SATHYABAMA

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SCHOOL OF ELECTRICAL AND ELECTRONICS

DEPARTMENT OF ELECTRONICS AND COMMMUNICATION ENGINEERING

UNIT - I
SIGNALS AND SYSTEMS – SECA1301

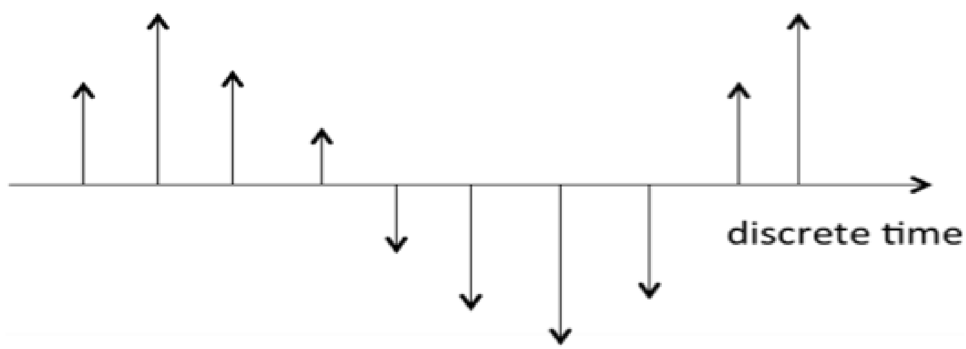
1.INTRODUCTION TO SIGNALS

Signal: Signals are represented mathematically as functions of one or more independent variables. It mainly focuses attention on signals involving a single independent variable. For convenience, this will generally refer to the independent variable as time. It is defined as physical quantities that carry information and changes with respect to time.
Ex: voice, television picture, telegraph.

Continuous Time signal – If the signal is defined over continuous-time, then the signal is a continuous- time signal.

Ex: Sinusoidal signal, Voice signal, Rectangular pulse function

Discrete Time signal – If the time t can only take discrete values, such as t=kTs is called Discrete Time signal

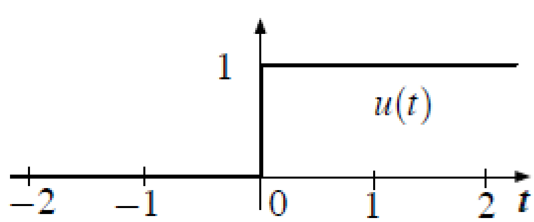


Unit Step Signal:

The Unit Step Signal u(t) is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

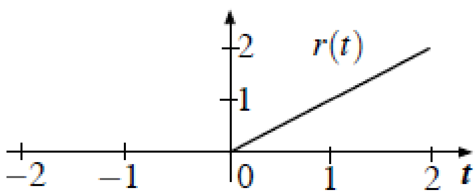
Graphically it is given by



Ramp Signal:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Graphically it is given by



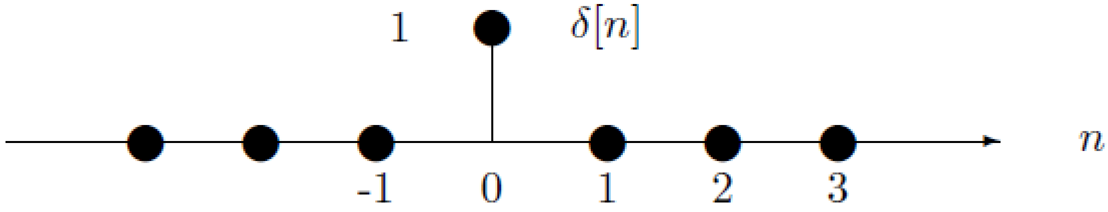
Pulse Signal:

A signal is having constant amplitude over a particular interval and for the remaining interval the amplitude is zero.

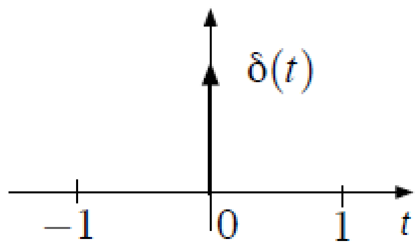
Impulse Signal:

$$\delta[n] \equiv \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

Impulse Signal DT representation



Impulse Signal CT representation



Exponential Signal:

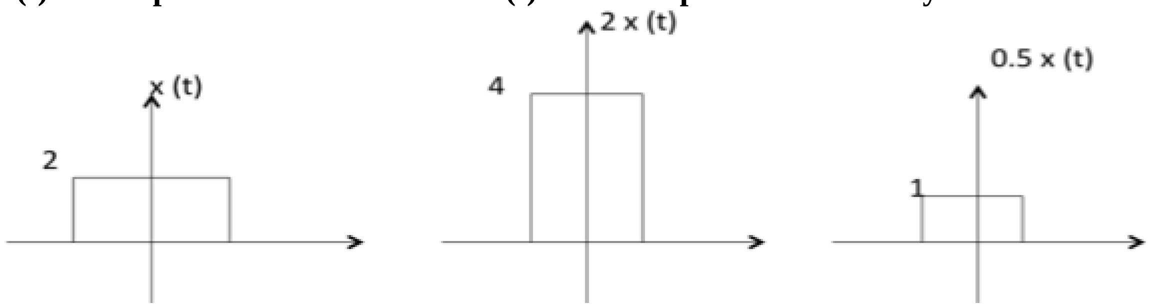
Exponential signal is of two types. These two type of signals are real exponential signal and complex exponential signalwhich are given below.

Real Exponential Signal: A real exponential signal is defined as $x(t)=Ae^{\sigma t}$
Complex exponential Signal: The complex exponential signal is given by $x(t)=Ae^{st}$ where $s=\sigma+j\omega$

Basic Operations on signals:

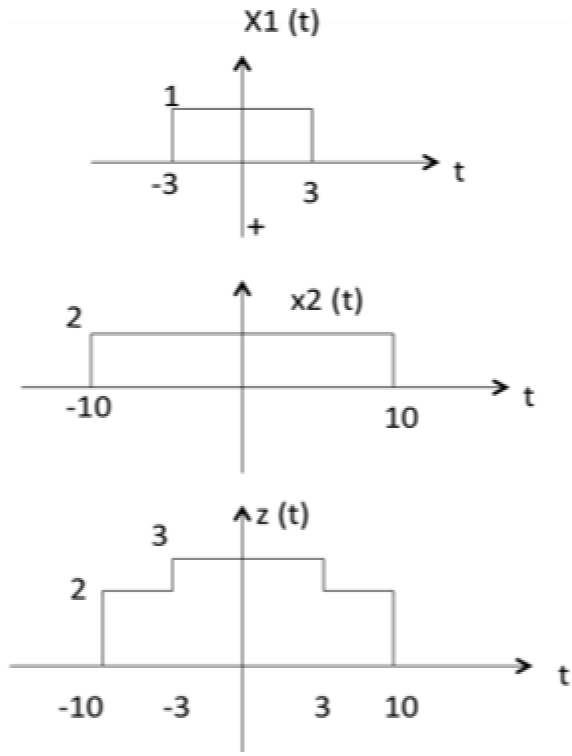
Several basic operations by which new signals are formed from given signals are familiar from the algebra and calculus of functions.

1. **Amplitude Scaling :y(t)= a x(t), where a is a real (or possibly complex) constant. C x(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.**



2. **Amplitude Shift: y(t)= x(t)+ b, where b is a real (or possibly complex) constant**

3. Signal Addition: y(t)= x1(t) + x2(t)



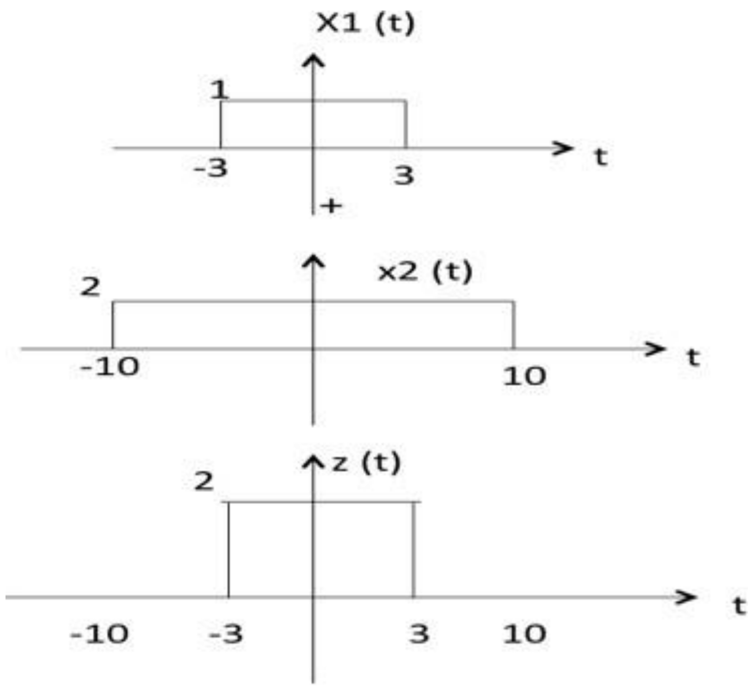
As seen from the diagram above,

-10 < t < -3 amplitude of z(t) = x1(t) + x2(t) = 0 + 2 = 2

-3 < t < 3 amplitude of z(t) = x1(t) + x2(t) = 1 + 2 = 3

3 < t < 10 amplitude of z(t) = x1(t) +x2(t) = 0 + 2 = 2

4. **Signal Multiplication: y(t)= x1(t). x2(t)**



As seen from the diagram above,

-10 < t < -3 amplitude of z (t) = x1(t) ×x2(t) = 0 ×2 = 0

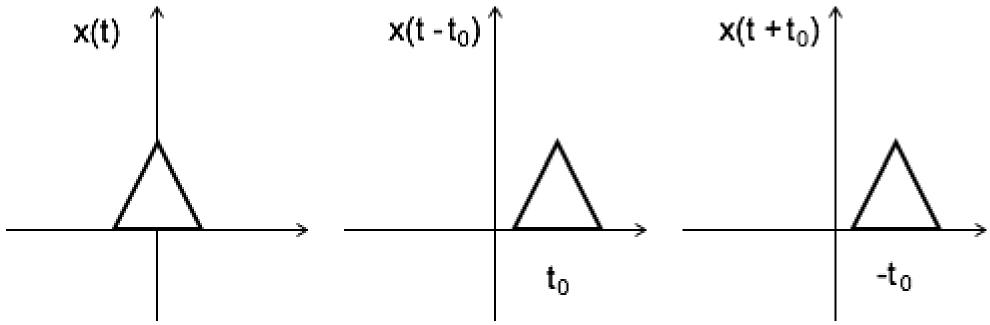
-3 < t < 3 amplitude of z (t) = x1(t) ×x2(t) = 1 ×2 = 2

3 < t < 10 amplitude of z (t) = x1(t) × x2(t) = 0 × 2 = 0

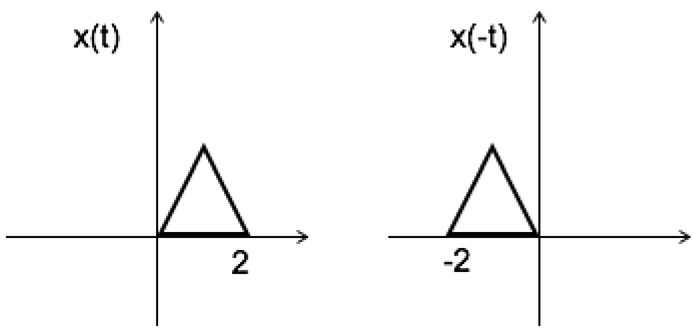
5. **Time Shift: If x(t) is a continuous function, the time-shifted signal is defined as** $y(t) = x(t - t_0)$.

If $t_0 > 0$, the signal is shifted to the right, and if $t_0 < 0$, the signal is shifted to the left.

$x(t \pm t_0)$ is time shifted version of the signal $x(t)$.
$x(t + t_0)$ →→ negative shift
$x(t - t_0)$ →→ positive shift



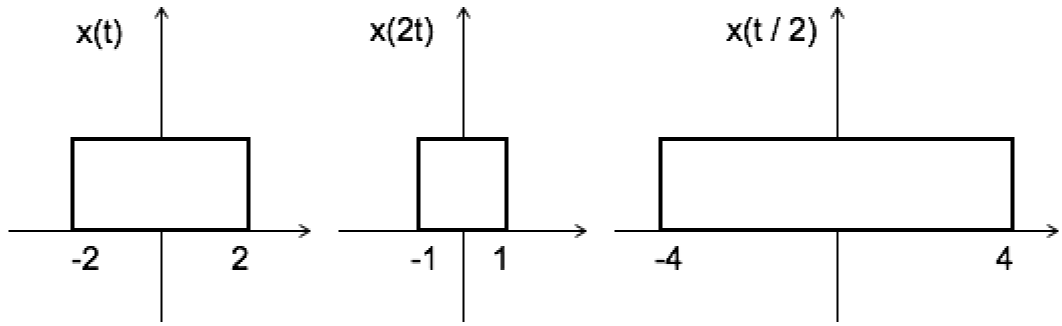
6. **Time Reversal: If $x(t)$ is a continuous function, the time-reversed signal is defined as $y(t) = x(-t)$. $x(-t)$ is the time reversal of the signal $x(t)$.**



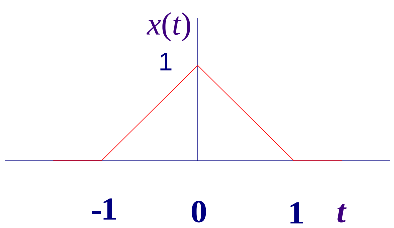
7. Time Scaling: If $x(t)$ is a continuous function, a time-scale version of this signal is defined as $y(t) = x(at)$. If $a>1$, the signal $y(t)$ is a compressed version of $x(t)$, i.e., the time interval is compressed to $\frac{1}{a}$. If $0<a<1$, the signal $y(t)$ is a stretched version of $x(t)$, i.e., the time interval is stretched by $\frac{1}{a}$. When operating on signals, the time-shifting operation must be performed first, and then the time-scaling operation is performed. $x(At)$ is time scaled version of the signal $x(t)$. where A is always positive.

$|A| > 1$ →→ Compression of the signal

$|A| < 1$ →→ Expansion of the signal



1. **A triangular pulse signal x(t) is depicted below.**



Sketch each of the following signals:

(a) $x(3t)$

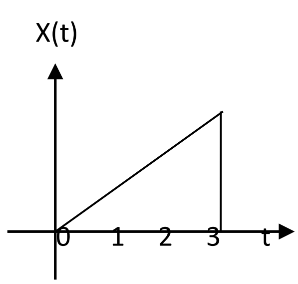
(b) $x(3t + 2)$

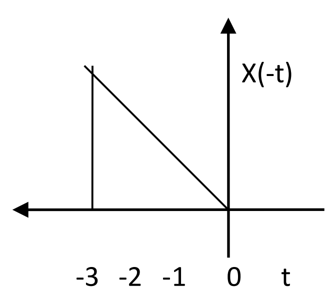
(c) $x(-2t - 1)$

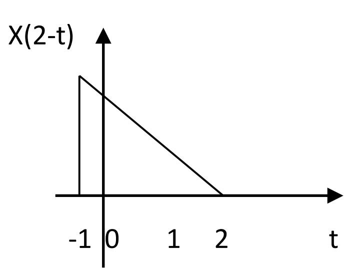
(d) $x(0.5t - 1)$

2. **Draw the waveform x(-t) and x(2-t) of the signal $x(t) = t \quad 0 \le t \le 3$**

$$0 \quad t > 3$$







Classification of DT and CT Signals:

1. **Even and Odd signal**
2. **Deterministic and Random Signal**
3. **Periodic and Aperiodic signal**
4. **Energy and Power signal**

Even and Odd Signal:

An even signal is any signal 'x' such that x(t) = x(-t). Odd signal is a signal 'x' for which

x(t) = -x(-t). The even and odd parts of a signal x(t) are
The even and odd parts of a signal are given by

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

Here $x_e(t)$ denotes the even part of signal x(t) and $x_o(t)$ denotes the odd part of signal x(t).

Deterministic Signal:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signals can be modeled exactly by a mathematical formula are known as deterministic signals.

Random (or) Nondeterministic Signals:

Nondeterministic signals and events are either random or irregular. Random signals are also called non deterministic signals are those signals that take random values at any given time and must be characterized statistically.Random signals, on the other hand, cannot be described by a mathematical equation they are modeled in probabilistic terms.

Periodic signal:

A CT signal x(t) is said to be periodic if it satisfies the following property: x(t)=x(t+T) at all time t, where T=Fundamental Time Interval (T=2π/ω)

Ex:

1. **x(t)=sin(4πt). It is periodic with period of 1/2**
2. **x(t)=cos(3πt). It is periodic with period of 2/3**

Aperiodic Signal:

A CT signal x(t) is said to be periodic if it satisfies the following property: x(t)≠x(t+T) at all time t, where T=Fundamental Time Interval

Energy Signal:

The Energy in the signal is defined as :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \, .$$

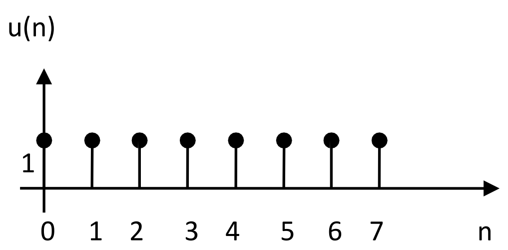
Power Signal:

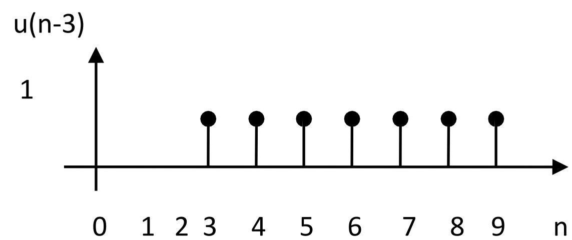
The Power in the signal is defined as

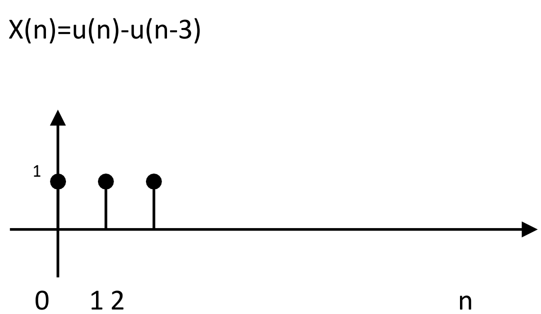
$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

If 0<E<∞ then the signal x(t) is called as Energy signal. However there are signals where this condition is not satisfied. For such signals we consider the power. If 0<P<∞ then the signal is called a power signal. Note that the power for an energy signal is zero (P=0) and that the energy for a power signal is infinite (E=∞). Some signals are neither energy nor power signals.

1. Draw the signal x(n) = u(n) – u(n-3)







2. **What is the total energy of the discrete time signal x(n) which takes the value of unity at n= - 1,0,1?**

Energy of the signal is given as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-1}^{1} |x(n)|^2$$

$$= |x(-1)|^2 + |x(0)|^2 + |x(1)|^2 = 3$$

3. Determine if the following signals are Energy signals, Power signals, or neither, and evaluate E and P for each signal $a(t) = 3\sin(2\pi t),\ -\infty < t < \infty$,

$$E_a = \int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2 dt$$

$$= 9\int_{-\infty}^{\infty} \frac{1}{2}\left[1 - \cos(4\pi t)\right] dt$$

$$= 9\int_{-\infty}^{\infty} \frac{1}{2} dt - 9\int_{-\infty}^{\infty} \cos(4\pi t) dt$$

$$= \infty \quad \text{J}$$

$$P_a = \frac{1}{1}\int_{0}^{1} |a(t)|^2 dt = \int_{0}^{1} |3\sin(2\pi t)|^2 dt$$

$$= 9\int_0^1 \frac{1}{2}\left[1 - \cos(4\pi t)\right]dt$$

$$= 9\int_0^1 \frac{1}{2}dt - 9\int_0^1 \cos(4\pi t)dt$$

$$= \frac{9}{2} - \left[\frac{9}{4\pi}\sin(4\pi t)\right]_0^1$$

$$= \frac{9}{2} \quad \mathrm{W}$$

So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. It is a power signal.

Real and Complex signals:

Exponential signal is of two types. These two type of signals are real exponential signal and complex exponential signalwhich are given below.

Real Exponential Signal: A real exponential signal is defined as x(t)=Ae$^{\sigma t}$

Complex exponential Signal:The complex exponential signal is given by x(t)=Aest where s=σ+jω

TEXT / REFERENCE BOOKS

1. **P.Ramesh Babu et al., "Signals and Systems", 4th Edition, Scitech Publishers, 2017.**
2. **Rodger E. Ziemer , William H Tranter, D. R. Fannin,"Signals and Systems: Continuous and Discrete", 4th Edition, Pearson Education India, 2014.**
3. **Haykin S. and Van Been B., "Signals and Systems", 2nd Edition, John Wiley and Sons, 2015.**
4. **H.P. Hsu, "Signals and Systems", 2nd Edition, Tata McGraw Hill, 2017.**

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UNIT - II
SIGNALS AND SYSTEMS – SECA1301

II. ANALYSIS OF CONTINUOUS TIME SIGNALS

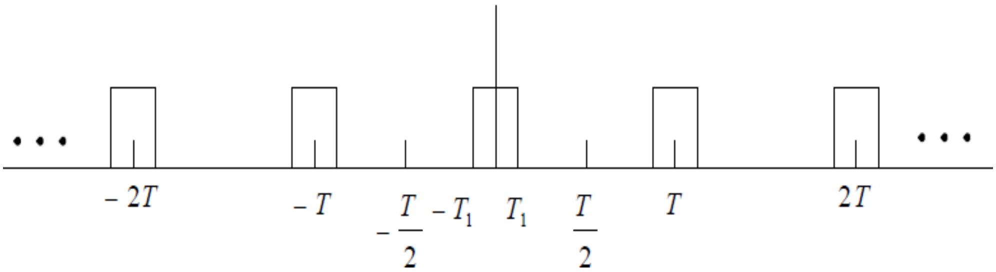
Continuous Time Fourier Transform

Any continuous time periodic signal x(t) can be represented as a linear combination of complex exponentials and the Fourier coefficients (or spectrum) are discrete. The Fourier series can be applied to periodic signals only but the Fourier transform can also be applied to non-periodic functions like rectangular pulse, step functions, ramp function etc. The Fourier transform of Continuous Time signals can be obtained from Fourier series by applying appropriate conditions.

The Fourier transform can be developed by finding Fourier series of a periodic function and the tending T to infinity.

Representation of Aperiodic signals: Starting from the Fourier series representation for the continuous-time periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{2.1}$$



The Fourier coefficients a_k for this square wave is

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}. \tag{2.2}$$

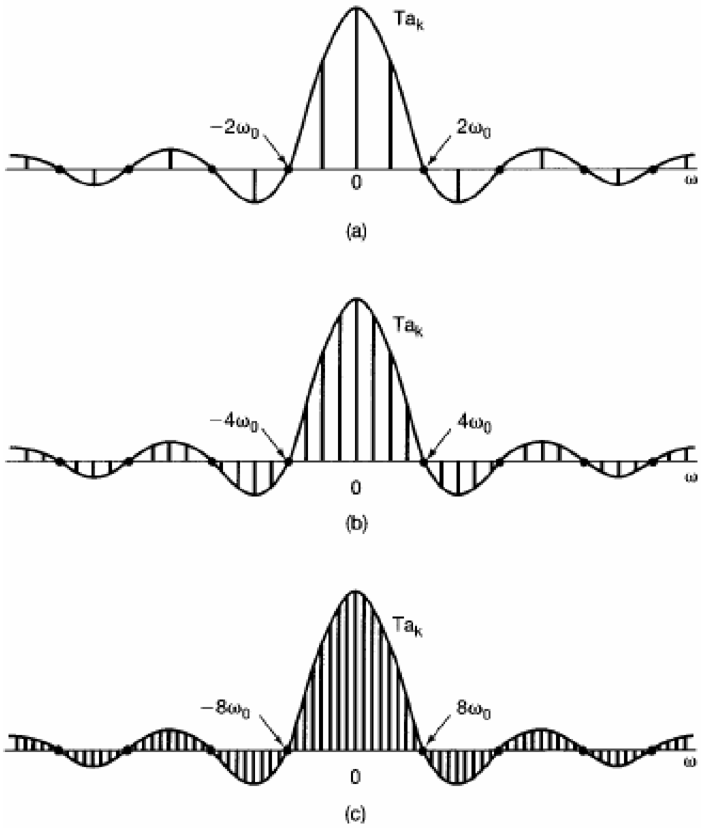
or alternatively 2.3

$$Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0}$$

where 2 sin(wT1) / w represent the envelope of *Ta*$_k$

When T increases or the fundamental frequency w $_0$ 2p / T decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.

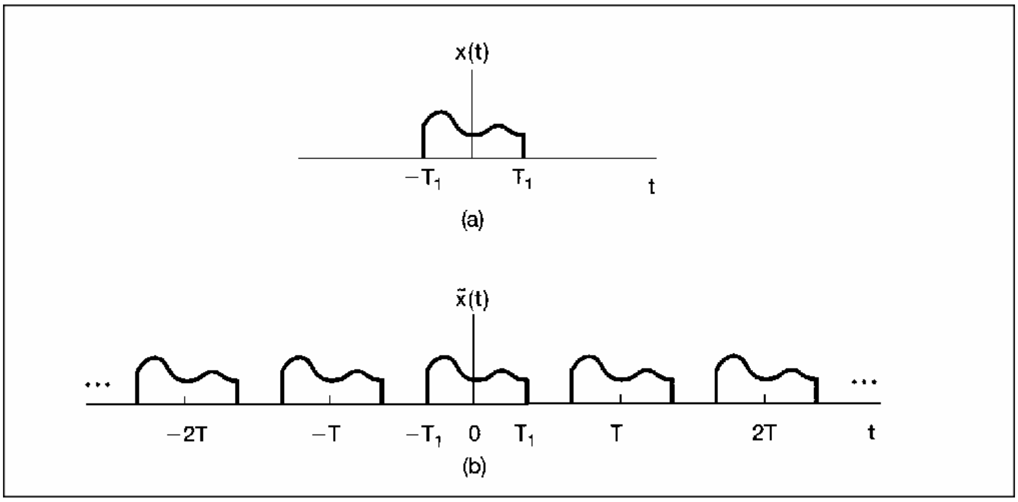
***Ta$_k$* becomes more and more closely spaced samples of the envelope, as T →∞, the Fourier series coefficients approaches the envelope function.**



This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals.

