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SCHOOL OF BIO & CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – I –Modelling and transport of bioprocesses – SCH1312

Unit I Momentum Transport

Introduction to Momentum Transport

Momentum transport deals with the transport of momentum which is responsible for flow in fluids. Momentum transport describes the science of fluid flow also called fluid dynamics. A few basic assumptions are involved in fluid flow and these are discussed below. No slip boundary condition

This is the first basic assumption used in momentum transport. It deals with the fluid flowing over a solid surface, and states that whenever a fluid comes in contact with any solid boundary, the adjacent layer of the fluid in contact with the solid surface has the same velocity as the solid surface. Hence, we assumed that there is no slip between the solid surface and the fluid or the relative velocity is zero at the fluid–solid interface. For example, consider a fluid flowing inside a stationary tube of radius *R* as shown in Fig 7.1. Since the wall of the tube at *r=R* is stationary, according to the no-slip condition implies that the fluid velocity at *r=R* is also zero.

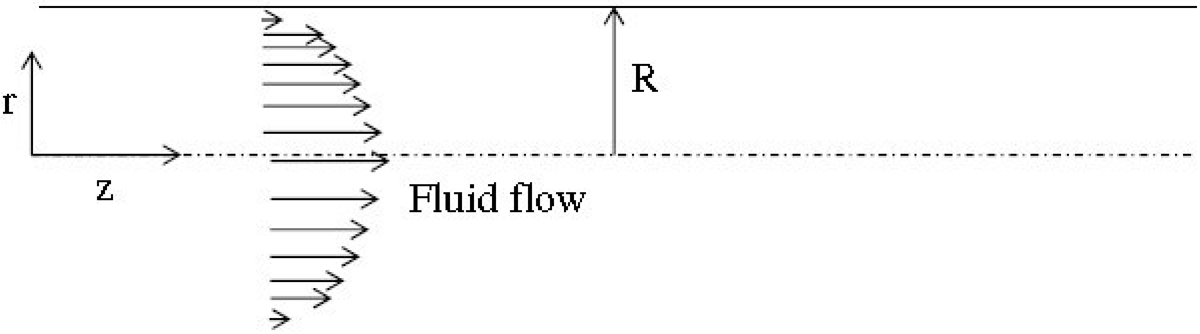
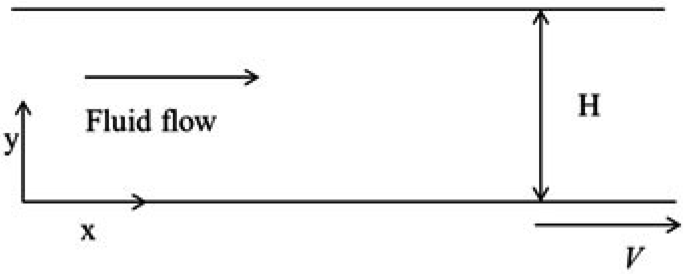


Fig .1 Fluid flow in a circular tube of radius R

In the second example as shown in Fig .2, there are two plates which are



separated by a Fig .2 Fig. 2 Two parallel plates at stationary condition distance *h*, and some fluid is present between these plates. If the lower plate is forced to move with a velocity *V* in x direction and the upper plate is held stationary, no-slip boundary conditions may be written as follows.

Thus, for Re <2100, we have laminar flow, i.e., no mixing in the radial direction leading to a thread like flow and for *Re >2100*, we have the turbulent flow, i.e., mixing in the radial direction between layers of fluid.

In laminar flow, the fluid flows as a stream line flow with no mixing between layers. In turbulent flow, the fluid is unstable and mixes rapidly due to fluctuations and disturbances in the flow. The disturbance might be present due to pumps, friction of the solid surface or any type of noise present in the system. This makes solving fluid flow problem much more difficult. To understand the difference in the velocity profile in two kinds of fluid flows, we consider a fluid flowing to a horizontal tube in *z* direction under steady state condition. Then, we can intuitively see the velocity profile may be shown below.

Fluid:

A fluid is a gas or liquid that, unlike a solid, flows to assume the shape of the container in which it is placed. This occurs because a fluid responds to a shear stress, or a force per unit area directed along the face of a cube of fluid, by flowing, rather than by an elastic displacement as in a solid.

TYPES OF FLUIDS BASED ON VISCOSITY:

The fluids may be classified into following five types:

1. Ideal fluid
2. Real fluid
3. Newtonian fluid
4. Non-Newtonian fluid
5. Ideal plastic fluid

Properties of fluid

Density: Density of a fluid is defined as the ratio of the mass of a fluid to its volume.

Specific Volume: Specific volume of a fluid is defined as the volume of a fluid occupied by a unit mass or volume per unit mass of a fluid.

Viscosity of liquid: Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of fluid.

$$y = 0,\ v_x = v$$

$$y = H,\ v_x = 0$$

Thus, every layer of fluid is moving at a different velocity. This leads to shear forces which are described in the next section.

Newton’s Law of Viscosity

Newton’s law of viscosity may be used for solving problem for Newtonian fluids. For many fluids in chemical engineering the assumption of Newtonian fluid is reasonably acceptable. To understand Newtonian fluid, let us consider

a hypothetical experiment, in which there are two infinitely large plates situated parallel to each other, separated by a distance h. A fluid is present between these two plates and the contact area between the fluid and the plates is *A*.
A constant force *F1* is now applied on the lower plate while the upper plate is held stationary. After steady state has reached, the velocity achieved by the lower plate is measured as V_1. The force is then changed, and the new velocity of the plate associated with this force is measured. The experiment is then repeated to take sufficiently large readings as shown in the following table. Table 1. Applied force vs velocity

$$\frac{F}{A} \propto \frac{dv_x}{dy}$$

$$\frac{F}{A} = \pm \mu \frac{dv_x}{dy}$$

surface on which this force is acting. The quantity or the velocity gradient is also called the shear rate. μ is a property of the fluid and is measured the resistance offered by the fluid to flow. Viscosity may be constant for many Newtonian fluids and may change only with temperature.

Thus, the Newton's law of viscosity, in its most basic form is given as

The shear stress is directly proportional to velocity gradient.
The constant of proportionality is known as coefficient of viscosity.

Variation of viscosity with respect to temperature

Viscosity of fluids is due to

1. cohesion between the fluid molecules and

2. transfer of momentum between the molecules.

Fluids are aggregations of molecules; widely spaced for a gas and closely spaced for liquids. Distance between the molecules is very large compared to the molecular diameter. The number of molecules involved is immense and the separation between them is normally negligible. Under these conditions, fluid can be treated as continuum and the properties at any point can be treated as bulk behavior of the fluids. For the continuum model to be valid, the smallest sample of matter of practical interest must contain a large number of molecules so that meaningful averages can be calculated. In the case of air at sea-level conditions, a volume of 10^{-9} mm^3 contains 3×10^7 molecules. In engineering sense, this volume is quite small, so the continuum hypothesis is valid.

If the *F/A* is plotted against *V/h*, we may observe that they lie on a straight line passing through the origin.

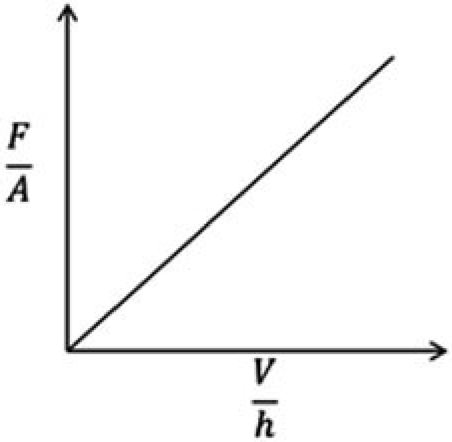


Fig 4 Shear stress vs. shear stain

Thus, it may be said that *F/A* is proportional to *v/h* for a Newtonian fluid.

$$\frac{F}{A} \propto \frac{v}{h}$$

It may be noted that it is the velocity gradient which leads to the development of shear forces. The above equation may be re-written as

$$\Rightarrow \frac{F_1}{A} \propto \frac{v_1 - 0}{h - 0} = \frac{\Delta v_x}{\Delta y}$$

In the limiting case, as *h* → *0*, we have

where, *μ* is a constant of proportionality, and is called as the viscosity of the fluid. The quantity *F/A* represents the shear forces/stress. It may be represented as $\tau_{yx.}$, where the subscript *x* indicates the direction of force and subscript *y* indicates the direction of outward normal of the

$$\tau_{yx} = \pm\mu\frac{dv_x}{dy}$$

Newton’s law of viscosity states that the shear stress is directly proportional to Velocity gradient.

Here, both ‘+’ or ‘–’ sign are valid. The positive sign is used in many fluid mechanics books whereas the negative sign may be found in transport phenomena books. If the positive sign is used then τ_{yx}. may be called the shear force while if the negative sign is used τ_{yx}. may be referred to as the momentum flux which flows from a higher value to a lower value.

$$\tau_{yx} = +\mu\frac{dv_x}{dy} \rightarrow \text{Shear force}$$

$$\tau_{yx} = -\mu\frac{dv_x}{dy} \rightarrow \text{Momentum flux}$$

The reason for having a negative sign for momentum flux in the transport phenomena is to have similarities with Fourier's law of heat conduction in heat transport and Ficks law of diffusion in mass transport. For example, in heat transport, heat flows from higher temperature to lower temperature indicating that heat flux is positive when the temperature gradient is negative. Thus,a minus sign is required in the Fourier's law of heat conduction. The interpretation of τ_{yx}. as the momentum flux is that *x* directed momentum flows from higher value to lower value in *y* direction.

Viscosity

Viscosity is a measure of flow resistance to the fluid. The viscosity is of two types dynamic or absolute viscosity and kinematic viscosity. Dynamic viscosity is simply the viscosity offered as resistance to fluid flow. Kinematic viscosity is defined as the ratio of absolute viscosity to density.

$$\mu = \frac{Force/Area}{\left[\frac{dv_x}{dy}\right]} = \frac{MLT^{-1}L^{-2}}{LT^{-1}L^{-1}} = ML^{-1}T^{-1}$$

The SI unit of viscosity is *kg/m.s* or *Pa.s*. In CGS unit is *g/cm.s* and is commonly known as poise (*P*). where *1 P = 0.1 kg/m.s*. The unit poise is also used with the prefix *centi-*, which refers to one-hundredth of a poise, i.e. *1 cP = 0.01 P*. The viscosity of air at 25^oC is *0.018 cP,* water at 25^oC is *1 cP* and for many polymer melts it may range from *1000* to *100,000 cP*, thus showing a long range of viscosity.

Laminar and turbulent flow

Fluid flow can broadly be categorized into two kinds: laminar and turbulent. In laminar flow, the fluid layers do not inter-mix, and flow separately. This is the flow encountered when a tap is just opened and water is allowed to flow very slowly. As the flow increases, it becomes much more irregular and the different fluid layers start mixing with each other leading to turbulent flow. Osborne Reynolds tried to distinguish between the two kinds of flow using an ingenious experiment and known as the Reynolds’s experiment. The basic idea behind this experiment is described below.

The experiment setup used for performing the Reynolds's experiment is shown in Fig. 7.5. The average velocity of fluid flow through the pipe diameter can be varied. Also, there is an arrangement to inject a colored dye at the center of the pipe. The profile of the dye is observed along the length of the pipe for different velocities for different fluids. If this experiment is performed, it may be seen that for certain cases the dye shows a regular thread type profile, which is seen at low fluid velocity and flow is called laminar flow. when the fluid velocity is increased the dye starts to mixed with the fluid and for larger velocities simply disappears. At this point fluid flow becomes turbulent.
For the variables average velocity of fluid $v_{z\ avg}$, pipe diameter D, fluid density ρ, and the fluid viscosity μ, Reynolds found a dimensionless group which could be used to characterize the type of fluid flow in the tube. This dimensionless quantity is known as the Reynolds number. From the experiment, It was observed that if $Re > 2100$, the dye simply disappeared and the flow has changed to laminar to turbulent flow.

For laminar flow, it is observed that fluid flows as smooth stream line and all other components of velocity are zero. Thus

$$v_z = v_z\left(r\right)$$

$$v_r = 0$$

$$v_\theta = 0$$

For turbulent flow, if we observe the fluid flows at a local point. It is observed that fluid flows in very random manner in all directions where these local velocities may be the function of any dimensions.

$$v_z = v_z(r, z, \theta, t)$$

$$v_r = v_r(r, z, \theta, t)$$

$$v_\theta = v_\theta(r, z, \theta, t)$$

Internal and external flows

Depending on how the fluid and the solid boundaries contact each other, the flow may be classified as internal flow or external flow. In internal flows, the fluid moves between solid boundaries as is the case when fluid flows in a pipe or a duct. In external flows, however, the fluid is flowing over an external solid surface, the example may be sited is the flow of fluid over a sphere as shown in Fig. 8.1.

On the other hand, if the velocity of fluid is very low the deviation due to disturbances may decay with time, and becomes negligible after that. Thus the flow remains in laminar region. Consider a practical example in which some cars are moving on the highway in the same direction but in the different lanes at different speeds. If suddenly, some obstacle comes on the road, then if the car's speed is sufficiently low, it can move on to other lane smoothly and come back to its original lane after the obstacle is crossed. This is the regular laminar case. On the other hand, if the car is moving at a high speed and suddenly encounters an obstacle, then the driver may lose control, and this car may move haphazardly and hit other cars and after that traffic may never return to normal traffic conditions. This is the turbulent case.

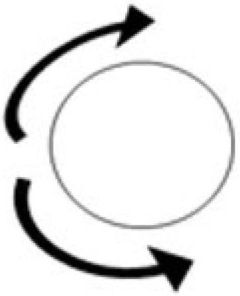


Fig 8.1 External flow around a sphere

Boundary layers and fully developed regions

Let us now consider the example of fluid flowing over a horizontal flat plate as shown in Fig.

The velocity of the fluid is V_∞ before it encounters the plate. As the fluid touches the plate, the velocity of the fluid layer just adjacent to the plate surface becomes zero due to the no slip boundary condition. This layer of fluid tries to drag the next fluid layer above it and reduces its velocity. As the fluid proceeds along the length of the plate (in x-direction), each layer starts to drag adjacent fluid layer but the effect of drag reduces as we go further away from the plate in y-direction. Finally, at some distance from the plate this drag effect disappears or becomes insignificant. This region where the velocity is changing or where the velocity gradients exists, is called the boundary layer region. The region beyond boundary layer where the velocity gradients are insignificant is called the potential flow region.

