SCHOOL OF MECHANICAL

DEPARTMENT OF MECHATRONICS

UNIT - I

Signals and Systems – SEC1208

I. Classification of Signals

9 hrs.

Continuous time signals (CT signals) and Discrete time signals (DT signals) - Basic operations on signals elementary signals- Step, Ramp, Pulse, Impulse, Exponential - Classification of CT and DT signals - Periodic, aperiodic signals-Deterministic and Random signals-even and odd signals - Real and Complex signals - Energy and power signals.

Signal: Signals are represented mathematically as functions of one or more independent variables. It mainly focuses attention on signals involving a single independent variable. For convenience, this will generally refer to the independent variable as time**.** It is defined as physical quantities that carry information and changes with respect to time.

Ex: voice, television picture, telegraph.

Continuous Time signal – If the signal is defined over continuous-time, then the signal is a continuous- time signal.

Ex: Sinusoidal signal, Voice signal, Rectangular pulse function

Discrete Time signal – If the time t can only take discrete values, such as t=kTs is called Discrete Time signal

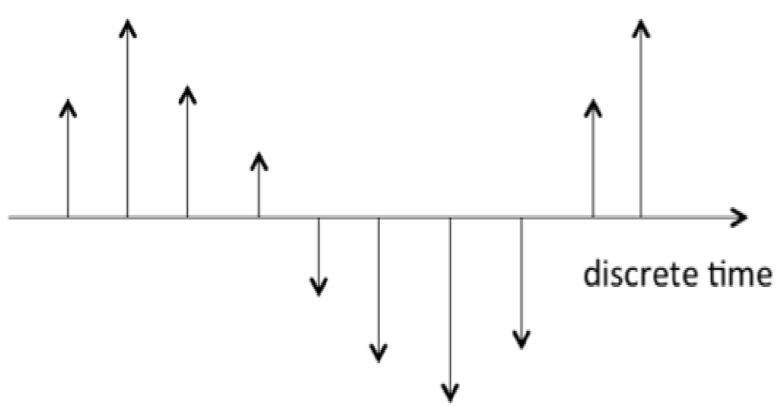


Fig.1.1 Discrete Time Signal

Unit Step Signal:

The Unit Step Signal u(t) is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Graphically it is given by

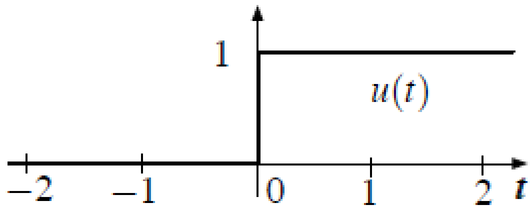


Fig.1.2 Unit Step Signal

Ramp Signal:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Graphically it is given by

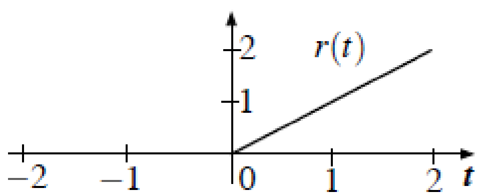


Fig.1.3 Ramp Signal

Pulse Signal:

A signal is having constant amplitude over a particular interval and for the remaining interval the amplitude is zero.

Impulse Signal:

$$\delta[n] \equiv \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

Impulse Signal DT representation

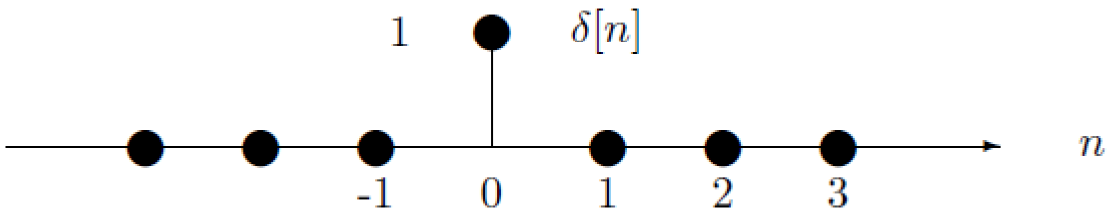


Fig.1.4 Impulse Signal DT Signal

Impulse Signal CT representation

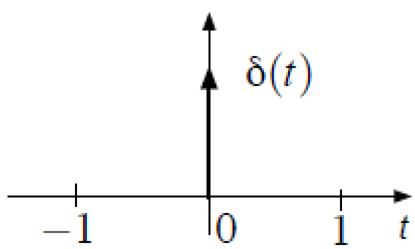


Fig.1.5 Impulse Signal CT Signal

Exponential Signal:

Exponential signal is of two types. These two type of signals are real exponential signal and complex exponential signal which are given below.

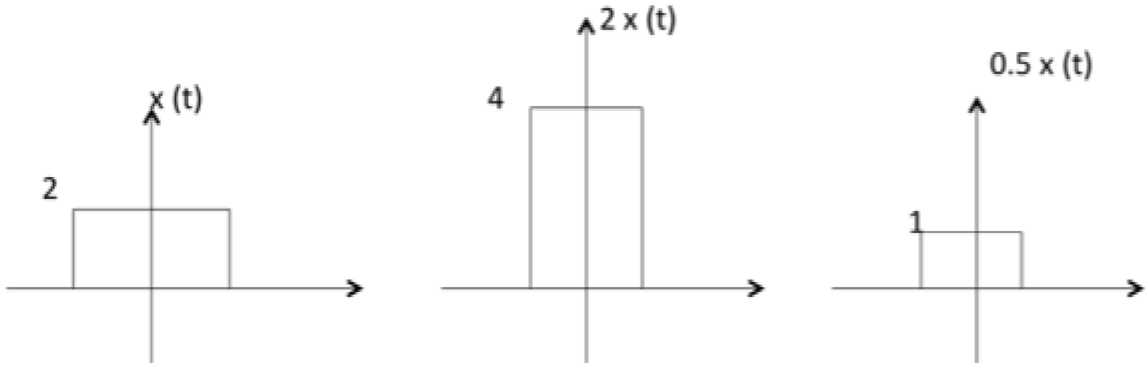
Real Exponential Signal: A real exponential signal is defined as $x(t)=Ae^{\sigma t}$

Complex exponential Signal: The complex exponential signal is given by $x(t)=Ae^{st}$ where $s=\sigma+j\omega$

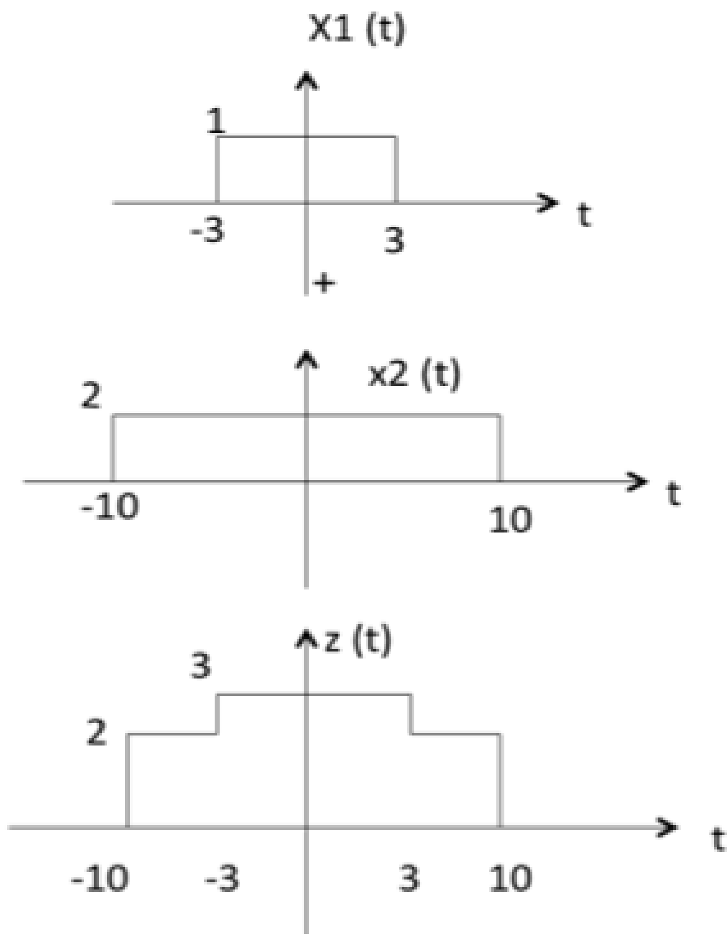
Basic Operations on signals:

Several basic operations by which new signals are formed from given signals are familiar from the algebra and calculus of functions.

1. Amplitude Scaling :y(t)= a x(t), where a is a real (or possibly complex) constant. C x(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.



2. Amplitude Shift: y(t)= x(t)+ b, where b is a real (or possibly complex) constant
3. Signal Addition: y(t)= x1(t) + x2(t)



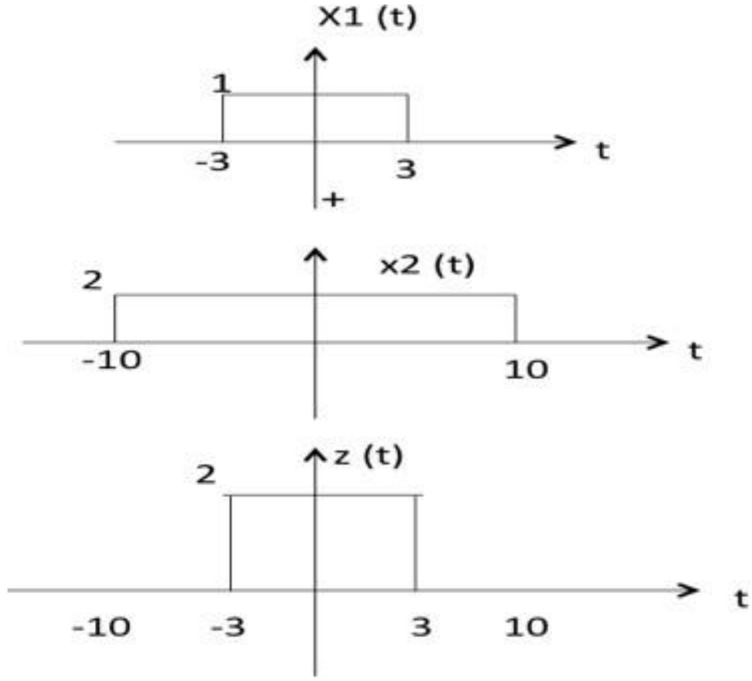
As seen from the diagram above,

-10 < t < -3 amplitude of z(t) = x1(t) + x2(t) = 0 + 2 = 2

-3 < t < 3 amplitude of z(t) = x1(t) + x2(t) = 1 + 2 = 3

3 < t < 10 amplitude of z(t) = x1(t) + x2(t) = 0 + 2 = 2

4. Signal Multiplication: y(t)= x1(t). x2(t)



As seen from the diagram above,

-10 < t < -3 amplitude of z (t) = x1(t) ×x2(t) = 0 ×2 = 0

-3 < t < 3 amplitude of z (t) = x1(t) ×x2(t) = 1 ×2 = 2 3

< t < 10 amplitude of z (t) = x1(t) × x2(t) = 0 × 2 = 0

5. Time Shift**:**

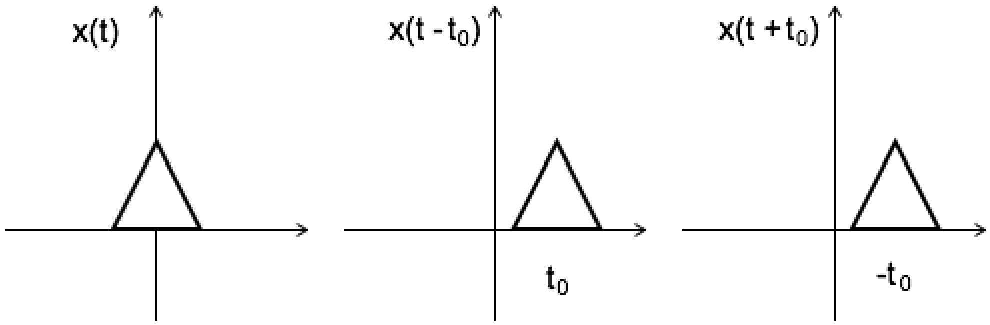
If x(t) is a continuous function, the time-shifted signal is defined as

$y(t) = x(t - t_0)$.

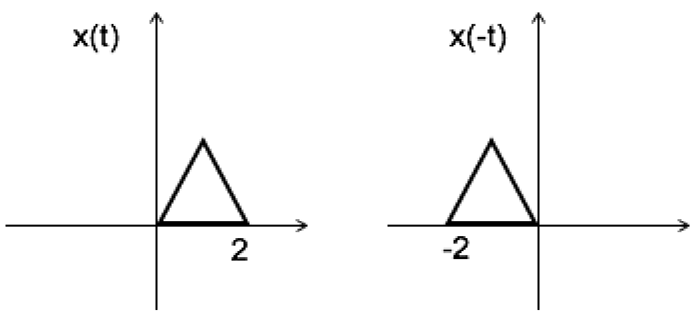
If $t_0 > 0$, the signal is shifted to the right, and if $t_0 < 0$, the signal is shifted to the left.

$x(t \pm t_0)$ is time shifted version of the signal x(t).

$x(t + t_0)$ →→ negative shift and $x(t - t_0)$ →→ positive shift



6. Time Reversal: If x(t) is a continuous function, the time-reversed signal is defined as y(t) = x(-t). x(-t) is the time reversal of the signal x(t).



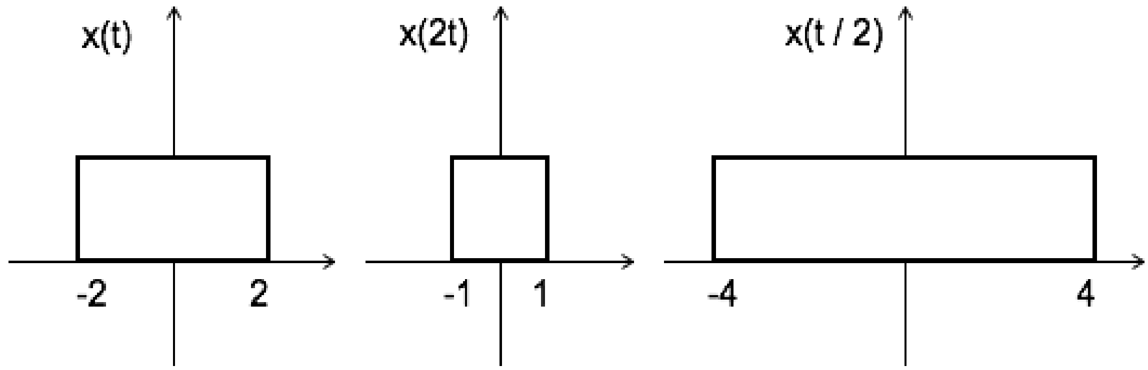
7. Time Scaling: If x(t) is a continuous function, a time-scale version of this signal is defined as y(t)= x(at). If a>1, the signal y(t) is a compressed version of x(t), i.e., the time interval is compressed to 1/a.

If 0<a< 1, the signal y(t) is a stretched version of x(t), i.e., the time interval is stretched by 1/a.

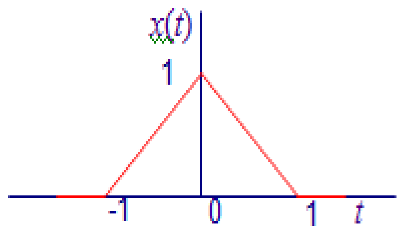
. When operating on signals, the time-shifting operation must be performed first, and then the time-scaling operation is performed. x(At) is time scaled version of the signal x(t). where, A is always positive.

$|A| > 1$ →→ Compression of the signal

$|A| < 1$ →→ Expansion of the signal



1. A triangular pulse signal x(t) is depicted below.



Sketch each of the following signals:

(a) $x(3t)$

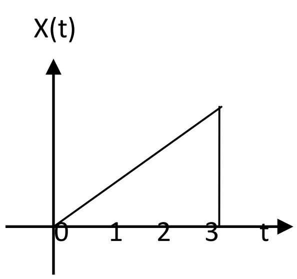
(b) $x(3t + 2)$

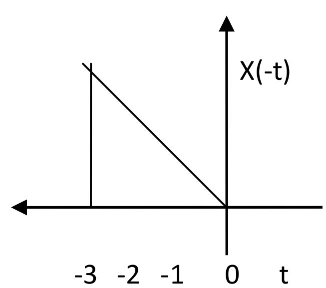
(c) $x(-2t - 1)$

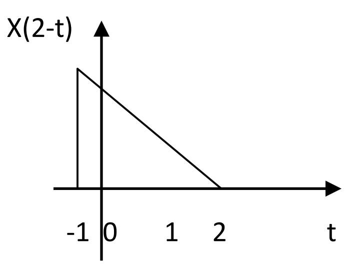
(d) $x(0.5t - 1)$

2. Draw the waveform x(-t) and x(2-t) of the signal $x(t) = t \quad 0 \le t \le 3$

$0 \quad t > 3$







Classification of DT and CT Signals:

1. Even and Odd signal
2. Deterministic and Random Signal
3. Periodic and Aperiodic signal
4. Energy and Power signal

Even and Odd Signal:

An even signal is any signal 'x' such that x(t) = x(-t). Odd signal is a signal 'x' for which x(t) = -x(-t).

The even and odd parts of a signal are given by

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Here Xe(t) denotes the even part of signal X(t) and Xo(t) denotes the odd part of signal X(t).

Deterministic Signal:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signals can be modeled exactly by a mathematical formula are known as deterministic signals.

Random (or) Nondeterministic Signals:

Non deterministic signals and events are either random or irregular. Random signals are also called non deterministic signals are those signals that take random values at any given time and must be characterized statistically. Random signals, on the other hand, cannot be described by a mathematical equation they are modeled in probabilistic terms.

Periodic signal:

A CT signal x(t) is said to be periodic if it satisfies the following property: x(t)=x(t+T) at all time t, where T=Fundamental Time Interval ($T=2\pi/\omega$)

Ex:

1. $x(t)=\sin(4\pi t)$. It is periodic with period of 1/2
2. $x(t)=\cos(3\pi t)$. It is periodic with period of 2/3

Aperiodic Signal:

A CT signal x(t) is said to be periodic if it satisfies the following property: $x(t) \neq x(t+T)$ at all time t, where T=Fundamental Time Interval

Energy Signal:

The Energy in the signal is defined as

$$E = \int_{-\infty}^{\infty} | x(t)|^2 \, dt \, .$$

Power Signal:

The Power in the signal is defined as

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} | x(t)|^2 \, dt$$

If $0<E<\infty$ then the signal x(t) is called as Energy signal. However there are signals where this condition is not satisfied. For such signals we consider the power. If $0<P<\infty$ then the signal is called a power signal. Note that the power for an energy signal is zero (P=0) and that the energy for a power signal is infinite ($E=\infty$). Some signals are neither energy nor power signals.

