure. Also shown are the discs of mass moments of inertia J_1 and J_2 and the applied torques M_{t1} and $M_{t2}$and and the rotational degrees of freedom θ_1 and θ_2 and The differential equations of rotational motion J_1 and $J_2$for the discs and can be derived as:

$$J_1\ddot{\theta}_1 = -k_{t1}\theta_1 + k_{t2}(\theta_2 - \theta_1) + M_{t1}$$
$$J_2\ddot{\theta}_2 = -k_{t2}(\theta_2 - \theta_1) - k_{t3}\theta_2 + M_{t2}$$

which upon rearrangement become

$$J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 = M_{t1}$$
$$J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 = M_{t2}$$

For the free-vibration analysis of the system, Eq. (5.19) reduces to

$$J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 = 0$$
$$J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 = 0$$

Figure 2 Torsional vibration of a two rotor system

Example

Find the natural frequencies and mode shapes for the torsional system shown in Fig. 5.9 for $J_1 = J_0$, $J_2 = 2J_0$, and $k_{t1} = k_{t2} = k_t$.

Solution: The differential equations of motion, reduce to (with $k_{t3} = 0$, $k_{t1} = k_{t2} = k_t$, $J_1 = J_0$, and $J_2 = 2J_0$):

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0 \qquad \text{(E.1)}$$

Rearranging and substituting the harmonic solution

$$\theta_i(t) = \Theta_i \cos(\omega t + \phi); \qquad i = 1, 2 \qquad \text{(E.2)}$$

gives the frequency equation:

$$2\omega^4 J_0^2 - 5\omega^2 J_0 k_t + k_t^2 = 0 \qquad \text{(E.3)}$$

The solution of Eq. (E.3) gives the natural frequencies

$$\omega_1 = \sqrt{\frac{k_t}{4J_0}(5 - \sqrt{17})} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k_t}{4J_0}(5 + \sqrt{17})} \qquad \text{(E.4)}$$

The amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = 2 - \frac{(5 - \sqrt{17})}{4}$$

Transverse vibration of beam with various boundary conditions

Consider the free-body diagram of an element of a beam shown in Fig. , where $M(x, t)$ is the bending moment, $V(x, t)$ is the shear force, and $f(x, t)$ is the external force per unit length of the beam. Since the inertia force acting on the element of the beam is

$$\rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

the force equation of motion in the z direction gives

$$-(V + dV) + f(x, t)\, dx + V = \rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

where ρ is the mass density and $A(x)$ is the cross-sectional area of the beam. The moment equation of motion about the y-axis passing through point O in Fig. leads to

$$(M + dM) - (V + dV)\, dx + f(x, t)\, dx \frac{dx}{2} - M = 0$$

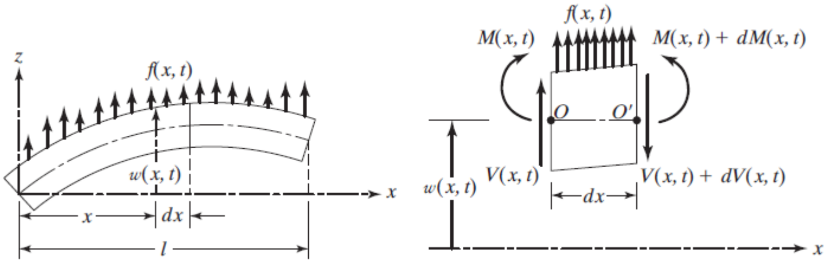

Figure 3 Transverse vibration of beam

By writing

$$dV = \frac{\partial V}{\partial x} dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x} dx$$

and disregarding terms involving second powers in dx, Eqs. can be written as

$$-\frac{\partial V}{\partial x}(x, t) + f(x, t) = \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t)$$

$$\frac{\partial M}{\partial x}(x, t) - V(x, t) = 0$$

By using the relation $V = \partial M / \partial x$ from Eq. becomes

$$-\frac{\partial^2 M}{\partial x^2}(x, t) + f(x, t) = \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t)$$

From the elementary theory of bending of beams (also known as the *Euler-Bernoulli* or *thin beam theory*), the relationship between bending moment and deflection can be expressed as

$$M(x, t) = EI(x) \frac{\partial^2 w}{\partial x^2}(x, t)$$

where E is Young's modulus and $I(x)$ is the moment of inertia of the beam cross section about the y-axis. Inserting Eq. we obtain the equation of motion for the forced lateral vibration of a nonuniform beam:

$$\frac{\partial^2}{\partial x^2}\left[EI(x) \frac{\partial^2 w}{\partial x^2}(x, t)\right] + \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

For a uniform beam, Eq. reduces to

$$EI \frac{\partial^4 w}{\partial x^4}(x, t) + \rho A \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

For free vibration, $f(x, t) = 0$, and so the equation of motion becomes

$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

where

$$c = \sqrt{\frac{EI}{\rho A}}$$

Since the equation of motion involves a second-order derivative with respect to time and a fourth-order derivative with respect to x, two initial conditions and four boundary conditions are needed for finding a unique solution for $w(x, t)$. Usually, the values of lateral displacement and velocity are specified as $w_0(x)$ and $\dot{w}_0(x)$ at $t = 0$, so that the initial conditions become

$$w(x, t = 0) = w_0(x)$$

$$\frac{\partial w}{\partial t}(x, t = 0) = \dot{w}_0(x)$$

The free-vibration solution can be found using the method of separation of variables as

$$w(x, t) = W(x)T(t)$$

Substituting and rearranging leads to

$$\frac{c^2}{W(x)}\frac{d^4W(x)}{dx^4} = -\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = a = \omega^2$$

where $a = \omega^2$ is a positive constant . Equation can be written as two equations:

$$\frac{d^4W(x)}{dx^4} - \beta^4 W(x) = 0$$

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$$

The solution of Eq. can be expressed as

$$T(t) = A \cos \omega t + B \sin \omega t$$

where A and B are constants that can be found from the initial conditions. For the solution of Eq. , we assume

$$W(x) = Ce^{sx}$$

where C and s are constants, and derive the auxiliary equation as

$$s^4 - \beta^4 = 0$$

The roots of this equation are

$$s_{1,2} = \pm\beta, \qquad s_{3,4} = \pm i\beta$$

Hence the solution of Eq. becomes

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

where C_1, C_2, C_3, and C_4 are constants. Equation can also be expressed as

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

or

$$W(x) = C_1(\cos \beta x + \cosh \beta x) + C_2(\cos \beta x - \cosh \beta x) + C_3(\sin \beta x + \sinh \beta x) + C_4(\sin \beta x - \sinh \beta x)$$

where C_1, C_2, C_3, and C_4, in each case, are different constants. The constants C_1, C_2, C_3, and C_4 can be found from the boundary conditions. The natural frequencies of the beam are computed from Eq. as

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

The function $W(x)$ is known as the *normal mode* or *characteristic function* of the beam and ω is called the *natural frequency of vibration*. For any beam, there will be an infinite number of normal modes with one natural frequency associated with each normal mode. The unknown constants C_1 to C_4 in Eq. and the value of β in Eq. can be determined from the boundary conditions of the beam as indicated below.

The common boundary conditions are as follows:

1. *Free end:*

$$\text{Bending moment} = EI \frac{\partial^2 w}{\partial x^2} = 0$$

$$\text{Shear force} = \frac{\partial}{\partial x}\left(EI \frac{\partial^2 w}{\partial x^2} \right) = 0$$

2. *Simply supported (pinned) end:*

$$\text{Deflection} = w = 0, \qquad \text{Bending moment} = EI \frac{\partial^2 w}{\partial x^2} = 0$$

3. *Fixed (clamped) end:*

$$\text{Deflection} = 0, \qquad \text{Slope} = \frac{\partial w}{\partial x} = 0$$

The frequency equations, the mode shapes (normal functions), and the natural frequencies for beams with common boundary conditions are given in Fig. We shall now consider some other possible boundary conditions for a beam.

4. *End connected to a linear spring, damper, and mass* : When the end of a beam undergoes a transverse displacement w and slope $\partial w/\partial x$. with velocity $\partial w/\partial t$ and acceleration $\partial^2 w/\partial t^2$, the resisting forces due to the spring, damper, and mass are proportional to w, $\partial w/\partial t$, and $\partial^2 w/\partial t^2$, respectively. This resisting force is balanced by the shear force at the end. Thus

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = a\left[kw + c\frac{\partial w}{\partial t} + m\frac{\partial^2 w}{\partial t^2}\right]$$

where $a = -1$ for the left end and $+1$ for the right end of the beam. In addition, the bending moment must be zero; hence

$$EI\frac{\partial^2 w}{\partial x^2} = 0$$

5. *End connected to a torsional spring, torsional damper, and rotational inertia* (Fig. 8.16(b)): In this case, the boundary conditions are

$$EI\frac{\partial^2 w}{\partial x^2} = a\left[k_t\frac{\partial w}{\partial x} + c_t\frac{\partial^2 w}{\partial x\partial t} + I_0\frac{\partial^3 w}{\partial x\partial t^2}\right]$$

where $a = +1$ for the left end and -1 for the right end of the beam, and

Commonly used boundary conditions for the transverse vibration of beam are as shown in Figure 4

