SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – I – SLOPE AND DEFLECTION OF BEAMS – SCIA1401

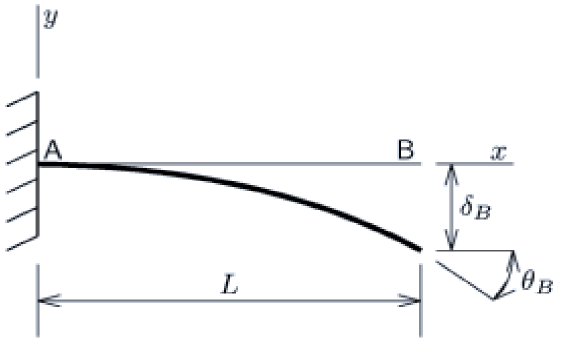
INTRODUCTION

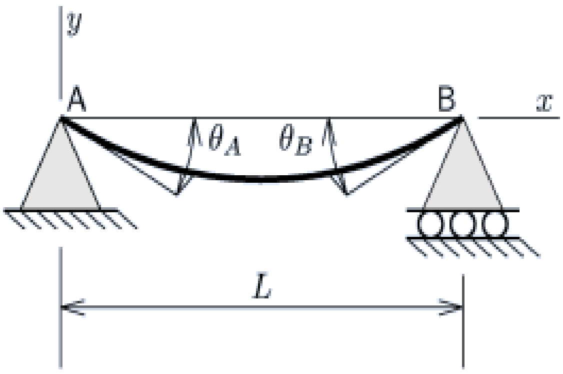
SLOPE OF A BEAM:

- ✓ slope at any section in a deflected beam is defined as the angle in radians which the tangent at the section makes with the original axis of the beam.
- ✓ slope of that deflection is the angle between the initial position and the deflected position.

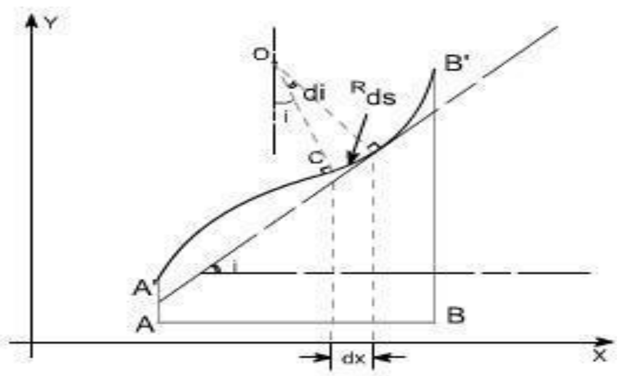
DEFLECTION OF A BEAM:

- ✓ The deflection at any point on the axis of the beam is the distance between its position before and after loading.
- ✓ When a structural is loaded may it be Beam or Slab, due the effect of loads acting upon it bends from its initial position that is before the load was applied. It means the beam is deflected from its original position it is called as Deflection.





BASIC DIFFERENTIAL EQUATION:



Consider a beam AB which is initially straight and horizontal when unloaded. If under the actionof loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holdsgood.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x andy, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Further, let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di.

But for the deflected shape of the beam the slope i at any point C is defined,

$\tan i = \frac{dy}{dx}$ ......(1) or $i = \frac{dy}{dx}$ Assuming $\tan i = i$

Futher

$ds = R\,di$

however,

$ds = dx$ [usually for small curvature]

Hence

$ds = dx = R\,di$

or $\boxed{\frac{di}{dx} = \frac{1}{R}}$

substituting the value of i, one get

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R}$ or $\frac{d^2 y}{dx^2} = \frac{1}{R}$

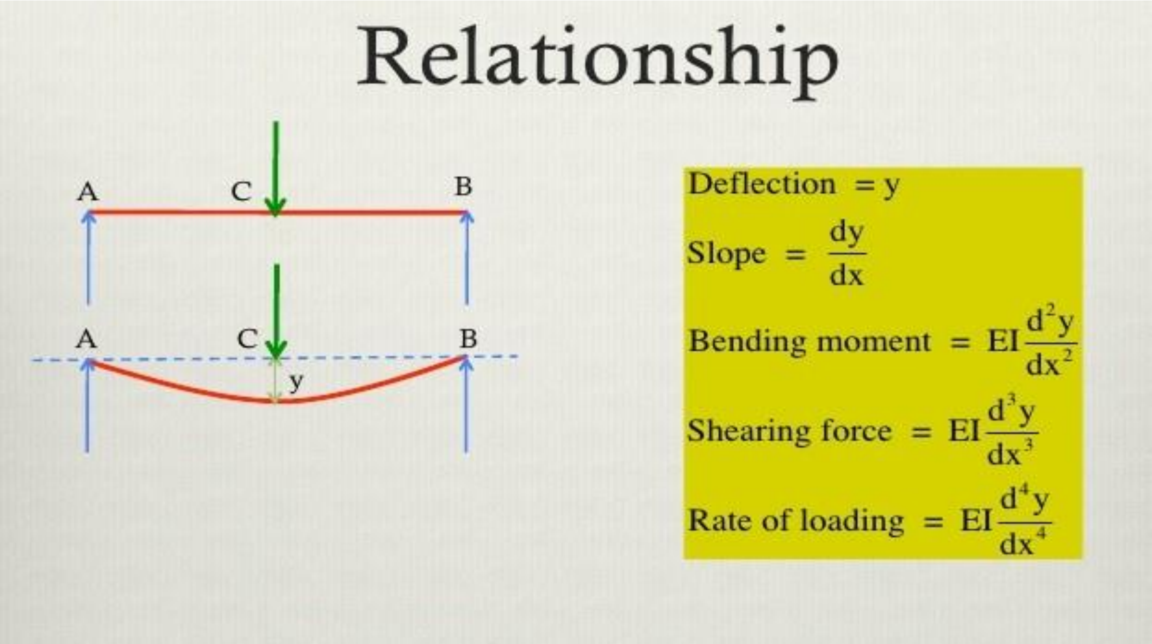
From the simple bending theory

$\frac{M}{I} = \frac{E}{R}$ or $M = \frac{EI}{R}$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2 y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry.



METHODS FOR FINDING THE SLOPE AND DEFLECTION OF BEAMS:

- Double integration method
- Moment area method
- Macaulay's method
- Conjugate beam method
- Strain energy method

DOUBLE INTEGRATION METHOD:

- ✓ The double integration method is a powerful tool in solving deflection and slope of a beam at any point because we will be able to get the equation of the elastic curve.
- ✓ This method entails obtaining the deflection of a beam by integrating the differential equation of the elastic curve of a beam twice and using boundary conditions to determine the constants of integration.
- ✓ The first integration yields the slope, and the second integration gives the deflection.

CONJUGATE BEAM:

- ✓ Conjugate beam is defined as the imaginary beam with the same dimensions (length) as that of the original beam but load at any point on the conjugate beam is equal to the bending moment atthat point divided by EI.
- ✓ Slope on real beam = Shear on conjugate beam
- ✓ Deflection on real beam = Moment on conjugate beam

PROPERTIES OF CONJUGATE BEAM:

- ✓ The length of a conjugate beam is always equal to the length of the actual beam.
- ✓ The load on the conjugate beam is the M/EI diagram of the loads on the actual beam.
- ✓ A simple support for the real beam remains simple support for the conjugate beam.
- ✓ A fixed end for the real beam becomes free end for the conjugate beam.
- ✓ The point of zero shear for the conjugate beam corresponds to a point of zero slope for the real beam.
- ✓ The point of maximum moment for the conjugate beam corresponds to a point of maximum deflection for the real beam.

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD:

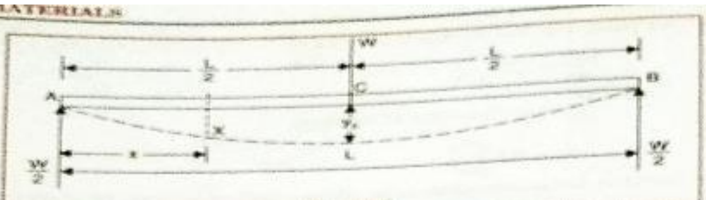



Fig. 12.3

Now $\qquad R_A = R_B = \frac{W}{2}$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x$$

$$= \frac{W}{2} \times x$$

(Plus sign is as B.M. for left portion at X is clockwise)

But B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2y}{dx^2} = \frac{W}{2} \times x \qquad \text{...(i)}$$

On integration, we get

$$EI \frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1 \qquad \text{...(ii)}$$

where C_1 is the constant of integration. And its value is obtained from boundary conditions. The boundary condition is that at $x = \frac{L}{2}$, slope $\left(\frac{dy}{dx}\right) = 0$ (As the maximum deflection is at the centre, hence slope at the centre will be zero). Substituting this boundary condition in equation (ii), we get

$$0 = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + C_1$$

or

$$C_1 = -\frac{WL^2}{16}$$

Substituting the value of C_1 in equation (ii), we get

$$EI \frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16} \qquad \text{...(iii)}$$

The above equation is known the *slope equation*. We can find the slope at any point on the beam by substituting the values of x. Slope is maximum at A. At A, $x = 0$ and hence slope at A will be obtained by substituting $x = 0$ in equation (iii).

$$\therefore \quad EI\left(\frac{dy}{dx}\right)_{\text{at }A} = \frac{W}{4}\times 0 - \frac{WL^2}{16}$$

$\left[\left(\frac{dy}{dx}\right)_{\text{at }A}\right.$ is the slope at A and is represented by $\left.\theta_A\right]$

$$EI \times \theta_A = -\frac{WL^2}{16}$$

or

$$\therefore \quad \theta_A = -\frac{WL^2}{16EI}$$

The slope at point B will be equal to θ_A, since the load is symmetrically applied.

$$\therefore \quad \theta_B = \theta_A = -\frac{WL^2}{16EI} \quad \ldots(12.6)$$

Equation (12.6) gives the slope in radians.

Deflection at any point

Deflection at any point is obtained by integrating the slope equation (*iii*). Hence integrating equation (*iii*), we get

$$EI \times y = \frac{W}{4}\cdot\frac{x^3}{3} - \frac{WL^2}{16}x + C_2 \quad \ldots(iv)$$

where C_2 is another constant of integration. At A, $x = 0$ and the deflection (y) is zero.

Hence substituting these values in equation (*iv*), we get

$$EI \times 0 = 0 - 0 + C_2$$

or

$$C_2 = 0$$

Substituting the value of C_2 in equation (*iv*), we get

$$EI \times y = \frac{Wx^3}{12} - \frac{WL^2 \cdot x}{16} \quad \ldots(v)$$

The above equation is known as *the deflection equation*. We can find the deflection at any point on the beam by substituting the values of x. The deflection is maximum at centre point C, where $x = \frac{L}{2}$. Let y_c represents the deflection at C. Then substituting $x = \frac{L}{2}$ and $y = y_c$ in equation (*v*), we get

$$EI \times y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16}\times\left(\frac{L}{2}\right)$$

$$= \frac{WL^3}{96} - \frac{WL^3}{32} = \frac{WL^3 - 3WL^3}{96}$$

$$= -\frac{2WL^3}{96} = -\frac{WL^3}{48}$$

$$\therefore \quad y_c = -\frac{WL^3}{48EI}$$

(Negative sign shows that deflection is downwards)

$$\therefore \quad \text{Downward deflection, } y_c = \frac{WL^3}{48EI} \quad \ldots(12.7)$$

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PROBLEMS:

1.A beam 6 m long, simply supported at its ends, is carrying a point load of 50 KN at its centre. The moment of inertia of the beam is 78 x 10^6 mm^4. If E for the material of the beam = 2.1 X 10^5 N/mm^2. calculate deflection at the centre of the beam and slope at the supports.

GIVEN DATA:

L = 6 m

W = 50 KN = 50 X 10^3 N

I = 78 X 10 mm^4

E = 2.1 X 10^5 N/mm^2

SOLUTION:

1. DEFLECTION AT THE CENTRE OF THE BEAM,

y_c = WL^3 / 48 EI

= 50000 X 6000^3 / (48 X 2.1 X 10^5 X 78 X 10^6)

= 13.736 mm.

2. SLOPE AT THE SUPPORTS,

$\Theta_A = \Theta_B = - WL^2 / 16\ EI$

$= 50000 \times 6000^2 / (16 \times 2.1 \times 10^5 \times 78 \times 10^6)$

= 0.06868 radians.

2. A beam carries 4 m long simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1° , find the deflection at the centre of the beam.

GIVEN DATA:

L = 4 m

$\Theta_A = \Theta_B = 1° = 1° \times (\pi /180) = 0.01745$ radians.

SOLUTION:

1. DEFLECTION AT THE CENTRE OF THE BEAM,

$\Theta_A = \Theta_B = - WL^2 / 16\ EI$

$0.01745 = WL^2 / 16\ EI$

$y_c = WL^3 / 48\ EI$

$= WL^2 / 16\ EI \times (L/3)$

$= 0.01745 \times (4000/3)$

= 23.26 mm.

3. A beam 3 m long, simply supported at its ends, is carrying a point load W at the centre. If the slope at the ends of the beam should not exceed 1° , find the deflection at the centre of the beam.

GIVEN DATA:

L = 3 m

$\Theta_A = \Theta_B = 1° = 1 \times (\pi /180) = 0.01745$ radians.

SOLUTION:

1.DEFLECTION AT THE CENTRE OF THE BEAM,

$\Theta_A = \Theta_B = - WL^2 / 16\ EI$

$0.01745 = WL^2 / 16\ EI$

$y_c = WL^3 / 48\ EI$

$= WL^2 / 16\ EI \times (L/3)$

$= 0.01745 \times (3000/3)$

= 17.45 mm.

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED WITH A UNIFORMLY DISTRIBUTED LOAD:

- ✓ A simply supported beam AB of length L and carrying a uniformly distributed load of w per unit length over the entire length is shown in fig.
- ✓ The reactions at A and B will be equal.
- ✓ Also, the maximum deflection will be at the centre of the beam.
- ✓ Each vertical reaction = (w X L)/2

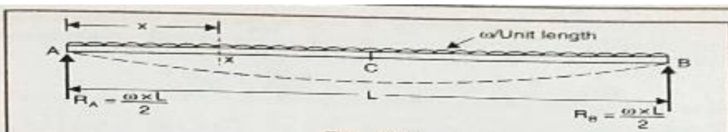

Fig. 12.6

$$\therefore \quad R_A = R_B = \frac{w \times L}{2}$$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x - w \times x \times \frac{x}{2} = \frac{w \cdot L}{2} \cdot x - \frac{w \cdot x^2}{2}$$

But B.M. at any section is also given by equation (12.3), as

$$M = EI\,\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\,\frac{d^2y}{dx^2} = \frac{w \cdot L}{2}\,x - \frac{w \cdot x^2}{2}$$

Integrating the above equation, we get

$$EI\,\frac{dy}{dx} = \frac{w \cdot L}{2} \cdot \frac{x^2}{2} - \frac{w}{2} \cdot \frac{x^3}{3} + C_1 \qquad \ldots(i)$$

where C_1 is a constant of integration.

Integrating the above equation again, we get

$$EI \cdot y = \frac{w \cdot L}{4} \cdot \frac{x^3}{3} - \frac{w}{6} \cdot \frac{x^4}{4} + C_1 x + C_2 \qquad \ldots(ii)$$

where C_2 is another constant of integration. Thus two constants of integration (*i.e.*, C_1 and C_2) are obtained from boundary conditions. The boundary conditions are :

(*i*) at $x = 0,\ y = 0$ and (*ii*) at $x = L,\ y = 0$

Substituting first boundary condition *i.e.*, $x = 0$, $y = 0$ in equation (*ii*), we get

$$0 = 0 - 0 + 0 + C_2 \quad \text{or} \quad C_2 = 0$$

Substituting the second boundary condition *i.e.*, at $x = L$, $y = 0$ in equation (*ii*), we get

$$0 = \frac{w \cdot L}{4} \cdot \frac{L^3}{3} - \frac{w}{6} \cdot \frac{L^4}{4} + C_1 \cdot L \qquad (C_2 \text{ is already zero})$$

$$= \frac{w \cdot L^4}{12} - \frac{w \cdot L^4}{24} + C_1 . L$$

or

$$C_1 = -\frac{wL^3}{12} + \frac{wL^3}{24} = -\frac{wL^3}{24}$$

Substituting the value of C_1 in equations (*i*) and (*ii*), we get

$$EI\,\frac{dy}{dx} = \frac{w \cdot L}{4} \cdot x^2 - \frac{w}{6}\,x^3 - \frac{wL^3}{24} \qquad \ldots(iii)$$

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and $$EI.y = \frac{w.L}{12}x^3 - \frac{w}{24}.x^4 + \left(-\frac{wL^3}{24}\right)x + 0 \quad (\because \ C_2 = 0)$$

or $$EIy = \frac{w.L}{12}x^3 - \frac{w}{24}.x^4 - \frac{wL^3}{24}x \qquad ...(iv)$$

Equation (*iii*) is known as *slope equation*. We can find the slope $\left(i.e., \text{the value of } \frac{dy}{dx}\right)$ at any point on the beam by substituting the different values of x in this equation.

Equation (*iv*) is known as *deflection equation*. We can find the deflection (*i.e.*, the value of y) at any point on the beam by substituting the different values of x in this equation.

Slope at the Supports

Let θ_A = Slope at support A. This is equal to $\left(\frac{dy}{dx}\right)_{at\ A}$

and θ_B = Slop at support $B = \left(\frac{dy}{dx}\right)_{at\ B}$

At A, $x = 0$ and $\frac{dy}{dx} = \theta_A$.

Substituting these values in equation (*iii*), we get

$$EI.\theta_A = \frac{wL}{4}\times 0 - \frac{w}{6}\times 0 - \frac{wL^3}{24}$$

$$= -\frac{wL^3}{24} = -\frac{WL^2}{24} \qquad (\because \ w.L = W = \text{Total load})$$

$$\therefore \quad \theta_A = -\frac{WL^2}{24EI} \qquad ...(12.12)$$

(Negative sign means that tangent at A makes an angle with AB in the anti-clockwise direction)

By symmetry, $$\theta_B = -\frac{WL^2}{24EI} \qquad ...(12.13)$$

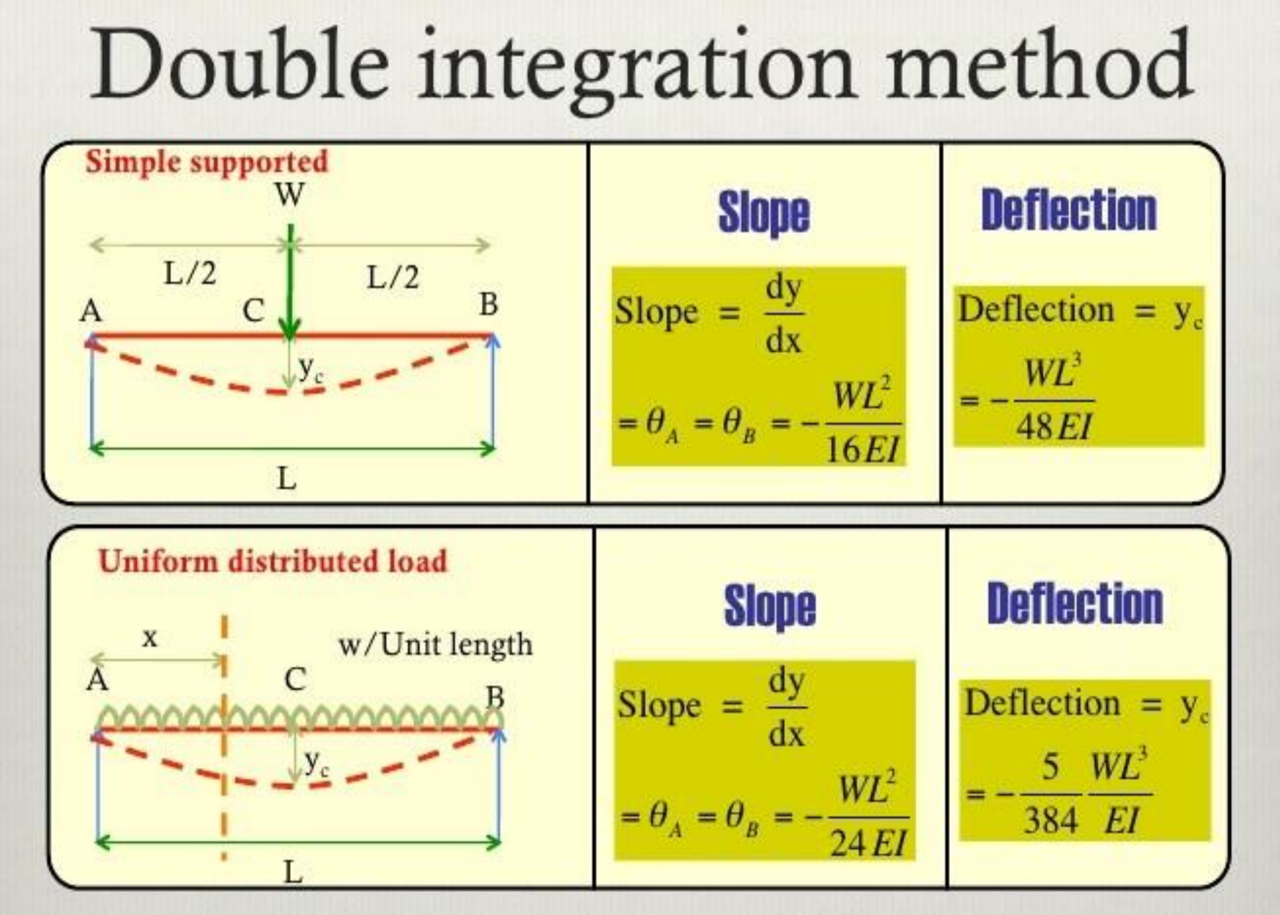
Maximum Deflection

The maximum deflection is at the centre of the beam *i.e.*, at point C, where $x = \frac{L}{2}$. Let y_C = deflection at C which is also maximum deflection. Substituting $y = y_C$ and $x = \frac{L}{2}$ in equation (*iv*), we get

$$EI.y_C = \frac{w.L}{12}.\left(\frac{L}{2}\right)^3 - \frac{w}{24}.\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}-\left(\frac{L}{2}\right)$$

$$= \frac{w.L^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48} = -\frac{5w.L^4}{384}$$

$$\therefore \quad y_C = -\frac{5}{384}.\frac{wL^4}{EI} = -\frac{5}{384}.\frac{W.L^3}{EI} \qquad (\because \ w.L = W = \text{Total load})$$



4. A beam of uniform rectangular section 200 mm wide and 300 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9 KN/m run over the entire span of 5 m. if the value of E for the beam material is 1 X 10^4 N/mm^2 , find the slope at the supports and maximum deflection.

GIVEN DATA:

L = 5 m = 5 X 10^3 mm

w = 9 KN/m = 9000 N/m

E = 1 X 10^4 N/mm^2

b = 200 mm

d = 300 mm

SOLUTION:

1. SLOPE AT THE SUPPORTS,

θ_A = - WL^2 / 24 EI

$$= \frac{45000 \times 5000^2}{24 \times 1 \times 10^4 \times 4.5 \times 10^8}$$

= **0.0104 radians.**

W = w.L = 9000 X 5 = 45000 N

I = bd^3/12 = 200 X 300^3 / 12

= 4.5 X 10^8 mm^4

2. MAXIMUM DEFLECTION,

$$y = \frac{5\,W\,L^3}{384\,E\,I}$$

$$= \frac{5 \times 45000 \times 5000^3}{384 \times 1 \times 10^4 \times 4.5 \times 10^8}$$

$$= \mathbf{16.27\ mm.}$$

5. A beam of length 5 m and of uniform rectangular section is simply supported at its ends. It carries a uniformly distributed load of 9 KN/m run over the entire length. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm^2 and central deflection is not to exceed 1 cm.

GIVEN DATA:

L = 5 m = 5 X 10^3 mm, w = 9 KN/m = 9000 N/m

∴ Total load, $W = w.L = 9 \times 5 = 45$ kN $= 45000$ N

Bending stress, $f = 7$ N/mm^2

Central deflection, $y_C = 1$ cm $= 10$ mm

Value of $E = 1 \times 10^4$ N/mm^2

Let b = Width of beam is mm

and d = Depth of beam in mm

∴ M.O.I., $I = \frac{bd^3}{12}$

Using equation (12.14), we get

$$y_C = \frac{5}{384} \cdot \frac{W \cdot L^3}{EI}$$

or $$10 = \frac{5}{384} \times \frac{45000 \times 5000^3}{1 \times 10^4 \times \left(\frac{bd^3}{12}\right)}$$

or $$bd^3 = \frac{5}{384} \times \frac{45000 \times 5000^3 \times 12}{1 \times 10^4 \times 10}$$

$$= 878.906 \times 10^7 \text{ mm}^4 \qquad \ldots(i)$$

The maximum bending moment for a simply supported beam carrying a uniformly distributed load is given by,

$$M = \frac{w \cdot L^2}{8} = \frac{W \cdot L}{8} \qquad (\because \quad W = w.L = \text{Total load})$$

$$= \frac{45000 \times 5}{8} \text{ Nm} = \frac{45000 \times 5}{8} \times 1000 \text{ Nmm}$$

$$= 28125000 \text{ Nmm}$$

Now using the bending equation as

$$\frac{M}{I} = \frac{f}{y}$$

or $$\frac{28125000}{\left(\frac{bd^3}{12}\right)} = \frac{7}{\left(\frac{d}{2}\right)} \qquad \left(\because \text{ Here } y = \frac{d}{2}\right)$$

or $$\frac{28125000 \times 12}{bd^3} = \frac{14}{d}$$

or $$bd^2 = \frac{28125000 \times 12}{14} = 24107142.85 \text{ mm}^3 \qquad \ldots(ii)$$

Dividing equation (*i*) by equation (*ii*), we get

$$d = \frac{838.906 \times 10^7}{24107142.85} = \mathbf{364.58\ mm.} \quad \textbf{Ans.}$$

Substituting this value of '*d*' in equation (*ii*), we get

$$b \times (364.58)^2 = 24107142.85$$

∴ $$b = \frac{24107142.85}{364.58^2} = \mathbf{181.36\ mm.} \quad \textbf{Ans.}$$

Problem 12.7. *A beam of length 5 m and of uniform rectangular section is supported at its ends and carries uniformly distributed load over the entire length. Calculate the depth of the section if the maximum permissible bending stress is 8 N/mm² and central deflection is not to exceed 10 mm.*

Take the value of $E = 1.2 \times 10^4$ *N/mm²*.

Sol. Given :

Length, $L = 5\text{ m} = 5000\text{ mm}$

Bending stress, $f = 8\text{ N/mm}^2$

Central deflection, $y_C = 10\text{ mm}$

Value of $E = 1.2 \times 10^4\text{ N/mm}^2$

Let W = Total load

and d = Depth of beam

The maximum bending moment for a simply supported beam carrying a uniformly distributed load is given by,

$$M = \frac{w.L^2}{8} = \frac{W.L}{8} \qquad (\because W = w.L) \qquad ...(i)$$

Now using the bending equation,

$$\frac{M}{I} = \frac{f}{y}$$

or
$$M = \frac{f \times I}{y} = \frac{8 \times I}{(d/2)} \qquad \left(\because y = \frac{d}{2}\right)$$

$\therefore$
$$M = \frac{16I}{d} \qquad ...(ii)$$

Equating the two values of B.M., we get

$$\frac{W.L}{8} = \frac{16I}{d}$$

or
$$W = \frac{16 \times 8I}{L \times d} = \frac{128I}{L \times d} \qquad ...(iii)$$

Now using equation (12.14), we get

$$y_C = \frac{5}{384} \times \frac{WL^3}{EI}$$

or
$$10 = \frac{5}{384} \times \frac{128I}{L \times d} \times \frac{L^3}{EI} \qquad \left(\because y_C = 10\text{ mm and } W = \frac{128I}{L \times d}\right)$$

$$= \frac{5}{384} \times \frac{128 \times L^2}{d \times E}$$

or
$$d = \frac{5}{384} \times \frac{128 \times L^2}{10 \times E} = \frac{5}{384} \times \frac{128 \times 5000^2}{10 \times 1.2 \times 10^4}$$

$= 347.2\text{ mm} = 34.72\text{ cm}$. **Ans.**

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH AN ECCENTRIC POINT LOAD

- SLOPE AT THE LEFT SUPPORT,

$$\theta_A = \frac{-W.a.b}{6EI.L}(a + 2b)$$

- MAXIMUM DEFLECTION,

$$y_{max} = \frac{W.b}{9\sqrt{3}EI.L}(a^2 + 2ab)^{3/2}$$

- DEFLECTION UNDER THE POINT LOAD,

$$y_C = \frac{Wa^2b^2}{3EIL}$$

6. Determine slope at the left support, deflection under the load and maximum deflection of a simply supported beam of length 5 m, which is carrying a point load of 5 KN at a distance of 3 m from the left end. Take E = 2 X 10^5 N/mm^2 and I = 1 X 10^8 mm^4.

GIVEN DATA:

L = 5 m = 5 X 10^3 mm

W = 5 KN = 5 X 10^3 N

I = 1 X 10^8 mm^4.

E = 2 X 10^5 N/mm^2

a = 3 m

b = L – a = 5 – 3 = 2 m = 2 X 10^3 mm

SOLUTION:

1. SLOPE AT THE LEFT SUPPORT,

$$\theta_A = \frac{-W \,.\, a.\, b}{6EI.L}\,(a + 2b)$$

= 0.00035 radians.

2. DEFLECTION UNDER THE POINT LOAD,

$$y_C = \frac{Wa^2b^2}{3EIL}$$

= 0.6 mm.

3. MAXIMUM DEFLECTION,

$$y_{max} = \frac{W.b}{9\sqrt{3}EI.L}(a^2 + 2ab)^{3/2}$$

= 0.6173 mm.

tion is expressed and in the manner in which the integrations are carried out.

12.7.1. Deflection of a Simply Supported Beam with an Eccentric Point Load. A simply supported beam AB of length L and carrying a point load W at a distance 'a' from left support and at a distance 'b' from right support is shown in Fig. 12.7. The reactions at A and B are given by,

$$R_A = \frac{W.b}{L} \quad \text{and} \quad R_B = \frac{W.a}{L}$$

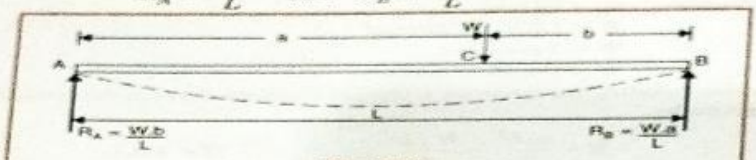


Fig. 12.7

The bending moment at any section between A and C at a distance x from A is given by,

$$M_x = R_A \times x = \frac{W.b}{L} \times x$$

The above equation of B.M. holds good for the values of x between 0 and 'a'. The B.M. at any section between C and B at a distance x from A is given by,

$$M_x = R_A.x - W \times (x - a)$$
$$= \frac{W.b}{L}.x - W(x - a)$$

The above equation of B.M. holds good for all values of x between $x = a$ and $x = b$.

The B.M. for all sections of the beam can be expressed in a single equation written as

$$M_x = \frac{W.b}{L}x \quad \vdots \quad - W(x - a) \qquad \text{...(i)}$$

Stop at the dotted line for any point in section AC. But for any point in section CB, add the expression beyond the dotted line also.

The B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2} \qquad \text{...(ii)}$$

Hence equating (*i*) and (*ii*), we get

$$EI\frac{d^2y}{dx^2} = \frac{W.b}{L}.x \quad \vdots \quad - W(x - a) \qquad \text{...(iii)}$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = \frac{W.b}{L}\frac{x^2}{2} + C_1 \quad \vdots \quad - \frac{W(x - a)^2}{2} \qquad \text{...(iv)}$$

where C_1 is a constant of integration. This constant of integration should be written after the first term. Also *the brackets are to be integrated as a whole*. Hence the integration of $(x-a)$ will be $\frac{(x-a)^2}{2}$ and not $\frac{x^2}{2} - ax$.

Integrating equation (*iv*) once again, we get

$$EIy = \frac{W.b}{2L}.\frac{x^3}{3} + C_1x + C_2 \;\vdots\; -\frac{W}{2}\frac{(x-a)^3}{3} \qquad ...(v)$$

where C_2 is another constant of integration. This constant is written after C_1x. The integration of $(x-a)^2$ will be $\left(\frac{x-a}{3}\right)^3$. This type of integration is justified as the constant of integrations C_1 and C_2 are valid for all values of x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(*i*) At $x = 0$, $y = 0$ and (*ii*) At $x = L$, $y = 0$

(*i*) At A, $x = 0$ and $y = 0$. Substituting these values in equation (*v*) upto dotted line only, we get

$$0 = 0 + 0 + C_2$$

$$\therefore \quad C_2 = 0$$

(*ii*) At B, $x = L$ and $y = 0$. Substituting these values in equation (*v*), we get

$$0 = \frac{W.b}{2L}.\frac{L^3}{3} + C_1 \times L + 0 - \frac{W}{2}\frac{(L-a)^3}{3}$$

($\because$ $C_2 = 0$. Here complete Eq. (*v*) is to be taken)

$$= \frac{W.b.L^2}{6} + C_1 \times L - \frac{W}{2}\frac{b^3}{3} \qquad (\because \; L - a = b)$$

$$\therefore \quad C_1 \times L = \frac{W}{6}.b^3 - \frac{W.b.L^2}{6} = -\frac{W.b}{6}(L^2 - b^2)$$

$$\therefore \quad C_1 = -\frac{W.b}{6L}(L^2 - b^2) \qquad ...(vi)$$

Substituting the value of C_1 in equation (*iv*), we get

$$EI\frac{dy}{dx} = \frac{W.b}{L}\frac{x^2}{2} + \left[-\frac{W.b}{6L}(L^2 - b^2)\right] \;\vdots\; -\frac{W(x-a)^2}{2}$$

$$= \frac{W.b.x^2}{2L} - \frac{W.b}{6L}(L^2 - b^2) \;\vdots\; -\frac{W(x-a)^2}{2} \qquad ...(vii)$$

Equation (*vii*) gives the slope at any point in the beam. Slope is maximum at A or B. To find the slope at A, substitute $x = 0$ in the above equation upto dotted line as point A lies in AC.

$$\therefore \quad EI.\theta_A = \frac{W.b}{2L} \times 0 - \frac{Wb}{6L}(L^2 - b^2) \qquad \left(\because \; \frac{dy}{dx} \text{ at } A = \theta_A\right)$$

$$= -\frac{Wb}{6L}(L^2 - b^2)$$

$$\therefore \quad \theta_A = -\frac{Wb}{6EIL}(L^2 - b^2) \qquad \text{(as given before)}$$

Substituting the values of C_1 and C_2 in equation (*v*), we get

$$EIy = \frac{W.b}{6L}.x^3 + \left[-\frac{Wb}{6L}(L^2 - b^2)\right]x + 0 \;\vdots\; -\frac{W}{6}(x-a)^3 \qquad ...(viii)$$

Equation (*viii*) gives the deflection at any point in the beam. To find the deflection y_c under the load, substitute $x = a$ in equation (*viii*) and consider the equation upto dotted line (as point C lies in AC). Hence, we get

$$EIy_c = \frac{W.b}{6L}.a^3 - \frac{W.b}{6L}(L^2 - b^2)a = \frac{W.b}{6L}.a(a^2 - L^2 + b^2)$$

$$= -\frac{W.a.b}{6L}(L^2 - a^2 - b^2)$$

$$= -\frac{W.a.b}{6L}[(a+b)^2 - a^2 - b^2] \qquad (\because \; L = a + b)$$

$$= -\frac{W.a.b}{6L}[a^2 + b^2 + 2ab - a^2 - b^2]$$

$$= -\frac{W.a.b}{6L}[2ab] = -\frac{Wa^2.b^2}{3L}$$

$$\therefore \quad y_c = -\frac{Wa^2.b^2}{3EIL} \qquad \text{...(same as before)}$$

Note. While using Macaulay's Method, the section x is to be taken in the last portion of the beam.