Real and Complex signals:

Exponential signal is of two types. These two type of signals are real exponential signal and complex exponential signal which are given below.

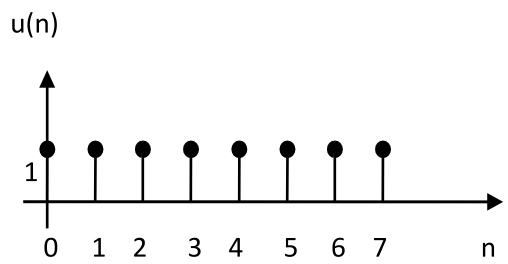
Real Exponential Signal:

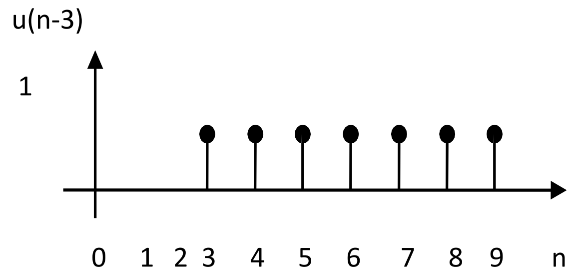
A real exponential signal is defined as $x(t)=Ae^{\sigma t}$

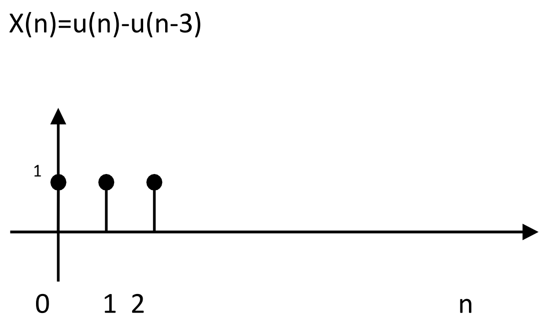
Complex exponential Signal:

The complex exponential signal is given by $x(t)=Ae^{st}$ where $s=\sigma+j\omega$

1. Draw the signal x(n) = u(n) – u(n-3)







2. What is the total energy of the discrete time signal x(n) which takes the value of unity at n= - 1,0,1?

Energy of the signal is given as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-1}^{1} |x(n)|^2$$

$$= |x(-1)|^2 + |x(0)|^2 + |x(1)|^2 = 3$$

3. Determine if the following signals are Energy signals, Power signals, or neither, and evaluate E and P for each signal.

$$E_a = \int_{-\infty}^{\infty} |a(t)|^2 \, dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2 \, dt$$

$$= 9\int_{-\infty}^{\infty} \frac{1}{2}[1-\cos(4\pi t)]dt$$

$$= 9\int_{-\infty}^{\infty} \frac{1}{2}dt - 9\int_{-\infty}^{\infty} \cos(4\pi t)dt$$

$$= \infty \quad \text{J}$$

$$P_a = \frac{1}{1}\int_{0}^{1} |a(t)|^2 \, dt = \int_{0}^{1} |3\sin(2\pi t)|^2 \, dt$$

$$= 9\int_{0}^{1} \frac{1}{2}[1-\cos(4\pi t)]dt$$

$$= 9\int_{0}^{0} \frac{1}{2}dt - 9\int_{0}^{1} \cos(4\pi t)dt$$

$$= \frac{9}{2} - \left[\frac{9}{4\pi}\sin(4\pi t)\right]_0^1$$

$$= \frac{9}{2} \quad \text{W}$$

So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. It is a power signal.

4. Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$

(b) $x(t) = \sin\frac{2\pi}{3}t$

(c) $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$

(d) $x(t) = \cos t + \sin\sqrt{2}t$

$(a)\quad x(t) = \cos\left(t + \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{\pi}{4}\right) \rightarrow \omega_0 = 1$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

$(b)\quad x(t) = \sin\frac{2\pi}{3}t \rightarrow \omega_0 = \frac{2\pi}{3}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

$(c)\quad x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t = x_1(t) + x_2(t)$

where $x_1(t) = \cos(\pi/3)t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

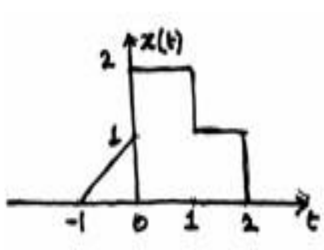
$(d)\quad x(t) = \cos t + \sin\sqrt{2}\,t = x_1(t) + x_2(t)$

where $x_1(t) = \cos t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin\sqrt{2}\,t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\,\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

Questions for Practice

PART - A

1. For the signal shown in Fig. 1, find x (2t + 3).



2. Sketch the following signals

   i)x(t)=4(t+3) ii)x(t) = -2r(t)

3. Define continuous time complex exponential signal.
4. Define unit impulse and unit step signal.
5. State the relationship between step, ramp and delta function (CT).

6. Define even and odd signal?

7. Determine whether the following signal is energy or power? $x(t) = e^{-2t}\, u(t)$

8. Find the fundamental period of the given signal $x(n)=\sin((6n\pi/7)+1)$.

9. Check whether the discrete time signal sin3n is periodic.

10. Define a random signal.

11. Determine the power and RMS value of the following signals $x(t)=10\cos 5t \cos 10t$.

12. Determine whether the following signal is energy or power? $x(n) = u(n)$

PART – B

1. Find the time period T of the following signal
   (i) $X(n) = \cos(n\pi/2) - \sin(n\pi/8) + 3\cos\{(n\pi/4) + (n/3)\}$

   (ii)Define and plot the following signals. Ramp, Step, Pulse, Impulse and Exponential signal.

2. (i) What is the periodicity of the signal $x(t) = \sin 100\pi t + \cos 150\pi t$?
   (ii)What are the basic continuous time signals? Draw any four Waveforms and write their equations.

3. Determine the energy of the discrete time signal.
   $$X(n) = \begin{cases} (1/2)^n, & n \geq 0 \\ 3^n, & n < 0 \end{cases}$$

4. Determine the even and odd component for the following signals.
   i)$x(t) = \cos t + \sin t + \cos t \sin t$
   ii)$x(n) = \{-2, 1, \underset{\uparrow}{2}, -1, 3\}$

5. Determine whether the following signals are periodic or not.
   i) $x(t) = 2\cos(10t+1) - \sin(5t-1)$
   ii)$x(n) = 12\cos(20n)$

6. Identify which of the following signals are energy signals, power signals and neither power nor energy signals.
   i)$x(t) = e^{-3t}u(t)$
   ii)$x(t) = \cos t$
   iii)$x(t) = t\, u(t)$

TEXT / REFERENCE BOOKS

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UNIT - III

Signals and Systems – SEC1208

Unit III - LINEAR TIME INVARIANT CONTINUOUS TIME SYSTEMS

9Hrs

Concept of CT systems - Linear Time invariant Systems - Basic properties of continuous time systems - Linearity, Invertilibity, Causality, Time invariance, Stability - Frequency response of LTI systems - Analysis and characterization of LTI systems using Laplace transform - Differential equation- Computation of impulse response, step response, natural response ,forced response and transfer function using Laplace transform –Convolution integral -Properties of convolution integral.

System

A system may be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends upon transfer function of the system. Mathematically, the functional relationship between input and output may be written as

$$y(t)=f[x(t)]$$

Types of system

Like signals, systems may also be of two types as under:

1. Continuous-time system
2. Discrete time system

Continuous time System

Continuous time system may be defined as those systems in which the associated signals are also continuous. This means that input and output of continuous – time system are both continuous time signals.

For example: Audio, video amplifiers, power supplies etc., are continuous time systems.

Discrete time systems

Discrete time system may be defined as a system in which the associated signals are also discrete time signals. This means that in a discrete time system, the input and output are both discrete time signals.

For example, microprocessors, semiconductor memories, shift registers etc., are discrete time signals.

LTI system:

Systems are broadly classified as continuous time systems and discrete time systems. Continuous time systems deal with continuous time signals and discrete time systems deal with discrete time system. Both continuous time and discrete time systems have several basic properties. Out of these several basic properties of systems, two properties namely linearity and time invariance play a vital role in the analysis of signals and systems. If a system has both the linearity and time invariance properties, then this system is called linear time invariant (LTI) system.

Characterization of Linear Time Invariant (LTI) system

Both continuous time and discrete time linear time invariant (LTI) systems exhibit one important characteristics that the superposition theorem can be applied to find the response y(t) to a given input x(t).

Hence, following steps may be adopted to find the response of a LTI system using super position theorem:

1. Resolve the input function x(t) in terms of simpler or basic function like impulse function for which response can be easily evaluated.

2. Determine individually the response of LTI system for the simpler input impulse functions.

3. Using superposition theorem, find the sum of the individual responses, which will become the overall response y(t) of function x(t).

From the above discussions, it is clear that to find the response of a LTI system to any given function, first we have to find the response of LTI system input to an unit impulse called unit impulse response of LTI system.

Hence, the impulse response of a continuous time or discrete time LTI system is the output of the system due to an unit impulse input applied at time t=0 or n=0.

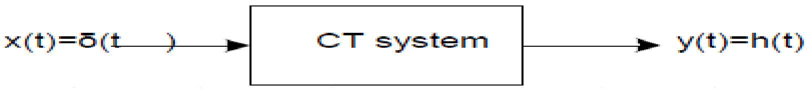


Fig. 3.1 Continuous System

Here, δ(t) is the unit impulse input in continuous time and h(t) is the unit impulse response of continuous time LTI system. Continuous time unit impulse response h(t) is the output of a continuous time system when applied input x(t) is equal to unit impulse function δ(t).

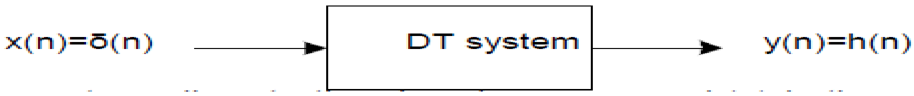


Fig. 3.2 Discrete System

Similarly, for a discrete time system, discrete time impulse response h(n) is the output of a discrete time system when applied input x(n) is equal to discrete time unit impulse function δ(n). Here, δ (n) is the unit impulse input in discrete time and h(n) is the unit impulse response of discrete time LTI system. Therefore, any LTI system can be completely characterized in terms of its unit impulse response.

Properties of Linear time invariant (LTI) system:-

The LTI systems have a number of properties not exhibited by other systems. Those are as under:

- Commutative property of LTI systems
- Distributive property of LTI systems
- Associative property of LTI systems
- Static and dynamic LTI systems
- Invertibility of LTI systems
- Causality of LTI systems
- Stability of LTI systems
- Unit-step response of LTI systems

Commutative property:

The commutative property is a basic property of convolution in both continuous and discrete time cases, thus, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are commutative. According to the property, for continuous time LTI system. The output is given by

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Or

$$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Thus, we can say that according to this property, the output of a continuous time LTI system having input x(t) and unit impulse h(t) is identical to the output of a continuous time LTI system having input h(t) and the unit impulse response x(t).

Distributive property:

The distributive property states that both convolution integral for continuous time LTI system and convolution sum for discrete time LTI system are distributive.

For continuous time LTI system, the distributive property is expressed as the output,

$$y(t) = x(t) * [h_1(t) + h_2(t)]$$

Or

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

Thus, the two continuous time LTI systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs and outputs are added as

$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

The output

$$y(t) = y_1(t) + y_2(t)$$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

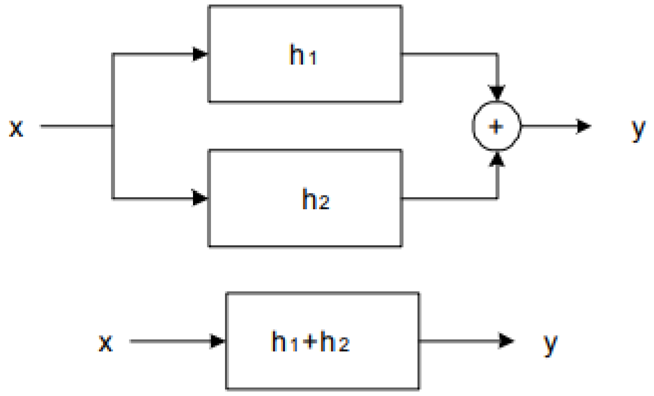


Fig.3.3 The distributive property of convolution integral for a parallel interconnection

Associative Property of LTI system:

According to associative property, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are associative.

For continuous time LTI system, according to associative property,

$$x*(h_1*h_2)=(x*h_1)*h_2 .$$

for both discrete-time and continuous-time systems.

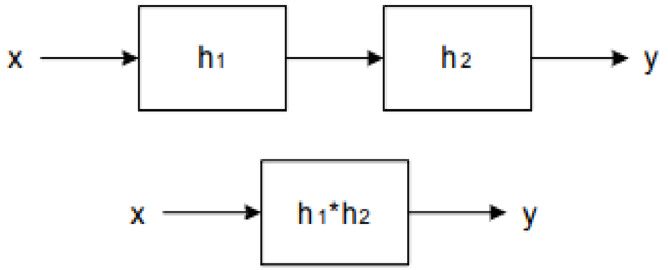


Fig 3.4 The associative property of convolution integral

Here we have y(t)=z(t) *h_2(t)But z(t)=x(t)*h1(t). Therefore y(t)=[x(t)*h_1(t)*h_2(t)]

Static and Dynamic property:

Static systems are also known as memory less systems. A system is known as static if its output at any time depends only on the value of the input at the same time. A continuous time system is memory less if its unit impulse response h(t) is zero for $t\neq 0$. These memory less LTI systems are characterized by y (t) =Kx (t) where K is constant.And its impulse response $h(t)=K\delta(t)$ .If K=1 ,then these systems are called identity systems.

Invertibility of LTI systems:

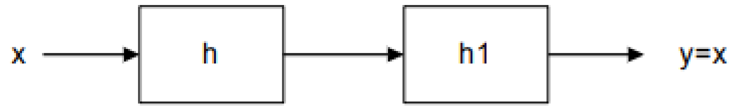


Fig.3.3 An inverse system for continuous time LTI system

A system is known as invertible only if an inverse system exists which, when cascade with the original system, produces and output equal to the input at first system. If an LTI system is invertible then it will have a LTI inverse system. This means that we have a continuous time LTI system with impulse response h(t) and its inverse system with impulse response $h_1(t)$ which results in an output equal to x(t). Cascade interconnection of original continuous time LTI system with its inverse system is given as identity system.

Thus, the overall impulse response of a system with impulse response h(t) cascaded with its inverse system with Impulse response $h_1(t)$ is given as $h(t)*h_1(t)=\delta(t)$.

Causality for LTI System

This property says that the output of a causal system depends only on the present and past values of the input to the system.A continuous time LTI system is called causal system if its impulse response h(t) is zero t<0. For a causal continuous time LTI system, convolution integral is given as

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{\tau=0}^{\infty} h(\tau)x(t-\tau)d\tau$$

For pure time shift with unit impulse response $h(t)=\delta(t-t_0)$ is a causal continuous time LTI system for $t\geq 0$. In this case time shift is known as a delay.

Stability for LTI systems

A stable system is a system which produces bounded output for every bounded input. Condition of Stability for continuous time LTI system:

Let us consider an input x(t) that is bounded in magnitude $|x(t)|<M$ for all values of t

Now, we apply this input to an continuous time LTI system with unit impulse response h(t). Output of this LTI system is determined by convolution integral and is given by

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Magnitude of output y(t) is given as

$$|y(t)| = \left|\int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| = \int_{\tau=-\infty}^{\infty} |h(\tau||x(t-\tau)|\, d\tau$$

$$|y(t)| = \left|\int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| = \int_{\tau=-\infty}^{\infty} |h(\tau||x(t-\tau)|\, d\tau$$

Substituting the value $|x(t-\tau) < M|$ for all values of τ and t, we get

$$|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)|M\, d\tau$$

Or

$|y(t)| \leq \int_{\tau=-\infty}^{\infty}|h(\tau)|\, d\tau$ for all values of t

From the above equation we can conclude that if the impulse response h(t) is absolutely integrable then output of a continuous time LTI system is bounded in magnitude, and thus, the system is bounded input, bounded output(BIBO) stable.

Unit step response of an LTI system:

Unit step response is the output of a LTI system for input is equal to unit step function or sequence. Unit step response of continuous time LTI system is found by convolution integral of u(t) with unit impulse response h(t) and is expressed as

$$g(t)=u(t)*h(t)=h(t)*u(t)$$

according to commutative property. Therefore, unit step response g(t) may be viewed as the response to the input h(t) of a continuous time LTI system with unit impulse response u(t).

Classification of CT LTI system:-

The systems are classified into two types: continuous time and discrete time systems, Now these two broad types of systems are further classified on the basis of system properties as under:

- Causal system and non causal system
- Time invariant and time variant system
- Stable and unstable system
- Linear and Non-linear system
- Static and Dynamic systems
- Invertible and noninvertible system

Causal systems and Non-causal systems

A system is causal if the response or output does not begin before the input function is applied. This means that if input is applied at $t=t_0$,then for causal system, output will depend on values of input x(t) for $t \leq t_0$.

Mathematically,

$y(t_0)=f[x(t),t \leq t_0]$.

In other words we can say that, the response or output of the causal system to an input does not depend on future values of that input but depends only on the present or past values of the input. This means that all the real-time systems are also causal systems since these systems cannot know the future values of the input signal when it constructs output signal. Thus, causal systems are physically realizable. For example a resister is a continuous time causal system because voltage across it is given by the expression v(t)=R.i(t) and output v(t),i.e., voltage depends only on the input i(t) i.e., current at the present time.

Time invariant and time variant system

A system is said to be time invariant if its input –output characteristics do not change with time. H{x(t)}=y(t) implies that ,H{x(t-t0)}=y(t-t0) for every input signal x(t) and every time shift t0A system is said to be time variant if its input- output characteristics changes with time.

Procedure to Test for Time Invariance:-

1. Delay the input signal by t0 units of time and determine the response of the system for this delayed input signal. Let this response be $y(t-t_0)$.