Based on this idea, we can derive the Fourier transform for aperiodic signals.

From this aperiodic signal, we construct a periodic signal (t) , shown in the figure below.



As $T \to \infty$, $\tilde{x}(t) = x(t)$, for any infinite value of t.

The Fourier series representation of $\tilde{x}(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \qquad 2.4$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt. \qquad 2.5$$

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, so we have

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt. \qquad 2.6$$

Define the envelope X (jw) of Ta_k as,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \qquad 2.7$$

we have for the coefficients a_k,

$$a_k = \frac{1}{T} X(jk\omega_0)$$

Then $\tilde{x}(t)$ can be expressed in terms of $X(j\omega)$, that is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 . \qquad 2.8$$

As $T \to \infty$, $\tilde{x}(t) = x(t)$ and consequently,

Equation 2.8 becomes representation of *x(t)*. In addition the right hand side of equation becomes an integral.

This results in the following Fourier Transform.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \mathit{Inverse\ Fourier\ Transform} \qquad 2.9$$

and

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \mathit{Fourier\ Transform} \qquad 2.10$$

Convergence of Fourier Transform

If the signal x(t) has finite energy, that is, it is square integrable,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty ,$$

Then we guaranteed that X(jw) is finite or equation 2.10 converges. If $e(t) = \tilde{x}(t) - x(t)$,

We have

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0 .$$

An alterative set of conditions that are sufficient to ensure the convergence:

Contition1: Over any period, x(t) must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty ,$$

Condition 2: In any finite interval of time, x(t) have a finite number of maxima and minima.

Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Continuous-Time Fourier Transform

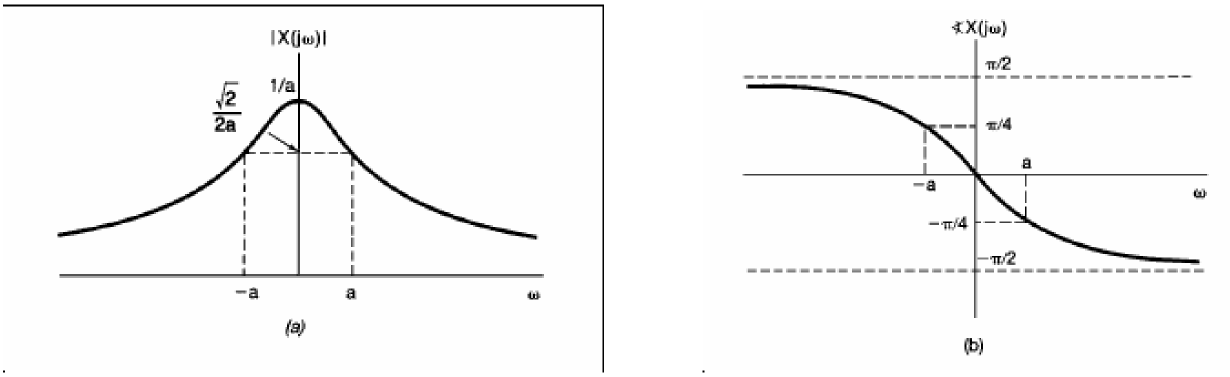
consider signal $x(t) = e^{-at}u(t)$, $a > 0$.

$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^\infty = \frac{1}{a + j\omega}, \qquad a > 0$$

If a is complex rather than real, we get the same result if Re{a}>0

The Fourier transform can be plotted in terms of the magnitude and phase, as shown in the figure below.

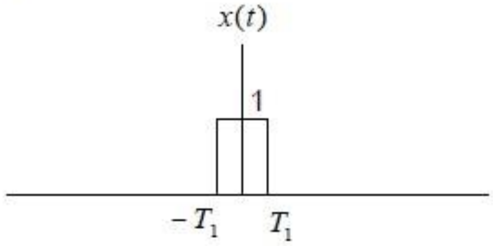
$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \qquad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

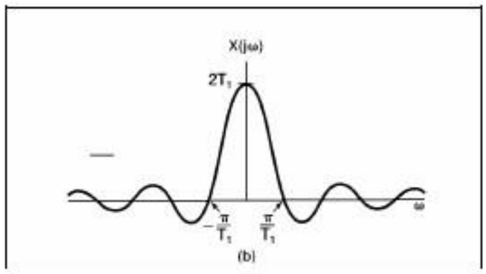


Example

Calculate the Fourier transform of the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}.$$





$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} 1e^{-j\omega t}dt = 2\frac{\sin \omega T_1}{\omega}.$$

The inverse Fourier Transform of the sinc function is

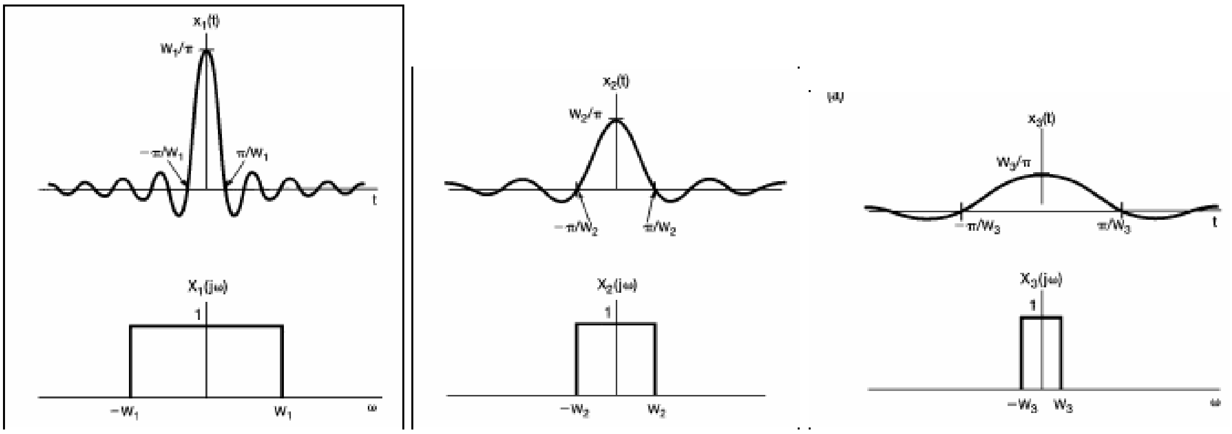
$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin Wt}{\pi t}$$

Comparing the results we have,

$$\text{Square wave} \underset{FT^{-1}}{\overset{FT}{\rightleftarrows}} \text{Sinc Function}$$

This means a square wave in the time domain, its Fourier transform is a sinc function. However, if the signal in the time domain is a sinc function, then its Fourier transform is a square wave. This property is referred to as Duality Property.

We also note that when the width of $X(jw)$ increases, its inverse Fourier transform x(t) will be compressed. When W → ∞, $X(jw)$ converges to an impulse. The transform pair with several different values of W is shown in the figure below.



The Fourier Transform for Periodic Signals

The Fourier series representation of the signal $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} .$$

It's Fourier transform is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) .$$

Properties of Fourier Transform

1. **Linearity**

If $x(t) \xleftrightarrow{F} X(jw)$ $y(t) \xleftrightarrow{F} Y(jw)$

then

$$ax(t) + by(t) \xleftrightarrow{F} aX(jw) + bY(jw)$$

2. **Time Shifting**

If $x(t) \xleftrightarrow{F} X(jw)$

Then

$$x(t-t_0) \xleftrightarrow{F} e^{-jwt_0} X(jw)$$

3. Conjugation and Conjugate Symmetry

If $x(t) \xleftrightarrow{F} X(jw)$

Then

$x^*(t) \xleftrightarrow{F} X^*(-jw)$

4. Differentiation and Integration

If $x(t) \xleftrightarrow{F} X(jw)$ then $\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\leftrightarrow} \frac{1}{j\omega} X(jw) + \pi X(0)\delta(\omega)$$

5. Time and Frequency Scaling

$x(t) \overset{F}{\leftrightarrow} X(j\omega)$

then,

$x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

From the equation we see that the signal is compressed in the time domain, the spectrum will be extended in the frequency domain. Conversely, if the signal is extended, the corresponding spectrum will be compressed.

If a = -1, we get from the above equation,

$X(-t) \overset{F}{\leftrightarrow} X(j\omega)$

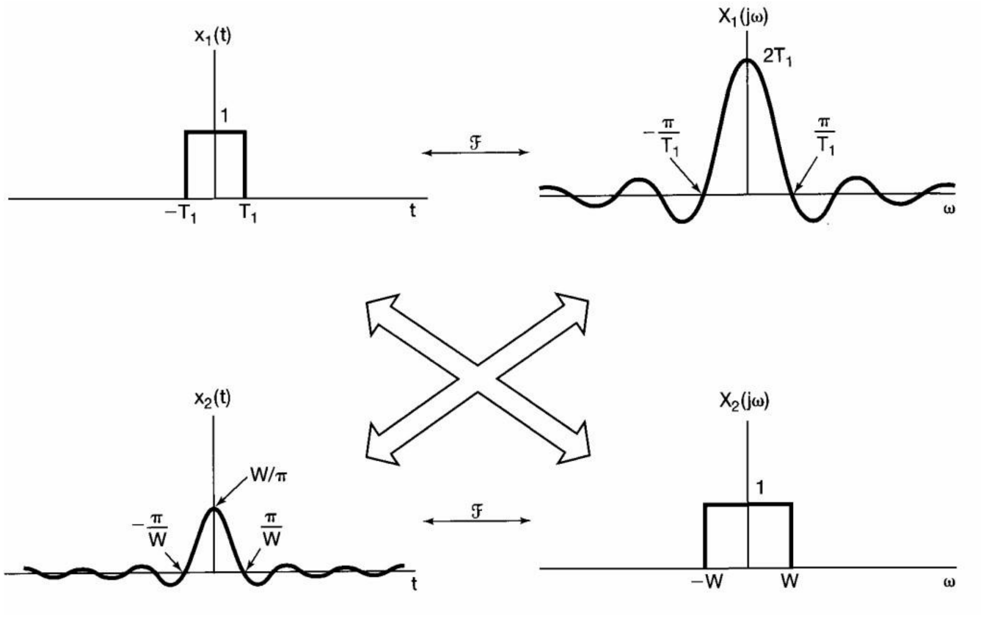
That is reversing a signal in time also reverses its Fourier transform.

6. **Duality**

The duality of the Fourier Transform can be demonstrated using the following example

$$x_1(t)=\begin{cases}1, & t < T_1\\ 0, & t > T_1\end{cases} \overset{F}{\leftrightarrow} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = \frac{\sin W T_1}{\pi t} \overset{F}{\leftrightarrow} X_2(j\omega) = \begin{cases}1, & |\omega| < W\\ 0, & |\omega| > W\end{cases}$$



For any transform pair, there is a dual pair with the time and frequency variables interchanged.

$$-jtx(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X(j(\omega - \omega_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{F} \int_{-\infty}^{\omega} x(\eta)d\eta$$

Parseval's Relation

If $x(t) \rightarrow X(j\omega)$,

We have, $$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(jw)|^2 d\omega$$

Parseval"s relation states that the total energy may be determined either by computing the energy per unit time $|x(t)|^2$ and integrating over all time or by computing the energy per unit frequency $|x(j\omega)|^2 / 2\pi$ and integrating over all

frequencies. For this reason, $x(j\omega)$ ^{2} is often referred toas the energy density spectrum.

The Parseval's theorem states that the inner product between signals is preserved in going from time to the frequency domain. This is interpreted physically as "Energy calculated in the time domain is sameas the energy calculated in the frequency domain"

The convolution properties

$$y\,(t\,)= h\,(t) * x\,(t)= Y\,(j\omega) = H\,j\omega\, X(j\omega)$$

The equation shows that the Fourier transform maps the convolution of two signals into product of their Fourier transforms.

***H(jω)*, the transform of the impulse response is the frequency response of the LTI system, which also completely characterizes an LTI system.**

Example

The frequency response of a differentiator.

$y(t)=\frac{dx\,(t)}{dt}$

From the differentiation property,

$Y(j\omega)= j\omega X\ j\omega$

The frequency response of the differentiator is,

$H(j\omega)=\frac{Y(j\omega\,)}{X(j\omega\,)}=j\omega$

The Multiplication Property

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Multiplication of one signal by another can be thought of as one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is often referred to as amplitude modulation.

Laplace Transform

The Laplace Transform is the more generalized representation of CT complex exponential signals. The Laplace transform provide solutions to most of the signals

and systems, which are not possible with Fourier method. The Laplace transform can be used to analyze most of the signals which are not absolutely integrable such as the impulse response of an unstable system. Laplace Transform is a powerful tool for analysis and design of Continuous Time signals and systems. The Laplace Transform differs from Fourier Transform because it covers a broader class of CT signals and systems which may or may not be stable.

Till now, we have seen the importance of Fourier analysis in solving many problems involving signals. Now, we shall deal with signals which do not have a Fourier transform. We note that the Fourier Transform only exists for signals which can absolutely integrated and have a finite energy. This observation leads to generalization of continuous-time Fourier transform by considering a broader class of signals using the powerful tool of "Laplace transform". With this introduction let us go on to formally defining both Laplace transform.

Definition of Laplace Transform

The Laplace transform of a function x(t) can be shown to be,

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

This equation is called the Bilateral or double sided Laplace transform equation.

$$x(t) = \int_{-\infty}^{\infty} X(s)e^{st}ds$$

This equation is called the Inverse Laplace Transform equation, x(t) being called the Inverse Laplace transform of X(s).

The relationship between x(t) and X(s) is

$x(t) \overset{LT}{} X(s)$

Region of Convergence (ROC):

The range of values for which the expression described above is finite is called as the Region of Convergence (ROC).

Convergence of the Laplace transform

The bilateral Laplace Transform of a signal x (t) exists if

X(s)= $\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

Substitute s=$\sigma + j\omega$

X(s) = $\int_{-\infty}^{\infty} x(t)e^{-\sigma t}\,e^{-j\omega t}\,dt$

Relationship between Laplace Transform and Fourier Transform

The Fourier Transform for Continuous Time signals is in fact a special case of Laplace Transform. This fact and subsequent relation between LT and FT are explained below.

Now we know that Laplace Transform of a signal 'x'(t)' is given by:

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

The s-complex variable is given by $s = \sigma + j\Omega$

But we consider σ = 0 and therefore „s" becomes completely imaginary. Thus we have s=jΩ. This means that we are only considering the vertical strip at σ = 0.

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$$

From the above discussion it is clear that the LT reduces to FT when the complex variable only consists of the imaginary part. Thus LT reduces to FT along the jΩ axis. (imaginary axis)

Fourier Transform of x (t) = Laplace Transform of $x(t)$ $_{s=j\Omega}$

Laplace transform becomes Fourier transform

if σ=0 and s=jω.

$$\mathbf{X(s)}\big|_{s=j\omega} = \mathbf{FT\{x(t)\}}$$

Example of Laplace Transform

(1) Find the Laplace transform and ROC of x(t) = $e^{-at}u(t)$

we notice that by multiplying by the term u(t) we are effectively considering the unilateral Laplace Transform whereby the limits tend from 0 to +∞

Consider the Laplace transform of x(t) as shown below

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a}; \text{ for } (s+a) > 0$$

(2) Find the Laplace transform and ROC of x(t)=-$e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} - e^{-at} e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a}; \text{ for } (s+a) < 0$$

If we consider the signals $e^{-at}u(t)$ and $-e^{-at}u(-t)$, we note that although the signals are differing, their Laplace Transforms are identical which is 1/(s+a). Thus we conclude that to distinguish L.T's uniquely their ROC's must be specified.

Properties of Laplace Transform

1. **Linearity**

If $x_1(t) \overset{L}{\leftrightarrow} X_1(s)$ with ROC R1 and $x_2(t) \overset{L}{\leftrightarrow} X_2(s)$ with ROC R2, then
$ax_1(t) + bx_2(t) \overset{L}{\leftrightarrow} aX_1(s) + bX_2(s)$ with ROC containing $R_1 \cap R_2$

The ROC of X(s) is at least the intersection of R_1 and R_2, which could be empty, in which case x(t) has no Laplace Transform.

2. **Differentiation in the time domain**

If $x(t) \overset{L}{\leftrightarrow} X(s)$ with ROC = R then $\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s)$ with ROC = R.
This property follows by integration by parts.

Hence, $\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s)$ The ROC of *sX(s)* includes the ROC of X(s) and may be larger.

3. **Time Shift**

If $x(t) \overset{L}{\leftrightarrow} X(s)$ with ROC = R then

$x(t-t_0) \overset{L}{\leftrightarrow} e^{-st_0}X(s)$ with ROC = R

4. **Time Scaling**

If $x(t) \overset{L}{\leftrightarrow} X(s)$ with ROC = R, then

$x(at) \overset{L}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$; ROC = $\frac{R}{a}$ ie., $\frac{s}{a} \epsilon\, R$

5. **Multiplication**

x(t) X y(t) $\overset{L}{\leftrightarrow} \frac{1}{2\pi j}[X(S) * Y(S)]$

6. **Time Reversal**

When the signal x(t) is time reversed(180° Phase shift)

X(-t) $\overset{L}{\leftrightarrow} X(-s)$

7. **Frequency Shifting**

$$\boldsymbol{e^{s_0 t} x(t) \overset{L}{\leftrightarrow} X(s - s_0)}$$

8. **Conjugation symmetry**

$x^*(t) \overset{L}{\leftrightarrow} x^*(-s)$

9. **Parseval's Relation of Continuous Signal**

It states that the total average power in a periodic signal x(t) equals the sum of the average in individual harmonic components, which in turn equals to the squared magnitude of X(s) Laplace Transform.

$$\int_0^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} |X(S)|^2 ds$$

10. Differentiation in Frequency

When x(t) is differentiated with respect to frequency then,

-t x(t) $\overset{L}{\leftrightarrow} \frac{dX(s)}{dS}$

11. Integration Property

When a periodic signal x(t) is integrated, then the Laplace Transform becomes,

$$\int_{-\infty}^{t} x(t)dt \overset{L}{\leftrightarrow} \frac{1}{S}X(S) + \frac{\int_{-\infty}^{\infty} x(\tau)d\tau}{S}$$

12. Convolution Property

$$x(t) * y(t) \overset{L}{\leftrightarrow} X(s).Y(S)$$

13. Initial Value Theorem

The initial value theorem is used to calculate initial value x(0^+) of the given sequence x(t) directly from the Laplace transform X(S). The initial value theorem does not apply to rational functions X(S) whose numerator polynomial order is greater than the denominator polynomial orders.

The initial value theorem states that,

$$\lim_{s\to\infty} SX\,S = X(0^+)$$

14. Final Value Theorem

It states that,

$$\lim_{s\to\infty} SX\,S = X(\infty)$$

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SCHOOL OF ELECTRICAL AND ELECTRONICS

DEPARTMENT OF ELECTRONICS AND COMMMUNICATION ENGINEERING

UNIT - III

SIGNALS AND SYSTEMS – SECA1301

III.LINEAR TIME INVARIANT CONTINUOUS TIME SYSTEMS

System

A system may be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends upon transfer function of the system. Mathematically, the functional relationship between input and output may be written as

$$y(t)=f[x(t)]$$

Types of system

Like signals, systems may also be of two types as under:

1. **Continuous-time system**
2. **Discrete time system**

Continuous time System

Continuous time system may be defined as those systems in which the associated signals are also continuous. This means that input and output of continuous – time system are both continuous time signals.

For example:

Audio, video amplifiers, power supplies etc., are continuous time systems.

Discrete time systems

Discrete time system may be defined as a system in which the associated signals are also discrete time signals. This means that in a discrete time system, the input and output are both discrete time signals.

For example, microprocessors, semiconductor memories, shift registers etc., are discrete time signals.

LTI system:-

Systems are broadly classified as continuous time systems and discrete time systems. Continuous time systems deal with continuous time signals and discrete time systems deal with discrete time system. Both continuous time and discrete time systems have several basic properties. Out of these several basic properties of systems, two properties namely linearity and time invariance play a vital role in the analysis of signals and systems. If a system has both the linearity and time invariance properties, then this system is called linear time invariant (LTI) system.

Characterization of Linear Time Invariant (LTI) system

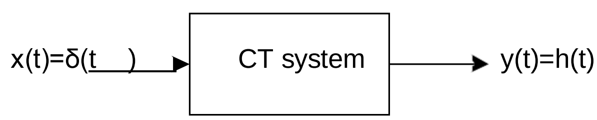
Both continuous time and discrete time linear time invariant (LTI) systems exhibit one important characteristics that the superposition theorem can be applied to find the response y(t) to a given input x(t).

Hence, following steps may be adopted to find the response of a LTI system using super position theorem:

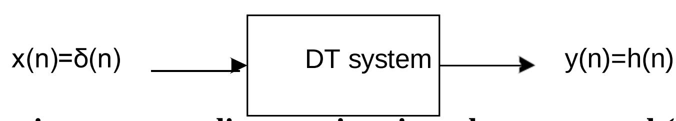
1. **Resolve the input function x(t) in terms of simpler or basic function like impulse function for which response can be easily evaluated.**
2. **Determine individually the response of LTI system for the simpler input impulse functions.**
3. **Using superposition theorem, find the sum of the individual responses, which will become the overall response y(t) of function x(t).**

From the above discussions, it is clear that to find the response of a LTI system to any given function, first we have to find the response of LTI system input to an unit impulse called unit impulse response of LTI system.

Hence, the impulse response of a continuous time or discrete time LTI system is the output of the system due to an unit impulse input applied at time t=0 or n=0.



Here, δ(t) is the unit impulse input in continuous time and h(t) is the unit impulse response of continuous time LTI system. Continuous time unit impulse response h(t) is the output of a continuous time system when applied input x(t) is equal to unit impulse function δ(t)



Similarly, for a discrete time system, discrete time impulse response h(n) is the output of a discrete time system when applied input x(n) is equal to discrete time unit impulse function δ(n). Here, δ (n) is the unit impulse input in discrete time and h(n) is the unit impulse response of discrete time LTI system. Therefore, any LTI system can be completely characterized in terms of its unit impulse response.

Properties of Linear time invariant (LTI) system:-

The LTI system has number of properties not exhibited by other systems. Those are as under:

- **Commutative property of LTI systems**
- **Distributive property of LTI systems**
- **Associative property of LTI systems**
- **Static and dynamic LTI systems**
- **Invertibility of LTI systems**
- **Causality of LTI systems**
- **Stability of LTI systems**
- **Unit-step response of LTI systems**

Commutative property:

The commutative property is a basic property of convolution in both continuous and discrete time cases, thus, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are commutative. According to the property, for continuous time LTI system. The output is given by

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Or

$$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Thus, we can say that according to this property, the output of a continuous time LTI system having input x(t) and unit impulse h(t) is identical to the output of a continuous time LTI system having input h(t) and the unit impulse response x(t).

Distributive property:

The distributive property states that both convolution integral for continuous time LTI system and convolution sum for discrete time LTI system are distributive.

For continuous time LTI system, the distributive property is

expressed as The output, $y(t) = x(t) * [h_1(t) + h_2(t)]$

Or

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

Thus, the two continuous time LTI systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs and outputs are added as

$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

The output $y(t) = y_1(t) + y_2(t)$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

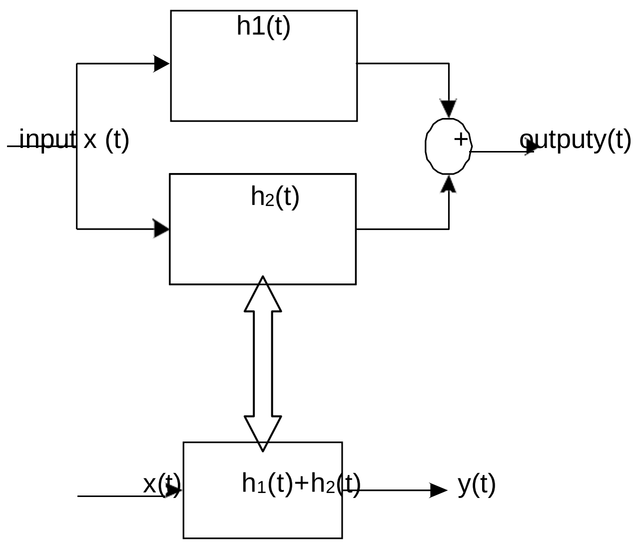


Fig 3.1The distributive property of convolution integral for a parallel interconnection of continuous time LTI system

Associative Property of LTI system:

According to associative property, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are associative.

For continuous time LTI system, according to associative property,

The output

$$\mathbf{y(t) = x(t) * [h_1(t) * h_2(t)]}$$

or

$$\mathbf{y(t) = [x(t) * h_1(t)] * h_2(t)}$$

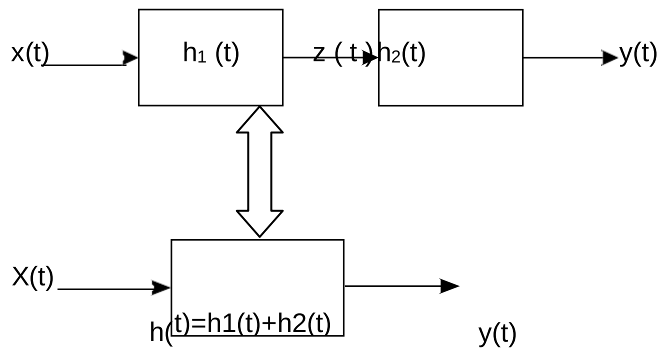


Fig 3.2: The associative property of convolution integral for a cascade interconnection of continuous time LTI systems

Here we have $y(t)=z(t) *h_2(t)$But $z(t)=x(t)*h1(t)$. Therefore $y(t)=[x(t)*h_1(t)*h_2(t)]$

Static and Dynamic property:

Static systems are also known as memory less systems. A system is known as static if its output at any time depends only on the value of the input at the same time. A continuous time system is memory less if its unit impulse response h(t) is zero for $t\neq0$. These memory less LTI systems are characterized by $y(t) = Kx(t)$ where K is constant. And its impulse response $h(t)=K\delta(t)$ .If K=1 ,then these systems are called identity systems.