Thus, we see that for laminar flow there is only one component of velocity present and it depends only on one coordinate whereas the solution of turbulent flow may be vary complex. For turbulent flow, one can ask the question that if the fluid is flowing in the z direction then why are the velocity components in r and θ direction non-zero? The mathematical answer for this question can be deciphered from the equation of motion. The equation of motion is a non-linear partial differential equation. This non-linear nature of the equation causes instability in the system which produces flow in other directions. The instability in the system may occur due to any disturbances or noise present in the environment.

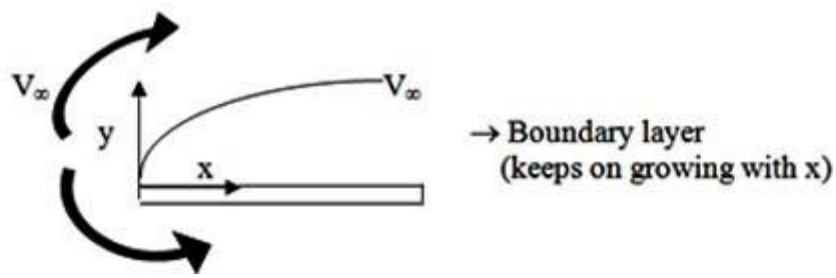


Fig 8.2 External flow over a flat plate

As depicted in Fig. 8.2, the boundary layer keeps growing along the x-direction, and may be referred to as the developing flow region. In internal flows (e.g. fluid flow through a pipe), the boundary layers finally merge after flow over a distance as shown in Fig. 8.3 below.

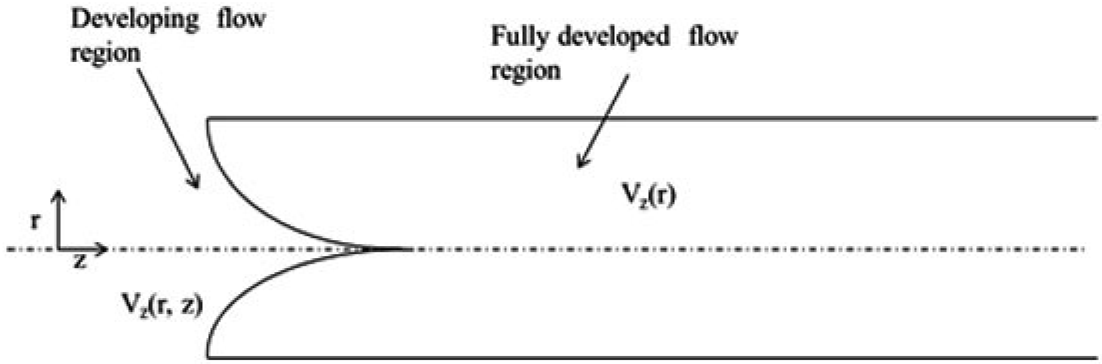


Fig 8.3 Developing flow and fully developed flow region

The region after the point at which the layers merge is called the fully developed flow region and before this it is called the developing flow region. In fact, fully developed flow is another important assumption which is taken for finding solution for varity of fluid flow problem. In the fully developed flow region (as shown in Figure 8.3), the velocity vz is a function of r direction only. However, the developing flow region, velocity vz is also changing in the z direction.

Main axioms of transport phenomena

The basic equations of transport phenomena are derived based on following five axioms.

- Mass is conserved, which leads to the equation of continuity.
- Momentum is conserved, which leads to the equation of motion.
- Moment of momentum is conserved leads to an important result that the 2nd order stress tensor $\underset{\sim}{\tau}$ is symmetric.

- Energy is conserved, which leads to equation of thermal energy.
- Mass of component i in a multi-component system is conserved, which leads to the convective diffusion equation.

The solution of equations, resulting from axiom 2, 4 and 5 leads to the solution of velocity, temperature and concentration profiles. Ones these profiles are known, all other important information needed can be determined. We first take the axiom -1. Other axioms will be taken up one by one letter on.
There are three types of control volumes (CV) which may be chosen for deriving the equations based these axioms.

- Rectangular shaped control volume fixed in space

In this case, the control volume is rectangular volume element and is fixed in space. This method is the easiest to understand but requires more number of steps.

- Irregular shaped control volume element fixed in space

In this case, the control volume can be of any shape, but it is again fixed in space. This method is somewhat more difficult than the previous method as it requires little better understanding vector analysis and surface and volume integrals.

- Material volume approach

In this case, the control volume can be of any shape but moves with the velocity of the flowing fluid. This method is most difficult in terms of mathematics, but requires least number of steps for deriving the equations.

All three approaches when applied to above axiom, lead to the same equations. In this web course, we follow the first approach. Other approaches may be found elsewhere.
Axioms-1

Mass is conserved

Consider a fluid of density ρ flowing with velocity $\underset{\sim}{v}$ as shown in Fig. 8.4. Here, ρ and $\underset{\sim}{v}$ are functions of space (x,y,z) and time (t). For conversion of mass, the rate of mass entering and leaving from the control volume (net rate of inflow) has to be evaluated and this should be equal to the rate of accumulation of mass in the control volume (CV). Thus, conservation of mass may be written in words as given below

$$\begin{pmatrix}\text{rate of accumulation of}\\ \text{mass in control volume}\end{pmatrix} = \begin{pmatrix}\text{rate of inflow of mass}\\ \text{into control volume}\end{pmatrix} - \left(\begin{matrix}\text{rate}\\ \text{out}\end{matrix}\right.$$

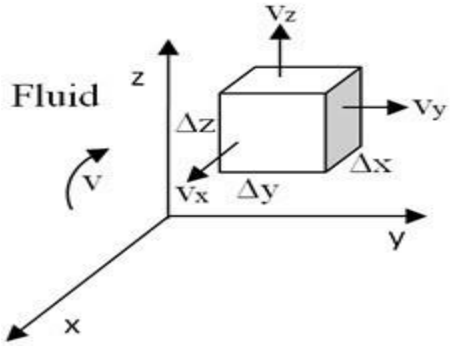


Fig 4 Fixed rectangular volume element through which fluid is flowing

The equation is then divided by the volume of the CV and converted into a partial differential equation by taking the limit as all dimensions go to zero. This limit effectively means that CV collapses to a point, thereby making the equation valid at every point in the system.

Let m and m+Δm be the mass of the control volume at time t and t+Δt respectively. Then, the rate of accumulation,

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

In order to evaluate the rate of inflow of mass into the control volume, we need to inspect how mass enters the control volume. Since the fluid velocity has three components vx, vy and vz, we need to identify the components which cause the inflow or the outflow at each of the six faces of the rectangular CV. For example, it is the component vx which forces the fluid to flow in the x direction, and thus it makes the fluid enter or exit through the faces having area ΔyΔz at x = x and x = x+Δx respectively. The component vy forces the fluid in y direction, and thus it makes the fluid enter or exit through the faces having area ΔxΔz at y = y and y = y+Δy respectively. Similarly, the component vz forces the fluid to flow in z direction, and thus it makes fluid enter or exit through the faces having area ΔxΔy at z = z and z = z+Δ z respectively.

The rate mass entering in x direction through the surface ΔyΔz is (ρvxΔyΔz|x), the rate of mass entering in y direction through the surface ΔxΔz is (ρvyΔxΔz|y) and the rate of mass entering from z direction through the surface ΔxΔy is (ρvzΔxΔy|z). In a similar manner, expressions for the rate of mass leaving from the control volume may be written.

Thus, the conservation of mass leads to the following expressions

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z) = \Big[\rho v_x\Delta y\Delta z\big|_x - \rho v_x\Delta y\Delta z\big|_{x+\Delta x} + \rho v_y\Delta x\Delta z\big|_y - \rho v_y\Delta x\Delta z\big|_{y+\Delta y} + \rho v_z$$

Dividing the Equation (8.1) by the volume ΔxΔyΔz, we obtain

$$\frac{\partial\rho}{\partial t} = \left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \Big[\frac{(\rho v_z)|_z - (\rho}{\Delta z}$$

Note that each term in Equation (8.2) has the unit of mass per unit volume per unit time. Now, taking the limits Δx→0, Δy→0 and Δz→0, we get

$$\frac{\partial\rho}{\partial t} = \lim_{\Delta x\to 0}\left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \lim_{\Delta y\to 0}\left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \lim_{\Delta z\to 0}\Big[$$

and using the definition of derivative, we finally obtain

$$\frac{\partial\rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right]$$

Equation (8.4) is applicable to each point of the fluid. Rearranging the terms, we get the equation of continuity, may be written as given below.

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

We need not to derive the equation of continuity again and again in other coordinate system (that is, spherical or cylindrical). The idea is to rewrite Equation (8.5) in vector and tensor form. Once it is written in this form, the same equation may be applied to other coordinate system as well. Thus, the Equation (8.5) may be rewritten in vector and tensor form as shown below.

$$\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}.(\rho\underset{\sim}{v}) = 0$$

Vector and tensor analysis of cylindrical and spherical coordinate systems is not done here, and can be looked up elsewhere. Thus, the final expressions in cylindrical and spherical coordinates are given as below.

cylindrical coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Spherical coordinates (r, θ, φ)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho v_\varphi)=0$$

Equation of continuity in terms of substantial derivative

The second term in Equation (8.6) may be broken into two parts as shown below. Partial derivative present in the Equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{\partial \rho}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In the above equation, the first two terms may be combined using the definition of substantial derivative to obtain the following equation.

$$\frac{D\rho}{Dt}+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In some cases, the fluid may be incompressible, i.e. density ρ is a constant with time as well as space coordinates. For example, water may be assumed as an incompressible fluid under isothermal conditions. In fact, all liquids may be assumed as incompressible fluids under isothermal conditions. For this special case, the equation of continuity may be further simplified as shown below

$$\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \quad (\rho \text{ is constant})$$

The above equation for an incompressible fluid does not mean that the system is under steady state conditions. The velocity of the fluid may still be a function of time. It only implies that if the velocity of the fluid changes in a particular direction (x, y or z) then it should also change in the other directions such that mass is conserved without changing its density. The equation of continuity provides additional information about the velocity profile and helps in solution of

EQUATION OF MOTION

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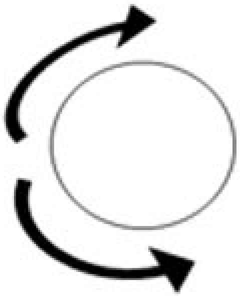

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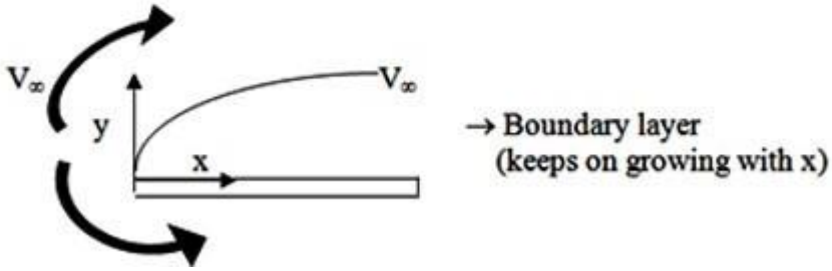

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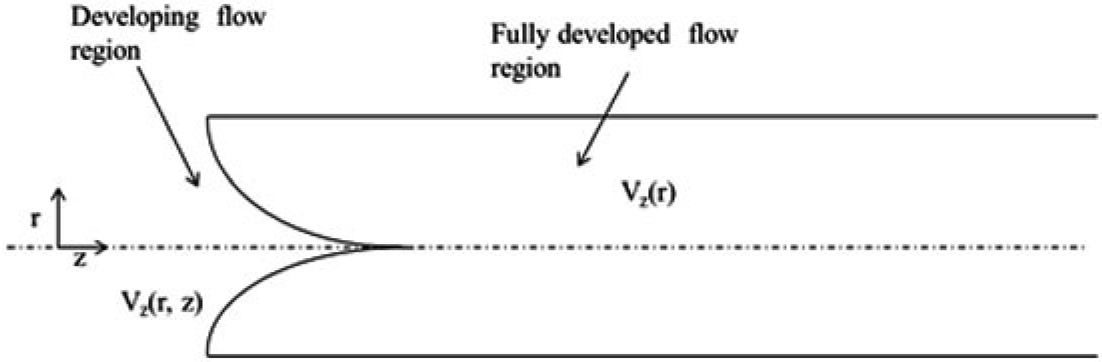

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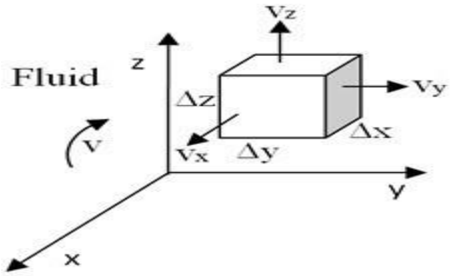


Fig 8.4 Fixed rectangular volume element through which fluid is flowing

The equation is then divided by the volume of the CV and converted into a partial differential equation by taking the limit as all dimensions go to zero. This limit effectively means that CV collapses to a point, thereby making the equation valid at every point in the system.

Let m and m+Δm be the mass of the control volume at time t and t+Δt respectively. Then, the rate of accumulation,

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

In order to evaluate the rate of inflow of mass into the control volume, we need to inspect how mass enters the control volume. Since the fluid velocity has three components vx, vy and vz, we need to identify the components which cause the inflow or the outflow at each of the six faces of the rectangular CV. For example, it is the component vx which forces the fluid to flow in the x direction, and thus it makes the fluid enter or exit through the faces having area ΔyΔz at x = x and x = x+Δx respectively. The component vy forces the fluid in y direction, and thus it makes the fluid enter or exit through the faces having area ΔxΔz at y = y and y = y+Δy respectively. Similarly, the component vz forces the fluid to flow in z direction, and thus it makes fluid enter or exit through the faces having area ΔxΔy at z = z and z = z+Δ z respectively.

The rate mass entering in x direction through the surface ΔyΔz is (ρvxΔyΔz|x), the rate of mass entering in y direction through the surface ΔxΔz is (ρvyΔxΔz|y) and the rate of mass entering from z direction through the surface ΔxΔy is (ρvzΔxΔy|z). In a similar manner, expressions for the rate of mass leaving from the control volume may be written.