Pinned-pinned	$\sin\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x]$	$\beta_1 l = \pi$ $\beta_2 l = 2\pi$ $\beta_3 l = 3\pi$ $\beta_4 l = 4\pi$
Free-free	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sin\beta_n x + \sinh\beta_n x + \alpha_n(\cos\beta_n x + \cosh\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l - \sinh\beta_n l}{\cosh\beta_n l - \cos\beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$ ($\beta l = 0$ for rigid-body mode)
Fixed-fixed	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sinh\beta_n x_n x - \sin\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \left(\frac{\sinh\beta_n l - \sin\beta_n l}{\cos\beta_n l - \cosh\beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Fixed-free	$\cos\beta_n l \cdot \cosh\beta_n l = -1$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x - \alpha_n(\cos\beta_n x - \cosh\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l + \sinh\beta_n l}{\cos\beta_n l + \cosh\beta_n l}\right)$	$\beta_1 l = 1.875104$ $\beta_2 l = 4.694091$ $\beta_3 l = 7.854757$ $\beta_4 l = 10.995541$
Fixed-pinned	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l - \sinh\beta_n l}{\cos\beta_n l - \cosh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$
Pinned-free	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x + \alpha_n \sinh\beta_n x]$ where $\alpha_n = \left(\frac{\sin\beta_n l}{\sinh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$ ($\beta l = 0$ for rigid-body mode)

Figure 4 Commonly used boundary conditions for transverse vibration of beam

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DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT 5-THEORY OF VIBRATIONS-SME1306

1 General comments

Vibration phenomena that might be modelled well using *linear vibration theory* include small amplitude vibrations of long, slender objects like long bridges, aeroplane wings, and helicopter blades; small rocking motions of ships in calm waters; the simplest whirling motions of flexible shafts, and so on. However, interactions between bridges and foundations, between wings/blades and air, between ships and waves, between shafts and bearings, and so on, are all *nonlinear*.

Nonlinear systems can display behaviours that linear systems cannot. These include:

(a) multiple steady state solutions, some stable and some unstable, in response to the same inputs,

(b) jump phenomena, involving discontinuous and significant changes in the response of the system as some forcing parameter is slowly varied,

(c) response at frequencies other than the forcing frequency,

(d) internal resonances, involving different parts of the system vibrating at different frequencies, all with steady amplitudes (the frequencies are usually in rational ratios, such as 1:2, 1:3, 3:5, etc.),

(e) self sustained oscillations in the absence of explicit external periodic forcing, and

(f) complex, irregular motions that are extremely sensitive to initial conditions (chaos).

Analytical intractability and limitations in computational resources make it difficult to systematically study the abovementioned phenomena in large systems (though harmonic balance is a useful technique; see below). For the most part, detailed studies of nonlinear vibrations are conducted using small systems (with perhaps just one or two degrees of freedom). A good qualitative understanding of the phenomena observed for the small system is invaluable when the same phenomena are subsequently encountered in larger systems.

The utility of precise numerical solutions remains high where appropriate. However, in nonlinear dynamics it is difficult to extract the qualitative essence from simulations alone. Therefore, an essential complement to all-numerical studies of large nonlinear systems is the analytical/theoretical study of simplified systems.

2 Analysis techniques

Three broad categories of techniques for analyzing nonlinear systems are:

(a) heuristic techniques like Galerkin methods, including harmonic balance
(b) asymptotic techniques, including the methods of averaging and multiple scales, and
(c) rigorous mathematical results about dynamical systems.

This introduction will concentrate on the first two categories.

2.1 Convergent, asymptotic, and heuristic

To make the later discussion more meaningful, let us distinguish between the terms convergent, asymptotic, and heuristic.

A *convergent* series dependent on a parameter (say, ϵ) is one where if we fix ϵ and take more and more terms, the sum converges to the correct answer. An *asymptotic* series dependent on a parameter (say, ϵ "small") is one where if we take a fixed number of terms and take ϵ smaller and smaller, the sum gets more and more accurate. Convergent series need not be asymptotic, and vice versa[1].

In harmonic balance, there is a periodic solution we wish to approximate. That periodic solution has a convergent Fourier series representation. However, in the application of harmonic balance with many terms, we obtain equally many coupled, usually nonlinear, equations in terms of the coefficients (see below). In practice, harmonic balance is often used with only a few harmonics, usually with excellent results but never any formal advance guarantees of how accurate the solution will be with a given number of terms included. In this sense, harmonic balance is a *heuristic* method.

We now discuss these methods in more detail.

2.2 Galerkin methods, and harmonic balance

The basic Galerkin method is now described using a simple boundary value problem,

$$\ddot{x} + x - 3t = 0\,, \text{ with } x(0) = x(\pi/2) = 0\,.$$

The exact solution is $x = 3t - \frac{3\pi}{2}\sin t$. As an approximation we assume, say, $x \approx \sum_{k=1}^{N} a_k \sin 2kt$. Substituting into the governing equation, we obtain a nonzero quantity $r(t)$ called the residual. We make $r(t)$ orthogonal to the assumed basis functions, i.e., set

$$\int_0^{\pi/2} r(t)\sin 2kt\, dt = 0\,, \text{ for } k = 1, 2, \cdots, N\,.$$

The above process, called a Galerkin projection, yields N equations for the N unknown a_k's, which upon solution give the approximate solution. The approximation to 3 terms is

$$x \approx -\sin 2t + \frac{1}{10}\sin 4t - \frac{1}{35}\sin 6t\,,$$

which has an error ≤ 0.024. More terms yield more accuracy.

[1]See, e.g., E. J. Hinch, *Perturbation Methods*, Cambridge University Press, 1991.

Note that for this linear ODE, the equations for the unknown a_k's are linear and algebraic, while for general nonlinear ODE's these will be nonlinear algebraic equations (see below). For partial differential equations in time and space, the approximation will typically be of the form $\sum_{k=1}^{N} a_k(t)\phi_k(x)$, where the ϕ_k are functions of space chosen to suit the problem (e.g., satisfy boundary conditions).

The technique of *Harmonic Balance* is a specialized application of the Galerkin method to find periodic solutions in vibration problems. There are several slightly different versions of the method. Here, we consider unforced, undamped, conservative problems, e.g.,

$$\ddot{x} + x^3 = 0\,. \tag{1}$$

We start with, say, $x \approx A\sin\omega t + B\sin 3\omega t$. Note that the unknown ω appears in the functions $\sin\omega t$ and $\sin 3\omega t$, and so there are actually three unknowns in the two term approximation. Substituting into the differential equation, multiplying in turn by $\sin\omega t$ and $\sin 3\omega t$, and integrating in each case from 0 to $2\pi/\omega$ and then equating to zero (the Galerkin projection), we obtain:

$$-A\omega^2 + 3A^3/4 - 3A^2B/4 + 3AB^2/2 = 0\,,$$

$$-9B\omega^2 - A^3/4 + 3A^2B/2 + 3B^3/4 = 0\,.$$

Treating the indeterminate A as a parameter, we obtain $\omega = 0.8869A$ and $B = -0.04482A$.

Variations of the above method are used as the problem changes.

Harmonic balance with a few terms usually gives good approximations to periodic solutions. For example, some numerical results for the above nonlinear oscillations of Eq. 1, as compared with the two term harmonic balance calculation given above, are shown in Fig. 1. Oscillations at four different amplitudes are shown, and the figure appears to have four different curves. Each of these curves is in fact two superimposed and nearly indistinguishable curves (one solid, one dash-dot). The small difference between the solid (numerical) and dash-dot (harmonic balance) is visible towards the right side of the figure (for larger t).

The results show that the two term harmonic balance solution is very accurate. The strong dependence of frequency on amplitude is also clearly seen.

2.3 A first look at asymptotic techniques

Asymptotic techniques depend on some parameter in the problem being very small (or very large, which is the same thing on taking reciprocals). In the limit as the small parameter becomes zero, the problem should be analytically tractable. The basic ideas can be demonstrated using the following root-finding example:

$$\epsilon\, x^6 + x - 1 = 0\,, \tag{2}$$

where $0 < \epsilon \ll 1$. If $\epsilon = 0$, $x = 1$ is the only root. For nonzero ϵ, that root is perturbed to

$$x = 1 - \epsilon + 6\epsilon^2 - 51\epsilon^3 + \mathcal{O}(\epsilon^4)\,.$$

The $\mathcal{O}(\epsilon^4)$ above represents a quantity that is no bigger than some finite constant times ϵ^4, as ϵ goes to zero.

For $\epsilon \neq 0$, Eq. 2 has five other "large" roots, obtainable via a singular perturbation scheme. One of them is

$$x = -\epsilon^{-1/5} - \frac{1}{5} + \frac{3}{25}\epsilon^{1/5} - \frac{14}{125}\epsilon^{2/5} + \mathcal{O}(\epsilon^{3/5})\,.$$

The two "asymptotic" approximations above are useful for sufficiently small ϵ.