Invertibility of LTI systems:

A system is known as invertible only if an inverse system exists which, when cascade with the original system, produces and output equal to the input at first system. If an LTI system is invertible then it will have a LTI inverse system. This means that we have a continuous time LTI system with impulse response h(t) and its inverse system with impulse response $h_1(t)$ which results in an output equal to x(t). Cascade interconnection of original continuous time LTI system with its inverse system is given as identity system.

Thus, the overall impulse response of a system with impulse response h(t) cascaded with its inverse system with Impulse response $h_1(t)$ is given as $h(t)*h_1(t)=\delta(t)$

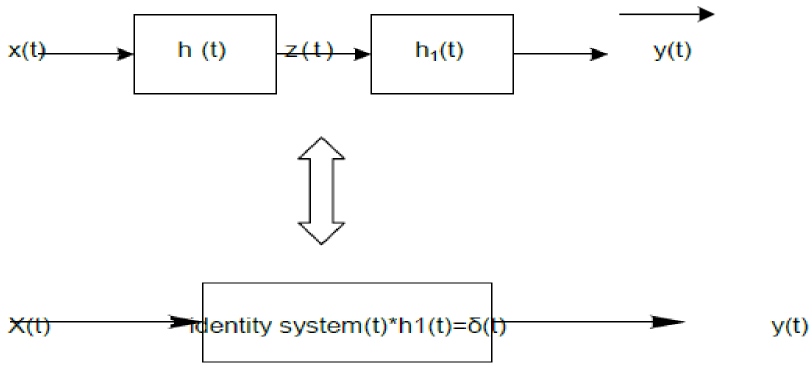


Fig3.3: An inverse system for continuous time LTI system

Causality for LTI System

This property says that the output of a causal system depends only on the present and past values of the input to the system.

A continuous time LTI system is called causal system if its impulse response h(t) is

zero t<0. For a causal continuous time

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{\tau=0}^{\infty} h(\tau)x(t-\tau)d\tau$$

LTI system, convolution integral is given as

For pure time shift with unit impulse response $h(t)=\delta(t-t_0)$ is a causal continuous time LTI system for $t\geq 0$. In this case time shift is known as a delay.

Stability for LTI systems

A stable system is a system which produces bounded output for every bounded input.

Condition of Stability for continuous time LTI system:

Let us consider an input x(t) that is bounded in magnitude |x(t)|<M for all values of t

Now, we apply this input to an continuous time LTI system with unit impulse response h(t).

Output of this LTI system is determined by convolution integral and is given by

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Magnitude of output y(t) is given as

$$|y(t)| = \left|\int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| = \int_{\tau=-\infty}^{\infty} |h(\tau||x(t-\tau)|\, d\tau$$

Substituting the value $|x(t-\tau) < M|$ for all values of τ and t, we get

$$|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)|M\, d\tau$$

Or

$|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau$ for all values of t

∞

From the above equation we can conclude that if the impulse response h(t) is absolutely integerable then output of a continuous time LTI system is bounded in magnitude, and thus, the system is bounded input, bounded output(BIBO) stable.

Unit step response of an LTI system:

Unit step response is the output of a LTI system for input is equal to unit step function or sequence. Unit step response of continuous time LTI system is found by convolution integral of u(t) with unit impulse response h(t) and is expressed as

g(t)=u(t)*h(t)=h(t)*u(t)

according to commutative property. Therefore, unit step response g(t) may be viewed as the response to the input h(t) of a continuous time LTI system with unit impulse response u(t).

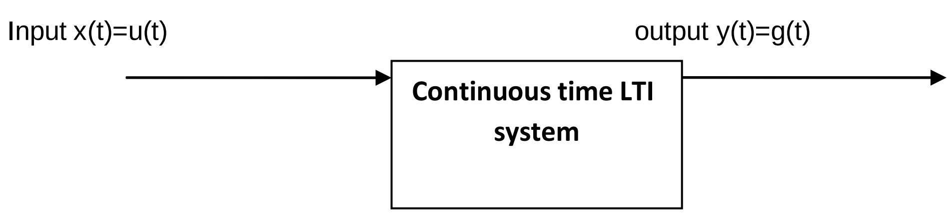


Fig 3.4: Continuous time LTI system

Classification of CT LTI system:-

The systems are classified into two types: continuous time and discrete time systems, Now these two broad types of systems are further classified on the basis of system properties as under:

- **Causal system and non causal system**
- **Time invariant and time variant system**
- **Stable and unstable system**
- **Linear and Non-linear system**
- **Static and Dynamic systems**
- **Invertible and noninvertible system**

Causal systems and Non-causal systems

A system is causal if the response or output does not begin before the input function is applied. This means that if input is applied at $t=t_0$,then for causal system, output will depend on values of input x(t) for $t \le t_0$.

Mathematically,

$y(t_0)=f[x(t),t \le t_0]$.

In other words we can say that, the response or output of the causal system to an input does not depend on future values of that input but depends only on the present or past values of the input.This means that all the real-time systems are also causal systems since these systems cannot know the future values of the input signal when it constructs output signal.Thus, causal systems are physically realizable. For example a resister is a continuous time causal systembecause voltage across it is given by the expression v(t)=R.i(t) and output v(t),i.e., voltage depends only on the input i(t) i.e., current at the present time.

Time invariant and time variant system

A system is said to be time invariant if its input –output characteristics do not change with time. H{x(t)}=y(t) implies that ,H{x(t-t0)}=y(t-t0) for every input signal x(t) and every time shift t0A system is said to be time variant if its input- output characteristics changes with time.

Procedure to Test for Time Invariance:-

1. **Delay the input signal by t0 units of time and determine the response of the system for this delayed input signal. Let this response be $y(t-t_0)$.**
2. **Delay the response of the system for undelayed input by t0 units of time. Let this delayed response be $y_d(t)$.**
3. **Check whether y(t-t0)=yd(t). If they are equal then the system is time invariant. Otherwise the system is time variant.**

Stable and unstable system

A system is called bounded input, bounded output(BIBO) stable if and only if every bounded input results in a bounded output. The output of such a system does not diverge or does not grow unreasonably large.

Condition of Stability for continuous time LTI system:

Let us consider an input x(t) that is bounded in magnitude

$|x(t)|<M<\infty$ for all values of t

Now, we apply this input to an continuous time LTI system with unit impulse response h(t).

Output of this LTI system is determined by convolution integral and is given by

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Magnitude of output y(t) is given as

$$|y(t)| = \left|\int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| = \int_{\tau=-\infty}^{\infty} |h(\tau||x(t-\tau)|\, d\tau$$

Substituting the value $|x(t-\tau) < M|$ for all values of τ and t, we get

$$|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)|M\, d\tau$$

Or

$|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$ for all values of t

From the above equation we can conclude that if the impulse response h(t) is absolutely integerable then output of a continuous time LTI system is bounded in magnitude, and thus, the system is bounded input, bounded output(BIBO) stable.

The systems not satisfying the above conditions are unstable.

Linear and nonlinear system

A linear system is one that satisfies the superposition principle. The principles of superposition requires that the response of the system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses of the system to each of the individual input signals.

A system is linear if

$H\{a_1x_1(t)+a_2x_2(t)\}=a_1H\{x_1(t)\}+a_2H\{x_2(t)\}$ for any arbitrary input sequences $x_1(t)$ and $x_2(t)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the super position principle as given by the above definition, then the system is nonlinear.

Static and dynamic system

A continuous time system is called static or memory less if its output at any instant t depends on present input but not on the past or future samples of the input. These systems contain no energy storage elements. This means that the equation relating its output signal to its input signal contains no derivative, integrals or signal delays.

As an example, consider the system described by the following relationship

$Y(t)=x^2(t)$ this system is memory less because the value of the output signal y(t) at time t depends only on the present value of the input signal x(t).In any other case the system is said to be dynamic or to have memory. Dynamic systems have one or more energy storage elements. Input output relationship of a dynamic continuous time system is described by its differential equation.

Invertible and non invertible system

A system is said to be invertible if there is a one to one correspondence between its input and output signals. If a system is invertible, then an inverse system exists. The cascading of and invertible system and its inverse system is equivalent to the identity system.

The frequency response of an inverse system is basically reciprocal of the frequency response of the original system or invertible system.

An example of an invertible continuous time system is given by

$y_1(t)$=3x(t)and its inverse system will be given by $y_2(t) = \frac{1}{3} y_1(t)$

Convolution integral:-

The output of any general input may be found by convolving the given input signal x(t) with the LTI systems unit impulse response h(t).

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Or

$$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Properties of convolution integral:

- **Commutative property**
- **Distributive property**
- **Associative property**

Commutative property:

The commutative property is a basic property of convolution in both continuous and discrete time cases, thus, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are commutative. According to the property, for continuous time LTI system. The output is given by

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Or

$$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Thus, we can say that according to this property, the output of a continuous time LTI system having input x(t) and unit impulse h(t) is identical to the output of a continuous time LTI system having input h(t) and the unit impulse response x(t).

The distributive property:

The distributive property states that both convolution integral for continuous time LTI system and convolution sum for discrete time LTI system are distributive.
For continuous time LTI system, the distributive property is

expressed as The output, $y(t) = x(t) * [h_1(t) + h_2(t)]$

Or

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

Thus, the two continuous time LTI systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs and outputs are added as

$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

The output $y(t) = y_1(t) + y_2(t)$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

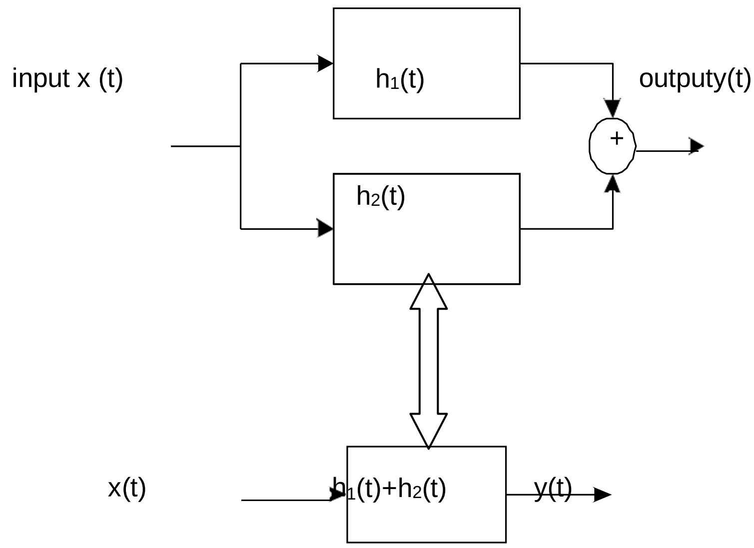


Fig3.1: The distributive property of convolution integral for a parallel interconnection of continuous time LTI system

Associative Property of LTI system:

According to associative property, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are associative.

For continuous time LTI system, according to associative property,

The output

$$y(t) = x(t) * [h_1(t) * h_2(t)]$$

or

$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

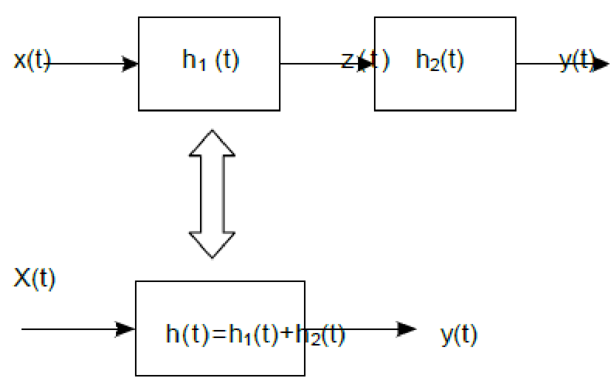


Fig 3.2:The associative property of convolution integral for a cascade interconnection of continuous time LTI systems

Here we have $y(t)=z(t) * h_2(t)$. But $z(t)=x(t)*h1(t)$. Therefore $y(t)=[x(t)*h_1(t)*h_2(t)]$

Linear constant coefficient differential equation:

The continuous time linear time invariant (LTI) systems are described by their linear constant coefficient differential equations. For this, let us consider a first or~~der~~ ~~d~~ifferential equation as under

$$\frac{dy(t)}{dt} + Ay(t) = x(t)$$

Where x (t) and y(t) are the input and output of the continuous time LTI system. A is a constant value. The first order differential equation can be extended for higher order differential equations. A general Nth order linear constant coefficient differential equation can be given by

$$\sum_{k=0}^{N} A_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} \frac{d^k}{dt^k} x(t)$$

The complete solution of differential eq~~ua~~tion consis~~ts~~ of the sum of particular solution $y_p(t)$ and homogenous solution $y_h(t)$.

The homogeneous solution of a differential equation is possible by substituting

$$\sum_{k=0}^{N} A_k \frac{d^k}{dt^k} y(t) = 0$$

This solution to differential equation is also known as natural response of the system.

A particular case of differential equation is determined by putting N=0, we obtain

$$y(t) = \frac{1}{A_0} \sum_{k=0}^{M} \frac{d^k}{dt^k} x(t)$$

Transfer function:

Transfer functions are commonly used in the analysis of systems such as single-input single-output filters, typically within the fields of signal processing, communication theory, and control theory. The term is often used exclusively to refer to linear, time-invariant systems (LTI). Most real systems have non-linear input/output characteristics, but many systems, when operated within nominal parameters have behavior that is close enough to linear that LTI system theory is an acceptable representation of the input/output behavior.

The descriptions below are given in terms of a complex variable, $s = \sigma + j \cdot \omega$, which bears a brief explanation. In many applications, it is sufficient to define $\sigma = 0$ (and $s = j \cdot \omega$), which reduces the Laplace transforms with complex arguments to Fourier transforms with real argument ω. The applications where this is common are ones where there is interest only in the steady-state response of an LTI system, not the fleeting turn-on and turn-off behaviors or stability issues. That is usually the case for signal processing and communication theory.

Thus, for continuous-time input signal x(t) and output y(t0, the transfer function H(s) is the linear mapping of the Laplace transform of the input, X(s)=L{x(t)}, to the Laplace transform of the output Y(s)=L{y(t)}:

Y(s)=H(s)X(s)

$$H(s) = \frac{Laplace\ transform\ of\ output}{laplace\ transform\ of\ input}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{L\{y(t)\}}{L\{x(t)\}}$$

Conditions required for transfer function:

(i) System should be in unloaded condition (initial conditions are zero)

(ii) The system should be linear time invariant.

Impulse Response:

In signal processing, the impulse response, of a dynamic system is its output when presented with a brief input signal, called an impulse. More generally, an impulse response refers to the reaction of any dynamic system in response to some external change. In both cases, the impulse response describes the reaction of the system as a function of time. In all these cases, the dynamic system and its impulse response may be actual physical objects, or may be mathematical systems of equations describing such objects. Since the impulse function contains all frequencies, the impulse response defines the response of a linear time- invariant system for all frequencies. The impulse can be modeled as a Dirac delta function for continuous- time systems, or as. The Dirac delta represents the limiting case of a pulse made very short in time while maintaining its area or integral. While this is impossible in any real system, it is a useful idealization. In Fourier theory, such an impulse comprises equal portions of all possible excitation frequencies, which makes it a convenient test probe. Any system in a large class known as linear, time-invariant (LTI) is completely characterized by its impulse response. That is, for any input, the output can be calculated in terms of the input and the impulse response. The impulse response of a linear transformation is the image of Dirac's delta function under the transformation, analogous to the fundamental solution of a partial differential operator. It is usually easier to analyze systems using transfer functions as opposed to impulse responses. The transfer function is the Laplace transform of the impulse response. The Laplace transform of a system's output may be determined by the multiplication of the transfer function with the input's Laplace transform in the complex plane, also known as the frequency domain. An inverse Laplace transform of this result will yield the output in the time domain. To determine an output directly in the time domain requires the convolution of the input with the impulse response. When the transfer function and the Laplace transform of the input are known, this convolution may be more complicated than the alternative of multiplying two functions in the domain. It is obtained by taking inverse Laplace transform of transfer function H(s).

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{Y(s)}{X(s)}\right\}$$

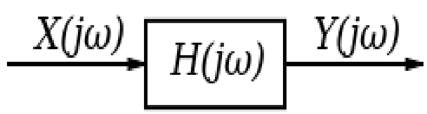
Frequency Response:

Frequency response is the quantitative measure of the output spectrum of a system or device in response to a stimulus, and is used to characterize the dynamics of the system. It is a measure of magnitude and phase of the output as a function of frequency, in comparison to the input.It is obtained from the transfer function by substituting s=jω in transfer function.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Systems respond differently to inputs of different frequencies. Some systems may amplify components of certain frequencies, and attenuate components of other frequencies. The way that the system output is related to the system input for different frequencies is called the *frequency response* of the system.

The frequency response is the relationship between the system input and output in the Fourier Domain.



In this system, X(jω) is the system input, Y(jω) is the system output, and H(jω) is the frequency response. We can define the relationship between these functions as:

Y(jω)=H(jω)X(jω)

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega)$$

Since the frequency response is a complex function, we can convert it to polar notation in the complex plane. This will give us a magnitude and an angle.

Amplitude Response:

For each frequency, the magnitude represents the system's tendency to amplify or attenuate the input signal.

$$A(\omega) = |H(j\omega)|$$

Phase Response:

The phase represents the system's tendency to modify the phase of the input sinusoids.

$$\Phi(\omega) = \angle H(j\omega)$$

The phase response, or its derivative the group delay, tells us how the system delays the input signal as a function of frequency.

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SCHOOL OF ELECTRICAL AND ELECTRONICS

DEPARTMENT OF ELECTRONICS AND COMMMUNICATION ENGINEERING

UNIT - IV
SIGNALS AND SYSTEMS – SECA1301

IV ANALYSIS OF DISCRETE TIME SIGNALS AND LTI DISCRETE TIME SYSTEMS

Discrete Time Fourier Transform (DTFT)

The discrete•time Fourier transform (DTFT) of a real, discrete•time signal x[n] is a complex•valued function defined by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

for any (integer) value of n.

Inverse Discrete Time Fourier Transform (IDTFT)

The function X ($e^{j\omega}$) or X (ω) **is called the Discrete•Time Fourier Transform(DTFT) of the discrete•time signal x(n). The inverse DTFT is defined by the following integral:**

$$x(n) = \frac{1}{2}\pi \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Properties of DTFT

Property	Periodic signal	Fourier Series Coefficients
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k \cdot e^{-jk\left(\frac{2\pi}{N}\right) n_0}$
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Frequency Shifting	$e^{jMw_0 n} x[n]$	a_{k-M}
First Difference	$x[n] - x[n-1]$	$\left(1 - e^{-jk\left(2\pi/N\right)}\right) a_k$
Conjugate Symmetry for Real Signals	x[n] real	$a_k = a^*_{-k}$
Real & Even Signals	x[n] real and even	a_k real and even
Real & Odd signals	x[n] real and odd	a_k purely imaginary and odd
Even-Odd Decomposition Of Real Signals	$x_e[n] = Ev\{x[n]\}$ $[x[n] real]$ $x_o[n] = Od\{x[n]\}$ $[x[n] real]$	$\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
Parseval's Relation	$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$	

1. Find the DTFT of an impulse function which occurs at time zero.

$$x[n] = \delta[n]$$
$$X\left(e^{jw}\right) = \sum \delta[n] e^{-jwn} = 1$$

$$\delta[n] \overset{F}{\Leftrightarrow} 1$$

$$\delta[n-1] \overset{F}{\Leftrightarrow} (1) \cdot e^{-jw(1)}$$

Discrete Fourier Transform

The DFT is used to convert a finite discrete time sequence x (n) to an N point frequency domain sequence denoted by X (K). The N point DFT of a finite duration sequence x (n) is defined as

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \quad \text{for } K=0, 1, 2, \ldots N-1$$

The discrete Fourier transform (DFT) is the Fourier transform for finite-length sequences because, unlike the (discrete-space) Fourier transform, the DFT has a discrete argument and can be stored in a finite number of infinite word-length locations. Yet, it turns out that the DFT can be used to exactly implement convolution for finite-size arrays

Inverse Discrete Fourier Transform

The IDFT is used to convert the N point frequency domain sequence X (K) to an N point time sequence. The IDFT of the sequence X (K) of length N is defined as

$$x(n) = 1/N \sum_{K=0}^{N-1} X(K)\, e^{+j2\pi nk/N} \quad \text{for } n=0, 1, 2, \ldots N-1$$

Properties of DFT

1. **Periodicity: X (K+N) =X (K) for all K.**
2. **Linearity: DFT[a1 x1 (n)+a2 x2(n)]=a1 X1 (K)+a2 X2 (K)**
3. **DFT of time reversed sequence: DFT[x(N-n)]=X(N-K)**
4. **Circular convolution :DFT[x1(n)*x2(n)]=X1(K) X2(K)**
5. **Shifting: If DFT {x (n)} =X (K), then DFT{x (n-no)} =X (K) $e^{-j2\pi n_o k/N}$**
6. **Symmetry property**
   Re[X(N-k)]=ReX(k)

This implies that amplitude has symmetry Im[X(N-k)]= - Im[X(k)]

This implies that the phase spectrum is antisymmetric.

7. **If x[n] is an even function xe[n] then**

$$F\left[x_e[n]\right] = X_e(k) = \sum_{n=0}^{N-1} x_e[n] \cos(k\Omega nT)$$

This implies that the transform is also even

8.If x[n] is odd function xo[n] than

$$F[x_o[n]] = X_o(k) = -j\sum_{n=0}^{N-1} x_o[n]\sin(k\Omega nT)$$

This implies that the transform is purely imaginary and odd

9.Parseval's Theorem

The normalized energy in the signal is given by either of the following expressions

$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

10. Delta Function

$$F[\delta(nT)] = 1$$

11. Unit step function

$$F[u[n]] = \frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty} \pi\delta(w+2\pi k)$$

$$F\left[e^{jw_0 n}\right] = \sum_{k=-\infty}^{\infty} 2\pi\delta(w-w_0+2\pi k)$$

12. **Fourier transform of a CT complex exponential is interpreted as an impulse at w=w₀. For discrete•time we expect something similar but difference is that DTFT is periodic in w with period 2π. This says that FT of x[n] should have impulses at w₀, w₀ ±2π, w₀±4π etc.**

$$\alpha^n u[n] \quad (|n|<1) \quad \overset{F}{\Leftrightarrow} \quad \frac{1}{1-\alpha e^{-jw}}$$

13. **Linear cross•correlation of two data sequences or series may be computed using DFTs. The linear cross correlation of two finite•length sequences x1[n] and x2[n] each of length N is defined to be:**

$$r_{x_1x_2}(j) = \frac{1}{N}\sum_{n=-\infty}^{\infty} x_1(n)x_2(n+j) \qquad ,-\infty \le j \le \infty$$

Circular correlation of finite length periodic sequences $x_{1p}[n]$ and $x_{2p}[n]$ is described as:

$$r_{cx_1x_2}(j) = \frac{1}{N}\sum_{n=0}^{N-1} x_{1p}(n)x_{2p}(n+j) \qquad , j = 0,\dots\dots,(N-1)$$

This circular correlation can be evaluated using DFTs as shown below:

$$r_{cx_1x_2}(j) = F^{-1}\left[X_1^*(k)X_2(k)\right]$$

The circular correlation can be converted into a linear correlation by using augmenting zeros. If the sequences are x1[n] of length N1 and x2[n] of length N2, then their linear correlation will be of length N1+N2•1.

To achieve this x1[n] is replaced by x1a[n] which consists of x1[n] with (N2•1) zeros added and x2[n] is augmented by (N1•1) zeros to become x2a[n].

$$\Rightarrow \quad r_{x_1x_2}(j) = F^{-1}\left[X_{1a}^*(k)X_{2a}(k)\right]$$

1. **Find the DFT of the following signal x(n)=δ(n)**

$$X(K) = \sum_{n=0}^{N•1} x(n)\, e^{•j2\pi nk/N} \text{ for } K=0, 1, 2\ldots.,N•1$$

$$X(K) = \sum_{n=0}^{N•1} \delta(n) e^{•j2\pi nk/N} \text{ for } K=0, 1, 2,\ldots N•1$$

$$X(K) = 1$$

2. **Consider a length-N sequence defined for n = 0,1,2,……,(N-1) where**

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$ Find the DFT of the given sequence.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,2,\ldots\ldots.,(N-1)$$

The N•point DFT is equal to $= 1$

Basic Principles of Z Transform:

The z•transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete•time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics.

These discrete models are solved with difference equations in a manner that is analogous to solving continuous models with differential equations. The role played by the z•transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations.

Types of Z Transform Unilateral Z•transform

Alternatively, in cases where x[n] is defined only for n ≥ 0, the single-sided or unilateral Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

In signal processing, this definition can be used to evaluate the Z•transform of the unit impulse response of a discrete•time causal system

Bilateral Z•transform

The bilateral or two•sided Z•transform of a discrete•time signal x[n] is the formalpower series X(Z) defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where n is an integer and z is, in general, a complex number:

$z = A e^{j\phi} = A (\cos \phi + j \sin \phi)$

where A is the magnitude of z, j is the imaginary unit, and ϕ is the complex argument (also referred to as angle or phase) in radians.