Thus, the conservation of mass leads to the following expressions

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z) = \Big[\rho v_x\Delta y\Delta z\big|_x - \rho v_x\Delta y\Delta z\big|_{x+\Delta x} + \rho v_y\Delta x\Delta z\big|_y - \rho v_y\Delta x\Delta z\big|_{y+\Delta y} + \rho v_z$$

Dividing the Equation (8.1) by the volume ΔxΔyΔz, we obtain

$$\frac{\partial\rho}{\partial t} = \left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \left[\frac{(\rho v_z)|_z - (\rho}{\Delta z}\right.$$

Note that each term in Equation (8.2) has the unit of mass per unit volume per unit time. Now, taking the limits Δx→0, Δy→0 and Δz→0, we get

$$\frac{\partial\rho}{\partial t} = \lim_{\Delta x\to 0}\left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \lim_{\Delta y\to 0}\left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \lim_{\Delta z\to 0}\Big[$$

and using the definition of derivative, we finally obtain

$$\frac{\partial\rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right]$$

Equation (8.4) is applicable to each point of the fluid. Rearranging the terms, we get the equation of continuity, may be written as given below.

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

We need not to derive the equation of continuity again and again in other coordinate system (that is, spherical or cylindrical). The idea is to rewrite Equation (8.5) in vector and tensor form. Once it is written in this form, the same equation may be applied to other coordinate system as well. Thus, the Equation (8.5) may be rewritten in vector and tensor form as shown below.

$$\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}.(\rho\underset{\sim}{v}) = 0$$

Vector and tensor analysis of cylindrical and spherical coordinate systems is not done here, and can be looked up elsewhere. Thus, the final expressions in cylindrical and spherical coordinates are given as below.

cylindrical coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Spherical coordinates (r, θ, φ)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho v_\varphi)=0$$

Equation of continuity in terms of substantial derivative

The second term in Equation (8.6) may be broken into two parts as shown below. Partial derivative present in the Equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{\partial \rho}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In the above equation, the first two terms may be combined using the definition of substantial derivative to obtain the following equation.

$$\frac{D\rho}{Dt}+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In some cases, the fluid may be incompressible, i.e. density ρ is a constant with time as well as space coordinates. For example, water may be assumed as an incompressible fluid under isothermal conditions. In fact, all liquids may be assumed as incompressible fluids under isothermal conditions. For this special case, the equation of continuity may be further simplified as shown below

$$\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \quad (\rho \text{ is constant})$$

The above equation for an incompressible fluid does not mean that the system is under steady state conditions. The velocity of the fluid may still be a function of time. It only implies that if the velocity of the fluid changes in a particular direction (x, y or z) then it should also change in the other directions such that mass is conserved without changing its density. The equation of

continuity provides additional information about the velocity profile and helps in solution of equation of motion.
Solution of momentum transport problem by shell momentum balances

Here, we solve few simple problems of fluid mechanics with simple geometries by using the shell momentum balance approach. This will lead to greater understanding of various terms involved in the application of conservation of momentum in fluid given in Equation (9.1)

Flow through circular tube

Flow of fluids through a circular tube is a common problem, encountered frequently in different fields of engineering. Consider an incompressible, Newtonian fluid, flowing through a horizontal circular tube as shown in Fig. (10.1). Assume that the fluid flow is laminar and under steady state conditions. Determine the velocity profile and average velocity of the fluid using shell momentum balance approach.

Assumptions

- Fluid density and viscosity are constants.
- System is in steady state.
- Laminar flow (simple shear flow).
- Newton's law of viscosity is applicable.
- Fully developed flow.

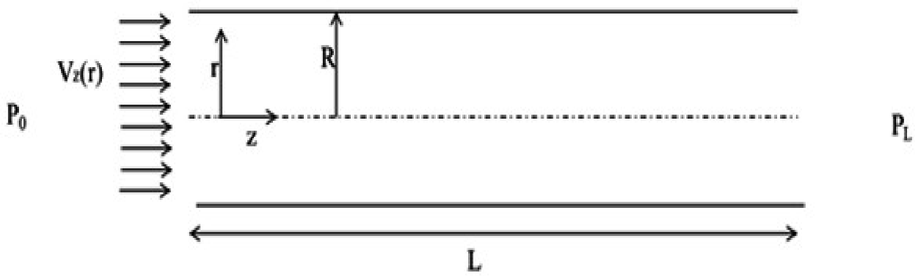


Fig Laminar flow in a horizontal pipe
Intuitively guess the velocity profile
Since the flow is steady and laminar, we may intuitively say that the velocities in r direction and θ direction are zero. Due to steady state conditions, the fluid velocity in z direction, vz, is not dependent on time t. Furthermore, due to the axisymmetric geometry fluid flow the velocity vz is

independent of θ. Thus,

$$v_r = 0,\ v_\theta = 0,\ v_z = v_z(r, z, \cancel{\theta}, \cancel{t})$$

$$v_z = v_z(r, z)$$

By applying the equation of continuity in cylindrical coordinates

$$\frac{\partial v_z}{\partial z} = 0$$

Hence,

$$v_z = v(r)$$

Since the fluid is flowing in z direction, we may conclude the following.

- Since vr=0, r directional momentum balance is not important.
- Since vθ=0 , θ directional momentum balance is again not important.
- Since vz≠0, z directional momentum balance is most important.

Equation for circular pipe:

The control volume should be decided very carefully. The geometry and size of the control volume should be taken according to the geometry of the system and based on the conditions given in the problem. In this case, the geometry of the pipe is cylindrical, hence we use the cylindrical control volume. The fluid is flowing in the z direction but velocity is changing only in r direction. Therefore, the control volume is taken in such a way that the variable thickness of the control volume is in the r direction. As the flow is not dependent on z and θ coordinates, we may choose any dimension in z or θ directions. This means that z may be any length. It may be L/4, L/2 or L. In a similar manner, any value of θ may be taken. It may be 2π or π or π/2 or π/4. However, in the r direction, we need to take the differential thickness dr. These arguments leads us to a control volume as shown in Fig. (10.2). The length of the cylindrical shell is L which is equal to length of pipe and thickness is dr.

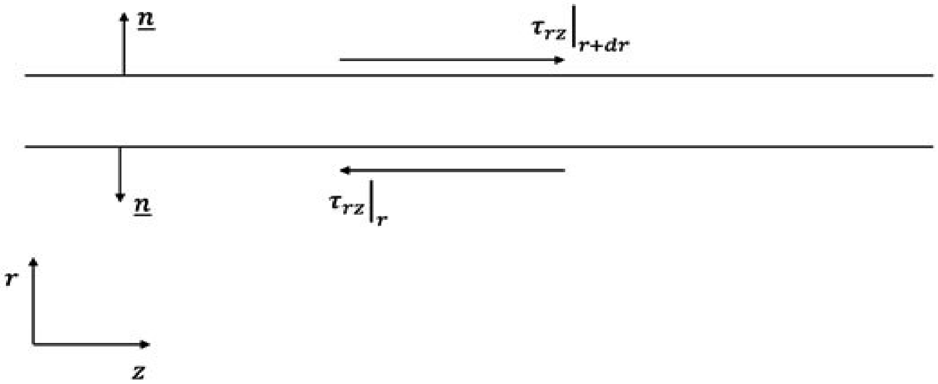


Fig 10.2 Control volume for flow through pipe

Momentum balance

As discussed earlier, the shear stress/forces may be written in two ways:

- Taking shear stress as actual shear forces.
- Taking shear stress as momentum flux.

Here, we show that both methods lead to the same final results for velocity profile.

Momentum balance using shear stress as shear force

Momentum flux entering the control volume by convection

=

$$(\rho v_z^2 2\pi r dr)\big|_{z=0}$$

Momentum flux leaving the control volume by convection

=

$$(\rho v_z^2 2\pi r dr)\big|_{z=L}$$

Since the pipe is horizontal, the force due to gravity is zero. No other body forces are acting on the control volume.

Surface forces

- Pressure force: Fluid is flowing in z direction only. So pressure forces which are working on the surface normal to z direction are

Pressure force at z=0 is

$$P_0 2\pi r dr\Big|_{z=0}$$

Pressure force at z=L is

$$- P_L 2\pi r dr\Big|_{z=L}$$

- Shear forces: The shear stress tensor in cylindrical coordinate is given below.

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

Among all 9 components the first column of stresses are important for r directional flow, the second column of stresses are important for θ directional flow, and the third column are important for z directional flow. Since the fluid is flowing in the z direction, only the third column needs to be considered. Since the Velocity gradient is present only in the r-direction, only τ_{rz} needs to be considered, the remaining two terms are not significant. Now, we need to decide the direction in which the shear forces are acting. Recall

$$\underset{\sim}{T}_n = \underset{\sim}{\delta}_n . \underset{\sim}{\tau}$$

Where the unit vector $\underset{\sim}{\delta}_n$ is the outer normal of a surface and if it is in positive direction then $\underset{\sim}{T}_n$ is also positive while if it is in negative direction then $\underset{\sim}{T}_n$ is shown as negative direction. Therefore, τ_{rz} (as a force) is positive at r+dr and negative at r as shown in Fig. 10.2.(Note: the first index, z, in τ_{rz} from right to left indicates the direction of force and second index, r, indicates the surface on which it acts).

Accumulation term: Due to steady state system, the rate of accumulation of momentum equals to zero .

General momentum balance is given below

$$\begin{pmatrix} \text{rate of accummulation} \\ \text{of momentum in CV} \end{pmatrix} = \begin{pmatrix} \text{rate of momentum} \\ \text{entering CV} \\ \text{by convection} \end{pmatrix} - \begin{pmatrix} \text{rate of momentum} \\ \text{leaving CV} \\ \text{by convection} \end{pmatrix} + \left(\sum \text{applied forces}\right)$$

or in this case

$$0 = (rv_z^2 2\pi r dr)|_{z=0} - (rv_z^2 2\pi r dr)|_{z=L} + 0 + P_0 2\pi r dr$$
$$- P_L 2\pi r dr + (\tau_{rz} 2\pi r L)|_{r+dr} - (\tau_{rz} 2\pi r L)|_r$$

Since the velocity is constant along the axial direction as shown in Equation (10.2), the first two terms in Equation (10.8) are cancel out and we are left with following Equation.

$$0 = 2\pi L\left[\left(\tau_{rz} r\right)|_{r+dr} - \left(\tau_{rz} r\right)|_r\right] + P_0 2\pi r dr - P_L 2\pi r dr$$

Dividing by $2\pi r dr$, we have

$$0 = \frac{P_0 - P_L}{L} + \frac{(\tau_{rz} r)|_{r+dr} - (\tau_{rz} r)|_r}{r dr}$$

As dr→0, the Equation (10.10) may be rewritten as given below.

(Note that, τ_{rz} is a function of r only which means we get the total derivative instead of the partial derivative.)

$$\frac{d(\tau_{rz} r)}{dr} = \frac{-r\,(P_0 - P_L)}{L}$$

Further integrating the Equation (10.11) once with respect to the variable r, we obtain

$$\tau_{rz} r = -\frac{r^2}{2}\frac{(P_0 - P_L)}{L} + c_1$$

or

$$\tau_{rz} = -\frac{r}{2}\frac{(P_0 - P_L)}{L} + \frac{c_1}{r}$$

Here, c1 is a constant of integration. Equation (10.12) shows that if r=0, the value of τ_{rz} will be infinite, which is physically not possible. Therefore, c1 must be zero. Hence,

$$\tau_{rz} = -\frac{r}{2}\frac{(P_0 - P_L)}{L}$$

Now, by applying Newton’s law of viscosity, and taking τ_{rz} as force, we obtain

$$\tau_{rz} = +\mu\frac{dv_z}{dr} = -\frac{r}{2}\frac{(P_0 - P_L)}{L}$$

Momentum balance using shear stress as momentum flux

Now, we will employ the second method where shear force are considered as momentum flux. To indicate the direction of momentum flux, we draw the arrow in r direction and find where this arrow enters the control volume and also leaves the control volume as shown in Fig (10.3). Thus, the momentum flux enters the control volume through the surface 2 π rL at r=r and leaves through the surface 2 π rL at r=r+dr.

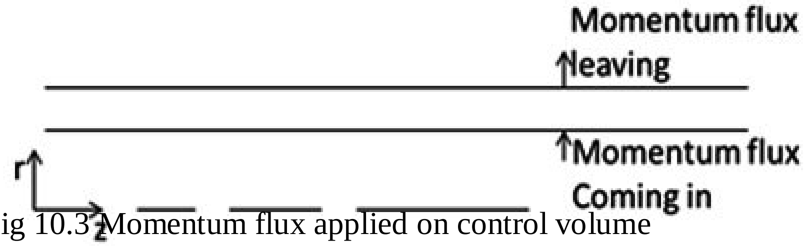


Fig 10.3 Momentum flux applied on control volume

Thus,

Momentum flux at r = r is

$$(\tau_{rz} 2\pi rL)\big|_r$$

Momentum flux at r = r +Δr is

$$(\tau_{rz} 2\pi rL)\big|_{r+dr}$$

(Note: when we consider τ_{rz} as the momentum flux, first index, z, indicates the direction of momentum flux, while the second index, r, indicates the direction of flow of momentum flux from higher to lower value. Subsequently, it will become clear that if we follow the coordinate system’s directions and assume momentum is flowing in this direction, the sign convention for momentum flux is automatically taken place.)

In this case, momentum balance in Equation (9.2) may be modified as shown below

$$\begin{pmatrix}\text{rate of accummulation}\\ \text{of momentum in CV}\end{pmatrix}=\begin{pmatrix}\text{rate of momentum}\\ \text{entering CV}\\ \text{by convection}\end{pmatrix}-\begin{pmatrix}\text{rate of momentum}\\ \text{leaving CV}\\ \text{by convection}\end{pmatrix}+(\sum$$

$$+\begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{entering CV}\end{pmatrix}-\begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{leaving CV}\end{pmatrix}$$

Here, the shear stress are taken into account as momentum flux. The pressure and gravity are the only applied forces.

Substituting various terms in above equation, we obtain

$$0 = 0 + (P_0 - P_L)\ 2\pi r dr + (\tau_{rz} 2\pi r L)\big|_r - (\tau_{rz} 2\pi r L)\big|_{r+dr}$$

Dividing by $2\pi r dr$, weobtain

$$0 = (\tau_{rz} r)\big|_r - (\tau_{rz} r)\big|_{r+dr} + \frac{(P_0 - P_L)}{L}$$

Again as dr→0 Equation (10.17) leads to

$$0 = -\frac{d(\tau_{rz}\mathrm{r})}{rdr} + \frac{(\mathrm{P}_0 - \mathrm{P}_L)}{L}$$

or

$$\frac{d(\tau_{rz}\mathrm{r})}{dr} = \frac{(\mathrm{P}_0 - \mathrm{P}_L)}{L} r$$

By integrating the Equation (10.18), we have

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r + \frac{c_1}{r}$$

As we discussed earlier, c1 should be zero. Therefore,

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Now applying Newton’s law of viscosity where shear stress is taken as momentum flux, we obtain

$$\mu \frac{dv_z}{dr} = -\frac{(P_0 - P_L)}{2L} r$$

Equation (10.14) and (10.20) are identical and hence show that both methods finally lead to the same result.
To obtain velocity profile we further integrating the Equation (10.21)

$$v_z = -\frac{(P_0 - P_L)}{2\mu L} \frac{r^2}{2} + c_2$$

Here c2 is the second constant of integration which may be determined by using appropriate boundary condition.

