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**SCHOOL OF BUILDING AND ENVIRONMENT**

**DEPARTMENT OF CIVIL ENGINEERING**

# UNIT – I - INTRODUCTION – SCIA5202

# SCIA5202 - Structural Dynamics

## UNIT I

**Introduction**

The dynamic behavior of structures is an important topic in many fields. Aerospace engineers must understand dynamics to simulate space vehicles and airplanes, while mechanical engineers must understand dynamics to isolate or control the vibration of machinery and in civil engineering, an understanding of structural dynamics is important in the design and retrofit of structures to withstand severe dynamic loading from earthquakes, hurricanes, and strong winds, or to identify the occurrence and location of damage within an existing structure.

The basis for structural dynamics is Newton's Laws of motion which are assumed known, however they are shortly repeated below.

**Law 1** A body at rest (or moving in a straight line at a particular speed) will remain at rest (or continue moving in a straight line at that speed) unless acted on by a nonzero net force.

**Law 2** When a body is acted on by a nonzero net force, the net force is equal to the timerate of change of the body's linear momentum. For a constant mass

**Law 3** For every action, there is an equal and opposite reaction. Newton's laws are valid in any inertial frame of reference i.e. a space-time coordinate system that neither rotates nor accelerates.

$$F = \frac{d}{dx} mv = ma$$

**Fundamental difference between static and dynamic load:**

A static load is doesn't vary with respect to time, but a dynamic load will vary with respect to time. The responses of a structure are static for static load and for a dynamic load; the responses of a structure are dynamic in nature.

$$\boxed{F = k.u} \qquad \boxed{F\ t = k.u(t)}$$

Where,

F → applied force/load on the structures.

F (t) → applied dynamic force/load on the structures.

k→ stiffness of the structures.

u→ a response of the structures.

u (t) → Dynamic response of the structures.

We can find the responses of the structure due to the dynamic load in two senses

- Deterministic approach
- Non-deterministic approach

**Deterministic approach:**

We can use this approach to calculate the dynamic response of the structure, when the time variant of the force is completely defined. In this analysis we can get the defined or exact values of structural responses.

**Non-deterministic approach**

We can use this approach to calculate the dynamic response of the structure, when the time variant of the force is not completely defined. In this analysis we can't get the defined or exact values of structural responses. (We can get only the static response of the structures).

**Types of dynamic loads**

- Periodic loads
- Non periodic loads
- Impulse / Pulse loads
- Transient load

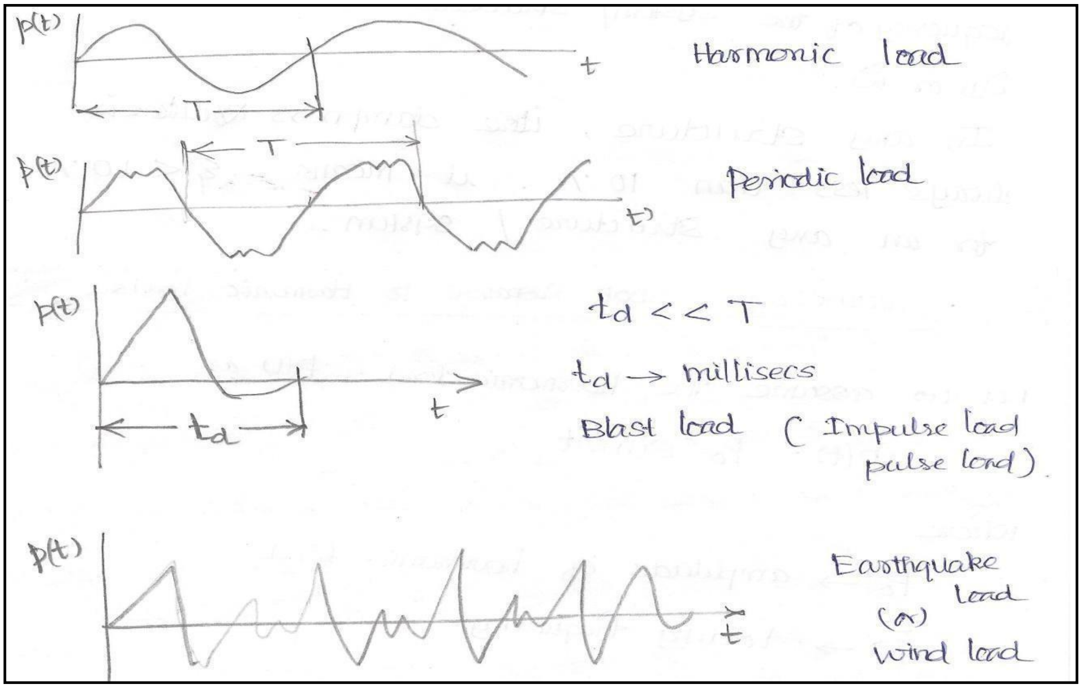



**Figure 1 Types of Dynamic loads**

**<u>Mass discretization / Discretization of mass</u>**

We can discretize the mass of the structure in two senses in the dynamic analysis. Either we can lump the mass of the structure at a point or we can also consider the mass is to be distributed continuously along the structure.

- Lump mass (Simple analysis)
- Continuously distributed mass (Complex analysis)

**<u>Degree of freedom</u>**

It is the number of independent displacement quantities.

**<u>Essential characteristics of dynamic loading</u>**

**<u>Time variant loading</u>**

1. Due to time variant loading on structures, we can get time variant response of thestructures.
2. In static analysis, time is constant.

**System should have significant inertia component**

According to Newton's second law,

Inertia force / Unbalanced (F) = mass (m) × acceleration (a)

$$F = m \times a$$

$$F = m \times \frac{dv}{dt}$$

If force F, changing with respect to time then

$$\frac{dF}{dt} = \frac{d}{dt}\left(m \times \frac{dv}{dt}\right)$$

$$P(t) = m \times \frac{d^2V}{dt^2}$$

**D'Alemerts Principle**

It is stated about dynamic equilibrium. According to this principle, the body is in dynamicequilibrium, the inertia force of the body will acts opposite direction of the body displacement.

**Mathematical model of single degree of freedom**

Assume a model is having single degree of freedom (direction of motion is indicated in thefigure)

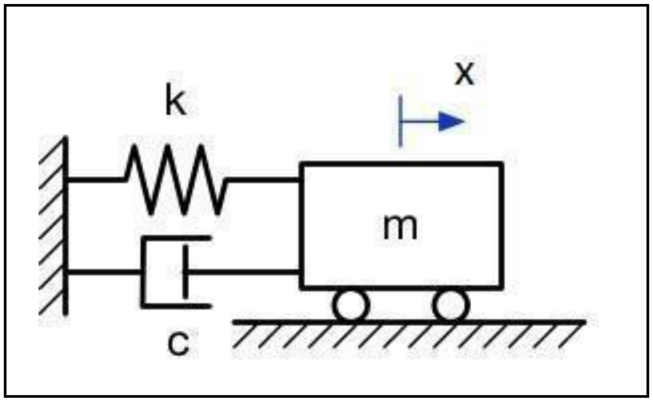



**Figure 2 Mathematical model of SDOF system**

In the assumed model, the mass body can move only in one direction (front and back)From the fig

- Mass (m) indicates the presence of inertia force.
- Spring (k) implementing an elastic restoring force and it can be representing by stiffness.
- damping component in the system, which can be indicated as “C”.
- Excitation force is a dynamic force and it can be represented as p (t).

Due to an external force p (t), the mass component can able to displace and due to the attached spring, the mass body will get back to its original position. But due to this displacement some friction force will generate between wheel and the surface. Due to this frictional force, energy dissipations will occur. So, it can be measured by providing

**Table 1 Factors influencing Dynamic Analysis**

| Elements | Un-damped free vibration | Damped free vibration | Un-damped forced vibration | Damped forced vibration |
|---|---|---|---|---|
| Mass (m) | √ | √ | √ | √ |
| Stiffness (k) | √ | √ | √ | √ |
| Damping (c) | X | √ | x | √ |
| Dynamic load p(t) | X | X | √ | √ |

**Equation of motion**

In order to carry out a dynamic analysis of a given structure it is necessary to establish theequations of motion for the actual degrees of freedom. The coupling between the degrees of freedom and thus the dynamic behavior of the structure will be determined by the following parameters

1. Stiffness properties of the system (k)
2. Size and distribution of masses (m)
3. Damping in the system (c)
4. Load intensity and distribution as a function of time (p(t))

The stiffness properties are known from static analysis where they commonly are described by the stiffness matrix. In a static analysis it is assumed that masses are displaced so slowly that no significant inertia loads are generated. However, in a dynamic analysis the accelerations generate inertia forces acting on all masses in the structure. The inertia forces can be determined using Newton's (second) law.

All real structures possess a certain amount of damping that will reduce vibrations. The damping is typically due to energy dissipation in the material itself (e.g. plastic deformations in steel, cracking in concrete) and friction in connections between different parts of a structure. This type of damping is called internal damping in opposition to external damping that is due to energy dissipation to the surroundings or in the external forcing mechanism. As damping mechanisms are difficult to describe in detail, it is often assumed that the effect of damping can be considered by a viscous damping model where the damping forces are determined as a damping factor times the velocity.

## Different methods to write an equation of motion

We can write the equation of motion for any degree of models (single degree model / multi degree model). Numbers of methods are available to write the equations of motion

1. Simple harmonic motion method (SHM)
2. Newton's law of motion
3. Energy method
4. Rayleigh method
5. D Alembert'sprinciple

## Simple harmonic motion method (SHM)

They are two main characteristics in Simple harmonic motion method.

- Acceleration will be always proportional to the distance or displacement of the body measured along the path of motion(x a $x$). A body will always direct towards its oppositedirection of motion.

$$x \quad a \; -x$$

$$x \quad = -\omega x$$

$$x \quad + \omega x = 0$$

**Note**: Negative sign represented as body will travels to opposite direction of motion

## The general equation of motion for an un-damped free vibration of SDOF

$$mx + kx = 0$$

$$\omega^2 = \frac{k}{m}$$

| Components | Formula | Units |
|---|---|---|
| Un-damped natural Frequency or angular Frequency of the structure | $\omega = \sqrt{\frac{k}{m}}$ <br> $\omega = 2\pi f$ | $rad/sec$ |
| Stiffness of the structure $k$ | $F = ku$ | $N/m$ |
| Mass of the structure $m$ | | kg |
| cyclic frequency | $F = \frac{1}{T}$ | $cycle/sec$ |
| Time period | T (sec) | s |

## General classifications of dynamics system / models:

- Un-damped free vibration system/model
- Un-damped forced vibration system/model
- Damped free vibration system/model
- Damped forced vibration system/model

## Derive the response of un-damped free vibration of SDOF system

Consider a single degree of un-damped system with free vibration

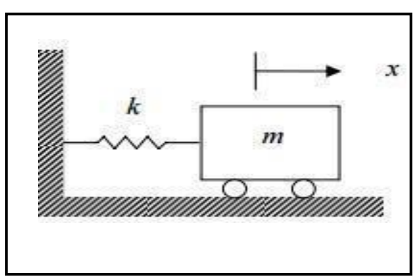



**Figure 3 Mathematical model of SDOF system with Un-damped Condition**

Eqn of motion of un-damped free vibration,

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \omega^2 x = 0$$

Second order ODE (ordinary differential

eqn) Auxiliary eqn for an above ODE,

$$D^2 + \omega^2 = 0$$

$$\therefore x(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

Where $C_1$ and $C_2$ can be found by using an initial conditions of the system.

Initial conditions

At $t = 0$; $x(t) = x(0)$

$$x(0) = C_1$$

At $t = 0$; $\dot{x}(t) = \dot{x}(0)$

$$C_2 = \frac{\dot{x}\,0}{\omega}$$

By substituting the values of $C_1$ and $C_2$ to an above expression

$$\boxed{x\,t = x\,0 \cos \omega t + \frac{\dot{x}\,0}{\omega} \sin \omega t}$$

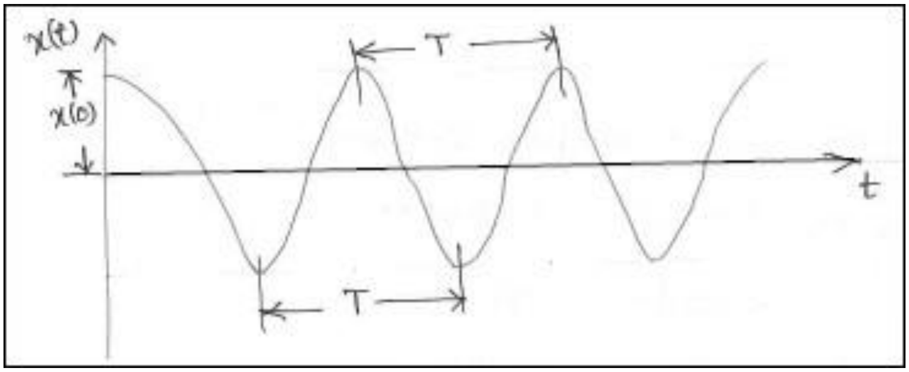



**Figure 4 Response of SDOF system with Un-damped free vibration**

**Damped free vibration of SDOF system**

**Damping**

Damping is a dissipation of energy from a vibrating structure. The damping coefficient is representing as "c" in the following mode

**Types of Damping:**

1. Viscous damping.
2. Coulomb or Dry Friction Damping.
3. Material or Solid or Hysteretic Damping.
4. Magnetic Damping.

**Viscous damping**

Viscous damping is the dissipation of energy that due to the movement of bodies in a fluid medium.