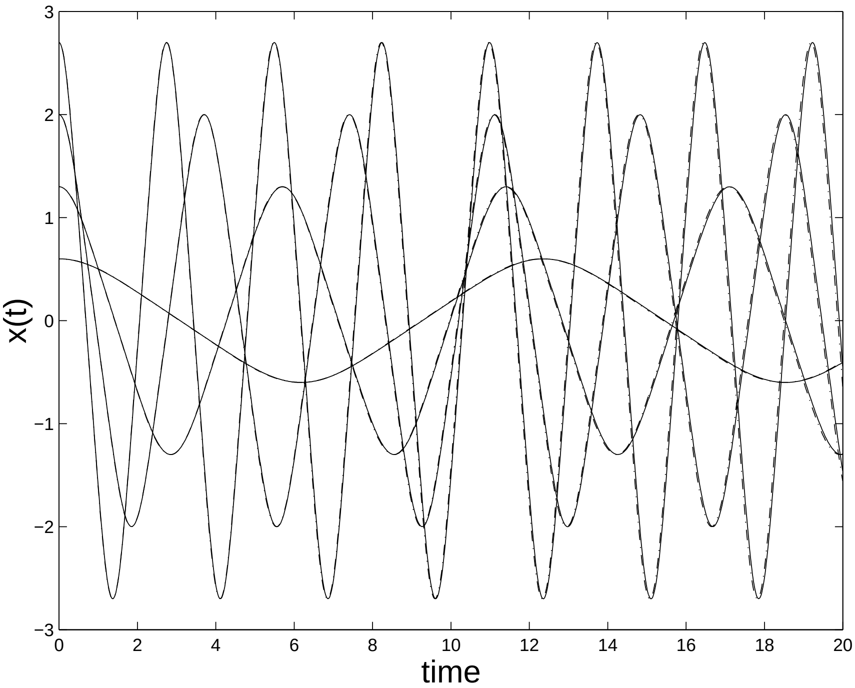

Figure 1: Solutions for Eq. 1. Solid line: numerical. Dashdot: harmonic balance (can be viewed as slightly distinct from solid line, for larger times).

2.4 Averaging and multiple scales

The method of *averaging* is a specialized asymptotic technique for systems of the form

$$\dot{x} = \epsilon f(x,t)\,, \;\; \epsilon \ll 1. \tag{3}$$

Here, we assume $f(x,t) = f(x,t+T)$ for all x,t. An approximation to the solution is found by solving the simpler equation

$$\dot{x} = \epsilon f_0(x)\,, \;\text{ where } f_0(x) = \frac{1}{T}\int_0^T f(x,t)\,dt\,.$$

Nonlinear oscillators, e.g.,

$$\ddot{x} + x = \epsilon \dot{x}(1-x^2)\,, \tag{4}$$

are not directly amenable to averaging; but they can be put in that form via a change of variables to $x = A(t)\sin(t+\phi(t))$, along with the added constraint equation $\dot{x} = A(t)\cos(t+\phi(t))$. In this form, the asymptotic method of averaging has been widely used to study a variety of weakly nonlinear oscillators that are slightly perturbed versions of the harmonic oscillator ($\ddot{x} + x = 0$).

For illustration, Eq. 4 yields the two equations

$$\dot{A} = \epsilon\,\Big(A/2 - A^3/8 + A\cos(2t+2\phi)/2 + A^3\cos(4t+4\phi)/8\Big)\;,$$

$$\dot{\phi} = \epsilon\,\Big(-\sin(2t+2\phi)/2 + A^2\sin(2t+2\phi)/4 - A^2\sin(4t+4\phi)/8\Big)\;.$$

Finally, by first order averaging (higher order averaging is possible, but not done here), we get

$$\dot{A} = \epsilon\left(A/2 - A^3/8\right), \text{ and } \dot{\phi} = 0\,.$$

The above two equations show that $A = 0$ is an unstable equilibrium; all other solutions slowly but eventually approach $A = 2$ (assuming $A > 0$); and the phase of the oscillation remains steady, at least at first order.

The method of *multiple scales*, also applicable to Eq. 3, involves an additional issue, namely the identification and removal of secular terms, as illustrated below for Eq. 4 using two time scales.

Let t be the actual time; and $\tau = \epsilon\, t$ be a slow time. Assume $x = x(t, \tau)$. Now

$$\dot{x} = \frac{\partial x}{\partial t} + \epsilon\,\frac{\partial x}{\partial \tau}\,, \text{ and } \ddot{x} = \frac{\partial^2 x}{\partial t^2} + 2\epsilon\,\frac{\partial^2 x}{\partial \tau \partial t} + \mathcal{O}(\epsilon^2)\,.$$

Using subscripts t and τ to denote partial derivatives with respect to these quantities, we have

$$x_{tt} + x = \epsilon\left\{-2x_{\tau t} + x_t\left(1 - x^2\right)\right\} + \mathcal{O}(\epsilon^2)\,.$$

Assuming a solution of the form $x = x_0 + \epsilon x_1 + \cdots$, we obtain

$$x_{0,tt} + x_0 = \epsilon\left\{-x_{1,tt} - x_1 - 2x_{0,\tau t} + x_{0,t}\left(1 - x_0^2\right)\right\} + \mathcal{O}(\epsilon^2)\,.$$

Collecting terms, at leading order we obtain

$$x_{0,tt} + x_0 = 0\,,$$

which has the general solution $x_0 = A(\tau)\sin(t + \phi(\tau))$. Substituting this at the next order we obtain (dropping the explicit dependence of A and ϕ on τ, and using primes to denote a τ-derivative)

$$x_{1,tt} + x_1 = A^3\cos(3t + 3\phi)/4 + (-2A' + A - A^3/4)\cos(t + \phi) + 2A\phi'\sin(t + \phi)\,.$$

In the above equation, the solution for x_1 *can* contain $t\sin(t + \phi)$ and $t\cos(t + \phi)$ (effectively the same as $t\sin t$ and $t\cos t$). These *secular* terms make the approximation break down by the time $t = \mathcal{O}(1/\epsilon)$. The validity of the expansion can be extended by removing the secular terms, which can be done here by requiring that the coefficients of the sine and cosine in the forcing be zero, i.e., $-2A' + A - A^3/4 = 0$ and $2A\phi' = 0$. Noting that $\dot{A} = \epsilon A'$, etc., we find the evolution of A and ϕ are governed, at this order of approximation, by the same equations as obtained by averaging:

$$\dot{A} = \epsilon\,(A/2 - A^3/8)\,, \text{ and } \dot{\phi} = 0\,. \tag{5}$$

3 The phase plane

Our study of entrainment in section 9 will involve the use of a popular and powerful idea from nonlinear dynamics: the idea of the *phase space*. The essential idea is described below.

Consider a system of two equations

$$\dot{x} = f(x, y)\,, \text{ and } \dot{y} = g(x, y)\,.$$

Sometimes, instead of plotting x and y individually versus t, we just plot x versus y. If, say, x rises monotonically from 0 to 1 as t increases, while y rises from -1 to 3 during the same time, then on

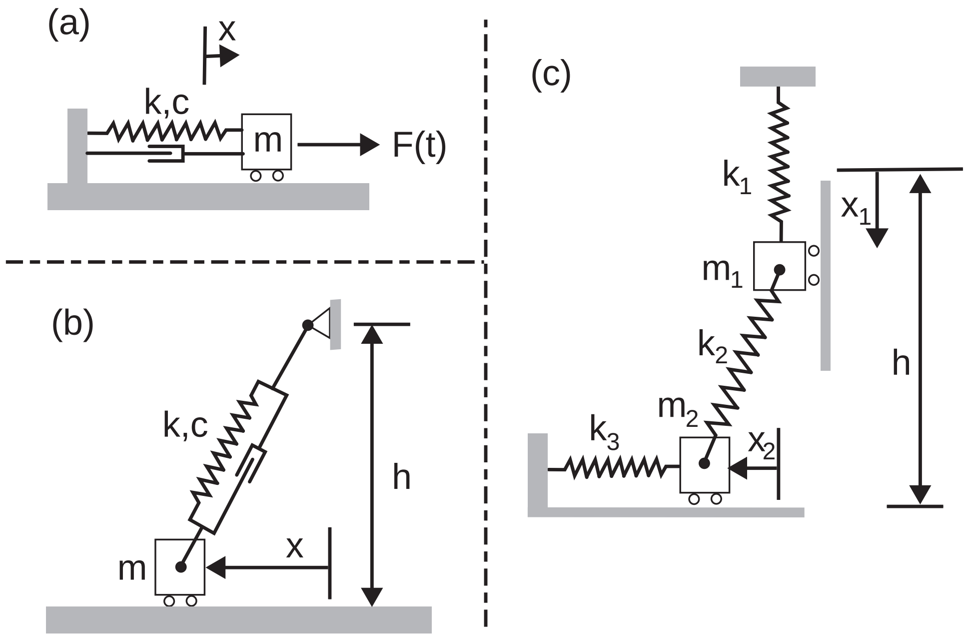

Figure 2: (a) A linear, damped, forced system. (b) A nonlinear system. The spring has a free length $L_0 > h$. (c) A nonlinear two degree of freedom system. Mass m_1 is constrained to move frictionlessly in the vertical direction, while mass m_2 moves in the horizontal direction. Gravity is neglected, for simplicity.

the x versus y plane we have a single curve that goes from the point $(0, -1)$ to $(1, 3)$. The (x, y) plane is called the phase plane. In a more general case, with n dependent variables, we would have an n-dimensional *phase space*.

Looking at solutions in the phase space has the disadvantage of losing detailed information about the exact way in which x and y vary with time. However, it has the obvious advantage of reducing the dimensionality of the system by one: the solution goes from a curve in the three-dimensional (x, y, t) space to the two-dimensional (x, y) plane. In addition, there are other advantages involving geometrical ideas about various types of solutions and how they behave. For example, if x and y approach constant values, then the graphs of x and y versus t are horizontal lines; but in the phase plane, the graph of x versus y approaches a point. Similarly, if x and y are periodic functions with some period T, and with some phase difference between them, then in the phase plane we see a closed curve. Interested readers will find many excellent books available on nonlinear dynamics, and topics touched upon in these notes are discussed properly in such books. A representative sample of references is provided at the end.