Boundary condition

By no-slip boundary condition
vz=0 at r=R

$$0 = -\frac{(P_0 - P_L)}{4\mu L} R^2 + c_2$$

$$c_2 = \frac{(P_0 - P_L)}{4\mu L} R^2$$

Substituting the value of c2 in Equation (10.22), we finally get

$$v_z = \frac{(P_0 - P_L)}{4\mu L}(R^2 - r^2)$$

Note: c1 can also be calculated by using the boundary condition in terms of velocity vz: i.e., vz is finite at r=0

or $\left.\frac{dv_z}{dr}\right|_{r=0} = 0$ (since the velocity profile is symmetric about r=0).

Thus, the velocity profile for flow through pipe is given by the following expression

$$v_z = \frac{(P_0 - P_L)}{4\mu L} R^2\left(1 - \frac{r^2}{R^2}\right)$$

The maximum velocity of the fluid will be exhibited at the centre of the pipe and is given by

$$v_{z\max} = v_z\,|_{r=0} = \frac{(P_0 - P_L)}{4\mu L} R^2$$

Alternatively, the velocity profile may also be expressed in terms of the maximum velocity as

$$v_z = v_{z,\max}\left(1 - \frac{r^2}{R^2}\right)$$

The average velocity of the fluid in the pipe is the average of all local velocities. Thus, this may be calculated by estimating the volumetric flow rate through the pipe and then dividing it by the cross sectional area of the pipe. The total volumetric flow in the system is

$$Q = \int dQ$$

where, dQ is the volumetric flow rate from small cylindrical strip of thickness dr.

$$= \int v_z 2\pi r dr$$

By substituting the value of v z from equation (10.27), we have

$$= \int v_{z\,max} \left[1 - \frac{r^2}{R^2} \right] 2\pi r dr$$

By integrating the equation (10.30) from r=0 to r=R, we obtain

$$= \int_0^R 2\pi\, v_{z\,max} \left[r - \frac{r^3}{R^2} \right] dr$$

or

$$= 2\pi v_{z\,max} \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R$$

Thus,

$$Q = \pi v_{z\,max} \frac{R^2}{2}$$

and average velocity is

$$v_{z,avg} = \frac{Q}{A_c}$$

$$= \frac{\pi\, v_{z,max} \frac{R^2}{2}}{\pi R^2}$$

or

$$v_{z,avg} = \frac{v_{z,max}}{2}$$

The velocity profile for laminar flow in a circular tube is shown in Fig. 10.5.

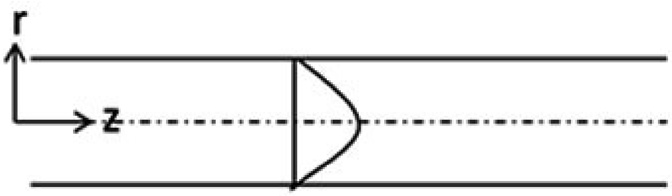


Fig 10.5 Velocity profile in horizontal pipe

We can also find the radial distance at which the local velocity of fluid flow equals the average velocity. For this, substitute $v_z = v_{z,avg} = \frac{v_{z,\max}}{2}$ into Equation (10.26), we obtain

$$1 - \frac{r^2}{R^2} = \frac{1}{2}$$

$$\frac{r^2}{R^2} = \frac{1}{2}$$

$$\text{or, } r = \frac{R}{\sqrt{2}}$$

Finally, the volumetric flow rate in terms of pressure drop is as follows

$$Q = v_{zavg} \pi R^2 = \frac{\pi(\mathrm{P}_0 - \mathrm{P}_\mathrm{L})}{8\mu L} R^4$$

$$Q = \frac{\pi(\mathrm{P}_0 - \mathrm{P}_\mathrm{L})}{128 u L} D^4$$

Equation (10.36) is known as the Hagen – Poiseuille equation. Thus, if the pressure drop is given, we can calculate the volumetric flow rate in the pipe and vice-versa. This equation can also be used for the calculation of viscosity in capillary flow viscometer. However, it may be noted that Hagen – Poiseuille equation is valid only for fully developed laminar flow. Therefore, when this equation is used for various calculations there may be some errors due to developing and exiting flow at both ends of the pipe. Hence, this equation has to be modified for real situations.

Friction factor

The friction factor is a dimensionless number, which provides an idea about the magnitude of shear stress produced by a solid boundary as fluid flows. This is defined as the ratio of shear stress at the wall and the kinetic energy head of the fluid, $\frac{1}{2} \rho v^2_{zavg}$. Here, ρ is the density and v_{zavg} is the average velocity of fluid. The friction factor is thereby defined as

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2_{zavg}}$$

where, τ_w is the shear force per unit area on the wall of the tube. This may be calculated as shown below

$$\tau_w = -\left(-\tau_{rz}\big|_{r=R}\right)$$

Here, first minus sign is used as the inside surface of the tube wall has outer normal in the negative r direction and second minus sign is used because τ_{rz} is treated here as momentum flux. If τ_{rz} is treated as actual shear force then positive sign would have to be taken. For fully developed laminar flow, the velocity profile is parabolic and is given by

$$v_z = v_{zmax}\left[1 - \frac{r^2}{R^2}\right]$$

Evaluating the velocity gradient at the wall (r=R), we have

$$\left.\frac{dv_z}{dr}\right|_{r=R} = -\frac{2v_{z\,max}}{R}$$

Thus, the shear stress considered as momentum flux is given by

$$\tau_{rz}\big|_{r=R} = -\mu\left.\frac{dv_z}{dr}\right|_{r=R} = +\frac{2\mu v_{zmax}}{R}$$

or

$$\tau_w = \tau_{rz}\big|_{r=R} = \frac{2\mu v_{zmax}}{R} = \frac{2\mu(P_0 - P_L)}{4\mu L}\frac{R^2}{R} = \frac{(P_0 - P_L)}{2L}R$$

The friction factor may now be calculated as shown below

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2_{zavg}} = \frac{\frac{(P_0 - P_L)}{2L}R}{\frac{1}{2}\frac{\rho(P_0 - P_L)}{8\mu L}R^2} = \frac{8\mu}{\rho R v_{zavg}}$$

or

$$f = \frac{16\mu}{\rho D v_{zavg}} = \frac{16}{Re}$$

Equation (11.7) shows that the friction factor in laminar flow region depends only on the Reynolds number. Clearly, the friction factor is also a dimensionless number.

Friction factor in turbulent flow

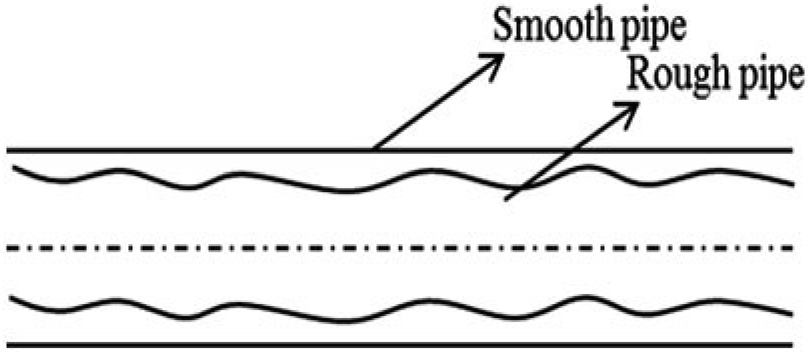
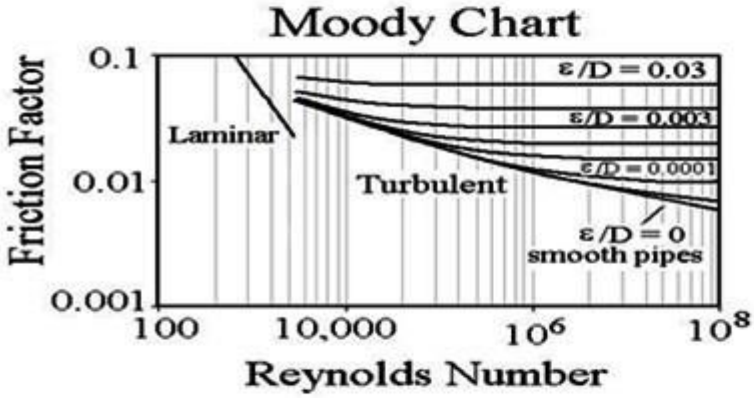


Fig 11.1 Smooth and rough surface of pipe

In turbulent flow, the friction factor also depends on the surface of the pipe. A rough pipe leads to higher turbulence than a smoother pipe, so that the friction factor for smoother pipes is less than that for rougher pipes. The ratio of surface roughness height (∈) to pipe diameter (D) is used to quantify the “roughness” of the pipe surface. In practice, the shear stress on the wall may be calculated by measuring the pressure drop across the pipe for a given flow rate. Thus, friction factor may be calculated as the function of Reynolds number and plotted on a log-log plot for a given surface roughness. The curves are different for different surface roughness as shown in figure. (11.2). The collection of these f-Re plots is called Moody Chart as shown in figure below, and can be used for estimating the friction factor for given flow parameters.



Ref: http://www.brighthub.com/engineering/civil/articles

Solution of some more fluid flow problems by shell momentum balance approach

In this section, we solve a few more fluid mechanics problems in simple geometries using the shell momentum balance approach. The detail procedure, which was also used in previous example, is outlined below.

1) Make a diagram of the flow geometry with the appropriate coordinate system

2) Specify all necessary assumptions

3) Intuitively assume the velocity profile

This is an important step for solving these problems. In laminar flow, the fluid flows in parallel layer without mixing. Thus, it is easy to guess the non-zero components of velocities by intuition.

4) Apply of the equation of continuity to modify the velocity profile

5) Determine the non-zero shear stress component(s)

Since the shear stress components depend on the velocity profile, the non-zero shear stress components may now be determine.

6) Determine control volume and make shell momentum balance for the control volume

Draw control volume in system diagram according to system shape, size and problem statement. The selection of proper control volume is very important to solve problem correctly. The control volume should be select in such way that it can be easily integrated for whole system. The differential length of control volume should be taken in direction of changing velocity.
Write momentum balance equation for the control volume. The shear stress may be considered as shear force or as momentum flux, both provide the same results as shown in previous example . Write down all surface and body force acting on the fluid carefully. Finally obtain a appropriate

differential equation and integrate.

7) Boundary conditions

Use appropriate boundary conditions which help us to determine the constant of integration in above step.

Falling film on an inclined flat surface

An inclined surface of length L and width W is situated at an angle B to the vertical direction as shown in Fig. (11.3). A Newtonian fluid is freely falling on the surface as a film of thickness δ. Assuming the flow to be laminar, determine the velocity profile, flow rate and shear force on the surface by the fluid.

Solution

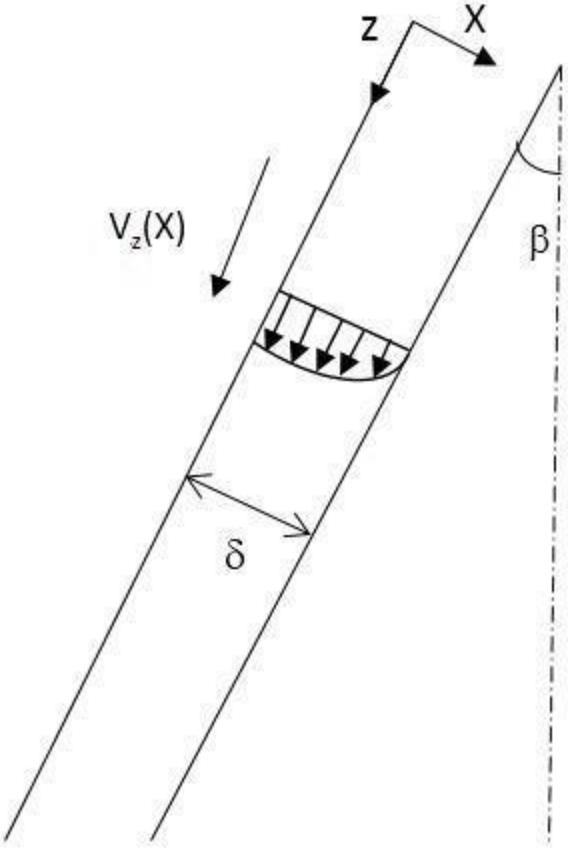


Fig 11.3 Laminar flow on an inclined surface

Assumptions

- Constant density, viscosity
- Steady state
- Laminar flow (simple shear flow)
- Fully developed flow
- Newton's law of viscosity is applicable

Assume velocity profile
The fluid is flowing in the z direction, hence only the z component of velocity is non-zero. Thus, we may assume

$$v_x = 0, \quad v_y = 0$$

an

$$v_z = v_z(x, y, z, t)$$

We may further assume that vz does not depends upon y coordinate. Since the flow is steady, vz does not depend on time. Thus,

$$v_z = v_z(x, z)$$

Using the equation of continuity in the cartesian coordinates for constant fluid density, we have

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

which reduces to

$$\frac{\partial v_z}{\partial z} = 0$$

Equation (11.11) indicates that vz does not depend on the z coordinate. Thus,

$$v_z = v_z(x)$$

There are nine components of the shear stress as shear force or momentum flux, namely

$$\tau_{xx}\,\tau_{xy}\,\tau_{xz} \rightarrow \text{acting on } x \text{ surface}$$
$$\tau_{yx}\,\tau_{yy}\,\tau_{yz} \rightarrow \text{acting on } y \text{ surface}$$
$$\tau_{zx}\,\tau_{zy}\,\tau_{zz} \rightarrow \text{acting on } z \text{ surface}$$

Since vz is only the non-zero velocity, and also it is the function of x coordinate , τ_{zx} is the only significant component of shear stress and we need to write momentum balance only in z direction. Because the pressure is same at both ends of the inclined plane, there is no pressure force on the fluid. Now, we can solve this problem by assuming shear stress as a shear force or shear stress as momentum flux.

Assuming τ_{xz} as momentum flux
Draw a control volume of length L, width W and differential thickness dx.