**Coulomb or Dry Friction Damping**

The damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces that are either dry or have insufficient lubrication.

**Material or Solid or Hysteretic Damping**

The damping caused by the friction between the internal Planes that slip or slide as the material deforms is called hysteresis or solid or structural damping.

**Magnetic Damping**

A phenomenon that has been observed for many years by which vibrating, oscillating or rotating conductors are slowly be brought to rest in the presence of a magnetic field.

The damping effects of magnetic induction are also proportional to the speed of the moving object hence making the braking phenomenon extremely smooth. It is hence the objective of this project to further investigate the aforementioned damping effects in the case of rotating discs, with the focus being not on the strength of the magnetic field or the speed of the disc, but on the various possible orientations of the applied magnetic field in relation to the disc.

**<u>Derive the response of damped free vibration of SDOF system</u>**

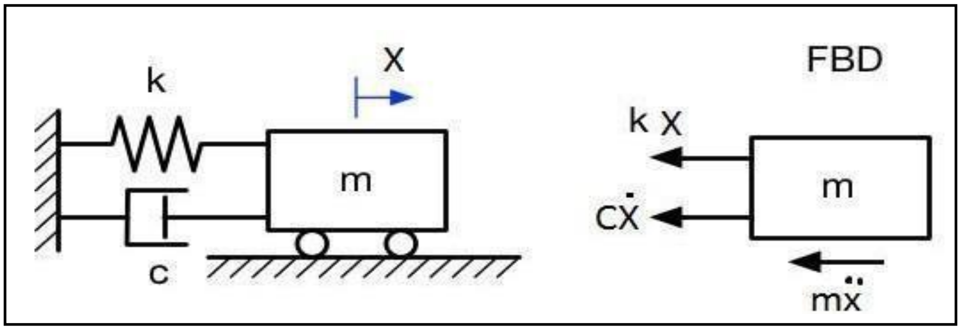



**Figure 5 Mathematical model of SDOF system with damped free vibration**

Equation of motion of damped free vibration:

$$mx + cx + kx = 0$$

$$x + \frac{c}{m}x + \frac{k}{m}x = 0$$

$$x + \frac{c}{m}x + \omega^2 x = 0$$

$$D^2 + D\frac{c}{m} + \omega^2 = 0$$

By solving this, we can get

The roots are

$$\boxed{\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2}}$$

**Critical damping system**

Set the discriminant to zero, the roots are real and equal.

$$\frac{c}{2m}^{2} - \omega^2 = 0$$

$$\frac{c}{\frac{2}{2m}} = \omega^2$$

$$\boxed{c_{cr} = 2m\omega}$$

$$\boxed{\therefore c_{cr} = 2\sqrt{km}}$$

where, $c_{cr} \rightarrow critical\ damping$

In this case the roots are real and equal

$$\alpha_1 = \alpha_2 = \alpha = -\frac{c_{cr}}{2m}$$

$$\boxed{\therefore x(t) = A + Bt\, e^{-\frac{c_{cr}}{2m}t}}$$

where, A and B are constants. It can be found by using initial conditions.

**Note**: It is observed that damping component will reduces the responses of the structures in terms of exponentially.

## Under damping system

Discriminant ≠ 0; The roots are real and imaginary.

$$\left(\frac{c}{2m}\right)^2 - \omega^2 < 0$$

$$\boxed{\alpha_{1}, \alpha_{2} = -\frac{c}{2m} \pm i\omega\sqrt{1-\xi^2}} \qquad \boxed{\alpha_{1}, \alpha_{2} = -\frac{c}{2m} \pm i\omega_D}$$

$$\omega_D \rightarrow Damping\ frequency = \omega\sqrt{1-\xi^2}$$

$$\boxed{\xi \rightarrow Damping\ ratio = \frac{C}{C_{cr}}}$$

The roots are real and imaginary $\alpha \pm i\beta$

$$Complementary\ Function = e^{\alpha t}\left(C1 \cos \beta t + C2 \sin \beta t\right)$$

$$\boxed{x(t) = e^{-\frac{c}{2m}t}\left(C_1 \cos \omega_D t + C_2 \sin \omega_D t\right)}$$

Where, $C1$ and $C2$ are constant and it can be found by using an initial conditions.

**Initial conditions**

At $t = 0$ ; displacement $x(t) = x(0)$

$$C_1 = x(0)$$

At $t = 0$ ; velocity $\dot{x}(t) = \dot{x}(0)$

$$C_2 = \frac{\dot{x}(0) + \xi\omega x(0)}{\omega_D}$$

Disturbance of damped free system,

$$\boxed{x(t) = e^{-\frac{c}{2m}t}\left(x(0) \cos \omega_D t + \frac{\dot{x}(0) + \xi\omega x(0)}{\omega_D} \sin \omega_D t\right)}$$

**Note**: response $x(t)$ is exponentially decaying

**Determinations of peaks of the damped free vibration system:**

At $t = 0$; $x(t) = x(0)$

At $t = \frac{\pi}{\omega_D}$;

$$x(t) = e^{-\frac{c}{2m}t}\left[x(0)\cos\omega_D t + \frac{\dot{x}(0) + \xi\omega x(0)}{\omega_D}\sin\omega_D t\right]$$

$$x\left(\frac{\pi}{\omega_D}\right) = e^{-\frac{c\pi}{2m\,\omega_D}}\left[x(0)\cos\omega_D\frac{\pi}{\omega_D} + \frac{\dot{x}(0) + \xi\omega x(0)}{\omega_D}\sin\omega_D\frac{\pi}{\omega_D}\right]$$

$$\boxed{x\left(\frac{\pi}{\omega_D}\right) = -x(0).e^{-\frac{c\pi}{2m\,\omega_D}}}$$

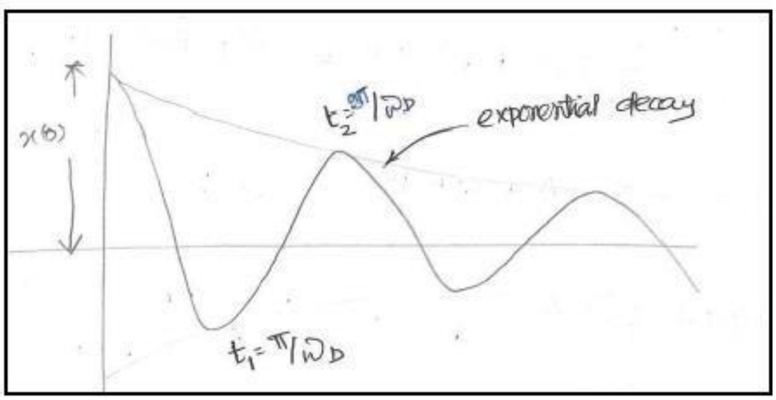



**Figure 6 Effect of damping in the system**

Let us consider $x0$, $x1$ and $x2$ are represents the peaks of the disturbances along the positive sideof the curve and similarly $x{-}1 and\ x{-}2$ are represents the peak of the disturbance along the negative side of the curve.

$$x(0) = x_0$$

$$x_1 = x_0 . e^{-\frac{c}{2m}\frac{2\pi}{\omega_D}}$$

Taking natural log on both sides

$$\frac{x_0}{x_1} = e^{\frac{c}{2m}\frac{2\pi}{\omega_D}}$$

$$\ln\frac{x_0}{x_1} = \ln e^{\frac{c}{2m}\frac{2\pi}{\omega_D}}$$

$$\ln\frac{x_0}{x_1} = \frac{c}{2m}\frac{2\pi}{\omega_D}$$

$$\ln\frac{x_0}{x_1} = \xi\omega.\frac{2\pi}{\omega_D} \quad \therefore \xi = \frac{c}{2m\omega}$$

$$\boxed{\text{Logarithmic decrement} = \delta = \ln\frac{x_0}{x_1} = 2\pi\varepsilon\frac{\omega}{\omega_D} = \frac{2\pi\varepsilon}{\sqrt{1-\xi^2}} \cong 2\pi\varepsilon}$$

**Calculation of *damping ratio* ($\xi$) from twosuccessive peaks of a time series**

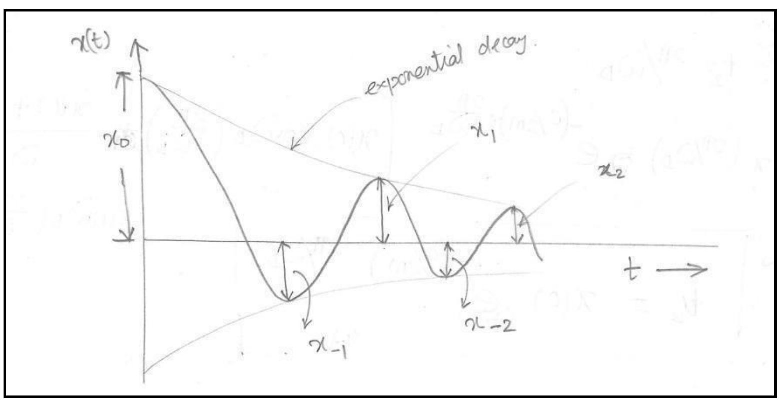
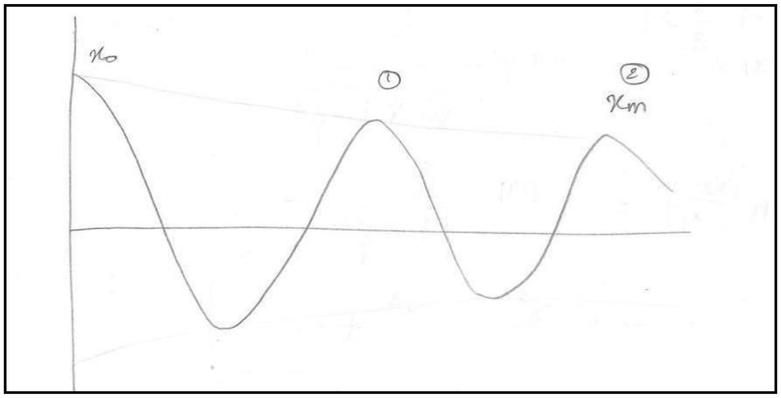



**Figure 7 Decaying Curve**

In general, $\frac{x_n}{x_{n+1}} = 1 + 2\pi\xi$

$$\boxed{\xi = \frac{1}{2\pi}\left(\frac{x_n}{x_{n+1}} - 1\right)}$$

**Calculation of $damping\ ratio$ ($\xi$) from any two peaks of a time series**

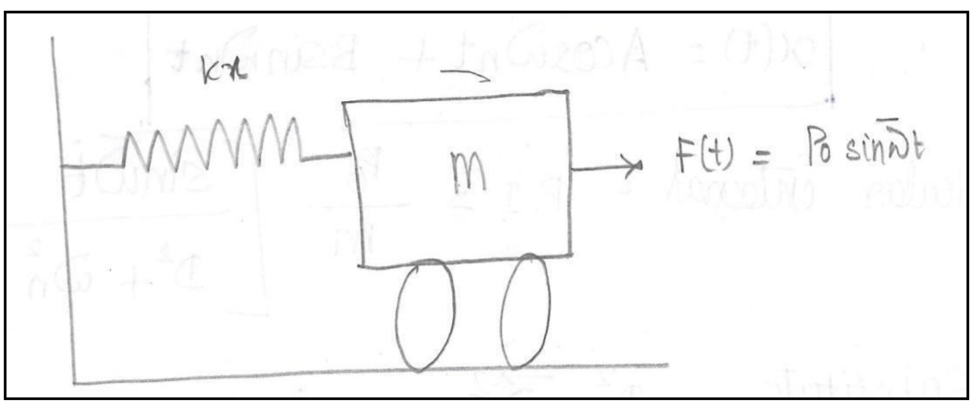



Where

$x0$ Is a reference peak.

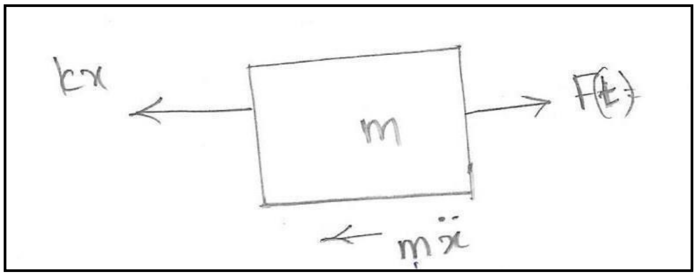
$$\xi = \frac{1}{2\pi m} \frac{x_n}{x_{n+m}} - 1$$

Derivation of Response of an Un-damped forced vibration of SDOF system




Let us consider a system with un-damped forced vibration condition. External force ($F\ t = P0 \sin \omega\ t$) is acting on the mass body (m). Based on the boundary conditions we are assuming that, because of an applied external force, the mass body (m) is displacing along only one direction. (SDOF system).where, $P0\ and\ \omega$are peak amplitude and frequency of an appliedforce $F\ t$.