2. Delay the response of the system for undelayed input by t0 units of time. Let this delayed response be $y_d(t)$.

3. Check whether y(t-t0)=yd(t). If they are equal then the system is time invariant. Otherwise the system is time variant.

Stable and unstable system

A system is called bounded input, bounded output(BIBO) stable if and only if every bounded input results in a bounded output. The output of such a system does not diverge or does not grow unreasonably large.

Condition of Stability for continuous time LTI system:

Let us consider an input x(t) that is bounded in magnitude

$$|x(t)|<M<\infty \text{ for all values of } t$$

Now, we apply this input to an continuous time LTI system with unit impulse response h(t).

Output of this LTI system is determined by convolution integral and is given by

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Magnitude of output y(t) is given as

$$|y(t)| = \left|\int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| = \int_{\tau=-\infty}^{\infty} |h(\tau||x(t-\tau)|\, d\tau$$

Substituting the value $|x(t-\tau) < M|$ for all values of τ and t, we get

$$|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)|M\, d\tau$$

Or

$|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$ for all values of t

From the above equation we can conclude that if the impulse response h(t) is absolutely integerable then output of a continuous time LTI system is bounded in magnitude, and thus, the system is bounded input, bounded output(BIBO) stable.

The systems not satisfying the above conditions are unstable.

Linear and nonlinear system

A linear system is one that satisfies the superposition principle. The principle of superposition requires that the response of the system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses of the system to each of the individual input signals.

A system is linear if

$$H\{a_1x_1(t)+a_2x_2(t)\}=a_1H\{x_1(t)\}+a_2H\{x_2(t)\}$$

for any arbitrary input sequences $x_1(t)$ and $x_2(t)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the super position principle as given by the above definition, then the system is nonlinear.

Static and dynamic system

A continuous time system is called static or memory less if its output at any instant t depends on present input but not on the past or future samples of the input. These systems contain no energy storage elements. This means that the equation relating its output signal to its input signal contains no derivative, integrals or signal delays.

As an example, consider the system described by the following relationship

$Y(t)=x^2(t)$ this system is memory less because the value of the output signal y(t) at time t depends only on the present value of the input signal x(t).In any other case the system is said to be dynamic or to have memory. Dynamic systems have one or more energy storage elements. Input output relationship of a dynamic continuous time system is described by its differential equation.

Invertible and non invertible system

A system is said to be invertible if there is a one to one correspondence between its input and output signals. If a system is invertible, then an inverse system exists. The cascading of and invertible system and its inverse system is equivalent to the identity system.

The frequency response of an inverse system is basically reciprocal of the frequency response of the original system or invertible system.

An example of an invertible continuous time system is given by

$y_1(t)=3x(t)$and its inverse system will be given by $y_2(t) = \frac{1}{3}y_1(t)$

Convolution integral:-

The output of any general input may be found by convolving the given input signal x(t) with the LTI systems unit impulse response h(t).

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Or

$$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Properties of convolution integral:

- Commutative property
- Distributive property
- Associative property

Commutative property:

The commutative property is a basic property of convolution in both continuous and discrete time cases, thus, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are commutative. According to the property, for continuous time LTI system. The output is given by

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Or

$$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Thus, we can say that according to this property, the output of a continuous time LTI system having input x(t) and unit impulse h(t) is identical to the output of a continuous time LTI system having input h(t) and the unit impulse response x(t).

The distributive property:

The distributive property states that both convolution integral for continuous time LTI system and convolution sum for discrete time LTI system are distributive.

For continuous time LTI system, the distributive property is expressed as

The output, $y(t) = x(t) * [h_1(t) + h_2(t)]$

Or

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

Thus, the two continuous time LTI systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs and outputs are added as

$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

The output $y(t) = y_1(t) + y_2(t)$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

Associative Property of LTI system:

According to associative property, both convolution integral for continuous time LTI systems and convolution sum for discrete time LTI systems are associative.

For continuous time LTI system, according to associative property,
The output

$$y(t) = x(t) * [h_1(t) * h_2(t)]$$

or

$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

Here we have $y(t)=z(t) *h_2(t)$. But $z(t)=x(t)*h1(t)$. Therefore $y(t)=[x(t)*h_1(t)*h_2(t)]$

Linear constant coefficient differential equation:

The continuous time linear time invariant (LTI) systems are described by their linear constant coefficient differential equations. For this, let us consider a first order differential equation as under

$$\frac{dy(t)}{dt} + Ay(t) = x(t)$$

Where x (t) and y(t) are the input and output of the continuous time LTI system. A is a constant value. The first order differential equation can be extended for higher order differential equations.

A general Nth order linear constant coefficient differential equation can be given by

$$\sum_{k=0}^{N} A_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} \frac{d^k}{dt^k} x(t)$$

The complete solution of differential equation consists of the sum of particular solution $y_p(t)$ and homogenous solution $y_h(t)$.

The homogeneous solution of a differential equation is possible by substituting

$$\sum_{k=0}^{N} A_k \frac{d^k}{dt^k} y(t) = 0$$

This solution to differential equation is also known as natural response of the system.
A particular case of differential equation is determined by putting N=0, we obtain

$$y(t) = \frac{1}{A_0} \sum_{k=0}^{M} \frac{d^k}{dt^k} x(t)$$

Transfer function:

Transfer functions are commonly used in the analysis of systems such as single-input single-output filters, typically within the fields of signal processing, communication theory, and control theory. The term is often used exclusively to refer to linear, time-invariant systems (LTI). Most real systems have non-linear input/output characteristics, but many systems, when operated within nominal parameters have behavior that is close enough to linear that LTI system theory is an acceptable representation of the input/output behavior.The descriptions below are given in terms of a complex variable, $S=\sigma + j\omega$ which bears a brief explanation. In many applications, it is sufficient to define $\sigma = 0$, which reduces the Laplace transforms with complex arguments to Fourier transforms with real argument ω. The applications where this is common are ones where there is interest only in the steady-state response of an LTI system, not the fleeting turn-on and turn-off behaviors or stability issues. That is usually the case for signal processing and communication theory.

Thus, for continuous-time input signal x(t) and output y(t0, the transfer function H(s) is the linear mapping of the Laplace transform of the input, X(s)=L{x(t)}, to the Laplace transform of the output Y(s)=L{y(t)}:

$$Y(s)=H(s)X(s)$$

$$H(s) = \frac{Laplace\ transform\ of\ output}{laplace\ transform\ of\ input}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{L\{y(t)\}}{L\{x(t)\}}$$

Conditions required for transfer function:

(i) System should be in unloaded condition (initial conditions are zero)

(ii) The system should be linear time invariant.

Impulse Response:

In signal processing, the impulse response, of a dynamic system is its output when presented with a brief input signal, called an impulse. More generally, an impulse response refers to the reaction of any dynamic system in response to some external change. In both cases, the impulse response describes the reaction of the system as a function of time. In all these cases, the dynamic system and its impulse response may be actual physical objects, or may be mathematical systems of equations describing such objects. Since the impulse function contains all frequencies, the impulse response defines the response of a linear time- invariant system for all frequencies. The impulse can be modeled as a Dirac delta function for continuous- time systems, or as. The Dirac delta represents the limiting case of a pulse made very short in time while maintaining its area or integral. While this is impossible in any real system, it is a useful idealization. In Fourier theory, such an impulse comprises equal portions of all possible excitation frequencies, which makes it a convenient test probe. Any system in a large class known as linear, time-invariant (LTI) is completely characterized by its impulse response. That is, for any input, the output can be calculated in terms of the input and the impulse response. The impulse response of a linear transformation is the image of Dirac's delta function under the transformation, analogous to the fundamental solution of a partial differential operator. It is usually easier to analyze systems using transfer functions as opposed to impulse responses. The transfer function is the Laplace transform of the impulse response. The Laplace transform of a system's output may be determined by the multiplication of the transfer function with the input's Laplace transform in the complex plane, also known as the frequency domain. An inverse Laplace transform of this result will yield the output in the time domain. To determine an output directly in the time domain requires the convolution of the input with the impulse response. When the transfer function and the Laplace transform of the input are known, this convolution may be more complicated than the alternative of multiplying two functions in the domain. It is obtained by taking inverse Laplace transform of transfer function H(s).

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{Y(s)}{X(s)}\right\}$$

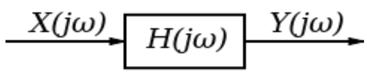
Frequency Response:

Frequency response is the quantitative measure of the output spectrum of a system or device in response to a stimulus, and is used to characterize the dynamics of the system. It is a measure of magnitude and phase of the output as a function of frequency, in comparison to the input.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

It is obtained from the transfer function by substituting s=jω in transfer function.

Systems respond differently to inputs of different frequencies. Some systems may amplify components of certain frequencies, and attenuate components of other frequencies. The way that the system output is related to the system input for different frequencies is called the frequency response of the system.The frequency response is the relationship between the system input and output in the Fourier Domain.



In this system, X(jω) is the system input, Y(jω) is the system output, and H(jω) is the frequency response. We can define the relationship between these functions as:

$$Y(j\omega)=H(j\omega)X(j\omega)$$

Since the frequency response is a complex function, we can convert it to polar notation in the complex plane. This will give us a magnitude and an angle.

Amplitude Response:

For each frequency, the magnitude represents the system's tendency to amplify or attenuate the input signal.

$$A(\omega) = |H(j\omega)|$$

Phase Response:

The phase represents the system's tendency to modify the phase of the input sinusoids.

$$\Phi(\omega) = \angle H(j\omega)$$

The phase response, or its derivative the group delay, tells us how the system delays the input signal as a function of frequency.

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SCHOOL OF MECHANICAL

DEPARTMENT OF MECHATRONICS

UNIT - IV

Signals and Systems – SEC1208

Unit IV – ANALYSIS OF DISCRETE TIME SYSTEMS

9Hrs

Spectrum of DT signals, Discrete Time Fourier Transform (DTFT)- Properties of DTFT - z-transform -Basic properties of Z transform - Region of convergence - Properties of ROC - Poles and Zeros - Inverse z-transform using Contour integration - Residue Theorem, Power Series expansion and Partial fraction expansion-Relation between DTFT and Z transform.

Discrete Time Fourier Transform (DTFT)

The discrete - time Fourier transform (DTFT) of a real, discrete - time signal x[n] is a complex - valued function defined by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

for any (integer) value of n.

Inverse Discrete Time Fourier Transform (IDTFT)

The function X ($e^{j\omega}$) or X (□) is called the Discrete - Time Fourier Transform (DTFT) of the discrete - time signal x(n). The inverse DTFT is defined by the following integral

$$x(n) = \frac{1}{2}\pi \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Find the DTFT of an impulse function which occurs at time zero.

$$x[n] = \delta[n]$$

$$X\left(e^{jw}\right) = \sum \delta[n] e^{-jwn} = 1$$

$$\delta[n] \overset{F}{\Longleftrightarrow} 1$$

$$\delta[n-1] \overset{F}{\Longleftrightarrow} (1) \cdot e^{-jw(1)}$$

Properties of DTFT

Property	Periodic signal	Fourier Series Coefficients
Linearity	$Ax[n]+By[n]$	Aa_k+Bb_k
Time Shifting	$x[n-n_0]$	$a_k \cdot e^{-jk\left(\frac{2\pi}{N}\right)n_0}$
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Frequency Shifting	$e^{jMw_0n}x[n]$	a_{k-M}
First Difference	$x[n]-x[n-1]$	$\left(1-e^{-jk\left(2\pi/N\right)}\right)a_k$
Conjugate Symmetry for Real Signals	x[n] real	$a_k=a^*_{-k}$
Real & Even Signals	x[n] real and even	a_k real and even
Real & Odd signals	x[n] real and odd	a_k purely imaginary and odd
Even-Odd Decomposition Of Real Signals	$x_e[n]=Ev\{x[n]\}$ $[x[n]real]$ $x_o[n]=Od\{x[n]\}$ $[x[n]real]$	$\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
Parseval's Relation	$\frac{1}{N}\sum_{n=\langle N\rangle}\lvert x[n]\rvert^2=\sum_{k=\langle N\rangle}\lvert a_k\rvert^2$	

Discrete Fourier Transform

The DFT is used to convert a finite discrete time sequence x (n) to an N point frequency domain sequence denoted by X (K). The N point DFT of a finite duration sequence x (n) is defined as

$$X(K)=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \text{ for } K=0,1,2,\ \ldots\ldots\ldots N-1$$

The discrete Fourier transform (DFT) is the Fourier transform for finite-length sequences because, unlike the (discrete-space) Fourier transform, the DFT has a discrete argument and can be stored in a finite number of infinite word-length locations. Yet, it turns out that the DFT can be used to exactly implement convolution for finite-size arrays

Inverse Discrete Fourier Transform

The IDFT is used to convert the N point frequency domain sequence X (K) to an N point time sequence. The IDFT of the sequence X (K) of length N is defined as

$$x(n) = 1/N \sum_{K=0}^{N-1} X(K)\, e^{+j2\pi nk/N} \quad \text{for } n=0, 1,2,\ldots\ldots N-1$$

Properties of DFT

1. Periodicity: X (K+N) =X (K) for all K.
2. Linearity: DFT[a1 x1 (n)+a2 x2(n)]=a1 X1 (K)+a2 X2 (K)
3. DFT of time reversed sequence: DFT[x(N-n)]=X(N-K)
4. Circular convolution :DFT[x1(n)*x2(n)]=X1(K) X2(K)
5. Shifting: If DFT {x (n)} =X (K), then DFT{x (n-no)} =X (K) e $^{-j2\pi no\ k/N}$
6. Symmetry property Re[X(N-k)]=ReX(k)

   This implies that amplitude has

   symmetry Im[X(N-k)]= - m[X(k)]

   This implies that the phase spectrum is anti symmetric.
7. If x[n] is an even function xe[n] then

   $$F[x_e[n]] = X_e(k) = \sum_{n=0}^{N-1} x_e[n]\cos(k\Omega nT)$$

   This implies that the transform is also even
8. If x[n] is odd function xo[n] than

   $$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

   This implies that the transform is purely imaginary and odd
9. Parseval's Theorem

   The normalized energy in the signal is given by either of the following expressions

   $$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

10. Delta Function

$$F[\delta(nT)] = 1$$

11. Unit step function

$$F[u[n]] = \frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty} \pi\delta(w+2\pi k)$$

12. Fourier transform of a CT complex exponential is interpreted as an impulse at $w=w_0$. For discrete-time we expect something similar but difference is that DTFT is periodic in w with period 2π. This says that FT of x[n] should have impulses at w_0, $w_0 \pm 2\pi$, $w_0 \pm 4\pi$ etc.

$$r_{x_1x_2}(j) = \frac{1}{N}\sum_{n=-\infty}^{\infty} x_1(n)x_2(n+j) \qquad ,-\infty \le j \le \infty$$

13. Linear cross-correlation of two data sequences or series may be computed using DFTs. The linear cross correlation of two finite-length sequences x1[n] and x2[n] each of length N is defined to be:

$$r_{x_1x_2}(j) = \frac{1}{N}\sum_{n=-\infty}^{\infty} x_1(n)x_2(n+j) \qquad ,-\infty \le j \le \infty$$

Circular correlation of finite length periodic sequences x1p[n] and x2p[n] is described as:

$$r_{cx_1x_2}(j) = \frac{1}{N}\sum_{n=0}^{N-1} x_{1p}(n)x_{2p}(n+j) \qquad , j = 0,\dots\dots,(N-1)$$

This circular correlation can be evaluated using DFTs as shown below:

$$r_{cx_1x_2}(j) = F^{-1}\left[X_1^*(k)X_2(k)\right]$$

The circular correlation can be converted into a linear correlation by using augmenting zeros. If the sequences are x1[n] of length N1 and x2[n] of length N2, then their linear correlation will be of length N1+N2-1.

To achieve this x1[n] is replaced by x1a[n] which consists of x1[n] with (N2-1) zeros added and x2[n] is augmented by (N1-1) zeros to become x2a[n].

$$\Rightarrow \qquad r_{x_1x_2}(j) = F^{-1}\left[X_{1a}^*(k)X_{2a}(k)\right]$$

1. Find the DFT of the following signal

x(n)=δ(n)

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \text{ for } K=0,\ 1,\ 2\ldots.,N-1$$

$$X(K) = \sum_{n=0}^{N-1} \delta(n) e^{-j2\pi nk/N} \text{ for } K=0,\ 1,\ 2,\ldots N-1$$

$$X(K) = 1$$

2. Consider a length - N sequence defined for n = 0,1,2,……,(N-1) where

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$ Find the DFT of the given sequence.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,2,\ldots\ldots,(N-1)$$

The N point DFT is equal to = 1

Basic Principles of Z Transform:

The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete-time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics.

These discrete models are solved with difference equations in a manner that is analogous to solving continuous models with differential equations. The role played by the z-transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations.

Types of Z Transform

Unilateral Z-transform

Alternatively, in cases where x[n] is defined only for $n \geq 0$, the single sided or unilateral Z- transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

In signal processing, this definition can be used to evaluate the Z-transform of the unit impulse response of a discrete time causal system

Bilateral Z-transform

The bilateral or two sided Z-transform of a discrete time signal x[n] is the formal power series X(Z) defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Where,

n is an integer and z is, in general, a complex number:

$$z = A\, e^{j\phi} = A\,(\cos \phi + j \sin \phi)$$

where,

A is the magnitude of z, j is the imaginary unit, and ϕ is the complex argument (also referred to as angle or phase) in radians.

Inverse Z Transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

1. The z-transform of the sequence Xn =cos(an) find its Z transform

$$Z[x_n] = Z[\cos(an)] = Z\left[\frac{1}{2}e^{ian} + \frac{1}{2}e^{-ian}\right]$$

$$= \frac{1}{2}Z\left[e^{ian}\right] + \frac{1}{2}Z\left[e^{-ian}\right] = \frac{1}{2}\frac{z}{z - e^{ia}} + \frac{1}{2}\frac{z}{z - e^{-ia}}$$

$$= \frac{1}{2}\left(\frac{z\left(z - e^{-ia}\right)}{(z - e^{ia})(z - e^{-ia})} + \frac{z\left(z - e^{ia}\right)}{(z - e^{ia})(z - e^{-ia})}\right)$$

$$= \frac{z\left(2z - e^{ia} - e^{-ia}\right)}{2(z - e^{ia})(z - e^{-ia})} = \frac{z\left(z - \frac{e^{ia}+e^{-ia}}{2}\right)}{(z - e^{ia})(z - e^{-ia})}$$

$$= \frac{z(z - \cos(a))}{(z - e^{ia})(z - e^{-ia})} = \frac{z(z - \cos(a))}{z^2 - 2z\cos(a) + 1}$$

2. Find the z-transform of the unit pulse or impulse sequence .

$$x_n = \delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = Z[x_n] = \sum_{n=0}^{\infty} x_n z^{-n} = 1 + \sum_{n=1}^{\infty} 0\, z^{-n} = 1$$

.