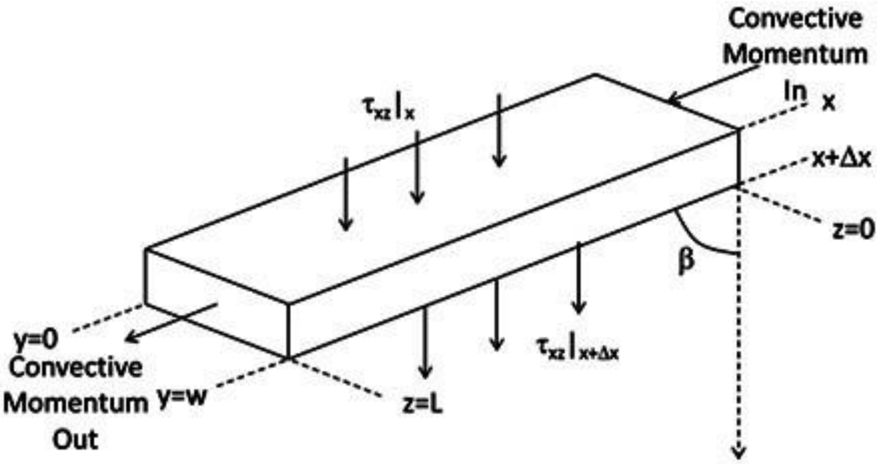


Fig 11.3 Control volume for falling film problem
Momentum balance in x direction
Rate of momentum flux entering CV due to viscous transport at

$$x = LW\tau_{xz}\big|_x$$

Rate of momentum flux leaving CV due to viscous transport at

$$x+\Delta x = LW\tau_{xz}\big|_{x+\Delta x}$$

Gravity force acting on fluid in z direction

$$= (LW\Delta x)(\rho g \cos\beta)$$

Rate of momentum flux entering in CV due to convective transport

$$= (\rho v_z W dx \; v_z)\big|_{z=0}$$

Rate of momentum flux leaving from CV due to convective transport

$$= (\rho v_z W dx \; v_z)\big|_{z=L}$$

Now, when above terms are substituted for z-momentum balance, we obtain

$$(\rho v_z w dx v_z)\big|_{z=0} - (\rho v_z w dx \, v_z)\big|_{z=L} + LW(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Since the velocity vz does not depends on z coordinate, the first two terms cancel out and we obtain

$$LW(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Dividing Equation (11.19) by volume of the control volume (LWΔx), we have

$$\frac{(\tau_{xz}|_x - \tau_{xz}|_{x+dx})}{\Delta x} + (\rho g \cos\beta) = 0$$

As Δx→0 , The Equation (11.20)simplified to

$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta$$

The Newton’s law of viscosity (here, shear stress is defined as momentum flux) is given by

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$

$$\frac{d}{dx}\left(-\mu \frac{dv_z}{dx}\right) = \rho g \cos\beta$$

or

$$-\mu \frac{d^2 v_z}{dx^2} = \rho g \cos\beta$$

or

$$\frac{d^2 v_z}{dx^2} = -\left(\frac{\rho g \cos\beta}{\mu}\right)$$

By integrating the Equation (11.25), we have

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos\beta}{\mu}\right) x + c_1$$

or

$$v_z = -\left(\frac{\rho g \cos\beta}{\mu}\right) \frac{x^2}{2} + c_1 x + c_2$$

The above equation requires two boundary conditions for determining c1 and c2.
Boundary conditions
1 At x=0 the liquid surface is in contact with air where the shear stresses at both gas liquid phases should be equal. Thus,

$$\tau_{xz\,(air)}\Big|_{x=0} = \tau_{xz\,(liquid)}\Big|_{x=0}$$

Since both may be assumed Newtonian fluids, we have

$$\mu_g \rho_g \left.\frac{dv_{z(air)}}{dx}\right|_{x=0} = \mu\rho \left.\frac{dv_z}{dx}\right|_{x=0}$$

where ρg is the density and µg is the viscosity of air. Thus

$$\left.\frac{dv_z}{dx}\right|_{x=0} = \frac{\mu_g \rho_g}{\mu\rho} \left.\frac{dv_{z(air)}}{dx}\right|_{x=0}$$

Since, µg and ρg is much smaller than µ and ρ, and Equation (11.30) may be approximately written as

$$\left.\frac{dv_z}{dx}\right|_{x=0} = 0$$

Substituting above boundary condition in Equation (11.26), we obtain

$$c_1 = 0$$

2. At x=δ no slip boundary condition may be applied, i.e., at

$$x = \delta \; , v_z = 0$$

Thus, from Equation (11.27), we get

$$0 = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2} x + c_2$$

or

$$c_2 = \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2}$$

Finally the velocity profile is obtained as

$$v_z = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{x^2}{2} + \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2}$$

or

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left(1 - \left(\frac{x}{\delta}\right)^2\right)$$

Falling film "Assuming τ_{xz} as shear force"

Now, we again solve the same problem (falling film over an inclined plane) by treating shear stress as a shear force. For this purpose, we take the same control volume as before.
For momentum balance in z direction, all terms are same as before except the terms for shear forces. Here, τ_{xz} represents the force in z direction acting on the surfaces which have normal in x direction. Shear force is positive if the outward normal is in positive direction and negative if normal is in negative direction. Thus,

shear force at x=x is
$-LW\tau_{xz}\big|_x$

Shear force at x=x+Δx is
$+LW\tau_{xz}\big|_{x+\Delta x}$
;

The z momentum balance for this case is as follows

$$LW(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Dividing Equation (12.3) by the volume of control volume WLΔx, we have

$$\frac{(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x)}{\Delta x} + (\rho g \cos\beta) = 0$$

As Δx→0 Equation (12.4) leads to

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos\beta$$

Now, substituting the Newton's law of viscosity for shear stress as a force

$$\tau_{xz} = \mu \frac{dv_z}{dx}$$

Therefore,

$$\mu \frac{d^2 v_z}{dx^2} = -\rho g \cos\beta$$

Equations (11.24) and (12.7) are the same, which show that both approaches provide the same answer.

Maximum velocity

It is clear from Equation (11.37) that the maximum velocity is given by

$$v_{z,max} = \frac{\rho g \delta^2 \cos\beta}{2\mu}$$

Average velocity and volumetric flow rate of falling film

vz is the linear velocity in z direction. Hence, the volumetric flow rate can be determined by integrating it over the cross section of flow (Wδ).Thus ,

$$Q = \int_0^W \int_0^\delta v_z \, dx\, dy$$

From Equation (11.37), we get

$$Q = \int_0^W \int_0^\delta \frac{\rho g \delta^2 \cos\beta}{2\mu} \left(1 - \left(\frac{x}{\delta} \right)^2 \right) dx dy$$

By integrating Equation (12.10), we find

$$Q = \frac{W \rho g \delta^3 \cos\beta}{3\mu}$$

To obtain the average velocity, we divide the volumetric flow rate by the cross sectional area.

$$\langle v_z \rangle_{avg} = \frac{Q}{\int_0^W \int_0^\delta dx\,dy}$$

or

$$\langle v_z \rangle_{avg} = \frac{\rho g \delta^2 \cos\beta}{3\mu}$$

Equation (12.12) may also be written as

$$\langle v_z \rangle_{avg} = \frac{2}{3} v_{z,max}$$

Force acting on solid surface due to the fluid

$$F = \int_0^L \int_0^W +\left(+\tau_{xz}\big|_{x=\delta}\right) dy\,dz$$

(Note: in Equation (12.14), first '+' sign shows the direction of the normal of the inclined surface and second '+' sign is taken since shear stress is defined as shear force). Thus,

$$F = \rho g \delta L W \cos\beta$$

In this lecture, we have once again seen that the shear stress tensor may be assumed as a shear force or as a momentum flux. In either case, we finally obtain the same expression for the velocity profile. The only difference is that when we treat shear stress as a shear force, it is included in the summation of all forces term in the momentum balance equation, while when we treat shear stress as momentum flux, it is written as momentum entering and leaving by the viscous transport. From now onwards, we will treat shear stress as momentum flux as it is more consistent with what we see in heat transfer as Fourier's law of heat conduction and in mass transfer as Fick's law of diffusion. Thus, in transport phenomena (Momentum transport, Heat transport, and Mass transport) for the basic transport laws we have minus sign in front the relevant gradient implying fluxes flow from higher values to lower values.

Falling film on the outside of a circular shell

In an experiment, a fluid flows upward through a small circular shell and then flows downward out side the tube under laminar conditions as shown in Fig. 12.2. We need to set up a relevant momentum balance and determine the velocity profile, mass flow rate and the force acting on outer surface of the tube.

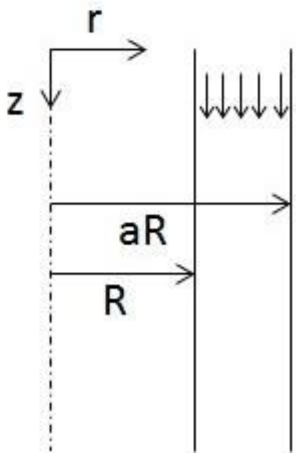


Fig 12.2 Falling film outside the circular tube

Assumptions

- Density and viscosity are constants.
- Steady state.
- Fully developed laminar flow.
- Newton’s law of viscosity is applicable.

Non-zero velocities

Fluid is flowing in the z direction due to gravity. There is no driving force in the θ direction and a solid surface is present in the r direction. Therefore, we may intuitively assume that

$$v_z = v_z(r,z)$$

$$v_\theta = 0$$

$$v_r = 0$$

Now, using the equation of continuity in cylindrical coordinate system, we have

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

or

$$\rho \frac{\partial}{\partial z}(v_z)=0$$

From Equation (12.18), we obtain that vz is independent to z. Therefore,

$$v_z = v_z(r)$$

Choose a control volume in the film of differential thickness dr and length L (it is a cylindrical shell).

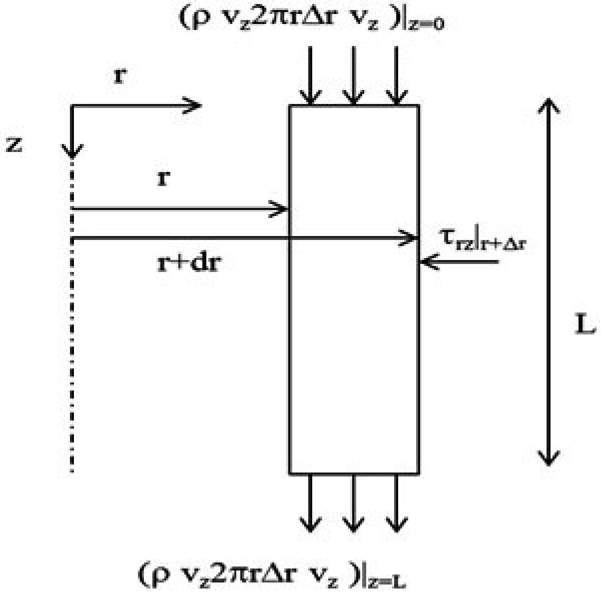


Fig 12.3 Control volume for falling film outside the circular tube

There are nine components of shear stress tensor. Since the fluid is flowing in z direction and it is a function of r only, we may argue as before τ_{rz} is the only important component of the shear stress tensor. The other components are insignificant for momentum balance in z direction. The momentum balance in z-direction is given below.

Momentum balance for control volume
Convective momentum entering the control volume at z=0 is

$$(\rho v_z 2\pi r \Delta r v_z)\big|_{z=0}$$

Convective momentum leaving the control volume at z=L is

$$(\rho v_z 2\pi r \Delta r v_z)\big|_{z=L}$$

Shear stress as momentum flux entering the control volume at r= r is

$$(2\pi r L\tau_{rz})\Big|_{r=r}$$

Shear stress as momentum flux entering the control volume at r= r + Δr is

$$(2\pi r L\tau_{rz})\Big|_{r=r+\Delta r}$$

{Note: If you consider shear stress as momentum flux, then it always flows in the positive direction of axes}

Fluid is flowing only due to gravity and may be written as

$$(2\pi r\Delta r L\rho g)$$

Substituting above terms, we obtain

$$(\rho v_z 2\pi r D r v_z)\Big|_{z=0} - (\rho v_z 2\pi r\Delta r v_z)\Big|_{z=L} + (2\pi r L\tau_{rz})\Big|_{r=r} - (2\pi r L\tau_{rz})\Big|_{r=r+\Delta r} + 2\pi r\Delta r L\rho g = 0$$

Since velocity, vz, is not dependent on the z, the first two terms in above equation are equal and cancel out, leaving the following equation for momentum balance.

$$(2\pi r L\tau_{rz})\Big|_{r=r} - (2\pi r L\tau_{rz})\Big|_{r=r+\Delta r} + 2\pi r\Delta r L\rho g = 0$$

Dividing Equation (12.26) by volume of control volume $2\pi r\Delta r L$, weobtain

$$\frac{(r\tau_{rz}\big|_r - r\tau_{rz}\big|_{r+\Delta r})}{r\Delta r} = -\rho g$$

As dr→0, Equation (12.27) reduces to

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = -\rho g$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-\mu\frac{dv_z}{dr}\right)\right) = \rho g$$

After integration we obtain

$$\frac{dv_z}{dr} = -\frac{\rho g r}{2\mu} + \frac{c_1}{r}$$

and

$$v_z = -\frac{\rho g r^2}{4\mu} + c_1 \ln r + c_2$$

Here, c_1 and c_2 are the constants of integration.
Boundary conditions

1. r=aR we have the air water interface where we may assume that

$$\frac{dv_z}{dr} = 0$$

(The explanation is given earlier in Lecture 11.)
Substituting the above boundary condition, we obtain

$$c_1 = \frac{\rho g a^2 R^2}{2\mu}$$

2. At r = R, no slip boundary condition is applicable. Thus,

$$v_z\big|_{r=R} = 0$$

Using this boundary condition, we obtain

$$c_2 = \frac{\rho g R^2}{4\mu} - c_1 \ln R$$

or,

$$c_2 = \frac{\rho g R^2}{4\mu} - \frac{\rho g a^2 R^2}{2\mu} \ln R$$

Therefore, the velocity profile is given by

$$v_z = -\frac{\rho g r^2}{4\mu} + \frac{\rho g a^2 R^2}{2\mu} \ln r + \frac{\rho g a^2 R^2}{4\mu} - \frac{\rho g a^2 R^2}{2\mu} \ln R$$

or

$$v_z = \frac{\rho g R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right]$$

Maximum velocity

At r = aR, the velocity is maximum. Thus,

$$v_{z,max} = \frac{\rho g R^2}{4\mu}\left[1-\alpha^2+2\alpha^2 \ln a\right]$$

Equation of motion

In this section, we derive the equation of motion, which may be used for solving any fluid mechanics problem. This equation is based on axiom 2, i.e., the momentum is conserved. We consider a control volume having volume $\Delta x,\Delta y,\Delta z$ fixed in space.