Inverse Z Transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

1. **The z•transform of the sequence** $x_n = \cos(an)$
   find its Z transform

$$\mathcal{Z}[x_n] = \mathcal{Z}[\cos(an)] = \mathcal{Z}\left[\frac{1}{2}e^{ian} + \frac{1}{2}e^{-ian}\right]$$

$$= \frac{1}{2}\mathcal{Z}\left[e^{ian}\right] + \frac{1}{2}\mathcal{Z}\left[e^{-ian}\right] = \frac{1}{2}\frac{z}{z-e^{ia}} + \frac{1}{2}\frac{z}{z-e^{-ia}}$$

$$= \frac{1}{2}\left(\frac{z\left(z-e^{-ia}\right)}{(z-e^{ia})(z-e^{-ia})} + \frac{z\left(z-e^{ia}\right)}{(z-e^{ia})(z-e^{-ia})}\right)$$

$$= \frac{z\left(2z-e^{ia}-e^{-ia}\right)}{2(z-e^{ia})(z-e^{-ia})} = \frac{z\left(z-\frac{e^{ia}+e^{-ia}}{2}\right)}{(z-e^{ia})(z-e^{-ia})}$$

$$= \frac{z(z-\cos(a))}{(z-e^{ia})(z-e^{-ia})} = \frac{z(z-\cos(a))}{z^2-2z\cos(a)+1}$$

2. **Find the z•transform of the unit pulse or impulse sequence** $x_n = \delta[n] = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$

$$X(z) = \mathcal{Z}[x_n] = \sum_{n=0}^{\infty} x_n z^{-n} = 1 + \sum_{n=1}^{\infty} 0\,z^{-n} = 1$$.

3. **The z•transform of the unit•step sequence** $x_n = u[n] = \begin{cases} 1 & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$ is $X(z) = \dfrac{z}{z-1}$

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

.

4. **The z•transform of the sequence** $x_n = b^n$ is $X(z) = \dfrac{z}{z-b}$.

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{b^n}{z^n}\right) = \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

$$= \frac{1}{1-\frac{b}{z}} = \frac{z}{z-b}$$

5. **The z•transform of the exponential** $x_n = e^{an}$ **sequence is** $X(z) = \frac{z}{z - e^a}$.

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} e^{an} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{an}}{z^n}\right) = \sum_{n=0}^{\infty} \left(\frac{e^a}{z}\right)^n$$

$$= \frac{1}{1 - \frac{e^a}{z}} = \frac{z}{z - e^a}$$

6.

Prove $Z[\{a^n\}] = \frac{z}{z-a}$

Proof

$$Z[\{a^n\}] = \sum_{0}^{\infty} a^n z^{-n} = \sum_{0}^{\infty} \left[\frac{a}{z}\right]^n = 1 + \left[\frac{a}{z}\right] + \left[\frac{a}{z}\right]^2 + \left[\frac{a}{z}\right]^3 + \left[\frac{a}{z}\right]^4 + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}, \quad \left|\frac{a}{z}\right| < 1 \quad \text{i.e; } |z| > |a|$$

	Sequence	z - transform
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$
4	$b^{n-1} u[n-1]$	$\frac{1}{z-b}$
5	e^{an}	$\frac{z}{z-e^a}$
6	n	$\frac{z}{(z-1)^2}$
7	n^2	

		$\frac{z(z+1)}{(z-1)^3}$
8	$b^n n$	$\frac{bz}{(z-b)^2}$
9	$e^{an} n$	$\frac{z e^a}{(z-e^a)^2}$
10	$\sin(an)$	$\frac{\sin(a)\, z}{z^2 - 2\cos(a)\, z + 1}$
11	$b^n \sin(an)$	$\frac{\sin(a)\, b z}{z^2 - 2\cos(a)\, b z + b^2}$
12	$\cos(an)$	$\frac{z(z - \cos(a))}{z^2 - 2\cos(a)\, z + 1}$
13	$b^n \cos(an)$	$\frac{z(z - b\cos(a))}{z^2 - 2\cos(a)\, b z + b^2}$

Properties of Z•Transform

Z•Transform has the following properties:

1. Linearity Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \overset{\text{Z.T}}{\longleftrightarrow} a\,X(Z) + b\,Y(Z)$$

2. Time Shifting Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then Time shifting property states that

$$x(n-m) \overset{\text{Z.T}}{\longleftrightarrow} z^{-m} X(Z)$$

3.Multiplication by Exponential Sequence Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n \, . \, x(n) \xleftrightarrow{\text{Z.T}} X(Z/a)$$

4. **Time Reversal Property**

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then time reversal property states that

$$x(-n) \xleftrightarrow{\text{Z.T}} X(1/Z)$$

5. **Differentiation in Z•Domain OR Multiplication by n Property**

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then multiplication by n or differentiation in z•domain property states that

$$n^k x(n) \xleftrightarrow{\text{Z.T}} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

6. **Convolution Property**

Then convolution property states that

$$x(n) * y(n) \xleftrightarrow{\text{Z.T}} X(Z).Y(Z)$$

7. **Correlation Property**

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \xleftrightarrow{\text{Z.T}} X(Z).Y(Z^{-1})$$

8. InitialValueandFinalValueTheorems

Initial value and final value theorems of z•transform are defined for causal signal.

Initial Value Theorem
For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z\to\infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z•transform

Final Value Theorem
For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z\to 1}[z-1]X(z)$$

This is used to find the final value of the signal without taking inverse z•transform.

Region of Convergence (ROC) of Z•Transform

The range of variation of z for which z•transform converges is called region of convergence of z• transform.

Properties of ROC of Z•Transforms

- **ROC of z•transform is indicated with circle in z•plane.**
- **ROC does not contain any poles.**
- **If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z• plane except at z = 0.**
- **If x(n) is a finite duration anti•causal sequence or left sided sequence, then the ROC is entire z• plane except at z = ∞.**
- **If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.**
- **If x(n) is a infinite duration anti•causal sequence, ROC is interior of the circle with radius a. i.e. |z|< a.**
- **If x(n) is a finite duration two sided sequence, then the ROC is entire z•plane except at z = 0 & z= ∞.**

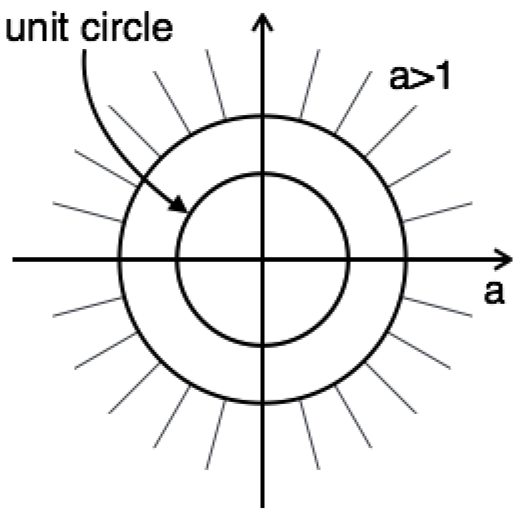
The concept of ROC can be explained by the following example:

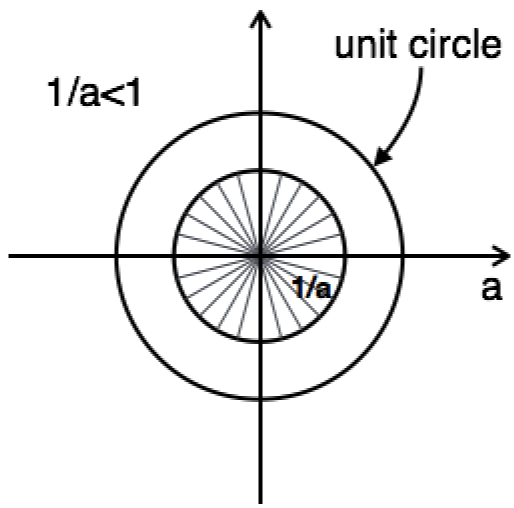
Example 1: Find z•transform and ROC of

$$a^n u[n] + a^- nu[-n-1]$$

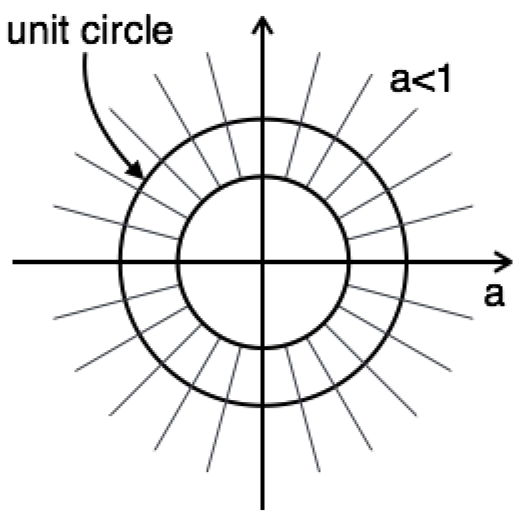
$$Z.T[a^n u[n]] + Z.T[a^{-n} u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

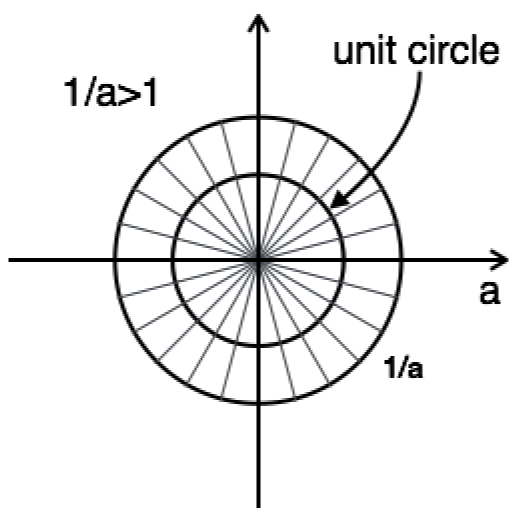
The plot of ROC has two conditions as a > 1 and a < 1, as the value of 'a' is not known.





In this case, there is no combination ROC.





Here, the combination of ROC is from

$$a < |z| < \frac{1}{a}$$

Hence for this problem, z•transform is possible when a < 1.

$$\mathcal{Z}[x[n]] = \frac{1}{1-az^{-1}} - \frac{1}{1-a^{-1}z^{-1}} = \frac{a^2-1}{a}\frac{z}{(z-a)(z-1/a)}$$

1. **The Z transform of a right sided signal** $x[n] = a^n u[n]$ **is**

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

For this summation to converge, i.e., for $X(z)$ to exist, it is necessary to have $|az^{-1}| < 1$, i.e., the ROC is $|z| > |a|$. As a special case when $\underline{a = 1}$, $x[n] = u[n]$ and we have

$$\mathcal{Z}[u[n]] = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

2. **The Z•transform of a left sided signal** $x[n] = -a^n u[-n-1]$ is:

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= 1 - \sum_{n=0}^{\infty}(a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

For the summation above to converge, it is required that $|a^{-1}z| < 1$, i.e., the ROC is $|z| < |a|$.

Comparing the two examples above we see that two different signals can have identical z•transform, but with different ROCs.

3. **Find the inverse of the given z•transform** $X(z) = 4z^2 + 2 + 3z^{-1}$
Comparing this with the definition of z•transform:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}=x[-2]z^2+x[-1]z^1+x[0]+x[1]z^{-1}+x[2]z^{-2}$$

we get $x[n]=4\delta[n+2]+2\delta[n]+3\delta[n-1]$

In general, we can use the time shifting property

$$\mathcal{Z}[\delta[n+n_0]]=z^{n_0}$$

to inverse transform the $X(z)$ **given above to** $x[n]$ **directly.**

Zeros and Poles of Z•Transform

All z•transforms in the above examples are rational, i.e., they can be written as a ratio of polynomials of variable z **in the general form**

$$X(z)=\frac{N(z)}{D(z)}=\frac{\sum_{k=0}^{M}b_kz^k}{\sum_{k=0}^{N}a_kz^k}=\frac{b_M}{a_N}\frac{\prod_{k=1}^{M}(z-z_{z_k})}{\prod_{k=1}^{N}(z-z_{p_k})}$$

where $N(z)$ is the numerator polynomial of order M with roots $z_{z_k},(k=1,2,\cdots,M)$, and $D(z)$ is the denominator polynomial of order N with roots $z_{p_k},(k=1,2,\cdots,N)$.

In general, we assume the order of the numerator polynomial is lower than that of the denominator polynomial, i.e., $M<N$. If this is not the case, we can always expand $X(z)$ into multiple terms so that $M<N$ is true for each of terms.

The zeros and poles of a rational $X(z)=N(z)/D(z)$ **are defined as:**

- **Zero**: Each of the roots of the numerator polynomial z_z for which $X(z)\Big|_{z=z_z}=X(z_z)=0$ is a zero of $X(z)$.

If the order of $D(z)$ exceeds that of $N(z)$ (i.e., $N > M$), then $X(\infty) = 0$, i.e., there is a zero at infinity:

$$\left.\frac{b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}\right|_{z\to\infty} = 0$$

- **Pole**: Each of the roots of the denominator polynomial z_p for which $\left.X(z)\right|_{z=z_p} = X(z_p) = \infty$ is a pole of $X(z)$.

If the order of $N(z)$ exceeds that of $D(z)$ (i.e., $M > N$), then $X(\infty) = \infty$, i.e, there is a pole at infinity:

$$\left.\frac{b_2 z^2 + b_1 z + b_0}{a_1 z + a_0}\right|_{z\to\infty} \to \infty$$

Most essential behavior properties of an LTI system can be obtained graphically from the ROC and the zeros and poles of its transfer function $H(z)$ on the z•plane

Inverse Z Transform using Contour Integration Method

1. $$F(z) = \frac{z\left(z^2 - 2z - 1\right)}{\left(z^2 + 1\right)^2}$$

$$F(z) = 2z\frac{(z-1)^2 - (z-1)(z-2) - (z-2)}{(z-1)(z-2)^2}$$

$$F(z) = 2z\frac{z^2 - 2z + 1 - z^2 + 3z - 2 - z + 2}{(z-1)(z-2)^2}$$

$$F(z) = 2z\frac{1}{(z-1)(z-2)^2}$$

2. **Evaluate the inverse z transform of integral.** $X(z) = \frac{1}{1 - az^{-1}}, |z| > |a|$ **using the complex inversion**

Long Division Method

The z•transform is a power series expansion,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \cdots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

where the sequence values $x(n)$ are the coefficients of z^{-n} in the expansion. Therefore, if we can find the power series expansion for $X(z)$, the sequence values $x(n)$ may be found by simply picking off the coefficients of z^{-n}.

1. Sometimes the inverse transform of a given $X(z)$ can be obtained by long division.

$$X(z) = \frac{1}{1 - az^{-1}}$$

By a long division, we get

$$1 \div (1 - az^{-1}) = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

which converges if the ROC is $|z| > |a|$, i.e., $|az^{-1}| < 1$ and we get

$$x[n] = a^n u[n]$$

. Alternatively, the long division can also be carried out as:

$$1 \div (-az^{-1} + 1) = -a^{-1}z - a^{-2}z^2 - \cdots$$

which converges if the ROC is $|z| < |a|$, i.e., $|a^{-1}z| < 1$ and we get

$$x[n] = -a^n u[-1 - n]$$

2. **To understand how an inverse Z Transform can be obtained by long division, consider the function**

$$F(z) = \frac{z}{z - 0.5}$$

If we perform long division

$$\begin{array}{r l}
 & 1 + 0.5z^{-1} + 0.25z^{-2} + \cdots \\
\cline{2-2}
z - 0.5 \big) & z \\
 & \underline{z - 0.5} \\
 & \quad 0.5 \\
 & \quad 0.5 - \underline{0.25z^{-1}} \\
 & \qquad 0.25z^{-1} \\
 & \qquad \underline{0.25z^{-1} - 0.125z^{-2}} \\
 & \qquad \vdots
\end{array}$$

we can see that

$$F(z) = 1 + 0.5z^{-1} + 0.25z^{-2} + \cdots.$$

So the sequence f[k] is given by

$$f = \{1, 0.5, 0.25, \cdots\}$$

Upon inspection

$$f[k] = 0.5^{k}$$

3. Find the Inverse Z Transform using Long Division Method

$$F(z) = \frac{2z^2 + z}{z^2 - 1.5z + 0.5}$$

$$\begin{array}{r l}
 & 2 + 4z^{-1} + 5z^{-2} + \cdots \\
\cline{2-2}
z^2 - 1.5z + 0.5 \big) & 2z^2 + z \\
 & 2z^2 - \underline{3z + 1} \\
 & \qquad 4z - 1 \\
 & \qquad 4z - \underline{6 + 2z^{-1}} \\
 & \qquad\qquad 5 - 2z^{-1} \\
 & \qquad\qquad \underline{5 - 7.5z^{-1} + 2.5} \\
 & \qquad\qquad \vdots
\end{array}$$

$$F(z) = 2 + 4z^{-1} + 5z^{-2} + \cdots$$

and the sequence f[k] is given by $f = \{2, 4, 5, \cdots\}$

4. $E(z) = \dfrac{0.5}{(z-1)(z-0.6)}$

$$
\begin{array}{r|l}
 & 0.5z^{-2} + 0.8z^{-3} + 0.98z^{-4} + \ldots \\
z^2 - 1.6z + 0.6 & 0.5 \\
 & \underline{0.5 - 0.8z^{-1} + 0.3z^{-2}} \\
 & 0.8z^{-1} - 0.3z^{-2} \\
 & \underline{0.8z^{-1} - 1.28z^{-2} + 0.48z^{-3}} \\
 & 0.98z^{-2} - 0.48z^{-3}
\end{array}
$$

$e(0) = 0, e(1) = 0, e(2) = 0.5, \ldots$

Inverse Z Transform using Residue Method:
Find the solution using the formula

$$y[n] = Z^{-1}[Y(z)] = \sum_{i=1}^{k} \mathrm{Res}[Y(z)\, z^{n-1}, z_i]$$

where $z_1, z_2, \ldots, z_k$ **are the poles of** $f(z) = Y(z)\, z^{n-1}$ **.**

Partial fraction method
Inverse Z Transform by Partial Fraction Expansion

This technique uses Partial Fraction Expansion to split up a complicated fraction into forms that are in the Z Transform table. As an example consider the function

$$F(z) = \frac{2z^2 + z}{z^2 - 1.5z + 0.5}$$

For reasons that will become obvious soon, we rewrite the fraction before expanding it by dividing the left side of the equation by "z."

$$\frac{F(z)}{z} = \frac{2z + 1}{z^2 - 1.5z + 0.5}$$

Now we can perform a partial fraction expansion

$$\frac{F(z)}{z} = \frac{2z+1}{z^2 - 1.5z + 0.5}$$
$$= \frac{2z+1}{(z-1)(z-0.5)}$$
$$= \frac{A}{z-1} + \frac{B}{z-0.5}$$
$$= \frac{6}{z-1} + \frac{-4}{z-0.5}$$

These fractions are not in our table of Z Transforms. However if we bring the "z" from the denominator of the left side of the equation into the numerator of the right side, we get forms that are in the table of Z Transforms; this is why we performed the first step of dividing the equation by "z"

$$F(z) = 6\frac{z}{z-1} - 4\frac{z}{z-0.5}$$

So

$$f[k] = 6u[k] - 4 \cdot 0.5^k$$

or

$$f = \{2, 4, 5, 5.5, \cdots\}$$

LTI – DT Systems:

A DT System which satisfies Linearity and time invariance property is called LTI DT systems. LTI systems comprise a very important class of systems, and they can be described by a standard mathematical formalism.

Characterization using difference equation:

Systems described by constant-coefficient, linear difference equations are LTI systems. In exploring this fact, it is important to keep in mind that our default setting is that all signals are defined for $-\infty < n < \infty$. The difference equation is a formula for computing an output sample at time n based on past and present input samples and past output samples in the time domain. The difference equation is as follows:

$$\begin{aligned} y(n) \quad &= b_0\, x(n) + b_1\, x(n-1) + \cdots + b_M\, x(n-M) \\ &\qquad -a_1\, y(n-1) - \cdots - a_N\, y(n-N) \\ &= \sum_{i=0}^{M} b_i\, x(n-i) - \sum_{j=1}^{N} a_j\, y(n-j) \end{aligned}$$

$b_i, i = 0, 1, 2, \ldots, M$

Where x(n) is the input signal and y(n) is the output signal and constants

$a_i, i = 1, 2, \ldots, N$

, **are called the coefficients. We have a system whose input and output signals are related by**

$$y[n] + ay[n-1] = bx[n]\ ,\ -\infty < n < \infty$$

where a and b are real constants. This is called afirst-order, constant-coefficient, linear difference equation. Given an input signal x[n], this can be viewed as an equation that must be solved for y[n] for each input signal x[n] thereis a unique solution for the output signal y[n].

$$y[n] = \sum_{k=-\infty}^{\infty} (-a)^{n-k} bx[k]$$

Example 1: Find the solution to the following difference equation by using the z-transform x(k+2)+3x(k+1)+2x(k)=0, x(0)=0,x(1)=1

Solution:

Take the z-transform of both side of the equation, we get $z^2 X(z) - z^2x(0) - z \cdot x(1) + 3z \cdot X(z) - 3z \cdot x(0) + 2X(z) = 0$

Substituting in the initial conditions and simplifying gives

$$X(z) = \frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)} = \frac{z}{z+1} - \frac{z}{z+2}$$

Take the inverse Z transform of above equation we get,

$$x(k) = \mathrm{Z}^{-1}[X(z)] = \mathrm{Z}^{-1}\left[\frac{z}{z-(-1)}\right] - \mathrm{Z}^{-1}\left[\frac{z}{z-(-2)}\right] = (-1)^k - (-2)^k, \quad k = 0, 1, 2, \ldots$$

Example 2: Using the z-transform to solve the following difference equation $x(k + 2) + 0.4x(k + 1) - 0.32x(k) = u(k)$, where $x(0) = 0$ and $x(1) = 1$. The input $u(k)$ is a unit step input, i.e. $u(k) = 1$, for $k \geq 0$.

Solution:

Take the z-transform of the difference equation we get

$$z^2X(z) - z^2x(0) - z\cdot x(1) + 0.4\cdot z\cdot X(z) - 0.4\cdot z\cdot x(0) - 0.32\cdot X(z) = \frac{z}{z-1}$$

Substituting the initial conditions and simplifying, we obtain

$$X(z) = \frac{z^2}{(z-1)(z^2+0.4z-0.32)} = \frac{z^2}{(z-1)(z+0.8)(z-0.4)}$$

The partial fraction expansion of the solution X(z) is

$$X(z) = 0.926\frac{z}{z-1} - 0.3704\frac{z}{z+0.8} - 0.5556\frac{z}{z-0.4}$$

The corresponding time sequence can be obtained by taking the inverse z-transform of the above equation:

$x(k) = 0.926 - 0.3704(-0.8)^k - 0.5556(0.4)^k$, for $k = 0, 1, 2, \ldots$

Convolution Sum: To each LTI system there corresponds a signal h[n] such that the input-output behavior of the system is described by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

This expression is called the convolution sum representation for LTI systems. In addition, the shifting property easily shows that h[n] is the response of the system to a unit-pulse input signal.

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=\sum_{k=-\infty}^{\infty} g[k]h[n-k]=h[n]$$

Thus the input-output behavior of a discrete-time, linear, time-invariant system is completely described by the unit-pulse response of the system. If h[n] is known, then the response to anyinput can be computed from the convolution sum.
The system response to this inputsignal is given by

$$\hat{y}[n]=\sum_{k=-\infty}^{\infty} \hat{x}[k]\,h[n-k]$$
$$=\sum_{k=-\infty}^{\infty} x[k-n_o]\,h[n-k]$$

To rewrite this expression, change the summation index from k to 1 = k — N, to obtain

$$\hat{y}[n]=\sum_{l=-\infty}^{\infty} x[l]\,h[n-n_o-l]$$
$$=y[n-n_o]$$

The convolution representation for linear, time-invariant systems can be developed by adopting a particular representation for the input signal and then enforcing the properties of linearity and time invariance on the correspondingresponse

Example, if n = 3, then the right side is evaluated by the sifting property to verify

$$\sum_{k=-\infty}^{\infty} x[k]h[3-k]=x[3]$$

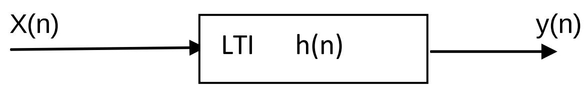
We can use this signal representation to derive an LTI system representation as follows. The response of an LTI system to a unit pulse input, x[n] = u[n], is given the special notation y[n] = h[n]. Then by time invariance, the response to a k—shifted unit pulse, u[n] = δ[n — k] = h[n — k]. Furthermore, by linearity, the response to a linear combination of shifted unit pulses is the linear combination of the responses to the shifted unit pulses. That is, the response to x[n], as written above, is

$$y[n]= \sum_{k=-\infty}^{\infty} x[k]\, h[n - k]$$

Thus have arrived at the convolution sum representation for LTI systems. The convolution representation follows directly from linearity and time invariance An alternate expression for the convolution sum is obtained by changing the summation variable from k to 1 = n - k:

$$y[n]= \sum_{k=-\infty}^{\infty} h[l]\, x[n - l]$$

It is clear from the convolution representation that if the unit-pulse response of an LTI system is known, then we can compute the response to any other input signal by evaluating the convolution sum. Indeed, we specifically label LTI systems with the unit-pulse response in drawing block diagrams, as shown below



Example: convolve the following signals using matrix method x(n)={ 1 1 2 2}, h(n)={1 2 3 4} Ans: y(n)={1 3 7 12 14 14 8}

Properties of Convolution – Interconnections of DT LTI Systems

Convolution of two signals given by

$$y[n] = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For any n, the value of y[n] in general depends on all values of the signals x[n] and h[n], y[n] = x[n]* h[n], for example, y[2] = x[2]* h[2] .

- Commutativity: Convolution is commutative. That is, x(n) * h(n) = h(n) * x(n)

$$\sum_{k-\infty}^{\infty} x[k]\, h[n-k]= \sum_{k-\infty}^{\infty} h[k]\, x[n-k] \text{ , for all n}$$

$$(x * h)[n]= \sum_{k=-\infty}^{\infty} x[k]h[n-k]= \sum_{q=\infty}^{\infty} x[n-q]h[q]$$

$$= \sum_{q=-\infty}^{\infty} h[q]x[n-q]= h(n)* x(n)$$

Using this result, there are two different ways to describe in words the role of the unit-pulse response values in the input-output behavior of an LTI system. The value of h[n – k] determineshow the n^{th} value of the output signal depends on the k^{th} value of the input signal. Or, the value of h[q] determines how the value of y[n] depends on the value of x[n – q].