4 Multiple solutions

A damped linear system, such as sketched in Fig. 2(a), governed by the linear differential equation

$$m\ddot{x} + c\dot{x} + kx = f(t)\,,$$

has a uniquely defined long term behaviour (after transients die out). For example, consider

$$\ddot{x} + 0.3\,\dot{x} + x = \sin 3.2\,t\,. \tag{6}$$

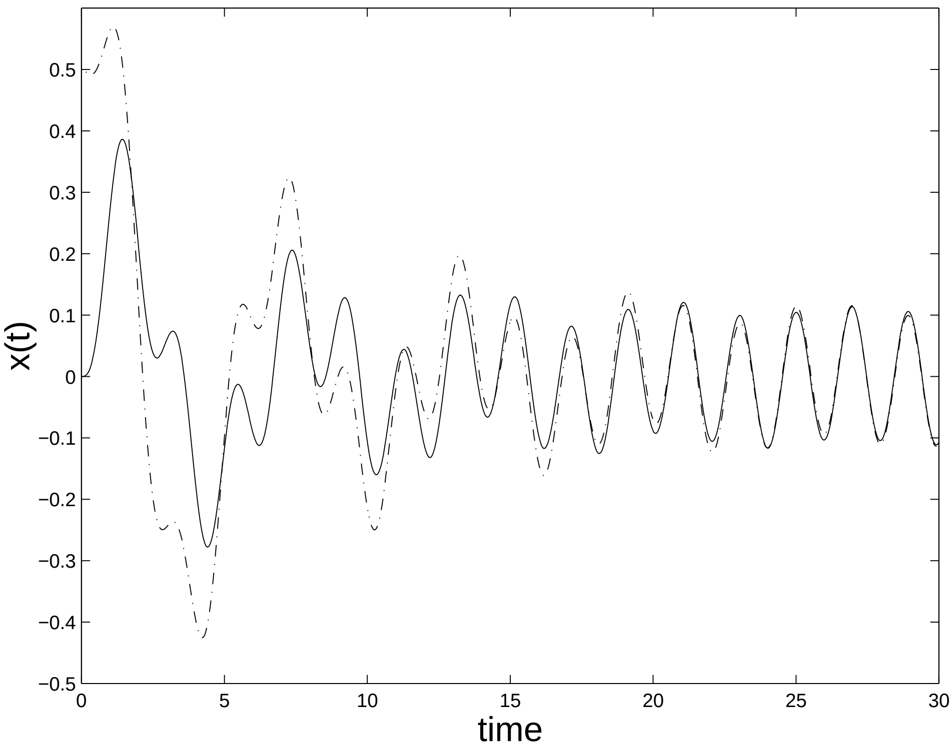

Figure 3: Solutions for Eq. 6 converge to the same long-time behaviour regardless of initial conditions.

Two different solutions, for two different initial conditions, are shown to converge to the same "long-time" solution in Fig. 3.

In contrast, consider the system shown in Fig. 2(b), with the spring's free length L_0 greater than h. Now it is clear that this nonlinear system will have three equilibrium positions: one at $x = 0$, which will be unstable, while one stable position at some nonzero positive x, and another (reflected) one for negative x. This simple example shows that it is possible for general deterministic nonlinear systems to have more than one steady state solution in response to the same inputs (but, of course, with different initial conditions).

This system is not analyzed here in detail; other examples of multiple solutions will soon be analyzed.

In practical engineering, examples of multiple solutions are encountered in a variety of situations. A few examples are provided below.

- *Buckling.* Beyond a certain load, the structure has more than one equilibrium; the nominal equilibrium loses stability, and new stable equilibrium positions appear. This is related to the system in Fig. 2(b).

- *Whirling of shafts* at, near and possibly beyond critical speed. A non-whirling solution still exists, but is now unstable.

- *Resonances* in nonlinear systems. When the forcing frequency is near the linear natural frequency, there can be more than one possible stable steady state solution. This example will be covered again under "jumps".

- *Machine tool chatter.* Under certain operating conditions, the cutting tool might chatter a lot (poorer surface finish) or very little: there is more than one stable steady state solution.
- *Systems with dry friction.* Some systems with dry friction, for small forcing near resonance, can have two solutions: one with large amplitude, and one without vibrations.

5 Forced vibrations (via harmonic balance)

Consider the damped nonlinear forced system given by

$$\ddot{x} + c\dot{x} + x + ax^3 - F \sin \omega t = 0\,. \tag{7}$$

We will study this system using single term harmonic balance. Let us assume $x \approx A \sin \omega t + B \cos \omega t$. The assumption is that the solution is dominated by a response at the same frequency, though not at the same phase, as the forcing. The assumption is exactly true for the linear system (with $a = 0$), and approximately true for reasonable values of a and most values of ω. This single harmonic approximation is sufficient for the purposes of this section.

Substituting into the equation of motion and using some trigonometric identities such as $\sin^3 x = (3 \sin x - \sin 3x)/4$, we obtain

$$\begin{aligned}
-\omega^2 A \sin \omega t - \omega^2 B \cos \omega t + c\omega A \cos \omega t - c\omega B \sin \omega t + A \sin \omega t + B \cos \omega t - \tfrac{1}{4} a A^3 \sin 3\omega t \\
\cdots + \tfrac{3}{4} a A^3 \sin \omega t + \tfrac{3}{4} a A^2 B \cos \omega t - \tfrac{3}{4} a A^2 B \cos 3\omega t + \tfrac{3}{4} a A B^2 \sin 3\omega t + \tfrac{3}{4} a A \sin \omega t B^2 \\
\cdots + \tfrac{1}{4} a B^3 \cos 3\omega t + \tfrac{3}{4} a B^3 \cos \omega t - F \sin \omega t = \text{ negligible terms.}
\end{aligned}$$

Multiplying by $\sin \omega t$ or $\cos \omega t$, integrating w.r.t. t from 0 to $2\pi/\omega$, and then setting them equal to zero, is equivalent to simply picking out the coefficients of $\sin \omega t$ or $\cos \omega t$, respectively, and setting them equal to zero. This gives:

$$-A\omega^2 - cB\omega + A + \frac{3}{4} a A^3 + \frac{3}{4} a A B^2 - F = 0\,,$$

$$-B\omega^2 + cA\omega + B + \frac{3}{4} a A^2 B + \frac{3}{4} a B^3 = 0\,.$$

The solutions to the two simultaneous equations above provide a fairly accurate picture of the dynamics of the system in Eq. 7.

5.1 Unforced, undamped case

If we put $c = 0$ and $F = 0$, then we obtain an approximate solution to the unforced, undamped system, for which

$$B = 0\,, \text{ and } \omega = \frac{1}{2}\sqrt{4 + 3aA^2}\,.$$

The above (approximate) result tells us that for undamped, unforced periodic oscillations the frequency of oscillations depends on the amplitude. The graph of A versus ω (i.e., with amplitude along the vertical axis) is usually called a "backbone curve" because of its shape. In this system, the strength of the nonlinearity is measured by the single quantity a, and so it is not surprising that the amplitude dependence of the frequency (which happens only for nonlinear systems) involves a-dependence as well. It is usual to call the case of $a > 0$ a *stiffening* nonlinearity, and the case $a < 0$ a *softening* nonlinearity. In the presence of a stiffening nonlinearity, frequency increases with amplitude; in the case of a softening nonlinearity, frequency decreases with amplitude.

5.2 Forced, damped case

In the general case, if we select a certain forcing amplitude F and angular frequency ω, then we can in principle solve for A and B (and hence the response) in terms of a and c. In practice, it is convenient to solve the equations numerically. Some specific results are shown in the three plots of Fig. 4, where frequency ω is plotted along the horizontal axes and amplitude of response (taken to mean $\sqrt{A^2 + B^2}$ from the harmonic balance equations) is plotted along the vertical axes.

Fig. 4(a) shows the effect of nonlinearity. For $a = 0$, we have the familiar linear resonance curve. For increasing a while holding all other things constant, the resonance curve leans over to the right (for a stiffening nonlinearity; if we took $a < 0$ it would lean over to the left).

Fig. 4(b) shows the effect of varying damping c, while holding all other things fixed. Since a is fixed, the backbone curve is fixed. It is seen that the hump in the amplitude versus frequency curve follows the backbone curve in each case – hence the importance of the backbone curve. All other things held fixed, decreasing c raises the hump, i.e., raises the maximum response amplitude possible with a given amplitude of harmonic forcing. If we allow both c and F to become very small, the amplitude-frequency curve follows the backbone curve even more closely (not surprising, because the backbone curve is obtained by setting $c = 0$, $F = 0$).

Finally, Fig. 4(c) shows the effect of increasing forcing amplitude while holding other things constant. It is seen that the amplitude versus frequency curve has a hump that leans over to the right; it would lean to the left if the nonlinearity was of the opposite sense, i.e., a was negative. It is seen that for relatively small damping, the hump in the amplitude versus frequency plot follows the backbone curve. Larger F leads to a higher hump.

A few further remarks may be made about the response of this simple nonlinear system. For very high frequencies of forcing, inertia dominates and the amplitude of motion is very small; in such cases, the x^3 nonlinearity is insignificant because $|x^3| \ll |x|$, and the system behaves essentially like a linear system. The most visible qualitative difference between the linear and nonlinear system is in the leaning over of the hump near resonance; this is not surprising because large amplitude motions are (in this system, though not for all systems) the reason for nonlinear terms to become important[2].

It is also clear that for $F = 2$ and $\omega = 3$, say, there are three different possible amplitudes of response (thus, multiple solutions in response to the same input forcing). Of these, it is possible to show that the smallest and largest amplitude solutions are stable, while the intermediate amplitude solution is unstable (this stability issue will not be discussed fully in these lectures, but a limited study will be presented below).