According to the momentum conservation equation,

Rate of accumulation of control volume = Net rate of inflow of momentum by convection + net rate of momentum by viscous transport + pressure forces + gravity forces

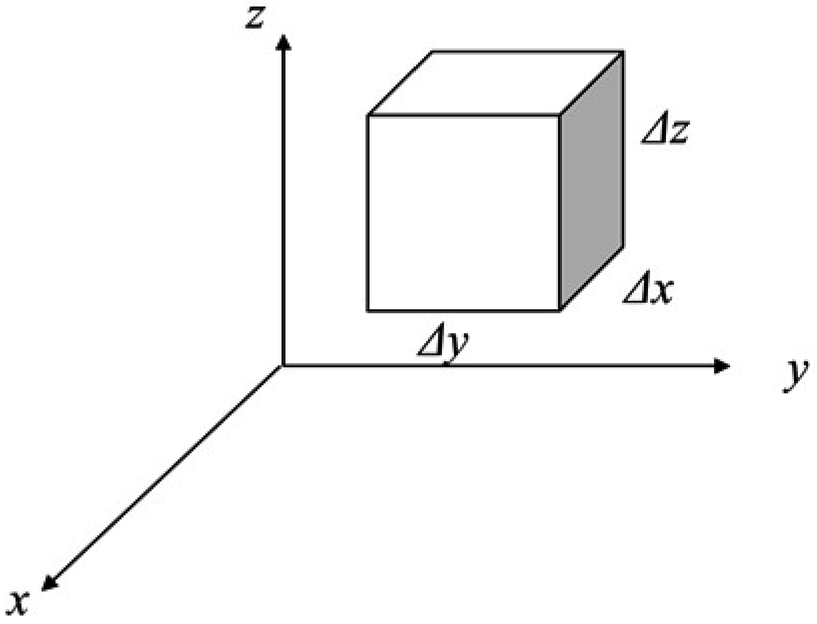


Fig 15.1 Cubical control volume fixed in space
Momentum balance in x direction

Rate of accumulation of x directed momentum in control volume

$$= \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z v_x)$$

Net rate of inflow of x directed momentum into CV by convection from x-phases

$$\left[(\rho v_x \Delta z \Delta y) v_x\right]_x - \left[(\rho v_x \Delta z \Delta y) v_x\right]_{x+\Delta x}$$

Net rate of inflow of x-momentum into CV from y-phases

$$= \left[(\rho v_y \Delta x \Delta z) v_x\right]_y - \left[(\rho v_y \Delta x \Delta z) v_x\right]_{y+\Delta y}$$

Net rate of inflow of x-momentum into CV from z-phases

$$= \left[(\rho v_z \Delta x \Delta y) v_x\right]_z - \left[(\rho v_z \Delta x \Delta y) v_x\right]_{.z+\Delta z}$$

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

In all shear stress component, the second index shows the direction of momentum flux and first index shows the direction in which the momentum is flowing. For example, τ_{yx} denotes the x directed momentum flowing in y direction. Therefore, the x directed momentum fluxes are τ_{xx}, τ_{yx} and τ_{zx}. Thus,

Net rate of inflow of x directed momentum by viscous transport from x phase are

$$= (\tau_{xx}\Delta z\Delta y)|_x - (\tau_{xx}\Delta z\Delta y)|_{x+\Delta x}$$

Net rate of inflow of x directed momentum by viscous transport from y phase are

$$= (\tau_{yx}\Delta x\Delta z)|_y - (\tau_{yx}\Delta x\Delta z)|_{y+\Delta y}$$

Net rate of inflow of x directed momentum by viscous transport from z phase are

$$=(\tau_{zx}\Delta x\Delta y)|_z - (\tau_{zx}\Delta x\Delta y)|_{z+\Delta z}$$

Net pressure force in x direction =

$$(P\Delta y\Delta z)|_x - (P\Delta y\Delta z)|_{x+\Delta x}$$

Gravity force in x direction =

$$(\rho\Delta x\Delta y\Delta z)\, g_x$$

Adding all the above terms and dividing by the volume of control volume Δx,Δy,Δz and finally taking the limits,
Δx→0, Δy→0,and Δz→0, we obtain

$$\frac{\partial(\rho v_x)}{\partial t}=-\left[\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}\right]-\left[\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right]-\frac{\partial p}{\partial x}+\rho g$$

The above equation is the x component of equation of motion in cartesian coordinate system. Similarly, for y-direction

$$\frac{\partial(\rho v_y)}{\partial t}=-\left[\frac{\partial(\rho v_x v_y)}{\partial x}+\frac{\partial(\rho v_y v_y)}{\partial y}+\frac{\partial(\rho v_z v_y)}{\partial z}\right]-\left[\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right]-\frac{\partial}{\partial}$$

and for z-direction

$$\frac{\partial(\rho v_z)}{\partial t}=-\left[\frac{\partial(\rho v_x v_z)}{\partial x}+\frac{\partial(\rho v_y v_z)}{\partial y}+\frac{\partial(\rho v_z v_z)}{\partial z}\right]-\left[\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right]-\frac{\partial}{\partial}$$

The above three equation may be combined in vector tensor form as

$$\frac{\partial(\rho\underset{\sim}{v})}{\partial t}=-\underset{\sim}{\nabla}.(\rho\underset{\sim}{v}\underset{\sim}{v})-\underset{\sim}{\nabla}.\underset{\sim}{\tau}-\underset{\sim}{\nabla}p+\rho\underset{\sim}{g}$$

In above form, the equation of motion may be used in any coordinate system.
Equation (15.14) may be written in substantial derivative form as shown below

$$\frac{\partial(\rho\underset{\sim}{v})}{\partial t}+\underset{\sim}{\nabla}.(\rho\underset{\sim}{v}\underset{\sim}{v})=-\underset{\sim}{\nabla}.\underset{\sim}{\tau}-\underset{\sim}{\nabla}p+\rho\underset{\sim}{g}$$

if $\underset{\sim}{x}$ and $\underset{\sim}{z}$ are the two vectors. We may use the following vector identity.

$$\nabla.(\underset{\sim}{x}\underset{\sim}{z})=\underset{\sim}{x}.\underset{\sim}{\nabla}\underset{\sim}{z}+\underset{\sim}{z}(\underset{\sim}{\nabla}.\underset{\sim}{x})$$

Now, replace $\underset{\sim}{x}$ by $\rho\underset{\sim}{v}$ and $\underset{\sim}{z}$ by $\underset{\sim}{v}$, then we have

$$\nabla.(\underset{\sim}{x}\underset{\sim}{z})=\nabla.(\rho\underset{\sim}{v}\underset{\sim}{v})=\rho\underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v}+\underset{\sim}{v}(\underset{\sim}{\nabla}.\rho\underset{\sim}{v})$$

also,

$$\frac{\partial(\rho \underset{\sim}{v})}{\partial t} = \rho \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \frac{\partial \rho}{\partial t}$$

After substituting Equations, the equation of motion reduces to

$$\underset{\sim}{v} \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underset{\sim}{v}}{\partial t} + \rho \underset{\sim}{v}.\underset{\sim}{\nabla} \underset{\sim}{v} + \underset{\sim}{v}(\underset{\sim}{\nabla}.\rho \underset{\sim}{v}) = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

Rearranging the terms on the left hand side, we have

$$\rho \left[\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} \underset{\sim}{v} \right] + \underset{\sim}{v} \left[\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}.\rho \underset{\sim}{v} \right] = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

But from the equation of continuity

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}.\rho \underset{\sim}{v} = 0$$

Hence,

$$\rho \left[\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} \underset{\sim}{v} \right] = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

or

$$\rho \frac{D\underset{\sim}{v}}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

Equations (15.20) and (15.21) are the generalized form of equation of motion without any assumption and may be applied to any coordinate system. The detailed form of this equation in cartesian, cylindrical and spherical coordinate system is given in Appendix-3.
Navier Stokes Equation for incompressible Newtonian fluid

The equation of motion may be further simplified by substituting the Newton's law of viscosity for the momentum flux term appearing in the equation of motion.
For a one-dimensional system where

vy=0,vz=0

we have seen that the Newton's law of viscosity may be written as,

$$\tau_{yx} = -\mu \frac{\partial v_x}{\partial y}$$

where, τ_{yx} represents x directed momentum flowing in the y direction However, in general, for a three dimensional flow, all 9 components of shear stress may be important. Thus,

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Here, τ_{xx} , τ_{yy} and τ_{zz} are the normal stresses and the remaining are shear stress.
Axiom 3: Moment of momentum is conserved

This axiom 3 leads to a very simple conclusion that the shear stress tensor is symmetric in nature. The derivative itself is lengthy and is not reproduced here. $\underset{\sim}{\tau}$ is symmetric implies that

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

Newton’s law of viscosity may now be generalized as given below. Again, the basis for this representation is not shown here, but it may be found in any standard books in fluid mechanics.

$$\underset{\sim}{\tau} = -\mu\underset{\sim}{\Delta} + \frac{2}{3}\mu(\underset{\sim}{\nabla}.\underset{\sim}{v})\delta_{ij}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_j$$

where,

$$\underset{\sim}{\Delta} = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j$$

Hence, we have the nine components of shear stress as

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3}\mu \underset{\sim}{\nabla}.\underset{\sim}{v}$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3}\mu \underset{\sim}{\nabla}.\underset{\sim}{v}$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3}\mu \underset{\sim}{\nabla}.\underset{\sim}{v}$$

$$\tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = -\mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)$$

$$\tau_{xz} = \tau_{zx} = -\mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

The detail form of Newtons law of viscosity in all coordinate system is given in Appendix- 01. Now, consider the situation when an incompressible fluid is flowing only in x direction and depends on y coordinate only. In such a case, we have $v_x = v_x(y)$, $v_y = 0$ and $v_z = 0$. We can easily see that for this case,

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = 0$$

$$\tau_{yz} = \tau_{zy} = 0$$

$$\tau_{xz} = \tau_{zx} = 0$$

and only significant components of stress are τ_{xy} and τ_{yx}. Also, the expression for τ_{yx} is the same as given earlier as Newton's law of viscosity. For rectangular coordinate system, substituting the value of $-\underset{\sim}{\nabla}.\underset{\sim}{\tau}$ in the x component of equation of motion, we obtain

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_x = -\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$$

Assuming that ρ and μ are constant, we obtain

$$-\underset{\sim}{\nabla}.\underset{\sim}{\tau} = -\left[\frac{\partial}{\partial x}\left(-2\mu\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(-\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right)\right]$$

or

$$= \mu\left[\frac{\partial}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial}{\partial y}\frac{\partial v_x}{\partial y} + \frac{\partial}{\partial y}\frac{\partial v_y}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial}{\partial z}\frac{\partial v_z}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_x}{\partial x}\right]$$

or

$$= \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial}{\partial x}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]\right]$$

But from equation of continuity for an incompressible fluid, we have

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Therefore,

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_x = \mu\left[\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right]$$

$$= \mu\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]v_x$$

or

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_x = \mu\underset{\sim}{\nabla}^2 v_x$$

or

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_y = \mu\underset{\sim}{\nabla}^2 v_y$$

similarly

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_z = \mu\underset{\sim}{\nabla}^2 v_z$$

Therefore, in vector and tensor form

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right) = \mu\underset{\sim}{\nabla}^2 \underset{\sim}{v}$$

Thus, the equation of motion reduce to

$$\rho\frac{D\underset{\sim}{v}}{Dt} = -\underset{\sim}{\nabla}p + \mu\underset{\sim}{\nabla}^2\underset{\sim}{v} + \rho\underset{\sim}{g}$$

Equation (16.25) is known as the Navier Stokes equation and is used for solving problems involving Newtonian fluids of constant density and viscosity. For non-Newtonian and compressible fluids, the generalized form of equation of motion given earlier must be used. The detailed forms of the equations of motion along with Navier Stokes equations in cartesian, cylindrical and spherical coordinates are given in the Appendix-03.
Solution of momentum transport problems using Navier Stokes equation

In this section, transport problems involving Newtonian fluids are solved by making use of the equation of motion or Navier Stokes equation. We will firstly solve the falling film problem and flow through a circular tube for comparing the solutions obtained earlier by using the shell momentum balance method. We will then proceed to solve some more fluid mechanics problems.

Falling film on an inclined surface

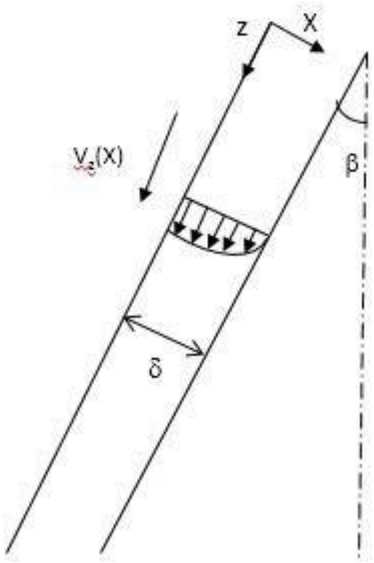


Fig 17.1 Falling film on inclined surface

This problem was solved earlier by the shell momentum balance technique. We will now try to solve this problem by using the Navier Stokes equations.
We are again required to make the same necessary assumptions as done earlier using the shell momentum balance technique. We postulate the non- zero components of the velocity and from there, determine the non-zero components of the shear stress tensor. These steps are the same as earlier and lead us to conclude that $v_z = v_z(r)$ and τ_{rz} is the only important component of shear stress. We now use the Navier Stokes equation in cartesian coordinates as given in Appendix-03.

x component is

$$\rho g_x = 0$$

y component is

$$\rho g_y = 0$$

z component is

$$\mu \frac{d^2 v_z}{dx^2} + \rho g_z = 0$$

where

$$g_z = g\cos\beta$$

Integrating Equation (17.3), we have

$$\frac{dv_z}{dx} = -(g\cos\beta)x + c_1$$

and

$$v_z = -(g\cos\beta)x + c_1 x + c_2$$

The boundary conditions are also the same as used earlier,
at

$$x = 0,\ \left.\frac{dv_z}{dx}\right|_{x=0} = 0$$

and
at

$$x = \delta,\ v_z = 0$$

This leads to the solution for velocity profile, as

$$v_z = -\frac{g\delta^2\cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right]$$

which is same as obtain earlier using shell momentum balance approach.
Fluid flow through a vertical tube
A Newtonian fluid is flowing inside a vertical tube having circular cross section due to pressure difference and gravity. Solve the problem using the Navier Stokes equations.