- Associativity:Convolution is associative. That is,

$$(x * h1 * h2))[n] = ((x * h1) * h_2)[n]$$

- Distributivity: Convolution is distributive (with respect to addition). That is, $(x * (h1 + h_2))[n] = (x * h_i)[n] + (x * h_2)[n]$

For any constant b,((bx)* h)[n] = b (x* h)[n]

- Shifting Property:This is simply a restatement of the time-invariance property. For any integer n_o , if $i[n] = x[n — n_o]$, then

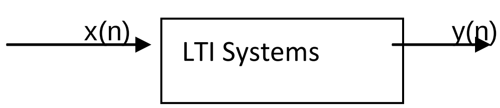
$$h[n] = (x * h)[n — n_o]$$

- Identity: It is worth noting that the "star" operation has the unit pulse as an

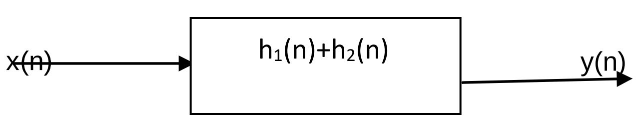
identity element. Namely,
(x* S)[n] = x[n]

This can be interpreted in system-theoretic terms as the fact that the identity system, y[n] = x[n] has the unit-pulse response h[n] = δ[n] .Also we can write (δ* δ)[n]= g[n]. The unit pulse is the unit-pulse response of the system whose unit-pulse response is a unit pulse.

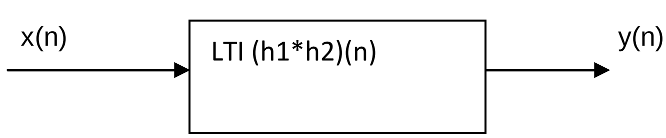
These algebraic properties of the mathematical operation of convolution lead directly to methods for describing the input-output behavior of interconnections of LTI systems. For example,



has the same input-output behavior as the system



both have the same input-output behavior as the system



Transfer Function and Impulse Response Sequence

The transfer function for the continuous-time system relates the Z transform of the continuous-time output to that of the continuous-time input. For discrete-time systems, the transfer function relates the z- transform of the output at the sample instance to that of the sampled input. Consider a linear time- invariant discrete-time system characterized by the following linear difference equation:

$$\begin{aligned} y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) \\ = b_0 u(k) + b_1 u(k-1) + b_2 u(k-2) + \cdots + b_n u(k-n) \end{aligned}$$

where u(k) and y(k) are the system input and output, respectively, at the k^{th} sample instances. If we take the z-transform and by using the time shift property of the z-transform, we obtain

$$\begin{aligned} Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_n z^{-n} Y(z) \\ = b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \cdots + b_n z^{-n} U(z) \end{aligned}$$

or

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}\right) \cdot Y(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}\right) \cdot U(z)$$

which can be written as

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \cdot U(z) = G(z) \cdot U(z)$$

where

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

Consider the response of the linear discrete-time system described by Equation, initially atrest (y(k) = 0, k < 0), when the input u(k) is the Kronecker delta function δ0(k), i.e.

$$u(k) = \delta_0(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

Since

$$U(z) = \mathsf{Z}\left[u(k)\right] = \mathsf{Z}\left[\delta_0(k)\right] = 1$$

then

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} = G(z)$$

Thus, G(z) is the z-transform of the response of the system to the Kronecker delta function input. The function G(z) is called the transfer function of the discrete-time system. In the above derivation, the role of the Kronecker delta function in discrete-time system is similar to that of the unit impulse function (the Dirac delta function) in continuous-time systems.The inverse transform of G(z) as given by Eq.

$$g(k) = \mathsf{Z}^{-1}\left[G(z)\right] = \mathsf{Z}^{-1}\left[\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}\right]$$

is called the impulse response function (sequence).The system described by the difference equation

$$\begin{aligned} &y(k+n) + a_1 y(k+n-1) + a_2 y(k+n-2) + \cdots + a_n y(k) \\ &\qquad = b_0 u(k+n) + b_1 u(k+n-1) + b_2 u(k+n-2) + \cdots + b_n u(k) \end{aligned}$$

where the system is initially at rest (y(k) = 0, k < 0) and the input u(k) = 0, for k < 0, can be represented by the transfer function G(z).

Example Consider the difference equationy(k + 2) + a y(k +1) + a y(k) = b u(k + 2) + b u(k +1) + b u(k). Assuming that the system is initially at rest and u(k) = 0 for k < 0, find the transfer function.

Solution:

The z-transform of the difference equation is

$$\begin{aligned} &\left[z^2 Y(z) - z^2 y(0) - z \cdot y(1)\right] + a_1\left[zY(z) - z \cdot y(0)\right] + a_2 Y(z) \\ &\qquad = b_0\left[z^2 U(z) - z^2 u(0) - z \cdot u(1)\right] + b_1\left[zU(z) - z \cdot u(0)\right] + b_2 U(z) \end{aligned}$$

Collect common terms

$$\begin{aligned} &\left(z^2 + a_1 z + a_2\right) \cdot Y(z) = \left(b_0 z^2 + b_1 z + b_2\right) \cdot U(z) \\ &\qquad + z^2\left[y(0) - b_0 u(0)\right] + z\left[y(1) + a_1 y(0) - b_0 u(1) + b_1 u(0)\right] \end{aligned}$$

Hence

$$Y(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \cdot U(z) + \frac{\left[y(0) - b_0 u(0)\right]z^2 + \left[y(1) + a_1 y(0) - b_0 u(1) + b_1 u(0)\right]z}{z^2 + a_1 z + a_2}$$

To determine the initial conditions y(0) and y(1), we substitute k = −2 into the original difference equation and obtain

$$y(0) + a_1 y(-1) + a_2 y(-2) = b_0 u(0) + b_1 u(-1) + b_2 u(-2),$$

which implies

$$y(0) = b_0 \cdot u(0)$$

By substitute k = −1 into the original difference equation and obtain

$$y(1) + a_1 y(0) + a_2 y(-1) = b_0 u(1) + b_1 u(0) + b_2 u(-1),$$

which implies y(1) =-a y(0)+ b u(1)+ b u(0)

By substituting Equations we get

$$Y(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \cdot U(z)$$

Hence, if both y(k) and u(k) are zero for k < 0, then the system's input and output are related by the above Equation.The transfer function G(z) = Y(z) / U(z) can be written as

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$

The above Equation is the same transfer function for the system described by the difference equation y(k) + a y(k − 1) + a y(k − 2) = b u(k) + b u(k − 1) + b u(k −2).

Causality and Stability of LTI Systems

A DT system is said to be a causal if the output of the system at any time depends only on the present input, past input but does not depend on future input and output

Ex: y(n)=x(n), x(n-1), x(n-2)….

A system is said to be stable if and only if every bounded input produces a bounded output condition for stability is given by

$$\sum_{-\infty}^{\infty} |h\, n| < \infty$$

Computation of Impulse response and Transfer function using Z Transform.
Example: Assuming that the system is initially at rest, find the impulse response of the following discrete time system:

y(k + 3) = 2u(k + 3) − u(k + 2) + 4u(k +1) + u(k). Find the transfer function
Solution:
Transfer function of the system can be written as

$$G(z) = \frac{2z^3 - z^2 + 4z + 1}{z^3} = 2 - z^{-1} + 4z^{-2} + z^{-3}$$

The impulse response of the system with zero initial condition is then the inverse z-transform of the pulse transfer function,

$$g(k) = \mathrm{Z}^{-1}\left[G(z)\right] = \mathrm{Z}^{-1}\left[2 - z^{-1} + 4z^{-2} + z^{-3}\right] = 2\cdot\delta_0(k) - \delta_0(k-1) + 4\cdot\delta_0(k-2) + \delta_0(k-3)$$

Hence g(0) = 2, g(1) = −1, g(2) = 4, g(3) = 1, g(k) = 0, for k >3

Frequency Response of Discrete-Time Systems

In order for systems to possess a steady-state response to a sinusoidal input, in must be stable (all the poles of the transfer function must lie within the unit circle of the complex z plane). Let the system of interest be defined by

$$G(z) = \frac{Y(z)}{U(z)} = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

where pi are the complex poles of the system. We further assume that the system is stable, i.e. pi <1 for all i.

Let the input to the system be a cosine sequence of radian frequency ω, i.e.

$$u(k) = A\cos(\omega kT) = \frac{A}{2}\left(e^{j\omega kT} + e^{-j\omega kT}\right)$$

The corresponding z-transform of the input sequence is

$$U(z) = \frac{A}{2}\left(\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{-j\omega T}}\right)$$

Substituting the input equations the output Y(z) is given by

$$Y(z) = G(z)\cdot U(z) = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}\cdot\frac{A}{2}\left(\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{-j\omega T}}\right)$$

A partial fraction expansion of the above equation can be written as

$$Y(z) = B\frac{z}{z-e^{j\omega T}} + C\frac{z}{z-e^{-j\omega T}} + \sum_{i=1}^{n} D_i\frac{z}{z-p_i}$$

Each term in the summation on the right hand side of equation yields a time domain sequence of the form Di (pi) k, which if pi <1 will vanish when k gets larger and hence does not contribute to the steady-state response. The coefficients B and C in Eq. can be evaluated by the following formula

$$B = \frac{z - e^{j\omega T}}{z} Y(z)\Big|_{z=e^{j\omega T}} \quad \text{and} \quad C = \frac{z - e^{-j\omega T}}{z} Y(z)\Big|_{z=e^{-j\omega T}}$$

j T

Substituting Y(Z) in the above formula gives

$$B = \frac{z - e^{j\omega T}}{z} Y(z)\Big|_{z=e^{j\omega T}} = \frac{A}{2}\left[1 + \frac{z - e^{j\omega T}}{z - e^{-j\omega T}}\right] G(z)\Big|_{z=e^{j\omega T}} = \frac{A}{2} G\left(e^{j\omega T}\right)$$

$$C = \frac{z - e^{-j\omega T}}{z} Y(z)\Big|_{z=e^{-j\omega T}} = \frac{A}{2}\left[\frac{z - e^{-j\omega T}}{z - e^{j\omega T}} + 1\right] G(z)\Big|_{z=e^{-j\omega T}} = \frac{A}{2} G\left(e^{-j\omega T}\right)$$

Thus the steady state response YSS(s) is

$$Y_{SS}(z) = \frac{A}{2}\left[G\left(e^{j\omega T}\right)\frac{z}{z - e^{j\omega T}} + G\left(e^{-j\omega T}\right)\frac{z}{z - e^{-j\omega T}}\right]$$

Since G(z) is a rational function of the complex variable z, G(e $^{j\omega T}$) is a complex number that can be written in polar forma as

$$G\left(e^{j\omega T}\right) = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{j\angle G\left(e^{j\omega T}\right)} = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{j\phi}$$

where φ is the phase angle of the complex number G(e jωT). With similar reasoning, G(e $^{-j\omega T}$) will have the same magnitude and conjugate phase angle as G(e $^{j\omega T}$), i.e.

$$G\left(e^{-j\omega T}\right) = \left|G\left(e^{-j\omega T}\right)\right| \cdot e^{j\angle G\left(e^{-j\omega T}\right)} = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{-j\phi}$$

Substituting the values, the steady-state response can be written as

$$Y_{SS}(z) = \frac{A}{2} \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \left[e^{j\phi}\frac{z}{z - e^{j\omega T}} + e^{-j\phi}\frac{z}{z - e^{-j\omega T}}\right]$$

Taking inverse z-transform of the above equation, we can obtain the time sequence of the steady state sinusoidal response to be

$$y(k) = \frac{A}{2} \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \left[e^{j\phi}\left(e^{j\omega T}\right)^k + e^{-j\phi}\left(e^{-j\omega T}\right)^k\right] = A \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \frac{1}{2}\left(e^{j(\omega kT+\phi)} + e^{-j(\omega kT+\phi)}\right)$$

Using the Euler identity, the above equation can be further simplified and the steady-state sinusoidal response is

$$y(k) = A \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \cos(\omega kT + \phi), \text{ where } \phi = \angle G\left(e^{j\omega T}\right).$$

From the above Equationwe see that, similar to the continuous-time case, the steady-state response of the system G(z) to a sinusoidal input is still sinusoidal with the same frequency but scaled in amplitude and shifted in phase. The amplitude of the steady-state response is scaled by a factor of G|$e^{jwt|}$,which will be referred to as the system gain associated with G(z) at frequency ω . The complex function of ω, G(e $^{j\omega T}$), is called the frequency response function of the system G(z). The frequency response function of a system can be obtained by replacing the z-transform complex variable z with e $^{j\omega T}$, i.e.

$$G\left(e^{j\omega T}\right) = G(z)\big|_{z=e^{j\omega T}} = G\left(\cos(\omega T) + j\sin(\omega T)\right).$$

As in the continuous-time case, we are usually interested in the magnitude and phasecharacteristics of this function as a function of frequency. It is interesting to note that the DC gain of the system corresponds to the magnitude of the frequency response function at ω = 0,

DC Gain= $G\left(e^{j\omega T}\right)\big|_{\omega=0} = G(z)\big|_{z=1} = G(1)$

This is slightly different from the continuous-time case where the DC gain is evaluated by substituting the Laplace variable s by 0.

Example: Find the frequency response for the discrete-time system described by the following difference equation:

y(k) = e^{-2T} y(k −1) + u(k), where T =π/5

Solution:

The impulse transfer function of the system can be found by taking the z-transform of thedifference equation and assuming zero initial conditions

$$Y(z) = e^{-2T} z^{-1} Y(z) + U(z)$$

which implies

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 - e^{-2T} z^{-1}} = \frac{z}{z - e^{-2T}}$$

The frequency response of the system is

$$G(e^{j\omega T}) = \frac{e^{j\omega T}}{e^{j\omega T} - e^{-2T}}$$

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SCHOOL OF ELECTRICAL AND ELECTRONICS

DEPARTMENT OF ELECTRONICS AND COMMMUNICATION ENGINEERING

UNIT - V
SIGNALS AND SYSTEMS – SECA1301

V.REAL TIME APPLICATIONS OF SIGNALS AND SYSTEMS

Signal modelling

- **Modelling of signal is basically mathematical representation of signal.**
- **Fourier series, Fourier transform are kind of signal models.**

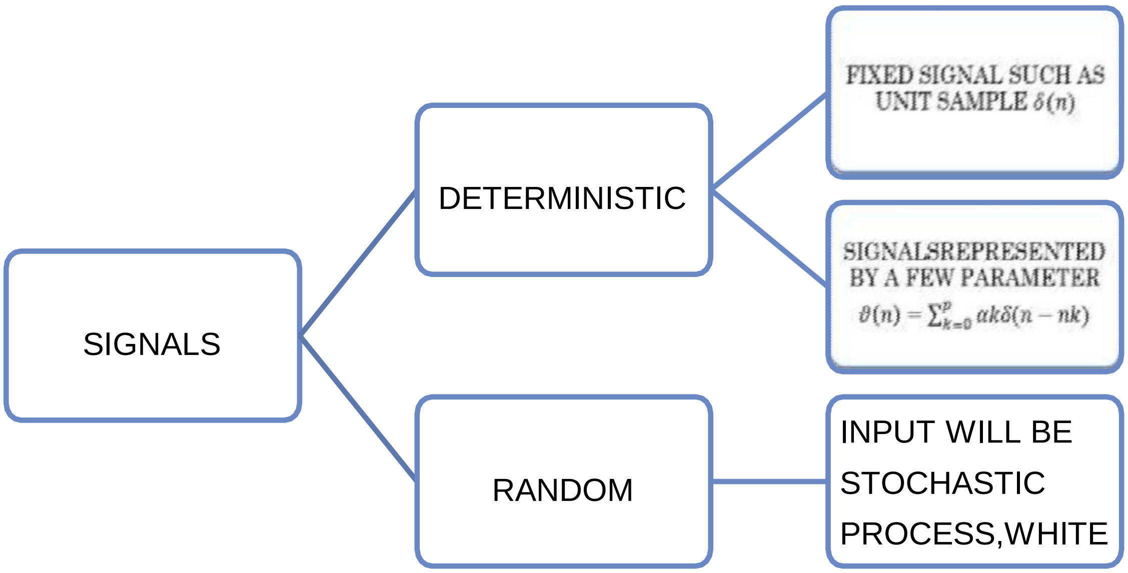


Figure 1: Signal classification

Speech Signal

Time domain

Energy

The amplitude of the speech signal varies over time due to the type of speech sound and to supra segmental factors. The short-time energy is a convenient way to represent these amplitude variations. For a digital signal

$$E = \sum_{n} [s(n)w(n)]^2$$

where w(n) is a window function, which is nonzero over only the analysis time interval. That is, w(n) selects the interval to be analyzed. (For energy determination, the shape of w(n) is not critical j it is often a rectangular window, i.e., 1 over its nonzero extent. For spectral analysis (we shall note that the shape of wen is important.) Because of the squaring operation, the energy can be sensitive to large signal levels, and the computation can produce overflow on short word length computers.

For analog signals, it is more convenient to measure magnitude, by means of a rectifier and low pass filter. Note that the impulse response of the low pass filter corresponds to the window function w(n) . Energy or magnitude provides a primary basis for separating speech from silence and, because voiced sounds are generally higher in amplitude than unvoiced, for distinguishing between voiced and unvoiced intervals. They are also useful for delineating syllables and for locating the syllable nucleus.

Zero Crossings

It is well known that infinitely clipped speech is intelligible (although distorted), so there must be useful information in the zero-crossings of the signal. The zero-crossing rate (ZCR) is obtained by counting the number of times the signal changes sign during a fixed-length analysis interval. This rate gives a very rough measure of the frequency content of the signal, as may be appreciated by noting that a sinusoid of frequency F gives an average ZCR of 2F sec- 1 ZCR is sometimes used after broad bandpass filters for a rough formant estimation, but the fact that the frequency ranges of the formants overlap limits the accuracy of these estimates. The ZCR also helps distinguish between speech and non speech and between voiced and unvoiced sounds (the mean ZCR for voiced sounds is about 1500 sec- 1 and for unvoiced sounds 5000 sec- 1, but the distributions overlap (3)).

Pitch Period Estimation

Measurement of the pitch period (or equivalently the fundamental frequency) of voiced speech and separation of voiced from unvoiced intervals are problems for which many techniques have been devised. Some of these techniques are oriented toward the time domain, and these will be described here; others are frequency-domain oriented.

Autocorrelation

The short-time autocorrelation of the signal,

$$\varnothing(k) = \sum_{n} s(n)s(n+k), \quad k=1,2,3,\ldots$$

peaks when K is equal to the pitch period, even if the signal is only approximately periodic, and this is the basis for autocorrelation methods. But the autocorrelation function also contains other information, enough to represent the entire spectrum. Autocorrelation peaks due to formant structure can be as large as peaks due to approximate periodicity. Therefore, before autocorrelating, it is useful to lowpass filter the speech (to remove the higher formants) and/or to flatten the spectrum (to remove formant information). Spectral flattening can be done by center-clipping (or linear prediction inverse filtering.

FREQUENCY DOMAIN METHODS

The representation of signals as the sum of sinusoids or complex exponentials often leads to convenient solutions and insight into physical phenomena. For linear systems, (such as our model of speech production), it is convenient to determine the system response to such a representation. A spectral representation often displays characteristics not evident in the waveform, and we know that a kind of spectral analysis is performed in the ear. For these reasons, frequency domain methods are important in speech analysis.

It is with frequency-domain methods, in which the shape of the spectrum is relevant, that the shape of the short-time window function is important. Windowing is multiplication of the waveform and window function, which appears in the frequency-domain as convolution of their Fourier transforms. Therefore, a narrow main lobe of the window's transform (to minimize "smearing" of the spectrum) and low side lobes (to minimize "leakage" from other parts of the spectrum) are desirable. In these respects, the rectangular window is inferior to smooth windows such as the Hann, Hamming, or Parzen windows (3-6). In analog filter bank spectral analysis , the window function is determined by the shape of the filter bandpass characteristics (a wider bandwidth yields sharper time resolution). Additional time-averaging is usually introduced by the smoothing of the

envelope detector following each filter.

Music signal Analysis and modelling

Pitch

Most musical instruments—including string-based instrumentssuch as guitars, violins,and pianos,as well as instruments based on vibrating air columns such as flutes, clarinets, and trumpets—are explicitly constructed to allow performers to produce sounds with easily controlled, locally stable funda- mental periods. Such a signal is well described as a harmonic series of sinusoids at multiples of a fundamental frequency, and results in the percept of a musical note (a single perceived event) at a clearly defined pitch in the mind of the listener. With the exception of unpitched instruments like drums, and a few inharmonic instruments such as bells, the periodicity of individual musical notes is rarely ambiguous, and thus equating the perceived pitch with fundamental frequency is common.

Harmony

While sequences of pitches create melodies—the "tune" of a music, and the only part reproducible by a monophonic instrument such as the voice—another essential aspect of much music is harmony, the simultaneous presentation of notes at different pitches. Different combinations of notes result in different musical colors or "chords," which remain recognizable regardless of the instrument used to play them. Consonant harmonies (those that sound "pleasant") tend to involve pitches with simple frequency ratios, indicating many shared harmonics. Fig. 2 shows middle C (262 Hz), E (330 Hz), and G (392 Hz) played on a piano; these three notes together form a C Major triad, a common harmonic unit in western music. The figure shows both the spectrogram and the chroma representation. The ubiquity of simultaneous pitches, with coincident or near-coincident harmonics, is a major challenge in the automatic analysis of music audio: note that the chord in Fig. is an unusually easy case to visualize thanks to its simplicity and long duration, and the absence of vibrato in piano notes.

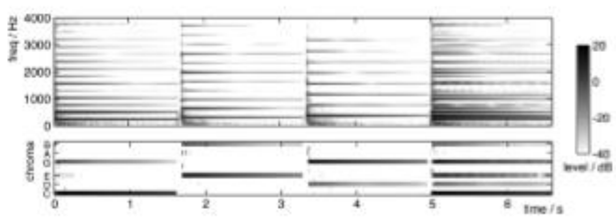


Figure 2. Frequency Spectrum

Medical uses of signal analysis and modelling

Biomedical signal analysis allows us to extract meaningful information from biological processes, thus enabling the assessment, characterization, and understanding of their originating mechanisms. Biomedical signals, however, are nonstationary and have statistics that change over time. As a result, conventional frequency-domain signal analyses have their limitations and a more powerful analysis technique is needed, particularly one capable of characterizing the changes in spectral content over time. In this article, we propose one such spectrotemporal representation—the modulation spectrogram. We start by presenting the theoretical foundation behind the technique and compare three of the most utilized approaches to calculate the spectrotemporal representation, namely the short-time Fourier transform, the continuous wavelet transform, and the Hilbert transform. An open-source amplitude modulation analysis toolbox is presented to allow the reader to explore amplitude modulation analysis of different biomedical signals. Lastly, to illustrate the advantages of the

modulation spectrum analysis over conventional frequency-domain tools, several biomedical applications are described, ranging from detecting breathing rate from ECG signal, to improved Alzheimer's disease diagnosis using novel features extracted from electroencephalograms, to artifact removal of wearable electrocardiograms. It is hoped that this article and its companion open-source toolkit will allow readers to quickly witness the advantages of the described spectrotemporal representation and explore novel biomedical signal analysis applications.

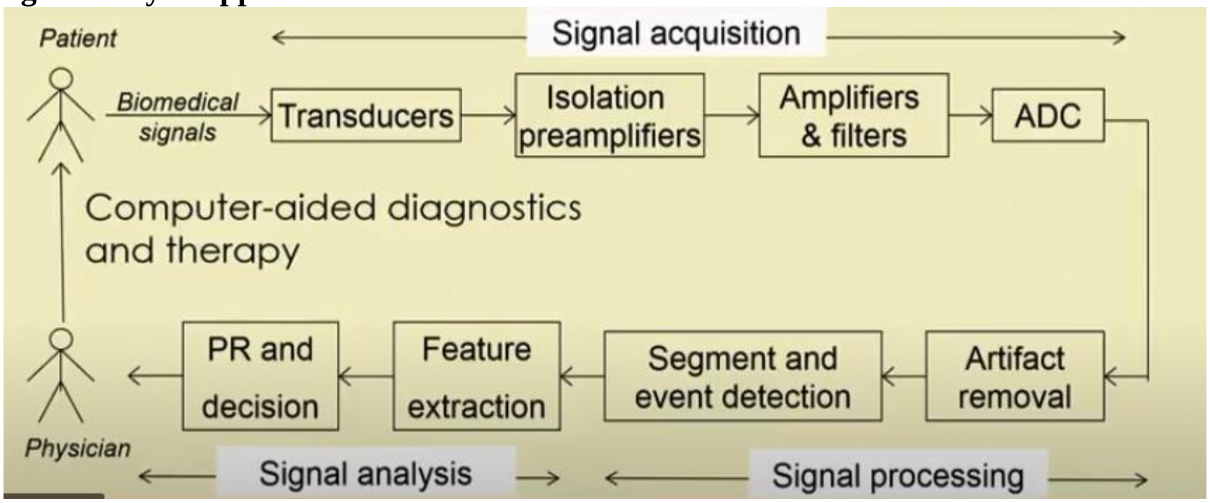


Figure 3. Medical Signal Processing and Analysis

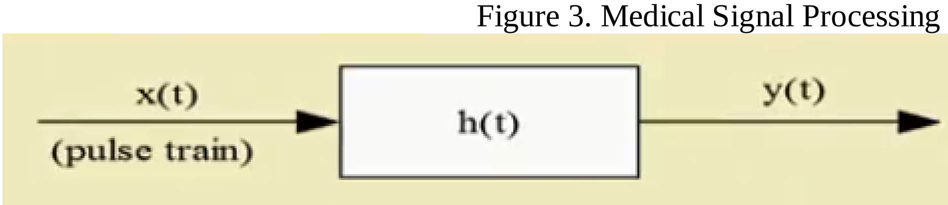


Figure 4. Pulse train

Motor Unit Firing Pattern

$$y(t) = \int_0^t h(t-\tau)x(\tau)d\tau ;$$

where y(t) is the SMUAP train;
h(t) is the SMUAP impulse response and
x(t) is the pulse train

Applications of Fourier Transform

- **The Fourier transform has many applications in any field of physical science that uses sinusoidal signals, such as engineering, physics, applied mathematics, and chemistry, will make use of Fourier series and Fourier transforms**

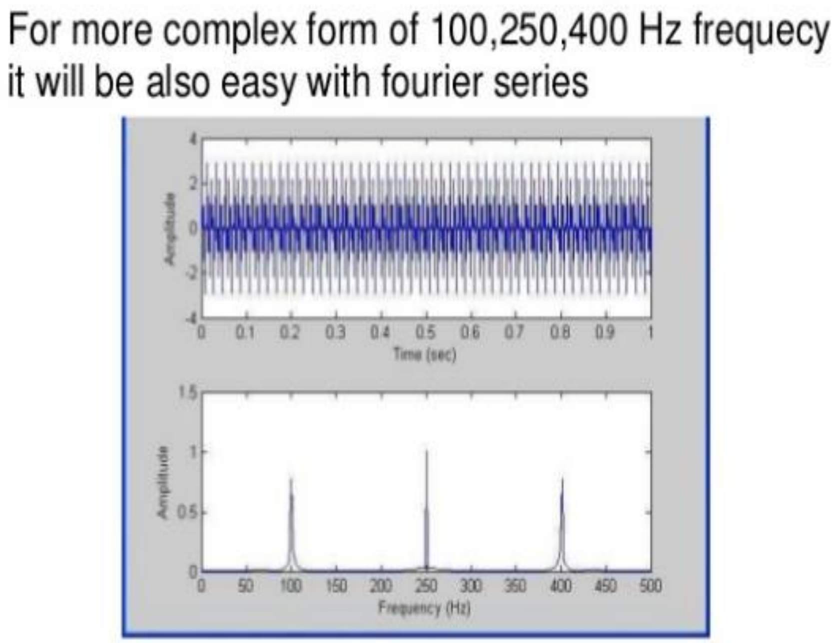


Figure No 5 FT of a signal with multiple frequencies

Telecommunication

- **It's a obvious for that there would be digital telecom without Nyquist-Shannon theorem and Fourier Transform. Both are extremely important. Fourier transform has greatly improved the way we are sending/collecting data. Note that mp3 file format uses transform. Transforms are used in file compression: for instance cosinus transform is also used in jpeg file compression. There are numerous applications in telecoms, and therefore I decided not to dwell on them and go further.**

Automotive

- **A car is going to be have an engine test stand. A computer with multiple microphone is going to measure voice of the engine the car produces. The software analyses it and gives output that shows us how devastated engine is and what is its real distance. This is comes in handy.**

- **A system of Vibro-acoustic signal analysis (acceleration of the unsprung mass of passenger's car suspension actuated to vibration by harmonic, kinematic vibration) that analized the suspension and adjusted it automatically to the current road condition.**

- **Adaptive cruise control. The cruise control is a systems that controls your speed car. There are many improvements to this system. One is that it uses FFT while controlling speed.**

Voice recognition system. Driving a car requires a lot of attention. Sometimes it's difficult to put the hands off the steering wheel and change the radio station. If we have to push one button, it is far less problematic than choosing a street number in a GPS receiver. The

automotive companies are working on a voice recognition systems

The Fourier transform of the square wave generates a frequency spectrum that presents the magnitude of the harmonics that make up the square wave (the phase is also generated, but is typically of less concern and therefore is often not plotted).