Problem 12.8. *A beam of length 6 m is simply supported at its ends and carries a point load of 40 kN at a distance of 4 m from the left support. Find the deflection under the load and maximum deflection. Also calculate the point at which maximum deflection takes place. Given M.O.I. of beam = 7.33×10^7 mm^4 and E = 2×10^5 N/mm^2.*

Sol. Given :

Length, $L = 6$ m $= 6000$ mm

Point load, $W = 40$ kN $= 40{,}000$ N

Distance of point load from left support, $a = 4$ m $= 4000$ mm

$\therefore$ $b = L - a = 6 - 4 = 2$ m $= 2000$ mm

Let y_c = Deflection under the load

y_{max} = Maximum deflection

Using equation $y_c = -\frac{W.a^2.b^2}{3EIL}$

$$\therefore \quad y_c = -\frac{40000 \times 4000^2 \times 2000^2}{3 \times 2 \times 10^5 \times 7.33 \times 10^7 \times 6000}$$

$= -$ **9.7 mm. Ans.**

Problem 12.9. *A beam of length 6 m is simply supported at its ends and carries two point loads of 48 kN and 40 kN at a distance of 1 m and 3 m respectively from the left support. Find :*

(*i*) *deflection under each load,*

(*ii*) *maximum deflection, and*

(*iii*) *the point at which maximum deflection occurs.*

Given E = 2×10^5 N/mm^2 and I = 85×10^6 mm^4.

Sol. Given :

$$I = 85 \times 10^5 \text{ mm}^4 \; ; E = 2 \times 10^5 \text{ N/mm}^2$$

First calculate the reactions R_A and R_B.

Taking moments about A, we get

$$R_B \times 6 = 48 \times 1 + 40 \times 3 = 168$$

$$\therefore \quad R_B = \frac{168}{6} = 28 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load} - R_B = (48 + 40) - 28 = 60 \text{ kN}$$

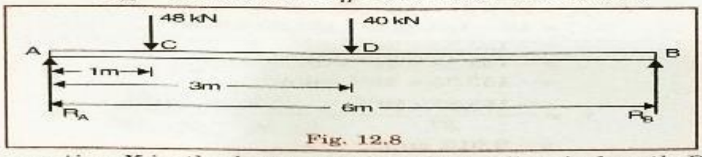


Fig. 12.8

Consider the section X in the last part of the beam (*i.e.*, in length DB) at a distance x from the left support A. The B.M. at this section is given by,

$$EI\frac{d^2y}{dx^2} = R_A.x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

$$= 60x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = \frac{60x^2}{2} + C_1 \;\vdots\; -48\frac{(x-1)^2}{2} \;\vdots\; -40\frac{(x-3)^2}{2}$$

$$= 30x^2 + C_1 \;\vdots\; -24(x-1)^2 \;\vdots\; -20(x-3)^2 \qquad ...(i)$$

Integrating the above equation again, we get

$$EIy = \frac{30x^3}{3} + C_1x + C_2 \;\vdots\; \frac{-24(x-1)^3}{3} \;\vdots\; \frac{-20(x-3)^3}{3}$$

$$= 10x^3 + C_1x + C_2 \;\vdots\; -8(x-1)^3 \;\vdots\; -\frac{20}{3}(x-3)^3 \qquad ...(ii)$$

To find the values of C_1 and C_2, use two boundary conditions. The boundary conditions are:

(*i*) at $x = 0$, $y = 0$, and (*ii*) at $x = 6$ m, $y = 0$.

(*i*) Substituting the first boundary condition *i.e.*, at $x = 0$, $y = 0$ in equation (*ii*) and considering the equation upto first dotted line (as $x = 0$ lies in the first part of the beam), we get

$$0 = 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(*ii*) Substituting the second boundary condition *i.e.*, at $x = 6$ m, $y = 0$ in equation (*ii*) and considering the complete equation (as $x = 6$ lies in the last part of the beam), we get

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3 \qquad (\because \; C_2 = 0)$$

or

$$0 = 2160 + 6C_1 - 8 \times 5^3 - \frac{20}{3} \times 3^3$$

$$= 2160 + 6C_1 - 1000 - 180 = 980 + 6C_1$$

$$\therefore \quad C_1 = \frac{-980}{6} = -163.33$$

Now substituting the values of C_1 and C_2 in equation (*ii*), we get

$$EIy = 10x^3 - 163.33x \;\vdots\; -8(x-1)^3 \;\vdots\; -\frac{20}{3}(x-3)^3 \qquad ...(iii)$$

(*i*) (*a*) *Deflection under first load i.e., at point C*. This is obtained by substituting $x = 1$ in equation (*iii*) upto the first dotted line (as the point C lies in the first part of the beam). Hence, we get

$$EI.\,y_c = 10 \times 1^3 - 163.33 \times 1$$
$$= 10 - 163.33 = -153.33 \text{ kNm}^3$$
$$= -153.33 \times 10^3 \text{ Nm}^3$$
$$= -153.33 \times 10^3 \times 10^9 \text{ Nmm}^3$$
$$= -153.33 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_c = \frac{-153.33 \times 10^{12}}{EI} = \frac{-153.33 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} \text{ mm}$$

$$= -9.019 \text{ mm.} \quad \textbf{Ans.}$$

(Negative sign shows that deflection is downwards).

(*b*) *Deflection under second load i.e. at point D*. This is obtained by substituting $x = 3$ m in equation (*iii*) upto the second dotted line (as the point D lies in the second part of the beam). Hence, we get

$$EI.y_D = 10 \times 3^3 - 163.33 \times 3 - 8(3-1)^3$$
$$= 270 - 489.99 - 64 = -283.99 \text{ kNm}^3$$
$$= -283.99 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_D = \frac{-283.99 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -16.7 \text{ mm.} \quad \textbf{Ans.}$$

(*ii*) *Maximum Deflection*. The deflection is likely to be maximum at a section between C and D. For maximum deflection, $\frac{dy}{dx}$ should be zero. Hence equate the equation (*i*) equal to zero upto the second dotted line.

$$\therefore \quad 30x^2 + C_1 - 24(x-1)^2 = 0$$

or $\quad 30x^2 - 163.33 - 24(x^2 + 1 - 2x) = 0 \qquad (\because \; C_1 = -163.33)$

or $\quad 6x^2 + 48x - 187.33 = 0$

The above equation is a quadratic equation. Hence its solution is

$$x = \frac{-48 \pm \sqrt{48^2 + 4 \times 6 \times 187.33}}{2 \times 6} = 2.87 \text{ m.}$$

(Neglecting – ve root)

Now substituting $x = 2.87$ m in equation (*iii*) upto the second dotted line, we get maximum deflection as

$$EIy_{max} = 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2.87 - 1)^3$$
$$= 236.39 - 468.75 - 52.31$$
$$= 284.67 \text{ kNm}^3 = -284.67 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_{max} = \frac{-284.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -16.745 \text{ mm.} \quad \textbf{Ans.}$$

MOMENT AREA METHOD:

✓ **MOHR’S THEOREM – I:**

The change of slope between any two points is equal to the net area of the B.M. diagram between these points divided by EI.

✓ **MOHR’S THEOREM – II:**

The total deflection between any two points is equal to the moment of the area of B.M. diagram between the two points about the last point divided by EI.

MOHR’S THEOREMS IS USED FOR FOLLOWING CASES:

- ✓ Problems on Cantilevers
- ✓ Simply supported beams carrying symmetrical loading
- ✓ Fixed beams

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD:

Fig. 12.19 (*a*) shows a simply supported *AB* of length *L* and carrying a point load *W* at the centre of the beam *i.e.*, at point *C*. The B.M. diagram is shown in Fig. 12.19 (*b*). This is a case of symmetrical loading, hence slope is zero at the centre *i.e.*, at point *C*.

But the deflection is maximum at the centre.

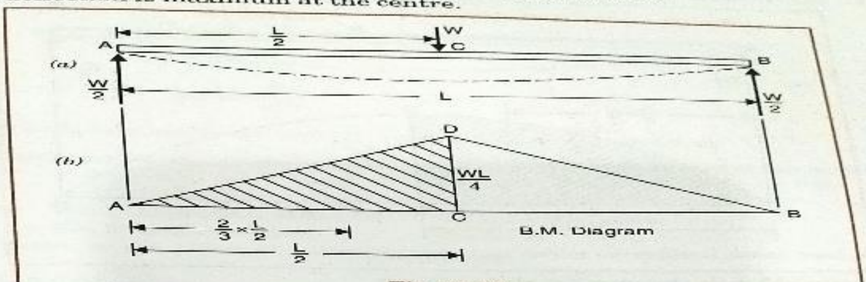


Fig. 12.19

Now using Mohr’s theorem for slope, we get

Slope at $$A = \frac{\text{Area of B.M. diagram between } A \text{ and } C}{EI}$$

But area of B.M. diagram between A and C

$$= \text{Area of triangle } ACD$$

$$= \frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4} = \frac{WL^2}{16}$$

$\therefore$ Slope at A or θ_A $$= \frac{WL^2}{EI}$$

Now using Mohr’s theorem for deflection, we get from equation (12.17) as

$$y = \frac{A\bar{x}}{EI}$$

where A = Area of B.M. Diagram between A and C

$$= \frac{WL^2}{16}$$

$\bar{x}$ = Distance of C.G. of area A from A

$$= \frac{2}{3} \times \frac{L}{2} = \frac{L}{3}$$

$$\therefore \quad y = \frac{\frac{WL^2}{16} \times \frac{L}{3}}{EI} = \frac{WL^3}{48EI}.$$

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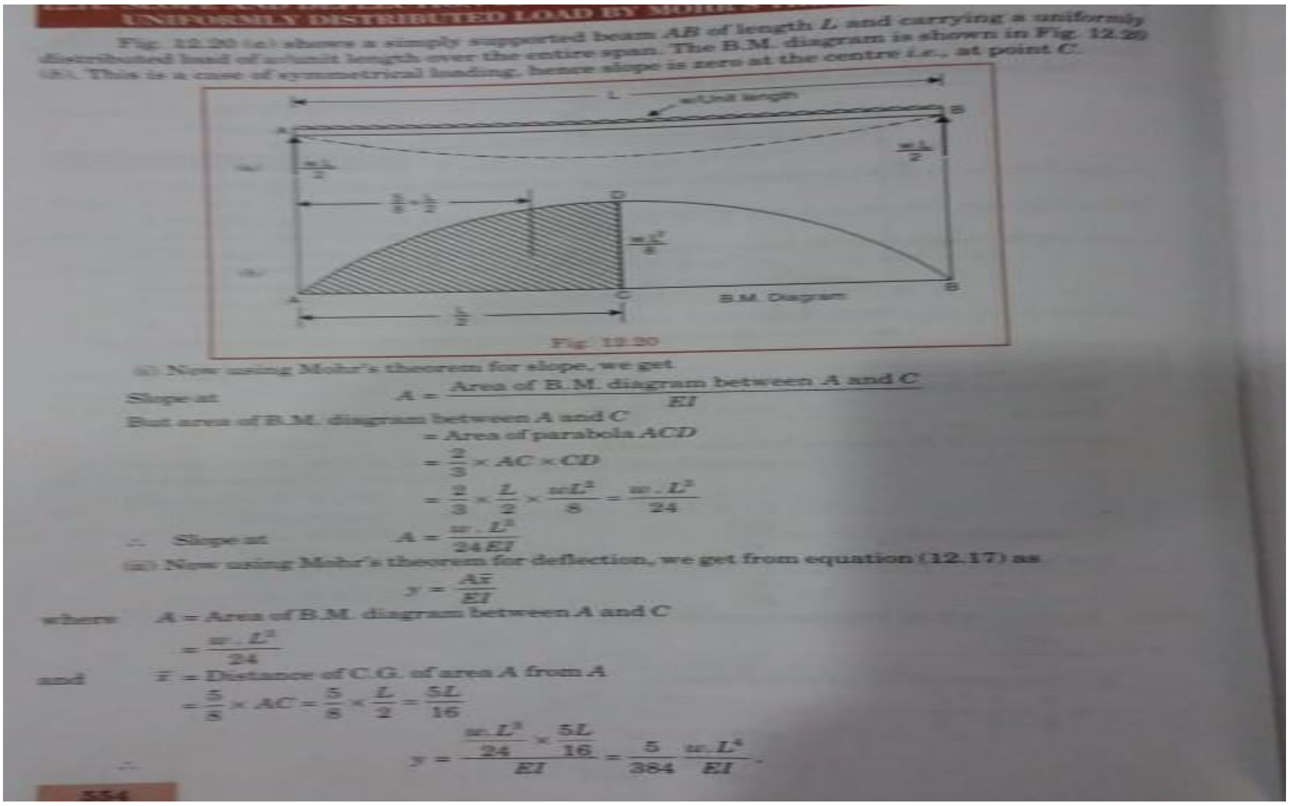
SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED WITH A UNIFORMLY DISTRIBUTED LOAD:



UNIFORMLY DISTRIBUTED LOAD BY MOHR'S ...

Fig. 12.20 (a) shows a simply supported beam AB of length L and carrying a uniformly distributed load of w/unit length over the entire span. The B.M. diagram is shown in Fig. 12.20 (b). This is a case of symmetrical loading, hence slope is zero at the centre *i.e.*, at point C.

Fig. 12.20

(i) Now using Mohr's theorem for slope, we get

Slope at $A = \dfrac{\text{Area of B.M. diagram between } A \text{ and } C}{EI}$

But area of B.M. diagram between A and C

$$= \text{Area of parabola } ACD$$

$$= \frac{2}{3} \times AC \times CD$$

$$= \frac{2}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{w \cdot L^3}{24}$$

$\therefore$ Slope at $A = \dfrac{w \cdot L^3}{24EI}$

(ii) Now using Mohr's theorem for deflection, we get from equation (12.17) as

$$y = \frac{A\bar{x}}{EI}$$

where A = Area of B.M. diagram between A and C

$$= \frac{w \cdot L^3}{24}$$

and $\bar{x}$ = Distance of C.G. of area A from A

$$= \frac{5}{8} \times AC = \frac{5}{8} \times \frac{L}{2} = \frac{5L}{16}$$

$$\therefore \quad y = \frac{\frac{w \cdot L^3}{24} \times \frac{5L}{16}}{EI} = \frac{5}{384} \frac{w \cdot L^4}{EI}.$$

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CONJUGATE BEAM METHOD:

- **CONJUGATE BEAM:**
  - Conjugate beam is an imaginary beam of length equal to that of the original beam but for which the load diagram is the M/EI diagram.
  - **NOTE 1 :**
  - The slope at any section of the given beam is equal to the shear force at the corresponding section of the conjugate beam.
  - **NOTE 2 :**
  - The deflection at any section for the given beam is equal to the bending moment at the corresponding section of the conjugate beam.

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD:

- A simply supported beam AB of length L carrying a point load W at the centre C.
- The B.M at A and B is zero and at the centre B.M will be WL/4.
- Now the conjugate beam AB can be constructed.

- ✓ The load on the conjugate beam will be obtained by dividing the B.M at that point by EI.
- ✓ The shape of the loading on the conjugate beam will be same as of B.M diagram.
- ✓ The ordinate of loading on conjugate beam will be equal to M/EI = WL/4EI.

EI EI 4EI

load diagram for conjugate beam is shown in Fig. 14.1 (c).

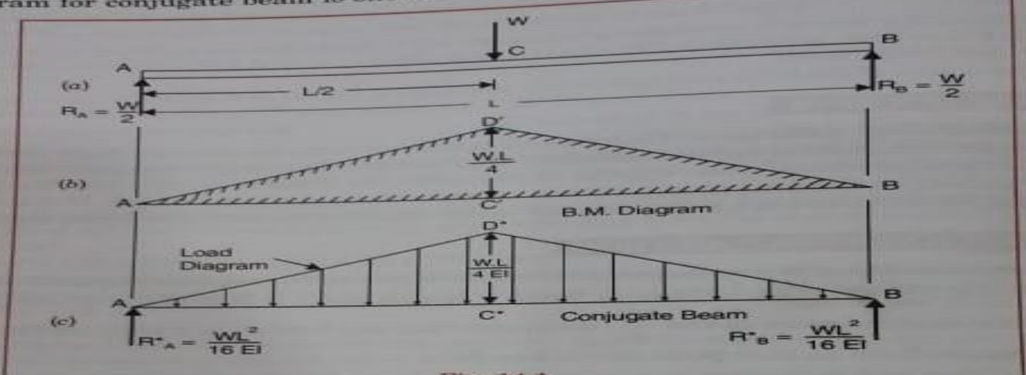

Fig. 14.1

Let R_A^* = Reaction at A for conjugate beam

R_B^* = Reaction at B for conjugate beam

Total load on the conjugate beam [See Fig. 14.1 (c)]

= Area of the load diagram

$$= \frac{1}{2} \times AB \times C^*D^* = \frac{1}{2} \times L \times \frac{WL}{4EI}$$

$$= \frac{WL^2}{8EI}$$

Reaction at each support for the conjugate beam will be half of the total load

$$\therefore \quad R_A^* = R_B^* = \frac{1}{2} \times \frac{WL^2}{8EI} = \frac{WL^2}{16EI}$$

Let θ_A = Slope at A for the given beam *i.e.*, $\left(\frac{dy}{dx}\right)$ at A

y_c = deflection at C for the given beam.

Then according to conjugate beam method,

θ_A = Shear force at A for the conjugate beam

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$= R_A^*$ (∵ S.F. at A for conjugate beam $= R_A^*$)

$$= \frac{WL^2}{16EI}$$

y_C = B.M. at C for the conjugate beam [See Fig. 14.1 (c)]

$= R_A^* \times \frac{L}{2}$ – Load corresponding to AC^*D^*

× Distance of C.G. of AC^*D^* from C

$$= \frac{WL^2}{16EI} \cdot \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4EI}\right) \times \left(\frac{1}{3} \times \frac{L}{2}\right)$$

$$= \frac{WL^3}{32EI} - \frac{WL^3}{96EI} = \frac{3WL^3 - WL^3}{96EI}$$

$$= \frac{WL^3}{48EI}.$$

Problem 14.1. *A simply supported beam of length 5 m carries a point load of 5 kN at a distance of 3 m from the left end. If $E = 2 \times 10^5$ N/mm² and $I = 10^8$ mm⁴, determine the slope at the left support and deflection under the point load using conjugate beam method.*

Sol. Given :

Length, $L = 5$ m

Point load, $W = 5$ kN

Distance AC, $a = 3$ m

Distance BC, $b = 5 - 3 = 2$ m

Value of $E = 2 \times 10^5 \text{ N/mm}^2 = 2 \times 10^5 \times 10^6 \text{ N/m}^2$

$= 2 \times 10^5 \times 10^3 \text{ kN/m}^2 = 2 \times 10^8 \text{ kN/m}^2$

Value of $I = 1 \times 10^8 \text{ mm}^4 = 10^{-4} \text{ m}^4$

Let R_A = Reaction at A

and R_B = Reaction at B.

Taking moments about A, we get

$$R_B \times 5 = 5 \times 3$$

$$\therefore \quad R_B = \frac{5 \times 3}{5} = 3 \text{ kN}$$

and R_A = Total load $- R_B = 5 - 3 = 2$ kN

The B.M. at $A = 0$

B.M. at $B = 0$

B.M. at $C = R_A \times 3 = 2 \times 3 = 6$ kNm.

Now B.M. diagram is drawn as shown in Fig. 14.3 (*b*).

Now construct the conjugate beam as shown in Fig. 14.3 (*c*). The vertical load at C^* on conjugate beam

$$= \frac{\text{B.M. at } C}{EI} = \frac{6 \text{ kNm}}{EI}$$

Now calculate the reaction at A^* and B^* for conjugate beam

Let R_A^* = Reaction at A^* for conjugate beam

R_B^* = Reaction at B^* for conjugate beam.

Taking moments about A^*, we get

$R_B^* \times 5$ = Load on $A^*C^*D^*$ × distance of C.G. of $A^*C^*D^*$ from A^*

+ Load on $B^*C^*D^*$ × Distance of C.G. of $B^*C^*D^*$ from A^*

$$= \left(\frac{1}{2} \times 3 \times \frac{6}{EI}\right) \times \left(\frac{2}{3} \times 3\right) + \left(\frac{1}{2} \times 2 \times \frac{6}{EI}\right) \times \left(3 + \frac{1}{3} \times 2\right)$$

$$= \frac{18}{EI} + \frac{6}{EI} \times \frac{11}{3} = \frac{8}{EI} + \frac{22}{EI} = \frac{40}{EI}$$

$$\therefore \quad R_B^* = \frac{40}{EI} \times \frac{1}{5} = \frac{8}{EI}$$

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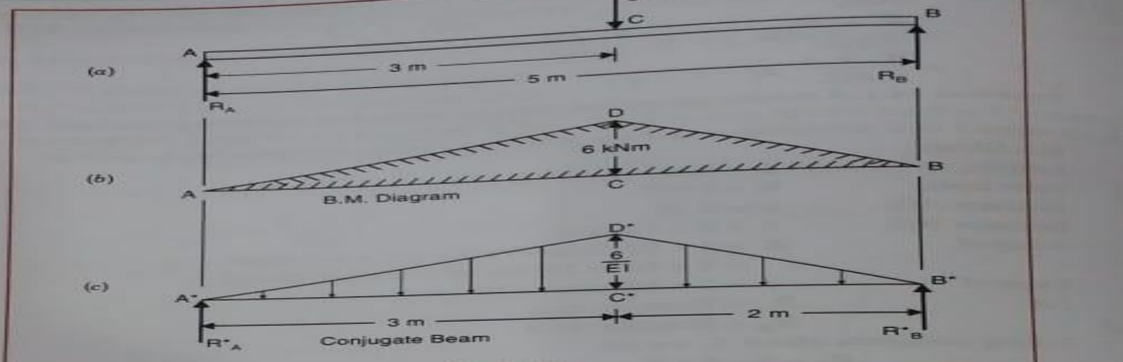

Fig. 14.3

$\therefore$ $R_A^* = $ Total load (*i.e.*, load $A^*B^*D^*$) $- R_B^*$

$$= \left(\frac{1}{2}\times 5\times\frac{6}{EI}\right)-\frac{8}{EI}$$

$$= \frac{15}{EI}-\frac{8}{EI}=\frac{7}{EI}$$

Let θ_A = Slope at A for the given beam *i.e.*, $\left(\frac{dy}{dx}\right)$ at A

y_C = Deflection at C for the given beam

Then according to conjugate beam method,

θ_A = Shear force at A^* for conjugate beam $= R_A^*$

$$= \frac{7}{EI}=\frac{7}{2\times10^8\times10^{-4}} \quad (\because\ E = 2\times10^8\ \text{kN/m}^2 \text{ and } I = 10^{-4}\ \text{m}^4)$$

$= 0.00035$ **radians. Ans.**

y_C = B.M. at C^* for conjugate beam

$= R_A^* \times 3 -$ Load $A^*C^*D^* \times$ Distance of C.G. of $A^*C^*D^*$ from C^*

$$= \frac{7}{EI}\times3-\left(\frac{1}{2}\times3\times\frac{6}{EI}\right)\times\left(\frac{1}{3}\times3\right)$$

$$= \frac{21}{EI}-\frac{9}{EI}=\frac{12}{EI}$$

$$= \frac{12}{2\times10^8\times10^{-4}}=\frac{6}{10^4}\ \text{m}=\frac{6\times1000}{10000}\ \text{mm} = \mathbf{0.6\ mm.}\ \textbf{Ans.}$$

Problem 14.2. *A simply supported beam of length 4 m carries a point load of 3 kN at a distance of 1 m from each end. If E = 2 × 10⁵ N/mm² and I = 10⁸ mm⁴ for the beam, then using conjugate beam method determine : (i) slope at each end and under each load (ii) deflection under each load and at the centre.*

Sol. Given :

Length, $L = 4$ m

Value of $E = 2\times10^5\ \text{N/mm}^2 = 2\times10^5\times10^6\ \text{N/m}^2$

$= 2\times10^5\times10^3\ \text{kN/m}^2 = 2\times10^8\ \text{kN/m}^2$

Value of $I = 10^8\ \text{mm}^4 = \frac{10^8}{10^{12}}\ \text{m}^4 = 10^{-4}\ \text{m}^4.$

As the load on the beam is symmetrical as shown in Fig. 14.4 (a), the reactions R_A and R_B will be equal to 3 kN.

Now B.M. at A and B are zero.

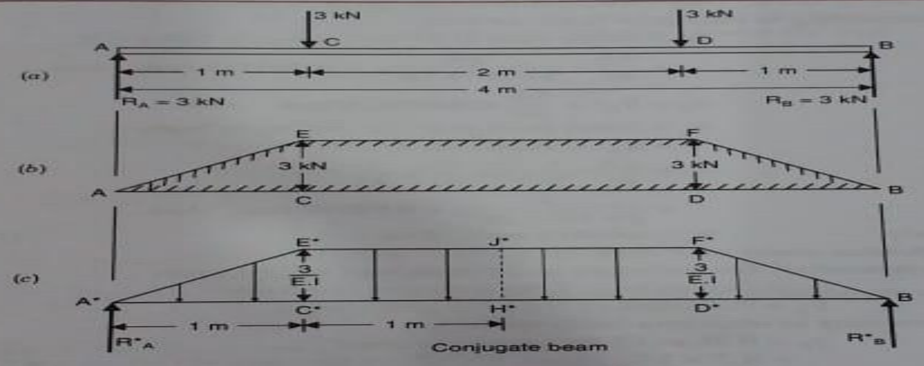

Fig. 14.4

B.M. at $C = R_A \times 1 = 3\times1 = 3$ kNm

B.M. at $D = R_B \times 1 = 3\times1 = 3$ kNm

Now B.M. diagram can be drawn as shown in Fig. 14.4 (b).

Now by dividing the B.M. at any section by EI, we can construct the conjugate beam as shown in Fig. 14.4 (c). The loading are shown on the conjugate beam.

Let R_A^* = Reaction at A^* for the conjugate beam and

R_B^* = Reaction at B^* for conjugate beam

The loading on the conjugate beam is symmetrical

$\therefore$ $R_A{}^* = R_B{}^*$ = Half of total load on conjugate beam

$= \frac{1}{2}$ [Area of trapezoidal $A^*B^*F^*E^*$]

$$= \frac{1}{2}\left[\frac{(E^*F^* + A^*B^*)}{2} \times E^*C^*\right]$$

$$= \frac{1}{2}\left[\frac{(2+4)}{2} \times \frac{3}{EI}\right] = \frac{4.5}{EI}$$

(i) Slope at each end and under each load

Let θ_A = Slope at A for the given beam *i.e.*, $\left(\frac{dy}{dx}\right)$ at A

θ_B = Slope at B for the given beam

θ_C = Slope at C for the given beam and

θ_D = Slope at D for the given beam

Then according to conjugate beam method,

θ_A = Shear force at A^* for conjugate beam = $R_A{}^*$

$$= \frac{4.5}{EI} = \frac{4.5}{2 \times 10^8 \times 10^{-4}} = \mathbf{0.000225\ rad.}$$ **Ans.**

$$\theta_B = R_B{}^* = \frac{4.5}{EI} = \mathbf{0.000225\ rad.}$$ **Ans.**

θ_C = Shear force at C^* for conjugate beam

= $R_A{}^*$ – Total load $A^*C^*D^*$

$$= \frac{4.5}{EI} - \frac{1}{2} \times 1 \times \frac{3}{EI} = \frac{3}{EI}$$

$$= \frac{3}{2 \times 10^8 \times 10^{-4}} = \mathbf{0.00015\ rad.}$$ **Ans.**

Similarly, θ_D = **0.00015 rad. Ans.** (By symmetr

13.1. INTRODUCTION

Cantilever is a beam whose one end is fixed and other end is free. In this chapter we shall discuss the methods of finding slope and deflection for the cantilevers when they are subjected to various types of loading. The important methods are (*i*) Double integration method (*ii*) Macaulay's method and (*iii*) Moment-area-method. These methods have also been used for finding deflections and slope of the simply supported beams.

13.2. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a point load at the free end B is shown in Fig. 13.1. AB shows the position of cantilever before any load is applied whereas AB' shows the position of the cantilever after loading.

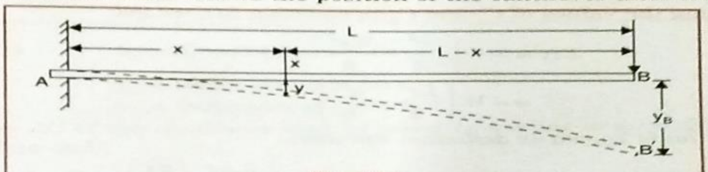

Fig. 13.1

Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -W(L-x) \qquad \text{(Minus sign due to hogging)}$$

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = -W(L-x) = -WL + W.x$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \qquad ...(i)$$

Integrating again, we get

$$EIy = -WL\frac{x^2}{2} + \frac{W}{2}\frac{x^3}{3} + C_1x + C_2 \qquad ...(ii)$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0, y = 0$ (*ii*) $x = 0, \frac{dy}{dx} = 0$

[At the fixed end, deflection and slopes are zero]

(*i*) By substituting $x = 0, y = 0$ in equation (*ii*), we get

$$0 = 0 + 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(*ii*) By substituting $x = 0, \frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = 0 + 0 + C_1 \quad \therefore \quad C_1 = 0$$

Substituting the value of C_1 in equation (*i*), we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2}$$

$$= -W\left(Lx - \frac{x^2}{2}\right) \qquad ...(iii)$$

Equation (*iii*) is known as *slope equation*. We can find the slope at any point on the cantilever by substituting the value of x. The slope and deflection are maximum at the free end. These can be determined by substituting $x = L$ in these equations.

Substituting the values of C_1 and C_2 in equation (*ii*), we get

$$EIy = -WL\frac{x^2}{2} + \frac{Wx^3}{6} \qquad (\because \quad C_1 = 0, C_2 = 0)$$

$$= -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \qquad ...(iv)$$

Equation (*iv*) is known as *deflection equation*.

Let θ_B = slope at the free end B *i.e.*, $\left(\frac{dy}{dx}\right)$ at $B = \theta_B$ and

y_B = Deflection at the free end B

(*a*) Substituting θ_B for $\frac{dy}{dx}$ and $x = L$ in equation (*iii*), we get

$$EI.\theta_B = -W\left(L.L - \frac{L^2}{2}\right) = -W.\frac{L^2}{2}$$

$$\therefore \quad \theta_B = -\frac{WL^2}{2EI} \qquad ...(13.1)$$

Negative sign shows that tangent at B makes an angle in the anti-clockwise direction with AB

$$\therefore \quad \theta_B = \frac{WL^2}{2EI} \qquad ...(13.1A)$$

(b) Substituting y_B for y and $x = L$ in equation (iv), we get

$$EI.y_B = -W\left(L.\frac{L^2}{2} - \frac{L^3}{6}\right) = -W\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -W.\frac{L^3}{3}$$

$$\therefore \quad y_B = -\frac{WL^3}{3EI} \quad \text{...(13.2)}$$

(Negative sign shows that deflection is downwards)

$$\therefore \quad \text{Downward deflection, } y_B = \frac{WL^3}{3EI} \quad \text{...(13.2 A)}$$

13.3. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point A and free at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig. 13.2.

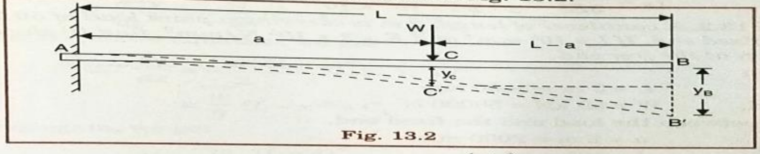

Fig. 13.2

Let θ_C = Slope at point C i.e., $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C

y_B = Deflection at point B

The portion AC of the cantilever may be taken as similar to a cantilever in Art. 13.1 (i.e., load at the free end).

$$\therefore \quad \theta_C = +\frac{Wa^2}{2EI} \quad \text{[In equation (13.1 A) change } L \text{ to } a\text{]}$$

and $$y_C = \frac{Wa^3}{3EI} \quad \text{[In equation (13.2 A) change } L \text{ to } a\text{]}$$

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore

Slope at C = slope at B

or $$\theta_C = \theta_B = \frac{Wa^2}{2EI} \quad \text{...(13.3)}$$

Now from Fig. 13.2, we have

$$y_B = y_C + \theta_C(L - a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a) \quad \left(\because \theta_C = \frac{Wa^2}{2EI}\right) \quad \text{...(13.4)}$$

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PROBLEMS:

1.A cantilever of length 3 m is carrying a point load of 25 KN at the free end. If I = 10^8 mm^4 and E = 2.1 X 10^5 N/mm^2, find the slope and deflection at the free end.

GIVEN DATA:

L = 3 m = 3000 mm

W = 25 KN = 25000 N

I = 10^8 mm^4

E = 2.1 X 10^5 N/mm^2

SOLUTION:

1. SLOPE AT THE FREE END,

$$\theta_B = WL^2 / 2\,EI = \frac{25000 \times 3000^2}{2 \times 2.1 \times 10^5 \times 10^8}$$

= 0.005357 radians.