According to newton’s law;




An unbalanced driven force, $F = m \times a = F\ t - kx$

$$mx\ \ = F\ \ t\ \ - kx$$

$$mx + kx = F\ t$$

By solving second order differential eqn

$$D^2 + \omega^2 = 0$$

Complementary function, x t = A cos $\omega t$ + $B$ sin $\omega t$

$$\text{Particular integral,x(t)} = \frac{P0}{m} \frac{\sin \omega t}{D^2 + \omega^2}$$

Total Response of an Un-damped Forced vibration system

$$\boxed{\text{x t} = \text{A} \cos \omega t + B \sin \omega t + \frac{P_0 \sin \omega t}{k\ 1 - r^2}}$$

Note:

- In an above equation, Constants A and B can be derived from an initial conditions.
- $r = \frac{\omega}{\omega}$, where “r” is a resonance frequency of a structure.
- $\omega$ is a frequency of an applied force, $F\ t$ and it does not depends on any initial conditions. This part of an equation is called as steady state portion.

- $\omega$ is a frequency of structural vibration and it is depends on initial conditions. This part ofan equation is called as transient state portion.

$$\boxed{x(t) = \underbrace{A\cos\omega t + B\sin\omega t}_{\text{Transient Portion}} + \underbrace{\frac{P_0 \sin\omega t}{k(1-r^2)}}_{\text{Steady Portion}}}$$

Initial condition, $@t = 0$ ; $x(t) = x_0$; and $x(t) = x(0)$

By using initial conditions an above equation can be reduced to

Total response of a structure, x (t)

$$x(t) = x_0 \cos\omega t + \left(\frac{x_0}{\omega} - \frac{P_0}{k} \times \frac{r}{1-r^2}\right)\sin\omega t + \frac{P_0 \sin\omega t}{k(1-r^2)}$$

For the forced vibration, there is no initial displacement and initial velocity.

$$\therefore x_0 = x(0) = 0$$

$\therefore$ *Above equation can be reduced to*

$$x(t) = -\frac{P_0}{k} \times \frac{r}{1-r^2}\sin\omega t + \frac{P_0}{k}\frac{\sin\omega t}{1-r^2}$$

$$\boxed{x(t) = \frac{P_0}{k} \times \frac{1}{1-r^2}\left(\sin\omega t - r\sin\omega t\right)}$$

Note: the transient part of the response is negligible and it is an unsteady portion, so we can

$$x(t)_{Steady\ state} = \frac{P_0}{k} \times \frac{1}{1-r^2} \sin \omega t$$

skip that part.

Note: From an above equation, it is informed that, the response of the system $x(t)$ is inversely proportional to resonance factor "r".

If $r = 1 \therefore \frac{\omega}{\omega} = 1$ then the magnitude of the response of the system is infinite.

Similarly we can derive the response of forced vibrations of damped system by following thesame procedures which we mentioned above.

**Response of forced vibrations of damped system**

Eqn of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega t$$

The response of damped system with forced condition $x(t)$

$$x(t) = e^{-\xi\omega_D t}(C_1 \sin \omega_D + C_2 \cos \omega_D) + \frac{P_0/k}{(1-r^2)^2 + (2\xi r)^2}\left[(1-r^2)\sin \omega t + 2\xi r \cos\omega t\right]$$

An above eqn can gave the steady state response of damped system with forced vibration. We can still reduce an above eqn. we can skip the transient state of response in the response equation to simplify the analysis. This assumption won't create any significant effect in the total response of the structure.

$$\therefore x(t)_{Steady\ state} = x(t)$$

$$x(t) = \frac{P_0/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega t - \theta)$$

$$Phase\ angle\ (\theta) = \tan^{-1}\frac{2\xi r}{1-r^2}$$

Where we can re-write the response equation as

$$x(t) = D.\frac{P_0}{k}\sin(\omega t - \theta)$$

$$X_{max} = D.\frac{P_0}{k} \quad \therefore F(t) = P_0 \sin \omega t$$

$$\frac{x_{max}}{x} = D \qquad \therefore P_0 = kx$$

$F(t) \rightarrow Applied\ dynamic\ force\ or\ External\ Dynamic\ force$

**Dynamic amplification factor (D)**

$$D = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

The dynamic factor will depends on both the parameters, $\xi$ and $r$

In reality, dampness of any structure is less than 10%

$$\therefore 2\xi r \ll small$$

$$\therefore (2\xi r)^2 \cong 0 \qquad \therefore r = \frac{\omega}{\omega}$$

Dynamic amplification factor can be reduced to, $D = \frac{1}{1-r^2}$

The value of "D" is depends on the resonance of the system

$$if\ r = 0 \Rightarrow D = 1$$

$$if\ r = 1 \Rightarrow D = \frac{1}{2\xi}$$

$$if\ r = \infty \Rightarrow D = \frac{1}{\infty} = 0$$

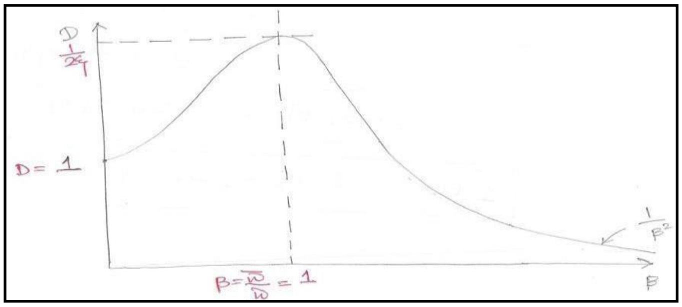



**Figure 8 Dynamic amplification curve**

**<u>Duhamel's Integration</u>**

The impulse loading is a load, which is applied over a short duration of time. The corresponding impulse of this type of loading is defined as the product of the force and the time of its duration. For an example, the impulse of the force $F\ r$ depicted in the figure at time $r$ during an interval $dr$ is represented by shaded area and it is equal to $F\ r\ .\ dr$

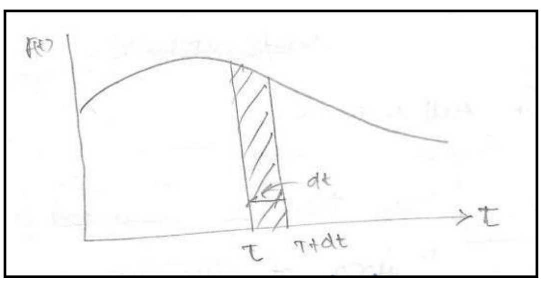



**Figure 9 Impulse loading - Duhamel's Integral**

This impulse is acting on a body of mass „m" produces a change in velocity, which can be determine from newton"s law of motion.

The impulse force produces the change in velocity $m \frac{dv}{d\tau} = F\ \tau$ .

By rearranging the eqn, we can get as $dv = \frac{F\ \tau\ .d\tau}{m}$

Consider an impulse $F\ \tau\ .\ d\tau$ acting on the structure represented by an un-damped single oscillator. The incremental velocity $dv$ is taken as an initial velocity and initial displacement $x0$ as zero at time„t".

$$x\ t = x\ 0 \cos \omega t + \frac{x\ 0}{\omega} \sin \omega t$$

$$dx\ t = \frac{F\ \tau\ .\ d\tau}{m\ \omega} \sin \omega\ t - \tau$$

An above eqn is showing single impulse by the sum, we can get full response of the structure. he general solution for total response of structure because of the simple oscillation is

$$\boxed{x\ t = x\ 0 \cos \omega t + \frac{x\ 0}{\omega} \sin \omega t + \frac{1}{m\omega} \int_0^t F\ \tau\ . \sin \omega\ t - \tau\ .\ d\tau}$$

SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

# UNIT –II -MUTI-DEGREE-OF FREEDOM SYSTEMS – SCIA5202

## Introduction

All the real structures are actually infinite degree of freedom systems. However, for the purposeof studying their dynamic behavior we can idealize them into single or multi degree of freedom systems. If mass of the structure can be lumped at different locations in the structure then the dynamic behavior consists of same number of modes of vibration. Such structure is called a multi degree of freedom system. Typically in a multi-storey building, masses of different elements can be lumped at center of mass at eachslab level.

## Formulation of MDOF equations of motion

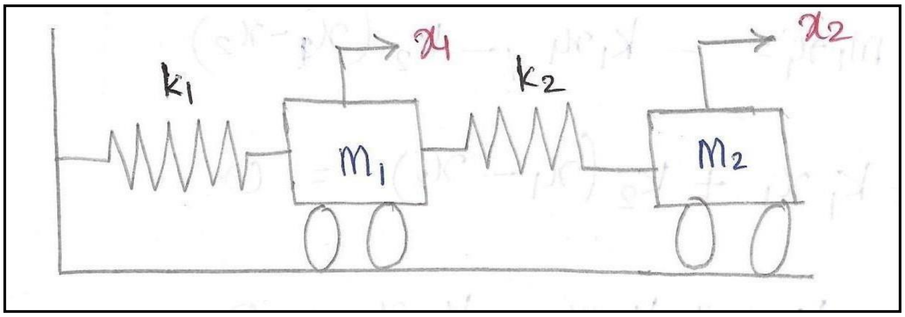



**Figure 1Mathematical model of two degree of freedom system**

Let us consider a mathematical model of two degree of freedom system. Where, $m_1$ and $m_2$ are two mass bodies. These mass bodies are connected with two stiffness and the stiffness of the springs are representedas $k_1$ and $k_2$.

The displacement of the mass bodies' $m_1$ and $m_2$ are represented as $x_1$ and $x_2$.

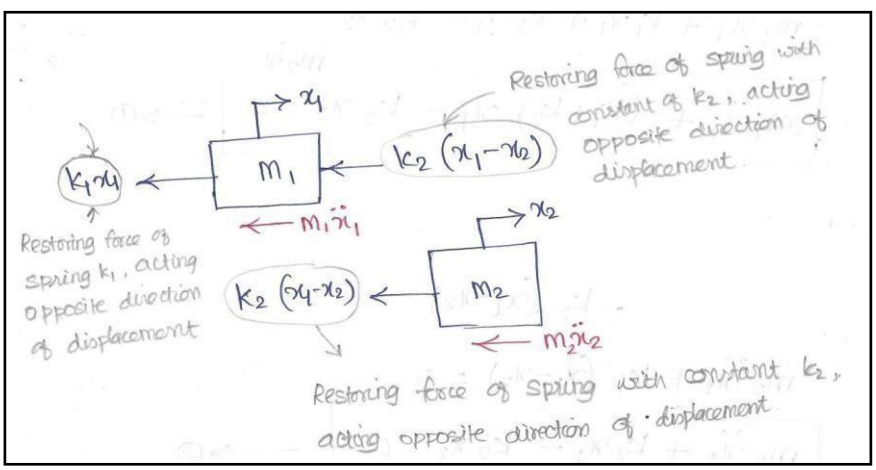



**Figure 2 FBD of two degree of freedom system**

$m_1 x_1 \rightarrow Inertial\ force\ of\ mass\ body\ m_1, acting\ opposite\ to\ the\ direction\ of\ displacement.$

$m_2 x_2 \rightarrow Inertial\ force\ of\ mass\ body\ m_2, acting\ opposite\ to\ the\ direction\ of\ displacement.$

According to newton’s second law,

First DOF

$$m_1x_{\ 1} = -k_1x_1 - k_2\ x_1 - x_2$$

$$\boxed{m_1x_{\ 1} + x_1\ k_1 - k_2 + k_2x_2 = 0}$$

Second DOF

$$m_2x_{\ 2} = -k_2\ x_2\ - x_1$$

$$\boxed{m_2x_{\ 2} + k_2x_2 - k_2x_1 = 0}$$

- For the lumped mass system, the masses in the mass matrix as a diagonal matrix

$$m\ = \begin{matrix} m1 & 0 \\ 0 & m2 \end{matrix}$$

- Stiffness matrix is square/symmetric and the diagonally dominant matrix

$$k = \begin{matrix} k1\ + k2 & -k2 \\ -k2 & k2 \end{matrix}$$

  Where, diagonally dominant means, the diagonal elements in the stiffness matrix isgreater than any other elements in the matrix.
- The size of the “k” matrix will be an order of n. Where, n is called as degree of freedom.