3. The z-transform of the unit-step sequence .

$x_n = u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ is $X(z) = \dfrac{z}{z-1}$

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

4. The z-transform of the sequence .

$$x_n = b^n \quad \text{is} \quad X(z) = \frac{z}{z-b}$$

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{b^n}{z^n}\right) = \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

$$= \frac{1}{1-\frac{b}{z}} = \frac{z}{z-b}$$

Properties of Z-Transform

Z-Transform has the following properties:

1. Linearity Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \xleftrightarrow{\text{Z.T}} a\,X(Z) + b\,Y(Z)$$

2. Time Shifting Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then Time shifting property states that

$$x(n-m) \xleftrightarrow{\text{Z.T}} z^{-m} X(Z)$$

3. Multiplication by Exponential Sequence Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n \,.\, x(n) \xleftrightarrow{\text{Z.T}} X(Z/a)$$

4. Time Reversal Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then time reversal property states that

$$x(-n) \overset{\text{Z.T}}{\longleftrightarrow} X(1/Z)$$

5. Differentiation in Z-Domain OR Multiplication by n Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \overset{\text{Z.T}}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

6. Convolution Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z)$$

7. Correlation Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z^{-1})$$

8. Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z \to 1} [z - 1] X(z)$$

This is used to find the final value of the signal without taking inverse z-transform.

Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z- transform.

Properties of ROC of Z-Transforms

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z- plane except at z = 0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z- plane except at z = ∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z|< a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

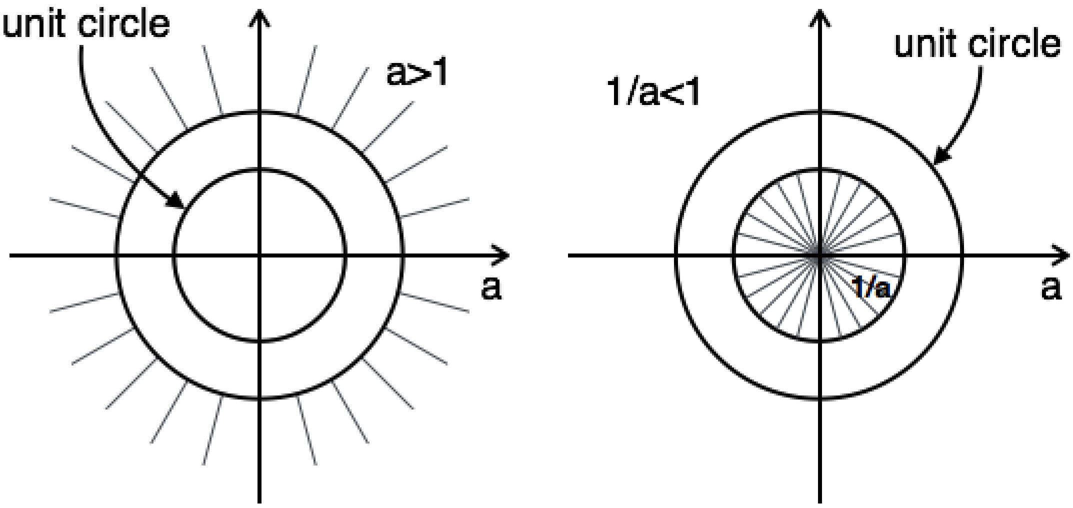
The concept of ROC can be explained by the following example:

Example 1: Find z-transform and ROC of

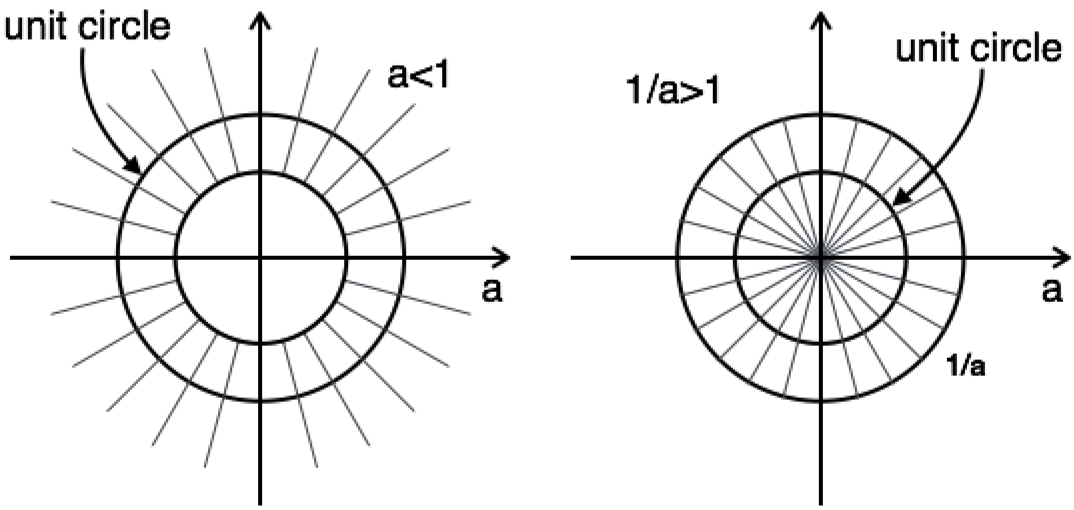
$$a^n u[n] + a^- nu[-n-1]$$

$$Z.T[a^n u[n]] + Z.T[a^{-n} u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

The plot of ROC has two conditions as a > 1 and a < 1, as the value of ‘a’ is not known.



In this case, there is no combination ROC.



Here, the combination of ROC is from

$$a < |z| < \frac{1}{a}$$

Hence for this problem, z-transform is possible when a < 1.

$$\mathcal{Z}[x[n]] = \frac{1}{1-az^{-1}} - \frac{1}{1-a^{-1}z^{-1}} = \frac{a^2-1}{a} \frac{z}{(z-a)(z-1/a)}$$

Determine the $\mathcal{Z}$-transform and their ROC of the following discrete time signals.

a) Given that, $x(n) = \{\underset{\uparrow}{3}, 2, 5, 7\}$

i.e., $x(0) = 3$; $x(1) = 2$; $x(2) = 5$; $x(3) = 7$; and $x(n) = 0$ for $n < 0$ and for $n > 3$.

By the definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range $n = 0$ to 3, hence the limits of summation is changed to $n = 0$ to $n = 3$.

$$\therefore X(z) = \sum_{n=0}^{3} x(n)\, z^{-n}$$

$$= x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3}$$

$$= 3 + 2z^{-1} + 5z^{-2} + 7z^{-3}$$

$$= 3 + \frac{2}{z} + \frac{5}{z^2} + \frac{7}{z^3}$$

In $X(z)$, when $z = 0$, except the first terms all other terms will become infinite. Hence $X(z)$ will be finite for all values of z, except $z = 0$. Therefore, the ROC is entire z-plane except $z = 0$.

b) Given that, $x(n) = \{6, 4, 5, \underset{\uparrow}{3}\}$

i.e, $x(-3) = 6$; $x(-2) = 4$; $x(-1) = 5$; $x(0) = 3$; and $x(n) = 0$ for $n < -3$ and for $n > 0$.

By the definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range $n = -3$ to 0, hence the limits of summation is changed to $n = -3$ to 0.

$$\therefore X(z) = \sum_{n=-3}^{0} x(n)\, z^{-n}$$

$$= x(-3)\, z^{3} + x(-2)\, z^{2} + x(-1)\, z + x(0)$$

$$= 6z^{3} + 4z^{2} + 5z + 3$$

In $X(z)$, when $z = \infty$, except the last term all other terms become infinite. Hence $X(z)$ will be finite for all values of z, except $z = \infty$. Therefore, the ROC is entire z-plane except $z = \infty$.

Determine the Z-transform and their ROC of the following discrete time signals.

a) Given that, x(n) = u(n)

The u(n) is a discrete unit step signal, which is defined as,

$$u(n) = 1 \; ; \; \text{for } n \geq 0$$
$$= 0 \; ; \; \text{for } n < 0$$

By the definition of Z-transform,

Infinite geometric series sum formula
$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \quad ; \quad \text{if, } 0 < \lvert C \rvert < 1$

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}}$$

Using infinite geometric series sum formula

$$= \frac{1}{1-1/z} = \frac{z}{z-1}$$

Here the condition for convergence is, $0 < |z^{-1}| < 1$.

$$\therefore \; |z^{-1}| < 1 \;\Rightarrow\; \frac{1}{|z|} < 1 \;\Rightarrow\; |z| > 1$$

The term $|z| = 1$ represents a circle of unit radius in z-plane. Therefore, the ROC is exterior of unit circle in z-plane.

Find the one sided Z-transform of the discrete time signals generated by mathematically sampling the following continuous time signal

Given that, $x(t) = t^2$

The discrete time signals is generated by replacing t by nT, where T is the sampling time period.

$$\therefore \quad x(n) = (nT)^2 = n^2T^2 = n^2\, g(n)$$

$$\text{where, } g(n) = T^2$$

By the definition of one sided Z-transform we get,

$$G(z) = \mathcal{Z}\{g(n)\} = \mathcal{Z}\{T^2\} = \sum_{n=0}^{\infty} T^2 z^{-n} = T^2 \sum_{n=0}^{\infty} (z^{-1})^n = T^2\left(\frac{1}{1-z^{-1}}\right) = \frac{T^2 z}{z-1}$$

By the property of Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2 g(n)\} = \left(-z\frac{d}{dz}\right)^2 G(z) = -z\frac{d}{dz}\left(-z\frac{d}{dz}G(z)\right)$$

$$= -z\frac{d}{dz}\left(-z\frac{d}{dz}\frac{T^2 z}{z-1}\right) = -z\frac{d}{dz}\left(-z \times \frac{(z-1)\,T^2 - T^2 z}{(z-1)^2}\right)$$

$$= -z\frac{d}{dz}\left(\frac{z\,T^2}{(z-1)^2}\right) = -z \times \frac{(z-1)^2\,T^2 - zT^2 \times 2(z-1)}{(z-1)^4}$$

$$= -z \times \frac{(z-1)\,(zT^2 - T^2 - 2zT^2)}{(z-1)^4} = -z \times \frac{-zT^2 - T^2}{(z-1)^3} = \frac{zT^2(z+1)}{(z-1)^3}$$

Find the one sided Z-transform of the following discrete time signals.

a) $x(n) = n\,a^{(n-1)}$ b) $x(n) = n^2$

Solution

a) Given that, $x(n) = n\,a^{(n-1)}$

Let, $x_1(n) = a^n$

By definition of one sided Z-transform,

$$X_1(z) = \sum_{n=0}^{\infty} x_1(n)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(a\,z^{-1}\right)^n$$

$$= \frac{1}{1 - a\,z^{-1}} = \frac{z}{z-a}$$

Using infinite geometric series sum formula

Let, $x_1(n-1) = a^{n-1}$

By shifting property,

$$\mathcal{Z}\{x_1(n-1)\} = z^{-1} X_1(z) = z^{-1}\frac{z}{z-a} = \frac{1}{z-a}$$

Given that, $x(n) = n\,a^{n-1}$

$$\mathcal{Z}\{x(n)\} = \mathcal{Z}\{n\,a^{n-1}\} = \mathcal{Z}\{n\,x_1(n-1)\} = -z\frac{d}{dz}X_1(z)$$

If $\mathcal{Z}\{x(n)\} = X(z)$

then $\mathcal{Z}\{n\,x(n)\} = -z\dfrac{d}{dz}X(z)$

$$= -z\frac{d}{dz}\frac{1}{z-a} = -z \times \frac{-1}{(z-a)^2} = \frac{z}{(z-a)^2}$$

---

b) Given that, $x(n) = n^2$

Let us multiply the given discrete time signal by a discrete unit step signal,

$\therefore\ x(n) = n^2\,u(n)$

Note : Multiplying a one sided sequence by u(n) will not alter its value.

By the property of Z-transform, we get,

$$\mathcal{Z}\{n^m u(n)\} = \left(-z\frac{d}{dz}\right)^m U(z)$$

$$\text{where, } U(z) = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

$$\therefore\ -z\frac{d}{dz}U(z) = -z\left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right] = -z\left[\frac{z-1-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$\left(-z\frac{d}{dz}\right)^2 U(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}U(z)\right]$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = -z\left(\frac{(z-1)^2 - z \times 2(z-1)}{(z-1)^4}\right)$$

$$= -z\left(\frac{(z-1)(z-1-2z)}{(z-1)^4}\right) = -z\left(\frac{-(z+1)}{(z-1)^3}\right) = \frac{z(z+1)}{(z-1)^3}$$

$$\therefore\ \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2 u(n)\} = \left(-z\frac{d}{dz}\right)^2 U(z) = \frac{z(z+1)}{(z-1)^3}$$

Zeros and Poles of Z -Transform

The complex variable, z is defined as,

$$z = u + jv$$

where, u = Real part of z

v = Imaginary part of z

Hence the z-plane is a complex plane, with u on real axis and v on imaginary axis

In the z-plane, the zeros are marked by small circle " " and the poles are marked by letter "X".

For example consider a rational function of z shown below.

$$X(z) = \frac{1.25 - 1.25\,z^{-1} + 0.2\,z^{-2}}{2 + 2\,z^{-1} + z^{-2}}$$

$$= \frac{1.25\left(1 - z^{-1} + \frac{0.2}{1.25}\,z^{-2}\right)}{2\left(1 + z^{-1} + \frac{1}{2}\,z^{-2}\right)} = \frac{0.625\,(1 - z^{-1} + 0.16\,z^{-2})}{(1 + z^{-1} + 0.5\,z^{-2})}$$

$$= \frac{0.625\,z^{-2}(z^2 - z + 0.16)}{z^{-2}(z^2 + z + 0.5)} = \frac{0.625\,(z - 0.8)\,(z - 0.2)}{(z + 0.5 + j0.5)\,(z + 0.5 - j0.5)}$$

The roots of quadratic, $z^2 - z + 0.16 = 0$ are,

$$z = \frac{1 \pm \sqrt{1 - 4 \times 0.16}}{2} = \frac{1 \pm 0.6}{2} = 0.8,\ 0.2$$

$$\therefore \quad z^2 - z + 0.16 = (z - 0.8)\,(z - 0.2)$$

The roots of quadratic, $z^2 + z + 0.5 = 0$ are,

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 0.5}}{2} = \frac{-1 \pm j}{2} = -0.5 \pm j\,0.5$$

$$\therefore \quad z^2 + z + 0.5 = (z + 0.5 + j0.5)\,(z + 0.5 - j0.5)$$

The zeros of X(z) are roots of numerator polynomial, which has two roots.

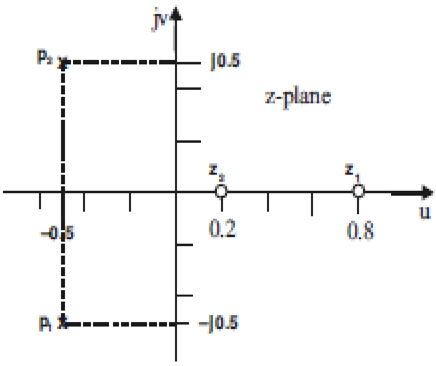
Therefore, the zeros of X(z) are,

$$z_1 = 0.8, \quad z_2 = 0.2$$

The poles of X(z) are roots of denominator polynomial, which has two roots.

Therefore, the poles of X(z) are,

$$p_1 = -0.5 - j0.5, \quad p_2 = -0.5 + j0.5$$



Inverse Z-Transform

Let X(z) be Z-transform of the discrete time signal x(n).The inverse Z-transform is the process of recovering the discrete time signal x(n) from its Z-transform X(z). The signal x(n) can be uniquely determined from X(z) and its ROC.

The inverse Z-transform can be determined by the following three methods.

1. Direct evaluation by contour integration (or residue method)
2. Partial fraction expansion method.
3. Power series expansion method.

Determine the inverse Z-transform of the function, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$ by the following three methods and prove that the inverse Z-transform is unique.

1. Residue Method
2. Partial Fraction Expansion Method
3. Power Series Expansion Method

Solution

Method-1 : Residue Method

Given that, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \dfrac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)} = \dfrac{3z^2 + 2z + 1}{z^2 - 3z + 2}$.

Let us divide the numerator polynomial by denominator polynomial and express X(z) as shown below.