2. DEFLECTION AT THE FREE END,

$$y_B = W L^3 / 3\,EI = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8}$$

= 10.71 mm

2. A cantilever of length 3 m is carrying a point load of 50 KN at a distance of 2 m from the fixed end. If I = 10^8 mm^4 and E = 2 X 10^5 N/mm^2, find the slope and deflection at the free end.

GIVEN DATA:

L = 3 m = 3000 mm

W = 50 KN = 50000 N

I = 10^8 mm^4

E = 2 X 10^5 N/mm^2

SOLUTION:

1. SLOPE AT THE FREE END,

$$\theta_B = Wa^2 / 2\,EI$$

$$= \frac{50000 \times 2000}{2 \times 2 \times 10^5 \times 10^8}$$

= 0.005 radians

2. DEFLECTION AT THE FREE END,

$$y_B = W a^3 / 3\,EI + W a^2 / 2\,EI\,(L - a)$$

$$= \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8}\,(3000 - 2000)$$

= 6.67 + 5

= 11.67 mm.

CANTILEVER BEAM WITH A UDL:

- A cantilever beam AB of length L fixed at the point A and free at the point B and carrying a UDL of w per unit length over the whole length.
- Consider a section X, at a distance x from the fixed end A.
- The bending moment at this section is given by,

$$M_x = \frac{-\,w\,(L - x)\,(L - x)}{2}$$

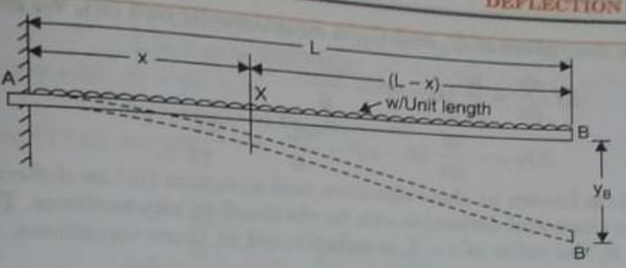


Fig. 13.3

But B.M. at any section is also given by equation (12.3) as

$$M = EI\,\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\,\frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2$$

Integrating the above equation, we get

$$EI\,\frac{dy}{dx} = -\frac{w}{2}\,\frac{(L-x)^3}{3}(-1) + C_1$$

$$= \frac{w}{6}(L-x)^3 + C_1 \qquad \ldots(i)$$

Integrating again, we get

$$EIy = \frac{w}{6}\cdot\frac{(L-x)^4}{4}(-1) + C_1x + C_2$$

$$= -\frac{w}{24}(L-x)^4 + C_1x + C_2 \qquad \ldots(ii)$$

Their values are obtained from boundary condi-

...(ii)

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0$, $y = 0$ and (*ii*) at $x = 0$, $\frac{dy}{dx} = 0$ (as the deflection and slope at fixed end A are zero).

(*i*) By substituting $x = 0$, $y = 0$ in equation (*ii*), we get

$$0 = -\frac{w}{24}(L-0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$$

$$\therefore \qquad C_2 = \frac{wL^4}{24}$$

(*ii*) By substituting $x = 0$ and $\frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = \frac{w}{6}(L-0)^3 + C_1 = \frac{wL^3}{6} + C_1$$

$$\therefore \qquad C_1 = -\frac{wL^3}{6}$$

Substituting the values of C_1 and C_2 in equations (*i*) and (*ii*), we get

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \quad \text{...(iii)}$$

and

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \quad \text{...(iv)}$$

Equation (*iii*) is known as *slope equation* and equation (*iv*) as *deflection equation*. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point *B*, the value of $x = L$ is substituted in these equations.

Let θ_B = Slope at the free end *B* *i.e.*, $\left(\frac{dy}{dx}\right)$ at *B*

y_B = Deflection at the free end *B*.

From equation (*iii*), we get slope at *B* as

$$EI.\theta_B = \frac{w}{6}(L-L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$$

$$\therefore \quad \theta_B = -\frac{wL^3}{6EI} = -\frac{WL^2}{6EI} \quad (\because \quad W = \text{Total load} = w.L) \quad \text{...(13.5)}$$

From equation (*iv*), we get the deflection at *B* as

$$EI.y_B = -\frac{w}{24}(L-L)^4 - \frac{wL^3}{6}\times L + \frac{wL^4}{24}$$

$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$$

$$\therefore \quad y_B = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI} \quad (\because \quad W = w.L)$$

$\therefore$ Downward deflection at *B*,

$$y_B = \frac{wL^4}{8EI} = \frac{WL^3}{8EI} \quad \text{...(13.6)}$$

PROBLEMS:

3. A cantilever of length 2.5 m carries a uniformly distributed load of 16.4 KN per metre length. If I = 7.95 X 10^7 mm^4 and E = 2 X 10^5 N/mm^2, determine the deflection at the free end.

GIVEN DATA:

L = 2.5 m = 2500 mm

w = 16.4 KN/m, W = w X L = 16.4 X 2.5 = 41000 N

I = 7.95 X 10^7 mm^4

E = 2 X 10^5 N/mm^2

SOLUTION:

1. DEFLECTION AT THE FREE END,

$$y_B = WL^3/8EI = \frac{41000 \times 2500^3}{8 \times 2 \times 10^5 \times 7.95 \times 10^7}$$

= 5.036 mm.

4. A cantilever of length 3 m carries a uniformly distributed load over the entire length. If the deflection at the free end is 40 mm, find the slope at the free end.

GIVEN DATA:

$L = 3\ m = 3000\ mm$

$y_B = 40\ mm$

SOLUTION:

1. SLOPE AT THE FREE END,

$y_B = WL^3/8EI$

$$40 = \frac{WL^2 \times L}{8\ EI} = \frac{WL^2 \times 3000}{8\ EI}$$

$$\frac{WL^2}{EI} = \frac{40 \times 8}{3000}$$

Slope at the free end,

$$\theta_B = WL^2 / 6\ EI = WL^2 / EI \times (1/6)$$

$$= \frac{40 \times 8}{3000} \times (1/6)$$

$$\mathbf{= 0.01777\ rad.}$$

5. A cantilever 120 mm wide and 200 mm deep is 2.5 m long. What is the uniformly distributed load which the beam can carry in order to produce a deflection of 5 mm at the free end? Take E = 200 GN/m^2.

GIVEN DATA:

$L = 2.5\ m = 2500\ mm$

$E = 200\ GN/m^2 = 2 \times 10^5\ N/mm^2$

$b = 120\ mm$

$d = 200\ mm$

$y_B = 5\ mm$

$I = bd^3/12 = 120 \times 200^3 / 12$

$= 8 \times 10^7\ mm^4$

SOLUTION:

1. UDL,

$W = w \times L = 2.5 \times w = 2.5\ w\ N.$

$y = WL^3/8EI$

$$5 = \frac{2.5\ w \times 2500^3}{8 \times 2 \times 10^5 \times 8 \times 10^7}$$

or
$$w = \frac{5 \times 8 \times 2 \times 10^5 \times 8 \times 10^7}{2.5 \times 2500^3} = 16384 \text{ N/m}$$
$$= \mathbf{16.384 \text{ kN/m.}} \quad \textbf{Ans.}$$

13.5. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the fixed end, is shown in Fig. 13.4.

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero. The deflected shape of the cantilever is shown by $AC'B'$ in which portion $C'B'$ is straight.

Let θ_C = Slope at C, *i.e.*, $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C, and

y_B = Deflection at point B.

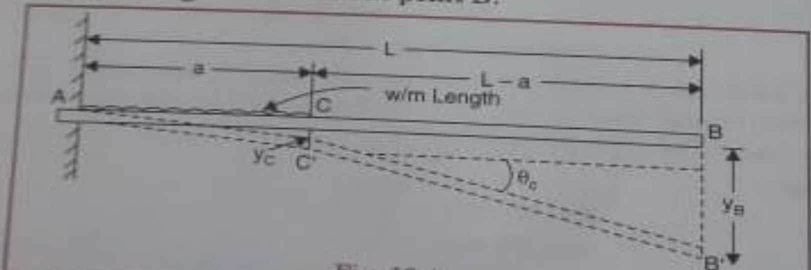


Fig. 13.4

The portion AC of the cantilever may be tak...

The upward deflection of point B due to upward uniformly distributed load acting on the portion AC = upward deflection of C + slope at $C \times CB$

$$= \frac{w(L-a)^4}{8EI} + \frac{w\,.\,(L-a)^3}{6EI} \times a \quad (\because \; CB = a)$$

$\therefore$ Net downward deflection of the free end B is given by

$$y_B = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right] \quad ...(13.10)$$

Problem 13.5. *Determine the slope and deflection of the free end of a cantilever of length 3 m which is carrying a uniformly distributed load of 10 kN/m over a length of 2 m from the fixed end.*

Take $I = 10^8$ mm^4 and $E = 2 \times 10^5$ N/mm^2.

Sol. Given :

Length, $L = 3$ m = 3000 mm

U.d.l., $w = 10$ kN/m = 10000 N/m = $\frac{10000}{1000}$ N/mm = 10 N/mm

Length of u.d.l. from fixed end, $a = 2$ m = 2000 mm.

Value of $I = 10^8$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

Let θ_B = Slope of the free end and

y_B = Deflection at the free end.

(*i*) Using equation (13.7), we have

$$\theta_B = \frac{wa^3}{6EI} = \frac{10 \times 2000^3}{6 \times 2 \times 10^5 \times 10^8} = \mathbf{0.00066.} \quad \textbf{Ans.}$$

(*ii*) Using equation (13.8), we get

$$y_B = \frac{wa^4}{8EI} + \frac{w\,.\,a^3}{6EI}(L-a)$$

$$= \frac{10 \times 2000^4}{8 \times 2 \times 10^5 \times 10^8} + \frac{10 \times 2000^3}{6 \times 2 \times 10^5 \times 10^8} \times (3000 - 2000)$$

$$= 1 + 0.67 = \mathbf{1.67 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 13.6. *A cantilever of length 3 m carries a uniformly distributed load of 10 kN/m over a length of 2 m from the free end. If $I = 10^8$ mm^4 and $E = 2 \times 10^5$ N/mm^2; find : (i) slope at the free end, and (ii) deflection at the free end.*

Sol. Given :

Length, $L = 3$ m = 3000 mm

U.d.l., $w = 10$ kN/m = 10000 N/m = $\frac{10000}{1000}$ N/mm = 10 N/mm

Length of u.d.l. from free end, $a = 2$ m = 2000 mm

Value of $I = 10^8$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

Let θ_B = Slope at the free end *i.e.*, $\left(\frac{dy}{dx}\right)$ at B and

y_B = Deflection at the free end.

(*i*) Using equation (13.9), we get

$$\theta_B = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI}$$

$$= \frac{10 \times 3000^3}{6 \times 2 \times 10^5 \times 10^8} - \frac{10(3000-2000)^3}{6 \times 2 \times 10^5 \times 10^8}$$

$$= 0.00225 - 0.000083 = \mathbf{0.002167\ rad.} \quad \textbf{Ans.}$$

(*ii*) Using equation (13.10), we get

$$y_B = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]$$

$$= \frac{10 \times 3000^4}{8 \times 2 \times 10^5 \times 10^8} - \left[\frac{10(3000-2000)^4}{8 \times 2 \times 10^5 \times 10^8} + \frac{10(3000-2000)^3}{6 \times 2 \times 10^5 \times 10^8} \times 2000\right]$$

$$= 5.0625 - [0.0625 + 0.1667] = \mathbf{4.8333\ mm.} \quad \textbf{Ans.}$$

Problem 13.7. *A cantilever of length 3 m carries two point loads of 2 kN at the free end and 4 kN at a distance of 1 m from the free end. Find the deflection at the free end. Take $E = 2 \times 10^5\ N/mm^2$ and $I = 10^8\ mm^4$.*

Sol. Given :

Length, L = 3 m = 3000 mm

Load at free end, W_1 = 2 kN = 2000 N

Load at a distance one m from free end,

W_2 = 4 kN = 4000 N

Distance AC, a = 2 m = 2000 mm

Value of $E = 2 \times 10^5$ N/mm²

Value of $I = 10^8$ mm⁴

Let y_1 = Deflection at the free end due to load 2 kN alone

y_2 = Deflection at the free end due to load 4 kN alone.

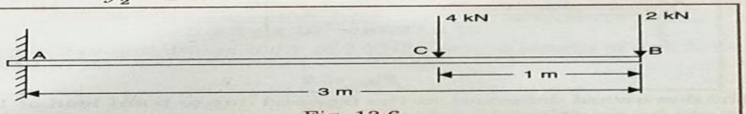


Fig. 13.6

Downward deflection due to load 2 kN alone at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{2000 \times 3000^3}{3 \times 2 \times 10^5 \times 10^8} = 0.9\ \text{mm}.$$

Downward deflection at the free end due to load 4 kN (*i.e.*, 4000 N) alone at a distance 2 m from fixed end is given by (13.4) as

$$y_2 = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a)$$

$$= \frac{4000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{4000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000-2000)$$

$$= 0.54 + 0.40 = 0.94\ \text{mm}$$

∴ Total deflection at the free end

$$= y_1 + y_2$$

$$= 0.9 + 0.94 = \mathbf{1.84\ mm.} \quad \textbf{Ans.}$$

Problem 13.8. *A cantilever of length 2 m carries a uniformly distributed load of 2.5 kN/m run for a length of 1.25 m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and $E = 1 \times 10^4\ N/mm^2$.*

Sol. Given :

Length, L = 2 m = 2000 mm

U.d.l., w = 2.5 kN/m = 2.5 × 1000 N/m

$$= \frac{2.5 \times 1000}{1000}\ \text{N/mm} = 2.5\ \text{N/mm}$$

Point load at free end, W = 1 kN = 1000 N

Distance AC, a = 1.25 m = 1250 mm

Width, b = 12 cm

Depth, d = 24 cm

Value of $$I = \frac{bd^3}{12} = \frac{12 \times 24^3}{12}$$

$$= 13824\ \text{cm}^4 = 13824 \times 10^4\ \text{mm}^4 = 1.3824 \times 10^8\ \text{mm}^4$$

Value of $E = 1 \times 10^4$ N/mm²

Let y_1 = Deflection at the free end due to point load 1 kN alone

y_2 = Deflection at the free end due to u.d.l. on length AC.

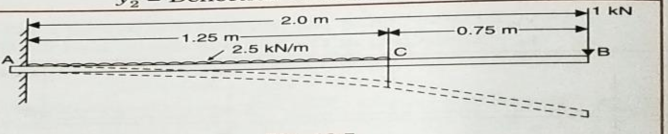


Fig. 13.7

(*i*) Now the downward deflection at the free end due to point load of 1 kN (or 1000 N) at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{1000 \times 2000^3}{3 \times 10^4 \times 1.3824 \times 10^8} = 1.929\ \text{mm}.$$

(*ii*) The downward deflection at the free end due to uniformly distributed load of 2.5 N/mm on a length of 1.25 m (or 1250 mm) is given by equation (13.8) as

$$y_2 = \frac{wa^4}{8EI} + \frac{w \cdot a^3}{6EI}(L-a)$$

$$= \frac{2.5 \times 1250^4}{8 \times 10^4 \times 1.3824 \times 10^8} + \frac{2.5 \times 1250^3}{6 \times 10^4 \times 1.3824 \times 10^8}(2000 - 1250)$$

$$= 0.5519 + 0.4415 = 0.9934$$

$\therefore$ Total deflection at the free end due to point load and u.d.l.

$$= y_1 + y_2 = 1.929 + 0.9934 = \mathbf{2.9224\ mm.} \quad \textbf{Ans.}$$

Problem 13.9. *A cantilever of length 2 m carries a uniformly distributed load 2 kN/m over a length of 1 m from the free end, and a point load of 1 kN at the free end. Find the slope and deflection at the free end if $E = 2.1 \times 10^5$ N/mm² and $I = 6.667 \times 10^7$ mm⁴.*

Sol. Given : (See Fig. 13.8)

Length, $L = 2$ m $= 2000$ mm

U.d.l., $w = 2$ kN/m $= \frac{2 \times 1000}{1000}$ N/mm $= 2$ N/mm

Length BC, $a = 1$ m $= 1000$ mm

Point load, $W = 1$ kN $= 1000$ N

Value of $E = 2.1 \times 10^5$ N/mm²

Value of $I = 6.667 \times 10^7$ mm⁴.

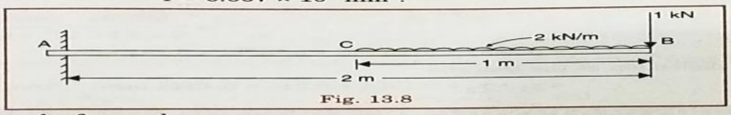


Fig. 13.8

(i) Slope at the free end

Let θ_1 = Slope at the free end due to point load of 1 kN *i.e.*, 1000 N

θ_2 = Slope at the free end due to u.d.l. on length BC.

The slope at the free end due to a point load of 1000 N at B is given by equation (13.1 A) as

$$\theta_1 = \frac{WL^2}{2EI} \qquad (\because \ \theta_B = \theta_1 \text{ here})$$

$$= \frac{1000 \times 2000^2}{2 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.0001428 \text{ rad.}$$

The slope at the free end due to u.d.l. of 2 kN/m over a length of 1 m from the free end is given by equation (13.9) as

$$\theta_2 = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI} \qquad (\because \ \theta_B = \theta_2 \text{ here})$$

$$= \frac{2 \times 2000^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000 - 1000)^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}$$

$$= 0.0001904 - 0.000238 = 0.0001666 \text{ rad.}$$

$\therefore$ Total slope at the free end

$$= \theta_1 + \theta_2 = 0.0001428 + 0.0001666 = \mathbf{0.0003094\ rad.} \quad \textbf{Ans.}$$

(ii) Deflection at the free end

Let y_1 = Deflection at the free end due to point load of 1000 N

y_2 = Deflection at the free end due to u.d.l. on length BC.

The deflection at the free end due to point load of 1000 N is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} \qquad (\because \text{ Here } y_1 = y_B)$$

$$= \frac{1000 \times 2000^3}{3 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.1904 \text{ mm.}$$

The deflection at the free end due to u.d.l. of 2 N/mm over a length of 1 m from the free end is given by equation (13.10) as

$$y_2 = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]$$

$$= \frac{2 \times 2000^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \left[\frac{2(2000-1000)^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} + \frac{2(2000-1000)^3 \times 1000}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}\right]$$

$$= 0.2857 - [0.01785 + 0.0238] = 0.244 \text{ mm}$$

$\therefore$ Total deflection at the free end

$$= y_1 + y_2 = 0.1904 + 0.244 = \mathbf{0.4344\ mm.} \quad \textbf{Ans.}$$

REFERENCE BOOKS:

1. Bansal R.K., “Strength of Materials”, 5th Edition, Laxmi Publications, 2013.
2. Ramamrutham R., “Strength of Materials”,14th Edition, Dhanpat Rai Publications, 2011.
3. Punmia B.C., “Strength of Materials”, Laxmi Publications, 2011.
4. Subramanian R., “Strength of Materials - Theory and Problems”, Oxford University Press, 2010.
5. Rajput R.K., “Strength of Materials”, S. Chand and Company Ltd., New Delhi, 2007.
6. Timoshenko S.P. and Gere J.M., “Mechanics of Materials”, 6th Edition, Tata McGraw Hill.

QUESTION BANK:

1. What are the methods used for determining slope and deflection?
2. What is the slope and deflection equation for simply supported beam carrying UDL through out the length?
3. What is a Macaulay’s method?
4. What is moment area method?
5. Define : Conjugate beam.
6. Find the slope and deflection of a simply supported beam carrying a point load at the centre using moment area method.
7. Distinguish between actual beam and conjugate beam.
8. A beam 4m long, simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1^{o}, find the deflection at the centre of the beam.
9. A cantilever of length 2 m carries a point load of 30 KN at the free end and another load of 30 KN at its centre. If EI = 1013 N.mm2 for the cantilever then determine slope and deflection at the free end by moment area method.
10. Determine slope at the left support, deflection under the load and maximum deflection of a simply supported beam of length 10 m, which is carrying a point load of 10 kN at a distance of 6 m from the left end. Take E = 2 x 105 N/mm2 and I = 1 x 108 mm4.
11. A cantilever of length 3 m is carrying a point load of 25 KN at the free end. If I = 108 mm4 and E = 2.1 X 105 N/mm2, then determine slope and deflection of the cantilever using conjugate beam method.
12. A simply supported beam of length 5 m carries a point load of 5 kN at a distance of 3m from the left end. If E = 2 x 105 N/mm2 and I = 108 mm4, determine the slope at the left support and deflection under the point load using conjugate beam method.

SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – II – STRAIN ENERGY – SCIA1401

INTRODUCTION

STRAIN ENERGY:

- ✓ Strain energy is defined as the energy stored in a body due to deformation.
- ✓ Strain energy is one of fundamental concepts in mechanics and its principles are widely used in practical applications to determine the response of a structure to loads.
- ✓ strain energy is equal to the work done by the point load.
- ✓ The unit of strain energy is N-m or Joules.

RESILIENCE:

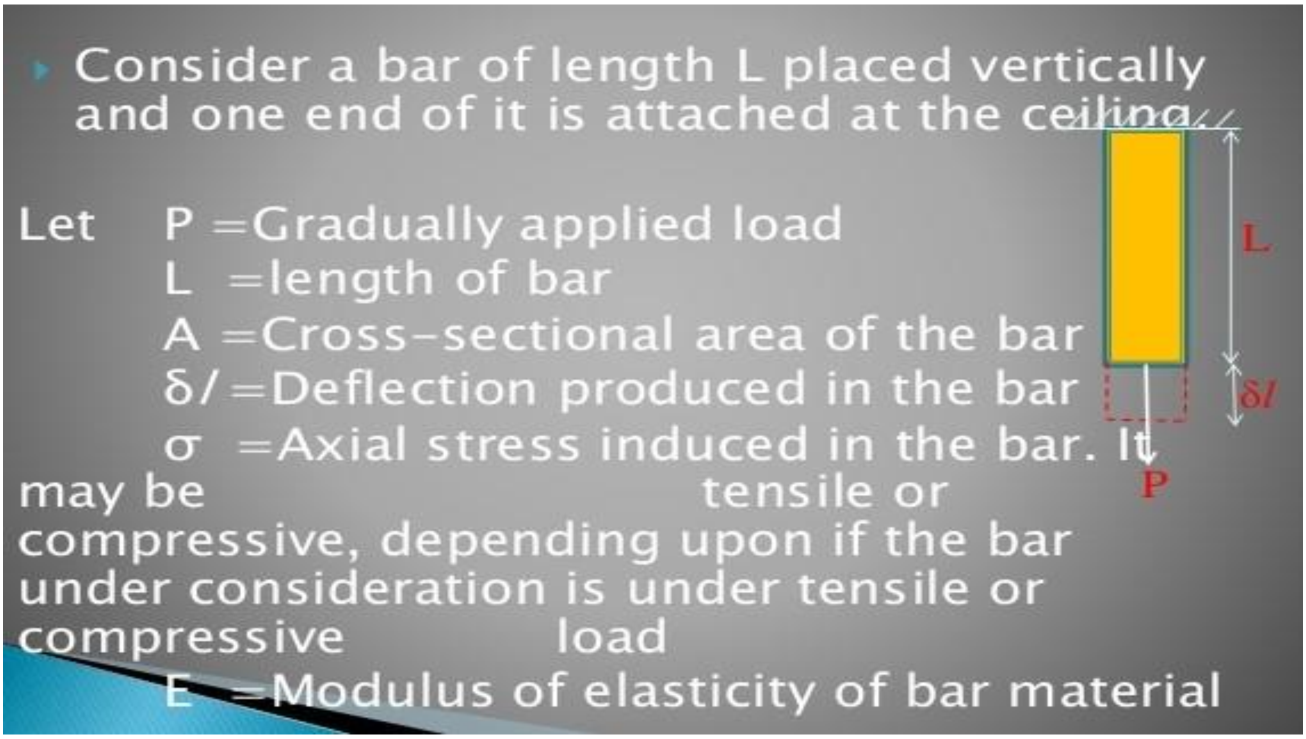
PROOF RESILIENCE:

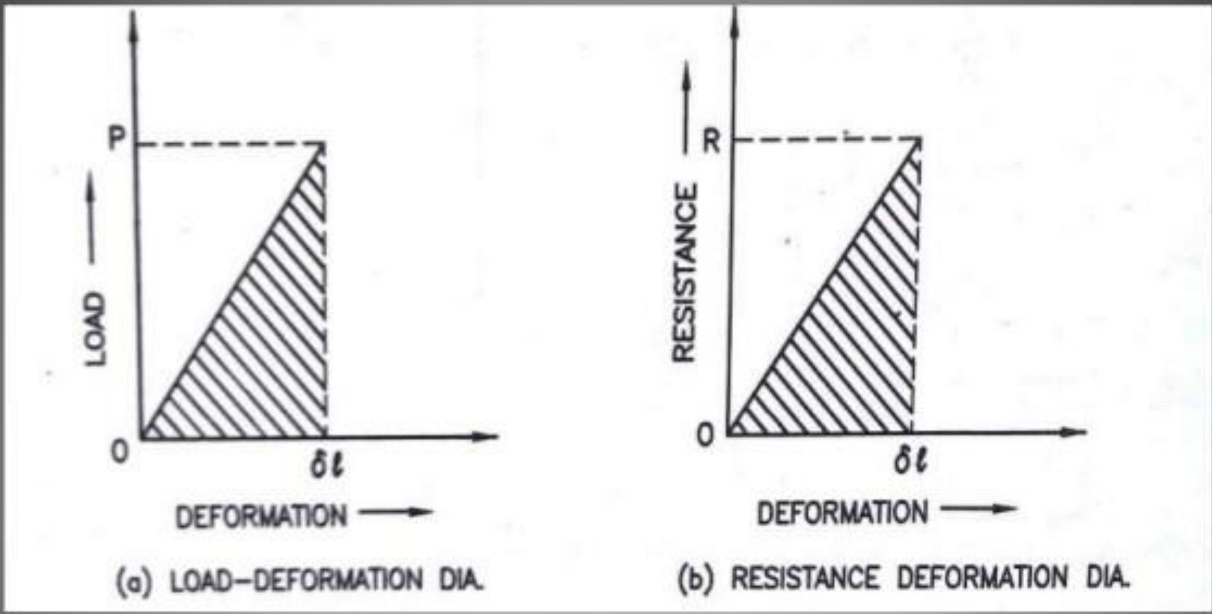
- ✓ Proof resilience is defined as the maximum energy that can be absorbed up to the elastic limit, without creating a permanent distortion.

MODULUS OF RESILIENCE:

- ✓ The modulus of resilience is defined as the maximum energy that can be absorbed per unit volume without creating a permanent distortion.

STRAIN ENERGY DUE TO GRADUALLY APPLIED LOAD





Work done on the bar = Area of the load - deformation diagram

$$= \frac{1}{2} \times P \times \delta l$$

Work Stored in the bar

= Area of the resistance - Deformation diagram

$$= \frac{1}{2} \times R \times \delta l$$

$$= \frac{1}{2} \times (\sigma \times A) \times \delta l \ldots (2)$$

Now,

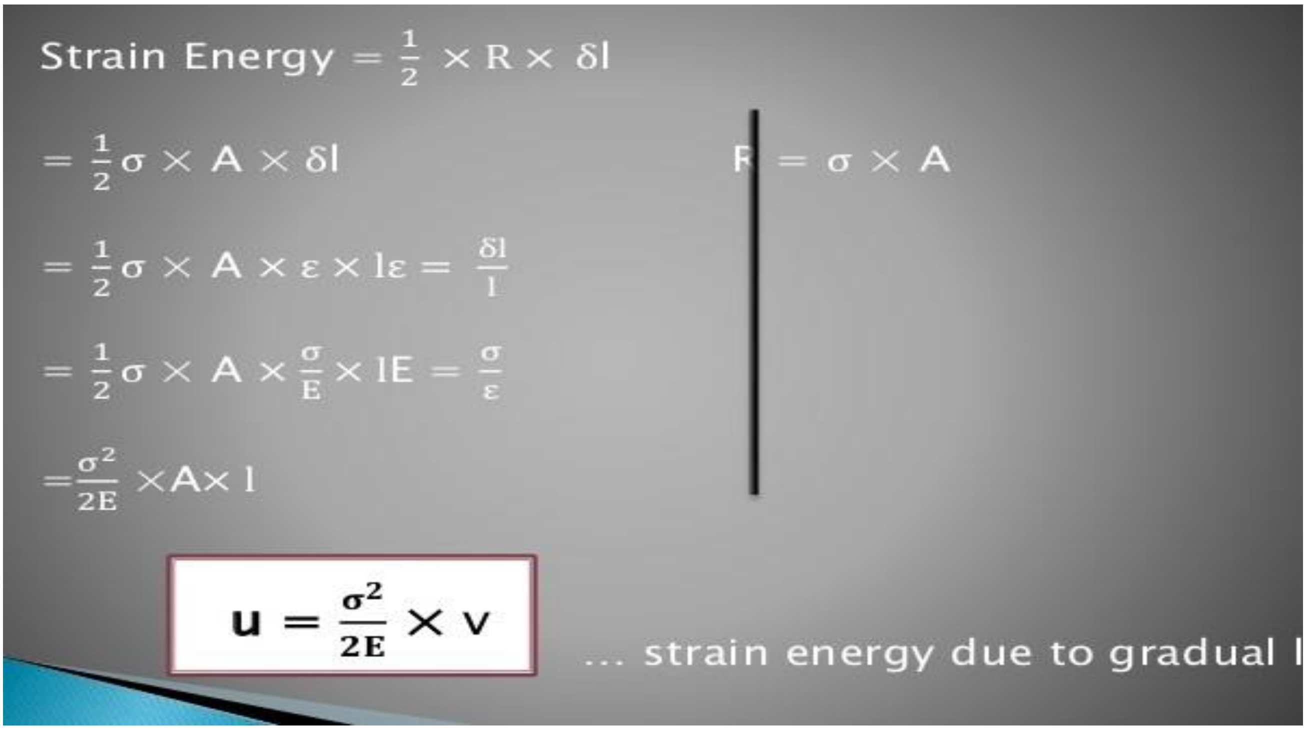
Work done = Work stored

$$\therefore \frac{1}{2} P \times \delta l = \frac{1}{2} \sigma \times A \times \delta l$$

$$\therefore P = \sigma \times A$$

$$\boxed{\therefore \sigma = \frac{P}{A}}$$

..... stress due to gradual load.



PROBLEMS:

1.A tensile load of 60 KN is gradually applied to a circular bar of 4 cm diameter and 5 m long. If the value of E = 2 X 10^5 N/mm^2. Determine stretch in the rod, stress in the rod and strain energy absorbed by the rod.

GIVEN DATA:

Gradually Load, P = 60 KN = 60 X 1000 N

Diameter, d = 4 cm = 40 mm.