The Fourier transform can also be used to analyze non-periodic functions such as transients (e.g. impulses) and random functions. With the advent of the modern computer the Fourier transform is almost always computed using the Fast Fourier Transform (FFT) computer algorithm in combination with a window function.

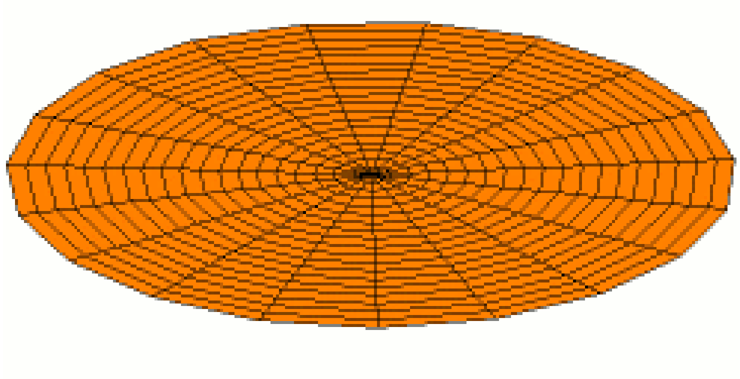


Figure 6. Magnitude plot

Hearing devices

- **Beltone hearing device**
- **You have from 9 up to 16 directed microphones, it minimizes and filters out the noise of a street. It is self adjusting – this means you get different voice intensification /amplification in different environments. This means that no mater where you are – in laboratory or in a public place, you can always talk and hear the person you are talking with. If you have a better one device – you can talk with your phone despite having the disability. This allows people for normality in their lives. Everything is packed in a small device, which you can hardly see.**
- **Voice recognition**
- **There are attempts to perform automatic voice recognition system in customer support. The goal is to eliminate the extremely boring procedure in veryfing the right user asking questions like: what is mother's maiden name, what is your postal code etc. The system may verify the user by analyzing its voice using Fourier Transform. It is right research direction in my opinion and will greatly facilitate our live.**
- **Military**
- **Submarine requires extremely reliable communication**

- **The main point of Fourier Transform in military applications is: perform reliable and safe communication. Using fourier transform you can decode your message in a special way.**

- **cosinus tranform is used in jpeg file (->data compression). But the point is to sent a message under the noise level.**

Medical Imaging

Selected methods of image processing have been applied to expert system supporting the process of identifying medical images. Fourier transform is well known tool for many applications in the processing of images in many fields of science and technology, also in medicine. In this case the image processing consists in spatial frequencies analysis of Fourier transforms of medical images. Skin lesions are studied on the base of their images and it seems that Fourier transformation is the right toll for such research. The distributions of selected colors in Fourier transform images are studied.

NOISE REMOVAL IN medical IMAGES

The use of the FT and its inverse to remove unwarranted information from an image. (a) An image obtained by adding the sine wave and chest radiography images together with its equivalent Fourier spectrum in (b). The unwanted interference caused by the sinusoidal brightness pattern can be removed by editing the spatial frequency information as shown by the blackened areas in (c). The inverse FT then recovers the original chest image largely undistorted as shown in (d). Further refinement of the editing process would allow complete restoration of the image quality, in theory.

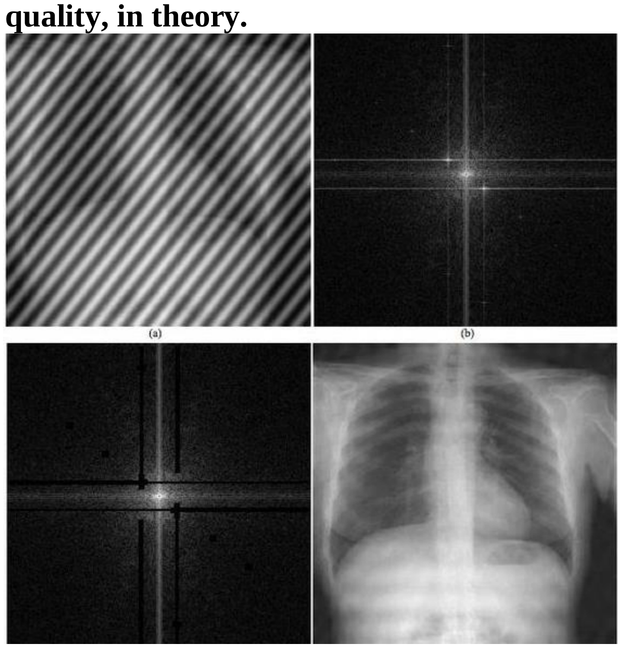


Figure 7. Noise removal using FT

MEDICAL SIGNALS

The EEG rhythms capture the nonlinear complex dynamic behavior of the brain system and the nonstationary nature of the EEG signals. This method analyzes common frequency components in multichannel EEG recordings, using the filter bank signal processing. The mean frequency (MF) and RMS bandwidth of the signal are estimated by applying Fourier-

transform-based filter bank processing on the EEG rhythms, which we refer intrinsic band functions, inherently present in the EEG signals. The MF and RMS bandwidth estimates, for the different classes (e.g., ictal and seizure-free, open eyes and closed eyes, inter-ictal and ictal, healthy volunteers and epileptic patients, inter-ictal epileptogenic and opposite to epileptogenic zone) of EEG recordings, are statistically different and hence used to distinguish and classify the two classes of signals using a least-squares support vector machine classifier.

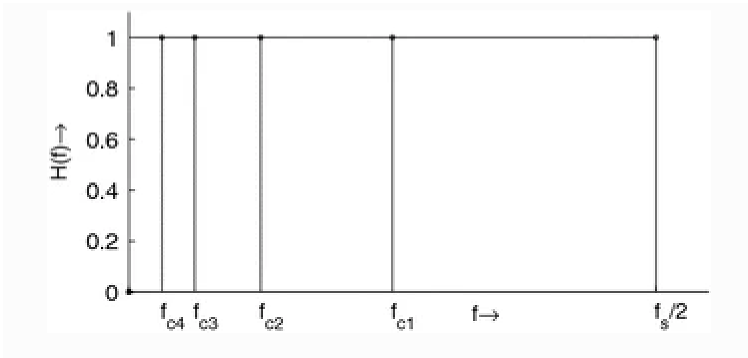


Figure 7. Fourier transform of EEG signal

- **In recent years, detection of cardiovascular abnormalities in patients can be achieved by using electrocardiogram (ECG) recording. Initially, an effective FFT is used to extract the feature points in ECG signals, such as PQRST wave's amplitude and wave function and then the proposed multi-objective genetic algorithm is used to classify the abnormality of heart patient. Basically, the ECG behaviour depends on various factors such as age, physical condition of patients and the surrounding environment.**

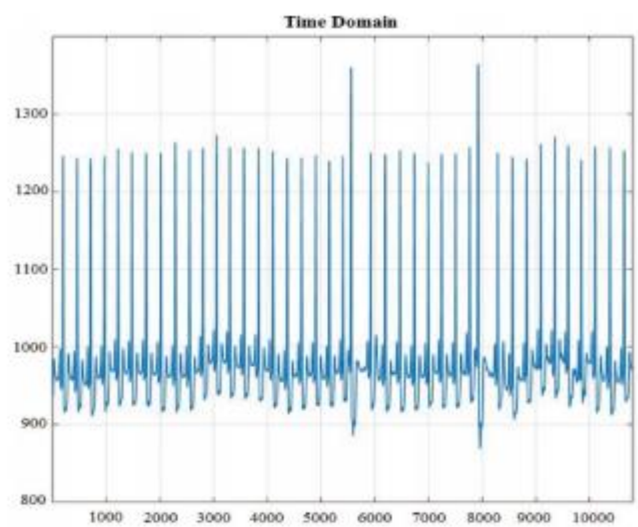


Figure 8 ECG wave

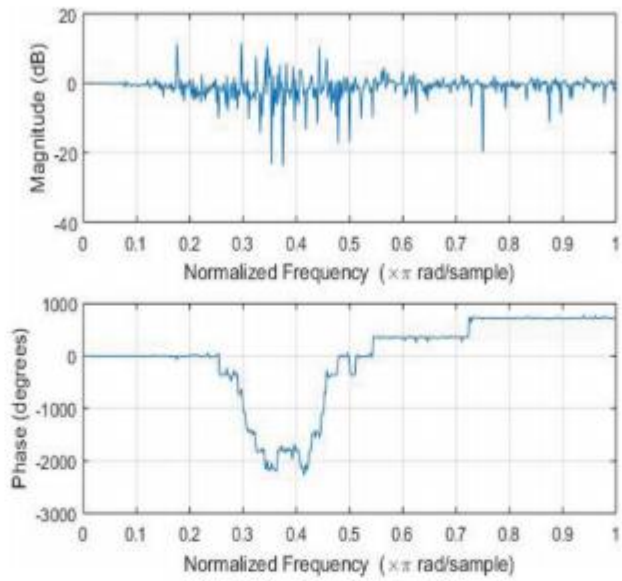


Figure 9 Magnitude and Phase spectrum

The Speech Communication Process

Speech is a complex sound produced by the human vocal apparatus, which consists of organs primarily used for breathing and eating: the lungs, trachea, larynx, throat, mouth, and nose. The source of energy for the production of these sounds is the reservoir of air in the lungs. Sound is generated in two ways. If air is forced through the larynx with the vocal folds appropriately positioned and tensioned, it sets them into oscillation, so that they release puffs of air in a quasi-periodic fashion at rates of about 80 to 200 Hz for male speakers, and at faster rates for women and children. This glottal source is rich in harmonics, and it excites the acoustic resonances of the vocal tract above the larynx, which filter the sound. These sharp resonances, called formants, are determined by the shape of the throat, mouth, and, if the velum (soft palate) is open, the nasal cavity. It is through manipulation of the vocal tract shape by the articulators (tongue, jaw, lips, and velum) that we control the formant frequencies that differentiate the various voiced speech sounds. These are the vowels, nasal consonants, liquids (/rl and Ill), and glides (/wl and Iy/). Only the lowest three or four formants, up to about 4 kHz, are perceptually significant. The other sound source used in speech is turbulent noise, produced by forcing air through some constriction (such as between the tongue and teeth in th-sounds) or by an abrupt release of pressure built up at some point of closure in the vocal tract (such as behind the lips in aIp/).

The spectral peaks associated with these fricated sounds generally lie between 2 and 8 kHz and are primarily determined by the position and the shape of the constriction. Some sounds, the voiced fricatives, such as Izl and lvi, have both voiced and turbulent excitation. Distinguishable differences in the voice signal can be produced by quite small changes in the way the vocal tract is manipulated, so a very large number of sounds can be produced. For the communication of language, however, only a restricted number of sounds, or more accurately, sound classes, are used. Words in English are made up of approximately 40 of these phonemes, which correspond roughly to the pronunciation symbols in a dictionary. Since there are relatively few of these elemental sound units, many speech recognition systems choose phonemes or phoneme-like units as the units of recognition. This concept of elemental units does not reduce speech recognition to the sequential recognition of 40 or so fixed pat terns. The acoustic realizations of phonemes in real speech are drastically different from their characteristics in isolated environments. When phonemes are strung together into words, the acoustic characteristics of successive phonemes become overlapped, due to the dynamics of the articulators and the tendency for an articulator to anticipate the position it will next assume. (For example, when you say the It I of the word "tin", the lips are spread apart, but in "twin", they are pursed together. Since the lips are not used for the articulation of the It I , they are free to assume the position for the following phoneme.)

These context effects, along with effects due to linguistic stress, speed of talking, the size of a particular speaker's vocal tract, and variation in the way a speaker says a given word from repetition to repetition make the decoding of the speech signal a nontrivial problem. To be sure, the speech wave does contain much information directly related to the phonemes intended by the speaker, but the details of the coding are complex . The phonemes themselves can be grouped into classes according to the ways they can modify and affect one another when they are in proximity in normal speech. It is convenient to describe these classes in terms of about fifteen binary distinctive features (9-12), most of which have straightforward articulatory and acoustic attributes, such as voiced/unvoiced, nasal/nonnasal, fricated/unfricated, and tongue position: back/front, high/low, etc. Distinctive features have proved to be quite powerful in describing the phonological changes that underlie the contextual effects referred to above. For that reason, some speech recognition systems use them as recognition units or as ways of organizing their recognition of phoneme-sized units.

Analog and Digital Techniques

There is a major division of SSPFE techniques between analog and digital. Analog signal processing techniques are those performed on the (electrical analog of the) speech signal by means of electronic circuitry. This requires investment in special-purpose electronic equipment, which, although frequently modular, is somewhat inflexible, and furthermore, requires periodic calibration and adjustment. Digital signal analysis, on the other hand, can be performed on a general purpose digital computer, a microprocessor, or special purpose digital hardware. When implemented on a computer, such digital instrumentation is very flexible. Programming turns the computer into a custom-designed instrument; the characterh:tics of the instrument may be changed or even redesigned without recourse to a soldering iron or supply cabinet, and there is nothing that can go out of calibration. Furthermore, many of the digital techniques now available are simply not realizable in the analog domain. Perhaps the most notable drawback is that while analog instrumentation is inherently real-time, it is not always possible for digital techniques to be performed in real time. This is counterbalanced by the potentially far greater functionality of digital techniques. The same technological advances that have made computers faster and cheaperhave done the same for digital processing techniques, and we may expect this tendency to accelerate. The flexibility and capability of digital SSPFE methods, together with the increasing speed and decreaSing costs of computers and microprocessors, have established a trend of increaSing importance of digital techniques in all aspects of speech processing. It represents both analog and digital techniques, but the dominance of the digital domain is unavoidable. For a signal to be analyzed digitally, it must first be converted from the form of a continuous signal (an acoustic pressure wave or its electrical analog) to a digital signal. A digital signal is discrete in two ways: it consists of measurements of the speech signal amplitude of discrete, regularly spaced instants of time, and each of these samples has been quantized to a certain number of bits of precision. This transformation from a continuous waveform to a list of digital numbers is performed by an analog-to-digital converter. Nyquist's sampling theorem (see, for example, (3-6)) tells us that in order to capture the information contained in a bandwidth of B Hz, we must sample at a rate of a least 2B samples per second; furthermore, if the signal contains energy at frequencies higher than B Hz, the result will be distorted unless we first lowpass filter the signal at or below B Hz. For example,

an application in which the 0-5 kHz band is of interest may call for lowpassfiltering at or below 5 kHz and sampling at 10 ,000 samples per second, with each sample represented by a 12 bit number.

Time and Frequency Domain Approaches Another fundamental division of SSPFE approaches is between time domain and frequency domain approaches. Time domain approaches deal directly with the waveform of the signal, and these representations are often attractive because of their simplicity of implementation. In working with time domain representations, it must be noted that the shape of the waveform for voiced speech depends on (among other things) the pitch period, so we generally shouldn't pay attention to details of the waveform. Also, the perception of speech is only minimally dependent on the phase of the signal, so phase-independent measures are desirable. Frequency-domain approaches involve (explicitly or implicitly) some form of spectral representation. A spectral representation often displays characteristics not evident in the time domain, and we know that a kind of spectral analysis is performed in the ear. The above observations about pitch and phase have counterparts in the frequency domain. For voiced speech, the spectrum has both a fine and a coarse structure, due to the pitch harmonics and the vocal tract transfer function respectively. We use the magnitude or power spectrum, which does not contain phase information. A time/frequency dichotomy does not account for all SSPFE approaches. Some are hybrids of the two, and for others the concept of "time or frequency" seems ill-defined.

Analysis and Synthesis Of Signals

The task of an automatic speech recognition (ASR) system is to convert a speech signal into a linguistic representation of its content -- to "recognize" the linguistic message contained in the signal. A common aspect of all ASR systems is that they process the speech signal in some way in order to produce a representation of the signal that is better suited to the recognition process. This operation (or set of operations) is the function of the component that can be called "speech signal processing and feature extraction" (SSPFE). What do we mean by "speech signal processing" and by "feature extraction"? "Speech signal processing" refers to the operations we perform on the speech signal (e.g., filtering, digitization, spectral analysis, fundamental frequency estimation, etc.). "Feature extraction" is a pattern recognition term that refers to the characterizing measurements that are performed on a pattern (or signal); these measurements (" features") form the input to the classifier that "recognizes" the pattern. In many cases in speech recognition, it is impossible to separate

these functions; they are one and the same.

Audio/speech processing is an important area of research in Forensic and Defense (speaker recognition) and in search engines (retrieval). In both cases, it is necessary to extract the features which describe the speech signal. Though considerable research is carried out in this area, these techniques are text and language dependent. Extracting the text features is highly complex. Also language dependency is yet another bottleneck in speech processing. Hence major research work aims at developing efficient speaker recognition/speech retrieval technique from the speech samples irrespective of the speech content and language. Major research challenge lies in the selection of appropriate transform for decomposing the signals, statistical features for aggregating the co-efficient and a classifier for identifying the speaker.

In case of defense organizations, forensic departments and biometric enabled industries, speech signals are processed to identify a speaker. On the other hand, search engines like Google and Yahoo need speech signal retrieval from queries. Both these cases involve the following steps: speech signal database, signal preprocessing, outlier removal, feature extraction for speech characterization and decision making. In both cases, Speech characterization necessitates the researcher to understand physics behind the generation of speech signals and the anatomy of vocal tract.

Human vocal system consists of lungs, trachea, vocal folds, epiglottis, tongue, velum and nasal cavity. When there is an urge to speak, the nerve cells receive the commands from brain and based on that the vocal system generates the speech signal. Air from the lungs passes to the larynx through the trachea and hence vibrates the vocal chords. It is the vibration of the vocal chords that result in sound generation. In addition, position of the tongue also alters the voice signal. Human vocal system is as shown in Figure 9.1.

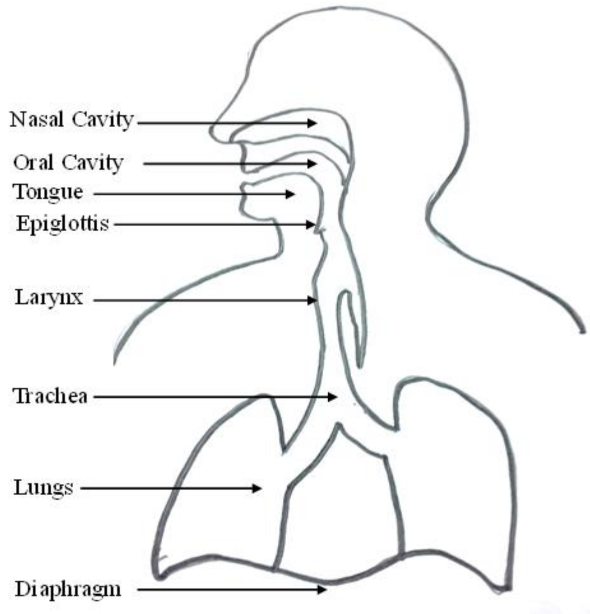


Figure 9.1 Human vocal system

Speech signal consists of voiced sound, unvoiced sound and silence (Figure 9.2). In case of voiced sound, vocal cords are opened and closed periodically. It chops the airflow into periodic pulses. Voiced sound occurs during vowel production. Unvoiced sound occurs during consonants and vocal cords are randomly opened and closed and it results in noisy turbulent signal. Silence occurs when neither voiced nor unvoiced sound is produced. In general, speech signal is defined as a one Dimensional signal s(t) where 't' denotes time and s(t) is the amplitude or intensity at 't'. If 's' and 't' are both finite, then it is called a digital speech signal.

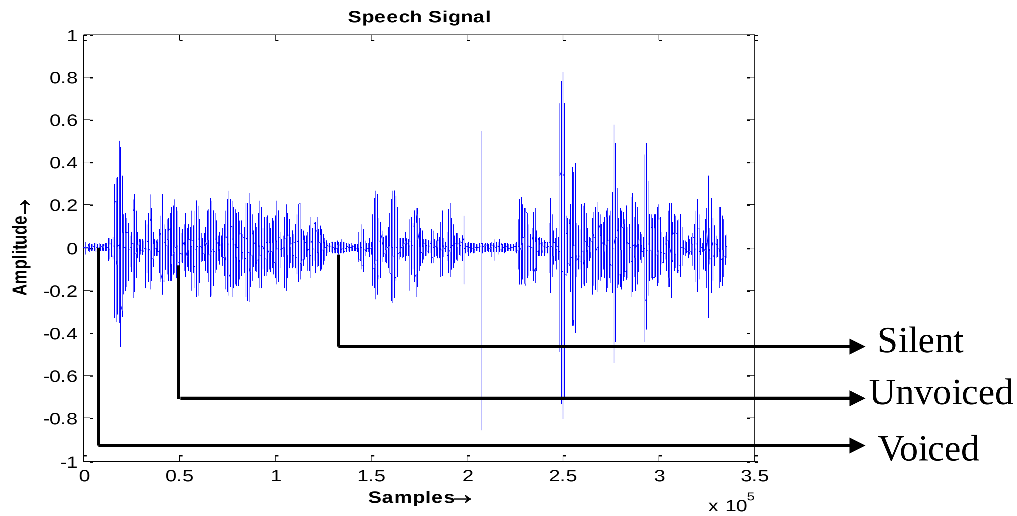


Figure 9.2 Voiced, unvoiced and silent regions in speech signal

Owing to the presence of voiced, unvoiced and silent samples in speech, speech signals are non-stationary in nature. However it can also be viewed as a series of short term (time) periodic signals. Energy (measure of amplitude of the signal) decreases as signal traverses from voiced to silent through unvoiced sound. Due to randomness in unvoiced sound, Zero Crossing Rate (ZCR) of unvoiced sound is higher than that of voiced sound. Autocorrelation is the measure of repetitiveness in the signal while pitch is an indicator of fundamental frequency. Having understood the nature of speech signals, the next task is to develop speaker identification and speech retrieval systems.

Speech Signal Feature Selection and Extraction

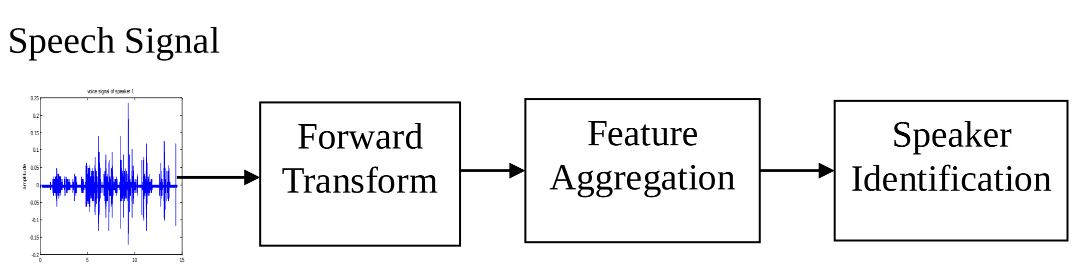


Figure 9.3 Block Diagram of Speaker Recognition System

Speaker Recognition system validates the identity of a person using the features extracted from their voice signals. The two phases of a speaker recognition system is the

feature extraction and speaker identification. In this chapter, various features selected for development of the proposed speaker recognition system, their significance and the various transforms used for extraction of those features are discussed. The block diagram of the proposed system is shown in Figure 9.3.