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 - 3z + 2} = 3 + \frac{11z - 5}{z^2 - 3z + 2}$$

$$= 3 + \frac{11z - 5}{(z-1)(z-2)}$$

$$\begin{array}{r|l} & 3 \\ \hline z^2 - 3z + 2 & 3z^2 + 2z + 1 \\ & 3z^2 - 9z + 6 \\ & (-) \quad (+) \quad (-) \\ \hline & \qquad 11z - 5 \\ \hline \end{array}$$

Let, $X_1(z) = 3$ and $X_2(z) = \dfrac{11z - 5}{z^2 - 3z + 2}$; $\therefore\ X(z) = X_1(z) + X_2(z)$

$$x(n) = Z^{-1}\{X(z)\} = Z^{-1}\{X_1(z)\} + Z^{-1}\{X_2(z)\}$$

$$= Z^{-1}\{3\} + Z^{-1}\{X_2(z)\}$$

$$= 3\,\delta(n) + \sum_{i=1}^{N}\left[(z - p_i)\,X_2(z)\,z^{n-1}\Big|_{z=p_i}\right]$$

Using residue theorem

$$= 3\,\delta(n) + (z-1)\frac{11z - 5}{(z-1)(z-2)}z^{n-1}\Bigg|_{z=1} + (z-2)\frac{11z - 5}{(z-1)(z-2)}z^{n-1}\Bigg|_{z=2}$$

$$= 3\,\delta(n) + \frac{11 - 5}{1 - 2}(1)^{n-1} + \frac{11 \times 2 - 5}{2 - 1}2^{n-1}$$

$$\therefore\ x(n) = 3\,\delta(n) - 6\,u(n-1) + 17(2)^{n-1}\,u(n-1) = 3\,\delta(n) + \left[-6 + 17(2)^{n-1}\right]u(n-1)$$

$$\text{When } n=0,\quad x(0) = 3-0+0 \qquad = 3$$

$$\text{When } n=1,\quad x(1) = 0-6+17\times 2^0 = 11$$

$$\text{When } n=2,\quad x(2) = 0-6+17\times 2^1 = 28$$

$$\text{When } n=3,\quad x(3) = 0-6+17\times 2^2 = 62$$

$$\text{When } n=4,\quad x(4) = 0-6+17\times 2^3 = 130$$

$$\therefore\ x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, \ldots\}$$

Method-2 : Partial Fraction Expansion Method

$$\text{Given that, } X(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{z^{-2}(3z^2+2z+1)}{z^{-2}(z^2-3z+2)} = \frac{3z^2+2z+1}{(z-1)(z-2)}.$$

$$\therefore\ \frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)}$$

$$\text{Let, } \frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$$

$$\text{Now, } A_1 = z\frac{X(z)}{z}\bigg|_{z=0} = z\,\frac{3z^2+2z+1}{z(z-1)(z-2)}\bigg|_{z=0} = \frac{1}{(-1)\times(-2)} = 0.5$$

$$A_2 = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = (z-1)\,\frac{3z^2+2z+1}{z(z-1)(z-2)}\bigg|_{z=1} = \frac{3+2+1}{1\times(1-2)} = -6$$

$$A_3 = (z-2)\frac{X(z)}{z}\bigg|_{z=2} = (z-2)\,\frac{3z^2+2z+1}{z(z-1)(z-2)}\bigg|_{z=2} = \frac{3\times 2^2+2\times 2+1}{2\times(2-1)} = 8.5$$

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}$$

$$\therefore\ X(z) = 0.5 - 6\frac{z}{z-1} + 8.5\frac{z}{z-2}$$

$$\boxed{\begin{array}{l} \mathcal{Z}\{\delta(n)\} = 1 \\ \mathcal{Z}\{u(n)\} = \dfrac{z}{z-1} \\ \mathcal{Z}\{a^n u(n)\} = \dfrac{z}{z-a} \end{array}}$$

On taking inverse Z-transform of X(z) we get,

$$x(n) = 0.5\,\delta(n) - 6\,u(n) + 8.5\,(2)^n\,u(n) = 0.5\,\delta(n) + [-6+8.5(2)^n]\,u(n)$$

$$\text{When } n=0,\quad x(0) = 0.5-6+8.5\times 2^0 = 3$$

$$\text{When } n=1,\quad x(1) = 0-6+8.5\times 2^1 = 11$$

$$\text{When } n=2,\quad x(2) = 0-6+8.5\times 2^2 = 28$$

$$\text{When } n=3,\quad x(3) = 0-6+8.5\times 2^3 = 62$$

$$\text{When } n=4,\quad x(4) = 0-6+8.5\times 2^4 = 130$$

$$\therefore\ x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, \ldots\}$$

Method-3 : Power Series Expansion Method

Given that, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$

Let us divide the numerator polynomial by denominator polynomial as shown below.

$$
\begin{array}{r|l}
 & 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ldots\ldots \\
\hline
1 - 3z^{-1} + 2z^{-2} & 3 + 2z^{-1} + z^{-2} \\
 & \underline{\underset{(-)}{3} \underset{(+)}{-\,9z^{-1}} \underset{(-)}{+\,6z^{-2}}} \\
 & \quad 11z^{-1} - 5z^{-2} \\
 & \quad \underline{\underset{(-)}{11z^{-1}} \underset{(+)}{-\,33z^{-2}} \underset{(-)}{+\,22z^{-3}}} \\
 & \qquad 28z^{-2} - 22z^{-3} \\
 & \qquad \underline{\underset{(-)}{28z^{-2}} \underset{(+)}{-\,84z^{-3}} \underset{(-)}{+\,56z^{-4}}} \\
 & \qquad\quad 62z^{-3} - 56z^{-4} \\
 & \qquad\quad \underline{\underset{(-)}{62z^{-3}} \underset{(+)}{-\,186z^{-4}} \underset{(-)}{+\,124z^{-5}}} \\
 & \qquad\qquad 130z^{-4} - 124z^{-5} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$\therefore\ X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ldots..$$

Let $x(n)$ be inverse Z-transform of $X(z)$.

Now, by definition of Z-transform,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

$$= \ldots..+ x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + \ldots\ldots$$

On comparing equations (1) and (2) we get,

$x(0) = 3$

$x(1) = 11$

$x(2) = 28$

$x(3) = 62$

$x(4) = 130$ and so on.

$$\therefore\ x(n) = \{\underset{\uparrow}{3},\ 11,\ 28,\ 62,\ 130,\ \ldots..\}$$

Conclusion : It is observed that the results of all the three methods are same.

2. Find the Inverse Z Transform using Long Division Method

$$F(z) = \frac{2z^2 + z}{z^2 - 1.5z + 0.5}$$

$$
\begin{array}{r|l}
 & 2 + 4z^{-1} + 5z^{-2} + \cdots \\
\hline
z^2 - 1.5z + 0.5 & 2z^2 + z \\
 & \underline{2z^2 - 3z + 1} \\
 & \quad 4z - 1 \\
 & \quad \underline{4z - 6 + 2z^{-1}} \\
 & \qquad 5 - 2z^{-1} \\
 & \qquad \underline{5 - 7.5z^{-1} + 2.5} \\
 & \qquad\quad \vdots
\end{array}
$$

$$F(z) = 2 + 4z^{-1} + 5z^{-2} + \cdots$$

and the sequence f[k] is given by

$$f = \{2, 4, 5, \cdots\}$$

Inverse Z Transform using Residue Method:

Find the solution using the formula

$$y[n] = Z^{-1}[Y(z)] = \sum_{i=1}^{k} \mathrm{Res}[Y(z)\, z^{n-1}, z_i]$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z) = Y(z)\, z^{n-1}$.

Partial fraction method

Inverse Z Transform by Partial Fraction Expansion

This technique uses Partial Fraction Expansion to split up a complicated fraction into forms that are in the Z Transform table. As an example consider the function

$$F(z) = \frac{2z^2 + z}{z^2 - 1.5z + 0.5}$$

For reasons that will become obvious soon, we rewrite the fraction before expanding it by dividing the left side of the equation by "z."

$$\frac{F(z)}{z} = \frac{2z + 1}{z^2 - 1.5z + 0.5}$$

Now we can perform a partial fraction expansion

$$\begin{aligned} \frac{F(z)}{z} &= \frac{2z+1}{z^2 - 1.5z + 0.5} \\ &= \frac{2z+1}{(z-1)(z-0.5)} \\ &= \frac{A}{z-1} + \frac{B}{z-0.5} \\ &= \frac{6}{z-1} + \frac{-4}{z-0.5} \end{aligned}$$

These fractions are not in our table of Z Transforms. However if we bring the "z" from the denominator of the left side of the equation into the numerator of the right side, we get forms that are in the table of Z Transforms; this is why we performed the

$$F(z) = 6\frac{z}{z-1} - 4\frac{z}{z-0.5}$$

first step of dividing the equation by "z"

So

$$f[k] = 6u[k] - 4 \cdot 0.5^k$$

or

$$f = \{2, 4, 5, 5.5, \cdots\}$$

PART - A

1. Distinguish between DFT and DTFT?
2. State and prove Parseval's relation for DFT.
3. Find the DFT of the sequence {0,1,0,1}
4. Define DFT and IDFT
5. Find IDFT of X(k) = {1, 0, 1, 0}.
6. Define Z transform
7. Mention the types of Z transform
8. Find the Z transform of u(n)
9. Define ROC
10. Mention the properties of ROC

PART – B

1. Explain the Properties of Z Transform
2. Explain the properties of DFT
3. Mention the Properties of DTFT
4. Find the one sided Z-transform of the following discrete time signals.
   a) $x(n) = n^2\, 2^n\, u(n)$ b) $x(n) = n(0.5)^{n+2}$ c) $x(n) = (0.5)^n\, [u(n) - u(n-2)]$
5. Determine the inverse Z-transform of the following functions.
   a) $X(z) = \dfrac{5z^2}{(z+1)\,(z+2)^2}$ **b)** $X(z) = \dfrac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$ **c)** $X(z) = \dfrac{z(z^2-1)}{(z^2+1)^2}$

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SCHOOL OF MECHANICAL

DEPARTMENT OF MECHATRONICS

UNIT - II

Signals and Systems – SEC1208

UNIT II – ANALYSIS OF CONTINUOUS TIME SIGNAL

9 Hrs

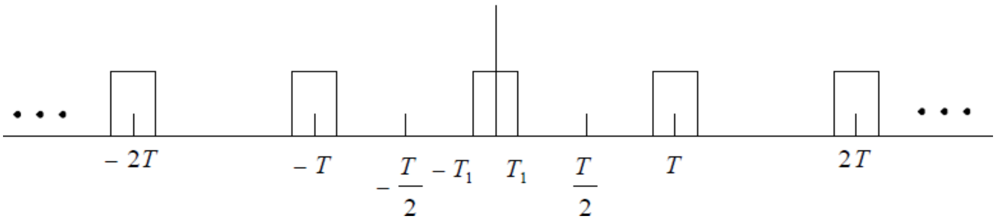
Continuous time Fourier Transform -Properties of CTFT-Inverse Fourier transform-unilateral and bilateral Laplace Transform analysis with examples - Basic properties - Parseval's relation - Convolution in time and frequency domain-Inverse Laplace transform using partial fraction expansion method - Relation between Fourier transform and Laplace transform-Fourier series analysis.

Continuous Time Fourier Transform

Any continuous time periodic signal x(t) can be represented as a linear combination of complex exponentials and the Fourier coefficients (or spectrum) are discrete. The Fourier series can be applied to periodic signals only but the Fourier transform can also be applied to non-periodic functions like rectangular pulse, step functions, ramp function etc. The Fourier transform of Continuous Time signals can be obtained from Fourier series by applying appropriate conditions. The Fourier transform can be developed by finding Fourier series of a periodic function and the tending T to infinity.

Representation of aperiodic signals: Starting from the Fourier series representation for the continuous-time periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{2.1}$$



The Fourier coefficients a_k for this square wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}. \tag{2.2}$$

or alternatively 2.3

$$Ta_k = \left. \frac{2\sin(\omega T_1)}{\omega} \right|_{\omega = k\omega_0}$$

Where, 2 sin(wT1) / w represent the envelope of Ta_k

When T increases or the fundamental frequency w $_0$ 2p / T decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.

Ta_k becomes more and more closely spaced samples of the envelope, as T $\rightarrow\infty$, the Fourier series coefficients approaches the envelope function.

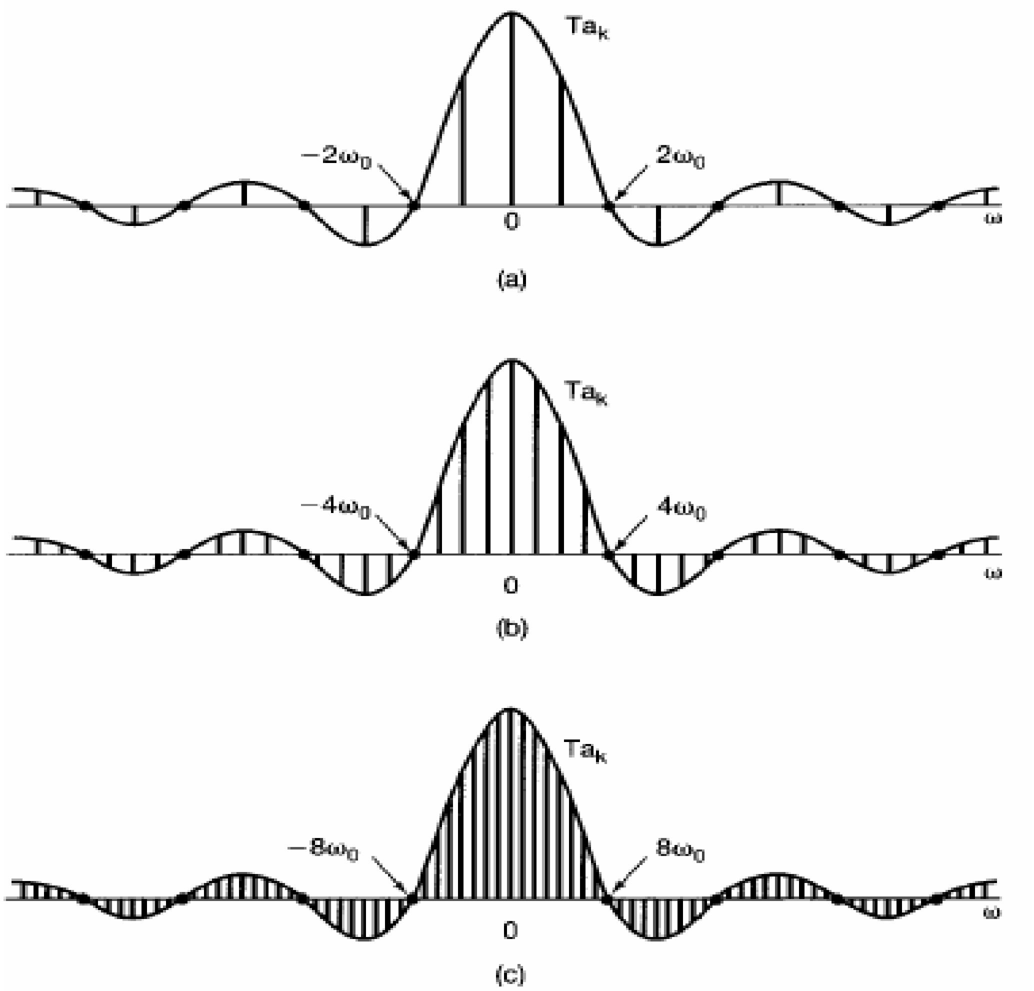


Fig. 2.1 Aperiodic Signals

This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals. Based on this idea, we can derive the Fourier transform for aperiodic signals.From this aperiodic signal, we construct a periodic signal (t) , shown in the figure below.

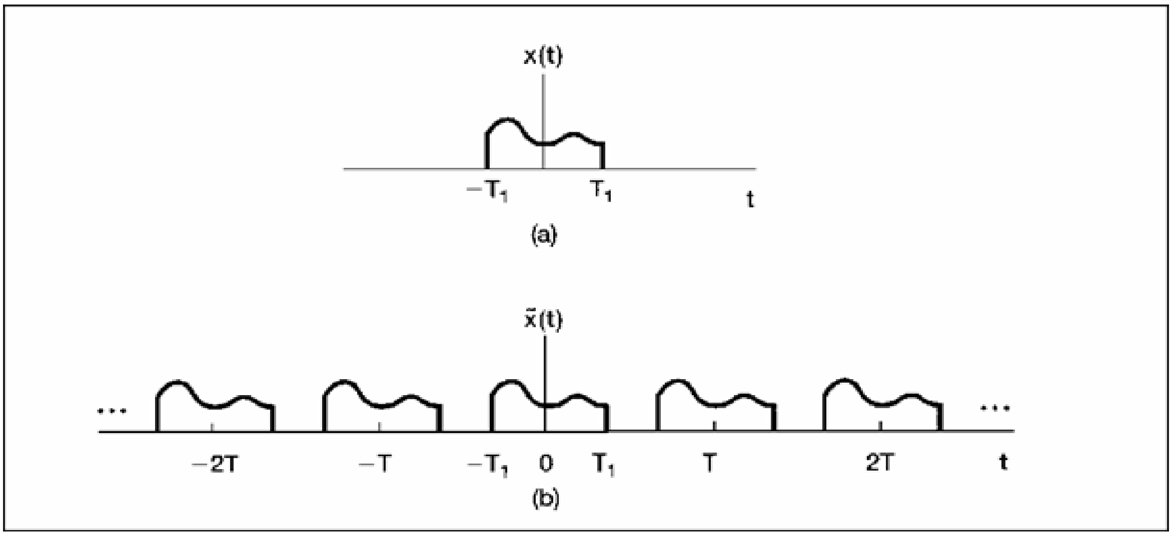


Fig. 2.2 Periodic Signals

As $T \to \infty$, $\tilde{x}(t) = x(t)$, for any infinite value of t

The Fourier series representation of $\tilde{x}(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \qquad 2.4$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt. \qquad 2.5$$

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, so we have

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt. \qquad 2.6$$

Define the envelope X (jw) of Ta_k as,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \qquad 2.7$$

we have for the coefficients a_k,

$$a_k = \frac{1}{T} X(jk\omega_0)$$

Then $\tilde{x}(t)$ can be expressed in terms of $X(j\omega)$, that is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 . \qquad 2.8$$

As $T \to \infty$, $\tilde{x}(t) = x(t)$ and consequently,

Equation 2.8 becomes representation of *x(t)*. In addition the right hand side of equation becomes an integral.

This results in the following Fourier Transform.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \textit{Inverse Fourier Transform} \qquad 2.9$$

and

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \textit{Fourier Transform}$$
2.10

Convergence of Fourier Transform

If the signal x(t) has finite energy, that is, it is square integrable,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty ,$$

Then we guaranteed that X(jw) is finite or equation 2.10 converges. If $e(t) = \tilde{x}(t) - x(t)$,

We have

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0 .$$

An alternative set of conditions that are sufficient to ensure the convergence:

Contition1: Over any period, x(t) must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty ,$$

Condition 2: In any finite interval of time, x(t) have a finite number of maxima and minima.

Condition 3:

In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Continuous-Time Fourier Transform

consider signal $x(t) = e^{-at}u(t)$, $a > 0$.

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a + j\omega}, \qquad a > 0$$

If a is complex rather than real, we get the same result if Re{a}>0

The Fourier transform can be plotted in terms of the magnitude and phase, as shown in the figure below.