Length, L = 5 m = 5000 mm

E = 2 X 10^5 N/mm^2

SOLUTION:

1.STRESS IN THE ROD,

$\sigma = P/A = 60000 / 400\pi$ $\quad\quad$ $A = (\pi \times 40^2)/4 = 400\pi$ mm^2

$= \mathbf{47.746\ N/\ mm^2}$

2. STRETCH IN THE ROD,

$x = (\sigma/E) \times L = (47.746/ 2 \times 10^5)/ 5000$

$= \mathbf{1.19\ mm.}$

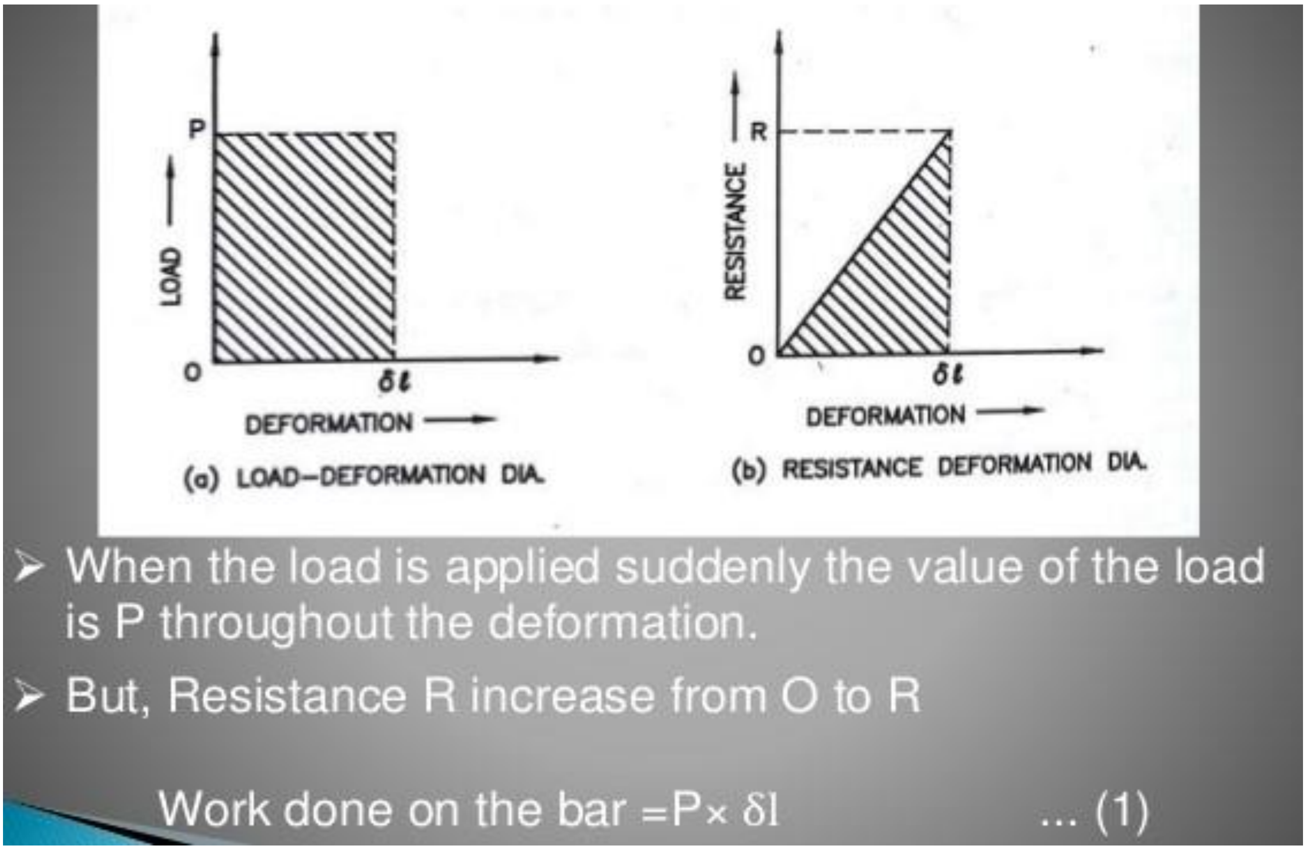
3. STRAIN ENERGY ABSORBED BY THE ROD,

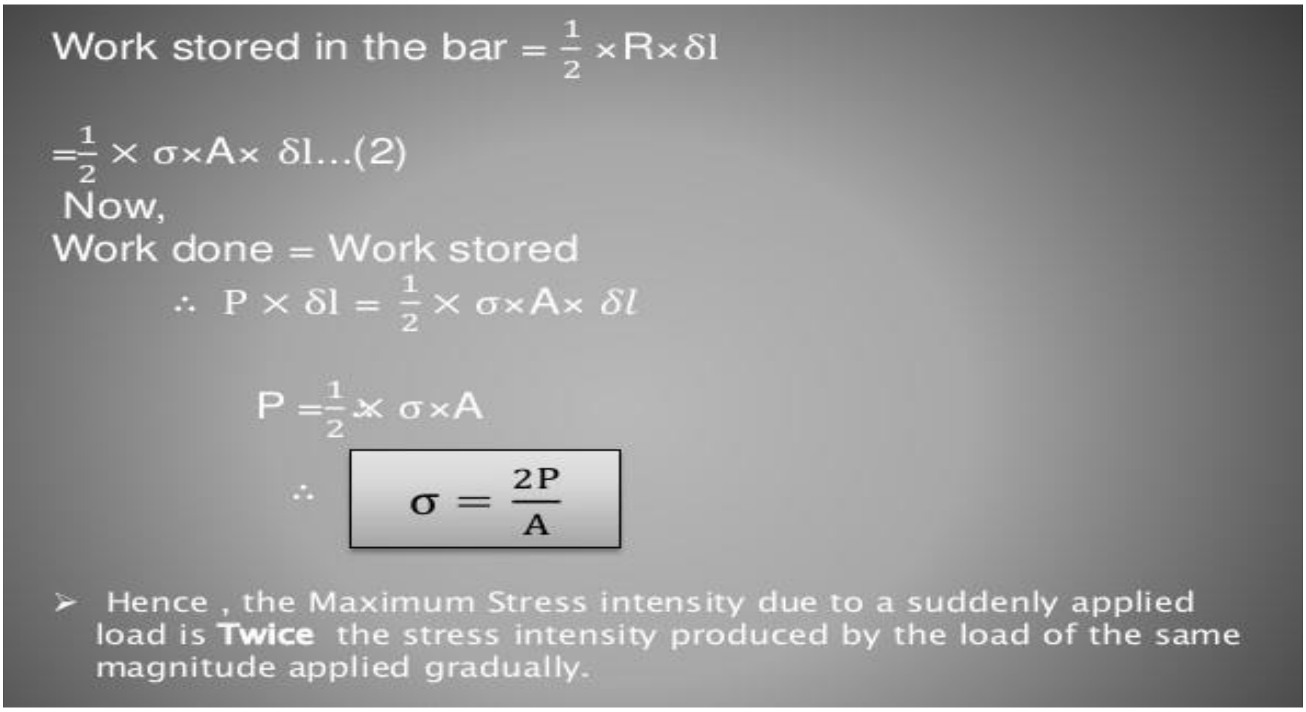
$$U = \frac{\sigma^2}{2E} \times V = \frac{47.746^2}{2 \times 2 \times 10^5} \times 2 \times 10^6 \pi \qquad V = A \times L = 400\pi \times 5000$$

$$= \mathbf{2 \times 10^6\ \pi\ mm^3}$$

$= \mathbf{35810\ N.mm = 35.81\ N.m}$

STRAIN ENERGY DUE TO SUDDENLY APPLIED LOAD





2.Calculate instantaneous stress produced in a bar 10 cm^2 in area and 3 m long by the sudden application of a tensile load of unknown magnitude, if the extension of the bar due to suddenly applied load is 1.5 mm. Also determine the suddenly applied load. Take E = 2 X 10^5 N/mm^2

GIVEN DATA:

Area, A = 10 cm^2 = 1000 mm^2

Length, L = 3 m = 3000 mm

Extension, x = 1.5 mm

E = 2 X 10^5 N/mm^2

SOLUTION:

1.STRESS IN THE ROD,

x = (σ/E) X L

1.5 = (σ / 2 X 10^5)/ 3000

$$\sigma = \frac{1.5 \times 2 \times 10^5}{3000}$$

$= \mathbf{100\ N/mm^2}$

2. SUDDENLY APPLIED LOAD,

$\sigma = 2 \times P/A$

$100 = 2 \times (P/1000)$

$$P = \frac{100 \times 1000}{2}$$

$= \mathbf{50000\ N = 50\ KN.}$

3. A steel rod is 2 m long and 50 mm in diameter. An axial pull of 100 KN is suddenly applied to the rod. Calculate the instantaneous stress induced and also the instantaneous elongation produced in the rod. Take E = 200 GN/m^2

GIVEN DATA:

Diameter, d = 50 mm

Length, L = 2 m = 2000 mm

Pull, P = 100 KN = 100 X 1000 N

$E = 200\ GN/m^2 = 2 \times 10^5\ N/mm^2$

SOLUTION:

1. **INSTANTANEOUS STRESS INDUCED,**

$\sigma = 2 \times P/A$ $\quad$ $A = (\pi \times 50^2)/4 = \mathbf{625\ \pi\ mm^2}$

$= 2 \times (100 \times 1000/ 625\ \pi)$

$= \mathbf{101.86\ N/mm^2}$

2. INSTANTANEOUS ELONGATION PRODUCED IN THE ROD,

$$\delta L = \frac{\sigma \times L}{E}$$

$$= \frac{101.86 \times 2000}{2 \times 10^5}$$

= **1.0186 mm.**

Problem 4.8. *The maximum stress produced by a pull in a bar of length 1 m is 150 N/mm². The area of cross-sections and length are shown in Fig. 4.3. Calculate the strain energy stored in the bar if E = 2 × 10⁵ N/mm².*

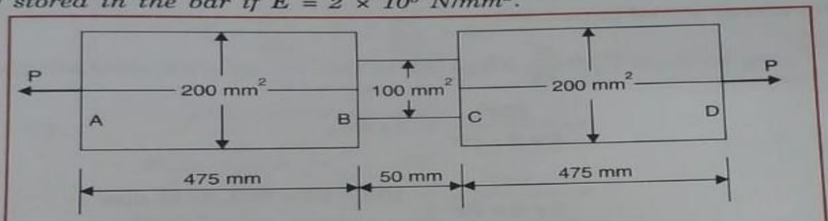


Fig. 4.3

Sol. Given :

Length of bar,	$L = 1 \text{ m} = 1000 \text{ mm}$
Max. stress,	$\sigma = 150 \text{ N/mm}^2$
Part *AB* : Length,	$L_1 = 475 \text{ mm}$
Area,	$A_1 = 200 \text{ mm}^2$
Part *BC* : Length,	$L_2 = 50 \text{ mm}$
Area,	$A_2 = 100 \text{ mm}^2$
Part *CD* : Length,	$L_3 = 475 \text{ mm}$
Area,	$A_3 = 200 \text{ mm}^2$
Value of	$E = 2 \times 10^5 \text{ N/mm}^2$

Axial pull *P* is not known. But stress is

Value of $E = 2 \times 10^5$ N/mm^2

In this problem, maximum stress is given. Axial pull P is not known. But stress is equal to load/area. As load (or axial pull) for the bar is same, hence stress will be maximum, when area will be minimum. Part BC is having less area and hence stress in part BC will be maximum. As parts AB and CD are having same areas, hence stresses in them will be equal.

Let σ_2 = Stress in part BC = 150 N/mm^2

σ_1 = Stress in part AB or in part CD

Now load = Stress × Area

or load $= \sigma_1 \times A_1 = \sigma_2 \times A_2$

$$\therefore \quad \sigma_1 = \frac{\sigma_2 A_2}{A_1} = \frac{150 \times 100}{200} = 75 \text{ N/mm}^2$$

Now strain energy stored in part AB,

$$U_1 = \frac{\sigma_1^2}{2E} \times V_1 \qquad \ldots$$

where V_1 = Volume of part AB

$= A_1 \times L_1 = 200 \times 475$

$= 95000$ mm^3

Substituting this value in equation (*i*), we get

$$U_1 = \frac{\sigma_1^2}{2E} \times 95000$$

$$= \frac{75^2}{2 \times 2 \times 10^5} \times 95000$$

$$= 1335.938 \text{ N-mm}$$

Strain energy stored in part BC,

$$U_2 = \frac{\sigma_2^2}{2E} \times V_2$$

$$= \frac{150^2}{2 \times 2 \times 10^5} \times A_2 \times L_2 \qquad (\because \quad V_2 = A_2 \times L_2)$$

$$= \frac{150^2}{2 \times 2 \times 10^5} \times 100 \times 50 = 281.25 \text{ N-mm}$$

Energy stored in part CD,

$$U_3 = \frac{\sigma_3^2}{2E} \times V_3 = 1335.938 \text{ N-mm} \quad (\because \quad V_3 = V_1, \sigma_3 = \sigma_1 \therefore U_3 = U_1)$$

$\therefore$ Total strain energy stored,

$$U = U_1 + U_2 + U_3 = 1335.938 + 281.25 + 1335.938 \text{ N-mm}$$

$= 2953.126$ **N-mm. Ans.**

Problem 4.6. *A tension bar 5 m long is made up of two parts, 3 metre of its length has a cross-sectional area of 10 cm² while the remaining 2 metre has a cross-sectional area of 20 cm². An axial load of 80 kN is gradually applied. Find the total strain energy produced in the bar and compare this value with that obtained in a uniform bar of the same length and having the same volume when under the same load. Take $E = 2 \times 10^5$ N/mm².*

Sol. Given :

Total length of bar, $L = 5 \text{ m} = 5000 \text{ mm}$

Length of 1st part, $L_1 = 3 \text{ m} = 3000 \text{ mm}$

Area of 1st part, $A_1 = 10 \text{ cm}^2 = 10 \times 100 \text{ mm}^2 = 1000 \text{ mm}^2$

∴ Volume of 1st part,

$$V_1 = A_1 \times L_1 = 1000 \times 3000 = 3 \times 10^6 \text{ mm}^3$$

Length of 2nd part, $L_2 = 2 \text{ m} = 2000 \text{ mm}$

Area of 2nd part, $A_2 = 20 \text{ cm}^2 = 20 \times 100 \text{ mm}^2 = 2000 \text{ mm}^2$

∴ Volume of 2nd part, $V_2 = 2000 \times 2000 = 4 \times 10^6 \text{ mm}^3$

Axial gradual load, $P = 80 \text{ kN} = 80 \times 1000 = 80000 \text{ N}$

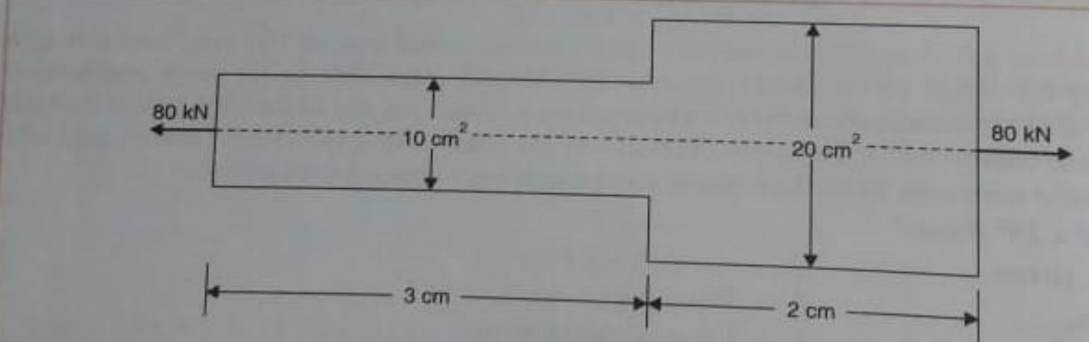


Fig. 4.2

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$

Stress in 1st part, $\sigma_1 = \dfrac{P}{A_1} = \dfrac{80000}{1000} = 80 \text{ N/mm}^2$

Stress in 2nd part, $\sigma_2 = \dfrac{P}{A_2} = \dfrac{80000}{2000} = 40 \text{ N/mm}^2$

Strain energy in 1st part,

$$U_1 = \frac{\sigma_1^2}{2E} \times V_1 = \frac{80^2}{2 \times 2 \times 10^5} \times 3 \times 10^6 = 48000 \text{ N-mm} = 48 \text{ N-m}$$

Strain energy in 2nd part,

$$U_2 = \frac{\sigma_2^2}{2E} \times V_2 = \frac{40^2}{2 \times 2 \times 10^5} \times 4000000 = 16000 \text{ N-mm} = 16 \text{ N-m}$$

∴ Total strain energy produced in the bar,

$$U = U_1 + U_2 = 48 + 16 = \mathbf{64 \text{ N-m.}} \quad \textbf{Ans.}$$

Strain energy stored in a uniform bar

Volume of uniform bar, $V = V_1 + V_2 = 3000000 + 4000000 = 7000000 \text{ mm}^2$

Length of uniform bar, $L = 5 \text{ m} = 5000 \text{ mm}$

Let A = Area of uniform bar

Then $V = A \times L$ or $7000000 = A \times 5000$

∴ $A = \dfrac{7000000}{5000} = 1400 \text{ mm}^2$

Stress in uniform bar, $\sigma = \dfrac{P}{A} = \dfrac{80000}{5000} = 57.143 \text{ N/mm}^2$

∴ Strain energy stored in the uniform bar,

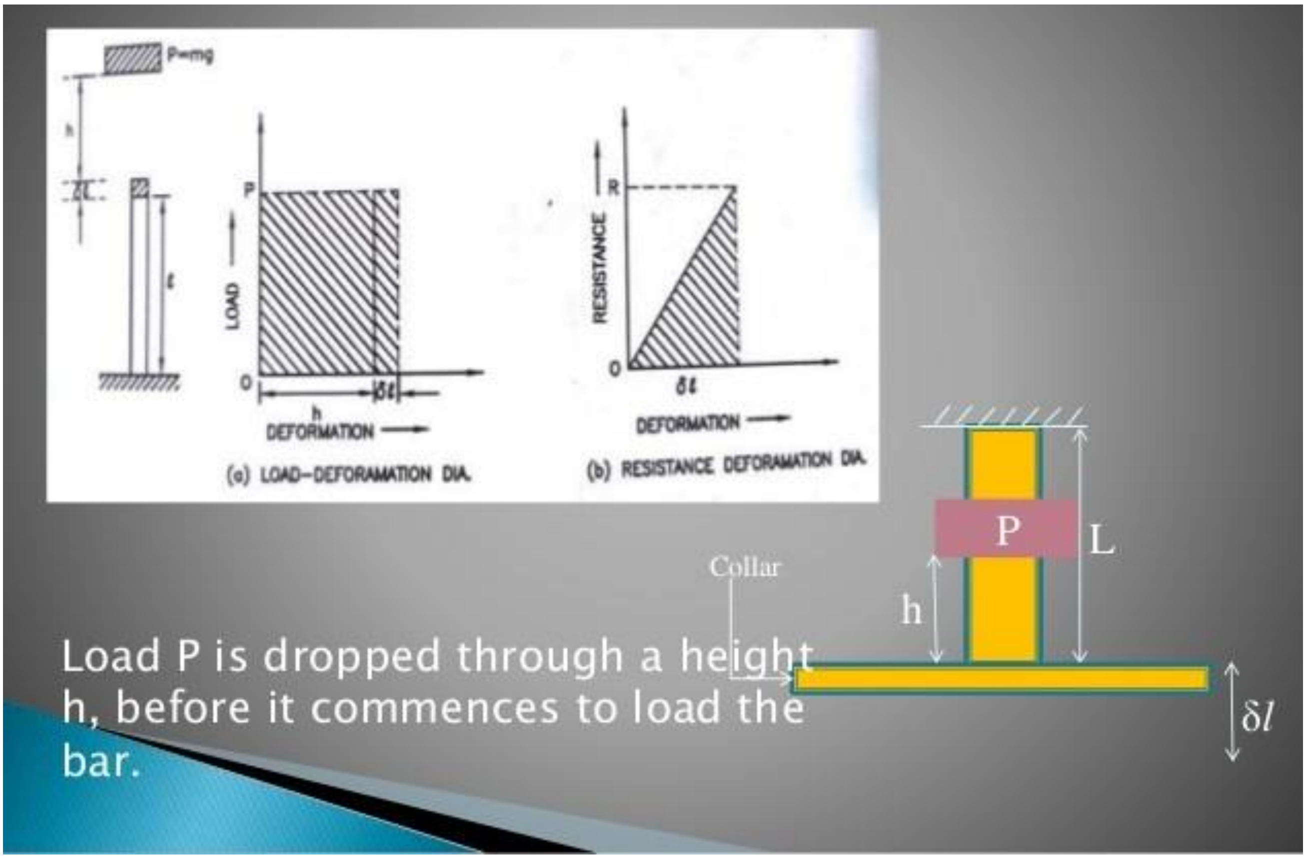
$$U = \frac{\sigma^2}{2E} \times V = \frac{57.143^2}{2 \times 2 \times 10^5} \times 7000000$$

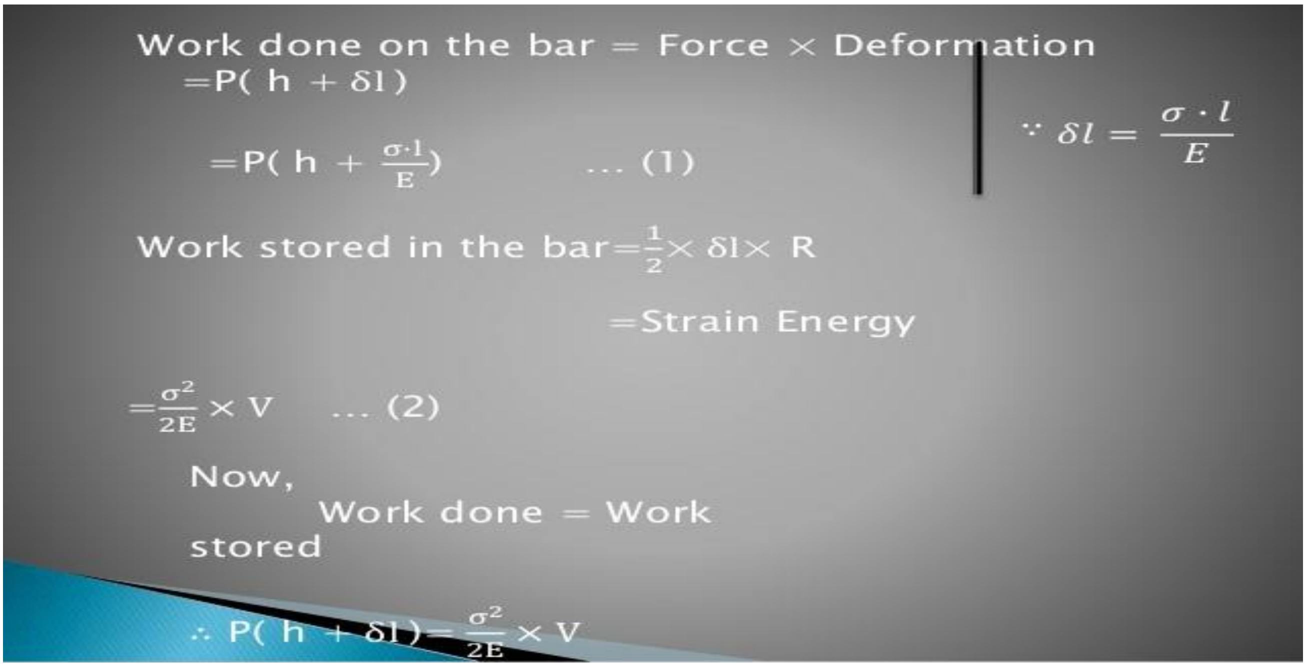
$$= 57143 \text{ N-mm} = 57.143 \text{ N-m}$$

∴ $\dfrac{\text{Strain energy in the given bar}}{\text{Strain energy in the uniform bar}} = \dfrac{64}{57.143} = \mathbf{1.12.}$ **Ans.**

...section 'A' and length 'L' hangs vertically, sub-

STRAIN ENERGY DUE TO IMPACT LOAD





$$\therefore P\left(h + \frac{\sigma l}{E}\right) = \frac{\sigma^2}{2E} \times A \times l$$

$$\therefore P \times h + \frac{p \times \sigma l}{E} = \frac{\sigma^2}{2E} \times A \times l$$

$$\therefore P \times h \times \frac{2E}{Al} + \frac{p \times \sigma l}{E} \times \frac{2E}{Al} = \sigma^2$$

$$\therefore \quad \frac{2EPh}{Al} + \frac{2P \times \sigma}{A} = \sigma^2$$

$$\therefore \sigma^2 - \frac{2P \times \sigma}{A} = \frac{2EPh}{Al}$$

$$\therefore \sigma^2 - \frac{2P \times \sigma}{A} + \frac{p^2}{A^2} = \frac{2EPh}{Al} + \frac{p^2}{A^2}$$

$$\therefore \left(\sigma - \frac{P}{A}\right)^2 = \frac{2EPh}{Al} + \frac{p^2}{A^2}$$

$$\therefore \left(\sigma - \frac{P}{A}\right) = \sqrt{\frac{2EPh}{Al} + \frac{p^2}{A^2}}$$

$$\boxed{\therefore \sigma = \frac{P}{A} + \sqrt{\frac{2EPh}{Al} + \frac{p^2}{A^2}}}$$

.. Stresses due to impact load

If load is applied suddenly, h = 0

$$\therefore \sigma = \frac{P}{A} + \sqrt{\frac{p^2}{A^2} + 0}$$

$$\boxed{\therefore \sigma = \frac{2P}{A}}$$

STRESS DUE TO IMPACT LOAD

$$\sigma = \frac{P}{A}\left(1+\sqrt{1+\frac{2AEh}{P.L}}\right)$$

- When δl is very small as compered to h , then

$$\text{Work done} = P \times h$$

$$\frac{\sigma^2}{2E} \times A \times l = P \times h$$

$$\sigma^2 = \frac{2EPh}{Al}$$

$$\boxed{\sigma = \sqrt{\frac{2EPh}{Al}}}$$

Problem 4.9. *A weight of 10 kN falls by 30 mm on a collar rigidly attached to a vertical bar 4 m long and 1000 mm² in section. Find the instantaneous expansion of the bar. Take E = 210 GPa. Derive the formula you use.*

Sol. Given :

Falling weight, $P = 10 \text{ kN} = 10{,}000 \text{ N}$

Falling height, $h = 30 \text{ mm}$

Length of bar, $L = 4 \text{ m} = 4000 \text{ mm}$

Area of bar, $A = 1000 \text{ mm}^2$

Value of $E = 210 \text{ GPa} = 210 \times 10^9 \text{ N/m}^2$

($\because$ G = Giga = 10^9 and Pa = Pascal = 1 N/m²)

$$= \frac{210 \times 10^9 \text{ N}}{10^6 \text{ mm}^2} \quad (\because \text{ 1 m} = 1000 \text{ mm and m}^2 = 10^6 \text{ mm}^2)$$

$$= 210 \times 10^3 \text{ N/mm}^2 = 2.1 \times 10^5 \text{ N/mm}^2$$

Let dL = Instantaneous elongation due to falling weight

σ = Instantaneous stress produced due to falling weight

Using equation (4.7), we get

$$\sigma = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2EAh}{P \times L}}\right)$$

$$= \frac{10000}{1000}\left(1 + \sqrt{1 + \frac{2 \times 2.1 \times 10^5 \times 1000 \times 30}{10000 \times 4000}}\right)$$

$$= 10\left(1 + \sqrt{1 + 315}\right) = 10\left(1 + \sqrt{316}\right)$$

$$= 10 \times 18.77 = 187.7 \text{ N/mm}^2$$

Now $$E = \frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{\left(\frac{\delta L}{L}\right)} \quad \text{or} \quad \frac{\delta L}{L} = \frac{\sigma}{E}$$

$\therefore$ $$\delta L = \frac{\sigma}{E} \times L = \frac{187.7 \times 4000}{2.1 \times 10^5} = \mathbf{3.575 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 4.10. *A load of 100 N falls through a height of 2 cm onto a collar rigidly attached to the lower end of a vertical bar 1.5 m long and of 1.5 cm² cross-sectional area. The upper end of the vertical bar is fixed.*

Determine :

(i) maximum instantaneous stress induced in the vertical bar,

(ii) maximum instantaneous elongation, and

(iii) strain energy stored in the vertical rod.

Take $E = 2 \times 10^5$ *N/mm²*.

Sol. Given :

Impact load, $P = 100$ N

Height through which load falls,

$h = 2$ cm $= 20$ mm

Length of bar, $L = 1.5$ m $= 1500$ mm

Area of bar, $A = 1.5\ \text{cm}^2 = 1.5 \times 100\ \text{mm}^2 = 150\ \text{mm}^2$

∴ Volume, $V = A \times L = 150 \times 1500 = 225000\ \text{mm}^3$

Modulus of elasticity, $E = 2 \times 10^5\ \text{N/mm}^2$

Let σ = Maximum instantaneous stress induced in the vertical bar,

δL = Maximum elongation, and

U = Strain energy stored.

(*i*) Using equation (4.7),

$$\sigma = \frac{P}{A}\left(1+\sqrt{1+\frac{2AEh}{P.L}}\right) = \frac{100}{150}\left(1+\sqrt{1+\frac{2\times150\times2\times10^5\times20}{100\times1500}}\right)$$

$$= \frac{100}{150}(1+\sqrt{1+8000}) = \mathbf{60.23\ N/mm^2}.\quad \textbf{Ans.}$$

(*ii*) Using equation (4.6),

$$\delta L = \frac{\sigma}{E}\times L = \frac{60.23\times1500}{2\times10^5} = \mathbf{0.452\ mm.}\quad \textbf{Ans.}$$

(*iii*) Strain energy is given by,

$$U = \frac{\sigma^2}{2E}\times V = \frac{60.23^2}{2\times2\times10^5}\times225000 = 2045\ \text{N-mm}$$

$$= \mathbf{2.045\ N\text{-}m.}\quad \textbf{Ans.}$$

Problem 4.11. *The maximum instantaneous extension, produced by an unknown falling weight through a height of 4 cm in a vertical bar of length 3 m and of cross-sectional area 5 cm^2, is 2.1 mm.*

Determine :

(i) the instantaneous stress induced in the vertical bar, and

(ii) the value of unknown weight. Take E = 2 × 10^5 N/mm^2.

Sol. Given :

Instantaneous extension, $\delta L = 2.1$ mm

Length of bar, $L = 3 \text{ m} = 3000 \text{ mm}$

Area of bar, $A = 5 \text{ cm}^2 = 500 \text{ mm}^2$

∴ Volume of bar, $V = 500 \times 3000 = 1500000 \text{ mm}^3$

Height through which weight falls, $h = 4 \text{ cm} = 40 \text{ mm}$

Modulus of elasticity, $E = 2 \times 10^5 \text{ N/mm}^2$

Let σ = Instantaneous stress produced, and

P = Unknown weight.

We know $E = \dfrac{\text{Stress}}{\text{Strain}}$ or Stress = E × Strain

$$\therefore \text{Instantaneous stress} = E \times \text{Instantaneous strain} = E \times \frac{\delta L}{L}$$

$$= 2 \times 10^5 \times \frac{2.1}{3000} \text{ N/mm}^2 = \mathbf{140 \text{ N/mm}^2}. \quad \text{Ans.}$$

Equating the work done by the falling weight to the strain energy stored, we get

$$P(h + \delta L) = \frac{\sigma^2}{2E} \times V$$

$$P(40 + 2.1) = \frac{140^2}{2 \times 2 \times 10^5} \times 1500000 = 73500$$

$$P = \frac{73500}{42.1} = \mathbf{1745.8 \text{ N.}} \quad \text{Ans.}$$

Note. The value of P can also be obtained by using equation (4.7).

Problem 4.12. *An unknown weight falls through a height of 10 mm on a collar rigidly attached to the lower end of a vertical bar 500 cm long and 600 mm² in section. If the maximum extension of the rod is to be 2 mm, what is the corresponding stress and magnitude of the unknown weight ? Take $E = 2.0 \times 10^5$ N/mm².*

Sol. Given :

Height through which the weight falls, $h = 10$ mm

Length of the bar, $L = 500 \text{ cm} = 5000 \text{ mm}$

Area of the bar, $A = 600 \text{ mm}^2$

Maximum extension, $\delta L = 2$ mm

Young's modulus, $E = 2.0 \times 10^5 \text{ N/mm}^2$

Let σ = Instantaneous stress produced in the bar, and

P = Weight falling on the collar.

We know $E = \dfrac{\text{Stress}}{\text{Strain}}$

$\therefore$ $$\text{Stress} = E \times \text{Strain} = E \times \frac{\delta L}{L} \qquad \left(\because \text{ Strain} = \frac{\delta L}{L}\right)$$

Substituting the known values, we get

$$\sigma = 2.0 \times 10^5 \times \frac{2}{5000} = \mathbf{80 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Value of weight falling on the collar

Using equation (4.7),

$$\sigma = \frac{P}{A}\left(1+\sqrt{1+\frac{2A.E.h}{P.L}}\right)$$

$$80 = \frac{P}{600}\left(1+\sqrt{1+\frac{2\times 600\times 2.0\times 10^5\times 10}{P\times 5000}}\right)$$

$$\frac{48000}{P} = 1+\sqrt{1+\frac{480000}{P}}$$

$$\frac{48000}{P}-1 = \sqrt{1+\frac{480000}{P}}$$

Squaring both sides,

$$\left(\frac{48000}{P}\right)^2+1-\frac{2\times 48000}{P}=1+\frac{480000}{P}$$

$$\frac{2304000000}{P^2}-\frac{96000}{P}=\frac{480000}{P} \text{ (cancelling 1 from both sides)}$$

$$\frac{2304000000}{P^2}=\frac{480000}{P}+\frac{96000}{P}=\frac{576000}{P}$$

$$\frac{2304000000}{P} = 576000 \qquad \left(\text{cancelling } \frac{1}{P} \text{ from both sides}\right)$$

$$P = \frac{2304000000}{576000} = 4000 \text{ N} = \mathbf{4 \text{ kN}}. \quad \textbf{Ans.}$$

576000

Problem 4.13. *A bar 12 mm diameter gets stretched by 3 mm under a steady load of 8000 N. What stress would be produced in the same bar by a weight of 800 N, which falls vertically through a distance of 8 cm onto a rigid collar attached at its end ? The bar is initially unstressed. Take E = 2.0 × 10⁵ N/mm².*

Sol. Given :

Dia. of bar,	d	$= 12$ mm
∵ Area of bar,	A	$= \frac{\pi}{4}(12)^2 = 113.1$ mm^2
Increase in length,	δL	$= 3$ mm
Steady load,	W	$= 8000$ N
Falling weight,	P	$= 800$ N
Vertical distance,	h	$= 8$ cm $= 80$ mm
Young's modulus,	E	$= 2.0 \times 10^5$ N/mm^2
Let	L	$=$ Length of the bar, and
	σ	$=$ Stress produced by the falling weight.

With steady load

We know

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{\left(\frac{\text{Steady load}}{\text{Area}}\right)}{\frac{\delta L}{L}}$$

$$2.0 \times 10^5 = \frac{\left(\frac{8000}{113.1}\right)}{\left(\frac{3}{L}\right)} = \frac{8000}{113.1} \times \frac{L}{3}$$

$$\therefore \quad L = \frac{2.0 \times 10^5 \times 113.1 \times 3}{8000} = 8482.5 \text{ mm}$$

Now using equation (4.7), we get

$$\sigma = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2AEh}{PL}}\right)$$

$$= \frac{800}{113.1}\left(1 + \sqrt{1 + \frac{2 \times 113.1 \times 2.0 \times 10^5 \times 80}{8.0 \times 8482.5}}\right) \text{N/mm}^2$$

$$= 7.0734(1 + \sqrt{1 + 533.33}) = 7.0734 \times 24.1155$$

$$= 170.578 \text{ N/mm}^2. \quad \textbf{Ans.}$$

$= 170.578\ N/mm^2$. Ans.

Problem 4.14. *A rod 12.5 mm in diameter is stretched 3.2 mm under a steady load of 10 kN. What stress would be produced in the bar by a weight of 700 N, falling through 75 mm before commencing to stretch, the rod being initially unstressed ? The value of E may be taken as $2.1 \times 10^5\ N/mm^2$.*

Sol. Given :

Dia. of rod, $d = 12.5$ mm

$\therefore$ Area of rod, $A = \frac{\pi}{4} \times 12.5^2 = 122.72\ mm^2$

Increase in length, $\delta L = 3.2$ mm

Steady load, $W = 10$ kN $= 10,000$ N

Falling load, $P = 700$ N

Falling height, $h = 75$ mm

Young's modulus, $E = 2.1 \times 10^5\ N/mm^2$

Let L = Length of the rod,

σ = Stress produced by the falling weight.

We know $E = \frac{\text{Stress}}{\text{Strain}}$

$$= \frac{\left(\frac{\text{Steady load}}{\text{Area}}\right)}{\left(\frac{\delta L}{L}\right)}$$

$$2.1 \times 10^5 = \frac{\left(\frac{10,000}{122.72}\right)}{\left(\frac{3.2}{L}\right)}$$

$$= \left(\frac{10,000}{122.72}\right) \times \left(\frac{L}{3.2}\right)$$

$$\therefore \quad L = \frac{2.1 \times 10^5 \times 122.72 \times 3.2}{10,000} = 8246.7 \text{ mm}$$

Now using equation (4.7), we get

$$\sigma = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2AEh}{P \times L}}\right)$$

$$= \frac{700}{122.72}\left(1 + \sqrt{1 + \frac{2 \times 122.72 \times 2.1 \times 10^5 \times 75}{700 \times 8246.7}}\right)$$

$$= \mathbf{153.74\ N/mm^2}. \quad \textbf{Ans.}$$

Problem 4.15. *A vertical round steel rod 1.82 metre long is securely held at its upper end. A weight can slide freely on the rod and its fall is arrested by a stop provided at the lower end of the rod. When the weight falls from a height of 30 mm above the stop the maximum stress reached in the rod is estimated to be 157 N/mm². Determine the stress in the rod if the load had been applied gradually and also the minimum stress if the load had fallen from a height of 47.5 mm.*

Take $E = 2.1 \times 10^5$ *N/mm²*.