Feature Selection

Speech waveform is produced by the human speech production system and the structure of their vocal tract.The voluntary movement of the anatomical structures present in our human speech production system generates an acoustic sound pressure wave called the speech waveform. Vocal folds, soft palate, tongue, teeth and lips are the finer anatomical components involved in the human speech production. These finer anatomical components are also called as articulators. Based on the movements and position of the articulators, the modulation occurs in the speech waveform. Larynx plays an important role in speech production as it provides the periodic excitation to the system for speech sound called as voice. Since the human system varies with time, the spectral characteristics of the speech waveform possess non stationary property. The prosodic features involved in the study of voicing are the fundamental period, fundamental frequency and pitch.Speech signal provides glottal information of the speaker, which by proper feature extraction gives the identity of the speaker. Features namely fundamental period, fundamental frequency, pitch and number of peaks for a smaller co articulation of the speakers along with a suitable feature extraction technique is responsible to explore the speaker identity. Significance of prosodic and statistical features for the representation of the speech signals is shown in Table 3.1.

As mentioned in Table 3.1, these features provide a holistic representation of the speech signal. These features can be used for both speaker recognition and speech ranking systems. In this research work, these features are determined on the actual coefficients and also on the transformed coefficients. Transforms are selected based on the appropriateness in describing the speech signal.

Transform Selection

As speech is a non-stationary signal and the information is present in low

frequencies, it necessitates a non-stationary tool that performs multi resolution analysis. Feature extraction on the components from non-stationary analysis is an efficient way to develop speaker recognition system. Significance of various transforms on speech signals is shown in Table 3.2.

Speech Feature Extraction

On the acquired speech signals, both prosodic and statistical features are determined from the transformed and time domain coefficients. These aggregated features are used for developiing automated speaker recognition system and feature based speech ranking system. The tree diagram in Figure 3.2 depicts the various features extracted on speech signals.

Table 3.1 Significance of Prosodic and Statistical Features

Sl. No.	Feature	Formula	Courtesy	Significance
1	Pitch f_0	$f0 = \frac{f_s}{T}$	www.indiana.edu/~acoustic/s302/phonation-chap4.pdf	For an adult male the vocal folds open and close completely 100 times in one second. For an adult female it is 200 times per second, for a child, it may be 300 times per second. Larger structures vibrate more slowly than smaller structures.
3	Energy E	$E = \sum_{n=0}^{N-1} x^2(n)$	iitg.vlab.co.in, (2011). Short Term Time Domain Processing of Speech.	The energy associated with voiced region is large compared to unvoiced region and silence region will not have least energy.
4	Zero Crossing Rate (ZCR)	$ZCR = n\_cross \cdot \frac{f_s}{N}$	www-gth.die.upm.es/partners/sony/Chapter7_b.doc	If the number of zero crossings are more in a given signal, then the signal is changing rapidly and accordingly the signal may contain high frequency information.

5	**Number of Peaks NOP)**	**Size(pks)**	**David GerhardTechnical Report TR-CS 2003-06November, 2003**	**Counting the number of positive peaks per second in the waveform determines the frequency of the waveform.**
6	**Mean**	$\mu = \frac{1}{N}\sum_{i=0}^{N-1} x_i$	**Digital Signal Processing: A Practical Guide for Engineers and Scientists**	**If N is small, the statistical noise in the calculated mean of the signal will be very large. In other words, you do not have access to enough data to properly characterize the signal. The larger the value of N, the smaller the expected error will become.**
7	**Standard Deviation**	$s = \sqrt{\frac{1}{N-1}\sum_{i}}$	**https://en.wikipedia. org/wiki/Standard_ deviation**	**Standard deviation is used to evaluate the differences between speaking styles. The higher the standard deviation, the more lively the voice, and the more pleasant and interesting to listen to**
8	**Skewness**	$s = \frac{E(x-\mu)}{\sigma^3}$	**www.itl.nist.gov/div898/handbook/eda/section3/eda35b.htm**	**Skewness is used to detect the polarity of the speech signal. It also provides vocal tract information and estimate on the glottal source. A positive skewness means the estimated glottal source will have a negative skewness. Higher skewness means higher excitation strength of the vocal tract**
Sl. No.	**Feature**	**Formula**	**Courtesy**	**Significance**
9	**Kurtosis**	$s = \frac{E(x-\mu)}{\sigma^4}$	**Ali Mansour et al (1998), “Kurtosis: Definition and properties”,**	**Kurtosis is negative during a silent period, and it becomes positive during the speech transient.**

			FUSION 98	
10	**Second Order Moment**	$m_2 = E(x-\mu)$	**http://www1.maths.leeds.ac.uk/applied/news.dir/issue2/hos_intro.html**	**Second order statistics is used to estimate the Number of harmonics present in the signal**
11	**Third Order Moment**	$m_3 = E(x-\mu)$	**http://www1.maths.leeds.ac.uk/applied/news.dir/issue2/hos_intro.html**	**Third order moment has the potential to capture nonlinear information and it is more immune to Gaussian noise**
12	**Spectral Flux**	$SF = \frac{}{(N-1}$ $\sum_{n=1}^{N-1}\sum_{k=1}^{M-1}\log(x$ $-\log(x(n-1,k)+\delta)$	**J. King Saud Univ., Vol. 19, Eng. Sci. (1), pp. 95-133, Riyadh (1427H./2006)**	**Music has a higher rate of change in the shape of the spectrum, and goes through more drastic frame-to-frame changes where speech typically has a more constant rate of change.**
13	**Spectral centroid**	$SC = \frac{\sum_k kX(}{\sum_k X(}$	**J. King Saud Univ., Vol. 19, Eng. Sci. (1), pp. 95-133, Riyadh (1427H./2006)**	**Spectral centroid describes the center of frequency at which most of the power in the signal is found. In speech signals the pitch of the audio signal stays in a more narrow range of low values. As a result, music has a higher spectral centroid than speech.**

Table 3.2 Significance of Transforms on Speech Signals

Sl.No.	Transform	Formula	Courtesy	Significance
1	**Stockwell Transform**	$S(\tau, f) = \int_{-\infty}^{\infty} h(t)\frac{\mid f \mid}{\sqrt{2\pi}} e^{-\frac{(\tau-t)^2 f^2}{2}} e^{-i2\pi ft} dt$	**Localization of the complex spectrum: The S Transform**	**S-transform uniquely combines a frequency dependent resolution of the time-frequency space with absolutely referenced local phase information.**
2	**Discrete Wavelet Transform**	$\tilde{x}\psi(a,b) = \frac{1}{\sqrt{a}} \int_{\infty}^{-\infty} x(t)\psi\left(\frac{t-b}{a}\right) dt$	**A Wavelet Tour of Signal Processing: The Sparse Way**	**DWT decomposes the signal into approximation coefficients and detailed coefficients Approximation coefficients correspond to low frequencies and detailed coefficients represent the**

				high frequency info. present in the signal.
3	**Cepstrum**	$C(n)=\sum_{n=0}^{N-1}\log\left(\left[\sum_{n=0}^{N-1}\left\|x(n)e^{-j\frac{2\pi}{N}kn}\right\|\right]\right)e^{j\frac{2\pi}{N}kn}$	**Digital Processing of Speech Signals**	**Cepstral analysis separates the speech signal into component representing excitation source and vocal tract impulse response. So it provides information about pitch and vocal tract configuration.**
4	**Autocorrelation**	$A(k)=\sum_{m=0}^{k-1}x(n)x(n+k)$	**Digital Processing of Speech Signals**	**One would expect exact similarity at a time lag of zero, with increasing dissimilarity as the time lag increases.**

5	**Empirical Mode Decomposition**	$x(t) = \sum_{j=1}^{n} c_j + r_n$	**Hilbert-Huang Transform and Its Applications**	**any complicated data set can be decomposed into a finite and often small number of components. EMD preserves the characteristics of the varying frequency.**

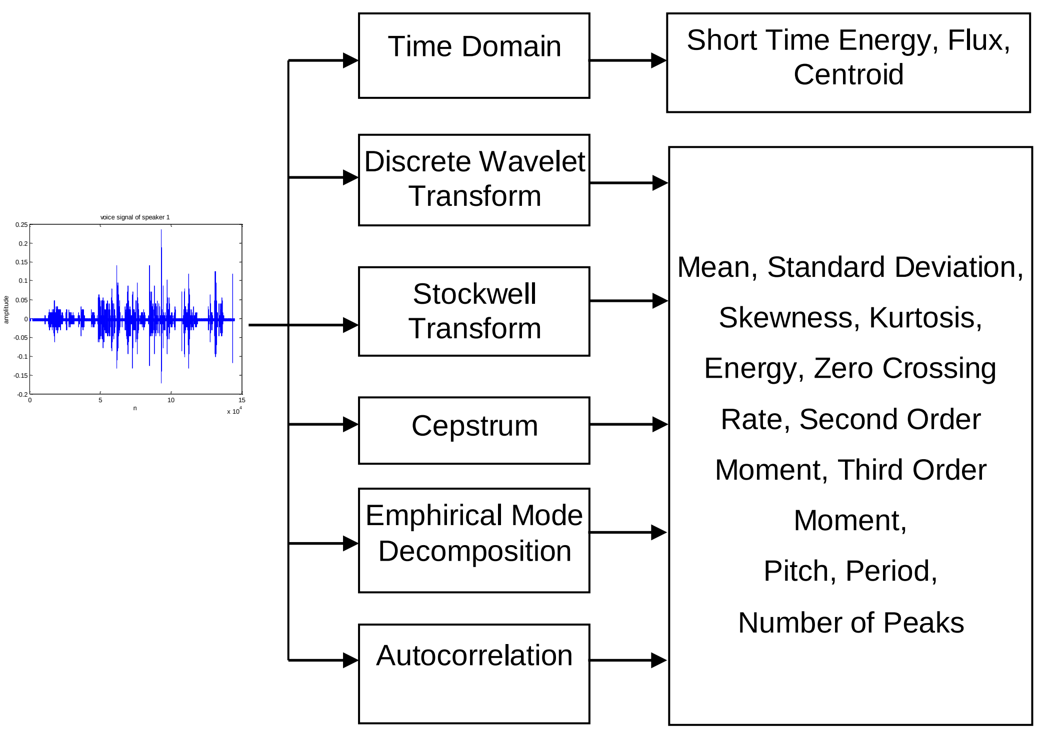


Figure 3.2 Feature Extraction on Speech Signals

Text Independent Speaker Recognition System with Ann

The robustness of any speaker recognition system depends on the appropriate selection of acoustic features, feature extraction technique and the classifier involved in the system. Initially text independent speaker recognition system using stockwell transform features and DWT features is proposed with Back Propagation Network (BPN) as classifier.

The block diagram of ANN based speaker recognition system using stockwell transform features is shown in Figure 5.1. Stockwell Transform is then applied on these signals to obtain amplitude localized to both time and frequency. These amplitudes are then aggregated using the statistical parameters namely mean, variance, Skewness and kurtosis. These features are then used to train and test the ANN based classifier. In the proposed work, Back Propagation Network (BPN) with three hidden layers and one output layer is chosen. The number of neurons in the hidden layer is 10,10,5 respectively and one neuron in the output layer. The learning parameter and the momentum parameters are 0.6 and 0.9 as these are the optimized values. An exemplar is created with mean, variance, skewness and kurtosis as input parameters and speaker code as output parameter. The speakers are coded from 1 to 10. Of the 100 exemplars generated, 50 are used for training and the other set of 50 is used for testing the Network.

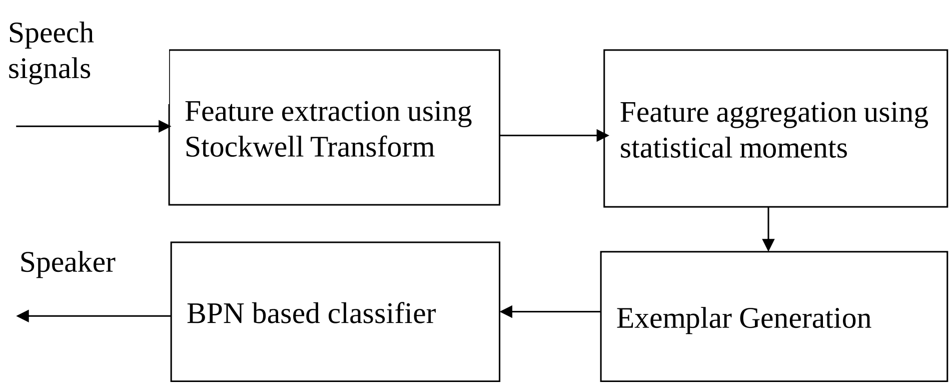


Figure 5.1 Block Diagram of ANN Based Speaker Recognition System using Stockwell Transform Features

Separate set of exemplars are used for training and testing the network. The performance of the proposed technique is measured in terms of relationship between the desired and the actual output.

Feature extraction on the components from nonstationary analysisis anefficient way to develop speaker recognition system. Discrete wavelet transform(DWT)is such a non stationary analysis tool for converting the signals from time domain to spectral domain. DWT decomposes the signal into approximation coefficients and detailed coefficients. Approximation coefficients correspond to low frequencies and detailed coefficients represent the high frequency information present in the signal. Since its formulation by Stephen Mallat, various wavelet packets are cited in the literature. Performance of wavelet transform based approaches is strongly dependent on the choice of the wavelet, order of the wavelet and level of decomposition. These coefficients are aggregated through statistical parameters namely, mean, variance, skewness, kurtosisetc. Figure 5.3 shows the various steps involved in an automatic speaker recognition system. The speech signals are storedin". Wav"format. DWT is then applied on these signals to obtain amplitude localized to both time and frequency. These amplitudes are then aggregated using the statistical parameters namely mean, variance, Skewness, kurtosis, ZCR, Second order moments, third order moments and energy.These features are then used to train and test the ANN based classifier.

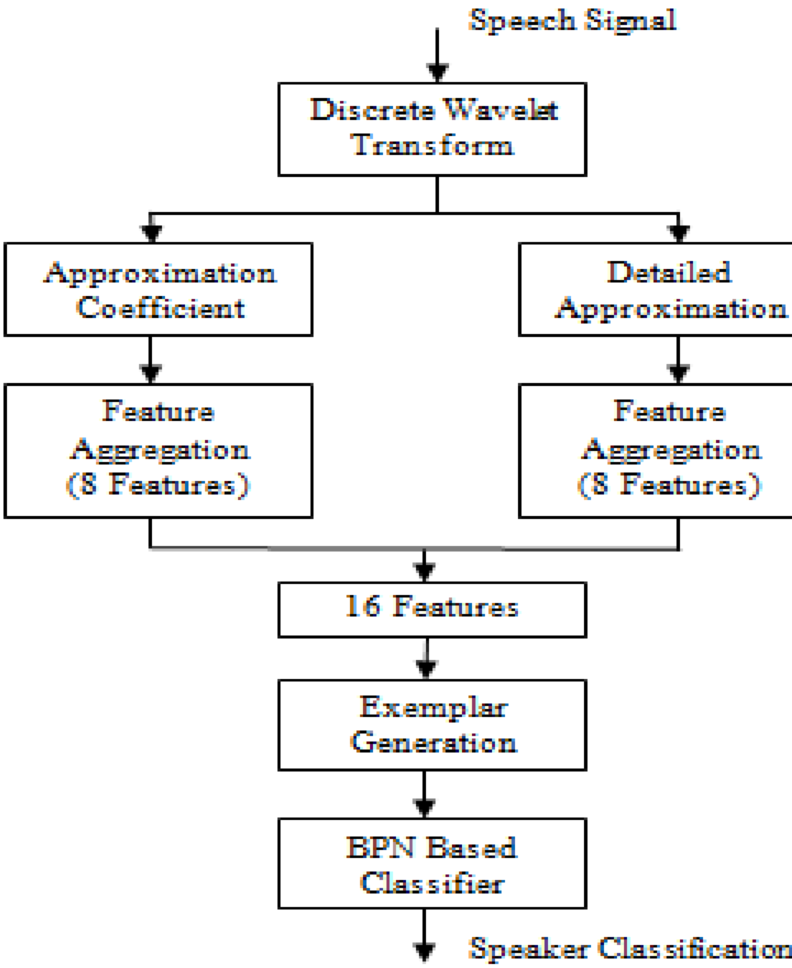


Figure 5.3 ANN Based Speaker Recognition System Using Discrete Wavelet Transform

Applications of Fourier Transform

Definition of Fourier Transform

The Fourier Transform is a generalization of the Fourier series.It only applies to continuous & a periodic functions.

We defined Fourier Transform of a piecewise continuous & absolutely integrable function x(t) by

$$X(m) = F\{x(t)\} = \int_{\infty}^{\infty} x(t).e^{-jmt}dt$$

Inverse Fourier Transform-

We define inverse Fourier Transform by using Fourier Transform

$$X(t) = F^{-1}\{X(m)\} = \frac{1}{2n}\int_{-\infty}^{\infty} X(m.e^{jmt}dm$$

Discrete Fourier Transform-

Let x[n]be a finite – length sequence of length N i.e x[n] = 0 outside the range$0 \leq n \leq N-1$
The Discrete Fourier Transform of x[n] ,denoted as X[k] , is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn},\ k= 0,1,\dots\dots\dots,N-1$$

Where W_N is the N^{th} root of unity given by

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

The inverse discrete Fourier transform is given by

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]W_N^{-kn} \quad , n = 0, 1, \dots,N-1$$

The Discrete Fourier Transform is closely related to Discrete Fourier series & the Fourier Transform. The Discrete Fourier transform is the appropriate Fourier representation for digital computer realization because it is discrete and of finite length in both time and frequency domain.

Also the Fast Fourier Transform computes DFT & produces exactly the same result as evaluating DFT definition directly .It is much faster than DFT.

The Fourier Transform method is applicable in many fields of science & technology such as

1) Application to IBVP
2) Circuit Analysis
3) Signal Analysis
4) Cell phones
5) Image Processing
6) Signal Processing& LTI system

Now we take brief overview of these applications

1. Application to Initial boundary value problems(IBVP) –

The solution of a IBVP consists of a partial differential equation together with boundary & initial conditions can be solved by Fourier Transform method. Here we solve the heat equation analytically by using boundary condition. In this case partial differential equations reduces to an Ordinary Differential Equations in Fourier Transform which is solved.

Now see the example

Example – Heat equation in one spatial dimension.

$$\frac{6T}{6t} = p\frac{6^2 T}{6s^2}$$

Where p is thermal diffusivity.

Open boundaries – T(x, t) defined on

$-\infty < x < +\infty \text{and } t \geq 0$

Also, require that $T(x, t) \rightarrow 0$ as $x \pm\infty$

Initial value problem: $T(x, t = 0) = \emptyset(x)$

Solution- Apply Fourier Transform to heat equation (at constant t)

$F[T_t] = F[p\,T_{xx}]$……………… 1

$\frac{6}{6t}F[T] = -k^2 p\,F[T]$………………2

Let us denote $r(k,t) = F[T(x,t)]$

$\frac{6}{6t}r(k,t) = -k^2 p\,r(k,t)$………..3

Equation 3 is now a simple ordinary differential equation .

Heat equation is much easier to solve in the Fourier domain.

The solution is

$r(k,t) = e^{-k^2pt}\, r(k,0)$………………..4

Still need to transform the initial condition

$T(x,0) = \emptyset(x)$

$F[\emptyset(x)] = F[T(x,t=0)] = r(k,0)$………..5

Combining equations 4 & 5

$r(k,t) = e^{-k^2pt}\, F[\emptyset(x)]$……………………6

In order to obtain solution in real space ,apply inverse Fourier Transform

$T(x,t) = F^{-1}[r(k,t)]$

$= F^{-1}[e^{-k^2pt} F[\emptyset(x)]]$………………7

However , we use convolution theorem on right hand side .

I recall this

$F[f * g] = F(f).F(g)$

Therefore ,

$f * g = F^{-1}[F(f).F(g)]$……......8

Now we apply this to equation 7

Let $F(f) = e^{-k^2pt}$ and $F(g) = F[\emptyset(x)]$.

It follows that $g = F^{-1}\{F[\emptyset(x)]\} = \emptyset(x)$by definition.

$f = F^{-1}[e^{-k^2pt}]$………………9

$= \frac{1}{\sqrt{2pt}} e^{-\frac{x^2}{4pt}}$……………………10

In last step we used inverse Fourier transform

Of a Gaussian.

Since $T(x,t) = f * g$………11

According to equations 7 & 8 we have

$T(x,t) = \frac{1}{\sqrt{2pt}} e^{-\frac{x^2}{4pt}} * \emptyset(x)$………. 12

$= \frac{1}{\sqrt{2\eta}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2pt}} e^{-(s-c)^2}/4pt\,.\emptyset(c)dc$…13

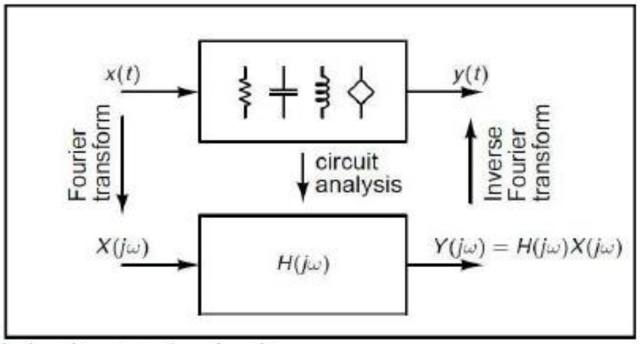
$= \frac{1}{\sqrt{4\eta pt}} \int_{-\infty}^{\infty} e^{-(s-c)^2}/4pt\;.\emptyset(c)dc$

This is the fundamental solution of the heat equation with open boundaries for an initial condition $T(x,t=0) = \emptyset(x)$

2.Circuit Analysis-

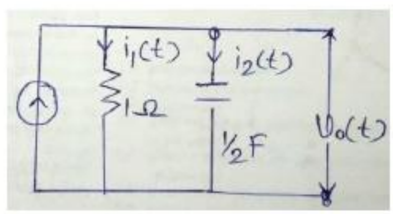
There are many linear circuits used in Electronic engineering field .These circuits include various components like capacitor, inductor ,resistor etc. Every Electronic circuit can be modelled using mathematical equations.

See this diagram



Where x(t)- actual signal applied as input to the circuit.
y(t) – output of the circuit
Now to perform frequency analysis of the circuit Fourier Transform is used.
Here we take one example.
In this example we have to find output voltage v_0 (t) by using Fourier transform.

Solution -



Where i(t) = current source
$i_1(t)$ = current flowing through resistance
$i_2(t)$ = current flowing through capacitor
$v_0(t)$ = output voltage
$i(t) = e^{-t}u(t)$
According to Kirchhoff's law
$i(t) = i_1(t) + i_2(t)$, $v_0(t) = i_2(t)$ and
$e^{-t} u(t) = v_0(t) + \frac{1}{2} v_0(t)$
By taking Fourier transform

$$\frac{}{jm+1} = v_0(jm) + \frac{1}{2}(jm)\, v_0(jm)$$

$$\frac{}{jm+1} = v_0(jm)\left[1 + \frac{1}{2} jm\right]$$

$$\frac{}{jm+1} = v_0(jm)\left[\frac{2+jm}{2}\right]$$

$$v_0(jm) = 2\left[\frac{1}{(jm+1)(2+jm)}\right] \ldots\ldots\ldots 1$$

By using partial fraction method,

$$\frac{1}{(jm+1)(2+jm)} = \frac{Æ}{(jm+1)} + \frac{B}{(2+jm)} \ldots\ldots\ldots\ldots 2$$

$$\frac{1}{(jm+1)(2+jm)} = \frac{Æ(2+jm)+B(jm+1)}{(jm+1)(2+jm)}$$

$$1 = A(2 + jm) + B(jm + 1) \ldots\ldots 3$$

Put $jm + 1 = 0 \rightarrow jm = -1$ inequation 3
we get A = 1.

Similarly if we put $2 + jm = 0 \rightarrow jm = -2$ in equation 3 we get $B = -1$.
Therefore equation 2 becomes

$$\frac{1}{(jm+1)(2+jm)} = \frac{1}{(jm+1)} + \frac{-1}{(2+jm)}$$

Put this in equation 1 we get

$$v_0(jm) = 2\left[\frac{1}{(jm+1)} + \frac{-1}{(2+jm)}\right]$$

Taking Inverse Fourier transform we get

$$F^{-1}[v_0(jm)] = 2F^{-1}\left\{\left[\frac{1}{(jm+1)} + \frac{-1}{(2+jm)}\right]\right\}$$

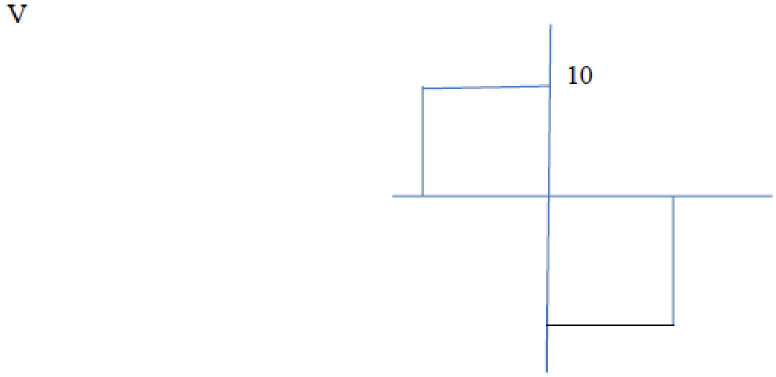
$$v_0(t) = 2\,[e^{-t}\,u(t) - e^{-2t}\,u(t)]$$

Since Fourier Transform helps us to analyse the behavior of circuit when different inputs are applied.

3. Signal Analysis-

Signal is the important part of any electronic circuit to design & analyze various electronic circuits. It is necessa do the signal analysis. Now I take example related to signals.

Here we have to find the magnitude and phase spectrum of the waveform shown in the figure below.