6 Jumps

Figure 4(a) also shows clearly the existence of multiple solutions for this problem. There are three vertical lines in the figure, marked 1, 2 and 3. For the solid line (marked 2), we see that there are three amplitudes possible (shown in the figure with heavy dots marked P, Q and R). Of these, the point Q is unstable, while P and R are stable. Though an analysis of the stabilities of these points is not conducted here, I mention that such stability analyses can be conducted using several techniques. These include direct numerical simulation; Floquet theory (a topic not covered in these lectures); and, under more limited circumstances (weak nonlinearity, light damping and small forcing), asymptotic techniques like the method of averaging or the method of multiple scales.

[2] Consider a "simply supported" slender rod supported on pins at its two ends. A small clearance will exist in the pin holes. For small amplitude vibrations, when the amplitude of vibrations is comparable to that clearance, the vibrations will in fact be strongly *nonlinear*. For somewhat larger amplitudes, nonlinearities will be unimportant. Finally, for very large amplitudes, nonlinearities will be important again.

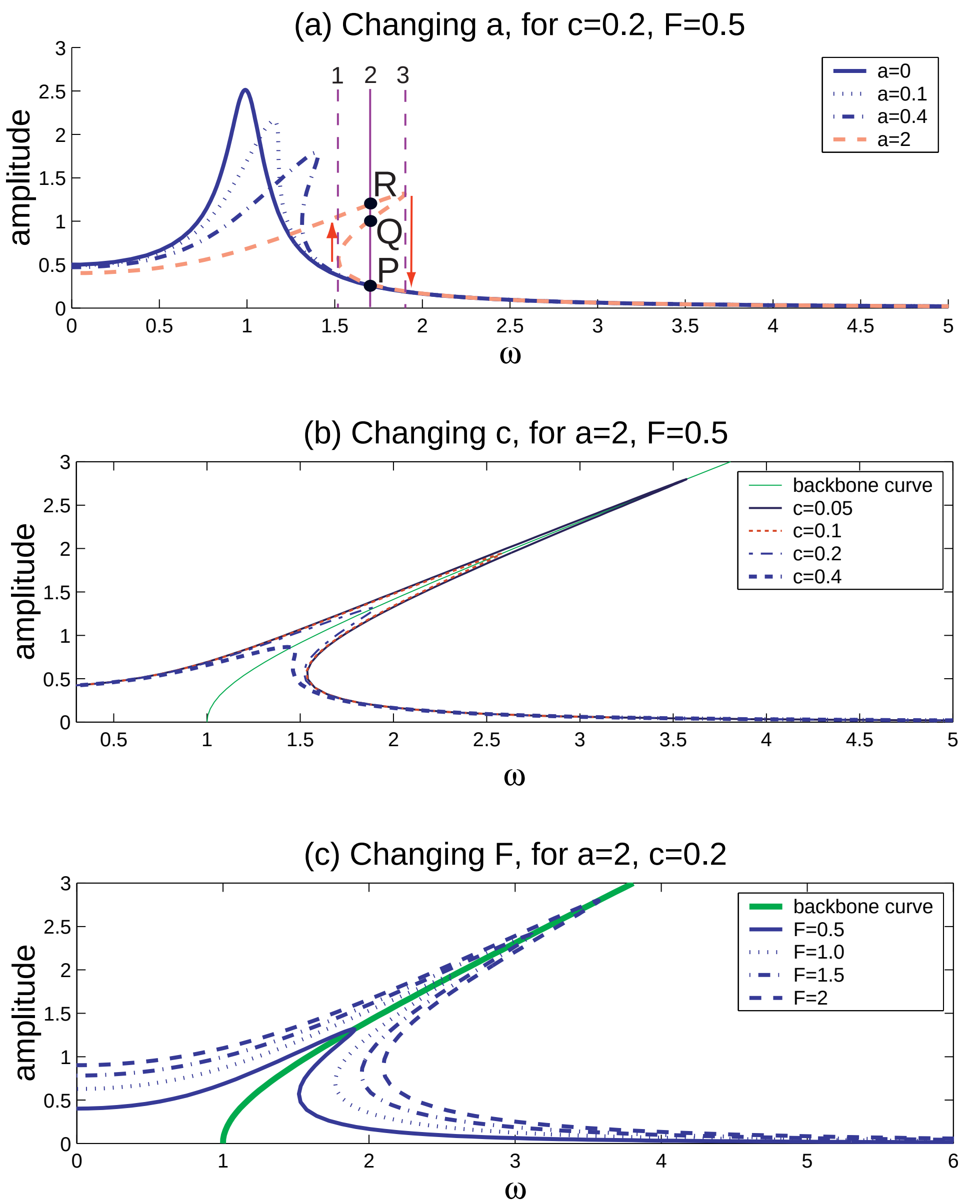

Figure 4: Forced vibrations of a nonlinear system, via harmonic balance. See text for details.

The figure also shows the phenomenon of *jumps*, which are discontinuous changes in the steady state response of a system as a parameter (here, forcing frequency) is slowly varied. Imagine that we start by forcing the system at a low frequency; there is a unique steady state periodic solution, on which the system response settles. As we raise the frequency quasistatically (very slowly; so slowly that there are no transients and we get a sequence of steady states), we eventually reach the first vertical dashed line (marked 1). Beyond this frequency, there are three possible solutions, but the system stays on the uppermost branch. Passing through point R, the system does not show any awareness of the alternative solutions at P and Q. Eventually reaching the vertical dashed line marked 3, the system response jumps to the lower branch, as indicated by the downward arrow. On further increasing the forcing frequency, the system has a unique, stable solution. Finally, if we now start decreasing the forcing frequency from some initially large value, then the system response stays on the lower branch as we cross line 3, and jumps up, as shown by the arrow, when we reach line 1. For frequencies between line 1 and line 3, if we start the system from arbitrary initial conditions, then which response the system chooses (upper or lower; not, for generic initial conditions, the intermediate unstable one) depends on initial conditions.

7 Harmonics and subharmonics

It is possible for the response of a nonlinear system to contain frequencies other than that of the forcing frequency. In fact, it is quite common for the response to have frequencies that are multiples of the forcing frequency. To see this through simple examples, consider the following system:

$$\ddot{x} + 0.05\,\dot{x}^3 + x^3 = \sin\omega t\,. \tag{8}$$

Two values of ω were chosen, based on a preliminary study using harmonic balance (details not given here), for detailed numerical study: these values are $\omega = 0.4$ and $\omega = 1.66$. Numerical results obtained are summarized in Fig. 5. In the figure, the numerically obtained power spectral density of the forcing is plotted for each case, and shows a single peak in each case (see Figs. 5 (a) and (d)). For both values of ω, the time series (direct numerical solution) settles down to qualitatively similar periodic solutions (see Figs. 5 (c) and (e)). For both cases, the power spectral density of the system response shows multiple peaks, at frequencies in the proportion $1:3:5:\cdots$. In other words, in the response in each case has content or "energy" at frequencies other than the forcing frequency. This feature is common in nonlinear systems. Note that in the harmonic ratios above, even numbers are missing. That is, no frequency component at twice the forcing frequency appears in the response. This is because the system chosen here has only odd order nonlinearities (cubic terms). In the presence of some even order nonlinear terms, even order harmonics would also be expected.

Finally, note that for $\omega = 0.4$ the fundamental frequency of the response equals the forcing frequency (compare the peaks in Figs. 5 (a) and (c)). This situation, with higher harmonics, is very common in nonlinear vibrations. In contrast, for $\omega = 1.66$, the fundamental frequency of the response is *one third* of the forcing frequency (compare the peaks in Figs. 5 (d) and (f)). The 1/3 frequency response is called a *subharmonic*, and occurs somewhat less frequently than higher harmonics.

The above example provides a simple picture of a relatively simple nonlinear system. For more general, higher dimensional systems under more complex forcing, many different modes as well as harmonics can interact in complex ways to produce responses that are difficult to characterize and understand. There is no simple yet general theory for such cases, and problems that arise need to be tackled on a case by case basis.