**Natural frequencies and mode shapes calculations of MDOF system by Eigen value solver method**

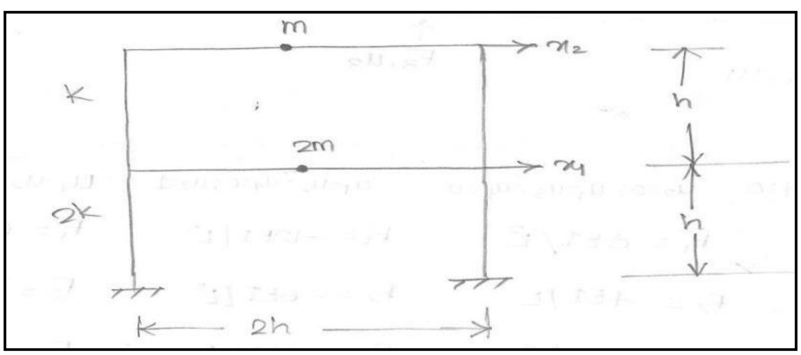



**Figure 3 Example problem**

Let us consider a single bay-two story framed structure. The stiffness of the both columns can be k and 2k.consider a bay width as 2h and story height of an each story as h. the lumped masses in the frames at two different points can be m and 2m. The excitation directions of the lumped masses are shown in the figure

Note:

Mass of an each floor level is lumped at one point. So the mass matrix of the structure is an ideal matrix.

$$\therefore \ m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$$

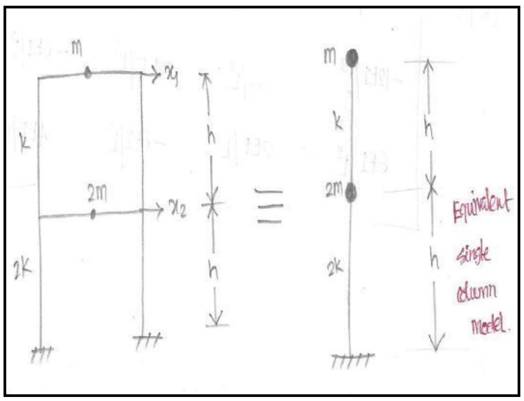



**Figure 4 Equivalent Single Column model**

The stiffness matrix of the frame can be written as

$$k_{2\times 2} = \begin{bmatrix} \frac{36EI}{h^3} & -\frac{12EI}{h^3} \\ -\frac{12EI}{h^3} & \frac{12EI}{h^3} \end{bmatrix}$$

Let us assume $P = \frac{12EI}{h^3}$

$$k_{2\times 2} = \begin{bmatrix} 3P & -P \\ -P & P \end{bmatrix}$$

$$k - \omega^2 m = 0$$

$$\begin{matrix} 3P & -P \\ -P & P \end{matrix} - \omega^2 \begin{matrix} 2m & 0 \\ 0 & m \end{matrix} = 0$$

After simplifying, we can get $2m^2\,\omega^4 - 5Pm\,\omega^2 + 2P^2 = 0$

- The natural frequencies from roots $\alpha 1$ $and$ $\alpha 2$ can be$\omega 1$ $and$ $\omega 2$.
- Natural frequencies of the system are called as Eigen values.
- Mode shapes of the structure are called as Eigen vectors.
- From the mode shape diagram only, it is possible to find basic natural frequency of a system.

**Mode shape calculation**

It can indicate the relative position of mass at any specific frequency of vibration.

$$A - \lambda\, x = 0$$

$(k - \omega^2 m)x = 0(k -$

$\omega^2 m)\phi = 0$

Note:

Mode shape is a relative displacement of lumped massed in the system for a corresponding frequency. So for an each frequency of vibration, we can get a corresponding mode shapes.

$$From\ \omega 1 \ \rightarrow we\ can\ get\ \phi 1$$

$$From\ \omega 2 \ \rightarrow we\ can\ get\ \phi 2$$

**Orthogonally conditions of mode shapes**

Let $\phi i$ $and$ $\phi j$ are two distinct mode shapes

Condition or 0rthogonality

$$\boxed{\begin{array}{ll} \phi_i{}^T\ \phi_j = 1 & for\ i = j \\ \phi_i{}^T\ \phi_j = 1 & for\ i \neq j \end{array}}$$

## Normalization of modes and weighted modes

Normalization is a process to make non orthogonal modes to an orthogonal modes. The resultant mode shape is called as weighted modes. We can normalize it by using mass matrix or stiffness matrix.

## To Prove an Orthogonal Property

Let us consider two mode shapes $\phi i \ and \ \phi j$

$$(k - \omega i^2 m)\phi i = 0$$

$k \quad \phi i \quad = \omega i^2 m \ \phi i$

Multiply $[\phi_j]^T$ on both sides

$$[\phi_j]^T k \quad \phi_i \quad = [\phi_j]^T \omega_i^2 m \ \phi_i$$

$$\boxed{[\phi_j]^T k \phi_i = \omega_i^2 [\phi_j]^T m \phi_i}$$

$$(k - \omega_j^2 m)\phi_j = 0$$

Multiply $[\phi_i]^T$ on both sides

$$[\phi_i]^T k \phi_j = [\phi_i]^T \omega_j^2 m \phi_j$$

take transpose on both sides

$$[[\phi_i]^T k \phi_j]^T = [[\phi_i]^T \omega_j^2 m \phi_j]^T$$

$$[\phi_j]^T [k]^T \phi_i = \omega_j^2 [\phi_j]^T m^T \phi_i$$

Where $[k]^T = k \ and \ m^T = [m]$ because $k \ and \ m$ are symmetrical matrix in nature.so we can rewrite it as

$$\boxed{[\phi_j]^T k \phi_i = \omega_j^2 [\phi_j]^T m \phi_i}$$

From top two eqn it is proved as $[\phi_j]^T k \phi_i = [\phi_j]^T m \phi_i = 0$,if it is true, then because $\omega_i \neq \omega_j$ .

## Approximate methods of extraction of Eigen values

In exact method difficulties arises in

- Solving roots of the characteristic equation. Except for very simple boundary conditions, one has to go for numerical solution.
- In determining the normal modes of the system
- Determination of steady state response

So for quick determination of the natural frequencies of a system, when a very accurate result is not of much importance one should go for an approximate method.

## Dunkerley's Method (Semi empirical) approximate solution

Let $W_1, W_2 \ldots W_n$ be the concentrated loads on the shaft due to masses $m_1, m_2 \ldots m_n$ and $\Delta_1, \Delta_2 \ldots \Delta_3$ are the static deflections of the shaft under each load. Also let the shaft carry a uniformly distributed mass of $m$ per unit length over its whole span and static deflection at the mid span dueto the load of this mass be $\Delta s$. Also

$\omega_n$ = Frequency of transverse vibration of the whole system.

$\omega_{ns}$ = Frequency with distributed load acting alone

$\omega_{n1}, \omega_{n2} \ldots\ldots$ = Frequency of transverse vibration when each of $W_1, W_2, W3 \ldots$ act alone.

According to Dunkerley's empirical formula

$$\frac{1}{\omega_n^2} = \frac{1}{\omega_{n1}^2} + \frac{1}{\omega_{n2}^2} + \ldots\ldots\ldots + \frac{1}{\omega_{ns}^2}$$

$$\frac{1}{f_n^2} = \frac{1}{f_{n1}^2} + \frac{1}{f_{n2}^2} + \ldots\ldots\ldots + \frac{1}{f_{ns}^2}$$

In the case of transverse vibration of beams, if $n$ functions are chosen for approximating the deflection $W(x)$, can be written as

$$w(x) = c_1 w_1(x) + c_2 w_2(x) + \cdots + c_n w_n(x)$$

Where, $w_1(x)$, $w_2(x)$, $\cdots w_n(x)$ are linear independent functions of the spatial coordinate $x$ which satisfy the boundary condition of the problem, and $c_1, c_2 \cdots, c_n$ are the coefficient to be found.

As the Rayleigh quotients have stationary value near the natural mode by differentiate by differentiating the Rayleigh quotient with respect to these coefficients will yield a set of homogeneous algebraic equations, which can be solved to obtain the frequencies.

**Matrix iteration method**

This method is used to determine the natural frequencies and mode shapes of a multi degree of freedom system. As it is known that for a multi degree of freedom system, the governing equation can be reduced to the eigenvalue problem given by

$$[A]\{X\} = \lambda\{X\} \tag{1}$$

Where $[A] = [M]^{-1}[K]$ is known as the dynamic matrix, $\lambda$ is the Eigen value and $\{X\}$ is the mode shape. From this equation it may be noted that any normal mode when multiplied with the dynamic matrix will reproduce itself. In matrix iteration method, assumed displacement of the masses is used to get the calculated displacement. This is repeated till equation (1) is satisfied.

**Steps used in the matrix iteration method.**

- Assume a value of the modal vector. (for example 3:2:1 for 3 dof system)
- Substitute the assumed value in left hand side of equation (1) and simplify to obtain a ratio (for example the obtained value is 4:3:1).
- If the value obtained in step II is same as the assumed value, then it is accepted as the correct modal value. Otherwise, the obtained value is substituted as the trial value and the second step is repeated till the correct modal value is obtained.
- After getting the modal values, from equation (1) the corresponding eigenvalue can be obtained.

In general matrix iteration method would converge to the fundamental mode. If the assumed system of displacements does not include the fundamental mode then the matrix iteration will converge to the next higher mode contained in the assumed system of displacements. Orthogonality principle is used to sweep out the unwanted modes from assumed displacements. In case of semi-definite systems, rigid body mode (zero frequency) is also present. For such cases constraint matrices can be constructed to sweep out rigid body component of the absolute motion.

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**SCHOOL OF BUILDING AND ENVIRONMENT**

**DEPARTMENT OF CIVIL ENGINEERING**

# UNIT –III -DYNAMIC RESPONSE OF MDOF SYSTEMS – SCIA5202

**UNIT III**

**SCIA5202 - Structural Dynamics**

## Normal co-ordinates

The eqn of MDOF system is given as follows

$$m \quad x \quad + \; c \;\; x \quad + \; k \quad x = \; P \, t$$

In an above eqn is a coupled eqn, so we need to uncouple it for easy analysis. The MDOF system having n degree of freedom is represented by n equations based on n independent coordinates of masses.

Coordinated have been chosen as the displacement of masses from the static equilibrium position. However other set of independent coordinate system can also be chosen by proper selection. It is also possible to select a set of coordinates such that the coupling of eqn can be removed as a result n set of coupled eqn are reduced to n set of an individual eqn.

Each of an individual eqnsis having unique amplitude, frequency and phase angles are called principal coordinates or normal coordinates.

$$x \;=\; \phi \;\; \varepsilon$$

$$where\ \phi = \phi^{1}\ \phi^{2}\ \phi^{3} \cdots\cdots \phi^{n}$$

Each column of $\phi$ represent one normal mode and $\varepsilon$ represents the principle coordinates.

## Mode super position technique

The normal coordinate transformation which serves to change the set of ‘N’ coupled equation of motion of a MDOF system into a set of ‘N’ uncoupled equation, in the basis of Mode superposition or Modal analysis method.

This method is reduces the problem of finding the response of a multi degree of freedom of system to the determination of response of SDOF system.

For a MDOF system under an un-damped condition, the equation of motion can be written as

$$m \quad x \quad + \; k \;\, x \;=\; P \, t$$

Such system of eqn contains number of variables which contains number of variables which are coupled or dependent. Those eqns should be uncoupled or independent to determine the response ofmulti degree of freedom system under forced vibration

The co-ordinates is to transform the coupled differentiate equations into uncoupledequation. It is called as Principle of co-ordinates.

The transformation equation is $x \quad = \phi \quad u \qquad \rightarrow 2$

Where: $x \rightarrow Displacement\ or\ Response\ of\ structures\ at\ any\ time.$

$\phi \rightarrow \ of\ structures\ at\ any\ time.$

$u \rightarrow Principle\ co-ordinates$

The transformation matrix can be rewritten as $x \quad = \phi \quad u \rightarrow 3$

Substitute the eqn 2 and 3 in eqn 1

$$m \quad \phi \quad u \qquad + \ k \ \phi \ u \ = \ P\ t$$

Pre multiply $\phi^T$ on both sides

$$\phi^T m \ \phi \ u + \phi^T k \ \phi \ u = \phi^T P \ t$$

$$M \quad u \qquad + \ K \ \ u \ = R$$

The uncoupled eqns are

$$m1 \quad u \qquad + \ \ k1 \quad u \ \ = R1$$

$$m2 \quad u \qquad + \ \ k2 \quad u \ \ = R1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$m1 \quad u \qquad\qquad + \ \ k1 \quad u \ \ = R1$$

To get the maximum value of u

$$u_{max} = \ DLF \ \ max \ \frac{R}{k}$$

$DLF \rightarrow Dynamic\ Load\ Factor$

**Numerical Integral Procedure**

Numerical method of integration, which is applied to linear and non- linear system of displacement, velocity and acceleration at any $i^{th}$ and $i + 1$ $^{th}$ time in the dynamic system.

Step 1: Form stiffness, mass and damping coefficient matrixes.Step

2: Determination of integral constants.

$$a_0 = \frac{1}{\alpha.\Delta t^2}$$

$$a_1 = \frac{\delta}{\alpha.\Delta t}$$

$$a_2 = \frac{1}{\delta.\Delta t}$$

$$a_3 = \frac{1}{2\alpha} - 1$$

$$a_4 = \frac{\delta}{\alpha} - 1$$

$$a_5 = \frac{\Delta t}{2}\frac{\delta}{\alpha} - 2$$

$$a_6 = \Delta t\ 1 - \delta$$

$$a_7 = \delta.\Delta t$$

Step 3: Formulate effective stiffness matrix

$$k = k + a_0\ m + a_1\ c$$

Step 4: Find an inverse of an effective stiffness matrix $k^{-1}$

Step 5: Effective load at $i + 1$ time

$$F\ i{+}1 = F\ i{+}1 + m\ a0xi - a2x\ i + a3x\ i + c\ a1xi + a4x\ i + a5x\ i$$

Step 6: Displacement at $i + 1$ time

$$\lambda^{i+1} = k^{-1}\ F^{i+1}$$

Step 7: Velocity at $i + 1$ time

$$x\ i{+}1 = x + a6xi + a7xi + 1$$

Step 8: Acceleration at $i + 1$ time

$$x_{i+1} = a_0 x_{i+1} - x_i - a_2 x_i - a_3 x_i$$

# MODE SUPERPOSITION

## INTRODUCTION

There are fundamentally two approaches that may be followed to compute the dynamic responseof structures:

- Time domain approach
- Frequency domain approach

The "time-domain involves solving the governing differential equation(s) wherein the loading function is a function of time "t" : p(t) and the in frequency domain approach the loading is discretized into a series of harmonic components (sines and cosines) by Fourier series / Fourier transformation and the dynamic response is expressed by superimposing the responses due to each harmonic component.