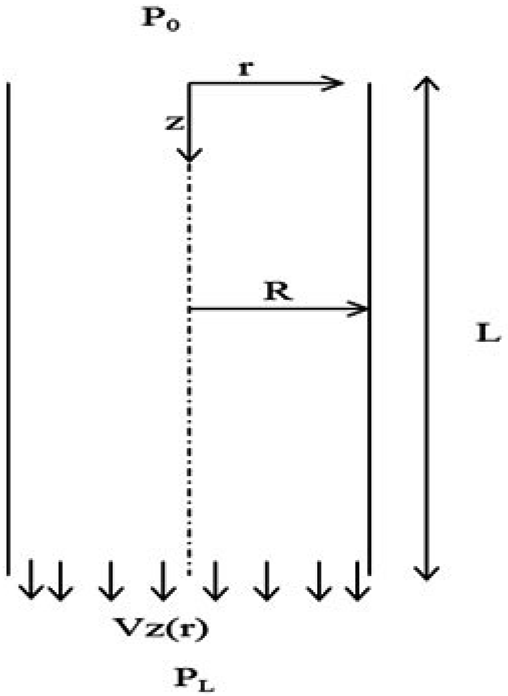


Fig 17.2 Flow through a vertical circular tube

A similar type of problem (for a horizontal pipe) was solved earlier using the shell momentum balance technique. Therefore, the initial steps are the same and include making appropriate assumptions and postulating the non- zero velocity components. As shown earlier, it leads to the conclusion that $v_z = v_z(r)$

Now using the Navier Stokes equation for cylindrical co-ordinates, after eliminating all zero terms, we have r- component of Navier Stokes equation

$$\frac{\partial P}{\partial r} = 0$$

θ- component

$$\frac{\partial P}{\partial \theta} = 0$$

z - component

$$-\frac{\partial P}{\partial z} + \mu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) + \rho g_z = 0$$

We can combine gravity and pressure forces as to rewrite Equation (17.11) as,

$$-\frac{\partial P_c}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=0$$

where, Pc is the effective pressure including the gravity, and is defined as,

$$P_c = P - \rho g_z z$$

Note that since pressure changes in only z direction and vz is a function of r only the partial derivative may be converted to total derivative. Furthermore, in Equation (17.12), the first term is only a function of z and the second term is only a function of r, i.e.,

$$F_1(z) + F_2(r) = 0$$

This leads to result that F1 and F2 both are constants as Equation (17.13) is true for all values of z and r.

$$F_1(z) = c_1;\ F_2(r) = -c_1$$

Therefore,

$$\frac{dP_c}{dz} = c_1$$

By integrating the Equation (17.15)

$$P_c = c_1 z + c_2$$

Boundary conditions are
at

$$z=0,\ P_c = P_{c0}$$

and
at

This leads to the following solution

$$z=L,\ P_c = P_{cL}$$

$$\frac{P_{c0} - P_{cL}}{L} = c_1$$

By substituting in Equation (17.12)

$$-\left(\frac{P_{c0} - P_{cL}}{L}\right) + \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0$$

or

$$v_z = -\left(\frac{P_{c0} - P_{cL}}{4\mu L}\right) r^2 + c_3 \ln r + c_4$$

Boundary conditions are
at r= 0, v_z is finite
and

at r = R,
$v_z = 0$

This leads to

$$v_z = \left(\frac{P_{c0} - P_{cL}}{4\mu L}\right) R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

which is again similar to what we have seen for a horizontal tube except for pressure difference term. In fact, it can be shown that the velocity profile given in Equation (17.22) is valid for any configuration, horizontal, vertical, or inclined, with effective pressure is defined as

$$P_c = P - \rho g_z z$$

$$T_z = \int_0^{2\pi}\int_0^R (-\tau_{\theta\phi})\big|_{\theta=\pi/2} r dr\, r\, d\phi = 2\pi\left(\frac{\mu\Omega}{\psi}\right)\int_0^R r^2 dr$$

$$v_\theta = v_\theta(r, z)$$

For simplifying the problem further, we may assume that for low shear rates

$$v_\theta = r f(z)$$

Using the θ component of the Navier – Stokes equation for cylindrical co-ordinate systems

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2} = 0$$

By substituting Equation (19.25), we get

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 f(z)\right)\right) + \frac{r\partial^2 f(z)}{\partial z^2} = 0$$

or

$$\frac{\partial^2 f(z)}{\partial z^2} = 0$$

$$f(z) = c_1 z + c_2$$

After integration, we finally obtain

where c_1 and c_2 are the integral constant.

Boundary conditions are

at $z = 0$, $v_\theta = 0$ or $f(z) = 0$

thus,

$$c_2 = 0$$

and at $z = B, v_\theta = v$ or

$$f(z) = v/r = \Omega$$

$$r\,\partial\theta$$

Now, the *z*-component of the torque exerted on the fluid by the upper rotating disc, may be calculated as

$$T_z = \int_0^{2\pi}\int_0^{R}\left(-\tau_{z\theta}\right)r\Big|_{z=B}\, r\,dr\,d\theta$$

or

$$= 2\pi\mu\,\frac{\Omega}{}\int^{R} r^{3}\,dr$$

$$T_z = \pi\mu\frac{\Omega}{2B}R^2$$

Thus, by plotting the angular velocity Ω vs torque T_z, the viscosity may be

Non-Newtonian fluids

Non-Newtonian fluids are the fluids which do not obey Newton’s law of viscosity. For describing Non-Newtonian fluids, let’s recall the Newton's law of viscosity experiment. There are two long parallel plate situated at distance h to each other. Top plate is stationary and bottom plate is moving with velocity $\underset{\sim}{v}$ as shown in Fig. (20.1).

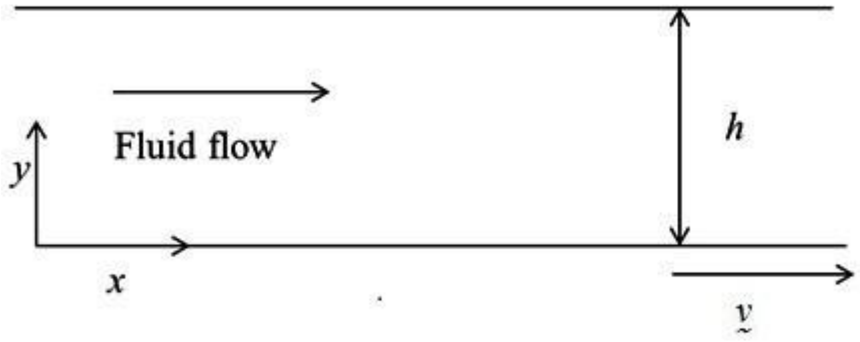


Fig 20.1 Non-Newtonian flow between two parallel plates

If a force, F, is applied to move plate, then (τ_{xy})

$$\tau_{xy} = \frac{F}{A}$$

and under steady state conditions when h is small and when

$$\frac{dv_x}{dy} = \frac{\underset{\sim}{v}}{h}$$

Now, we calculate τ_{xy} by repeating experiments for different applied forces and velocity achieved by the bottom plate and plotting a graph as shown in Fig. (20.2). Depending on the nature of fluid, different types of curves may be obtained.

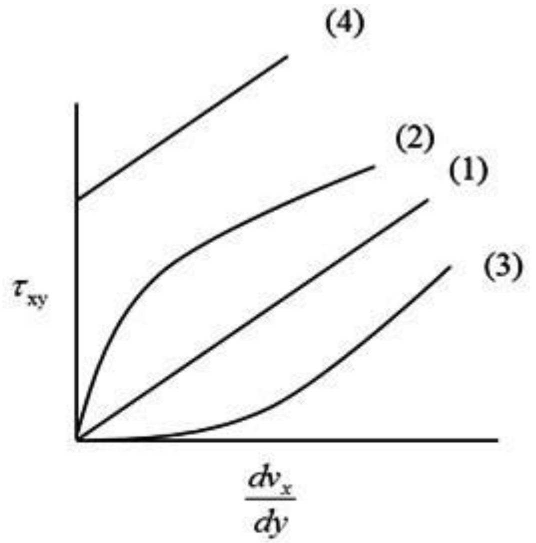


Fig 20.2 Shear stress vs. shear strain diagram for Newtonian and non-Newtonian fluids

Rheological behaviour of fluids

If fluid shows the behaviour like curve (1) then it is a Newtonian fluid. Other fluids are non-Newtonian fluids. Curve (2) represents a Pseudo-plastic fluid, curve (3) represents a Dilatant fluid, and curve (4) represents a Bingham plastic fluid. There are several Theoretical and empirical models available to describe the rheological behaviour of non-Newtonian fluids. Here, we discuss some of them, which come under the group of generalized Newtonian models. Basic equation for a generalized non-Newtonian fluid is given below

$$\tau_{yx} = -\eta \frac{dv_x}{dy}$$

Here, η is the apparent viscosity, which is clearly a function of shear rate as may be seen. Therefore,

$$\eta = f\left(\frac{dv_x}{dy}\right)$$

If the apparent viscosity increases with increase in shear rate, $\frac{dv_x}{dy}$, then the fluid is called Dilatant fluid and if it decreases with increase in shear rate, $\frac{dv_x}{dy}$ then fluid is called Pseudo-plastic fluid. Some fluids require a critical shear stress to initiate the flow. These fluids are called Bingham fluids. Some important rheological models for non-Newtonian fluids are given below.

1 Power Law or Ostwald De Waele model

Power law or Ostwald De Waele model is the most generalized model for non-Newtonian fluids. The expression of this model is given in Equation (20.3)

$$\tau_{yx} = -m\left|\left(\frac{dv_x}{dy}\right)\right|^{n-1}\frac{dv_x}{dy}$$

Here, apparent viscosity η is defined as,

$$\eta = m\left|\left(\frac{dv_x}{dy}\right)\right|^{n-1}$$

This is a two-parameter model where m and n are the two parameters.

If n = l then η =m

where m is similar to the viscosity of the fluid and model shows the Newtonian behaviour .

If n>1, then η increases with increasing shear rate and the model shows the Dilatant behaviour.

If n<1, then η decreases with increasing shear rate and the model shows the Pseudo-plastic behaviour.

Modulus sign

In power law model, modulus sign can be removed according to the value of shear rate.

1. If $\frac{dv_x}{dy}$ is positive, then

$$\eta = m\left(\frac{dv_x}{dy}\right)^{n-1}$$

2. If $\frac{dv_x}{dy}$ is negative, then

$$\eta = m\left(-\frac{dv_x}{dy}\right)^{n-1}$$

Several fluids do not show single type of rheological behaviour. They show Newtonian behaviour for a range of shear stress and Non-Newtonian behaviour for some other ranges of shear stresses. Several models have been suggested for these types of fluids. Some popular models like Eyring model, Ellis model, Reiner Philipp off model and Bingham Fluid model are discussed here.

2. Eyring model

Eyring model is a two-parameter model. The equation of Eyring model is as follow

$$\sinh\left(\frac{\tau_{yx}}{A}\right) = -\frac{1}{B}\frac{dv_x}{dy}$$

where A, B are the two parameters.

In Eyring model, if $\tau_{xy} \rightarrow 0$ which means very low shear forces, we have

$$\sinh\left\{\frac{\tau_{yx}}{A}\right\} \rightarrow \frac{\tau_{yx}}{A}$$

Therefore, as $\tau_{yx} \rightarrow 0$, the model shows Newtonian behaviour

$$\tau_{yx} = -\frac{A}{B}\frac{dv_x}{dy}$$

Here, viscosity = $\left(\frac{A}{B}\right)$

If τ_{yx} is very large, the model shows Non-Newtonian behaviour as shown Fig. 20.3

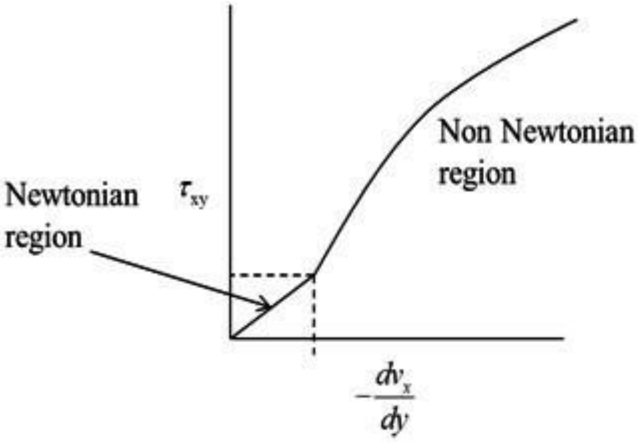


Fig 20.3 Shear stress vs. shear strain diagram for Eyring model

Therefore, Eyring model may be used for a fluid which shows Newtonian behaviour at low shear rates and non- Newtonian behaviour at high shear rates.

3. Ellis model

Ellis model is a three-parameter model. The equation of this model is as follows

$$-\frac{dv_x}{dy} = \left\{\varphi_o + \varphi_1 \left|\tau_{yx}\right|^{\alpha-1}\right\}\tau_{yx}$$

Here, φ_0, φ_1 and α are the three parameters .
Here, we consider some special cases,
1. If $\varphi_1 = 0$ then Equation (20.11) reduce to

$$\frac{dv_x}{dy} = -\varphi_0\tau_{yx}$$

or

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

which is same as Newton’s law of viscosity with $\left(\frac{1}{\varphi_0}\right)$ as the viscosity of the fluid.

2. If $\varphi_0 = 0$, then

$$-\frac{dv_x}{dy} = -\varphi_1 \,|\, \tau_{yx} \,|^{a-1}\, \tau_{yx}$$

which is similar to a Power law model

3. If α >1 and τ_{yx} is small then the second term is approximately zero and equation reduces to

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

which is similar to Newton’s law of viscosity.

4. If α <1 and τ_{yx} is very large, then again, second term is negligible and we have

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

Which again shows Newtonian behaviour. Therefore, Ellis model may be used for fluids which show Newtonian behaviour at very low and very high shear stresses, but non-Newtonian behaviour at intermediate value of shear stresses.