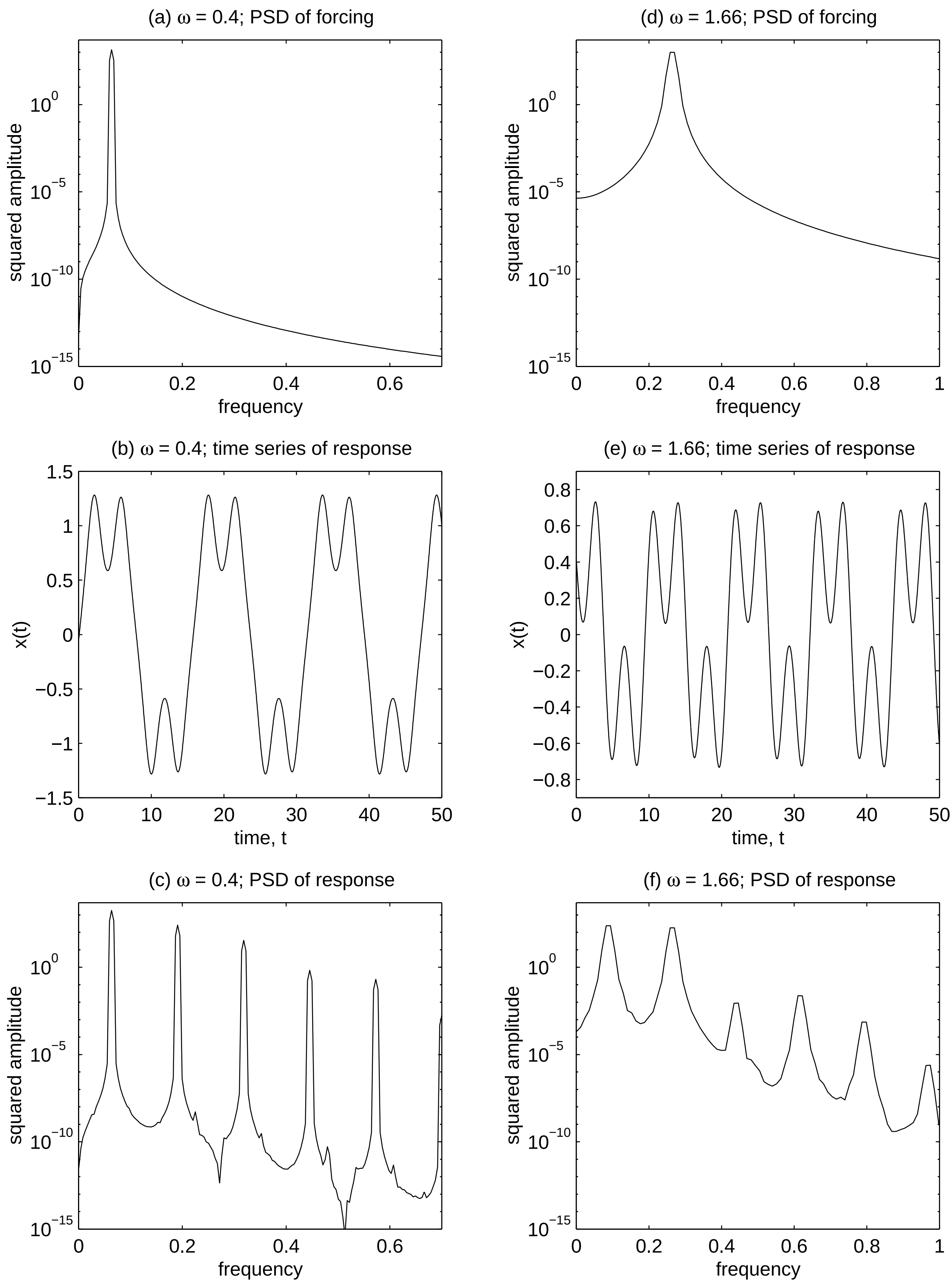

Figure 5: Numerical simulation results for Eq. 8. See text for details.

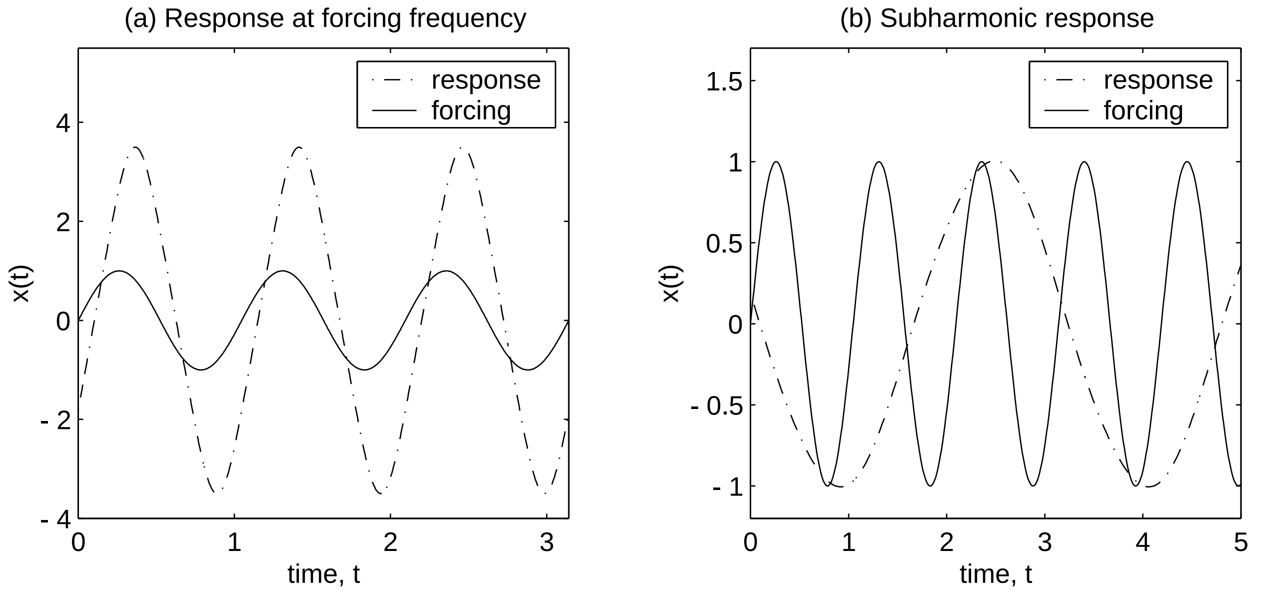

Figure 6: Numerical simulation results for Eq. 9. See text for details.

Before concluding this section, it is worth looking at another system with a clearer and more convincing subharmonic response. The equation

$$\ddot{x} + x + 4x^3 = \sin 6t$$

has the *exact* solution

$$x = -\sin 2t\,,$$

a subharmonic resonance. In this solution, the response has *no component* at the forcing frequency.

The same system also has a solution that is dominated by the forcing frequency. By first order harmonic balance, that solution is approximately $A \sin 6t$, with A satisfying

$$3A^3 - 35A - 1 = 0, \qquad \text{or} \qquad A \approx 3.43,\ -0.03,\ -3.40\,.$$

From our previous experience with forced vibrations (section 5), we can arrange the three solutions by magnitude, to get

$$|A| = 0.03,\ 3.40,\ 3.43\,;$$

and we expect that the intermediate solution (3.40) will turn out to be unstable.

To see that these solutions are meaningful, we add a little damping, and numerically integrate the equation

$$\ddot{x} + 0.03\dot{x} + x + 4x^3 = \sin 6t \tag{9}$$

for different initial conditons. Partial numerical results are shown in Fig. 6. For initial conditions that are sufficiently close to the steady state motion of interest, the numerical solution converges in each case to a steady state solution approximately equal to that expected from the above calculations (due to the introduction of small damping, the solutions are slightly different). In particular, Fig. 6(a) shows a solution at the forcing frequency, with an amplitude of roughly 3.4 (consistent with our undamped estimate of 3.43); while Fig. 6(b) shows a solution at 1/3 of the forcing frequency, with an amplitude of approximately 1.

8 Limit cycles

Some systems have *self-sustained* vibrations. These include squealing door hinges, electric wires whistling in the wind, and whirling shafts. These self-sustained vibrations are periodic motions that are locally unique, and which occur in the absence of external periodic forcing.

To study limit cycles, we will again use the van der Pol equation (Eq. 4), reproduced here as

$$\ddot{x} + x - \epsilon \dot{x}(1 - x^2) = 0\,. \tag{10}$$

Let us start with one-term harmonic balance,

$$x \approx A \sin \omega t\,. \tag{11}$$

Note that the cosine is not *explicitly* included here, because time t does not appear explicitly in the equation, and so we can choose $t = 0$ in such a way as to make the coefficient of the cosine equal to zero; however, for the same reasons, the cosine is implicitly included, i.e., the same solution form, on shifting time, automatically includes the cosine.

Substituting Eq. 11 into Eq. 10, we obtain

$$\left(-\omega^2 A + A\right) \sin \omega t + \epsilon\omega \left(\frac{A^3}{4} - A\right) \cos \omega t - \frac{\epsilon \omega A^3}{4} \cos 3\omega t = \text{ negligibly small terms.}$$

From the coefficient of $\sin \omega t$, we find that either $A = 0$ or $\omega = 1$. From the coefficient of $\cos \omega t$, we find that $A^3/4 - A = 0$, which means either $A = 0$ or $A = 2$ (we ignore the negative roots, which provide no new physical information).

Thus, with one term harmonic balance, we have partly verified the information obtained from first order averaging, or from the method of multiple scales, for the case $0 < \epsilon \ll 1$, in section 2.4, Eq. 5. However, Eq. 5 had in fact provided more information, which harmonic balance has not provided. Harmonic balance can find periodic solutions, but it cannot say whether they are stable or not. Equation 5, on the other hand, shows that $A = 0$ is unstable and $A = 2$ is stable, by the following simple analysis:

1. For the $A = 0$ case, we linearize for small A, and obtain

$$\dot{A} = \epsilon \frac{A}{2}\,,$$

   which shows that solutions grow exponentially; thus, the $A = 0$ solution is unstable.

2. For the $A = 2$ case, we let $A = 2 + B$, to obtain

$$\dot{B} = \epsilon \left(-B - \frac{3B^2}{4} - \frac{B^3}{8}\right)\,,$$

   which we linearize for small B to obtain

$$\dot{B} = -\epsilon B\,,$$

   which in turn shows that provided B (or the deviation from $A = 2$) is sufficiently small to start with, it decays exponentially to zero. Thus, the $A = 2$ solution is stable. (In fact, it is easy to show that all initial conditions other than $A = 0$ eventually settle on $A = 2$, but we skip that demonstration here.)

Note that the previous stability conclusions are exactly reversed if ϵ is negative instead of positive; while the harmonic balance results, blind to stability issues, remain unaffected.