Most common dynamic analysis procedures that may be classified under the time domainapproach include: Duhamel / convolution integral based procedure wherein the loading p(t) is considered to be a succession of short-duration impulses, and the free vibration response to each impulse becomes a separate contribution to the response history at any subsequent time.

## PRINCIPLE OF THE MODE SUPERPOSITION METHOD OF DYNAMIC ANALYSIS

Consider the 2 DOF systems shown in the figure below;

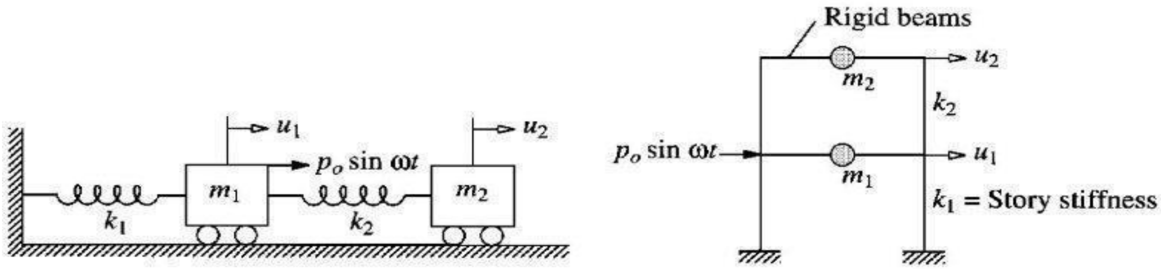


**Figure: 2 storey frame excited by a harmonic force (also idealized as a mass-spring system)**

The 2 storey frame with degrees of freedom u1 and u2 (neglecting rotational inertia) could be idealized in to the mass-spring system. The system is excited by a harmonic force: p1 (t) = P0sin ωt. Thus the equation of motion of the system may be written as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} \sin \omega t$$

It may be noted that the equations are coupled through the stiffness matrix i.e. the stiffness matrix is non-diagonal and hence one equation cannot be solved independent of the other and theequations must be solved simultaneously.

One would normally resort to the numerical approximation procedures like the explicit central difference method or the implicit Newmark's integration scheme in order to solve the coupled equations.

A simpler method to obtain the dynamic response of the frame is the mode superposition method which principally transforms the equations of motion from the physical coordinate system into another co-ordinate system known as the normal or the modal coordinate system and in this coordinate system the structural matrices i.e. the mass and the stiffness matrix are diagonal.

Therefore, the solution of the simultaneous coupled equations is completely avoided in this coordinate system and would solve one equation independent of the other. The solution of each equation now (after the problem is transformed into the normal coordinate system) becomes equivalent to solving an equation of a SDOF system. That is: instead of solving a coupled system with ‘N’ degrees of freedom in the physical coordinate system, one, instead solves ‘N’ uncoupled equations wherein each of the ‘N’ equations is independent of the other and thus solved as an equation of a SDOF system.

This coordinate system where the structural matrices are diagonal is called the “normal” coordinate system or the “modal” coordinate system.

**UNCOUPLING THE EQUATIONS OF MOTION FOR MODE SUPERPOSITION ANALYSIS**

Let us consider the same 2 DOF system shown in section or any other undamped system with ‘N’ DOF’s wherein the mass and / or stiffness matrices are non-diagonal i.e. the equations are coupled. The equation of motion for such a system can be written as;

$$\boxed{M\ddot{X} + C\dot{X} + KX = F(t)}$$

It should be noted that;

- M, C and K, in general, are non-diagonal
- Thus, the equations are coupled.

Suppose, we introduce a new set of dependent variables Z (t) using the transformation;

$$X(t) = T\,Z(t)$$

Where; T is an n x n transformation matrix to be selected.

Then;

$$M\ddot{X} + C\dot{X} + KX = F(t)$$
$$X(0) = X_0; \dot{X}(0) = X_0$$

These represent the displacement and velocity at t = 0; the initial conditions.Now, carrying out the transformation, that is;

$$
\begin{aligned}
& X(t) = TZ(t) \\
& \Rightarrow MT\ddot{Z}(t) + CT\dot{Z}(t) + KTZ(t) = F(t) \\
& \Rightarrow T^t MT\ddot{Z}(t) + T^t CT\dot{Z}(t) + T^t KTZ(t) = T^t F(t) \\
& \Rightarrow \bar{M}\ddot{Z}(t) + \bar{C}\dot{Z}(t) + \bar{K}Z(t) = \bar{F}(t)
\end{aligned}
$$

**The Eigen value problem**

Considering the seemingly unrelated problem of un-damped free vibration analysis;

$$M\ddot{X} + KX = 0$$

Let us assume that all the points of the structure oscillate harmonically at the same frequency;That is;

$$x_k(t) = r_k \exp(i\omega t); k = 1, 2, \cdots, n$$

OR;

$$X(t) = R\exp(i\omega t)$$

Where; R is an nx1 vector

$$\Rightarrow \dot{X}(t) = i\omega R\exp(i\omega t) \;\&\; \ddot{X}(t) = -\omega^2 R\exp(i\omega t)$$

Substituting this in the equation;

$$M\ddot{X} + KX = 0$$

$$\left[-\omega^2 MR + KR\right]\exp(i\omega t) = 0$$

$$\Rightarrow \left[-\omega^2 RM + KR\right] = 0$$

Therefore;

$$KR = w^2 MR$$

This is an algebraic Eigen value problem because:

Mathematically, we're looking for vectors R that solve this equation, as every positive definite matrix is invertible, any vector that solves that equation will also solve:

$$M^{-1}K\dot{R} = w^2 R$$

So then solving it is merely a task of finding the eigenvectors and respective eigenvalues. On the other hand, assuming you only have M and K, you can find multiple R and w that will satisfy this equation, assuming the matrix has a complete set of eigenvectors.

Hence R must an eigenvector of [M-1 K] with eigenvalue ω.

**Solving the eigenvalue problem**

Consider;

$$KR = \omega^2 MR$$

$$\text{Let } \left[K - \omega^2 M\right]^{-1} \text{ exist.}$$

That is;

$$\Rightarrow \left[K - \omega^2 M\right]^{-1} \left[K - \omega^2 M\right] R = 0$$

If [K – ω2 M] -1 exists, R = 0 is the solution.

$$\Rightarrow \left|K - \omega^2 M\right| = 0$$

Condition for existence of the non-trivial solution is that;[K

– ω2 M] -1 should not exist.

That is; if [K – ω2 M] -1 does not exist;

$$\Rightarrow \left|K - \omega^2 M\right| = 0$$

This is called the characteristic equation and this leads to the characteristic values;

$$\omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2 \; and \; associated \; eigen \; vactors, \quad R_1, R_2, \ldots \ldots, R_n$$

Orthogonality property of eigenvectors

Consider the $r^{th}$ and the $s^{th}$ Eigen pairs;

$$KR_r = \omega_r^2 MR_r \qquad (1)$$

$$KR_s = \omega_s^2 MR_s \qquad (2)$$

Multiplying (1) by Rst and multiplying 2 by Rrt, we get;

$$R_s^t KR_r = \omega_r^2 R_s^t MR_r \qquad (3)$$

$$R_r^t KR_s = \omega_s^2 R_r^t MR_s \qquad (4)$$

Transposing both sides of equation 4 we get;

$$R_s^t K^t R_r = \omega_s^2 R_s^t M^t R_r$$

Since, Kt = K and Mt = M; we get;

$$R_s^t KR_r = \omega_s^2 R_s^t MR_r \qquad (5)$$

Subtracting (3) from (5);

$$\left(\omega_r^2 - \omega_s^2\right) R_s^t MR_r = 0$$

That is;

$$R_s^t MR_r = 0 \qquad r \neq s$$

Substituting;

$$R_s^t M R_r = 0$$

In equation 3;

$$R_s^t K R_r = 0 \qquad r \neq s$$

Normalization

$$R_s^t M R_s = 1$$

$$R_s^t K R_s = \omega_s^2$$

**Conclusions because of the orthogonality property of eigenvectors**

The properties of Eigen vectors derived in section 3.3, lead to very important conclusions. Thatis;

If we introduce;

$$\Phi = \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}_{(n\times n)}$$

$$\Lambda = \text{Diag}\begin{bmatrix} \omega_1^2 & \omega_2^2 & \cdots & \omega_n^2 \end{bmatrix}$$

Then we have;

**Orthogonality relations**

$$\Phi^t M \Phi = I$$

$$\Phi^t K \Phi = \Lambda$$

T=φ

.

## **SOLVING THE UNDAMPED FORCED VIBRATION ANALYSIS IN THE MODAL (NORMAL) CO-ORDINATE SYSTEM**

Consider the un-damped forced vibration analysis of a MDOF System, the equation of motion of which is denoted by;

$$M\ddot{X} + KX = F\left(t\right)$$

We have the initial conditions given by;

$$X\left(0\right) = X_0; \dot{X}\left(0\right) = X_0$$

Carrying out the transformation, using the transformation matrix as φ which corresponds to the mode shapes obtained by considering the free vibration problem wherein all DOF oscillate simple harmonically with the same frequency;

$$X(t) = \Phi Z\left(t\right)$$

Substituting this transformation in the equation of motion;

$$\Rightarrow M\Phi\ddot{Z}\left(t\right) + K\Phi Z\left(t\right) = F(t)$$

Pre-multiplying by $\varphi^T$ and then normalizing;

$$\Rightarrow \Phi^t M\Phi\ddot{Z}\left(t\right) + \Phi^t K\Phi Z\left(t\right) = \Phi^t F(t)$$

$$\Rightarrow I\ddot{Z}(t)+\Lambda Z(t)=\bar{F}(t)$$

In the above equation, I corresponds to the identity matrix, Λ is the matrix whose diagonal terms correspond to the square of frequencies ω12, ω22, ω32,….

In the above, the equations of motion are uncoupled and each equation can be solved as a SDOF system;

$$\Rightarrow \ddot{z}_r+\omega_r^2 z_r = f_r(t); r=1,2,\cdots,n$$

The initial conditions: Z (0) and in the new coordinate system can be evaluated as below;

$$X(0)=\Phi Z(0)$$

$$\Phi^t MX(0)=\Phi^t M\Phi Z(0)=Z(0)$$

$$Z(0)=\Phi^t MX(0)\ \&\ \dot{Z}(0)=\Phi^t M\dot{X}(0)$$

Thus, initial conditions in the new coordinate system are obtained.

**Solving the equation of motions in the new coordinate system**

Thus, now the equation of motion of the MDOF system has been transformed to a new coordinate system wherein each equation could be solved independently as a SDOF system. The SDOF system equation may be solved through any of the techniques like;

Duhamel / convolution integral approach

Explicit central difference method Implicit

Newmark’s integration scheme.

If one were to solve the SDOF equation in the new coordinate system, using the Duhamel integral approach, then;

$$z_r(t) = z_r(0)\cos\omega_r t + \frac{\dot{z}_r(0)}{\omega_r}\sin\omega_r t + \int_0^t \frac{1}{\omega_r}\sin\omega_r(t-\tau)f_r(\tau)d\tau$$

In order to obtain the solution in the original / physical coordinate system,

$$X(t) = \Phi Z(t)$$

$$x_k(t) = \sum_{r=1}^{n} \Phi_{kr} z_r(t)$$

As seen from the above equation, the response corresponding to each DOF is obtainedsuperimposing the contribution from each mode. Hence, the name mode superposition.Thus;

$$= \sum_{r=1}^{n} \Phi_{kr} \left\{ z_r(0)\cos\omega_r t + \frac{\dot{z}_r(0)}{\omega_r}\sin\omega_r t + \int_0^t \frac{1}{\omega_r}\sin\omega_r(t-\tau)f_r(\tau)d\tau \right\}$$

**SUMMARY: MODE SUPERPOSITION ANALYSIS**

From the above, it can be summarized that;

1. Equations of motion for MDOF systems are generally coupled.
2. Coupling between co-ordinates is manifest in the form of structural matrices being non-diagonal
3. Coupling is not an intrinsic property of a vibrating system. It is dependent upon the co-ordinate system. This choice itself arbitrary.
4. Equation of motions are not unique. They depend upon the choice of the co-ordinate system.
5. Normal modes of vibration (mode shapes) are special undamped free vibration solutions such that the degrees of freedom of the structure oscillate simple harmonically at the same frequency with the ratio of the displacements between any two degrees of freedom being independent of time.
6. The frequencies at which normal mode oscillations are possible are called the natural frequencies.

7. The modal matrix (each column of the modal matrix corresponds to a mode shape) is orthogonal to the mass and the stiffness matrix.
8. The undamped normal modes in conjunction with proportional damping models (dampingis not discussed in this write-up) simplify vibration analysis considerably.