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \qquad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

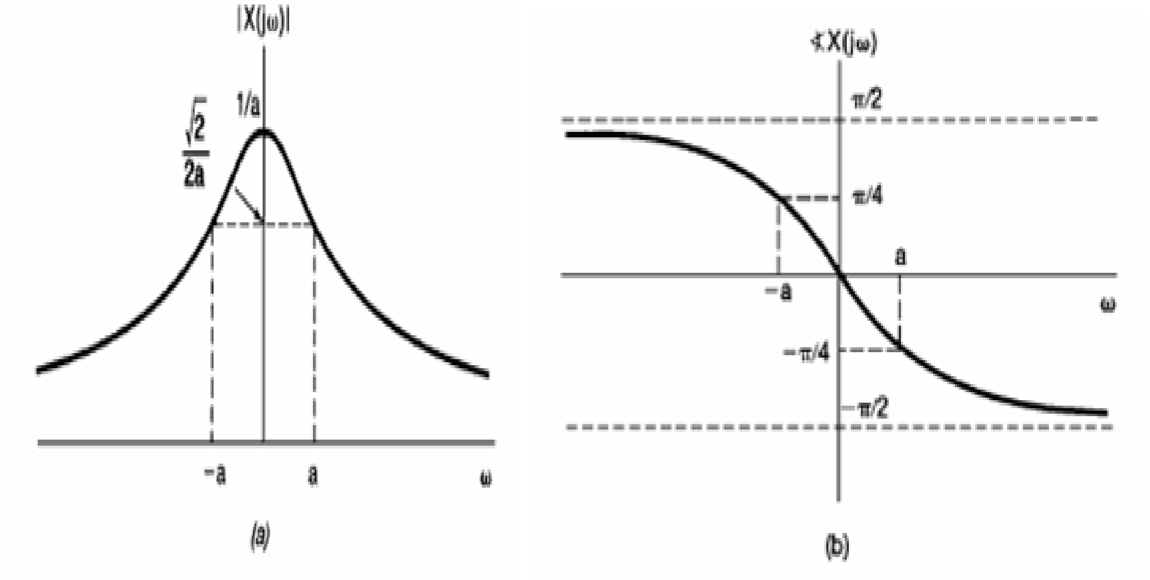
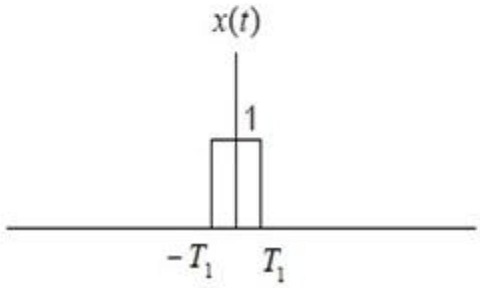


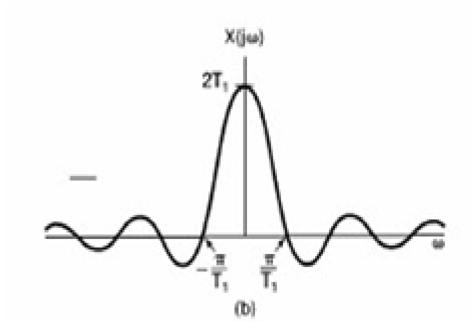
Fig. 2.3 Magnitude and Phase plot

Example:

Calculate the Fourier transform of the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}.$$





$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} 1e^{-j\omega t}dt = 2\frac{\sin \omega T_1}{\omega}.$$

The inverse Fourier Transform of the sinc function is

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{\sin Wt}{\pi t}$$

Comparing the results we have,

$$\text{Square wave} \underset{FT^{-1}}{\overset{FT}{\rightleftarrows}} \text{Sinc Function}$$

This means a square wave in the time domain, its Fourier transform is a sinc function. However, if the signal in the time domain is a sinc function, then its Fourier transform is a square wave. This property is referred to as **Duality Property.**

We also note that when the width of $X(jw)$ increases, its inverse Fourier transform x(t) will be compressed. When W $\rightarrow \infty$, $X(jw)$ converges to an impulse. The transform pair with several different values of W is shown in the figure below.

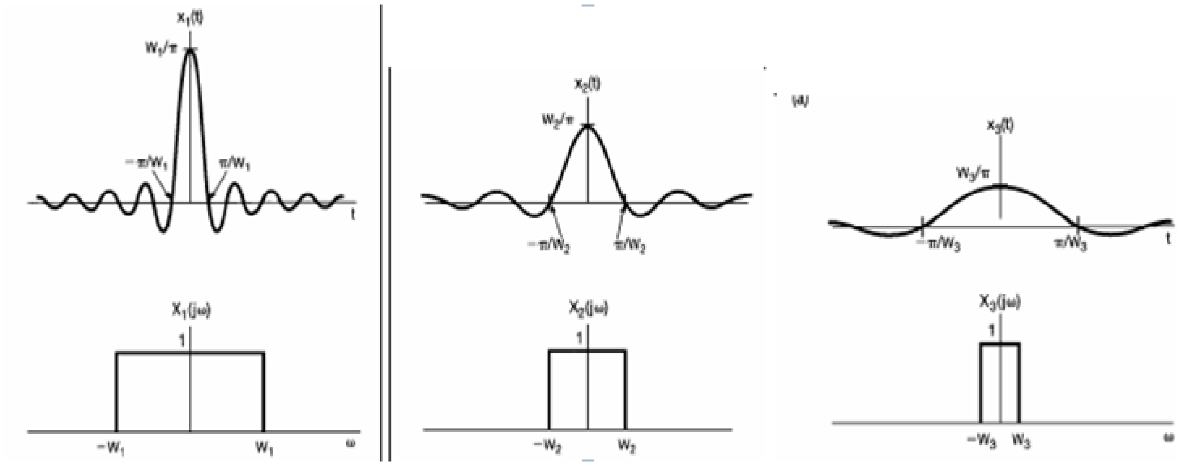


Fig. 2.3 Transform Pairs

The Fourier series representation of the signal $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \,.$$

It's Fourier transform is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \,.$$

Properties of Fourier Transform

1. Linearity

If $x(t) \xleftrightarrow{F} X(jw)$ $y(t) \xleftrightarrow{F} Y(jw)$

then

$ax(t) + by(t) \xleftrightarrow{F} aX(jw) + bY(jw)$

2. Time Shifting

If $x(t) \xleftrightarrow{F} X(jw)$

Then

$x(t-t_0) \xleftrightarrow{F} e^{-jwt_0} X(jw)$

3. Conjugation and Conjugate Symmetry

If $x(t) \overset{F}{\longleftrightarrow} X(jw)$

Then

$x^*(t) \overset{F}{\longleftrightarrow} X^*(-jw)$

4. Differentiation and Integration

If $x(t) \overset{F}{\longleftrightarrow} X(jw)$ *then* $\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\leftrightarrow} \frac{1}{j\omega} X(jw) + \pi X(0)\delta(\omega)$$

5. Time and Frequency Scaling

$x(t) \overset{F}{\leftrightarrow} X(j\omega)$

then,

$x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

From the equation we see that the signal is compressed in the time domain, the spectrum will be extended in the frequency domain. Conversely, if the signal is extended, the corresponding spectrum will be compressed.

If a = -1, we get from the above equation,

$X(-t) \overset{F}{\leftrightarrow} X(j\omega)$

That is reversing a signal in time also reverses its Fourier transform.

6. Duality

The duality of the Fourier Transform can be demonstrated using the following example.

$x_1(t) = \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases} \overset{F}{\leftrightarrow} X_1^{(j\omega)} = \frac{2 \sin \omega T_1}{\omega}$

$x_2(t) = \frac{sinWT_1}{\pi t} \overset{F}{\leftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$

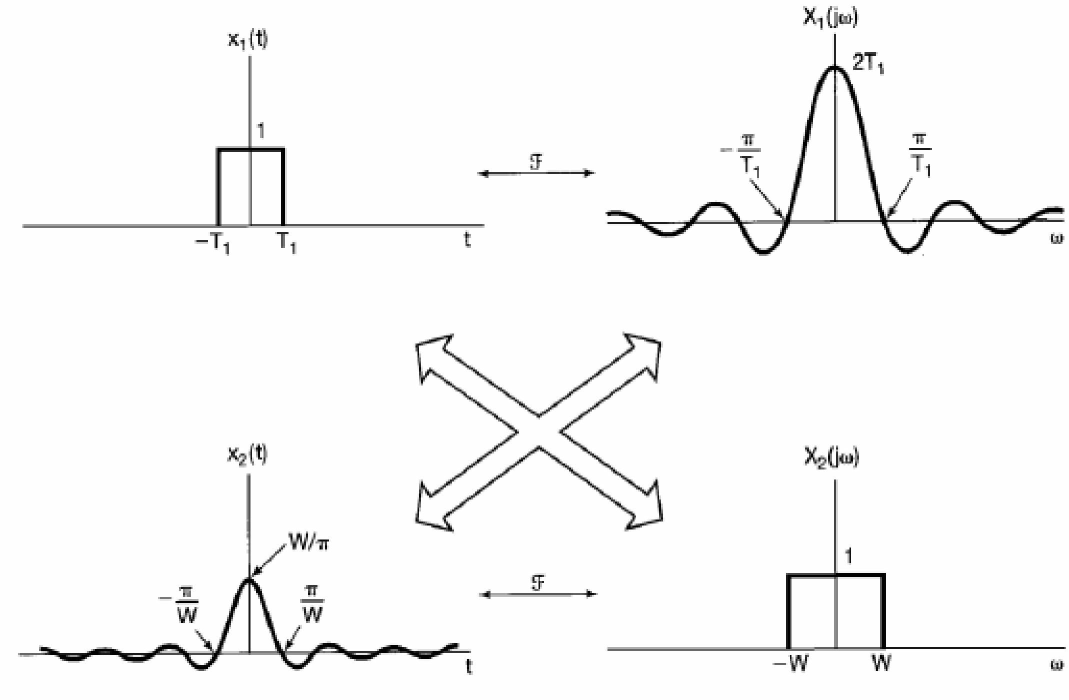


Fig. 2.4 Dual Pairs

For any transform pair, there is a dual pair with the time and frequency variables interchanged.

$$-jtx(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X(j(\omega-\omega_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{F} \int_{-\infty}^{\omega} x(\eta)d\eta$$

Parseval's Relation

If $x(t) \overset{F}{\leftrightarrow} X(j\omega)$,

We have,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(jw)|^2 \, d\omega$$

Parseval‟s relation states that the total energy may be determined either by computing the energy per unit time $x(t)^2$ and integrating over all time or by computing the energy per unit frequency $x(j\omega)^2/2\pi$ and integrating over all frequencies. For this reason, $(j\omega)^2$ is often referred to as the energy density spectrum.

The Parseval's theorem states that the inner product between signals is preserved in going from time to the frequency domain. This is interpreted physically as "Energy calculated in the time domain is same as the energy calculated in the frequency domain"

The convolution properties

$$y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

The equation shows that the Fourier transform maps the convolution of two signals into product of their Fourier transforms.

H(jω), the transform of the impulse response is the frequency response of the LTI system, which also completely characterizes an LTI system.

Example

The frequency response of a differentiator.

$$y(t)=\frac{dx(t)}{dt}$$

From the differentiation property,

$Y(j\omega)= j\omega X\ j\omega$

The frequency response of the differentiator is

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=j\omega$$

The Multiplication Property

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Multiplication of one signal by another can be thought of as one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is often referred to as amplitude modulation.

Laplace Transform

The Laplace Transform is the more generalized representation of CT complex exponential signals. The Laplace transform provide solutions to most of the signals and systems, which are not possible with Fourier method. The Laplace transform can be used to analyze most of the signals which are not absolutely integrable such as the impulse response of an unstable system. Laplace Transform is a powerful tool for analysis and design of Continuous Time signals and systems. The Laplace Transform differs from Fourier Transform because it covers a broader class of CT signals and systems which may or may not be stable.

Till now, we have seen the importance of Fourier analysis in solving many problems involving signals. Now, we shall deal with signals which do not have a Fourier transform. We note that the Fourier Transform only exists for signals which can absolutely integrated and have a finite energy. This observation leads to generalization of continuous-time Fourier transform by considering a broader class of signals using the powerful tool of "Laplace transform". With this introduction let us go on to formally defining both Laplace transform.

Definition of Laplace Transform

The Laplace transform of a function x(t) can be shown to be,

$$\mathbf{L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt}$$

This equation is called the Bilateral or double sided Laplace transform equation.

$$x(t) = \int_{-\infty}^{\infty} X(s) e^{st} ds$$

This equation is called the Inverse Laplace Transform equation, x(t) being called the Inverse Laplace transform of X(s). The relationship between x(t) and X(s) is

$$x(t) \xrightarrow{LT} X(s)$$

Region of Convergence (ROC):

The range of values for which the expression described above is finite is called as the Region of Convergence (ROC).

Convergence of the Laplace transform

The bilateral Laplace Transform of a signal x(t) exists if

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Substitute s=$\sigma + j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\, dt$$

Relationship between Laplace Transform and Fourier Transform

The Fourier Transform for Continuous Time signals is in fact a special case of Laplace Transform. This fact and subsequent relation between LT and FT are explained below. Now we know that Laplace Transform of a signal 'x'(t)' is given by

Now we know that Laplace Transform of a signal 'x'(t)' is given by:

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

The s-complex variable is given by $s = \sigma + j\Omega$

But we consider σ = 0 and therefore "s" becomes completely imaginary. Thus we have s=jΩ. This means that we are only considering the vertical strip at σ = 0.

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$$

From the above discussion it is clear that the LT reduces to FT when the complex variable only consists of the imaginary part. Thus LT reduces to FT along the jΩ axis. (imaginary axis).

Fourier Transform of x(t) = Laplace Transform of $x(t)_{\ s=j\Omega}$

Laplace transform becomes Fourier transform

if σ=0 and s=jω.

$$X(s)|_{s=j\omega}=FT\{x(t)\}$$

Example of Laplace Transform

(1) Find the Laplace transform and ROC of x(t) = $e^{-at}u(t)$

We notice that by multiplying by the term u(t) we are effectively considering the unilateral Laplace Transform whereby the limits tend from 0 to +∞

Consider the Laplace transform of x(t) as shown below

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a}\text{; for } (s+a) > 0$$

(2) Find the Laplace transform and ROC of x(t)=-$e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} - e^{-at}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a}\text{; for } (s+a) < 0$$

If we consider the signals $e^{-at}u(t)$ and $-e^{-at}u(-t)$, we note that although the signals are differing, their Laplace Transforms are identical which is 1/(s+a). Thus we conclude that to distinguish L.T's uniquely their ROC's must be specified.

Properties of Laplace Transform

1. Linearity

If $x_1(t) \overset{L}{\leftrightarrow} X_1(s)$ with ROC R1 and $x_2(t) \overset{L}{\leftrightarrow} X_2(s)$ with ROC R2, then
$ax_1(t) + bx_2(t) \overset{L}{\leftrightarrow} aX_1(s) + bX_2(s)$ with ROC containing $R_1 \cap R_2$

The ROC of X(s) is at least the intersection of R_1 and R_2, which could be empty, in which case x(t) has no Laplace Transform.

2. Differentiation in the time domain

If $x(t) \overset{L}{\leftrightarrow} X(s)$ with ROC = R then $\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s)$ with ROC = R.
This property follows by integration by parts.

Hence, $\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s)$ The ROC of *sX(s)* includes the ROC of X(s) and may be larger.

3. Time Shift

If $x(t) \overset{L}{\leftrightarrow} X(s)$ with ROC = R then

$x(t-t_0) \overset{L}{\leftrightarrow} e^{-st_0}X(s)$ with ROC = R

4. Time Scaling

If $x(t) \overset{L}{\leftrightarrow} X(s)$ with ROC = R, then

$x(at) \overset{L}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$; ROC = $\frac{R}{a}$ ie., $\frac{s}{a} \epsilon R$

5. Multiplication

$x(t) \times y(t) \overset{L}{\leftrightarrow} \frac{1}{2\pi j}[X(S) * Y(S)]$

6. Time Reversal

When the signal x(t) is time reversed(180° Phase shift)

$X(-t) \overset{L}{\leftrightarrow} X(-s)$

7. Frequency Shifting

$$e^{s_0 t}x(t) \overset{L}{\leftrightarrow} X(s - s_0)$$

8. Conjugation symmetry

$x^*(t) \overset{L}{\leftrightarrow} x^*(-s)$

9. Parseval's Relation of Continuous Signal

It states that the total average power in a periodic signal x(t) equals the sum of the average in individual harmonic components, which in turn equals to the squared magnitude of X(s) Laplace Transform.

$$\int_0^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} |X(S)|^2 ds$$

10. Differentiation in Frequency

When x(t) is differentiated with respect to frequency then,

$$-t\,x(t) \overset{L}{\leftrightarrow} \frac{dX(s)}{dS}$$

11. Integration Property

When a periodic signal x(t) is integrated, then the Laplace Transform becomes,

$$\int_{-\infty}^{t} x(t)dt \overset{L}{\leftrightarrow} \frac{1}{S}X(S) + \frac{\int_{-\infty}^{\infty} x(\tau)d\tau}{S}$$

12. Convolution Property

$$x(t) * y(t) \overset{L}{\leftrightarrow} X(s).Y(S)$$

13. Initial Value Theorem

The initial value theorem is used to calculate initial value $x(0^+)$ of the given sequence x(t) directly from the Laplace transform X(S). The initial value theorem does not apply to rational functions X(S) whose numerator polynomial order is greater than the denominator polynomial orders.

The initial value theorem states that,

$$\lim_{S\to\infty} SX\,S = X(0^+)$$

14. Final Value Theorem

It states that,

$$\lim_{S\to\infty} SX\,S = X(\infty)$$

Questions for Practice

Part A

1. Determine the Fourier transform of unit impulse signal.
2. Find the Fourier transform of signum function.
3. Find the Laplace transform of hyperbolic sine function.
4. What is Fourier series representation of a signal?
5. Write discrete time Fourier series pair.
6. List out the properties of Fourier series.
7. What is the need for transformation of signal?
8. Define Fourier Transform.
9. Define Laplace Transform.
10. Mention the properties of Laplace Transform.
11. How do we fine Fourier series coefficient for a given signal.
12. State time shifting property of CT Fourier series.
13. State time shifting property of DT Fourier series.
14. Determine the Fourier series coefficient of sinω_0n.
15. State conjugate symmetry of CT Fourier series.
16. Determine the Fourier series coefficient of cosω_0n.
17. Determine the Fourier Transform of x(t)=sin $\omega_0\, t$.
18. Determine the Fourier Transform of x(t)=cos $\omega_0 t$.
19. Determine the Fourier Transform of Step signal
20. Find the Laplace transform of the signal x(t) = e^{-bt}

Part B

1. State Parseval"s theorem for discrete time signal.
2. Find the FT of the following and sketch the magnitude and phase spectrum
   (i) x(t) = $\delta(t)$
   (ii) x(t) = $e^{-at}u(t)$
   (iii) x(t) = e^{-t}
   (iv) x(t) = $e^{2t}\; u\;\; t$
3. Find the Laplace transform of
   (i) x(t) = $\delta\;\; t$
   (ii) x(t) = u(t)
   (iii) x(t) = cos $\Omega_0\, t$
   (iv) x(t) = sin $\omega_t u(t)$
4. Determine initial and final value of a signal x(t) = sin $2t\;\, u(t)$
5. Determine the initial and final value of signal x(t) whose unilateral Laplace Transform is,X(S) = (2S+5)/S(S+3)
6. State and prove the properties of Fourier Transform.
7. State and prove the properties of Laplace Transform.