9 Entrainment

Recall the van der Pol equation encountered above:

$$\ddot{x} + x = \epsilon \dot{x} \left(1 - x^2\right) \,, \text{ where } 0 < \epsilon \ll 1 \,.$$

As shown above, this equation has a stable limit cycle of amplitude about 2, and angular frequency $1 + \mathcal{O}\left(\epsilon^2\right)$. Now consider a small perturbation where the "spring" is a little stiffer or a little softer,

$$\ddot{x} + x = \epsilon \left[\dot{x}\left(1 - x^2\right) + \Delta x\right] \,,$$

where the $\mathcal{O}(1)$ quantity Δ is called a detuning parameter. It may be expected that changing the spring stiffness a little, just in itself, does nothing except change the angular frequency of the limit cycle a little, so that it becomes $1 + \mathcal{O}(\epsilon)$. Indeed, on averaging the above equation, we find

$$\dot{A} = \epsilon \left(A/2 - A^3/8\right) \,, \text{ and } \dot{\phi} = -\epsilon \frac{\Delta}{2} \,.$$

The above nonzero $\dot{\phi}$ indicates that the period of the solution is now slightly different from 2π.

Now, consider the equation

$$\ddot{x} + x = \epsilon \left[\dot{x}\left(1 - x^2\right) + \Delta x + F \sin t\right] \,, \tag{12}$$

which represents a van der Pol oscillator periodic with forcing at a frequency slightly different from that of the unforced limit cycle[3].

What sort of behaviour can we expect? If F is small, then we expect the original limit cycle with period slightly different from 2π because of the detuning. If F is sufficiently large, perhaps the forcing will overwhelm the unforced dynamics, and the oscillation will phase-lock with the forcing, a phenomenon called *entrainment*. For intermediate values of F, perhaps some sort of transition region might be observed.

By first order averaging, we obtain:

$$\dot{A} = \epsilon \left(\frac{A}{2} - \frac{A^3}{8} - \frac{F\cos\phi}{2}\right) \,, \text{ and} \tag{13}$$

$$\dot{\phi} = \epsilon \left(-\frac{\Delta}{2} + \frac{F\sin\phi}{2A}\right) \,. \tag{14}$$

It will be more convenient for us to study the above "slow flow" (or averaged equations) after transforming to polar coordinates. Recall that the approximate solution is

$$A\sin(t+\phi) = A\cos\phi\,\sin t + A\sin\phi\,\cos t = C\sin t + D\cos t \,,$$

where

$$C = A\cos\phi \,, \quad D = A\sin\phi \,.$$

In terms of C and D, we get

$$\dot{C} = \epsilon \left(\frac{C}{2} - \frac{C^3}{8} + \frac{\Delta D}{2} - \frac{CD^2}{8} - \frac{F}{2}\right) \,, \text{ and} \tag{15}$$

$$\dot{D} = \epsilon \left(\frac{D}{2} - \frac{D^3}{8} - \frac{\Delta C}{2} - \frac{C^2 D}{8}\right) \,. \tag{16}$$

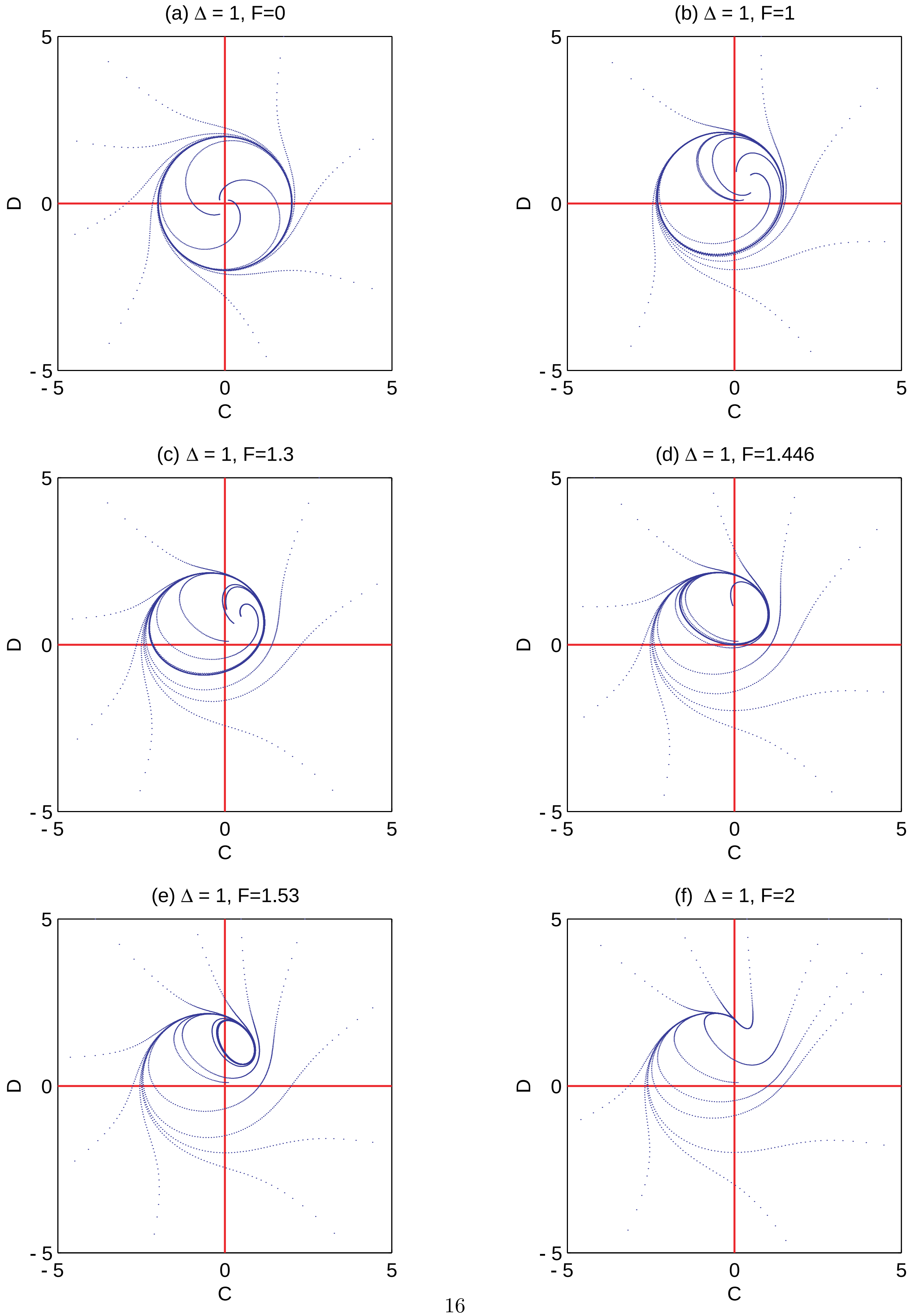

Figure 7: Phase plane for Eqs. 15 and 16. See text for details.

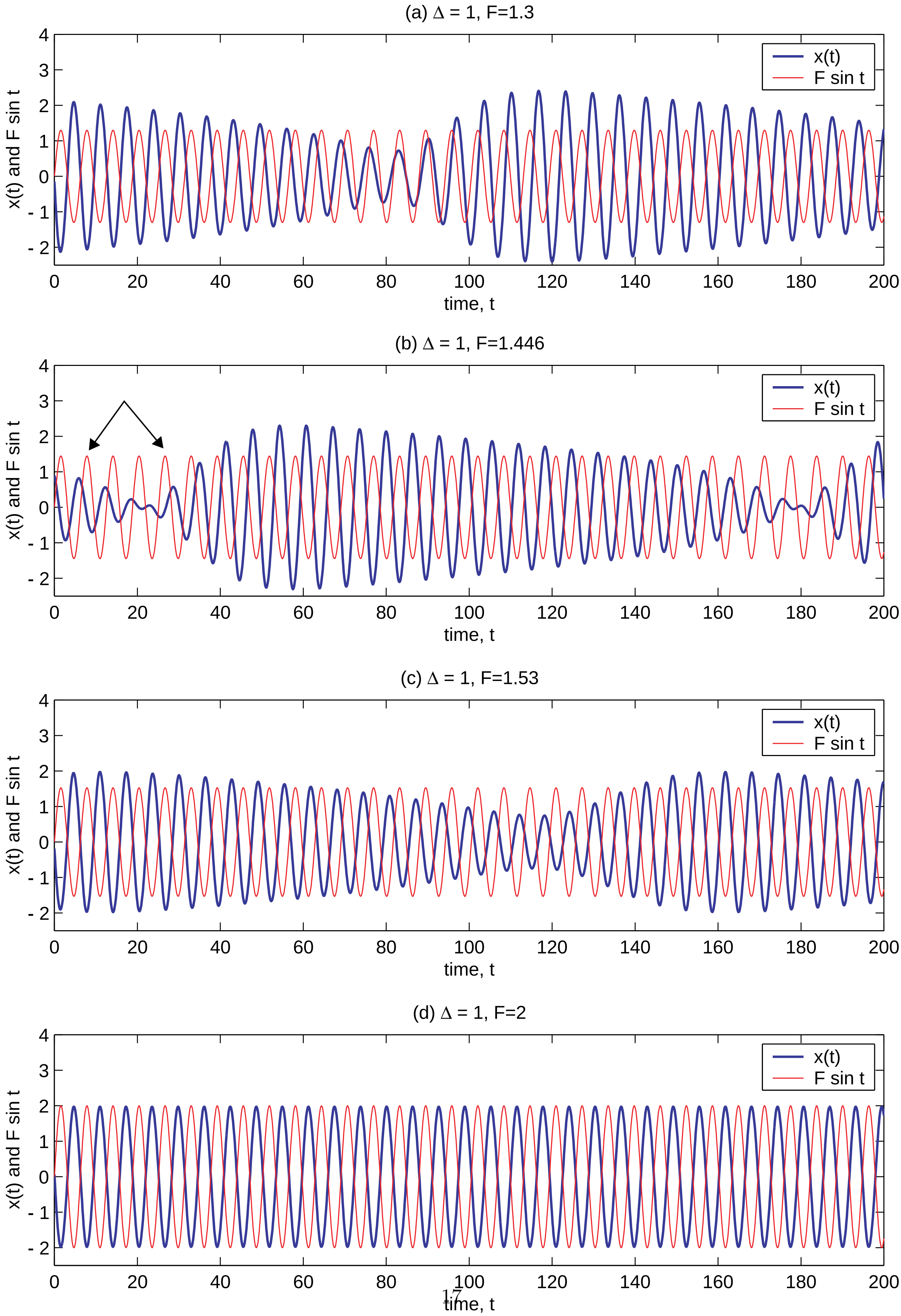

Figure 8: Numerical solutions of Eq. 12. See text for details.