**Dynamic Analysis of system with distributed Properties:**

The dynamic analysis of the structures, modeled as a lumped parameter system with discrete coordinates was presented in previous problems.

The modeling of structures with discrete coordinates provides a practical approach for the analysis of structures subjected to dynamic loads. However the results obtained from these discrete models can only give approximate solutions to the actual behavior of dynamic system which have continuous distributed properties and consequently an infinite number of degree of freedom.

In this present topic, we considered the dynamic theory of beams and rods having distributed mass and elasticity for which the governing equations of motion are partialdifferential equation.

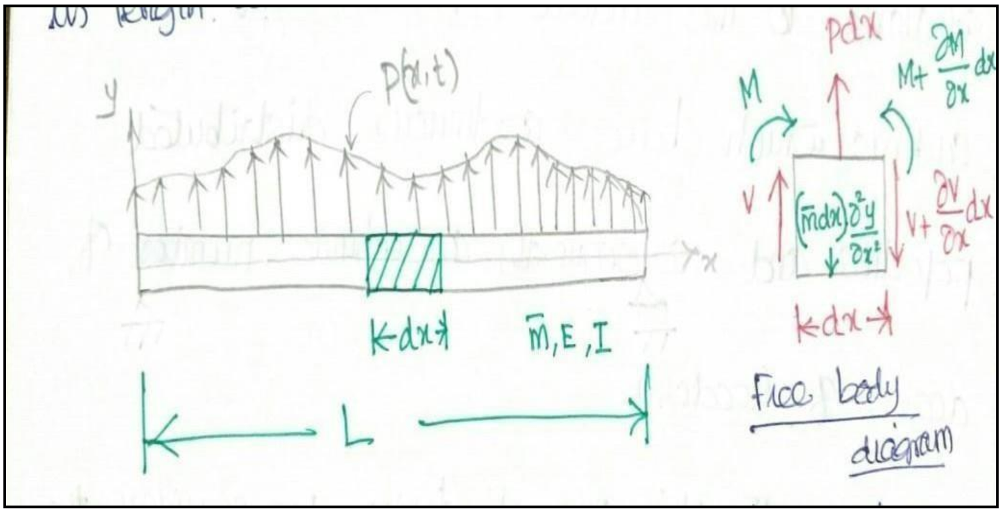


**Figure1 Flexural Member with distributed mass**

Flexural vibration of uniform beam

The study on flexural behavior of beam is generally based on simple bending theory. The method of analysis is known as Bernoulli-Euler's theory, which assumes that plane section remains plane during flexure

Let us consider a beam with varying load along its length (L) and consider a small segment inbeam of length "dx". Where $m$ is the mass per unit length and $P = P(x, t)$ is the load per unit length. The shear force and bending moment acting on the segment is shown in the figure.
Due to an equilibrium condition, the sum of the forces in the particular section is zero

$$-V + V + \frac{\partial V}{\partial x}dx - P(x, t)\,dx + m\,dx\frac{\partial^2 y}{\partial t^2} = 0$$

An above eqn can be reduced to

$$\boxed{\frac{\partial V}{\partial x} + m\frac{\partial^2 y}{\partial t^2} = P(x,t)}$$

From simple bending theory

$$M = EI\frac{\partial^2 y}{\partial x^2}$$

$$V = \frac{\partial M}{\partial x}$$

$$\therefore V = EI\frac{\partial^3 y}{\partial x^3}$$

$$\frac{dV}{dx} = EI\frac{\partial^4 y}{\partial x^4}$$

By substituting these terms into above expressions then we will get

$$\boxed{EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = P(x,t)}$$

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SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

# UNIT –IV - CONTINUOUS SYSTEMS – SCIA5202

Flexural vibration of uniform beam

The study on flexural behavior of beam is generally based on simple bending theory. The method of analysis is known as Bernoulli-Euler's theory, which assumes that plane section remains plane during flexure

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By substituting these terms into above expressions then we will get

$$\boxed{EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = P\ x, t}$$

An above expression is representing an eqn of motion of flexural member with forcedcondition. Similarly for the free condition of vibration, an above eqn can be modify as

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = 0$$

**Note:**

An above eqn have two variables (x and t), so by solving this we have to use method ofseparation of variables.

A solution of above eqn can be written as $\boxed{y = \varphi\, x \,.\, f(t)}$

$f(t) \rightarrow Function\ of\ t \qquad y = \varphi\ \ x\ .\, f(t)$

$$\frac{\partial^4 y}{\partial x^4} = \varphi^{IV} x \,.\, f(t)$$

$$\frac{\partial^2 y}{\partial t^2} = \varphi\ \ x\ .\, f^{II}(t)$$

By substituting these terms into above expressions then we will get

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = 0$$

$$EI\varphi^{IV}\ x\ .\, f\ \ t\ + m\varphi\ \ x\ .\, f^{II}\ t\ = 0$$

$$\frac{EI}{m} \times \frac{\varphi^{IV}\ x}{\varphi\ x} = -\frac{f^{II}\ t}{f\ t} = Constant$$

$$\frac{EI}{m} \times \frac{\varphi^{IV}\ x}{\varphi\ x} = -\frac{f^{II}\ t}{f\ \ t} = \omega^2$$

$$\frac{\varphi^{IV}\ x}{\varphi\ x} = \frac{m}{EI}\omega^2$$

$$\boxed{\frac{\varphi^{IV}\, x}{\varphi\, x} = a^4} \quad \boxed{\therefore a^4 = \frac{m}{EI}\omega^2}$$

By solving an above eqn, we can get

$$\cancel{\varphi\, x = C_1 \cosh ax + \sinh ax + C_2 \cosh ax - \sinh ax + C_3 \cos ax + C_4 \sin ax}$$

Similarly

$$-\frac{f^{II}\, t}{f\, t} = \omega^2$$

By solving an above eqn, we can get

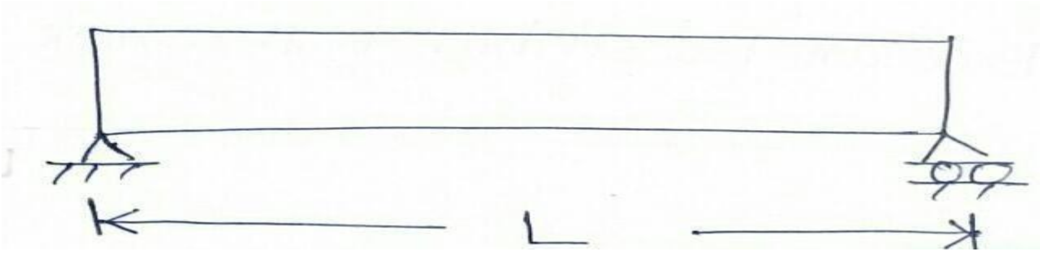
$$\boxed{f\ t = P \cos \omega t + Q \sin \omega t}$$

$$y = \varphi\ x\ .f(t)$$

$$\boxed{\begin{aligned} y = C_1 \cosh ax + \sinh ax + C_2 \cosh ax - \sinh ax + C_3 \cos ax + C_4 \sin ax\, .\, P \cos \omega t \\ + Q \sin \omega t \end{aligned}}$$

**FOR SIMPLY SUPPORTED BEAM:**

**Case (i): Derivation of $\varphi(x)$ by keeping $f(t)$ as constant**




Boundary conditions:

$$\boxed{\begin{matrix} y(0,t) = 0 \quad M(0,t) = 0 \\ y(L,t) = 0 \quad y(L,t) = 0 \end{matrix}}$$

By applying all the boundary condition, we can get a natural frequency of a system

$$\boxed{\omega_n = n^2\pi^2 \sqrt{\frac{EI}{mL}}}$$

By applying all the boundary condition, $\boldsymbol{\varphi(x)}$ *can be reduced as*

$$\boxed{\varphi(x) = A.\sin\left(\frac{n\pi}{L}x\right)}$$

An eqn "y" can be written as

$$y = \varphi(x).f(t)$$

$$\boxed{y = A.\sin\left(\frac{n\pi}{L}x\right).\left(P\cos\omega t + Q\sin\omega t\right)}$$

**Case (i): Derivation of $f(t)$**

Let us assume the distributed mass in a system can be doesn't change with respect to time. So $f(t)$ leads to 1.

Therefore an above expression can be written as

$$y = A.\sin\frac{n\pi}{L}x$$

Let us assume response amplitude can be 1 $(\therefore A = 1)$

Therefore normal of a system can be written as

$$\boxed{y_n = \sin\frac{n\pi}{L}x}$$

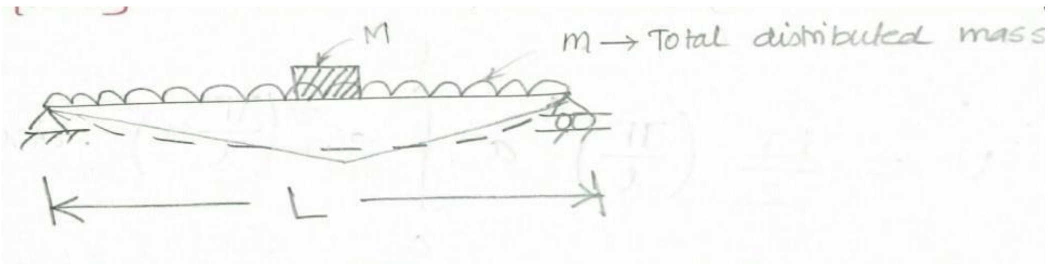
### Rayleigh's Method for Continuous Mass System

In Rayleigh method, the basic principle is the strain energy stored in a member/structure isequal to kinematic energy in a member/structure.

It is an approximate method to compute the fundamental frequency of the system.It can apply for both discrete and continuous system.

**Example:**

Determine the fundamental frequency of simply supported beam with total distributed massof "m" and a central load of M



Let us assume a deflection profile for this loading case can be $y = a \sin\frac{\pi}{L} x$

Boundary conditions

At $x = 0$; $y = 0$ $x = L$; $y = 0$

$$y = a \sin\frac{\pi}{L} x$$

$$\frac{dy}{dx} = \frac{\pi}{L} \times a \times \cos\frac{\pi}{L} x$$

$$\frac{d^2y}{dx^2} = -\left(\frac{\pi}{L}\right)^2 \times a \times \sin\frac{\pi}{L} x$$

By solving above eqn, we can get

$$\boxed{= \frac{EI\,\pi^4}{4\,L}\,a L^2}$$

<u>Calculation of kinetic Energy:</u>

It can be due to both cases

1. Due to central load "M"
2. Due to total distributed mass"m"

**Kinetic energy in a system because of central load "M"**

Let us consider maximum deflection at center as "a". The displacement profile can be

$$y = a.\sin \omega t$$

$$velocity = \frac{dy}{dx} = a\omega \cos \omega t$$

$$Max.\,Velocity = v = a\omega \cos \omega t$$

$$v_{max} = a\omega$$

$$K.E_{Due\ to\ M} = \frac{1}{2} m v^2$$

$$\boxed{K.E_{Due\ to\ M} = \frac{1}{2} \times M \times a\omega^2}$$

**Kinetic Energy due to self-weight "m"**

Mass per unit length= $m_x = \frac{m}{L}$

$$\boxed{K.E_{\,Due\ to\ Self\ wt} = \int_0^L \frac{1}{2}\,\omega^2\, m_x \,.\, y^2 \,.\, dx}$$

To reduce an above eqn, we can get

$$\boxed{K.E_{\,Due\ to\ Self\ wt} = \frac{m\omega^2 a^2}{4}}$$

Total Kinetic Energy in a system= $K.E_{\,Total} = K.E_{\,Due\ to\ M} + K.E_{\,Due}$

$$\boxed{K.E_{\,Total} = \frac{1}{2}\times M \times a\omega^2 + \frac{m\omega^2 a^2}{4}}$$

$$K.E_{\,Max\ in\ a\ system} = P.E_{\,Max\ in\ a\ system}$$

$$\frac{1}{2}\times M \times a\omega^2 + \frac{m\omega^2 a^2}{4} = \frac{EI}{4}\left(\frac{\pi}{L}\right)^4 a^2 L$$

By solving an above expression

$$\boxed{\omega^2 = \frac{EI\,\frac{L}{2}\left(\frac{\pi}{L}\right)^4}{M + \frac{m}{2}}}$$

SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

# UNIT –V -APPLICATIONS – SCIA5202

# SCIA5202 - Structural Dynamics

UNIT V

## BASE ISOLATION METHOD IN BUILDING

### INTRODUCTION

- Earthquake by itself, is not a disaster, it is natural phenomenon result from ground movement, sometimes violent.
- These produce surface waves, which cause vibration of the ground and structures standing on top.
- Depending on the characteristics of these vibrations, the ground may develop cracks, fissures and settlements.
- The possible risk of loss of life adds a very serious dimension to seismic design, putting a moral responsibility on structural engineers.
- In recent times, many new systems have been developed, either to reduce the earthquake forces acting on the structure or to absorb a part of seismic energy.
- One of the most widely implemented and accepted seismic protection systems is base isolation.