Solution – The equation of voltage waveform is given by

$$V(t) = \begin{cases} 10 & -T/2 \leq t \leq 0 \\ -10 & 0 \leq t \leq T/2 \\ 0 & \text{o.w} \end{cases}$$

$$V(m) = F[v(t)] = \int_{-\infty}^{\infty} v(t)\, e^{-jmt} dt$$

$$= \int_{-T/2}^{0} v(t)\, e^{-jmt} dt + \int_{0}^{T/2} v(t)\, e^{-jmt} dt$$

$$= \int_{-T/2}^{0} 10\, e^{-jmt} dt + \int_{0}^{T/2} -10\, e^{-jmt} dt$$

$$= \int_{-T/2}^{0} 10\, e^{-jmt} dt + \int_{0}^{T/2} -10\, e^{-jmt} dt$$

$$= \int_{0}^{T/2} 10\, e^{jmt} dt - \int_{0}^{T/2} 10\, e^{-jmt} dt$$

$$= 10 \int_{0}^{T} (e^{jmt} - e^{-jmt})\,dt$$

$$= 10 \times 2j \int_{0}^{T/2} \sin mt\,dt$$

$$= 20j \left[\frac{-\cos mt}{m}\right]_{0}^{T/2}$$

$$V(m) = \frac{20j}{m}\left[1 - \frac{\cos mT}{2}\right]$$

$$\text{Magnitude of spectrum} = \frac{20}{m}\left[1 - \frac{\cos mT}{2}\right]$$

$$\text{Also } \emptyset(m) = \tan^{-1}\left[\frac{INv(m)}{Rev(m)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{20}{m}\left(1-\cos\frac{mT}{2}\right)}{0}\right]$$

$$= \begin{cases} \pi/2 & m \text{ is } +ve \\ -\pi/2 & m \text{ is } -ve \end{cases}$$

To find out frequency components in the given signal Fourier Transform is used.

4. Cell phones-

Communication is all based on Mathematics .The communication includes automatic transmission of data over wires and radio circuits through signals .Cell phones are one of the most prominent communication device , the cell phone is dramatically changing the way people interact and communicate with each other.

The principle of Fourier Transform is used in signal ,such as sound produced by a musical instrument For e.g- piano, violin ,drum any sound recording can be represented as the sum of a collection of sine and cosine waves with various frequencies and amplitudes. This collection of waves can then be manipulated with relative ease. Our mobile phone has performing Fourier Transform. Every mobile device – such as netbook, tablet ,and phone have been built in high speed cellular connection , just like Fourier Transform. Humans very easily perform it mechanically everyday.

For ex. When you are in a room with a great deal of noise & you selectively hear your name above the noise, then you just performed Fourier transform.

5. Image Processing-

Fourier transform is used in a wide range of applications such as image analysis ,image filtering , image reconstruction and image compression.

The Fourier Transform is an important image processing tool which is used to decompose an image into its sine and cosine components.

The Fourier Transform is used if we want to access the geometric characteristics of a spatial domain image. Because the image in the Fourier domain is decomposed into its sinusoidal components, it is easy to examine or process certain frequencies of the image, thus influencing the geometric structure in the spatial domain.

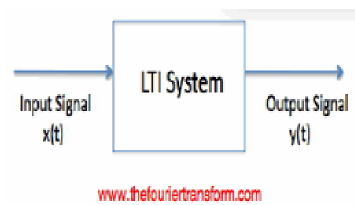
In most implementations the Fourier image is shifted in such a way that the DC-value (*i.e.* the image mean) *F(0,0)* is displayed in the center of the image. The further away from the center an image point is, the higher is its corresponding frequency.

6. Signal Processing& LTI system-

The Fourier Transform is extensively used in the field of Signal Processing. In fact, the Fourier Transform is probably the most important tool for analyzing signals in that entire field.

A signal is any waveform (function of time). This could be anything in the real world - an electromagnetic wave, the voltage across a resistor versus time, the air pressure variance due to your speech (i.e. a sound wave), or the value of Apple Stock versus time. Signal Processing then, is the act of processing a signal to obtain more useful information, or to make the signal more useful.

Suppose we have a box that accepts an input signal and produces an output signal from that. Such a box can be thought of as a system:



A System which takes an input signal and produces and output signal

when we view the Fourier Transform of the output, we now know how the system reacts toevery possiblefrequency.The reason for this goes back to the linearity of the Fourier Transform: the impulse in time can be thought of as an infinite sum of sinusoids at every possible frequency. The output result then is the sum of the responses to each frequency.

Fourier Transform visualize the affect of an LTI system simple and the analysis much easier.

The Fourier Transform is extensively used in LTI system theory, filtering and signal processing. In fact, the majority of the analysis takes place in the frequency domain, making the understanding of Fourier Theory indispensable.

Sampling Theorem

A signal has three properties like voltage or amplitude, frequency, phase. The signals are represented only in an analog form where the digital form of technology is not available. Analog signals are continuous in time and difference in voltage levels for different periods of the signal. Here, the main drawback of this is, the amplitude keeps on changing along with the period of the signal. This can be overcome by the digital form of signal representation. Here conversion of an analog form of the signal into digital form can be done using the sampling technique. The output of this technique represents the discrete version of its analog signal. Here in this article, you can find what is sampling theorem, definition, applications, and its types.

What is the Sampling Theorem?

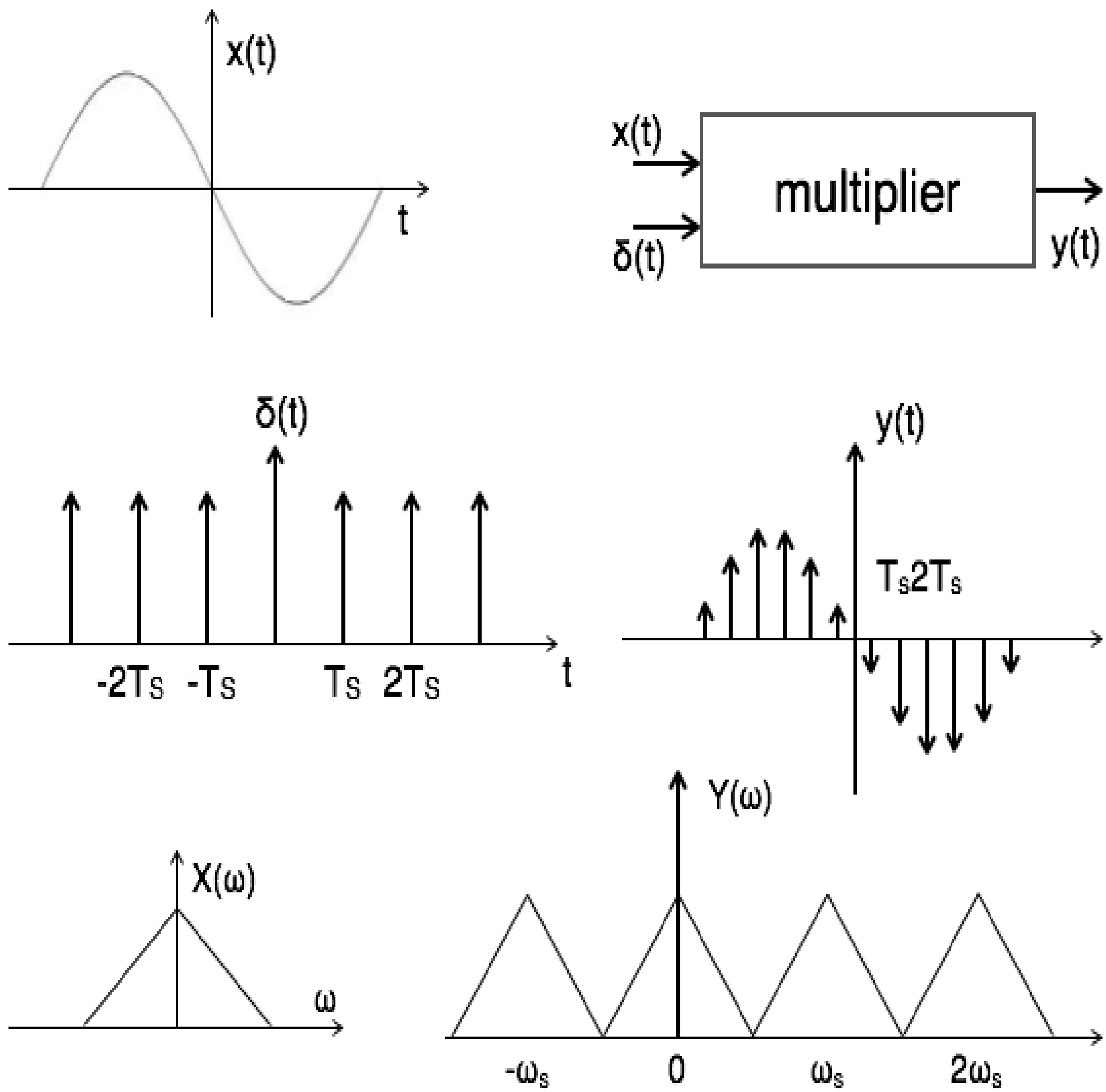
A continuous signal or an analog signal can be represented in the digital version in the form of samples. Here, these samples are also called as discrete points. In sampling theorem, the input signal is in an analog form of signal and the second input signal is a sampling signal, which is a pulse train signal and each pulse is equidistance with a period of "Ts". This sampling signal frequency should be more than twice of the input analog signal frequency. If this condition satisfies, analog signal perfectly represented in discrete form else analog signal may be losing its amplitude values for certain time intervals. How many times the sampling frequency is more than the input analog signal frequency, in the same way, the sampled signal is going to be a perfect discrete form of signal. And these types of discrete signals are well performed in the reconstruction process for recovering the original signal.

Statement: A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal. i. e.

$$f_s \geq 2f_m.$$

Proof: Consider a continuous time signal x(t). The spectrum of x(t) is a band limited to f_m Hz i.e. the spectrum of x(t) is zero for $|\omega|>\omega_m$.

Sampling of input signal x(t) can be obtained by multiplying x(t) with an impulse train δ(t) of period T_s. The output of multiplier is a discrete signal called sampled signal which is represented with y(t) in the following diagrams:



Here, you can observe that the sampled signal takes the period of impulse. The process of sampling can be explained by the following mathematical expression:

Sampled signal y(t)=x(t).δ(t)......(1)

The trigonometric Fourier series representation of δ (t) is given by

$$\delta(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_s t + b_n \sin n\omega_s t) \quad \ldots\ldots (2)$$

Where $a_0 = \frac{1}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)dt = \frac{1}{T_s}\delta(0) = \frac{1}{T_s}$

$$a_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\cos n\omega_s\, dt = \frac{2}{T_2}\delta(0)\cos n\omega_s 0 = \frac{2}{T}$$

$$b_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\sin n\omega_s t\, dt = \frac{2}{T_s}\delta(0)\sin n\omega_s 0 = 0$$

Substitute above values in equation 2.

$$\therefore\ \delta(t) = \frac{1}{T_s} + \Sigma_{n=1}^{\infty}(\frac{2}{T_s}\cos n\omega_s t + 0)$$

Substitute δ(t) in equation 1.

$$\rightarrow y(t) = x(t).\delta(t)$$

$$= x(t)[\frac{1}{T_s} + \Sigma_{n=1}^{\infty}(\frac{2}{T_s}\cos n\omega_s t)]$$

$$= x(t)[\frac{1}{T_s} + \Sigma_{n=1}^{\infty}(\frac{2}{T_s}\cos n\omega_s t)]$$

$$= \frac{1}{T_s}[x(t) + 2\Sigma_{n=1}^{\infty}(\cos n\omega_s t)x(t)]$$

$$y(t) = \frac{1}{T_s}[x(t) + 2\cos\omega_s t.\, x(t) + 2\cos 2\omega_s t.\, x(t) + 2\cos 3\omega_s t.\, x(t) \ldots\ldots]$$

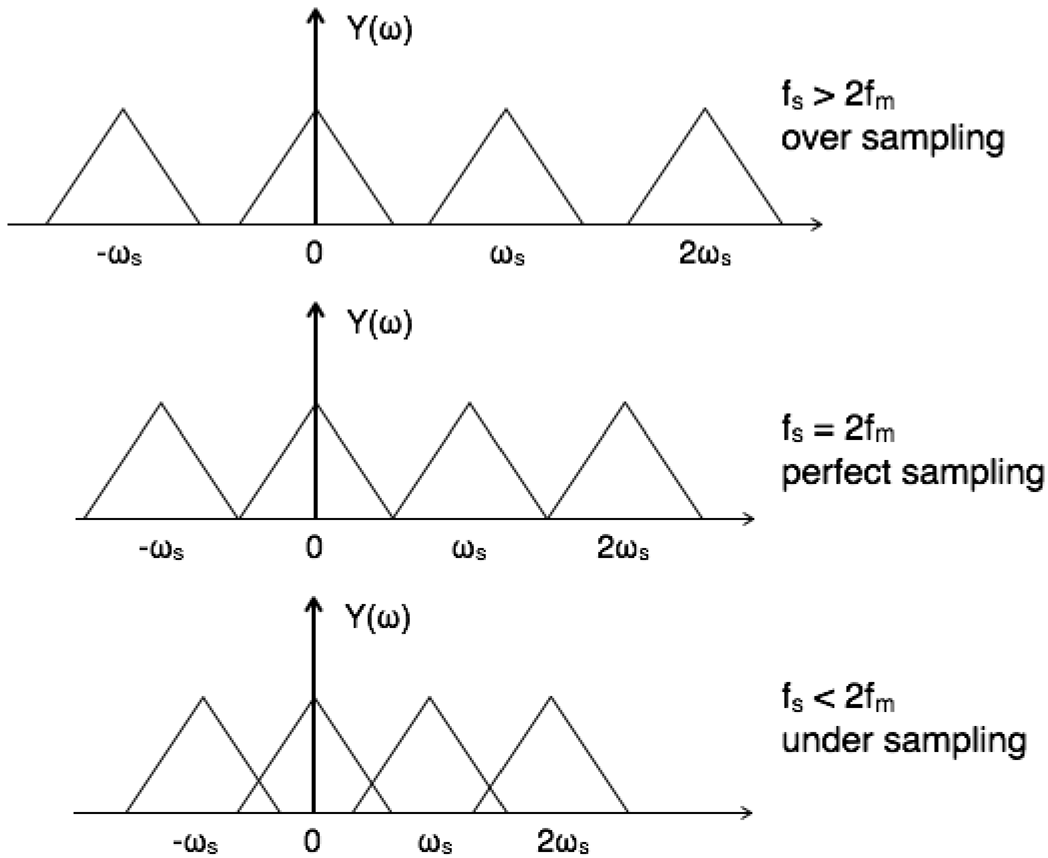
Take Fourier transform on both sides.

$$Y(\omega) = \frac{1}{T_s}[X(\omega) + X(\omega-\omega_s) + X(\omega+\omega_s) + X(\omega-2\omega_s) + X(\omega+2\omega_s) + \ldots]$$

$$\therefore\ Y(\omega) = \frac{1}{T_s}\Sigma_{n=-\infty}^{\infty}X(\omega - n\omega_s) \qquad where\ n = 0, \pm1, \pm2, \ldots$$

To reconstruct x(t), you must recover input signal spectrum X(ω) from sampled signal spectrum Y(ω), which is possible when there is no overlapping between the cycles of Y(ω).

Possibility of sampled frequency spectrum with different conditions is given by the following diagrams:



Aliasing Effect

The overlapped region in case of under sampling represents aliasing effect, which can be removed by

- **considering $f_s > 2f_m$**
- **By using anti aliasing filters.**

Applications

There are few applications of sampling theorem are listed below. They are

- **To maintain sound quality in music recordings.**
- **Sampling process applicable in the conversion of analog to discrete form.**
- **Speech recognition systems and pattern recognition systems.**
- **Modulation and demodulation systems**
- **In sensor data evaluation systems**
- **Radar and radio navigation system sampling is applicable.**
- **Digital watermarking and biometric identification systems, surveillance systems.**

Mathematical Models

• Mathematical models serve as tools in the analysis and design of complex systems

• A mathematical model is used to represent, in an approximate way, a physical process or system where measurable quantities are involved

• Typically a computer program is written to evaluate the mathematical model of the system and plot performance curves – The model can more rapidly answer questions about system performance than building expensive hardware prototypes

• Mathematical models may be developed with differing degrees of fidelity

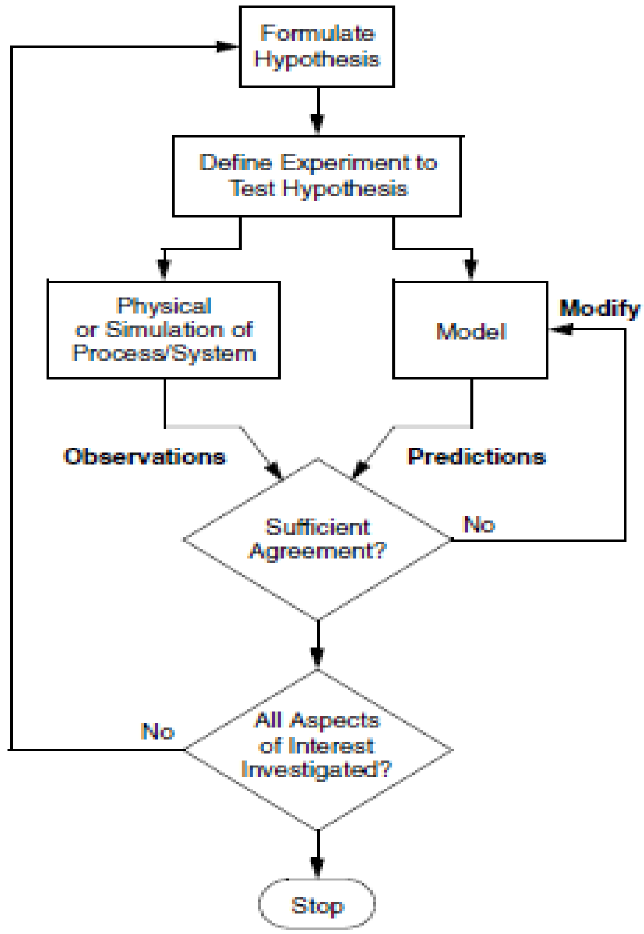
• A system prototype is ultimately needed, but a computer simulation model may be the first step in this process

• A computer simulation model tries to accurately represent all relevant aspects of the system under study

• Digital signal processing (DSP) often plays an important role in the implementation of the simulation model

• If the system being simulated is to be DSP based itself, the simulation model may share code with the actual hardware prototype

• The mathematical model may employ both deterministic and random signal models

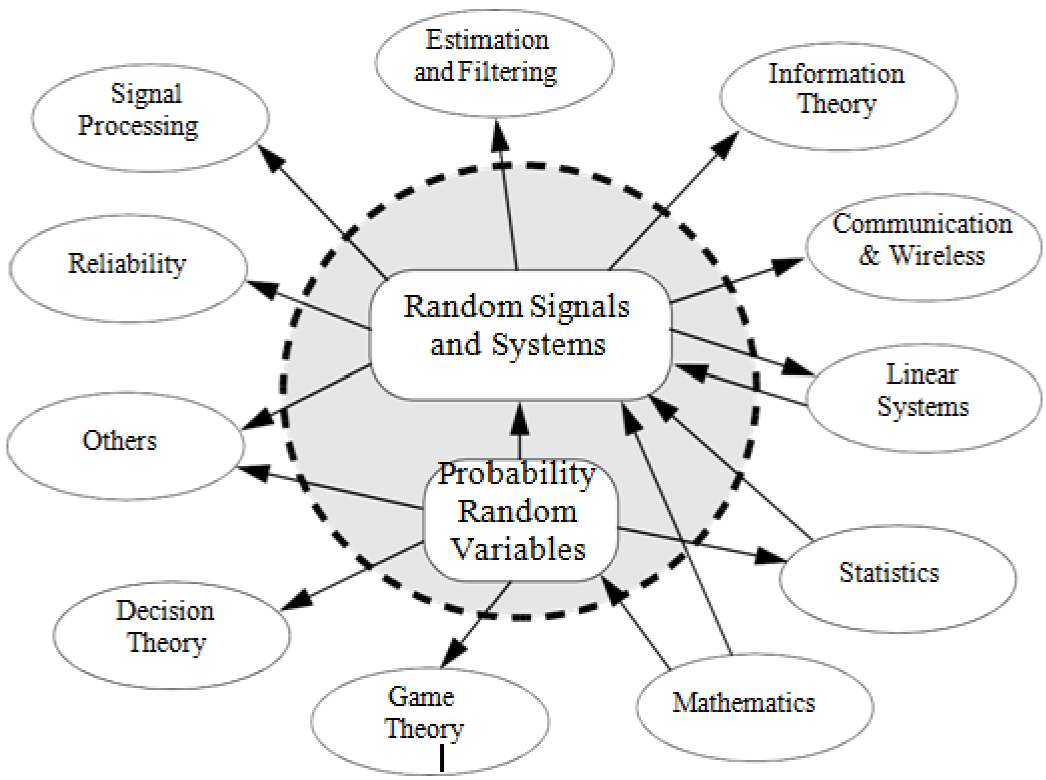


The Mathematical Modeling Process[1]

Engineering Applications

Communications, Computer networks, Decision theory and decision making, Estimation and filtering, Information processing, Power en- gineering, Quality control, Reliability, Signal detection, Signal and data processing, Stochastic systems, and others.

Relation to Other Subjects



Random Signals in Practice

- **A typical application of random signals concepts involves one or more of the following:**
  - **Probability**
  - **Random variables**
  - **Random (stochastic) processes**

Example : Modeling with Random Processes

- **Consider a random or stochastic process of the form**

x(t)* = *A* cos(2*π* f_c*t* + *θ*) + *n(t)

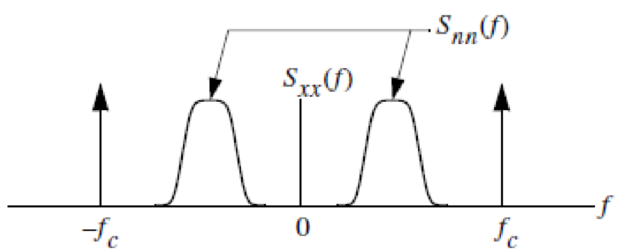
which is a sinusoidal carrier plus noise

- **In this example the carrier phase *θ* is modeled as a random vari- able and *n(t)* is modeled as an independent stationary random process**
- **We may be interested in how to recover the sinusoidal carrier from the noisy signal *x(t)***
- **The power spectral density of a random process allows us to see the spectral content of a signal**
- **The *power spectral density* of a *wide sense stationary* random process *x(t)* is given by the Fourier transform of the autocorre- lation function**

- In this case the power spectrum is

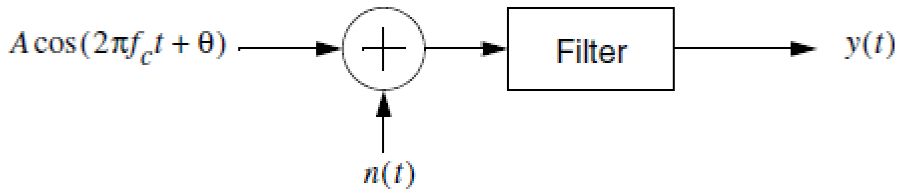
$$S_{xx}(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] + S_{nn}(f)$$

where $S_{nn}(f)$ is the power spectrum of the noise alone

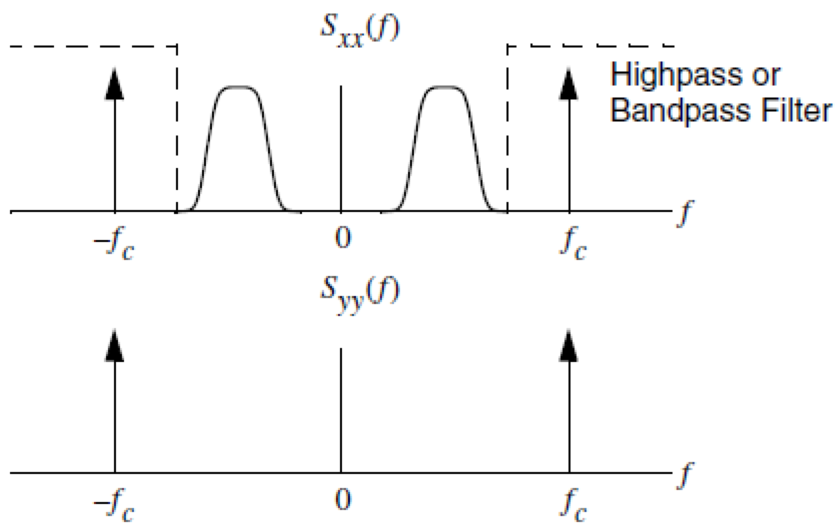


Power spectral density of $x(t)$

- To recover just the carrier from $x(t)$ we may pass $x(t)$ through a filter



Signal processing $x(t)$ to recover just the carrier signal



Filtering $x(t)$ to obtain $y(t)$ with spectrum $S_{yy}(f)$

Application Areas

- **Control**
- **Communications**
- **Signal Processing**

Control Applications

- **Industrial control and automation (Control the velocity or position of an object)**
- **Examples: Controlling the position of a valve or shaft of a motor**
- **Important Tools:**

– Time-domain solution of differential equations

– Transfer function (Laplace Transform)

– Stability

Communication Applications

- **Transmission of information (signal) over a channel**
- **The channel may be free space, coaxial cable, fiber optic cable**
- **A key component of transmission: Modulation (Analog and Digital Communication)**

Signal Processing Applications

- **Signal processing=Application of algorithms to modify signals in a way to make them more useful.**
- **Goals:**

– Efficient and reliable transmission, storage and display of information

– Information extraction and enhancement

- **Examples:**

– Speech and audio processing

– Multimedia processing (image and video)

– Underwater acoustic

– Biological signal analysis

Multimedia Applications

- **Compression: Fast, efficient, reliable transmission and storage of data**
- **Applied on audio, image and video data for transmission over the Internet, storage**
- **Examples: CDs, DVDs, MP3, MPEG4**
- **Mathematical Tools: Fourier Transform, Quantization, Modulation Biological Signal Analysis**
- **Examples:**

– Brain signals (EEG)

– Cardiac signals (ECG)

– Medical images (x-ray, PET, MRI)

- **Goals:**

– Detect abnormal activity (heart attack, seizure)

– Help physicians with diagnosis

- **Tools: Filtering, Fourier Transform**

TEXT / REFERENCE BOOKS

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