What do Eqs. 15 and 16 say?

We begin by looking at the case $\Delta = 0$, $F = 0$. In this case, the equations reduce to

$$\dot{C} = \epsilon \frac{C}{2}\left(1 - \frac{C^2}{4} - \frac{D^2}{4}\right), \text{ and} \quad (17)$$

$$\dot{D} = \epsilon \frac{D}{2}\left(1 - \frac{C^2}{4} - \frac{D^2}{4}\right). \quad (18)$$

By temporarily calling

$$E = \left(1 - \frac{C^2}{4} - \frac{D^2}{4}\right),$$

we find

$$\dot{C} = \epsilon \frac{EC}{2}, \text{ and } \dot{D} = \epsilon \frac{ED}{2}.$$

Thus, if $E > 0$, then in the (C, D) phase plane points move radially outwards; while if $E < 0$, points move radially inwards. By the definition of E, we conclude that all points move radially until they reach the circle $C^2 + D^2 = 4$, or (in terms of the original quantity) $A = 2$.

Now consider $F = 0$ but $\Delta \neq 0$. This causes the trajectories to spiral out (or in) instead of moving purely radially. The steady state solution now goes round and round on the circle with radius 2 and centre at the origin. The situation is shown using the (C, D) phase plane in Fig. 7(a).

Now, as we increase F, we find that the circle is shifted and deformed into a smaller closed curve; and the unstable equilibrium point shifts away from the origin as well. The situation is depicted in Figs. 7(b) through 7(e). Finally, for even larger values of F, the closed curve shifts to a point and merges with the unstable equilibrium point, which now becomes stable. The situation is shown in 7(f). At this point, the solution is periodic, and completely phase locked with the solution (entrained).

An interesting transition occurs at $F \approx 1.446$, when the origin leaves the closed curve. For F below this critical value, the closed curve encloses the origin, and therefore as the point in the (C, D) space slowly goes round and round the closed curve, the phase of the oscillations drifts further and further away from the forcing (gaining or losing 2π with every encircling of the origin). For the critical value of $F \approx 1.446$, there is a point when the oscillation amplitude becomes zero (when the (C, D) trajectory passes through the origin); and then as the oscillation grows again, there has been a sudden change in the phase by π. Finally, for F above this critical value, although the point in (C, D) space goes round and round on its closed curve or limit cycle, the phase angle oscillates between limits. For such F values, there is weak phase locking and the phase does not drift away. Finally, as mentioned above, for large enough F, the phase of the oscillation is exactly locked with the forcing. The above situations are also seen in numerical solutions of the original Eq. 12, as shown in Fig. 8. The figure shows the steady state solutions, after transients have died out, for the case $\epsilon = 0.1$ (recall that the averaging procedure is based on ϵ being small). A close look at the time histories of the vibration response ($x(t)$) and the forcing ($F \sin t$), for $\Delta = 1$ and various F values as given in the figure, shows that the predictions from a study of the averaged Eq. 15 and 16 are borne out completely. In particular, for $F = 1.446$, it is seen that the oscillation amplitude periodically comes to zero, as predicted; and, as shown by arrows towards the left of the figure, the response

[3] By getting a slightly slow or fast clock, as appropriate, we slightly change our definition of time so that the forcing has a period of 2π while the van der Pol oscillator's unperturbed limit cycle period is slightly different from 2π. We could have done the reverse, letting the unperturbed limit cycle period be 2π and the forcing period be slightly different from 2π. But the former is more convenient for averaging.

changes from being about $\pi/2$ behind the forcing to about $\pi/2$ ahead, every time the amplitude becomes small.

Note that harmonic balance, which only finds periodic solutions and says nothing about stability, would have only found the steady periodic solution even for small F values, and missed the stable, modulated solutions shown in Fig. 8 (corresponding to the limit cycles or closed curves of Fig. 7).

10 Modal interactions

The term *modal interaction* refers to a situation in which energy is exchanged between modes in a system. Thus, situations in which modal interactions occur are in distinction to the case for linear, completely diagonalizable systems in which the normal modes *do not* exchange energy. Here we will look at an example of modal interactions caused by nonlinear terms. Consider the system shown in Fig. 2(c). Let the free length of spring k_2 be h, as shown; let the equilibrium positions of masses m_1 and m_2 be at $x_1 = 0$ and $x_2 = 0$ respectively.

We now take $m_1 = m_2 = 1$, $k_1 = k_2 = k_3 = 1$, and $h = 1$. The equations of motion for the above system are strongly nonlinear; however, for small displacements, we retain linear and quadratic terms but drop third order and higher order terms. Then the equations of motion are

$$\ddot{x}_1 = -2x_1 + \frac{x_2^2}{2}\,, \text{ and} \tag{19}$$

$$\ddot{x}_2 = -x_2 + x_1 x_2\,. \tag{20}$$

Note that the linearized system is decoupled, and thus each degree of freedom by itself constitutes one vibration mode. The natural frequencies are: $\sqrt{2}$ for x_1 and 1 for x_2. Due to the presence of nonlinearities, however, the modes are coupled and energy exchange can occur between them.

For simplicity, we now add small amounts of forcing and damping to these systems, writing

$$\ddot{x}_1 = -2x_1 - c\dot{x}_1 + \frac{x_2^2}{2} + F\sin 2t\,, \text{ and} \tag{21}$$

$$\ddot{x}_2 = -x_2 - \frac{c}{3}\dot{x}_2 + x_1 x_2\,. \tag{22}$$

In the above, the choice of damping is not critical, but it is significant that the forcing frequency is twice the natural frequency of the x_2 mode. Note that the forcing is applied to mass m_1, and the natural frequency of the x_1 mode is $\sqrt{2}$, so the forcing frequency is *not* equal or close to the natural frequency of the forced mode.

On numerically solving the above equations for $c = 0.04$ and $F = 0.06$, we obtain the steady state responses shown in Fig. 9. It is seen that the response of x_2, i.e., the unforced mode, is at its natural frequency, which is *one half* of the forcing frequency; moreover, this response has a greater amplitude than that of the forced mode itself; and finally, the forced mode responds at the forcing frequency.

Since the unforced mode is damped, it dissipates energy; that energy comes from the forced mode, showing that the two modes are interacting. This system also provides an example of a motion where different parts of a system vibrate with different frequencies.

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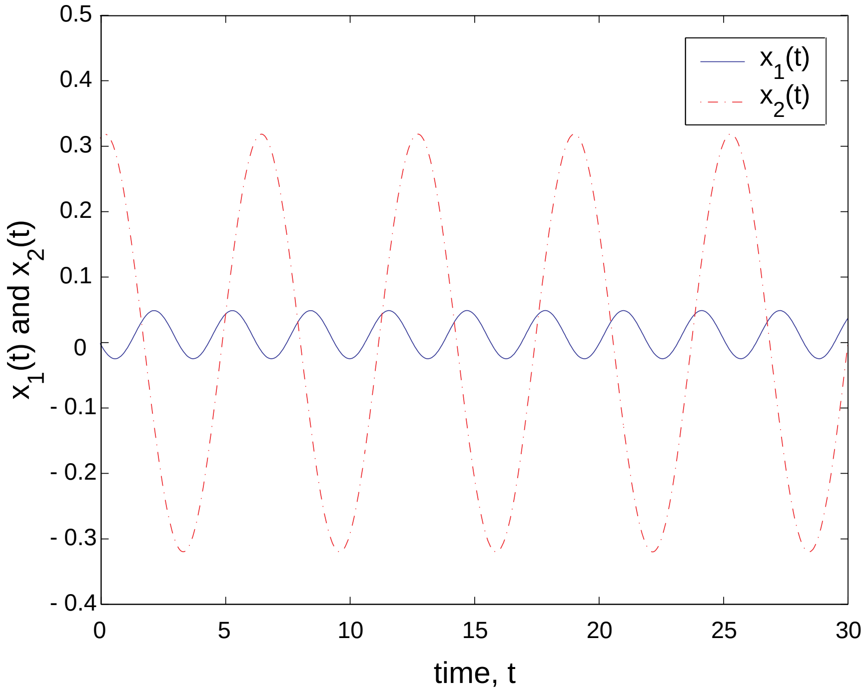

Figure 9: Numerical solutions of Eqs. 21 and 22 with $c = 0.04$ and $F = 0.06$.

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