Sol. Given :

Length of rod, $L = 1.82 \text{ m} = 1.82 \times 1000 = 1820 \text{ mm}$

Height through which load falls, $h = 30$ mm

Maximum stress induced in the rod, $\sigma = 157 \text{ N/mm}^2$

Modulus of elasticity, $E = 2.1 \times 10^5 \text{ N/mm}^2$

Let σ_1 = Stress induced in the rod if the load is applied gradually and

σ_2 = Maximum stress if the load had fallen from a height of 47.5 mm.

Strain energy stored in the rod when load falls through a height of 30 mm,

$$U = \frac{\sigma^2}{2E} \times \text{Volume} = \frac{157}{2 \times 2.1 \times 10^5} \times V$$

$$= 0.05868 \times V \text{ N-m}$$

The extension of the rod is given by equation (4.6),

$$\delta L = \frac{\sigma}{E} \times L$$

$$= \frac{157}{2.1 \times 10^5} \times 1820 = 1.36 \text{ mm}$$

$\therefore$ Total distance through which load falls

$$= h + \delta L = 30 + 1.36 = 31.36 \text{ mm}$$

$\therefore$ Work done by the falling load = Load × Total distance

$$= P \times 31.36$$

Equating the work done by the falling load to the strain energy stored, we get

$$P \times 31.36 = 0.05868 \times V$$

or
$$\frac{P}{V} = \frac{0.05868}{31.36} = 0.001871$$

or
$$\frac{P}{A.L} = 0.001871 \qquad (\because \; V = A.L)$$

or
$$\frac{P}{A} = 0.001871 \times L = 0.001871 \times 1820 = 3.4$$

1st Case. If the load had been applied gradually, the stress induced is given by,

$$\sigma_1 = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

$$= \mathbf{3.4\ N/mm^2.} \quad \textbf{Ans.}$$

2nd Case. If the load had fallen from a height of 47.5 mm.

Let σ_2 = Maximum stress induced.

Using equation (4.7), we get

$$\sigma_2 = \frac{P}{A}\left[1 + \sqrt{1 + \frac{2AEh}{P \times L}}\right] \qquad [\text{Here } \sigma = \sigma_2]$$

$$= 3.4\left[1 + \sqrt{1 + \frac{2 \times 2.1 \times 10^5 \times 47.5}{3.4 \times 1820}}\right] \qquad \left(\because \frac{P}{A} = 3.4, h = 47.5\right)$$

$$= 3.4\left(1 + \sqrt{1 + 3219.24}\right)$$

$$= \mathbf{196.64\ N/mm^2.} \quad \textbf{Ans.}$$

Problem 4.19. *A cage weighing 60 kN is attached to the end of a steel wire rope. It is lowered down a mine shaft with a constant velocity of 1 m/s. What is the maximum stress produced in the rope when its supporting drum is suddenly jammed ? The free length of the rope at the moment of jamming is 15 m, its net cross-sectional area is 25 cm² and E = 2 × 10⁵ N/mm². The self-weight of the wire rope may be neglected.*

Sol. Given :

Weight, $W = 60 \text{ kN} = 60{,}000 \text{ N}$

Velocity, $V = 1 \text{ m/s}$

Free length, $L = 15 \text{ m} = 15{,}000 \text{ mm}$

Area, $A = 25 \text{ cm}^2 = 25 \times 100 \text{ mm}^2$

Value of $E = 2 \times 10^5 \text{ N/mm}^2$

K.E. of the cage
$$= \frac{1}{2} mV^2 = \frac{1}{2}\left(\frac{W}{g}\right) \times V^2$$

$$= \frac{1}{2} \times \left(\frac{60{,}000}{9.81}\right) \times 1^2 \text{ N-m} = \frac{30000}{9.81} \text{ N-m}$$

$$= \frac{30000 \times 1000}{9.81} \text{ N-mm} \qquad ...(i)$$

This energy is to be absorbed (or stored) by the rope.

Let σ = Maximum stress produced in the rope when its supporting drum is suddenly jammed.

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But the maximum energy stored

$$= \frac{\sigma^2}{2E} \times A \times L = \frac{\sigma^2}{2 \times 2 \times 10^5} \times 2500 \times 15000 \text{ N-mm} \quad \ldots(ii)$$

But K.E. of the cage = Energy stored in the rope

$$\frac{30000 \times 1000}{9.81} = \frac{\sigma^2}{2 \times 2 \times 10^5} \times 2500 \times 15000$$

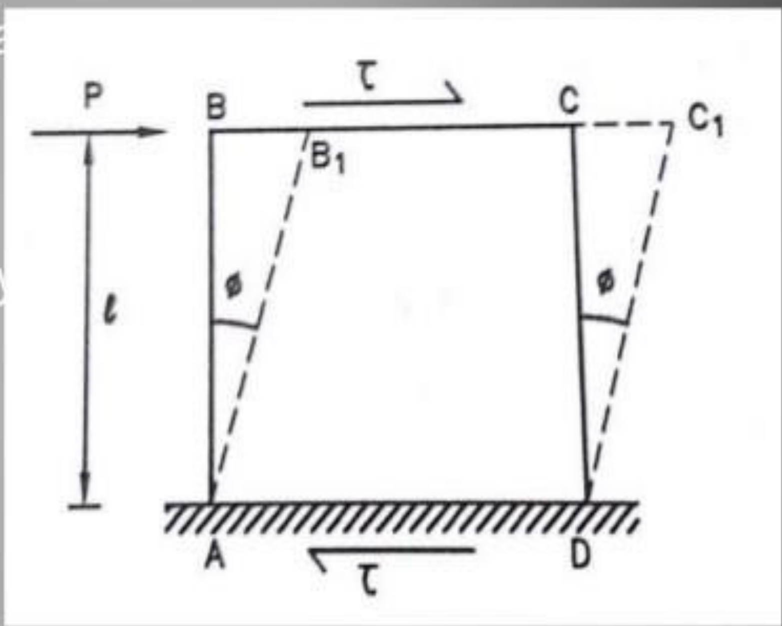
$$\sigma^2 = \frac{30000 \times 1000 \times 2 \times 2 \times 10^5}{9.81 \times 2500 \times 15000}$$

$$\sigma = \sqrt{\frac{30000 \times 1000 \times 2 \times 2 \times 10^5}{9.81 \times 2500 \times 15000}} = 180.61 \text{ N/mm}^2. \quad \textbf{Ans.}$$

STRAIN ENERGY DUE TO SHEAR STRESS

- If t is the uniform shear stress produce in the material by external forces applied within elastic limit, the energy Stored due to shear Loading is given by,

$$\boxed{u = \frac{\tau^2}{2G} \times V}$$



Where, t = shear stress
G = Modulus of rigidity

∴ Strain energy stored = $\frac{\quad}{2C}$

Problem 4.20. *The shear stress in a material at a point is given as 50 N/mm². Determine the local strain energy per unit volume stored in the material due to shear stress. Take $C = 8 \times 10^4$ N/mm².*

Sol. Given :

Shear stress, $\tau = 50$ N/mm²

Modulus of rigidity, $C = 8 \times 10^4$ N/mm²

Using equation (4.9),

$$\text{Strain energy} = \frac{\tau^2}{2C} \times \text{Volume} = \frac{50^2}{2 \times 8 \times 10^4} \times \text{Volume}$$

$$= 0.015625 \times \text{Volume}$$

∴ Strain energy per unit volume

$$= \frac{0.015625 \times \text{Volume}}{\text{Volume}} = \mathbf{0.015625\ N/mm^2.} \quad \textbf{Ans.}$$

STRAIN ENERGY IN SOLID SHAFT DUE TO TORSION

∴ Total strain energy in the shaft due to torsion,

$$U = \frac{\tau^2 . l}{2CR^2} \times \frac{\pi}{32} D^4$$

$$= \frac{\tau^2 . l}{2CR^2} \times \frac{\pi}{32} \times (2R)^4 \qquad (\because$$

$$= \frac{\tau^2 . l}{2CR^2} \times \frac{\pi}{32} \times 16R^4 = \frac{\tau^2}{4C} . \pi R^2 . l$$

$$= \frac{\tau^2}{4C} . V \qquad (\because \ \text{Volume}, V = \pi R^2 . l)$$

STRAIN ENERGY IN HOLLOW SHAFT DUE TO TORSION

Total strain energy in the hollow shaft due to torsion

Let D = Outer dia. of shaft

d = Inner dia. of shaft

J = Polar moment of inertia of hollow shaft

$$= \frac{\pi}{32}[D^4 - d^4]$$

Substituting the value of J in equation (i), we get

Total strain energy in the hollow shaft due to torsion,

$$U = \frac{\tau^2 . l}{2CR^2} \times \frac{\pi}{32}[D^4 - d^4]$$

$$= \frac{\tau^2 . l}{2C\left(\frac{D}{2}\right)^2} \times \frac{\pi}{32}[D^2 + d^2][D^2 - d^2] \qquad \left[\because R = \frac{D}{2}\right]$$

$$= \frac{\tau^2 . l \times 4}{2C . D^2} \times \frac{\pi}{32}[D^2 + d^2][D^2 - d^2]$$

$$= \frac{\tau^2}{4CD^2} \times \frac{\pi}{4} \times (D^2 - d^2) . l (D^2 + d^2)$$

$$= \frac{\tau^2}{4CD^2} \times V \times (D^2 + d^2) \qquad \left[\because V = \frac{\pi}{4}(D^2 - d^2) . l\right]$$

$$= \frac{\tau^2}{4CD^2}(D^2 + d^2) \times V \qquad \text{...(16.21)}$$

PROBLEMS:

Problem 16.28. *Determine the maximum strain energy stored in a solid shaft of diameter 10 cm and of length 1.25 m, if the maximum allowable shear stress is 50 N/mm². Take C = 8 × 10⁴ N/mm².*

Sol. Given :

Dia. of shaft, $D = 10$ cm

$\therefore$ Area of shaft, $A = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2 = 7854 \text{ mm}^2$

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Length of shaft, $l = 1.25 \text{ m} = 125 \text{ cm}$

$\therefore$ Volume of shaft, $V = A \times l = 78.54 \times 125 = 9817.5 \text{ cm}^3 = 9817.5 \times 10^3 \text{ mm}^3$

Maximum allowable shear stress,

$$\tau = 50 \text{ N/mm}^2$$

(shear stress is maximum on the surface of the shaft

Modulus of rigidity, $C = 8 \times 10^4 \text{ N/mm}^2$

Let U = Shear strain energy stored in the shaft.

Using equation (16.20), we get

$$U = \frac{\tau^2}{4C} \times V = \frac{50^2}{4 \times 8 \times 10^4} \times 9817.5 \times 10^3$$

$$= \mathbf{76699 \text{ N-mm.}} \quad \textbf{Ans.}$$

$= \mathbf{76699 \text{ N-mm.}}$ **Ans.**

Problem 16.29. *The external and internal diameters of a hollow shaft are 40 cm and 20 cm. Determine the maximum strain energy stored in the hollow shaft if the maximum allowable shear stress is 50 N/mm² and length of the shaft is 5 m. Take C = 8 × 10⁴ N/mm².*

Sol. Given :

External dia., $D = 40 \text{ cm} = 400 \text{ mm}$

Internal dia., $d = 20 \text{ cm} = 200 \text{ mm}$

$\therefore$ Area of cross-section, $A = \frac{\pi}{4}(40^2 - 20^2) = 942.47 \text{ cm}^2 = 94247 \text{ mm}^2$

Maximum allowable shear stress (*i.e.*, shear stress on the surface of the shaft),

$$\tau = 50 \text{ N/mm}^2$$

Length of shaft, $l = 5 \text{ m} = 500 \text{ cm}$

$\therefore$ Volume of hollow shaft,

$$V = A \times l = 942.47 \times 500 = 471235 \text{ cm}^3 = 471235 \times 10^3 \text{ mm}^3$$

Modulus of rigidity, $C = 8 \times 10^4 \text{ N/mm}^2$

Let U = Strain energy stored

Using equation (16.21), we have

$$U = \frac{\tau^2}{4CD^2}(D^2 + d^2) \times V$$

$$= \frac{50^2}{4 \times 8 \times 10^4 \times 400^2} \times (400^2 + 200^2) \times 471235 \times 10^3$$

$$= 4601900 \text{ N-mm} = \mathbf{4601.9 \text{ N-m.}} \quad \textbf{Ans.}$$

Problem 16.31. *A solid circular shaft of 10 cm diameter of length 4 m is transmitting 112.5 kW power at 150 r.p.m. Determine : (i) the maximum shear stress induced in the shaft and (ii) Strain energy stored in the shaft. Take $C = 8 \times 10^4$ N/mm^2.*

Sol. Given :

Dia. of shaft, $D = 10$ cm $= 100$ mm

Length of shaft, $l = 4$ m $= 4000$ mm

Power, $P = 112.5$ kW $= 112.5 \times 10^3$ W

Speed of shaft, $N = 150$ r.p.m.

Modulus of rigidity, $C = 8 \times 10^4$ N/mm^2

Let τ = Maximum shear stress induced in the shaft and

U = Strain energy stored in the shaft.

We know
$$P = \frac{2\pi NT}{60}$$

or
$$112.5 \times 10^3 = \frac{2\pi \times 150 \times T}{60}$$

$\therefore$
$$T = \frac{112.5 \times 10^3 \times 60}{2\pi \times 150} = 7159 \text{ N-m} = 7159000 \text{ N-mm}$$

But we know,
$$T^* = \frac{\pi}{16} \times \tau \times D^3$$

or
$$7159000 = \frac{\pi}{16} \times \tau \times 100^3$$

$\therefore$
$$\tau = \frac{7159000 \times 16}{\pi \times 10^6} = 36.5 \text{ N/mm}^2$$

Using equation (16.20) for strain energy,

$$U = \frac{\tau^2}{4C} \times \text{Volume of shaft}$$

$$= \frac{36.5^2}{4 \times 8 \times 10^4} \times \text{Volume of shaft}$$

$$= \frac{36.5^2}{4 \times 8 \times 10^4} \times \frac{\pi}{4} \times 100^2 \times 4000 \quad \left(\because \text{Volume} = \frac{\pi}{4} \times D^2 \times l\right)$$

$$= 130793 \text{ N-mm.} \quad \textbf{Ans.}$$

= 130793 N-mm

Problem 16.32. *A hollow shaft of internal diameter 10 cm, is subjected to pure torque and attains a maximum shear stress q on the outer surface of the shaft. If the strain energy stored in the hollow shaft is given by $\frac{\tau^2}{3C} \times V$, determine the external diameter of the shaft.*

Sol. Given :

Internal dia., $d = 10$ cm

Maximum shear stress $= \tau$

Strain energy stored, $U = \frac{\tau^2}{3C} \times V$ where V = Volume

Let D = External diameter of the hollow shaft.

Using equation (16.21) for the strain energy in hollow shaft

$$U = \frac{\tau^2}{4CD^2}(D^2 + d^2) \times V$$

Equating the two values of strain energy, we get

$$\frac{\tau^2}{4CD^2}(D^2 + d^2) \times V = \frac{\tau^2}{3C} \times V$$

$$\frac{D^2 + d^2}{4D^2} = \frac{1}{3} \qquad \left(\text{Cancelling } \frac{\tau^2}{C} \times V \text{ from both sides}\right)$$

$$3D^2 + 3d^2 = 4D^2$$

$$3d^2 = 4D^2 - 3D^2 = D^2$$

$$\frac{D^2}{d^2} = 3 \quad \text{or} \quad \frac{D}{d} = \sqrt{3} = 1.732$$

$$D = 1.732 \times d = 1.732 \times 10 = \mathbf{17.32\ cm.}$$ **Ans.**

CASTIGLIANO'S FIRST THEOREM:

- ✓ Castigliano's first theorem states that the partial derivative of the total strain energy in a structure with respect to a load is equal to the deflection of the point where the load is acting, the deflection being measured in the direction of the load.
- ✓ $\frac{dU}{dP_i} = \Delta_i$, $\frac{dU}{dM_j} = \Theta_j$

Where, U = Total strain energy

P_i & M_j = Loads

Δ_i & Θ_j = Deflections.

THEOREM USED IN THE FOLLOWING CASES:

- ✓ To determine the displacements of complicated structures.
- ✓ To find the deflection of beams due to shearing or bending if the total strain energy due to shearing forces or bending moments is known.
- ✓ To find the deflections of curved beams, springs etc.

BETTI'S THEOREM:

- ✓ Betti's theorem, also known as Maxwell–Betti reciprocal work theorem, discovered by Enrico Betti in 1872, states that for a linear elastic structure subject to two sets of forces $\{P_i\}$ i=1,2,...,n and $\{Q_j\}$, j=1,2,...,n, the work done by the set P through the displacements produced by the set Q is equal to the work done by the set Q through the displacements produced by the set P.

MAXWELL'S LAW OF RECIPROCAL DEFLECTION:

- ✓ The beam is not just deflected at the centre but all along its length.
- ✓ Let the deflection at a point D be δ_{DC}.
- ✓ Maxwell's reciprocal theorem says that the deflection at D due to a unit load at C is the same as the deflection at C if a unit load were applied at D.

  In our notation, $\delta_{CD} = \delta_{DC}$.

REFERENCE BOOKS:

1. Bansal R.K., “Strength of Materials”, 5th Edition, Laxmi Publications, 2013.
2. Ramamrutham R., “Strength of Materials”,14th Edition, Dhanpat Rai Publications, 2011.
3. Punmia B.C., “Strength of Materials”, Laxmi Publications, 2011.
4. Subramanian R., “Strength of Materials - Theory and Problems”, Oxford University Press, 2010.
5. Rajput R.K., “Strength of Materials”, S. Chand and Company Ltd., New Delhi, 2007.
6. Timoshenko S.P. and Gere J.M., “Mechanics of Materials”, 6th Edition, Tata McGraw Hill.

QUESTION BANK:

1. Define: strain energy.
2. Define the terms: Resilience & Modulus of Resilience.
3. Define: Proof resilience.
4. What are the different types of loads?
5. Define: Castigliano's theorem.
6. Define: maxwell's reciprocal theorem.
7. A steel rod is 2 m long and 50 mm in diameter. An axial pull of 100 KN is suddenly applied to the rod. Find the instantaneous stress induced. Take E = 200 GN/m2
8. The shear stress in a material at a point is given as 50 N/mm2. Find the strain energy per unit volume stored in the material due to shear stress. Take C = 8 X 104 N/mm2.
9. Define: Betties theorem.
10. A tensile load of 50 KN is gradually applied to a circular bar of 5 cm diameter and 4 m long. If E = 2 X 105 N/mm2, determine stretch in the rod, stress in the rod and strain energy absorbed by the rod.
11. A tension bar 5 m long is made up of two parts, 3 m of its length has a cross sectional area of 10 cm2 while the remaining 2 m has a cross sectional ara of 20 cm2. An axial load of 80 KN is gradually applied. Find the total strain energy produced in the bar and compare this value with that obtained in a uniform bar of the same length and having the same volume when under the same load. Take E = 2 X 105 N/mm2.
12. A load of 200 N falls through a height of 2.5 cm on to a collar rigidly attached to the lower end of a vertical bar 2 m long and of 3 cm2 cross sectional area. The upper end of the vertical bar is fixed. Determine maximum instantaneous stress induced in the vertical bar, maximum instantaneous elongation and strain energy stored in the vertical rod. Take E = 2 X 105 N/mm2
13. A rod 12.5 mm in diameter is stretched 3.2 mm under a steady load of 10 KN. Determine the stress would be produced in the bar by a weight of 700 N, falling through 75 mm before commencing to stretch, the rod being initially unstressed? Take E = 2.1 X 105 N/mm2.
14. The maximum instantaneous elongation produced by an unknown falling weight through a height of 4 cm in a vertical bar of length 5 m and of cross-sectional area 5 cm2 is 1.8 mm. Determine the instantaneous stress induced in the vertical bar and the values of unknown weight. Take E = 2 X 106 Kgf/cm2.

SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – III – THEORY OF COLUMNS – SCIA1401

INTRODUCTION

Any member subjected to axial compressive load is called a column or Strut.

A vertical member subjected to axial compressive load – COLUMN. (Eg: Pillars of a building)

An inclined member subjected to axial compressive load – STRUT.

A strut may also be a horizontal member.

Critical or Crippling or Buckling load – Load at which buckling starts.

CLASSIFICATION OF COLUMNS:

According to nature of failure – short, medium and long columns.

Short column – whose length is so related to its c/s area that failure occurs mainly due to direct compressive stress only and the role of bending stress is negligible.

Medium Column - whose length is so related to its c/s area that failure occurs by a combination of direct compressive stress and bending stress.

Long Column - whose length is so related to its c/s area that failure occurs mainly due to bending stress and the role of direct compressive stress is negligible.

Long Column :-

- When length of column is more as compared to its c/s dimension, it is called long column.

  $Le/k_{min} > 50$

For mild steel $\lambda > 80$ is called long column.

Short Column:-

- When length of column is less as compared to its c/s dimension, it is called Short column.

  $Le/k_{min} < 50$

FAILURE OF A COLUMN

The failure of a column takes place due to anyone of the following stresses set up in the columns:

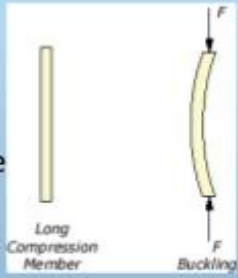
i. Direct compressive stresses

ii. Buckling stresses

iii. Combined direct compressive and buckling stresses.

BUCKLING LOAD

The minimum axial load at which the column tends to have lateral displacement or buckle is called buckling load.



FAILURE OF A SHORT COLUMN

A short column of uniform cross sectional area A, subjected to an axial compressive load P. The compressive stress induced is given by

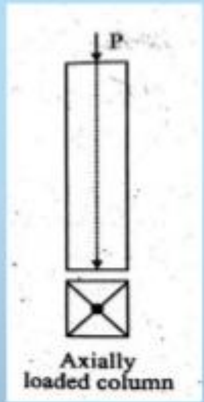
$$p = \frac{P}{A}$$

If the compressive load on the short column is gradually increased, a stage will reach when the column will be on point of failure by crushing. Let,

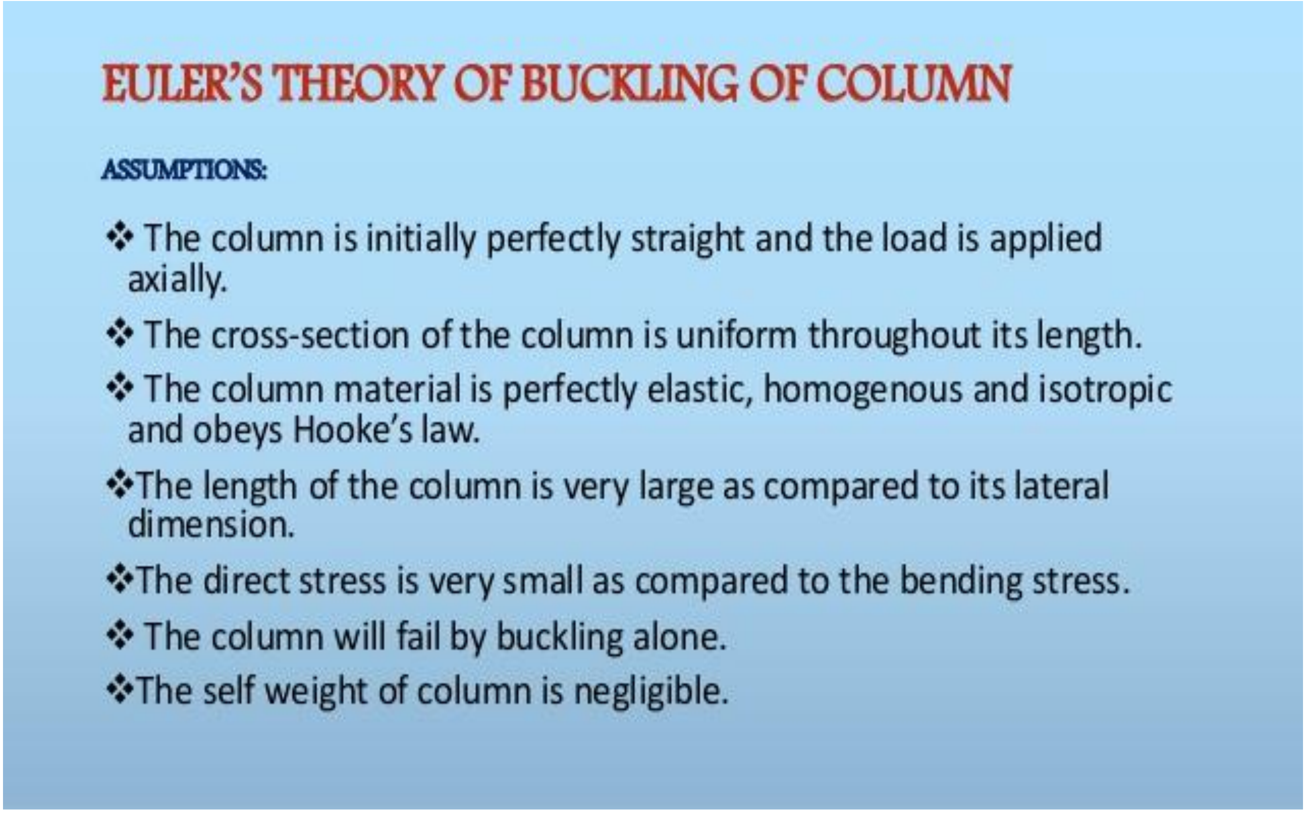
P_c = Crushing load, σ_c = Crushing stress

A= Area of cross-section

$$\text{Then , } \sigma = \frac{P_c}{A}$$



ASSUMPTIONS IN EULER'S COLUMN THEORY:



LIMITATIONS OF EULER'S FORMULA:

The general expression of bucking load for the long column as per Euler's theory is given as,

$P = \Pi^2 E I / L^2$

$\sigma = \Pi^2 E / (Le / k)^2$

We know that, Le / k = slenderness ratio.

LIMITATION 1:

The above formula is applied only for long columns

LIMITATION 2:

As the slenderness ratio decreases the crippling stress increases.

Consequently, if the slenderness ratio reaches to zero, then the crippling stress reaches infinity, practically which is not possible.

LIMITATION 3:

If the slenderness ratio is less than certain limit, then crippling stress is greater than crushing stress, which is not possible practically.

Therefore, up to limiting extent Euler's formula is applicable with crippling stress equal to crushing stress.

Euler's formula is applicable when the slenderness ratio is greater than or equal to 80.

Euler's formula is applicable only for long column.

Euler's formula is thus unsuitable when the slenderness ratio is less than a certain value.

SLENDERNESS RATIO:

Slenderness ratio is the ratio of the actual length of a column and the least radius of gyration of its cross section.

Slenderness Ratio = l/k.

EFFECTIVE LENGTH:

The effective length (Le) of a column is defined as the distance between successive inflection points or points of zero moment.

Effective length is also called equivalent length.

Crippling load for any type of end condition is given by,

$$P = \frac{\pi^2 EI}{Le^2}$$

CRIPPLING STRESS IN TERMS OF EFFECTIVE LENGTH AND RADIUS OF GYRATION:

$$P = \frac{\pi^2 EI}{Le^2} = \frac{\pi^2 E \text{ X } A k^2}{Le^2}$$

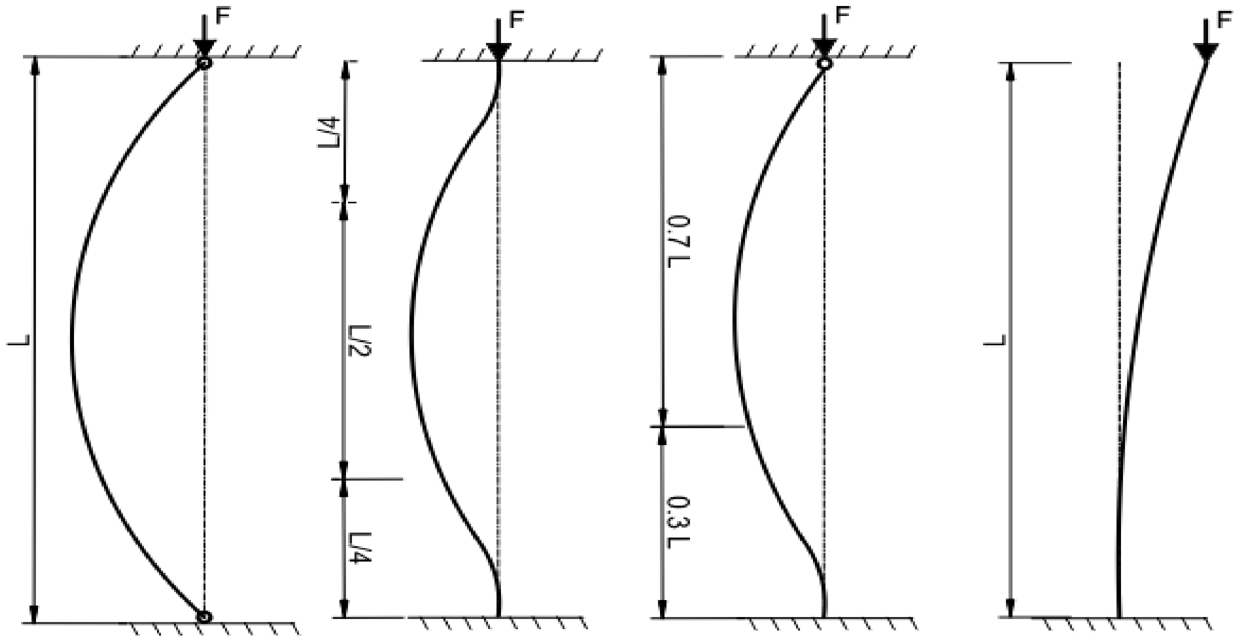
$$= \frac{\pi^2 E \text{ X } A}{Le^{2} / k^2} = \frac{\pi^2 E \text{ X } A}{(Le/ k)^2}$$

$$\text{Crippling stress} = \frac{\text{Crippling load}}{\text{Area}}$$

$$= \frac{\pi^2 E \text{ X } A}{A\,(Le/ k)^2} = \frac{\pi^2 E}{(Le/ k)^2}$$

END CONDITIONS FOR COLUMNS:

- ✓ Both ends are hinged or pinned.
- ✓ One end is free and the other end is fixed.
- ✓ Both ends are fixed.
- ✓ One end is fixed and the other end is pinned.



PROBLEMS:

1. **A solid round bar 3 m long and 5 cm in diameter is used as a strut. Determine the crippling load for all the end conditions. Take E = 2 X 10^5 N/mm^2.**

GIVEN DATA:

Length, L= 3 m = 3000 mm

Diameter, d = 5 cm = 50 mm I = π X d^4 /64 = π X 50^4 /64

E = 2 X 10^5 N/mm^2. = **30.68 X 10^4 mm^4.**

SOLUTION:

1. Crippling load for both ends hinged,

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2} = 67288 \text{ N} = \mathbf{67.288\ KN.}$$

2. Crippling load when one end is fixed and other end is free,

$$P = \frac{\pi^2 EI}{4l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{4 \times 3000^2}$$

$$= 16822 \text{ N} = \mathbf{16.822\ KN.}$$

3. Crippling load when both the ends are fixed,

$$P = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$$

$$= 269152 \text{ N} = \mathbf{269.152\ KN.}$$

2. Crippling load when one end is fixed and other end is hinged,

$$P = \frac{2\pi^2 EI}{l^2} = \frac{2 \times \pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$$

$$= 134576 \text{ N} = \mathbf{134.576\ KN.}$$

2. **A column of timber section 15 cm X 20 cm is 6 m long both ends being fixed. If E for timber = 17.5 KN/mm^2, determine crippling load and safe load for the column if factor of safety = 3.**

GIVEN DATA:

Length, L= 6 m = 6000 mm

Dimension of section, = 15 cm X 20 cm

E = 17.5 KN/mm^2.

Factor of safety = 3.

SOLUTION:

$$I_{XX} = \frac{15 \times 20^3}{12} = 10000 \text{ cm}^4 = \mathbf{10000 \times 10^4\ mm^4.}$$

$$I_{YY} = \frac{20 \times 15^3}{12} = 5625 \text{ cm}^4 = \mathbf{5625 \times 10^4\ mm^4.}$$

Value of I will be the least value of the two moment of inertia.

,

$I = I_{YY} = 5625\ cm^4 = \mathbf{5625 \times 10^4\ mm^4}$.

1. Crippling load for both ends fixed,

$$P = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 17.5 \times 10^3 \times 5625 \times 10^4}{6000^2}$$

$= 1079480\ N =$ **1079.480 KN.**

2. Safe load for the column,

Safe load = Crippling load / Factor of safety

= 1079.480 / 3

= **359.8 KN.**

3. A hollow mild steel tube 6 m long 4 cm internal diameter and 5 mm thick is used as a strut with both ends hinged. Find the crippling load and safe load taking factor of safety as 3. Take E = 2 X 10^5 N/mm^2.

GIVEN DATA:

Length, L= 6 m = 6000 mm

Internal diameter, d = 4 cm = 40 mm.

Thickness, t = 5 mm.

External diameter, D = d + 2t = 40 + 2 X 5 = 50 mm.

$E = 2 \times 10^5\ N/mm^2$.

Factor of safety = 3.

SOLUTION:

$$I_{XX} = I = \frac{\pi \times (D^4 - d^4)}{64} = \frac{\pi \times (50^4 - 40^4)}{64} = \mathbf{18.11 \times 10^4\ mm^4}.$$

1. Crippling load for both ends hinged,

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 18.11 \times 10^4}{6000^2}$$

$= 9929.9\ N =$ **9930 N**

2. Safe load for the column,

Safe load = Crippling load / Factor of safety

= 9930 / 3

= **3310 N.**

4. A simply supported beam of length 4 m is subjected to a uniformly distributed load of 30 KN/m over the whole span and deflects 15 mm at the centre. Determine the crippling loads when this beam is used as a column with the following conditions:

One end fixed and other end hinged.

Both the ends pin jointed.

GIVEN DATA:

Length, L= 4 m = 4000 mm

Uniformly distributed load, w = 30 KN/m = 30 N/mm.

Deflection at the centre, δ= 15 mm.