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SCHOOL OF MECHANICAL

DEPARTMENT OF MECHATRONICS

UNIT - V
Signals and Systems – SEC1208

Unit V - LINEAR TIME INVARIANT DISCRETE TIME SYSTEMS

9Hrs

Concept of LTI-DT systems - Properties and types of LTIDT systems- causality, stability, invertibility, time invariant, linearity -interconnection of LTI Systems -Difference equation - Computation of Impulse response, Frequency response, step response, natural response, forced response and Transfer function using Z Transform, Convolution Sum using matrix, graphical and tabulation method-Properties of convolution sum.

Discrete-Time LTI Systems

Consider a discrete-time system with input x[n] and output y[n]. First, define the impulse response of the system to be the output when x[n] = δ[n] (i.e., the input is an impulse function). Denote this impulse response by the signal h[n].

Now, consider an arbitrary signal x[n]. Recall from the sifting property of the impulse function that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k],$$

By the additivity property

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

The above is called the convolution sum; the convolution of the signals x[n] and h[n] is denoted by

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Thus we have the following very important property of discrete-time LTI systems: if x[n] is the input signal to an LTI system, and h[n] is the impulse response of the system, then the output of the system is

y[n] = x[n] * h[n]

Example - 1 Consider an LTI system with impulse response

$$h[n] = \begin{cases} 1 & \text{if } 0 \le n \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the input signal is

$$x[n] = \begin{cases} 1 & \text{if } 0 \le n \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Since both $x[k] = 0$ for $k < 0$ and $h[n-k] = 0$ for $k > n$,

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k].$$

Thus $y[n] = 0$ for $n < 0$. When $n = 0$ we have

$$y[0] = \sum_{k=0}^{0} x[k]h[-k] = x[0]h[0] = 1.$$

When $n = 1$ we have

$$y[1] = \sum_{k=0}^{1} x[k]h[1-k] = x[0]h[1] + x[1]h[0] = 2.$$

Similarly, $y[2] = 3$, $y[3] = 4$, $y[4] = 3$, $y[5] = 2$, $y[6] = 1$ and $y[n] = 0$ for $n \ge 7$

Example - 2 Consider an LTI system with impulse response h[n] = u[n].

Suppose the input signal is $x[n] = \alpha^n u[n]$ with $0 < \alpha < 1$. Then we have

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Since both $x[k] = 0$ for $k < 0$ and $h[n-k] = 0$ for $k > n$, we have

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for $n \ge 0$, and $y[n] = 0$ for $n < 0$.

Properties of Linear Time-Invariant Systems

In this section we will study some useful properties of the convolution operation; based on the previous section, this will have implications for the input-output behavior of linear time-invariant systems (h[n] for discrete-time systems and h(t) for continuous-time systems).

1. The Commutative Property

The first useful property of convolution is that it is commutative:

x[n] * h[n] = h[n] * x[n]

x(t) * h(t) = h(t) * x(t)

To see this, start with the definition of convolution and perform a change of variable by setting r = n - k. This gives

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n]$$

2. The Distributive Property

The second useful property of convolution is that it is distributive:

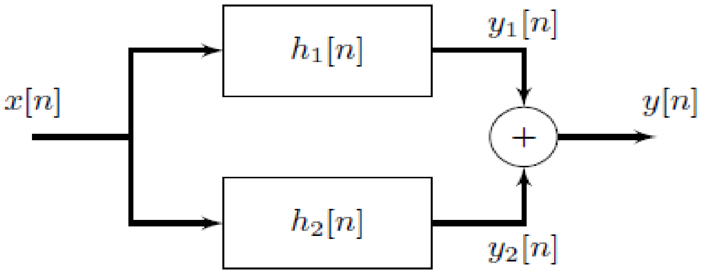
x[n] * (h1[n] + h2[n]) = x[n] * h1[n] + x[n] * h2[n]

x(t) * (h1(t) + h2(t)) = x(t) * h1(t) + x(t) * h2(t)

This property is easy to verify:

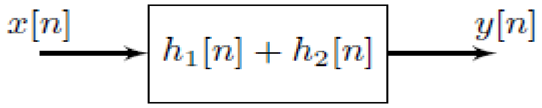
$$\begin{aligned} x[n] * (h_1[n] + h_2[n]) &= \sum_{k=-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k] \\ &= x[n] * h_1[n] + x[n] * h_2[n]. \end{aligned}$$

The distributive property has implications for LTI systems connected in parallel:



y[n] = y1[n] + y2[n] = x[n] * h1[n] + x[n] * h2[n] = x[n] * (h1[n] + h2[n])

The above expression indicates that the parallel interconnection can equivalently be viewed as x[n] passing through a single system whose impulse response is h1[n] + h2[n]:



3. The Associative Property

A third useful property of convolution is that it is associative:

x[n] * (h1[n] * h2[n]) = (x[n] * h1[n]) * h2[n]

x(t) * (h1(t) * h2(t)) = (x(t) * h1(t)) * h2(t)

In other words, it does not matter which order we do the convolutions. The above relationships can be proved by manipulating the summations (or integrals);

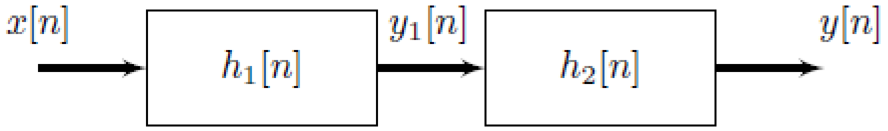


Fig. 3.1 A series interconnection of systems

Just as the distributive property had implications for parallel interconnections of systems, the associative property has implications for series interconnections of systems. Specifically, consider the series interconnection shown in Fig. 3.1. We have

y[n] = y1[n] * h2[n] = (x[n] * h1[n]) * h2[n] = x[n] * (h1[n] * h2[n])

Thus, the series interconnection is equivalent to a single system with impulse response h1[n] * h2[n], as shown in Fig. 3.2.

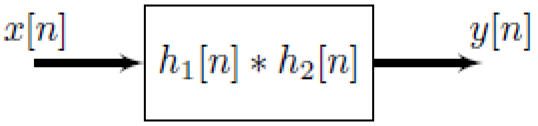


Fig. 3.2 The equivalent representation of a series interconnection.

Further note that since h1[n] * h2[n] = h2[n] * h1[n], we can also interchange the order of the systems in the series interconnection as shown in Fig. 3.3, without changing the overall input-output relationship between x[n] and y[n].

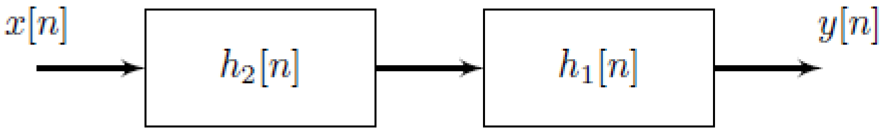


Fig. 3.3 Equivalent series representation

Figure 3.3: An equivalent series representation of the interconnection shown in Fig. 3.1.

4. Memory less LTI Systems

Let us now see the implications of the memory less property for LTI systems. Specifically, let h[n] (or h(t)) be the impulse response of a given LTI system.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k],$$

we see that $y[n]$ will depend on a value of the input signal other than at time-step n unless $h[k] = 0$ for all $k \neq 0$. In other words, for an LTI system to be memoryless, we require $h[n] = K\delta[n]$ for some constant K. Similarly, a continuous-time LTI system is memory less if and only if h(t) = Kδ(t) for some constant K. In both cases, all LTI memory less systems have the form

y[n] = Kx[n] or y(t) = Kx(t) for some constant K.

5. Invertibility of LTI Systems

Consider an LTI system with impulse response h[n] (or h(t)). Recall that the system is said to be invertible if the output of the system uniquely specifies the input. If a system is invertible, there is another system (known as the inverse system") that takes the output of the original system and outputs the input to the original system, as shown in Fig. 3.4.

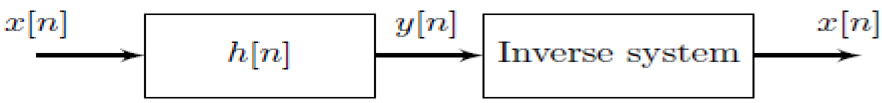


Fig.3.4 System in series with its inverse

Suppose the second system is LTI and has impulse response hI [n]. Then, by the associative property discussed earlier, we see that the series interconnection of the system with its inverse is equivalent (from an input-output sense) to a single system with impulse response h[n] * h_I [n]. In particular, we require

$$x[n] = x[n] * (h[n] * h_I[n])$$

for all input signals x[n], from which we have

$$h[n] * h_I[n] = \delta[n]$$

Suppose the second system is LTI and has impulse response h_I [n]. Then, by the associative property discussed earlier, we see that the series interconnection of the system with its inverse is equivalent (from an input-output sense) to a single system with impulse response h[n] * h_I [n]. In particular, we require

$$x[n] = x[n] * (h[n] * h_I[n])$$

for all input signals x[n], from which we have

$$h[n] * h_I[n] = \_[n]$$

In other words, if we have an LTI system with impulse response h[n], and another LTI system with impulse response h_I [n] such that h[n] * h_I [n] = δ[n], then those systems are inverses of each other. The analogous statement holds in continuous-time as well.

6. Causality of LTI Systems

Recall that a system is causal if its output at time t depends only on the inputs up to (and potentially including) t. To see what this means for LTI systems, consider the convolution sum.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k],$$

where the second expression follows from the commutative property of the convolution. In order for $y[n]$ to not depend on $x[n+1], x[n+2], \ldots$, we see that $h[k]$ must be zero for $k < 0$. The same conclusion holds for continuous-time systems, and thus we have the following: **A continuous-time LTI system is causal if and only if its impulse response $h(t)$ is zero for all $t < 0$. A discrete-time LTI system is causal if and only if its impulse response $h[n]$ is zero for all $n < 0$.**

Note that causality is a property of a system; however we will sometimes refer to a signal as being causal, by which we simply mean that its value is zero for n or t less than zero.

7. Stability of LTI Systems

To see what the LTI property means for stability of systems, consider again the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Note that

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |x[k]h[n-k]|$$
$$= \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]|.$$

Now suppose that $x[n]$ is bounded, i.e., there exists some $B \in \mathbb{R}_{\geq 0}$ such that $|x[n]| \leq B$ for all $n \in \mathbb{Z}$. Then the above expression becomes

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[n-k]|.$$

Thus, if $\sum_{k=-\infty}^{\infty} |h[n-k]| < \infty$ (which means that $h[n]$ is **absolutely summable**), then $|y[n]|$ will also be bounded for all n. It turns out that this is a necessary condition as well: if $\sum_{k=-\infty}^{\infty} |h[n-k]| = \infty$, then there is a bounded input that causes the output to be unbounded.

Example – 3 Test the stability of the following systems.

a) y(n) = cos[x(n)] b) y(n) = x(-n-2) c) y(n) = n x(n)

a) Given that, y(n) = cos [x(n)]

The given system is nonlinear system, and so the test for stability should be performed for specific inputs.

The value of cosθ lies between -1 to +1 for any value of θ. Therefore the output y(n) is bounded for any value of input x(n). Hence the given system is stable.

b) Given that, y(n) = x(-n-2)

The given system is time variant system, and so the test for stability should be performed for specific inputs.

The operations performed by the system on the input signal are folding and shifting. A bounded input signal will remain bounded even after folding and shifting. Therefore in the given system, the output will be bounded as long as input is bounded. Hence the given system is BIBO stable.

c) Given that, y(n) = n x(n)

The given system is time variant system, and so the test for stability should be performed for specific inputs.

<u>Case i :</u> If x(n) tends to infinity or constant, as "n" tends to infinity, then y(n) = n x(n) will be infinite as "n" tends to infinity. So the system is unstable.

<u>Case ii :</u> If x(n) tends to zero as "n" tends to infinity, then y(n) = n x(n) will be zero as "n" tends to infinity. So the system is stable.

8. Step Response of LTI Systems

Just as we defined the impulse response of a system to be the output of the system when the input is an impulse function, we define the step response of a system to be the output when the input is a step function u[n] (or u(t) in continuous-time). We denote the step response as s[n] for discrete-time systems and s(t) for continuous-time systems.

To see how the step response is related to the impulse response, note that

$$s[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k] = \sum_{k=-\infty}^{\infty} u[n-k]h[k] = \sum_{k=-\infty}^{n} h[k].$$

This is equivalent to $s[n] = s[n-1] + h[n]$. Thus, the step response of a discrete-time LTI system is the running sum of the impulse response.

For continuous-time systems, we have the same idea:

$$s(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau.$$

Differentiating both sides and applying the fundamental theorem of calculus, we have

$$\frac{ds}{dt} = h(t),$$

i.e., the impulse response is the derivative of the step response.

Differential and Difference Equation Models

For Causal LTI Systems As we have already seen in a few examples, many systems can be described using differential equation (in continuous-time) or difference-equation (in discrete time) models, capturing the relationship between the input and the output. For example, for a vehicle with velocity v(t) and input acceleration a(t), we have

$$dv/dt = a(t)$$

In discrete-time, consider a bank-account where earnings are deposited at the end of each month. Let the amount in the account at the end of month n be denoted by s[n]. Then we have

$$s[n] = (1 + r)s[n - 1] + x[n]$$

Where r is the interest rate and x[n] is the new amount deposited into the account at the end of month n.

Example - 4 Consider the differential equation

$$y''(t) + y'(t) - 6y(t) = x'(t) + x(t)$$

where $x(t) = e^{4t}u(t)$.

We first search for a homogeneous solution $y_h(t) = Ae^{mt}$ satisfying

$$y_h''(t)+y_h'(t)-6y_h(t) = 0 \Rightarrow m^2Ae^{mt}+mAe^{mt}-6Ae^{mt} = 0 \Rightarrow (m^2+m-6) = 0$$

This yields $m = -3$ or $m = 2$. Thus, the homogeneous solution is of the form

$$y_h(t) = A_1e^{-3t} + A_2e^{2t}$$

for some constants A_1 and A_2 that will be determined from the initial conditions.

To find a particular solution, note that for $t > 0$, we have $x'(t) + x(t) = 4e^{4t} + e^{4t} = 5e^{4t}$. Thus we search for a particular solution of the form $y_p(t) = Be^{4t}$ for $t > 0$. Substituting into the differential equation (3.1), we have

$$y_p''(t)+y_p'(t)-6y_p(t) = x'(t)+x(t) \Rightarrow 16Be^{4t}+4Be^{4t}-6Be^{4t} = 5e^{4t} \Rightarrow B = \frac{5}{14}.$$

Thus, $y_p(t) = \frac{5}{14}e^{4t}$ for $t > 0$ is a particular solution.

The overall solution is then of the form $y(t) = y_h(t) + y_p(t) = A_1e^{-3t} + A_2e^{2t} + \frac{5}{14}e^{4t}$ for $t > 0$. If we are told that the system is at rest until the input is applied, and that $y(0) = y'(0) = 0$, we have

$$y(0) = A_1 + A_2 + \frac{5}{14} = 0$$
$$y'(0) = -3A_1 + 2A_2 + \frac{20}{14} = 0.$$

Solving these equations, we obtain $A_1 = \frac{1}{7}$ and $A_2 = -\frac{1}{2}$. Thus, the solution is

$$y(t) = \left(\frac{1}{7}e^{-3t} - \frac{1}{2}e^{2t} + \frac{5}{14}e^{4t}\right) u(t).$$

Example – 5 Determine the response of first order discrete time system governed by the difference equation,

y(n) = - 0.5 y(n - 1) + x(n)

When the input is unit step, and with initial condition a) y(-1) = 0 b) y(-1) = 1/3.

Given that, $y(n) = -0.5\, y(n-1) + x(n)$

$$\therefore\ y(n) + 0.5\, y(n-1) = x(n) \qquad .....(1)$$

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when $x(n) = 0$.

$$\therefore\ y(n) + 0.5\, y(n-1) = 0 \qquad .....(2)$$

Put, $y(n) = \lambda^n$ in equation (2).

$$\therefore\ \lambda^n + 0.5\, \lambda^{(n-1)} = 0$$

$$\lambda^{(n-1)}(\lambda + 0.5) = 0 \quad \Rightarrow \quad \lambda = -0.5$$

The homogeneous solution $y_h(n)$ is given by,

$$y_h(n) = C\lambda^n = C(-0.5)^n \ ; \quad \text{for } n \geq 0 \qquad .....(3)$$

Particular Solution

Given that the input is unit step and so the particular solution will be in the form,

$$y(n) = K\, u(n) \qquad .....(4)$$

On substituting for $y(n)$ from equation (4) in equation (1) we get,

$$K\, u(n) + 0.5\, K\, u(n-1) = u(n) \qquad .....(5)$$

In order to determine the value of K, let us evaluate equation (5) for $n = 1$, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when $n = 1$, we get,

$$K + 0.5\, K = 1$$

$$1.5\, K = 1$$

$$\therefore \quad K = \frac{1}{1.5} = \frac{10}{15} = \frac{2}{3}$$

The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\, u(n) = \frac{2}{3}\, u(n) \ ; \quad \text{for all } n$$

$$= \frac{2}{3} \qquad ; \quad \text{for} \quad n \geq 0$$

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution.

$\therefore$ Response, $y(n) = y_h(n) + y_p(n)$

$$= C(-0.5)^n + \frac{2}{3} \quad ; \quad \text{for } n \geq 0 \qquad .....(6)$$

At n = 0, from equation (1), we get, $y(0) + 0.5\,y(-1) = 1$

$$\therefore\ y(0) = 1 - 0.5\,y(-1) \qquad ....(7)$$

At n = 0, from equation (6), we get, $y(0) = C + \frac{2}{3}$ .....(8)

On equating (7) and (8) we get, $C + \frac{2}{3} = 1 - 0.5\,y(-1)$

$$\therefore\ C = 1 - 0.5\,y(-1) - \frac{2}{3}$$

$$= \frac{1}{3} - 0.5\,y(-1) \qquad .....(9)$$

On substituting for C from equation (9) in equation (6) we get,

$$y(n) = \left(\frac{1}{3} - 0.5\,y(-1)\right)(-0.5)^n + \frac{2}{3}$$

a) When y(−1) = 0

$$y(-1) = 0$$

$$\therefore\ y(n) = \frac{1}{3}(-0.5)^n + \frac{2}{3} \quad ; \quad \text{for} \quad n \geq 0$$

b) When y(−1) = 1/3

$$y(-1) = \frac{1}{3}$$

$$\therefore\ y(n) = \left(\frac{1}{3} - 0.5 \times \frac{1}{3}\right)(-0.5)^n + \frac{2}{3}$$

$$= \frac{0.5}{3}(-0.5)^n + \frac{2}{3}$$

$$= \frac{1}{6}(-0.5)^n + \frac{2}{3} \ ; \ \text{for } n \geq 0$$

Discrete or Linear Convolution

The Discrete or Linear convolution of two discrete time sequences x1(n) and x2(n) is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)}$$

where, $x3(n)$ is the sequence obtained by convolving $x1(n)$ and $x2(n)$

m is a dummy variable

If the sequence $x1(n)$ has N1 samples and sequence $x2(n)$ has N2 samples then the output sequence

$x3(n)$ will be a finite duration sequence consisting of "N1+N2-1" samples. The convolution results in a non periodic sequence. Hence this convolution is also called aperiodic convolution.