### BASE ISOLATION:

- Base isolation is one of the most widely accepted seismic protection systems inearthquake prone areas.
- It mitigates the effect of an earthquake by essentially isolating the structure frompotentially dangerous ground motions.
- Seismic isolation is a design strategy, which uncouples the structure for the damagingeffects of the ground motion.
- The term isolation refers to reduced interaction between structure and the ground.
- When the seismic isolation system is located under the structure, it is referred as“base isolation”.
- The other purpose of an isolation system is to provide an additional means of energydissipation, thereby reducing the transmitted acceleration into the superstructure.
- The decoupling allows the building to behave more flexibly which improves itsresponse to an earthquake.

## CONCEPT OF BASE ISOLATION

- The concept of base isolation is explained through an example building resting onfrictionless rollers.
- When the ground shakes, the rollers freely roll, but the building above does not move.
- Thus, no force is transferred to the building due to shaking of the ground; simply, thebuilding does not experience the earthquake.
- A careful study is required to identify the most suitable type of device for a particularbuilding.
- Also, base isolation is not suitable for all buildings.
- Most suitable structures for base-isolation are
- Low to medium-rise buildings rested on hard soil underneath
- High-rise buildings or buildings rested on soft soil are not suitable for base isolation.

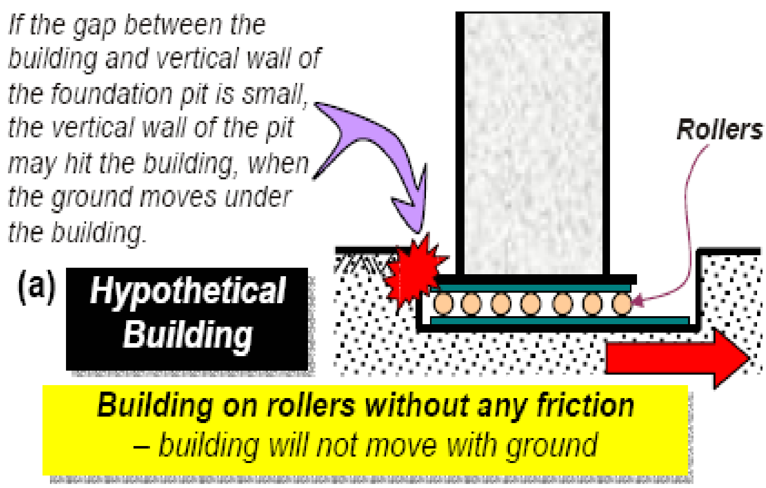



**Figure 1 Building base with bottom roller condition**

## PRINCIPLE OF BASE ISOLATION

- The fundamental principle of base isolation is to modify the response of the building so that the ground can move below the building without transmitting these motions into the building.
- A building that is perfectly rigid will have a zero period.
- When the ground moves the acceleration induced in the structure will be equal to the ground acceleration and there will be zero relative displacement between the structure and the ground.
- The structure and ground move the same amount.
- A building that is perfectly flexible will have an infinite period.
- For this type of structure, when the ground beneath the structure moves there will be zero acceleration induced in the structure and the relative displacement between the structure and ground will be equal to the ground displacement.
- So in flexible structures the structure will not move, the ground will.

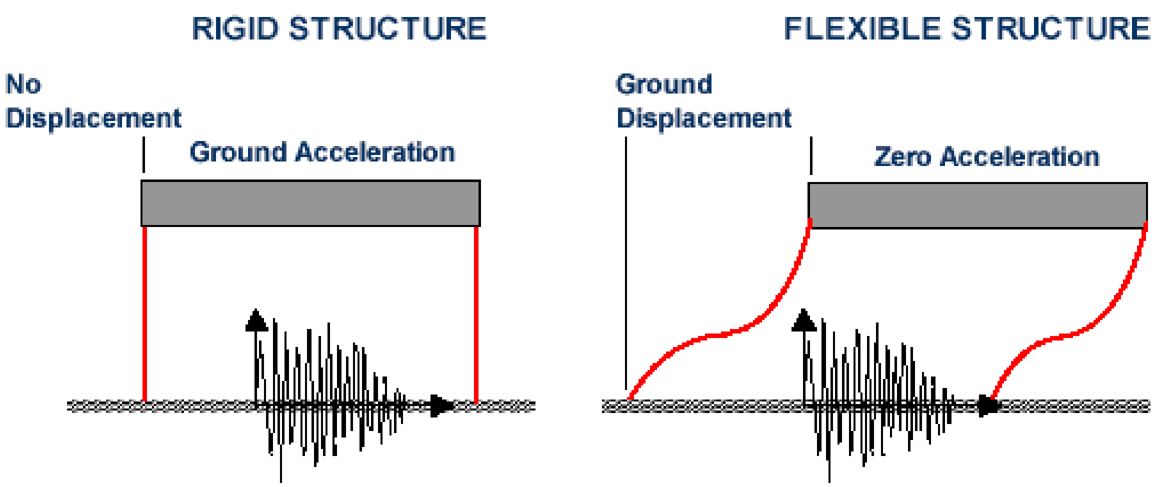



**Figure 2 Difference between Rigid and FlexibleStructure**

**FIXED BASE STRUCTURE VS ISOLATED BASE STRUCTURE**

- During earthquakes, the conventional structure without seismic isolation is subjected to substantial story drifts, which may lead to damage or even collapse of the building.
- Whereas the isolated structure vibrates almost like a rigid body with largedeformations or displacements restricted by the isolation bearings.
- The decoupling effect gives this extra advantage to isolated structures.
- The lateral forces of the isolated building are not only reduced in magnitude but also fairly redistributed over the floors, which further mitigates the overturning moment of the structure.

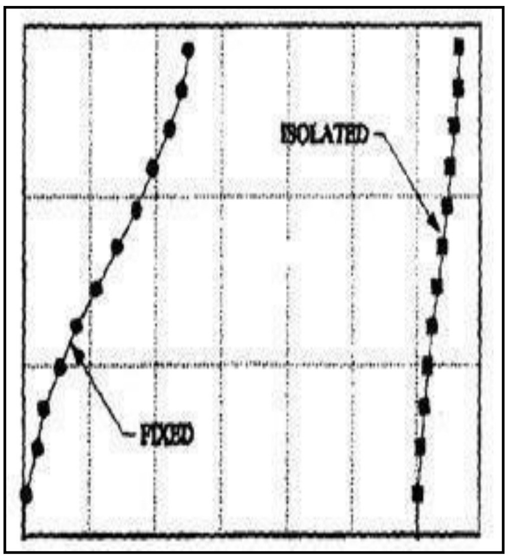



**Figure 3 Comparison of Fixed Base Structure VSIsolated Base Structure**

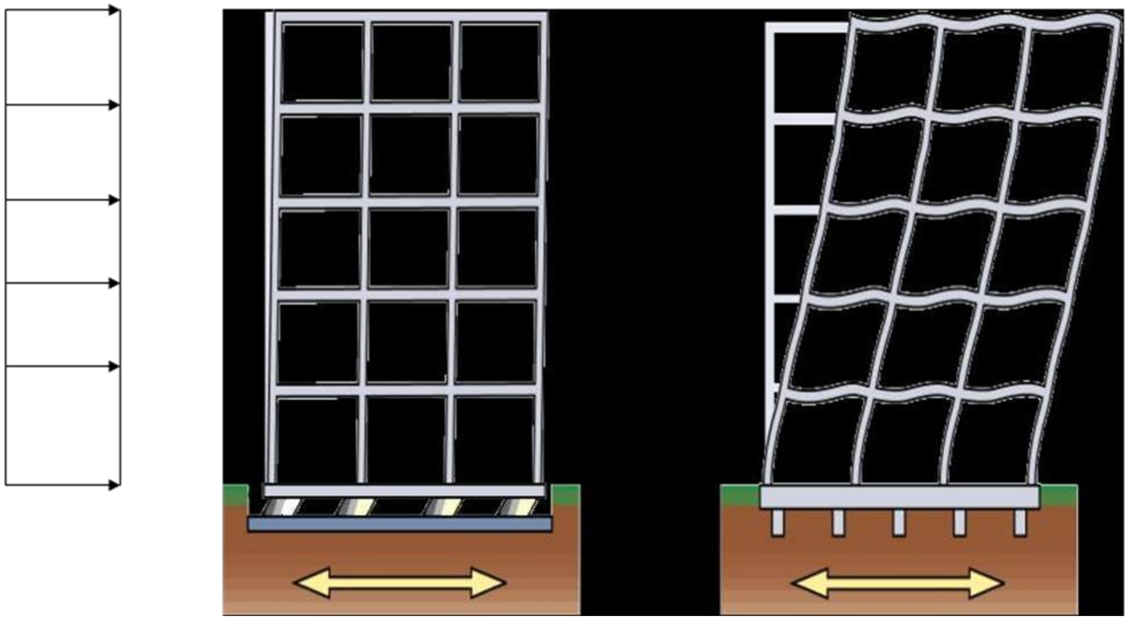


**Figure 4 Fixed Base Structure vs Isolated BaseStructure**

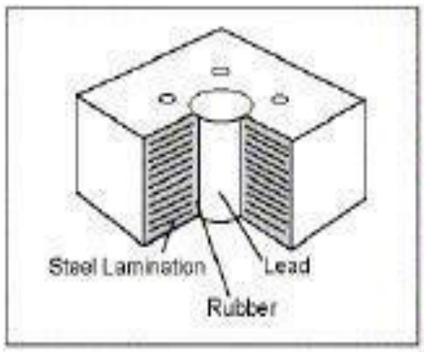
**ISOLATION COMPONENTS**

- **Elastomeric Isolators**
  - Natural Rubber Bearings
  - Low-Damping Rubber BearingsLead-
  - Rubber Bearings
  - High-Damping Rubber Bearings
- **Sliding Isolators**
  - Resilient Friction System Friction
  - Pendulum System

**Elastomeric Isolators**

- These are formed of horizontal layers of natural or synthetic rubber in thin layers bonded between steel plates.
- The steel plates prevent the rubber layers from bulging and so the bearing is able to support higher vertical loads with only small deformations.
- Plain elastomeric bearings provide flexibility but no significant damping and will move under service loads.

- Methods used to overcome these deficits include lead cores in the bearing, specially formulated elastomers with high damping and stiffness for small strains or other devices in parallel.



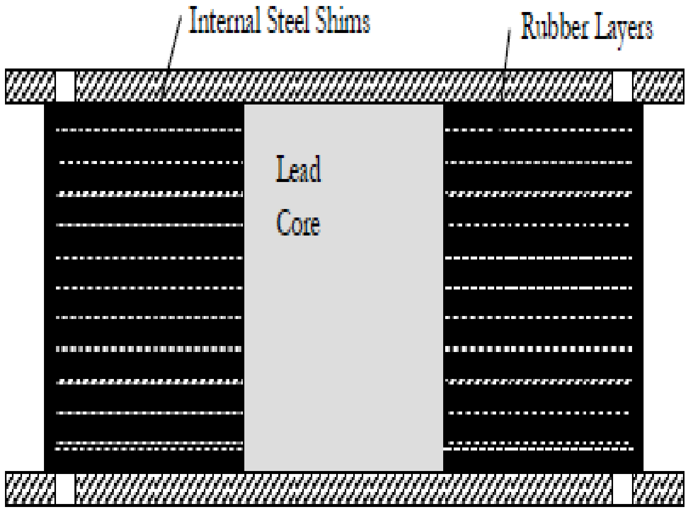

**Low Damping Natural or Synthetic Rubber Bearings**

- Elastomeric bearings use either natural rubber or synthetic rubber (such as neoprene),which have little inherent damping, usually 2% to 3% of critical viscous damping.
- For isolation they are generally used with special elastomer compounds (high dampingrubber bearings) or in combination with other devices (lead rubber bearings).
- They are also flexible at all strain levels.

**Lead Rubber Bearings**

- A lead-rubber bearing is formed of a lead plug force-fitted into a pre-formed hole in an elastomeric bearing.
- The lead core provides rigidity under service loads and energy dissipation under high lateral loads.
- Top and bottom steel plates, thicker than the internal shims, are used to accommodate mounting hardware. The entire bearing is encased in cover rubber to provide environmental protection.
- When subjected to low lateral loads (such as minor earthquake, wind or traffic loads) the lead rubber bearing is stiff both laterally and vertically.

- The lateral stiffness results from the high elastic stiffness of the lead plug and the vertical rigidity (which remains at all load levels) results from the steel-rubber construction of the bearing.



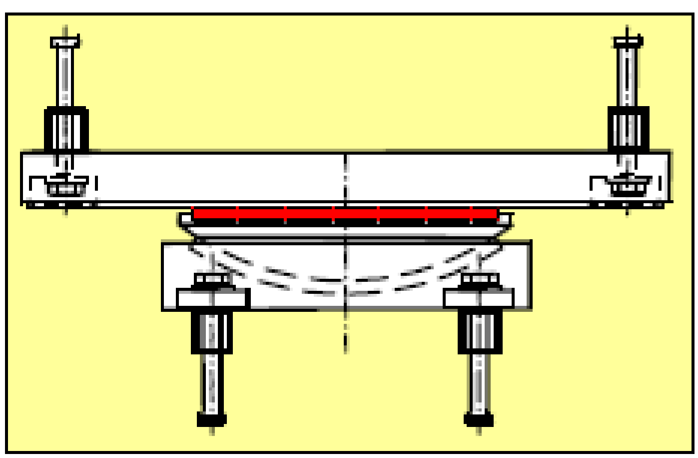

**Isolation Systems based on Sliding**

- The second most common type of isolation system uses sliding elements between the foundation and base of the structure.
- In this type of Isolation system, the sliding displacements are controlled by high-tension springs or laminated rubber bearings, or by making the sliding surface curved.
- These mechanisms provide a restoring force to return the structure to its equilibriumposition.