SOLUTION:

For a simply supported beam, carrying UDL over the whole span, the deflection at the centre is given by,

$$\delta = \frac{5\,w\,L^4}{384\,EI}$$

$$15 = \frac{5 \times 30 \times 4000^4}{384 \times EI}$$

EI = **0.66×10^{13} N.mm²**

1. Crippling load when one end is fixed and other end is hinged,

$$P = \frac{2\pi^2\,EI}{l^2} = \frac{2 \times \pi^2 \times 0.66 \times 10^{13}}{4000^2}$$

= **8224.5 KN.**

2. Crippling load for both ends hinged,

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 0.66 \times 10^{13}}{4000^2}$$

$$= \mathbf{4112.25\ KN.}$$

5. A solid round bar 4 m long and 5 cm in diameter was found to extend 4.6 mm under a tensile load of 50 KN. This bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.

GIVEN DATA:

Length, L= 4 m = 4000

Diameter, d = 5 cm = 50 mm.

Extension of bar, δL= 4.6 mm.

Tensile load, W = 50 KN.

SOLUTION:

$A = \pi \times d^2 /4 = \pi \times 50^2/4 =$ **$625\ \pi$ mm^2.**

$I = \pi \times d^4 /64 = \pi \times 50^4/64 =$ **30.68×10^4 mm^4.**

Young's Modulus, E = Tensile Stress/Tensile Strain

$$= \frac{\text{Tensile Load/Area}}{\text{Change in length of bar/Original length of bar}}$$

$$= \frac{W/A}{\delta L/L} = \frac{W \times L}{A \times \delta L}$$

$$= \frac{50000 \times 4000}{625\ \pi \times 4.6} = \mathbf{2.214 \times 10^4\ N/mm^2.}$$

1. Crippling load for both ends hinged,

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2.214 \times 10^4 \times 30.68 \times 10^4}{4000^2}$$

$$= 4189.99\ N = \mathbf{4190\ N}$$

2. Safe load,

Safe load = Crippling load / Factor of safety

= 4190 / 4

= **1047.5 N.**

6. A hollow alloy tube 5 m long with external diameter and internal diameters 40 mm and 25 mm respectively was found to extend 6.4 mm under a tensile load of 60 KN. Find the buckling load for the tube when used as a column with both ends pinned. Also find the safe load for the tube, taking a factor of safety = 4.

GIVEN DATA:

Length, L= 5 m = 5000 mm

External diameter, D = 40 mm.

Internal diameter, d = 25 mm.

Extension of bar, δL= 6.4 mm.

Tensile load, W = 60 KN.

Factor of safety = 4.

SOLUTION:

$$A = \frac{\pi \times (D^2 - d^2)}{4} = \frac{\pi \times (40^2 - 25^2)}{4} = \mathbf{766\ mm^2.}$$

$$I_{XX} = I = \frac{\pi \times (D^4 - d^4)}{64} = \frac{\pi \times (40^4 - 25^4)}{64} = \mathbf{106500\ mm^4.}$$

Young's Modulus, E = Tensile Stress/Tensile Strain

$$= \frac{\text{Tensile Load/Area}}{\text{Change in length of bar/Original length of bar}}$$

$$= \frac{W/A}{\delta L/L} = \frac{W \times L}{A \times \delta L}$$

$$= \frac{60000 \times 5000}{766 \quad\quad 6.4} = \mathbf{6.11945 \times 10^4\ N/mm^2.}$$

1. Crippling load for both ends pinned,

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 X 6.11945 X 10^4 X 106500}{5000^2}$$

$$= \mathbf{2573\ N}$$

2. Safe load,

Safe load = Crippling load / Factor of safety

= 2573 / 4

= **643.2 N.**

7. Calculate the safe compressive load on a hollow cast iron column (one end fixed and other hinged) of 15 cm external diameter, 10 cm internal diameter and 10 m in length. Use Euler's formula with a factor of safety of 5 and E = 95 KN/mm^2.

GIVEN DATA:

Length, L= 10 m = 10000 mm

External diameter, D = 15 cm = 150 mm

Internal diameter, d = 10 cm = 100 mm

Factor of safety = 5

E = 95 KN/mm^2

SOLUTION:

$$I = \frac{\pi X (D^4 - d^4)}{64} = \frac{\pi X (150^4 - 100^4)}{64} = \mathbf{1994.175 X 10^4 mm^4.}$$

1. Crippling load for one end fixed and other hinged,

$$P = \frac{2\pi^2 EI}{l^2} = \frac{2 X \pi^2 X 95 X 10^3 X 1994.175 X 10^4}{10000^2}$$

$$= 373950\ N = \mathbf{373.95\ KN.}$$

2. Safe load,

Safe load = Crippling load / Factor of safety

= 373.95 / 5

= **74.79 KN.**

8. Determine Euler's crippling load for an I-section joist 40 cm X 20 cm X 1 cm and 5 m long which is used as a strut with both ends fixed. Take E = 2.1 X 10^5 N/mm^2.

GIVEN DATA:

Dimension of I-section = 40 cm X 20 cm X 1 cm

Length, l = 5 m = 5000 mm

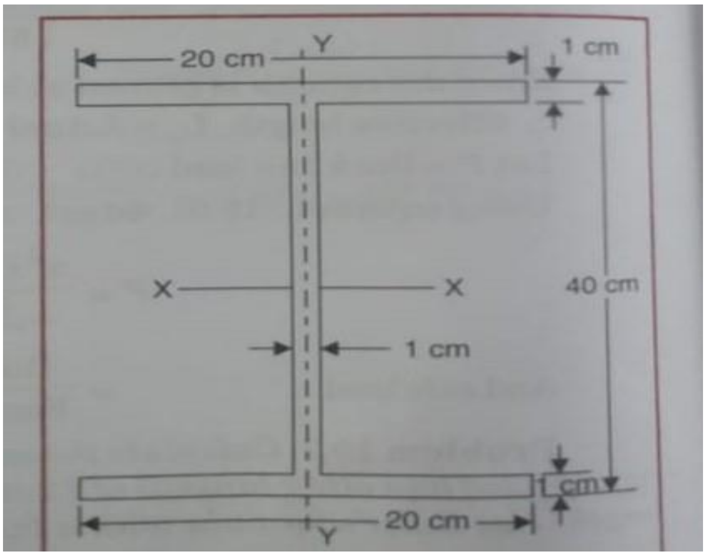
E = 2.1 X 10^5 N/mm^2.

SOLUTION:

I_{xx} = M.O.I of rectangle (20 X 40) – M.O.I of rectangle [(20 – 1) X (40 – 1 – 1)]

$$= \frac{b\,d^3}{12} - \frac{b_1\,d_1^3}{12} = \frac{20 \times 40^3}{12} - \frac{19 \times 38^3}{12}$$

= **19786 cm^4**



I_{YY} = M.O.I of rectangle (38 X 1) – M.O.I of 2 rectangles (1 X 20)

$$= \frac{d\,b^3}{12} - \frac{d_1\,b_1^3}{12} = \frac{38 \times 1^3}{12} - \frac{2 \times 1 \times 20^3}{12}$$

= **1336.5 cm^4**

Least value of the moment of inertia is about YY axis.

$I = I_{YY} = 1336.5\ cm^4 = 1336.5 \times 10^4\ mm^4$

1.Euler's crippling load for both ends fixed,

$$P = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 2.1 \times 10^5 \times 1336.5 \times 10^4}{3000^2}$$

$= 4432080$ N = **4432.08 KN.**

9. Using Euler's formula, calculate the critical stresses for a series of struts having slenderness ratio of 40,80,120,160 and 200 under the following conditions:

Both ends hinged

Both ends fixed. Take E = 2.05 X 10^5 N/mm^2.

GIVEN DATA:

slenderness ratio, l/k = 40,80,120,160 and 200

$E = 2.05 \times 10^5\ N/mm^2$.

SOLUTION:

1. Critical stress for both ends hinged,

$$\text{Critical stress} = \frac{\pi^2 E}{(L_e/k)^2} = \frac{\pi^2 E}{(l/k)^2}$$

When l/k = 40, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{\pi^2 \times 2.05 \times 10^5}{40^2}$

= **1264.54 N/mm^2.**

When l/k = 80, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{\pi^2 \times 2.05 \times 10^5}{80^2}$

$= 316.135\ N/mm^2$.

When l/k = 120, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{\pi^2 \times 2.05 \times 10^5}{120^2}$

$$= \mathbf{140.5\ N/mm^2.}$$

When l/k = 160, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{\pi^2 \times 2.05 \times 10^5}{160^2}$

$$= \mathbf{79.03\ N/mm^2.}$$

When l/k = 200, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{\pi^2 \times 2.05 \times 10^5}{200^2}$

$$= \mathbf{50.58\ N/mm^2.}$$

2. Critical stress for both ends fixed,

Critical stress $= \dfrac{\pi^2 E}{(L_e/k)^2} = \dfrac{\pi^2 E}{[(l/2)/k]^2} = \dfrac{4\pi^2 E}{(l/k)^2}$

When l/k = 40, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{4 \times \pi^2 \times 2.05 \times 10^5}{40^2}$

$$= \mathbf{5058.16\ N/mm^2.}$$

When l/k = 80, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{4 \times \pi^2 \times 2.05 \times 10^5}{80^2}$

$$= \mathbf{1264.54\ N/mm^2.}$$

When l/k = 120, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{4 \times \pi^2 \times 2.05 \times 10^5}{120^2}$

$$= \mathbf{562.02\ N/mm^2.}$$

When l/k = 160, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{4 \times \pi^2 \times 2.05 \times 10^5}{160^2}$

$$= \mathbf{316.135\ N/mm^2.}$$

When l/k = 200, critical stress $= \dfrac{\pi^2 E}{(l/k)^2} = \dfrac{4 \times \pi^2 \times 2.05 \times 10^5}{200^2}$

$$= \mathbf{202.32\ N/mm^2.}$$

RANKINE'S FORMULA:

$$P = \frac{\sigma_C \times A}{1 + a\,(L_e/k)^2}$$

Where, σ_C = Ultimate crushing stress

A = Area of cross section

a = Rankine's constant.

L_e = Effective length

k = Least radius of gyration.

S. No.	Material	σ_c in N/mm^2	a
1.	Wrought Iron	250	$\frac{1}{9000}$
2.	Cast Iron	550	$\frac{1}{1600}$
3.	Mild Steel	320	$\frac{1}{7500}$
4.	Timber	50	$\frac{1}{750}$

10. The external and internal diameter of a hollow cast iron column are 5 cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take σ_C = 550 N/mm^2 and a = 1/1600.

GIVEN DATA:

Length, L= 3 m = 3000 mm

External diameter, D = 5 cm = 50 mm.

Internal diameter, d = 4 cm = 40 mm.

Ultimate crushing stress, σ_C = 550 N/mm^2

Rankine's constant, a = 1/1600.

SOLUTION:

1.Cripling load for both ends fixed,

$A = \frac{\pi \times (D^2 - d^2)}{4} = \frac{\pi \times (50^2 - 40^2)}{4} = \mathbf{225\ \pi\ mm^2.}$

$I_{XX} = I = \frac{\pi \times (D^4 - d^4)}{64} = \frac{\pi \times (50^4 - 40^4)}{64} = \mathbf{57656\ \pi\ mm^4.}$

Least radius of gyration, k = I/A = 57656 π / 225 π = **16.007 mm.**

For both the ends are fixed, L = l/2 = 3000/2 = **1500 mm.**

$$P = \frac{\sigma_C \times A}{1 + a\,(L_e/k)^2} = \frac{550 \times 225\,\pi}{1 + 1/1600\,(1500/16.007)^2}$$

= **59918.3 N.**

11. A hollow cylindrical cast iron column is 4 m long with both ends fixed. Determine the minimum diameter of the column if it has to carry a safe load of 250 KN with a factor of safety of 5. Take the internal diameter as 0.8 times the external diameter. Take σ_C = 550 N/mm^2 and a = 1/1600.

GIVEN DATA:

Length, L= 4 m = 4000 mm

Internal diameter, d = 0.8 X D

Safe load = 250 KN

Ultimate crushing stress, σ_C = 550 N/mm^2

Rankine's constant, a = 1/1600, Factor of safety = 5.

SOLUTION:

Safe load = Crippling load / Factor of safety

Crippling load = Safe load X Factor of safety = 250 X 5 = **1250 KN.**

Area of column, $A = \frac{\pi}{4}[D^2 - (0.8D)^2]$

$$= \frac{\pi}{4}[D^2 - 0.64D^2] = \frac{\pi}{4} \times 0.36D^2 = \pi \times 0.09D^2$$

Moment of Inertia, $I = \frac{\pi}{64}[D^4 - (0.8D)^4] = \frac{\pi}{64}[D^4 - 0.4096D^4]$

$$= \frac{\pi}{64} \times 0.5904 \times D^4 = 0.009225 \times \pi \times D^4$$

But $I = A \times k^2$, where k is radius of gyration

$\therefore$ $$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.009225 \times \pi D^4}{\pi \times 0.09 \times D^2}} = 0.32D$$

Now using equation (19.9), $P = \dfrac{\sigma_c \, . \, A}{1 + a\left(\dfrac{L_e}{k}\right)^2}$

$$1250000 = \frac{550 \times \pi \times 0.09\,D^2}{1 + \dfrac{1}{1600} \times \left(\dfrac{2000}{0.32D}\right)^2} \qquad (\because \quad A = \pi \times 0.09D^2)$$

$$\frac{1250000}{550 \times \pi \times 0.09} = \frac{D^2}{1 + \dfrac{24414}{D^2}} \quad \text{or} \quad 8038 = \frac{D^2 \times D^2}{D^2 + 24414}$$

$8038D^2 + 8038 \times 24414 = D^4$ or $D^4 - 8038D^2 - 8038 \times 24414 = 0$

$D^4 - 8038\,D^2 - 196239700 = 0.$

The above equation is a quadratic equation in D^2. The solution is

$\therefore$ $$D^2 = \frac{8038 \pm \sqrt{8038^2 + 4 \times 1 \times 196239700}}{2}$$

$$\left(\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{8038 \pm \sqrt{646094 + 784958800}}{2} = \frac{8038 \pm 29147}{2}$$

$$= \frac{8038 + 29147}{2} \text{ (The other root is not possible)}$$

$$= 18592.5 \text{ mm}^2$$

$\therefore$ $$D = \sqrt{18592.5} = 136.3 \text{ mm}$$

$\therefore$ External diameter $= \mathbf{136.3}$ **mm. Ans.**

Internal diameter $= 0.8 \times 136.3 = \mathbf{109}$ **mm. Ans.**

12. A 1.5 m long column has a circular cross section of 5 cm diameter. One of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using, Rankine's formula, take σ_C = 560 N/mm^2 and a = 1/1600, Euler's formula, E = 1.2 X 10^5 N/mm^2.

GIVEN DATA:

Length, L= 1.5 m = 1500 mm

Diameter, d = 5 cm

Ultimate crushing stress, σ_C = 560 N/mm^2

Rankine's constant, a = 1/1600.

E = 1.2 X 10^5 N/mm^2.

Factor of safety = 3

SOLUTION:

$A = \pi \times d^2/4 = \pi \times 50^2/4 =$ **1963.5 mm^2.**

$I = \pi \times d^4/64 = \pi \times 50^4/64 =$ **30.7 X 10^4 mm^4.**

Least radius of gyration, k = I/A = 30.7 X 10^4 / 1963.5 = **12.5 mm.**

$L_e = 2\,l = 2 \times 1500 =$ **3000 mm**

1. Safe load using Rankine's formula,

$$P = \frac{\sigma_C \times A}{1 + a\,(L_e/k)^2} = \frac{560 \times 1963.5}{1 + 1/1600\,(3000/12.5)^2}$$

= **29717.8 N.**

Safe load = Crippling load / Factor of safety

= 29717.8 / 3

= **9905.9 N.**

2. Safe load using Euler's formula,

$$P = \frac{\pi^2 EI}{4l^2} = \frac{\pi^2 \times 1.2 \times 10^5 \times 30.7 \times 10^4}{4 \times 1500^2}$$

$= \mathbf{40399.5\ N}$

Safe load = Crippling load / Factor of safety

$= 40399.5 / 3$

$= \mathbf{13466.5\ N.}$

13. A hollow cast iron column 200 mm outside diameter and 150 mm inside diameter, 8 m long has both ends fixed. It is subjected to an axial compressive load. Taking a factor of safety as 6, $\sigma_C = 560\ N/mm^2$ and a = 1/1600. Determine the safe Rankine load.

GIVEN DATA:

Length, L= 8 m = 8000 mm, $L_e = l/2 = 8000/2 = 4000$ mm.

External diameter, D = 200 mm

Internal diameter, d = 150 mm

Ultimate crushing stress, $\sigma_C = 560\ N/mm^2$

Rankine's constant, a = 1/1600

Factor of safety = 6

SOLUTION:

1.Safe Rankine's load,

$$A = \frac{\pi \times (D^2 - d^2)}{4} = \frac{\pi \times (200^2 - 150^2)}{4} = \mathbf{13744\ mm^2.}$$

$$I_{XX} = I = \frac{\pi \times (D^4 - d^4)}{64} = \frac{\pi \times (200^4 - 150^4)}{64} = \mathbf{53689000\ mm^4.}$$

Least radius of gyration, k = I/A = 53689000/ 13744 = **62.5 mm.**

$$P = \frac{\sigma_C \times A}{1 + a (L_e/ k)^2} = \frac{560 \times 13744}{1 + 1/1600\ (4000/62.5)^2}$$

$= 2161977\ N = \mathbf{2161.977\ KN.}$

Safe load = Crippling load / Factor of safety

$= 2161.977 / 6$

$= \mathbf{360.3295\ KN.}$

14. A hollow C.I column whose outside diameter is 200 mm has a thickness of 20 mm. It is 4.5 m long and is fixed at both ends. Calculate the safe load by Rankine's formula using a factor of safety of 4. Calculate the slenderness ration and the ratio of Euler's and Rankine's critical loads. Take σ_C = 550 N/mm^2 and a = 1/1600 and E = 9.4 X 10^4 N/mm^2.

GIVEN DATA:

Length, L= 4.5 m = 4500 mm, L_e = l/2 = 4500/2 = 2250 mm.

External diameter, D = 200 mm, Thickness, t = 20 mm.

Internal diameter, d = D – 2 X t = 200 – 2 X 20 = 160 mm

Ultimate crushing stress, σ_C = 550 N/mm^2

Rankine's constant, a = 1/1600

Factor of safety = 4

E = 9.4 X 10^4 N/mm^2.

SOLUTION:

$$A = \frac{\pi \times (D^2 - d^2)}{4} = \frac{\pi \times (200^2 - 160^2)}{4} = \mathbf{11310\ mm^2.}$$

$$I_{XX} = I = \frac{\pi \times (D^4 - d^4)}{64} = \frac{\pi \times (200^4 - 160^4)}{64} = \mathbf{46370000\ mm^4.}$$

Least radius of gyration, k = I/A = 46370000 / 11310 = **64 mm.**

1.Slenderness ratio,

Slenderness ratio = l/k = 4500/64 = **70.30**

2. Safe load by Rankine's formula,

$$P = \frac{\sigma_C \times A}{1 + a\,(L_e/\,k)^2} = \frac{550 \times 11310}{1 + 1/1600\,(2250/64)^2}$$

$$= \mathbf{3511000\ N.}$$

Safe load = Crippling load / Factor of safety

$= 3511000 / 4$

$=$ **877750 N.**

3. Ratio of Euler's and Rankine's critical loads,

$$\text{Euler's critical load, } P = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 9.4 \times 10^4 \times 46370000}{4500^2}$$

$=$ **8497700 N.**

$$\frac{\text{Euler's critical load}}{\text{Rankine's critical load}} = \frac{8497700}{3511000} = \mathbf{2.42}$$

I.S CODE FORMULA

Slenderness, ratio = $\frac{L_e}{k}$	Safe stress, σ_c (N/mm^2)	Slenderness ratio = $\frac{L_e}{k}$	Safe stress, σ_c (N/mm^2)
0	1.25	150	47.4
10	124.6	160	42.3
20	123.9	170	37.3
30	122.4	180	33.6
40	120.3	190	30.0
50	117.2	200	27.0
60	113.0	210	24.3
70	107.5	220	21.9
80	100.7	230	19.9
90	92.8	240	18.1
100	84.0	250	16.6
110	75.3	300	19.9
120	67.1	350	7.6
130	59.7		
140	53.1		

15. Determine the safe load by I.S. code for a hollow cylindrical mild steel tube of 4 cm external diameter and 3 cm internal diameter when the tube is used as a column of length 2.5 m long with both ends hinged.

GIVEN DATA:

External diameter – 4 cm = 40 mm

Internal diameter – 3 cm = 30 mm

Length – 2.5 m = 2500 mm

SOLUTION:

1.Safe load by I.S. CODE formula,

$A = \frac{\pi \times (D^2 - d^2)}{4} = \frac{\pi \times (40^2 - 30^2)}{4} = \mathbf{549.77\ mm^2.}$

$I_{XX} = I = \frac{\pi \times (D^4 - d^4)}{64} = \frac{\pi \times (40^4 - 30^4)}{64} = \mathbf{85902.92\ mm^4.}$

To determine the safe load by I.S. code formula, first find the value of slenderness ratio. Then according to slenderness ratio, obtain the safe compressive stress from the table.

Now, slenderness ratio = l/k

k = I/A = 85902.92 / 549.77 = **12.5 mm.**

slenderness ratio = 2500/12.5 = **200.**

From table, corresponding to slenderness ratio of 200, the allowable compressive stress is 27 N/ mm^2.

$\boldsymbol{\sigma_C = 27\ N/mm^2.}$

Safe load for the column = σ_C X A

= 27 X 549.77

= **14843.79 N.**

SECANT FORMULA

Maximum stress

$$\boxed{\sigma_{\max} = \frac{P}{A}\left[1 + \frac{e.y}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right]}$$

Perry-Robertson Formula

The formula used for structural steelwork is the Perry-Robertson formula that represented as the average end stress to cause yield in a strut.

$$\sigma_p = \tfrac{1}{2}\left[\sigma_{\text{yield}} + (\eta+1)\sigma_E\right] - \sqrt{\left[\tfrac{1}{2}\sigma_{\text{yield}} + \tfrac{1}{2}(\eta+1)\sigma_E\right]^2 - \sigma_{\text{yield}}\sigma_E}$$

where:

σ_p = stress based on Perry-Robertson formula.

σ_E = Euler's stress

σ_{yield} = yield stress depending on the yield strength of material

η = constant depending on the material. For a brittle material η = 0.015L/r,

for a ductile material $\eta = 0.3\ (L/100r)^2$

COLUMNS WITH ECCENTRIC LOADING

$$\sigma_{max} = \sigma_0 + \sigma_b = \frac{P}{A} + \frac{P \times e \sec\left(l \times \sqrt{\frac{P}{EI}}\right)}{Z}$$

Problem 19.22. *A column of circular section is subjected to a load of 120 kN. The load is parallel to the axis but eccentric by an amount of 2.5 mm. The external and internal diameters of columns are 60 mm and 50 mm respectively. If both the ends of the column are hinged and column is 2.1 m long, then determine the maximum stress in the column. Take E = 200 GN/m².*

Sol. Given :

Load, $P = 120$ kN $= 120 \times 10^3$ N

Eccentricity, $e = 2.5$ mm $= 2.5 \times 10^{-3}$ m

$D = 60$ mm $= 0.06$ m, $d = 50$ mm $= 0.05$ m, $l = 2.1$ m

Both ends are hinged, $L_e = l = 2.1$ m

Value of $E = 200$ GN/m^2 $= 200 \times 10^9$ N/m^2

The maximum stress is given by equation (19.13) as

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)}{Z} \qquad \ldots(i)$$

where A = Area of section

$$= \frac{\pi}{4}[D^2 - d^2] = \frac{\pi}{4}[0.06^2 - 0.05^2]$$

$$= \frac{\pi}{4} \times 0.0011 = 8.639 \times 10^{-4}\ \text{m}^2$$

$$I = \text{Moment of inertia} = \frac{\pi}{64}(D^4 - d^4)$$

$$= \frac{\pi}{64}(0.06^4 - 0.05^4)\ \text{mm}^4$$

$$= \frac{\pi}{64}(1.296 \times 10^{-5} - 0.625 \times 10^{-5}) = 0.0329 \times 10^{-5}$$

Z = Section modulus

$$= \frac{I}{y} = \frac{\frac{\pi}{64}[D^4 - d^4]}{\left(\frac{D}{2}\right)} = \frac{\frac{\pi}{64}[0.06^4 - 0.05^4]}{0.03}$$

$$= \frac{\pi(1.296 \times 10^{-5} - 0.625 \times 10^{-5}]}{64 \times 0.03}$$

$$= 1.0975 \times 10^{-5} \text{ m}^3$$

$$\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right) = \sec\left(\frac{2.1}{2} \times \sqrt{\frac{120 \times 10^3}{200 \times 10^9 \times 0.0329 \times 10^{-5}}}\right)$$

$= \sec(1.4179 \text{ radians})$ (Here 1.4179 is in radians $= 1.4179 \times \frac{180}{\pi} = 81.239$)

$$= \sec(81.239) = 6.566$$

Substituting these values in equation (i) above, we get

$$\sigma_{max} = \frac{120 \times 10^3}{8.639 \times 10^{-4}} + \frac{(120 \times 10^3) \times (2.5 \times 10^{-3}) \times 6.566}{1.0975 \times 10^{-5}}$$

$$= 138.9 \times 10^6 + 179.48 \times 10^6 \text{ N/m}^2$$

$= \mathbf{318.38 \times 10^6\ N/m^2}$ **or** **318.38 N/mm² Ans.**

Problem 19.23. *If the given column of problem 19.22 is subjected to an eccentric load 100 kN and maximum permissible stress is limited to 320 MN/m², then determine the maximum eccentricity of the load.*

Sol. Given :

Data from problem 19.22

$D = 60$ mm $= 0.06$ m, $d = 50$ mm $= 0.05$ m, $l = 2.1$ m, $L_e = l = 2.1$ m,

$E = 200$ GN/m² $= 200 \times 10^9$ N/m²., $I = 0.0329 \times 10^{-5}$ m⁴,

$Z = 1.0975 \times 10^{-5}$ m³, $A = 8.639 \times 10^{-4}$ m²

Eccentric load, $P = 100$ kN $= 100 \times 10^3$ N

Max. stress, $\sigma_{max} = 320$ MN/m² $= 320 \times 10^6$ N/m²

Let e = Maximum eccentricity

Using equation (19.13), we get

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)}{Z}$$

Let us first find the value of $\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)$.

$$\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right) = \sec\left[\frac{2.1}{2} \times \sqrt{\frac{100 \times 10^3}{200 \times 10^9 \times 0.0329 \times 10^{-5}}}\right]$$

$$= \sec[1.294 \text{ rads}] = \sec\left(1.294 \times \frac{180^\circ}{\pi}\right)$$

$$= \sec(74.16^\circ) = 3.665$$

Substituting the known values in equation (*i*), we get

$$320 \times 10^6 = \frac{100 \times 10^3}{8.639 \times 10^{-4}} + \frac{(100 \times 10^3) \times e \times 3.665}{1.0975 \times 10^{-5}}$$

$$= 115.754 \times 10^6 + 33394\, e \times 10^6$$

$$320 = 115.754 + 33394 \times e$$

$$e = \frac{320 - 115.754}{33394} \text{ m} = 6.116 \times 10^{-3} \text{ m} = \mathbf{6.116\ mm.} \quad \text{Ans.}$$

REFERENCE BOOKS:

1. Bansal R.K., “Strength of Materials”, 5th Edition, Laxmi Publications, 2013.
2. Ramamrutham R., “Strength of Materials”,14th Edition, Dhanpat Rai Publications, 2011.
3. Punmia B.C., “Strength of Materials”, Laxmi Publications, 2011.
4. Subramanian R., “Strength of Materials - Theory and Problems”, Oxford University Press, 2010.
5. Rajput R.K., “Strength of Materials”, S. Chand and Company Ltd., New Delhi, 2007.
6. Timoshenko S.P. and Gere J.M., “Mechanics of Materials”, 6th Edition, Tata McGraw Hill.

QUESTION BANK:

1. Distinguish columns & struts.
2. List the assumptions made in Euler’s column theory.
3. What are the different end conditions of a column?
4. Define: slenderness ratio.
5. List the limitations of Euler's formula.
6. Define: Crippling load.
7. What is Equivalent length of a column?
8. What is a Rankine's constant? What is the approximate value of Rankine's constant for cast iron column?
9. Define: Safe load.
10. A column of timber section 10 cm X 15 cm is 5 m long both ends being fixed. If the young's modulus for timber = 17.5 KN/mm2. Determine crippling load and safe load for the column if factor of safety = 3.
11. A hollow mild steel tube 5 m long, 4 cm internal diameter and 5 mm thick is used as a strut with both ends hinged. Determine the crippling load and safe load taking factor of safety as 3.0. Take E = 2 X 105 N/mm2.
12. Determine Euler's crippling load for an I - section joist 30 cm X 15 cm X 2 cm and 5 m long which is used as a strut with both ends fixed. Take E = 2 X 105 N/mm2.
13. Determine the crippling load for a T - section of dimensions 12 cm X 12 cm X 2 cm and of length 6 cm when it is used as a strut with both of its ends hinged. Take E = 2 X 105 N/mm2.
14. Determine the ratio of buckling strengths of two columns one hollow and the other solid. Both are made of the same material and have the same length, cross sectional area and end conditions. The internal diameter of hollow column is 2/3rd of its external diameter.
15. A 2 m long column has a circular cross section of 6 cm diameter. One of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, determine the safe load using

    i) Rankine's formula taking yield stress = 550 N/mm2 and a = 1/1600 for pinned ends.

SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – IV – THIN CYLINDERS AND SPHERES, THICK CYLINDERS – SCIA1401

INTRODUCTION

THIN CYLINDER:

- ✓ The cylinder which have thickness is less than 1/10 to 1/20 of its Diameter, that cylinder is called as thin cylinder.
- ✓ Thin cylinder is only resist to the internal Pressure.
- ✓ Thin cylinder failure due to internal fluid pressure by the formation of circumferential stress and longitudinal stress.
- ✓ The internal pressure which is acting radially inside the thin cylinder is known as radial pressure in thin cylinder.

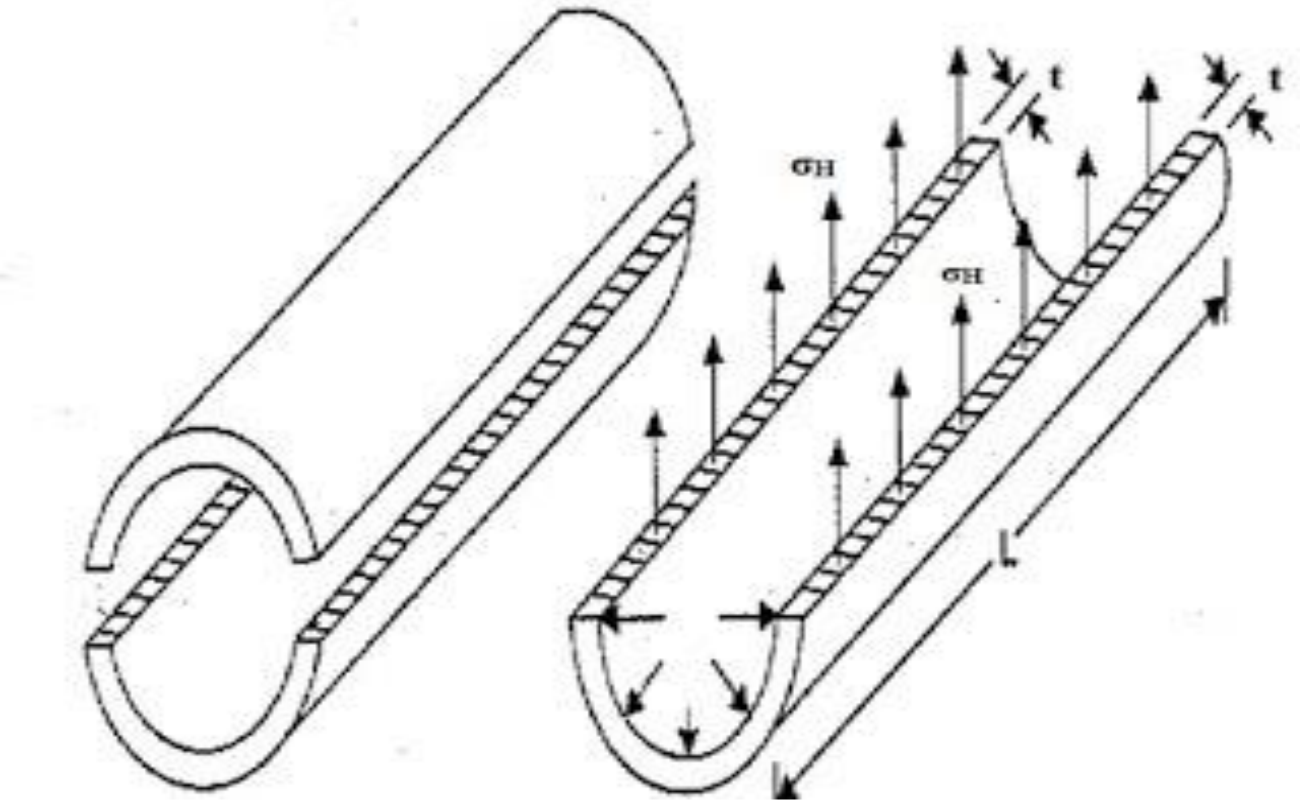
STRESSES IN A THIN CYLINDRICAL SHELL:

There will be two types of stresses, which will be developed in the wall of thin cylindrical shell and these stresses are as mentioned here.