The convolution relation of equation (6.29) can be symbolically expressed as

$x3(n) = x1(n) * x2(n) = x2(n) * x1(n)$

where, the symbol * indicates convolution operation.

Procedure for Evaluating Linear Convolution

1. Change of index : Change the index n in the sequences $x1(n)$ and $x2(n)$, to get the sequences $x1(m)$ and $x2(m)$.
2. Folding : Fold $x2(m)$ about $m = 0$, to obtain $x2(-m)$.
3. Shifting : Shift $x2(-m)$ by q to the right if q is positive, shift $x2(-m)$ by q to the left if q is negative to obtain $x2(q- m)$.
4. Multiplication : Multiply $x1(m)$ by $x2(q - m)$ to get a product sequence. Let the product sequence be $vq(m)$. Now, $vq(m) = x1(m) * x2(q- m)$.
5. Summation : Sum all the values of the product sequence $vq(m)$ to obtain the value of $x3(n)$ at $n = q$. [i.e., $x3(q)$].

Properties of Linear Convolution

The Discrete convolution will satisfy the following properties.

Commutative property : $x1(n) * x2(n) = x2(n) * x1(n)$

Associative property : $[x1(n) * x2(n)] * x3(n) = x1(n) * [x2(n) * x3(n)]$

Distributive property : $x1(n) * [x2(n) + x3(n)] = [x1(n) * x2(n)] + [x1(n) * x3(n)]$

Methods of Performing Linear Convolution

Method -1: Graphical Method

Let x1(n) and x2(n) be the input sequences and x3(n) be the output sequence.

1. Change the index "n" of input sequences to "m" to get x1(m) and x2(m).
2. Sketch the graphical representation of the input sequences x1(m) and x2(m).
3. Let us fold x2(m) to get x2(-m). Sketch the graphical representation of the folded sequence x2(-m).
4. Shift the folded sequence x2(-m) to the left graphically so that the product of x1(m) and shifted x2(-m) gives only one non-zero sample. Now multiply x1(m) and shifted x2(-m) to get a product sequence, and then sum-up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift x2(– m) of previous step to one position right and multiply the shifted sequence with x1(m) to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
6. To get subsequent samples of output sequence, the step-5 is repeated until we get a non-zero product sequence.

Method -2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representations of the sequences are employed instead of graphical representation. In tabular method, every input sequence, folded and shifted sequence is represented by a row in a table.

Method -3: Matrix Method

Let x1(n) and x2(n) be the input sequences and x3(n) be the output sequence. In matrix method one of the sequences is represented as a row and the other as a column as shown below.

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence x3(n). (The sums of the diagonal elements are shown below for reference).

Example – 6 Determine the response of the LTI system whose input x(n) and impulse response h(n) are given by,

$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$$

In this example the convolution operation is performed by three methods.

The Input sequence starts at n = 0 and the impulse response sequence starts at n = –1. Therefore the output sequence starts at n = 0 + (–1) = –1.

The input and impulse response consists of 4 samples, so the output consists of 4 + 4 – 1 = 7 samples.

Method 1 : Graphical Method

The graphical representation of x(n) and h(n) after replacing n by m are shown below. The sequence h(m) is folded with respect to m = 0 to obtain h(– m).

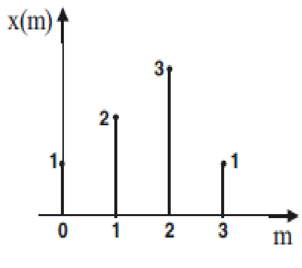


Fig 1 : Input sequence.

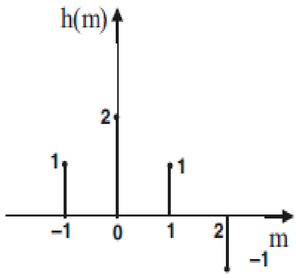


Fig 2 : Impulse response.

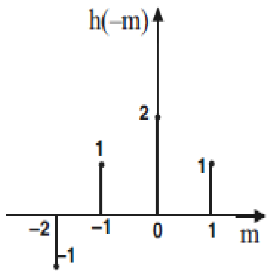


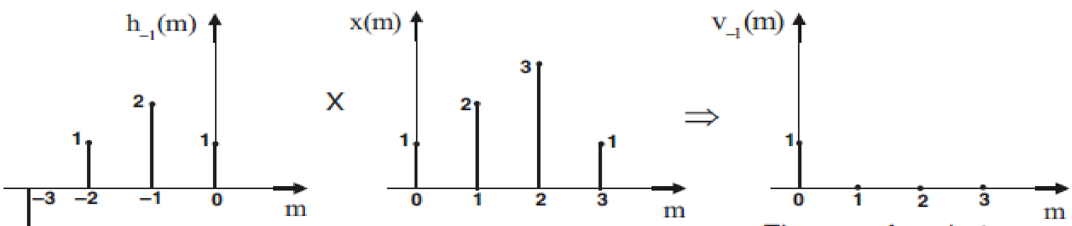
Fig 3 : Folded impulse response.

The samples of y(n) are computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m) \text{ ; where } h_n(m) = h(n-m)$$

The computation of each sample using the above equation are graphically shown in fig 4 to fig 10. The graphical representation of output sequence is shown in fig 11.

$$\text{When } n = -1 \text{ ; } y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$$



The sum of product sequence $v_{-1}(m)$ gives y(–1). ∴ y(–1) = 1

Fig 4 : Computation of y(–1).

When $n = 0$; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$

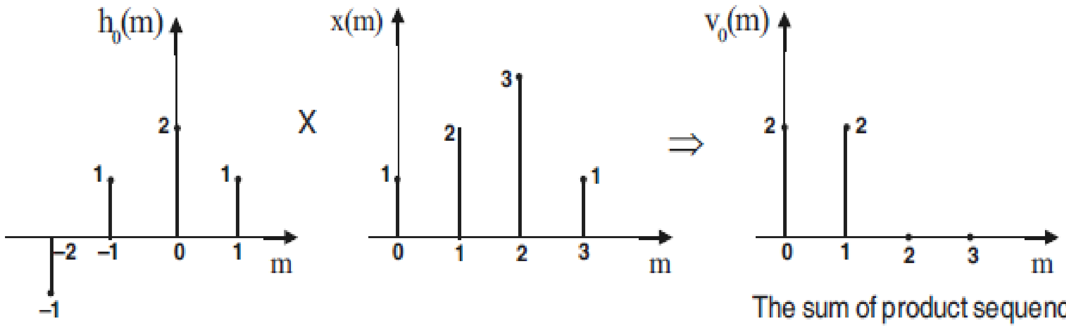


Fig 5 : Computation of y(0).

The sum of product sequence $v_0(m)$ gives $y(0)$. $\therefore\ y(0) = 2 + 2 = 4$

When $n = 1$; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$

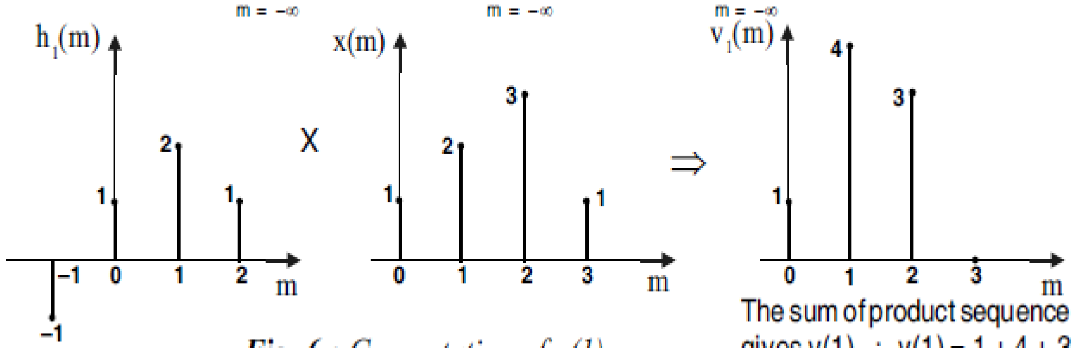


Fig 6 : Computation of y(1).

The sum of product sequence $v_1(m)$ gives $y(1)$. $\therefore\ y(1) = 1 + 4 + 3 = 8$

When $n = 2$; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$

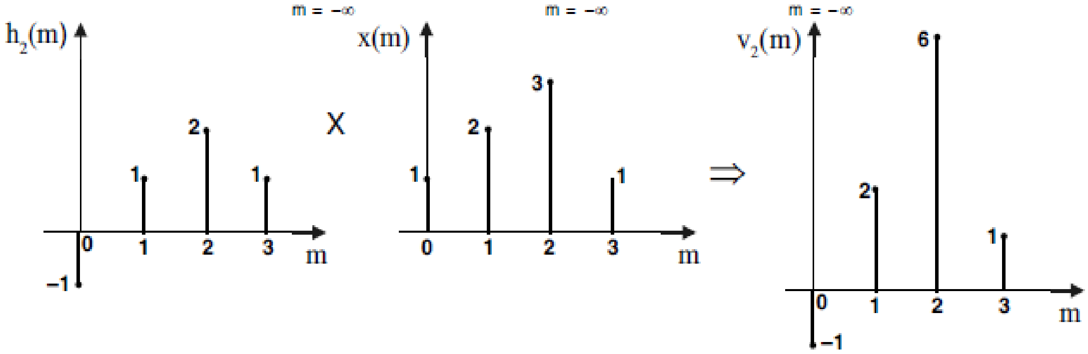
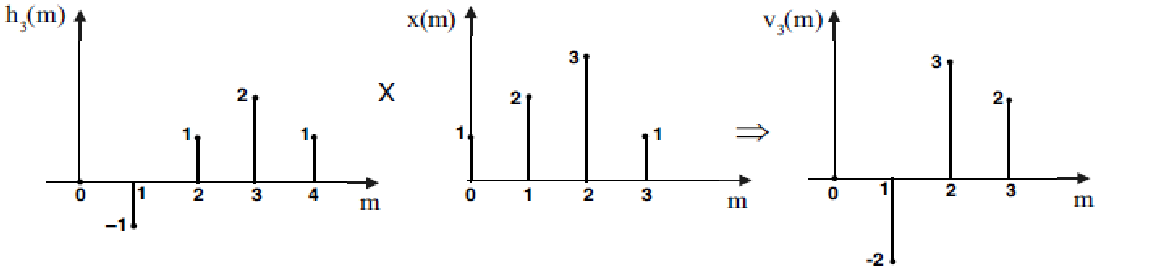


Fig 7 : Computation of y(2).

The sum of product sequence $v_2(m)$ gives $y(2)$. $\therefore\ y(2) = -1 + 2 + 6 + 1 = 8$

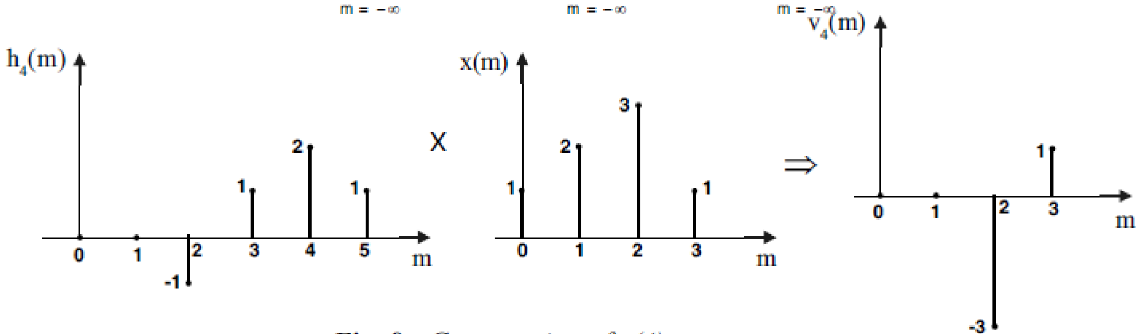
When $n = 3$; $y(3) = \sum_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum_{m=-\infty}^{+\infty} v_3(m)$



The sum of product sequence $v_3(m)$ gives $y(3)$. $\therefore$ $y(3) = -2 + 3 + 2 = 3$

Fig 8 : Computation of y(3).

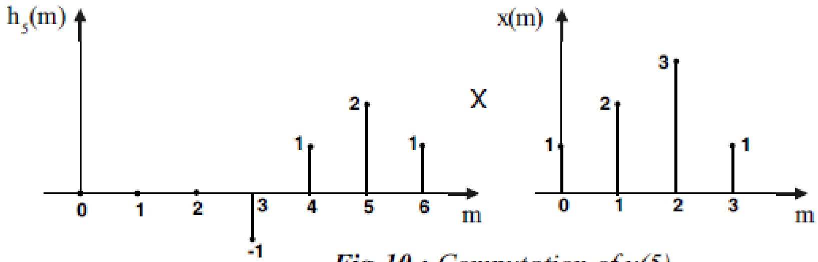
When $n = 4$; $y(4) = \sum_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum_{m=-\infty}^{+\infty} v_4(m)$

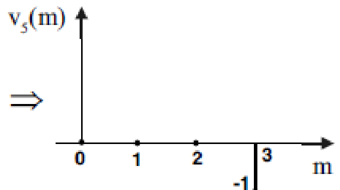


The sum of product sequence $v_4(m)$ gives $y(4)$. $\therefore$ $y(4) = -3 + 1 = -2$

Fig 9 : Computation of y(4).

When $n = 5$; $y(5) = \sum_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum_{m=-\infty}^{+\infty} v_5(m)$





The sum of product sequence $v_5(m)$ gives $y(5)$. $\therefore$ $y(5) = -1$

Fig 10 : Computation of y(5).

The output sequence, $y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

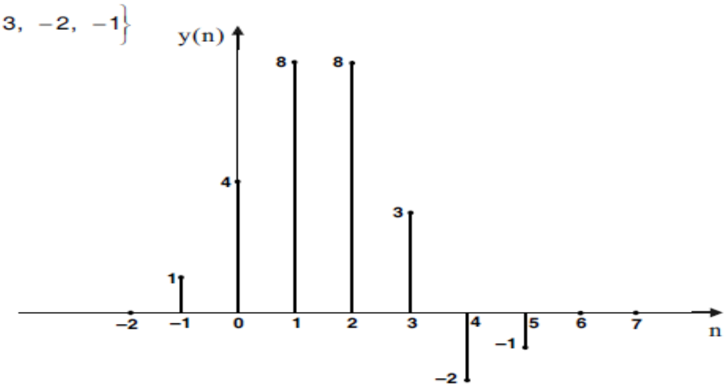


Fig 11 : Graphical representation of y(n).

Method - 2 : Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	−3	−2	−1	0	1	2	3	4	5	6
$x(m)$				1	2	3	1			
$h(m)$			1	2	1	−1				
$h(-m)$		−1	1	2	1					
$h(-1-m) = h_{-1}(m)$	−1	1	2	1						
$h(0-m) = h_0(m)$		−1	1	2	1					
$h(1-m) = h_1(m)$			−1	1	2	1				
$h(2-m) = h_2(m)$				−1	1	2	1			
$h(3-m) = h_3(m)$					−1	1	2	1		
$h(4-m) = h_4(m)$						−1	1	2	1	
$h(5-m) = h_5(m)$							−1	1	2	1

Each sample of $y(n)$ is computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m), \quad \text{where } h_n(m) = h(n-m)$$

To determine a sample of $y(n)$ at $n = q$, multiply the sequence $x(m)$ and $h_q(m)$ to get a product sequence (i.e., multiply the corresponding elements of the row $x(m)$ and $h_q(m)$). The sum of all the samples of the product sequence gives $y(q)$.

When $n = -1$; $y(-1) = \sum_{m=-3}^{3} x(m)\, h_{-1}(m)$ $\quad \boxed{\because \text{ The product is valid only for } m = -3 \text{ to } +3}$

$$= x(-3)\, h_{-1}(-3) + x(-2) h_{-1}(-2) + x(-1) h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1$$

The samples of $y(n)$ for other values of n are calculated as shown for $n = -1$.

When $n = 0$; $y(0) = \sum_{m=-2}^{3} x(m)\, h_0(m) = 0 + 0 + 2 + 2 + 0 + 0 = 4$

When $n = 1$; $y(1) = \sum_{m=-1}^{3} x(m)\, h_1(m) = 0 + 1 + 4 + 3 + 0 = 8$

When $n = 2$; $y(2) = \sum_{m=0}^{3} x(m)\, h_2(m) = -1 + 2 + 6 + 1 = 8$

When $n = 3$; $y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = 0 - 2 + 3 + 2 + 0 = 3$

When $n = 4$; $y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0 - 3 + 1 + 0 + 0 = -2$

When $n = 5$; $y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 0 - 1 + 0 + 0 + 0 = -1$

The output sequence, $y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

Method - 3 : Matrix Method

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).

$x(n)\downarrow$ \ $h(n)\rightarrow$	1	2	1	−1
1	1×1	1×2	1×1	$1\times(-1)$
2	2×1	2×2	2×1	$2\times(-1)$
3	3×1	3×2	3×1	$3\times(-1)$
1	1×1	1×2	1×1	$1\times(-1)$

$\Rightarrow$

$x(n)\downarrow$ \ $h(n)\rightarrow$	1	2	1	1
1	1	2	1	−1
2	2	4	2	−2
3	3	6	3	−3
1	1	2	1	−1

$y(-1) = 1$

$y(0) = 2 + 2 = 4$

$y(1) = 3 + 4 + 1 = 8$

$y(2) = 1 + 6 + 2 + (-1) = 8$

$y(3) = 2 + 3 + (-2) = 3$

$y(4) = 1 + (-3) = -2$

$y(5) = -1$

$\therefore y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

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