**Sliding isolator without Recentering capacity**

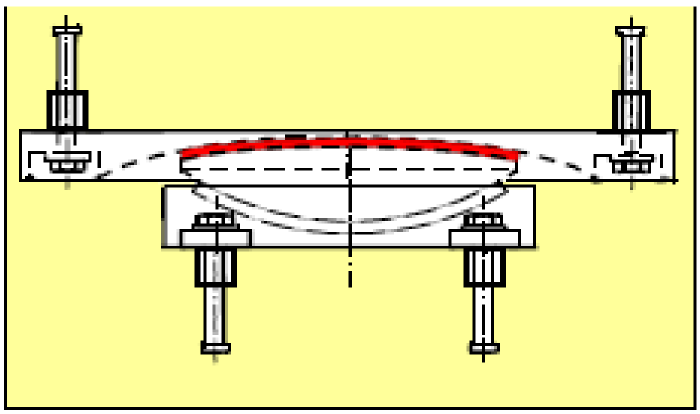
- This consists of a horizontal sliding surface, allowing a displacement and thus dissipating energy by means of defined friction between both sliding components and stainless steel.

- One particular problem with a sliding structure is the residual displacements that occurafter major earthquakes.



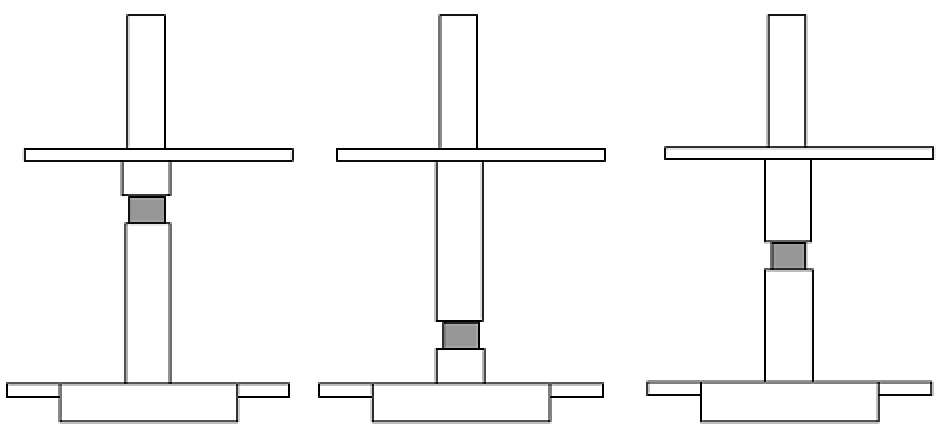
## Sliding Isolator with Recentering Capacity

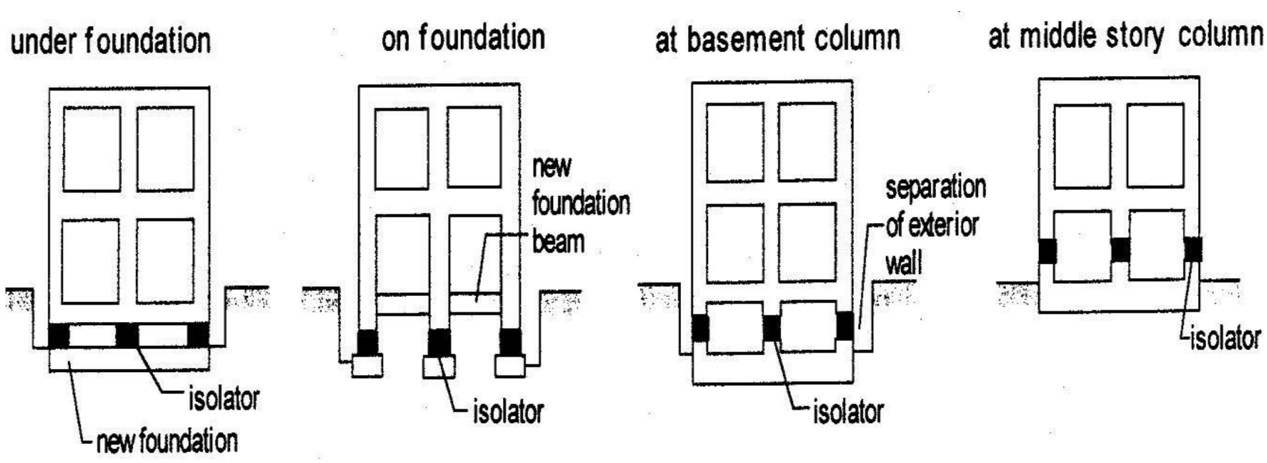
- Compared with sliding isolators, sliding isolation pendulum (SIPs) with recentering capacity have a concave sliding plate.
- Due to geometry, each horizontal displacement results in a vertical movement of theisolator.
- The potential energy, stored by the superstructure, which has been pushed to the top, automatically results in recentering the bearing into neutral position.
- They remain horizontally flexible, dissipate energy and recenter the superstructure intoneutral position.



## Isolator Locations

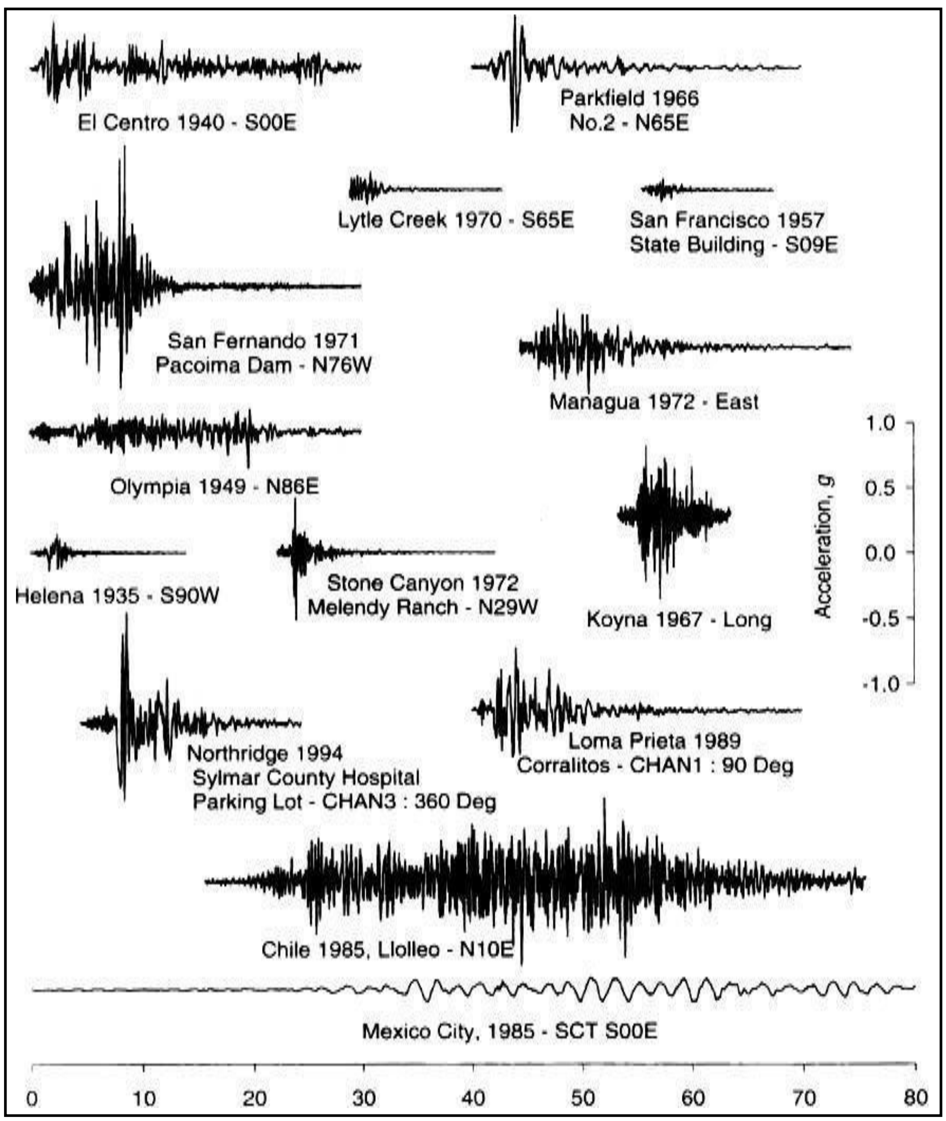
- The requirement for installation of a base isolation system is that the building be able tomove horizontally relative to the ground, usually at least **100 mm.**
- The most common configuration is to install a diaphragm immediately above theisolators.
- If the building has a basement then the options are to install the isolators at the top,bottom or mid-height of the basements columns and walls.






Response Spectrum

**<u>Typical Accelerograms</u>**



**Response Spectrum:**

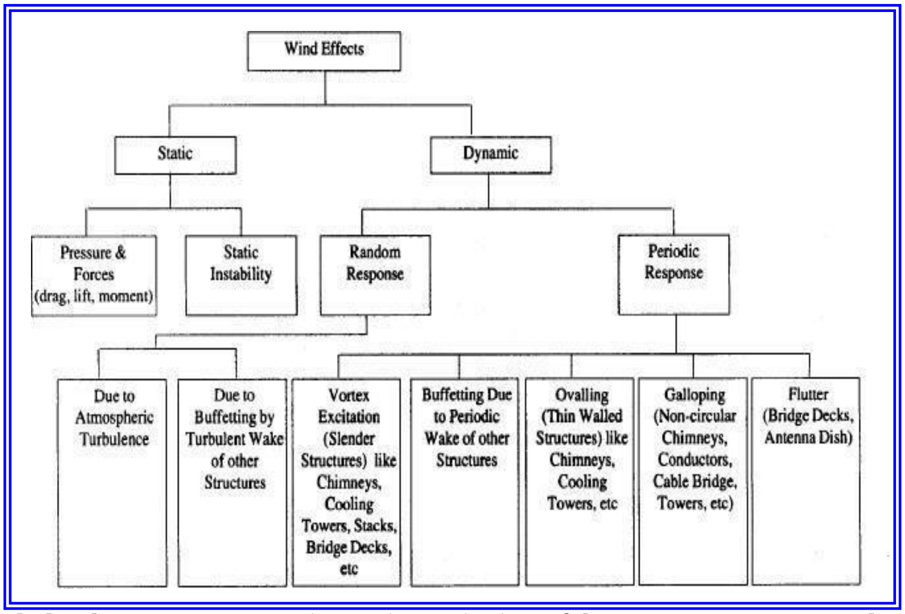
- If the ground moves as per the given accelerogram, what is the maximum response ofa single degree of freedom (SDOF) system (of given natural period and damping)?
  - Response may mean any quantity of interest, e.g., deformation, acceleration
  - Using a computer, one can calculate the response of SDOF system with time(time history of response)
  - Can pick maximum response of this SDOF system (of given T and damping)from this response time history
  - Repeat this exercise for different values of natural period.
  - For design, we usually need only the maximum response.
  - Hence, for future use, plot maximum response versus natural period (for agiven value of damping).
  - Such a plot of maximum response versus natural period for a given accelerogram is called **response spectrum.**

**RESPONSE SPECTRUM – IS 1893:2002**

- Different terms used in the code:

- Design Acceleration Spectrum (clause 3.5)
  - Response Spectrum (clause 3.27)
- Acceleration Response Spectrum (used in cl. 3.30)
  - Design Spectrum (title of cl. 6.4)
- Structural Response Factor
- Average response acceleration coefficient (see terminology of Sa/g on p. 11)

WIND LOAD ANALYSIS:




For tall, long span and slender structures a „dynamic analysis‟ of the structure is essential.
Wind Gusts cause fluctuating forces on the structure which induce large dynamic motions /oscillations.
The severity of the wind-induced dynamic motions /oscillations depends on - the naturalfrequency of vibration - the damping of the structure.
Dynamic motions are induced in both directions „along-wind‟ direction as well as „across-wind‟ direction.
The „along-wind‟ response of the structure is accounted for by a magnification factor („ gust factor‟) applied to static forces.
The “across-wind‟ response requires a separate “dynamic - analysis”.

**<u>Provisions of IS 875: 1987 (SP-64, 2001) can be broadly classified as:</u>**

- Computation of design wind speed based on wind zone, terrain category,topography and wind direction.
- Computation of design wind pressure.
- Computation of wind load using pressure coefficients. (Pressure coefficientsare applicable to design of structural elements like walls, roofs and cladding.)
- Computation of wind load using force coefficients. (Force coefficientsapplicable to the building frame / structural frameworks as a whole)
- Computation of along-wind forces using gust factor method to account fordynamic effect of wind.
- Evaluation of across-wind forces using wind tunnel - model analysis.

TEXT / REFERENCE BOOKS

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2. Anderson R.A., “Fundamentals of vibration”, Amerind Publishing Co.,1972.
3. Ray W. Clough and Joseph Penzien, “Dynamics of structures”,McGrawHill, New York, 1993