CIRCUMFERENTIAL STRESS OR HOOP STRESS:

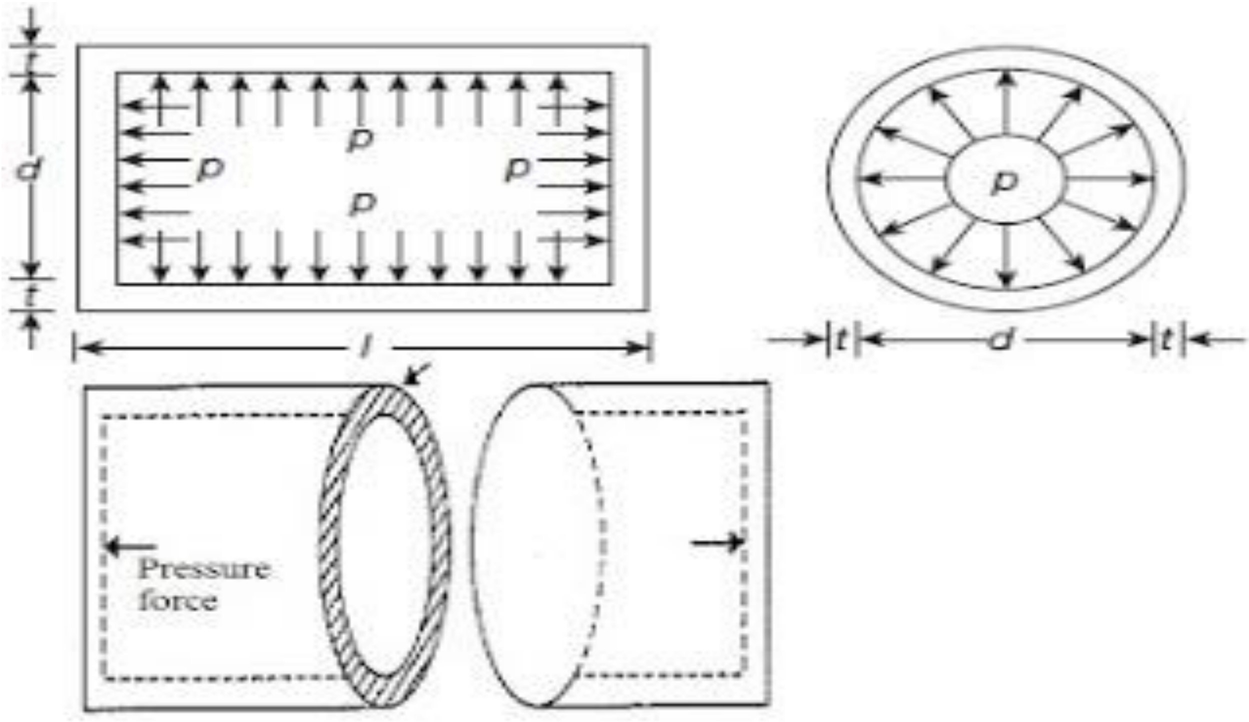
Stress acting along the circumference of thin cylinder will be termed as circumferential stress or hoop stress.

If fluid is stored under pressure inside the cylindrical shell, pressure will be acting vertically upward and downward over the cylindrical wall.



LONGITUDINAL STRESS:

- ✓ Stress acting along the length of thin cylinder will be termed as longitudinal stress.
- ✓ If fluid is stored under pressure inside the cylindrical shell, pressure force will be acting along the length of the cylindrical shell at its two ends.



EXPRESSION FOR CIRCUMFERENTIAL or HOOP STRESS:

$$\sigma_1 = \frac{p\,d}{2\,t}$$

Where, p – Internal fluid pressure

d – Internal diameter if the cylinder

t – Thickness of the wall of the cylinder

σ_1 – Circumferential or hoop stress in the material.

EXPRESSION FOR LONGITUDINAL STRESS:

$$\sigma_2 = \frac{p\,d}{4\,t}$$

σ_2 - Longitudinal stress in the material

Longitudinal stress= Half of the circumferential stress.

PROBLEMS

1.A cylindrical pipe of diameter 1.5 m and thickness 1.5 cm is subjected to an internal fluid pressure of 1.2 N/mm^2. Determine the longitudinal stress developed in the pipe and circumferential stress developed in the pipe.

GIVEN DATA:

Diameter - 1.5 m

Thickness - 1.5 cm = 0.015 m

Internal fluid pressure of 1.2 N/mm^2

SOLUTION:

1. LONGITUDINAL STRESS,

$$\sigma_2 = \frac{p\,d}{4\,t} = \frac{1.2 \times 1.5}{4 \times 0.015} = \mathbf{30\ N/mm^2}$$

2. CIRCUMFERENTIAL STRESS,

$$\sigma_1 = \frac{p\,d}{2\,t} = \frac{1.2 \times 1.5}{2 \times 0.015} = \mathbf{60\ N/mm^2}$$

2. A cylinder of internal diameter 2.5 m and of thickness 5 cm contains a gas. If the tensile stress in the material is not to exceed 80 N/mm^2, determine the internal pressure of the gas.

GIVEN DATA:

Internal diameter - 2.5 m

Thickness - 5 cm = 0.05 m

Tensile stress - 80 N/mm^2

As tensile stress is given, hence this should be equal to circumferential stress (σ_1), σ_1 = 80 N/mm^2

SOLUTION:

1.INTERNAL PRESSURE OF THE GAS,

$$\sigma_1 = \frac{p\,d}{2\,t}$$

$$p = \frac{\sigma_1 \times 2\,t}{d} = \frac{80 \times 2 \times 0.05}{2.5} = \mathbf{3.2\ N/mm^2}$$

3. A cylinder of internal diameter 0.5 m contains air at a pressure of 7 N/mm^2. If the maximum permissible stress induced in the material is 80 N/mm^2, find the thickness of the cylinder.

GIVEN DATA:

Internal diameter - 0.5 m

Pressure - 7 N/mm^2

Maximum permissible stress - 80 N/mm^2

As maximum permissible stress is given, hence this should be equal to circumferential stress (σ_1).

σ_1 - 80 N/mm^2

SOLUTION:

1.THICKNESS OF THE CYLINDER,

$$\sigma_1 = \frac{p\,d}{2\,t}$$

$$t = \frac{p\,d}{2 \times \sigma_1} = \frac{7 \times 0.5}{2 \times 80}$$

t = **0.021875 m or 2.18 cm.**

4. A thin cylinder of internal diameter 1.25 m contains a fluid at an internal pressure of 2 N/mm^2. Determine the maximum thickness of the cylinder if i) longitudinal stress is not to exceed 30 N/mm^2 ii) circumferential stress is not to exceed 45 N/mm^2.

GIVEN DATA:

Internal diameter, d - 1.25 m

Internal fluid pressure, p - 2 N/mm^2

Longitudinal stress, σ_2 = 30 N/mm^2

circumferential stress, σ_1 = 45 N/mm^2.

SOLUTION:

1.MAXIMUM THICKNESS OF THE CYLINDER if σ_1 = 45 N/mm^2,

$$\sigma_1 = \frac{p\,d}{2\,t}$$

$$t = \frac{p\,d}{2 X \sigma_1} = \frac{2 X 1.25}{2 X 45}$$

t = **0.0277 m or 2.77 cm.**

2. MAXIMUM THICKNESS OF THE CYLINDER if σ_2 = 30 N/mm^2,

$$\sigma_2 = \frac{p\,d}{4\,t}$$

$$t = \frac{p\,d}{4 X \sigma_1} = \frac{2 X 1.25}{4 X 30}$$

t = **0.0208 m or 2.08 cm.**

The longitudinal or circumferential stresses induced in the material are inversely proportional to the thickness of the cylinder. Hence the stress induced will be less if the value of ‘t’ is more. Hence take the maximum value of ‘t’.

t = **2.77 cm.**

5. A water main 80 cm diameter contains water at a pressure head of 100 m. If the weight density of water is 9810 N/m^3, find the thickness of the metal required for the water main. Given the permissible stress as 20 N/mm^2.

GIVEN DATA:

Diameter of main, d - 1.25 m

Pressure head of water, h = 100 m

Permissible stress, σ_1 = 20 N/mm^2

SOLUTION:

1.THICKNESS OF THE METAL,

б

Pressure of water inside the water main, $P = w \times h = 9810 \times 100$

$$= 981000 \text{ N/m}^2 = 0.981 \text{ N/mm}^2.$$

$$\sigma_1 = \frac{p\,d}{2\,t}$$

$$t = \frac{p\,d}{2 \times \sigma_1} = \frac{0.981 \times 80}{2 \times 20} = \mathbf{2\ cm.}$$

EFFICIENCY OF A JOINT

The cylindrical shells such as boilers are having two types of joints namely longitudinal joint and circumferential joint.

η_l - Efficiency of a longitudinal joint and

η_c - Efficiency of a circumferential joint.

Circumferential stress, $\sigma_1 = \dfrac{p\,d}{2\,t \times \eta_l}$

Longitudinal stress, $\sigma_2 = \dfrac{p\,d}{4\,t \times \eta_c}$

Efficiency of a joint means the efficiency of a longitudinal joint.

6. A boiler is subjected to an internal steam pressure of 2 N/mm^2. The thickness of boiler plate is 2 cm and permissible tensile stress is 120 N/mm^2. Find out the maximum diameter, when efficiency of longitudinal joint is 90 % and that of circumferential joint is 40 %.

GIVEN DATA:

Internal steam pressure, $p = 2$ N/mm^2

Thickness of boiler plate, $t = 2$ cm

Permissible tensile stress = 120 N/mm^2

$\eta_l = 90$ %

$\eta_c = 40$ %.

SOLUTION:

In case of a joint, the permissible stress may be circumferential stress or longitudinal stress.

1.MAXIMUM DIAMETER FOR CIRCUMFERENTIAL STRESS,

$$\sigma_1 = \frac{p\,d}{2\,t \times \eta_l}$$

$$120 = \frac{2 \times d}{2 \times 0.90 \times 2}$$

$$d = \frac{120 \times 2 \times 0.90 \times 2}{2} = \mathbf{216\ cm.}$$

2. MAXIMUM DIAMETER FOR LONGITUDINAL STRESS,

$$\sigma_2 = \frac{p\,d}{4\,t \times \eta_c}$$

$$120 = \frac{2 \times d}{4 \times 0.40 \times 2}$$

$$d = \frac{120 \times 4 \times 0.40 \times 2}{2} = \mathbf{192\ cm.}$$

The longitudinal or circumferential stresses induced in the material are directly proportional to diameter. Hence the stress induced will be less if the value of ‘d’ is less. Hence take the minimum value of d.

Maximum diameter of the boiler is equal to the minimum value of diameter.

Hence maximum diameter, **d = 192 cm.**

If d = 216 cm, σ_2 will be more than the given permissible stress.

$$\sigma_2 = \frac{p\,d}{4\,t \times \eta_c} = \frac{2 \times 216}{4 \times 2 \times 0.4} = \mathbf{135\ N/mm^2}$$

7. A cylinder of thickness 1.5 cm has to withstand maximum internal pressure of 1.5 N/mm^2. If the ultimate tensile stress in the material of the cylinder is 300 N/mm^2, factor of safety 3 and joint efficiency 80 %, determine the diameter of the cylinder.

GIVEN DATA:

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Thickness of cylinder, t = 1.5 cm

Internal pressure, p = 1.5 N/mm^2

Ultimate tensile stress = 300 N/mm^2

Factor of safety = 3

η = 80 %

Joint efficiency means the longitudinal joint efficiency, η_l = 80 %

SOLUTION:

$$\text{Working stress, } \sigma_1 = \frac{\text{Ultimate tensile stress}}{\text{Factor of safety}} = \frac{300}{3} = 100 \text{ N/mm}^2$$

$$\sigma_1 = \frac{p\,d}{2\,t \times \eta_l}$$

$$100 = \frac{1.5 \times d}{2 \times 0.80 \times 1.5}$$

$$d = \frac{100 \times 2 \times 0.80 \times 1.5}{1.5} = \mathbf{160\ cm\ or\ 1.6\ m.}$$

EFFECT OF INTERNAL PRESSURE ON THE DIMENSIONS OF A THIN CYLINDRICAL SHELL:

$$\text{Change in diameter, } \delta d = \frac{pd^2}{2tE}\left[1 - \frac{\mu}{2}\right]$$

$$\text{Change in length, } \delta L = \frac{pd\,L}{2tE}\left[\frac{1}{2} - \mu\right]$$

$$\text{Volumetric Strain, } \frac{\delta V}{V} = \frac{pd}{2tE}\left[\frac{5}{2} - 2\mu\right]$$

Change in volume, $\delta V = V\,[\,2\,e_1 + e_2]$

Change in volume, $\delta V = V \left[2 \frac{\delta d}{d} + \frac{\delta L}{L} \right]$

8. Calculate the change in diameter, change in length and change in volume of a thin cylindrical shell 100 cm diameter, 1 cm thick and 5 m long when subjected to internal pressure of 3 N/mm^2. Take the value of E = 2 X 10^5 N/mm^2 and poisson's ratio, μ = 0.3.

GIVEN DATA:

Diameter of shell, d = 100 cm

Thickness of shell, t = 1 cm

Length of shell, L = 5 m = 500 cm

Internal pressure, p = 3 N/mm^2

Young's modulus, E = 2 X 10^5 N/mm^2

Poisson's ratio, μ = 0.3.

SOLUTION:

Change in diameter, $\delta d = \frac{pd^2}{2tE} \left[1 - \frac{\mu}{2} \right]$

$$= \frac{3 \times 100^2}{2 \times 1 \times 2 \times 10^5} \left[\frac{1 - 0.3}{2} \right]$$

= **0.06375 cm.**

Change in length, $\delta L = \frac{pd\,L}{2tE} \left[\frac{1}{2} - \mu \right]$

$$= \frac{3 \times 100 \times 500}{2 \times 1 \times 2 \times 10^5} \left[\frac{1}{2} - 0.3 \right]$$

= **0.075 cm.**

Change in volume, $\delta V = V \left[2 \frac{\delta d}{d} + \frac{\delta L}{L} \right]$

$V = (\pi d^2/4) \times L = (\pi \times 100^2 /4) \times 500 = 3926990.817\ cm^3$

$$\delta V = 3926990.817 \times \left[\frac{2 \times 0.06375}{100} + \frac{0.075}{500} \right]$$

$= \mathbf{5595.96\ cm^3}$

9. A cylindrical thin drum 80 cm in diameter and 3 m long has a shell thickness of 1 cm. If the drum is subjected to an internal pressure of 2.5 N/mm^2. Determine the change in diameter, change in length and change in volume. Take the value of E = 2 X 10^5 N/mm^2 and poisson's ratio, μ = 0.25.

GIVEN DATA:

Diameter of drum, d = 80 cm

Thickness of shell, t = 1 cm

Length of shell, L = 3 m = 300 cm

Internal pressure, p = 2.5 N/mm^2

Young's modulus, E = 2 X 10^5 N/mm^2

Poisson's ratio, μ = 0.25.

SOLUTION:

Change in diameter, $\delta d = \frac{pd^2}{2tE} \left[1 - \frac{\mu}{2} \right]$

$$= \frac{2.5 \times 80^2}{2 \times 1 \times 2 \times 10^5} \left[\frac{1 - 0.25}{2} \right]$$

$= \mathbf{0.035\ cm.}$

Change in length, $\delta L = \frac{pd\,L}{2tE} \left[\frac{1}{2} - \mu \right]$

$$= \frac{2.5 \times 80 \times 300}{2 \times 1 \times 2 \times 10^5} \left[\frac{1}{2} - 0.25 \right]$$

= 0.0375 **cm.**

Change in volume, $\delta V = V\left(2\ \frac{\delta d}{d} + \frac{\delta L}{L}\right)$

$V = (\pi d^2/4) \times L = (\pi \times 80^2 /4) \times 300 = 1507964.473\ cm^3$

$\delta V = 1507964.473 \times \left(\frac{2 \times 0.035}{80} + \frac{0.0375}{300}\right)$

$=$ **$1507.96\ cm^3$**

10. A cylindrical vessel whose ends are closed by means of rigid flange plates is made of steel plate 3 mm thick. The length and the internal diameter of the vessel are 50 cm and 25 cm respectively. Determine the longitudinal and circumferential stresses in the cylindrical shell due to an internal fluid pressure of 3 $N/mm^{2.}$ Also calculate the increase in length, diameter and volume of the vessel. Take the value of E = 2 X 10^5 N/mm^2 and poisson's ratio, μ = 0.3.

GIVEN DATA:

Thickness, t = 3 mm = 0.3 cm

Length of cylindrical vessel, L = 50 cm

Internal diameter, d = 25 cm

Internal fluid pressure, p = 3 N/mm^2

Young's modulus, E = 2 X 10^5 N/mm^2

Poisson's ratio, μ = 0.25.

SOLUTION:

1.CIRCUMFERENTIAL STRESS,

$\sigma_1 = \frac{p\ d}{2\ t} = \frac{3 \times 25}{2 \times 0.3} =$ **$125\ N/mm^2$**

2. LONGITUDINAL STRESS,

$\sigma_2 = \frac{p\ d}{4\ t} = \frac{3 \times 25}{4 \times 0.3} =$ **$62.5\ N/mm^2$**

Change in diameter, $\delta d = \frac{pd^2}{2tE}\left(1 - \frac{\mu}{2}\right)$

$$= \frac{3 \times 25^2}{2 \times 0.3 \times 2 \times 10^5}\left(1 - \frac{0.3}{2}\right)$$

$$= \mathbf{0.0133\ cm.}$$

Change in length, $\delta L = \frac{pdL}{2tE}\left(\frac{1}{2} - \mu\right)$

$$= \frac{3 \times 25 \times 50}{2 \times 0.3 \times 2 \times 10^5}\left(\frac{1}{2} - 0.3\right)$$

$$= \mathbf{0.00625\ cm.}$$

Change in volume, $\delta V = V\left(\frac{2\,\delta d}{d} + \frac{\delta L}{L}\right)$

$$V = (\pi d^2/4) \times L = (\pi \times 25^2 /4) \times 50 = 24543.69\ cm^3$$

$$\delta V = 24543.69 \times \left(\frac{2 \times 0.0133}{25} + \frac{0.00625}{50}\right)$$

$$= \mathbf{29.18\ cm^3}$$

11. A cylindrical vessel is 1.5 m diameter and 4 m long is closed at ends by rigid plates. It is subjected to an internal pressure of 3 N/mm^2. If the maximum principal stress is not to exceed 150 N/mm^2, find the thickness of the shell. Assume E = 2 X 10^5 N/mm^2 and poisson's ratio, μ = 0.25. Find the changes in diameter, length and volume of the shell.

GIVEN DATA:

Diameter of vessel, d = 1.5 m = 1500 mm

Length of cylindrical vessel, L = 4 m = 4000 mm

Internal pressure, p = 3 N/mm^2

Maximum principal stress = 150 N/mm^2

Young's modulus, E = 2 X 10^5 N/mm^2

Poisson's ratio, μ = 0.25.

SOLUTION:

1.THICKNESS OF THE SHELL,

Maximum principal stress means circumferential stress.

$\sigma_1 = 150\ N/mm^2$

$$\sigma_1 = \frac{p\,d}{2\,t}$$

$$t = \frac{p\,d}{2 \times \sigma_1} = \frac{3 \times 1500}{2 \times 150}$$

t = **15 mm.**

2. CHANGES IN DIAMETER, LENGTH & VOLUME,

Change in diameter, $\delta d = \frac{pd^2}{2tE}\left(1 - \frac{\mu}{2}\right)$

$$= \frac{3 \times 1500^2}{2 \times 15 \times 2 \times 10^5}\left(1 - \frac{0.25}{2}\right)$$

= **0.984 mm.**

Change in length, $\delta L = \frac{pd\,L}{2tE}\left(\frac{1}{2} - \mu\right)$

$$= \frac{3 \times 1500 \times 4000}{2 \times 15 \times 2 \times 10^5}\left(\frac{1}{2} - 0.25\right)$$

= **0.75 mm.**

Volumetric Strain, $\frac{\delta V}{V} = \frac{pd}{2tE}\left(\frac{5}{2} - 2\mu\right)$

Change in volume, $\delta V = \frac{pd}{2tE}\left(\frac{5}{2} - 2\mu\right) \times V$

$$= \frac{3 \times 1500}{2 \times 2 \times 10^5 \times 15}\left(\frac{5}{2} - 2 \times 0.25\right) \times V$$

$V = (\pi d^2/4) \times L = (\pi \times 1500^2/4) \times 4000 = 7.0685 \times 10^9 \text{ mm}^3$

$$\delta V = \frac{3 \times 1500}{2 \times 2 \times 10^5 \times 15}\left(\frac{5}{2} - 2 \times 0.25\right) \times 7.0685 \times 10^9$$

$= \mathbf{10602875\ mm^3}$

WIRE WINDING OF THIN CYLINDERS:

- We know that the hoop stress is two times the longitudinal stress in a thin cylinder, when the cylinder is subjected to internal fluid pressure.
- Hence the failure of a thin cylinder will be due to hoop stress.
- Also, the hoop stress which is tensile in nature is directly proportional to the fluid pressure inside the cylinder.
- In case of cylinders which have to carry high internal fluid pressures, some methods of reducing the hoop stresses have to be devised.
- One method is to wind strong steel wire under tension on the walls of the cylinder.
- The effect of the wire is to put the cylinder wall under an initial compressive stress.

THIN SPHERICAL SHELLS:

- A thin spherical shell of internal diameter ‘d’ and thickness ‘t’ is subjected to an internal fluid pressure ‘p’.
- The fluid inside the shell has the tendency to split the shell into two hemi spheres along x-x axis.
- The force P which has a tendency to split the shell.

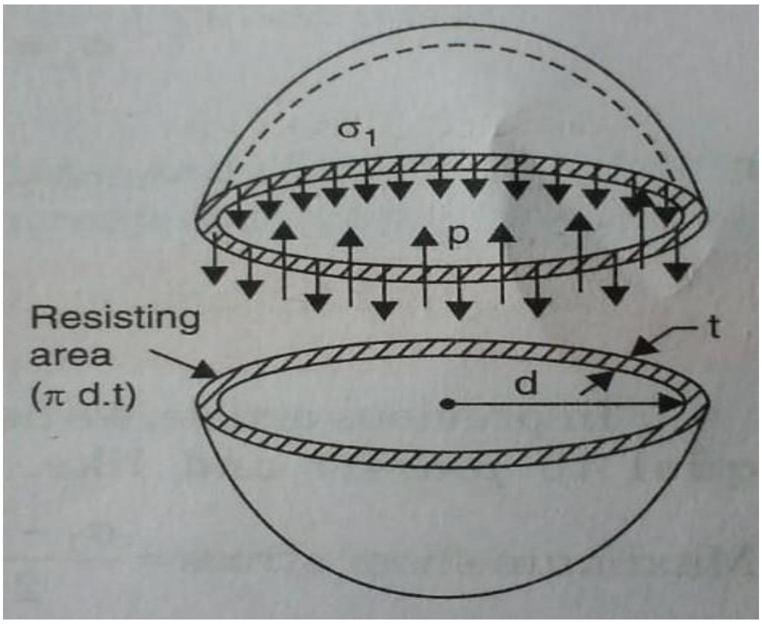
$$= p \times (\pi \times d^2/4)$$

The area resisting this force, $A = \pi \times d \times t$

Hoop stress induced in the material of the shell is given by,

$$\sigma_1 = \frac{\text{Force (P)}}{\text{Area resisting this force (A)}}$$

$$\sigma_1 = \frac{p \times (\pi \times d^2/4)}{\pi \times d \times t} = \frac{p\,d}{4\,t}$$



PROBLEMS:

1.A vessel in the shape of a spherical shell of 1.2 m internal diameter and 12 mm shell thickness is subjected to pressure of 1.6 N/mm^2 . Determine the stress induced in the material of the vessel.

GIVEN DATA:

Internal diameter of shell, d = 1.2 m = 1200 mm

Thickness of shell, t = 12 mm

Pressure, p = 1.6 N/mm^2

SOLUTION:

1. STRESS INDUCED IN THE MATERIAL OF THE VESSEL,

$$\sigma_1 = \frac{p\,d}{4\,t} = \frac{1.6 \text{ X } 1200}{4 \text{ X } 12} = \mathbf{40\ N/mm^2}$$

2.A spherical vessel 1.5 m diameter is subjected to an internal pressure of 2 N/mm^2. Find the thickness of the plate required if maximum stress is not to exceed 150 N/mm^2 and joint efficiency is 75 %.

GIVEN DATA:

Diameter of spherical vessel, d = 1.5 m = 1500 mm

Internal pressure = 2 N/mm^2

Maximum stress, σ_1 = 150 N/mm^2

Joint efficiency, η = 75 %.

SOLUTION:

$$\sigma_1 = \frac{p\,d}{4\,t \times \eta}$$

$$t = \frac{2 \times 1500}{4 \times 150 \times 0.75} = \mathbf{6.67\ mm}$$

CHANGE IN DIMENSIONS OF A THIN SPHERICAL SHELL DUE TO AN INTERNAL PRESSURE:

Strain, $\frac{\delta d}{d} = \frac{pd}{4tE}(1 - \mu)$

Change in diameter, $\delta d = \frac{pd}{4tE}(1 - \mu) \times d$

Volumetric strain, $\frac{\delta V}{V} = 3 \times \frac{pd}{4tE}(1 - \mu)$

Change in volume, $\delta V = 3 \times \frac{pd}{4tE}(1 - \mu) \times V$

3.A spherical shell of internal diameter 0.9 m and of thickness 10 mm is subjected to an internal pressure of 1.4 N/mm^2. Determine the increase in diameter and increase in volume. Take E = 2 X 10^5 N/mm^2 and poisson's ratio, μ = 1/3.

GIVEN DATA:

Internal diameter of spherical shell, d = 0.9 m = 900 mm

Thickness, t = 10 mm

Internal pressure, p = 1.4 N/mm^2

E = 2 X 10^5 N/mm^2

Poisson's ratio, μ = 1/3.

SOLUTION:

1. INCREASE IN DIAMETER,

Increase in diameter, $\delta d = \frac{pd}{4tE}(1 - \mu) \times d$

Increase in diameter, $\delta d = \frac{1.4 \times 900}{4 \times 10 \times 2 \times 10^5}(1 - (1/3)) \times 900$

$= \mathbf{0.0945\ mm.}$

2. INCREASE IN VOLUME,

Change in volume, $\delta V = 3 \times \frac{pd}{4tE}(1 - \mu) \times V$

Volume, $V = \pi \times d^3/6 = \pi \times 900^3/6 = \mathbf{381.7\ mm^3.}$

Change in volume, $\delta V = \frac{3 \times 1.4 \times 900}{4 \times 10 \times 2 \times 10^5}(1 - (1/3)) \times 381.7$

$= \mathbf{12028.5\ mm^3.}$

THICK CYLINDER:

- The cylinder which have Thickness is more than 1/20 of its diameter that Cylinder is called as thick Cylinder.
- If the ratio of thickness to internal diameter is more than 1/20, then cylindrical shell is known as thick cylinders.

STRESSES PRODUCED DUE TO INTERNAL FLUID PRESSURE:

- ✓ Radial pressure p (Compressive)
- ✓ Circumferential stress or Hoop stress σ_1 (Tensile)
- ✓ Longitudinal stress σ_2 (Tensile)

Radial Pressure: $p_x = \frac{b}{x^2} - a$ …………….(1)

Hoop Stress: $\sigma_x = \frac{b}{x^2} + a$ ………………..(2)

Above 2 equations are called Lame's equations.

The constants 'a' and 'b' are obtained from boundary conditions.

1. At $x = r_1$, $p_x = p_0$ or the pressure of fluid inside the cylinder.
2. At $x = r_2$, $p_x = 0$ or atmospheric pressure.

PROBLEMS:

1.Determine the maximum and minimum stress across the section of a pipe of 400 mm internal diameter and 100 mm thick, when the pipe contains a fluid at a pressure of 8 N/mm^2. Also sketch the radial pressure distribution and hoop stress distribution across the section.

GIVEN DATA:

Internal diameter, $d_1 = 400$ mm

Internal radius, $r_1 = 400/2 = 200$ mm

External diameter, $d_2 = 400 + 2 \times 100 = 600$ mm

External radius, $r_2 = 600/2 = 300$ mm

Fluid pressure, $p_0 = 8$ N/mm^2

SOLUTION:

1. MAXIMUM AND MINIMUM STRESS,

The radial pressure $p_x = \frac{b}{x^2} - a$ ……………(1)

Now apply the boundary conditions to the above equation.

The boundary conditions are:

1. At $x = r_1 = 200$ mm, $p_x = p_0 = 8$ N/mm^2
2. At $x = r_2 = 300$ mm, $p_x = 0$.

Substituting these boundary conditions in equation (1), we get

$$8 = \frac{b}{200^2} - a = \frac{b}{40000} - a \text{ …………(2)}$$

$$0 = \frac{b}{300^2} - a = \frac{b}{90000} - a \;\ldots\ldots\ldots\ldots..(3)$$

Subtracting equation 3 from equation 2, we get

$$8 - 0 = \frac{b}{40000} - a - \frac{b}{90000} + a$$

$$8 = \frac{9b - 4b}{360000} = \frac{5\,b}{360000}$$

$$b = \frac{360000 \text{ X } 8}{5} = \mathbf{576000}$$

Substituting this value in equation (3), we get

$$0 = \frac{576000}{90000} - a$$

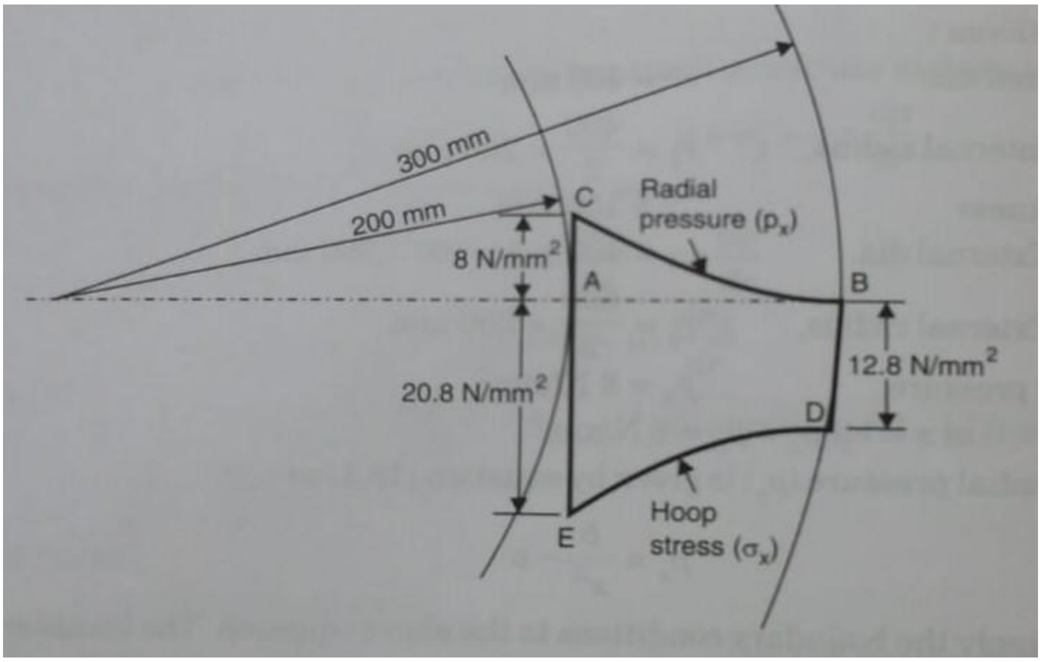
$$a = 576000/90000 = \mathbf{6.4}$$

The values of ‘a’ and ‘b’ are substituted in the hoop stress.

$$\text{Hoop Stress: } \sigma_x = \frac{b}{x^2} + a = \frac{576000}{x^2} + 6.4$$

$$\text{At } x = 200 \text{ mm, } \sigma_{200} = \frac{576000}{200^2} + 6.4 = \mathbf{20.8\ N/mm^2}$$

$$\text{At } x = 300 \text{ mm, } \sigma_{300} = \frac{576000}{300^2} + 6.4 = \mathbf{12.8\ N/mm^2}$$


300 mm
200 mm
C
Radial
pressure (p_x)
8 N/mm²
A
B
12.8 N/mm²
20.8 N/mm²
D
Hoop
stress (σ_x)
E

REFERENCE BOOKS:

1. Bansal R.K., “Strength of Materials”, 5th Edition, Laxmi Publications, 2013.
2. Ramamrutham R., “Strength of Materials”,14th Edition, Dhanpat Rai Publications, 2011.
3. Punmia B.C., “Strength of Materials”, Laxmi Publications, 2011.
4. Subramanian R., “Strength of Materials - Theory and Problems”, Oxford University Press, 2010.
5. Rajput R.K., “Strength of Materials”, S. Chand and Company Ltd., New Delhi, 2007.
6. Timoshenko S.P. and Gere J.M., “Mechanics of Materials”, 6th Edition, Tata McGraw Hill.

QUESTION BANK:

1. Distinguish thin and thick cylinder.
2. Name the stresses set up in a thin cylinder subjected to internal fluid pressure.
3. What are circumferential and longitudinal stresses?
4. What do you mean by Lame's equation?
5. Name the stresses set up in a thick cylinder subjected to internal fluid pressure.
6. A cylinder of internal diameter 2.5 m and of thickness 5 cm contains a gas. If the tensile stress in the material is not to exceed 80 N/mm2, find the internal pressure of the gas.
7. A vessel in the shape of a spherical shell of 1.2 m internal diameter and 12 mm shell thickness is subjected to pressure of 1.6 N/mm2. Find the stress induced in the material of the vessel.
8. A spherical vessel 1.5 m diameter is subjected to an internal pressure of 2 N/mm2. Find the thickness of the plate required if maximum stress is not to exceed 150 N/mm2 and joint efficiency is 75 %.
9. A cylinder pipe of diameter 2 m and thickness 2 cm is subjected to an internal fluid pressure of 1.5 N/mm2, find the longitudinal and circumferential stress developed in the pipe material.
10. A thin cylinder of internal diameter 2 m contains a fluid at an internal pressure of 3 N/mm2. Determine the maximum thickness of the cylinder if i)longitudinal stress is not to exceed 30 N/mm2 ii)circumferential stress is not to exceed 40 N/mm2.
11. A thin cylindrical shell of 120 cm diameter, 1.5 cm thick and 6 m long is subjected to internal fluid pressure of 2.5 N/mm2.If E = 2 X105 N/mm2 and poisson's ratio = 0.3, determine i) Change in diameter ii) change in length iii) change in volume.
12. Determine the thickness of metal necessary for a cylindrical shell of internal diameter 150 mm to withstand an internal pressure of 50 N/mm2. The maximum hoop stress in the section is not to exceeed 150 N/mm2.

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SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – V – THEORIES OF FAILURE – SCIA1401

INTRODUCTION

Theories of failure are those theories which help us to determine the safe dimensions of a machine component when it is subjected to combined stresses due to various loads acting on it during its functionality.

IMPORTANT THEORIES OF FAILURE:

Maximum principal stress theory (Rankine's theory)

Maximum principal strain theory (St. Venant's theory)

Maximum shear stress theory (Coulomb, Tresca and Guest's theory)

Maximum strain energy theory (Haigh's theory)

Maximum shear strain energy theory or Distortion energy theory (Huber von Mises and Hencky's theory)

MAXIMUM PRINCIPAL STRESS THEORY (RANKINE'S THEORY)

According to this theory, the failure of a material will occur when the maximum principal tensile stress (σ_1) in the complex system reaches the value of the maximum stress at the elastic limit in simple tension or the minimum principal stress (maximum principal compressive stress) reaches the value of the maximum stress at the elastic limit in simple compression.

Let in a complex three-dimensional stress system,

σ_1 ,σ_2 and σ_3 = principal stresses at a point in three perpendicular directions.

The stresses σ_1 and σ_2 are tensile and σ_3 is compressive.

Also, σ_1 is more than σ_2

σ_t^* = tensile stress at elastic limit in simple tension.

σ_c^* = compressive stress at elastic limit in simple compression.

$\sigma_1 > \sigma_t^*$ in simple tension.

$\sigma_3 > \sigma_c^*$ in simple compression.

This is the simplest and oldest theory of failure and is known as Rankine's theory.

If the maximum principal stress σ_1 is the design criterion, then maximum principal stress must not exceed the permissible stress σ_t for the given material.

Hence, $\sigma_1 = \sigma_t$

Where, σ_t = permissible stress and is given by

$\sigma_t = \sigma_t^*$ / factor of safety.

PROBLEMS:

1.The principal stresses at a point in an elastic material are 100 N/mm^2 (tensile), 80 N/mm^2 (tensile) and 50 N/mm^2 (compressive). If the stress at the elastic limit in simple tension is 200 N/mm^2 , determine whether the failure of material will occur according to maximum principal stress theory. If not, then determine the factor of safety.

GIVEN DATA:

σ_1 = 100 N/mm^2

σ_2 = 80 N/mm^2

σ_3 = 50 N/mm^2 = - 50 N/mm^2

σ_t^* = 200 N/mm^2

SOLUTION:

1.TO DETERMINE WHETHER FAILURE OF MATERIAL WILL OCCUR OR NOT:

From the three given stresses, the maximum principal tensile stress is σ_1 = 100 N/mm^2

And the stress at elastic limit in simple tension is σ_t^* = 200 N/mm^2

As **σ_1 is less than σ_t^*,** the **failure will not occur** according to maximum principal stress theory.

2.FACTOR OF SAFETY:

Using equation, $\sigma_1 = \sigma_t$ σ_t = 100 N/mm^2

$\sigma_t = \sigma_t^*$ / factor of safety

Factor of safety = σ_t^* / σ_t = 200/100 = **2.**

MAXIMUM PRINCIPAL STRAIN THEORY (ST. VENANT'S THEORY)

According to this theory, the failure of a material will occur when the maximum principal strain reaches strain due to yield stress in simple tension or when the minimum principal strain (maximum compressive strain) reaches the strain due to yield stress in simple compression.

Yield stress is the maximum stress at elastic limit.

Consider a three-dimensional stress system.

limit. Consider a three dimensional stress system.

Principal strain in the direction of principal stress σ_1 is,

$$e_1 = \frac{\sigma_1}{E} - \frac{\mu\,\sigma_2}{E} - \frac{\mu\,\sigma_3}{E}$$

$$= \frac{1}{E}\,[\sigma_1 - \mu(\sigma_2 + \sigma_3)]$$

Principal strain in the direction of principal stress σ_3 is

$$e_3 = \frac{1}{E}\,[\sigma_3 - \mu(\sigma_1 + \sigma_2)]$$

Strain due to yield stress in simple tension $= \frac{1}{E} \times$ yield stress in tension

$$= \frac{1}{E} \times \sigma_t^*$$

and strain due to yield stress in simple compression $= \frac{1}{E} \times \sigma_c^*$

where yield stress is the maximum stress at elastic limit.

According to this theory, the failure of the material will take place when

$$e_1 \geq \frac{\sigma_t^*}{E}$$

or

$$|\,e_3\,| \geq \frac{\sigma_c^*}{E}$$

Substituting the values of e_1 and e_3, we get the conditions of failure as :

(i) $$\frac{1}{E}\,[\sigma_1 - \mu(\sigma_2 + \sigma_3)] \geq \frac{1}{E} \times \sigma_t^* \qquad \text{...(24.2)}$$

or $$\sigma_1 - \mu(\sigma_2 + \sigma_3) \geq \sigma_t^*$$

(ii) $$\left|\frac{1}{E}\,[\sigma_3 - \mu(\sigma_1 + \sigma_2)]\right| \geq \frac{1}{E} \times \sigma_c^* \qquad \text{...[24.2(A)]}$$

or $$|\,[\sigma_3 - \mu(\sigma_1 + \sigma_2)]\,| \geq \sigma_c^*$$

For actual design (*i.e.*, where some quantity is to be calculated), instead of σ_t^* or σ_c^*, the permissible stress (σ_t or σ_c) in simple tension or compression should be used where

$$\sigma_t = \frac{\sigma_t^*}{\text{Safety factor}} \qquad \text{...[24.2(B)}$$

and

$$\sigma_c = \frac{\sigma_c^*}{\text{Safety factor}} \qquad \text{...[24.2(C}$$

Hence for design purpose, the equations (24.2) and [24.2(A)] becomes as

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \sigma_t \qquad \text{...[24.2(}$$

and

$$|\,[\sigma_3 - \mu(\sigma_1 + \sigma_2)]\,| = \sigma_c \qquad \text{...[24.2(}$$

... [24.2(E)] should be used for design purposes (where so

2.The principal stresses at a point in an elastic material are 200 N/mm^2 (tensile), 100 N/mm^2 (tensile) and 50 N/mm^2 (compressive). If the stress at the elastic limit in simple tension is 200 N/mm^2 , determine whether the failure of material will occur according to maximum principal strain theory. Take Poisson's ratio = 0.3.

GIVEN DATA:

σ_1 = 200 N/mm^2

σ_2 = 100 N/mm^2

σ_3 = 50 N/mm^2 = - 50 N/mm^2

σ_t^* = 200 N/mm^2

Poisson's ratio = 0.3.

SOLUTION:

1.TO DETERMINE WHETHER FAILURE OF MATERIAL WILL OCCUR OR NOT:

Out of three principal stresses, the maximum principal tensile stress is σ_1. Hence the maximum principal strain will be in the direction of σ_1. Let this strain is e_1. Hence maximum principal strain in the direction of σ_1 will be,

$$e_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)]$$

$$= \frac{1}{E}[200 - 0.3\{100 + (-50)\}] \qquad [\because \ \sigma_3 = -50]$$

$$= \frac{1}{E}[200 - 30 + 15] = \frac{185}{E} \qquad ...(i)$$

Strain due to stress at elastic limit in simple tension,

$$e_t^* = \frac{\sigma_t^*}{E} = \frac{200}{E} \qquad ...(ii)$$

According to maximum principal strain theory, the failure of a material occurs if the maximum principal strain (e_1) reaches the strain due to stress at elastic limit in simple tension. Here $e_1 < e_t^*$, hence failure will not occur. Ans.

3. Determine the diameter of a bolt which is subjected to an axial pull of 9 KN together with a transverse shear force of 4.5 KN using, i) Maximum principal stress theory and ii) maximum principal strain theory. Given the elastic limit in tension = 225 N/mm^2 , factor of safety = 3 and Poisson's ratio = 0.3.

GIVEN DATA:

Axial pull, P = 9 KN = 9000 N

Transverse shear force, F = 4.5 KN = 4500 N

$\sigma_t^* = 225\ N/mm^2$

Factor of safety = 3

Poisson's ratio = 0.3.

SOLUTION:

The permissible stress in tension is given by,

$\sigma_t = \sigma_t^*$ / factor of safety = 225/3 = **75 N/mm²**

The axial pull will produce tensile stress whereas transvers shear force will produce shear stress in the bolt.

Let us calculate these stresses.

Now, tensile stress, $\sigma = \frac{P}{A} = \frac{P}{\pi d^2/4} = \frac{4\,P}{\pi d^2}$

$$= \frac{4 \times 9000}{\pi d^2} = \mathbf{\frac{11459\ N/mm^2}{d^2}}$$

Now, shear stress, $\tau = \frac{F}{\pi d^2/4} = \frac{4F}{\pi d^2} = \frac{4 \times 4500}{\pi d^2} = \mathbf{\frac{5729.5\ N/mm^2}{d^2}}$

Now let us calculate the maximum and minimum principal stresses σ_1 and σ_2.
The principal stresses (maximum and minimum) in the bolt are given by

$$\sigma_1 \text{ and } \sigma_2 = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \qquad \text{[Refer to equation (3.16). Here } \sigma_x = \sigma,\ \sigma_y = 0]$$

$$= \frac{11459}{2 \times d^2} \pm \sqrt{\left(\frac{11459}{2 \times d^2}\right)^2 + \left(\frac{5729.5}{d^2}\right)^2}$$

$$= \frac{5729.5}{d^2} \pm \sqrt{\left(\frac{5729.5}{d^2}\right)^2 + \left(\frac{5729.5}{d^2}\right)^2}$$

$$= \frac{5729.5}{d^2} \pm \frac{5729.5}{d^2}\sqrt{(1+1)}$$

$$= \frac{5729.5}{d^2} \pm \frac{8103}{d^2}$$

$$\therefore \qquad \sigma_1 = \frac{5729.5}{d^2} + \frac{8103}{d^2} = \frac{13832.5}{d^2} \text{ N/mm}^2$$

$$\sigma_2 = \frac{5729.5}{d^2} - \frac{8103}{d^2} = \frac{-2373.5}{d^2} \text{ N/mm}^2$$

Hence the principal stresses in the bolt are :

$$\sigma_1, \sigma_2 \text{ and } 0 \quad \text{or} \quad \frac{13832.5}{d^2}, \frac{-2373.5}{d^2} \text{ and } 0.$$

1.DIAMETER OF BOLT ACCORDING TO MAXIMUM PRINCIPAL STRESS THEORY:

Here diameter of the bolt is to be calculated.

This becomes the case of design.

For the purpose of design, according to maximum principal stress theory, the maximum principal stress should not exceed the permissible stress σ_t in tension.

Here, the maximum principal stress is σ_1

$$\sigma_1 = \sigma_t$$

$$\frac{13832.5}{d^2} = 75$$

$$d^2 = \frac{13832.5}{75}$$

$$d = \sqrt{\frac{13832.5}{75}} = 13.58 \simeq \mathbf{13.6\ mm.}$$

(ii) Diameter of bolt according to maximum principal strain theory

The three principal stresses are :

$$\sigma_1 = \frac{13832.5}{d^2} \text{ N/mm}^2$$

$$\sigma_2 = \frac{-2373.5}{d^2} \text{ N/mm}^2 \quad \text{and} \quad \sigma_3 = 0$$

Here maximum principal stress is σ_1. Hence maximum strain will be in the direction of σ_1.

$$\therefore \quad \text{Maximum strain} = \frac{\sigma_1}{E} - \frac{\mu}{E}(\sigma_2 + \sigma_3)$$

$$= \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)]$$

$$= \frac{1}{E}\left[\frac{13832.5}{d^2} - 0.3\left(\frac{-2373.5}{d_2} + 0\right)\right]$$

$$= \frac{1}{E}\left[\frac{13832.5}{d^2} + \frac{712.05}{d^2}\right]$$

$$= \frac{1}{E} \times \frac{14544.55}{d^2} \quad \text{...(i)}$$

Maximum strain due to permissible stress in tension $= \frac{1}{E} \times \sigma_t$

$$= \frac{1}{E} \times 75 \qquad (\because \sigma_t = 75) \text{ ...(ii)}$$

For design purpose, (as here diameter is to be calculated), the maximum strain should be equal to the strain due to permissible stress in tension.

Hence equating the two values given by equations (*i*) and (*ii*), we get

$$\frac{1}{E} \times \frac{14544.55}{d^2} = \frac{1}{E} \times 75$$

or
$$d = \sqrt{\frac{14544.55}{75}} = \mathbf{13.92\ mm.}\ \text{Ans.}$$

MAXIMUM SHEAR STRESS THEORY (GUEST'S THEORY)

According to this theory, the failure of a material will occur when the maximum shear stress in a material reaches the value of maximum shear stress in simple tension at the elastic limit.

The maximum shear stress in the material is equal to half the difference between maximum and minimum principal stress.

If σ_1 , σ_2 and σ_3 are principal stresses at a point in a material for which σ_t^* is the principal stress in simple tension at elastic limit, then

Maximum shear stress in the material = Half of difference of maximum and minimum principal stresses.

$$= \frac{1}{2} [\sigma_1 - \sigma_3]$$

In case of simple tension, at the elastic limit the principal stresses are σ_t^*, 0, 0.

[In simple tension, the stress is existing in one direction only]

- Maximum shear stress in simple tension at elastic limit = Half of difference of maximum and minimum principal stresses.

$$= \frac{1}{2} [\sigma_t^* - 0] = \frac{1}{2} \sigma_t^*$$

For the failure of material,

$$\frac{1}{2} [\sigma_1 - \sigma_3] = \frac{1}{2} \sigma_t^* \quad \text{OR} \quad [\sigma_1 - \sigma_3] > \sigma_t^*$$

For actual design instead of σ_t^*, the allowable stress σ_t in simple tension should be considered.

$\sigma_t = \sigma_t^*$ / factor of safety

Hence for design purpose, the following equation should be used.

$(\sigma_1 - \sigma_3) = \sigma_t$

4.The principal stresses at a point in an elastic material are 200 N/mm^2 (tensile), 100 N/mm^2 (tensile) and 50 N/mm^2 (compressive). If the stress at the elastic limit in simple tension is 200 N/mm^2 , determine whether the failure of material will occur according to maximum shear stress theory.

GIVEN DATA:

$\sigma_1 = 200$ N/mm^2

$\sigma_2 = 100$ N/mm^2

$\sigma_3 = 50$ N/mm^2 = - 50 N/mm^2

$\sigma_t^* = 200$ N/mm^2

SOLUTION:

Maximum shear stress in the material = Half of difference of maximum and minimum principal stresses.

$$= \frac{1}{2} [\sigma_1 - \sigma_3] = \frac{1}{2} [200 - (-50)] = 250/2 = \mathbf{125\ N/mm^2}$$

Maximum shear stress in simple tension at elastic limit = Half of difference of maximum and minimum principal stresses.

$$= \frac{1}{2} \times \sigma_t^* = \frac{1}{2} \times 200 = \mathbf{100\ N/mm^2}$$

As maximum shear stress developed in the material is 125 N/mm^2 is more than the maximum shear stress at the elastic limit in simple tension is 100 N/mm^2 , hence **failure will occur.**

5.According to the theory of maximum shear stress, determine the diameter of a bolt which is subjected to an axial pull of 9 KN together with a transverse shear

force of 4.5 KN. Elastic limit in tension = 225 N/mm^2 , factor of safety = 3 and Poisson's ratio = 0.3.

GIVEN DATA:

Axial pull, P = 9 KN = 9000 N

Transverse shear force, F = 4.5 KN = 4500 N

$\sigma_t^* = 225$ N/mm^2

Factor of safety = 3

Poisson's ratio = 0.3.

SOLUTION:

1.DIAMETER OF BOLT ACCORDING TO MAXIMUM PRINCIPAL STRESS THEORY:

Permissible simple stress in tension, $\sigma_t = \sigma_t^* /3 = 225/3 =$ **75 N/mm^2**

The axial pull will produce tensile stress whereas transvers shear force will produce shear stress in the bolt.

Let us calculate these stresses.

Now, tensile stress, $\sigma = \dfrac{P}{A} = \dfrac{P}{\pi d^2/4} = \dfrac{4P}{\pi d^2}$

$$= \frac{4 \times 9000}{\pi d^2} = \mathbf{\frac{11459}{d^2}\ N/mm^2}$$

Now, shear stress, $\tau = \dfrac{F}{\pi d^2/4} = \dfrac{4F}{\pi d^2} = \dfrac{4 \times 4500}{\pi d^2} = \mathbf{\dfrac{5729.5}{d^2}\ N/mm^2}$

4

The maximum and minimum principal stresses in the bolt are given by

$$\sigma_1 \text{ and } \sigma_2 = \frac{1}{2} \times \sigma \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

[Refer to equation (3.16). Here $\sigma_x = \sigma$, $\sigma_y = 0$]

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \quad \text{and} \quad \sigma_2 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Hence the principal stresses in the bolt are :

$$\frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}, \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \text{ and } 0$$

Now apply the theory of maximum shear stress.

Maximum shear stress due to principal stresses

$$= \frac{1}{2}\left[\left(\frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}\right) - \left(\frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}\right)\right]$$

$$= \frac{1}{2}\left[2 \times \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}\right] = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

$$= \sqrt{\left(\frac{11459}{2 \times d^2}\right)^2 + \left(\frac{5729.5}{d^2}\right)^2} \qquad \left[\because \sigma = \frac{11459}{d^2} \text{ and } \tau = \frac{5729.5}{d^2}\right]$$

$$= \frac{8103}{d^2} \qquad \text{...}(i)$$

Max. shear stress in simple tension

$$= \frac{1}{2}[\sigma_t - 0] = \frac{\sigma_t}{2} = \frac{75}{2} = 37.5 \text{ N/mm}^2 \qquad \text{...}(ii)$$

Equating the two maximum shear stresses given by equations (*i*) and (*ii*), we get

$$\frac{8103}{d^2} = 37.5 \quad \text{or} \quad d = \sqrt{\frac{8103}{37.5}} = \mathbf{14.70\ mm.}$$ **Ans.**

MAXIMUM STRAIN ENERGY THEORY (HAIGH'S THEORY)

According to this theory, the failure of a material will occur when the total strain energy per unit volume in the material reaches the strain energy per unit volume of the material at the elastic limit in simple tension.

Strain energy in a body is equal to work done by the load in straining the material and is equal to ½ X P X δL.

$$\therefore \quad U = \text{Strain energy}$$

$$= \frac{1}{2} \times P \times \delta L$$

$$= \frac{1}{2} \times (\sigma \times A) \times (e) \times L \qquad \left[\because \ \sigma = \frac{P}{A} \ \therefore \ P = \sigma \times A \ \text{and} \ e = \frac{\delta L}{L} \ \therefore \ \delta L = e \times L\right]$$

$$= \frac{1}{2} \times \sigma \times e \times A \times L$$

$$= \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume} \qquad (\because \ A \times L = \text{volume})$$

$\therefore$ Strain energy per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$= \frac{1}{2} \times \sigma \times e \qquad \ldots(24.4)$$

For a three dimensional stress system, the principal stresses acting at a point are σ_1, σ_2 and σ_3. The corresponding strains are e_1, e_2 and e_3, where e_1 = principal strain in the direction of σ_1

Now, $$e_2 = \frac{\sigma_1}{E} - \frac{\mu}{E}(\sigma_2 + \sigma_3)$$

Similarly, $$e_2 = \frac{\sigma_2}{E} - \frac{\mu}{E}(\sigma_3 + \sigma_1)$$

and $$e_3 = \frac{\sigma_3}{E} - \frac{\mu}{E}(\sigma_1 + \sigma_2)$$

$$U = \frac{1}{2} \times \sigma_1 \times e_1 + \frac{1}{2} \times \sigma_2 \times e_2 + \frac{1}{2} \sigma_3 \times e_3$$

$$= \frac{1}{2} \sigma_1 \times \left[\frac{\sigma_1}{E} - \frac{\mu}{E}(\sigma_2 + \sigma_3)\right] + \frac{1}{2}\sigma_2 \times \left[\frac{\sigma_2}{E} - \frac{\mu}{E}(\sigma_3 + \sigma_1)\right] + \frac{1}{2}\sigma_3 \times \left[\frac{\sigma_3}{E} - \frac{\mu}{E}(\sigma_1 + \sigma_2)\right]$$

$$= \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad ...(24.5)$$

The strain energy per unit volume corresponding to stress at elastic limit in simple tension

$$= \frac{1}{2} \times \sigma_t^* \times e_t^* \qquad \text{[where } e_t^* = \text{strain due to } \sigma_t^*$$

$$= \frac{1}{2} \times \sigma_t^* \times \frac{\sigma_t^*}{E} \qquad \left.\therefore\ E = \frac{\sigma_t^*}{e_t^*} \text{ or } e_t^* = \frac{\sigma_t^*}{E}\right]$$

$$= \frac{1}{2E} \times (\sigma_t^*)^2 \quad ...(24.6)$$

For the failure of the material

$$\frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \geq \frac{1}{2E} \times (\sigma_t^*)^2$$

or

$$[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \geq (\sigma_t^*)^2 \quad ...(24.7)$$

For a two-dimensional stress system, $\sigma_3 = 0$. Hence above equation becomes as

$$\sigma_1^2 + \sigma_2^2 - 2\mu(\sigma_1\sigma_2) \geq (\sigma_t^*)^2 \quad ...(24.8)$$

For actual design (*i.e.*, when some quantity is to be calculated) in stead of σ_t^*, the allowable stress (σ_t) in simple tension should be considered where

$$\sigma_t = \frac{\sigma_t^*}{\text{Factor of safety}}$$

Hence for design, the following equation should be used

$$\sigma_1^2 + \sigma_2^2 - 2\mu(\sigma_1 \times \sigma_2) = \sigma_t^2 \quad ...(24.9$$

6. The principal stresses at a point in an elastic material are 200 N/mm^2 (tensile), 100 N/mm^2 (tensile) and 50 N/mm^2 (compressive). If the stress at the elastic limit in simple tension is 200 N/mm^2 , determine whether the failure of material will occur according to maximum principal strain theory. Take Poisson's ratio = 0.3.

GIVEN DATA:

σ_1 = 200 N/mm^2

σ_2 = 100 N/mm^2

σ_3 = 50 N/mm^2 = - 50 N/mm^2

σ_t^* = 200 N/mm^2

Poisson's ratio = 0.3.

SOLUTION:

1.TO DETERMINE WHETHER FAILURE OF MATERIAL WILL OCCUR OR NOT:

Total strain energy per unit volume in the material,

$$\frac{1}{2E}[\sigma_1 + \sigma_1 + \sigma_1 - 2\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

$$= \frac{1}{2E}[200^2 + 100^2 + (-50)^2 - 2 \times 0.3\,\{200 \times 100 + 100 \times (-50) + (-50) \times 200\}]$$

$$= \frac{1}{2E}[49500]$$

strain energy per unit volume of the material at the elastic limit in simple tension

$$\frac{1}{2E} \times \sigma_t^{*2}$$

$$= \frac{1}{2E} \times 200^2 = \frac{40000}{2E}$$

We find the total strain energy per unit volume in the material is **more** than the strain energy per unit volume at elastic limit in simple tension. Hence **failure will occur.**

7. According to maximum strain energy theory, determine the diameter of a bolt which is subjected to an axial pull of 9 KN together with a transverse shear force of 4.5 KN. Elastic limit in tension = 225 N/mm^2 , factor of safety = 3 and Poisson's ratio = 0.3.

GIVEN DATA:

Axial pull, P = 9 KN = 9000 N

Transverse shear force, F = 4.5 KN = 4500 N

$\sigma_t^* = 225$ N/mm^2

Factor of safety = 3

Poisson's ratio = 0.3.

SOLUTION:

1.DIAMETER OF BOLT ACCORDING TO MAXIMUM STRAIN ENERGY THEORY:

Permissible simple stress in tension, $\sigma_t = \sigma_t^* /3 = 225/3 = 75$ N/mm^2

The axial pull will produce tensile stress whereas transvers shear force will produce shear stress in the bolt.

Let us calculate these stresses.

Now, tensile stress, $\sigma = \frac{P}{A} = \frac{P}{\pi d^2/4} = \frac{4\,P}{\pi d^2}$

$$= \frac{4 \times 9000}{\pi d^2} = \frac{\mathbf{11459}}{\mathbf{d^2}} \ \mathbf{N/mm^2}$$

Now, shear stress, $\tau = \dfrac{F}{\pi d^2/4} = \dfrac{4F}{\pi d^2} = \dfrac{4 \times 4500}{\pi d^2} = \dfrac{\mathbf{5729.5}}{\mathbf{d^2}}$ **N/mm²**

$$\sigma_1 \text{ and } \sigma_2 = \frac{1}{2} \times \sigma \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

The value calculated

$$= \frac{11459}{2d^2} \pm \frac{8103}{d^2}$$

$$= \frac{5729.5}{d^2} \pm \frac{8103}{d^2}$$

$$\therefore \quad \sigma_1 = \frac{5729.5}{d^2} + \frac{8103}{d^2} = \frac{13832.5}{d^2} \text{ N/mm}^2$$

$$\sigma_2 = \frac{5729.5}{d^2} - \frac{8103}{d^2} = \frac{-2373.5}{d^2} \text{ N/mm}^2$$

Hence the principal stresses at the point are : σ_1, σ_2 and 0

$$\frac{13832.5}{d^2}, \ \frac{-2373.5}{d^2}, \ 0$$

$$\left[\frac{13832.5}{d^2}\right]^2 + \left[\frac{-2373.5}{d^2}\right]^2 - 2 \times 0.3 \left[\frac{13832.5}{d^2} \times \frac{(-2373.5)}{d^2}\right] = 75^2 \quad (\because \sigma_t = 75)$$

$$\frac{19134 \times 10^4}{d^4} + \frac{563.35 \times 10^4}{d^4} + \frac{1969.9 \times 10^4}{d^4} = 5625$$

$$\frac{21667.25 \times 10^4}{d^4} = 5625$$

$$d = \left(\frac{21667.25 \times 10^4}{5625}\right)^{1/4} = 10 \times (3.852)^{1/4}$$

$$= 10 \times 1.401 = \mathbf{14.01 \ mm.}$$ **Ans.**

MAXIMUM SHEAR STRAIN ENERGY THEORY (MISES-HENKY THEORY)

- According to this theory, the failure of a material will occur when the total shear strain energy per unit volume in the material reaches the value of shear strain energy per unit volume at the elastic limit in the simple tension.
- The total shear strain energy per unit volume due to principal stresses σ_1, σ_2 and σ_3 in a stressed material is given as,

$$\frac{1}{2C}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

The simple tension test is a uniaxial stress system which means the principal stresses are σ_1 ,0,0.

At the elastic limit the tensile stress in simple tension is σ_t^*

Hence at the elastic limit in simple tension test, the principal stresses are σ_t^* ,0,0.

The shear strain energy per unit volume at the elastic limit in simple tension will be.

$$\frac{1}{12C}\left[(\sigma_t^*-0)^2+(0-0)^2+(0-\sigma_t^*)^2\right]$$

$$\frac{1}{12C}\left[2 \text{ X } \sigma_t^{*2}\right]$$

For the failure of the material,

$$\frac{1}{12C}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] > \frac{1}{12C}\left[2 \text{ X } \sigma_t^{*2}\right]$$

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 > 2 \text{ X } \sigma_t^{*2}$$

For actual design instead of σ_t^*, the allowable stress σ_t in simple tension should be considered.

$\sigma_t = \sigma_t^*$ / factor of safety

Hence for design purpose, the following equation should be used.

$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2 \text{ X } \sigma_t^2$

8. The principal stresses at a point in an elastic material are 200 N/mm^2 (tensile), 100 N/mm^2 (tensile) and 50 N/mm^2 (compressive). If the stress at the elastic limit in simple tension is 200 N/mm^2 , determine whether the failure of material will occur according to maximum shear strain energy theory. Take Poisson's ratio = 0.3.

GIVEN DATA:

σ_1 = 200 N/mm^2

σ_2 = 100 N/mm^2

σ_3 = 50 N/mm^2 = - 50 N/mm^2

σ_t^* = 200 N/mm^2

Poisson's ratio = 0.3.

SOLUTION:

1.TO DETERMINE WHETHER FAILURE OF MATERIAL WILL OCCUR OR NOT:

The total shear strain energy per unit volume due to principal stresses σ_1 ,σ_2 and σ_3 in a stressed material is given by,

$$\frac{1}{12C} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{12C} [(200 - 100)^2 + \{100 - (-50)\}^2 + \{(-50) -200\}^2]$$

$$= \frac{\mathbf{1}}{\mathbf{12C}} \mathbf{X\ 95000}$$

The shear strain energy per unit volume at the elastic limit in simple tension will be,

$$\frac{1}{12C}\left[2 \times \sigma_t^{*2} \right] = \frac{1}{12C}\left[2 \times 200^2 \right] = \mathbf{\frac{1}{12C} \times 80000}$$

We find that total shear strain energy per unit volume due to principal stresses σ_1 ,σ_2 and σ_3 are **more** than the shear strain energy per unit volume at the elastic limit in simple tension.

Hence **failure will occur.**

9. According to maximum shear strain energy theory, determine the diameter of a bolt which is subjected to an axial pull of 9 KN together with a transverse shear force of 4.5 KN. Elastic limit in tension = 225 N/mm^2 , factor of safety = 3 and Poisson's ratio = 0.3.

GIVEN DATA:

Axial pull, P = 9 KN = 9000 N

Transverse shear force, F = 4.5 KN = 4500 N

σ_t^* = 225 N/mm^2

Factor of safety = 3

Poisson's ratio = 0.3.

SOLUTION:

1.DIAMETER OF BOLT ACCORDING TO MAXIMUM SHEAR STRAIN ENERGY THEORY:

Permissible simple stress in tension,$\sigma_t = \sigma_t^* /3$ = 225/3 = **75 N/mm^2**

The axial pull will produce tensile stress whereas transvers shear force will produce shear stress in the bolt.

Let us calculate these stresses.

Now, tensile stress, $\sigma = \dfrac{P}{A} = \dfrac{P}{\pi d^2/4} = \dfrac{4P}{\pi d^2}$

$$= \frac{4 \times 9000}{\pi d^2} = \frac{\mathbf{11459}}{\mathbf{d^2}}\ \mathbf{N/mm^2}$$

Now, shear stress, $\tau = \dfrac{F}{\pi d^2/4} = \dfrac{4F}{\pi d^2} = \dfrac{4 \times 4500}{\pi d^2} = \dfrac{\mathbf{5729.5}}{\mathbf{d^2}}\ \mathbf{N/mm^2}$

$$\sigma_1 \text{ and } \sigma_2 = \frac{1}{2} \times \sigma \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

The value calculated

$$= \frac{11459}{2d^2} \pm \frac{8103}{d^2}$$

$$= \frac{5729.5}{d^2} \pm \frac{8103}{d^2}$$

$$\therefore \quad \sigma_1 = \frac{5729.5}{d^2} + \frac{8103}{d^2} = \frac{13832.5}{d^2}\ \text{N/mm}^2$$

$$\sigma_2 = \frac{5729.5}{d^2} - \frac{8103}{d^2} = \frac{-2373.5}{d^2}\ \text{N/mm}^2$$

Hence the principal stresses at the point are : σ_1, σ_2 and 0

$$\frac{13832.5}{d^2},\ \frac{-2373.5}{d^2},\ 0$$

In this problem, diameter is to be calculated according to the maximum shear strain energy theory.

Hence the following equation is to be used.

$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2 \times \sigma_t^2$

$$\left[\frac{13832.5}{d^2}-\left(\frac{-2373.5}{d^2}\right)\right]^2+\left[\left(\frac{-2373.5}{d^2}-0\right)\right]^2+\left[0-\frac{13832.5}{d^2}\right]^2 = 2\times 75^2$$

$$\left[\frac{13832.5+2373.5}{d^2}\right]^2+\left[\frac{2373.5}{d^2}\right]^2+\left[\frac{-13832.5}{d^2}\right]^2 = 11250$$

$$\frac{26263\times 10^4}{d^4}+\frac{563.35\times 10^4}{d^4}+\frac{19134\times 10^4}{d^4} = 11250$$

$$\frac{45960.35\times 10^4}{d^4} = 11250$$

$$d^4 = \frac{45960.35\times 10^4}{11250} = 4.08536\times 10^4$$

$$d = [4.08536\times 10^4]^{1/4} = (4.08536)^{1/4}\times 10$$

$$= 1.4217\times 10 = \mathbf{14.217\ mm.}$$ **Ans.**

REFERENCE BOOKS:

1. Bansal R.K., “Strength of Materials”, 5th Edition, Laxmi Publications, 2013.
2. Ramamrutham R., “Strength of Materials”,14th Edition, Dhanpat Rai Publications, 2011.
3. Punmia B.C., “Strength of Materials”, Laxmi Publications, 2011.
4. Subramanian R., “Strength of Materials - Theory and Problems”, Oxford University Press, 2010.
5. Rajput R.K., “Strength of Materials”, S. Chand and Company Ltd., New Delhi, 2007.
6. Timoshenko S.P. and Gere J.M., “Mechanics of Materials”, 6th Edition, Tata McGraw Hill.

QUESTION BANK:

1. Name the important theories of failure.
2. Define: Maximum principal stress theory.
3. Define: Maximum principal strain theory.
4. Define: Maximum shear stress theory.
5. Define: Maximum strain energy theory.
6. Define: Maximum shear strain energy theory.
7. The principal stresses at a point in an elastic material are 22 N/mm2 tensile, 110 N/mm2 (tensile) and 55 N/mm2 (compressive). If the stress at the elastic limit in simple tension is 200 N/mm2, determine whether the failure of material will occur according to maximum principal stress theory. If not, then determine the factor of safety.
8. Determine the diameter of a bolt which is subjected to an axial pull of 9 kN together with a transverse shear force of 4.5 kN using Maximum principal strain theory. Given the elastic limit in tension = 225 N/mm2, factor of safety = 3 and Poisson's ratio = 0.3.
9. At a section of mild steel shaft, the maximum torque is 8437.5 Nm and maximum bending moment is 5062.5Nm. The diameter of shaft is 90 mm and the stress at the elastic limit in simple tension for the material of the shaft is 220 N/mm2.Determine whether the failure of the material will occur or not according to maximum shear stress theory. If not, then find the factor of safety.
10. According to the theory of maximum shear stress, determine the diameter of a bolt which is subjected to an axial pull of 9 kN together with a transverse shear force of 4.5 kN. Elastic limit in tension is 225 N/mm2, factor of safety = 3 and poisson's ratio = o.3.
11. The principal stresses at a point in an elastic material are 22 N/mm2 tensile, 110 N/mm2 (tensile) and 55 N/mm2 (compressive). If the stress at the elastic limit in simple tension is 220 N/mm2 and poisson's ratio = 0.3, determine whether the failure of material will occur or not according to maximum strain energy theory.
12. A cylindrical shell made of mild steel plate and 1.2 m in diameter is to be subjected to an internal pressure of 1.5 MN/m2. If the material yields at 200 MN/m2, calculate the thickness of the plate on the basis of the following three theories, assuming a factor of safety 3 in each case: (i) maximum principal stress theory, (ii) maximum shear stress theory, and (iii) maximum shear strain energy theory.