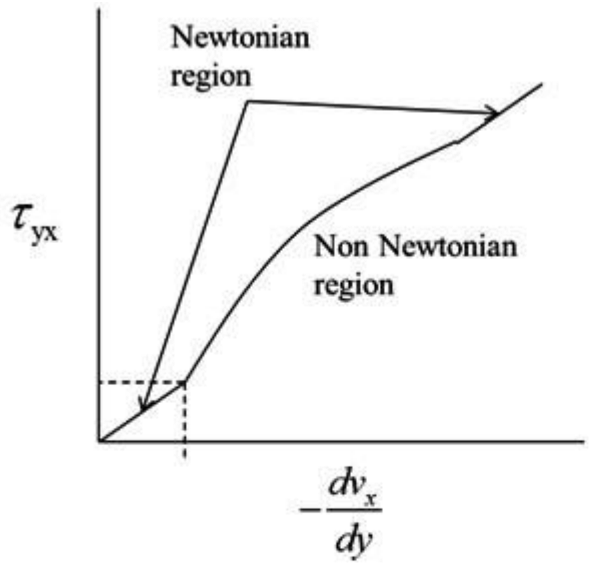


Fig 20.4 Shear stress vs. shear strain diagram for Ellis model

This type of behaviour may be shown by some polymer melts

4. Reiner Philipp off model

This is also a three-parameter model. The equation of Reiner Philipp off model is as follows,

$$-\frac{dv_x}{dy}=\left[\frac{1}{\mu_\infty+\dfrac{\mu_0-\mu_\infty}{1+\left(\dfrac{\tau_{yx}}{\tau_s}\right)^2}}\right]\tau_{yx}$$

where, μ_0 , μ_∞ and τ_s are the three parameters.

In Reiner Philipp off model, if τ_{yx} is very large, the equation reduces to,

$$-\frac{dv_x}{dy}=\frac{1}{\mu_\infty}\tau_{yx}$$

or

$$\tau_{yx} = -\mu_{\infty} \frac{dv_x}{dy}$$

which is same as the Newton's law of viscosity,

If τ_{yx} is very small then equation reduces to

$$-\frac{dv_x}{dy} = \frac{1}{\mu_0} \tau_{yx}$$

or

$$\tau_{yx} = -\mu_0 \frac{dv_x}{dy}$$

which is also same as the Newton's law of viscosity. Therefore, Reiner Philipp off model may be used for a fluid which shows Newtonian behaviour at very low and very high shear stresses but non-Newtonian behaviour for intermediate values of shear stress. Here, μ_0 and μ_{∞} represent the viscosity of fluid at very low and very high shear stress conditions respectively.

5. Bingham Fluid model

Bingham fluid is special type of fluid which require a critical shear stress to start the flow.
The equation of Bingham fluid model are given below

$$\tau_{yx} = -\left[\mu + \frac{\tau_0}{\left|\frac{dv_x}{dy}\right|}\right] \frac{dv_x}{dy}$$

if

$$|\tau_{yx}| > \tau_0$$

$$\frac{dv_x}{dy}=0$$

A typical shear stress vs. shear rate diagram for a Binghum model is shown below

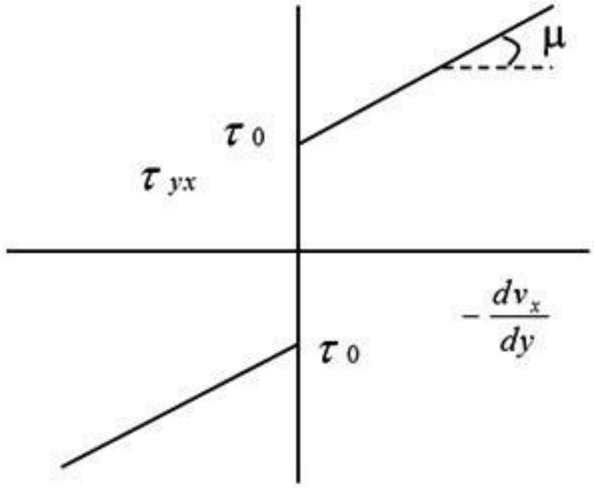


Fig 20.5 Shear stress vs. shear strain diagram for Bingham model

Momentum transport problems involving Power law and Bingham fluids:

In this section, we will solve fluid mechanics problem for Power law and Bingham plastic fluids. These problems have been earlier solved for Newtonian fluids. We have chosen the same problems here for better understanding.

Falling film on inclined plane

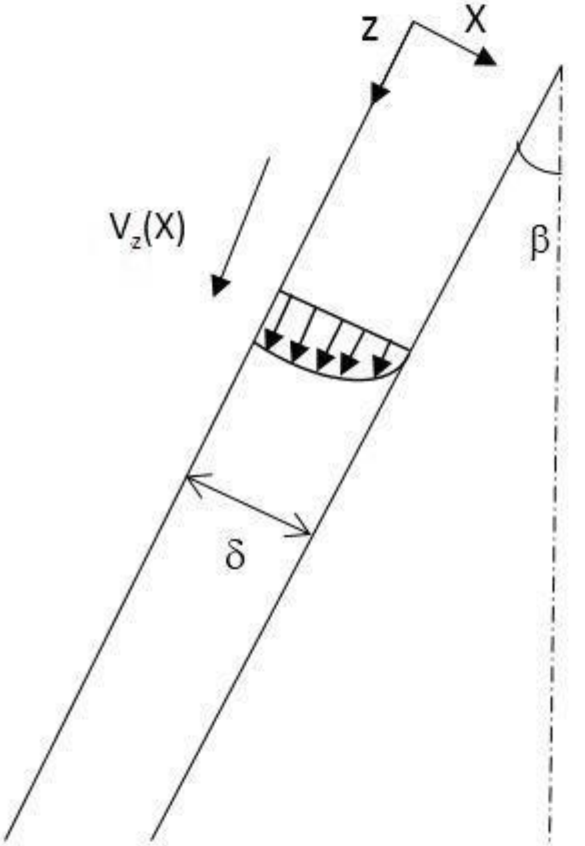


Fig 21.1 Falling film problem for non-Newtonian fluid

Initial steps, such as making appropriate assumptions, finding important velocity components, applying equation of continuity, and determining important shear stress components are similar as steps seen for Newtonian fluid in lecture 11 and 17. As before $v_z = v_z(x)$ and is the only non-zero velocity component and τ_{xz} is the only important shear stress component.

(Note: Since the forms of shear stress $\underset{\sim}{\tau}$ for Newtonian and non-Newtonian fluids are same, the only difference is the viscosity μ for Newtonian fluids and apparent viscosity η for non-Newtonian fluids and furthermore as non-zero components of velocities are also same, the same components of shear stress $\underset{\sim}{\tau}$ are significant for both Newtonian and non- Newtonian fluids.)

To solve the problem, we start with the generalized equation of motion in terms of $\underset{\sim}{\tau}$. Since the fluid is moving in z direction, discarding all terms which are zero, z-component of the equation of motion reduces to

$$-\frac{d\tau_{xz}}{dx}+\rho g_z=0$$

where

$$g_z = g\cos\beta$$

therefore,

$$\tau_{xz}=\rho g\cos\beta x+c_1$$

For Power law fluids

$$\tau_{xz}=-\eta\frac{dv_z}{dx}$$

$$\eta=m\left|\frac{dv_z}{dx}\right|^{n-1}$$

Since vz is decreasing with increasing value of x , the negative sign should be used for removing the modulus sign, i.e. ,

$$\tau_{xz}=-m\left(-\frac{dv_z}{dx}\right)^{n-1}\left(\frac{dv_z}{dx}\right)$$

or

$$\tau_{xz}=m\left(-\frac{dv_z}{dx}\right)^{n}$$

By substituting in above,

$$\left(-\frac{dv_z}{dx}\right)^n = \frac{\rho g \cos\beta}{m} x + c_1$$

By applying the boundary condition, at $x = 0, \quad \tau_{xz}\big|_{air} = \tau_{xz}\big|_{fluid}$ which simplifies to

$$\left.\frac{dv_z}{dx}\right|_{x=0} = 0$$

as disused in lecture 11

By substituting this boundary condition in Equation (21.8), we get $c_1 = 0$. Therefore,

$$\left(-\frac{dv_z}{dx}\right)^n = \frac{\rho g \cos\beta}{m} x$$

or

$$v_z = -\sqrt[n]{\frac{\rho g \cos\beta}{m}} \frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c_2$$

Here, c_2 is another integral constant.

Now, using the second boundary condition, at $x = \delta \quad , v_z = 0$, we finally obtain

$$v_z = \frac{n}{n+1} \sqrt[n]{\frac{\rho g \cos\beta}{m}} \delta^{\frac{n+1}{n}} \left[1 - \left(\frac{x}{\delta}\right)^{\frac{n+1}{n}}\right]$$

Tube flow problem for Power law fluid

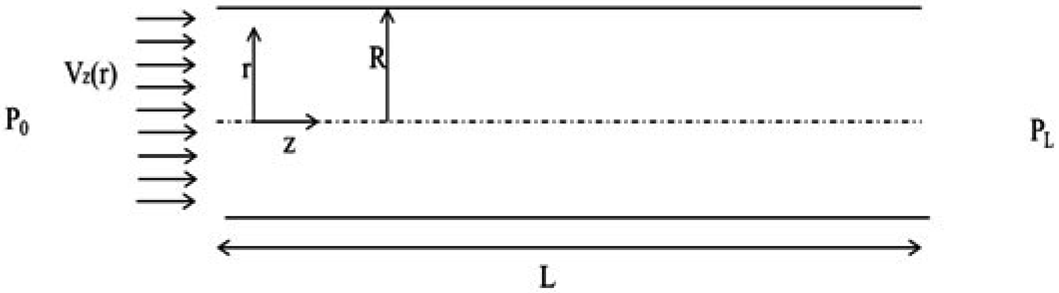


Fig 21.1 Flow through pipe for non-Newtonian fluid

As we discussed in lecture 10, the only non-zero component of velocity is vz, which depends on r only. The important component of shear stress is τ_{rz}.

By applying general equation of motion in cylindrical co-ordinate, we get

$$\frac{\partial P}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0$$

Equation (21.11) may be further simplified as before

$$\frac{P_0 - P_L}{L} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0$$

or

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L}r + \frac{c_1}{r}$$

By applying the boundary condition, at r=0, velocity is finite, we obtain

$$c_1 = 0$$

and for power law fluids

$$\tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^n$$

(Note: Since vz is decreasing with increasing value of r, the negative sign should be used for removing the modulus sign.)

By substituting Equation (21.13) to Equation (21.12), we get

$$m\left(-\frac{dv_z}{dr}\right)^n = \frac{(P_0-P_L)}{2L}r$$

Integrating above equation, we obtain

$$v_z = -\sqrt[n]{\frac{(P_0-P_L)}{2Lm}}\frac{r^{\frac{n+1}{n}}}{\frac{n+1}{n}}+c_2$$

Now, by applying the no-slip boundary condition at, $r = R, v_z = 0$, we obtain

$$c_2 = -\sqrt[n]{\frac{(P_0-P_L)}{2Lm}}\frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}}$$

Thus,

$$v_z = \sqrt[n]{\frac{(P_0-P_L)}{2Lm}}\frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}}\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

Equation (21.15) represents the velocity profile of freely falling film on an inclined surface for a Power law non-Newtonian fluid. If we substitute the n=1 and m=μ in this expression, we get Equation (10.25) which was derived earlier for a Newtonian fluid.

Tube Flow Problem for a Bingham Fluid

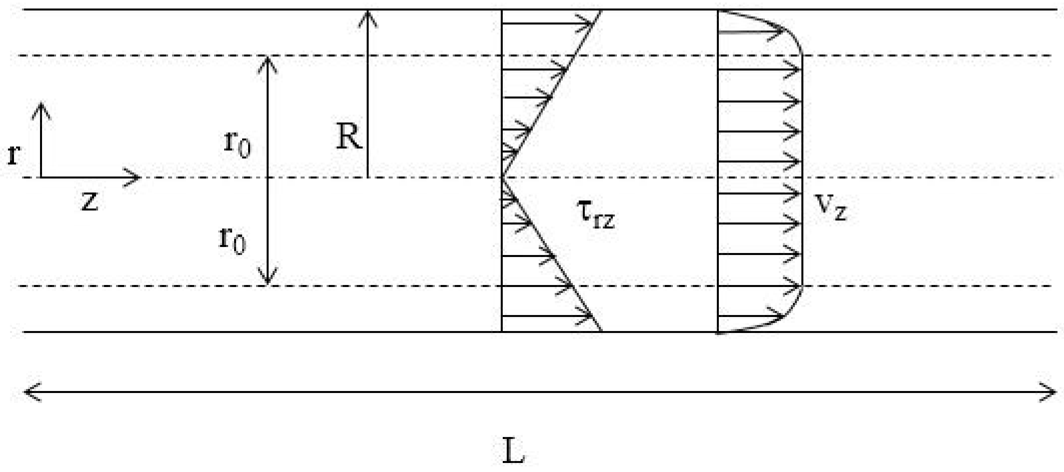


Fig 22.1 Flow through pipe for Bingham fluid

As mentioned in the previous lecture, the forms of shear stress $\underset{\sim}{\tau}$ for Newtonian and non-Newtonian fluids are the same. Therefore, Equation (21.12) is applicable for a Bingham fluids also, i.e.,

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Equations (20.19) and (20.20) may be written for this system

1. For $\tau_{rz} < \tau_0$ ($r < r_0$), where r_0 is to be determine latter,

$$\frac{dv_z}{dr} = 0$$

, or

$$v_z = v_0 \text{ (constant)}$$

2. For $\tau_{rz} \geq \tau_0$ ($r \geq r_0$)

$$\eta = \mu_0 + \frac{\tau_0}{\left|\frac{dv_z}{dr}\right|}$$

In Equation (22.2), $\frac{dv_z}{dr}$ is negative. Therefore, after removing the modulus sign, we obtain

$$\eta = \mu_0 - \frac{\tau_0}{\left(\frac{dv_z}{dr}\right)}$$

Thus,

$$\tau_{rz} = -\left\{\mu - \frac{\tau_0}{\frac{dv_z}{dr}}\right\}\frac{dv_z}{dr}$$

or

$$\tau_{rz} = -\mu\frac{dv_z}{dr} + \tau_0$$

Condition for movement of fluid

As we start to pressurize the fluid by imposing pressure difference , fluid does not move initially. As we continue to increase the pressure difference the fluid may start to move at some critical pressure difference ($P_{c0} - P_{cL}$). This critical value may be determined by setting $\tau_{rz}\big|_{r=R} = \tau_0$. Thus,

$$\tau_0 = \frac{(P_{c0} - P_{cL})}{2L}R$$

Thus, the fluid will flow if

$$\tau_0 \le \frac{(P_{c0} - P_{cL})}{2L}R$$

Suppose the pressure difference across the tube exceeds this critical value of pressure ($P_{c0} - P_{cL}$) then the fluid will start to flow. Now, under this condition we may calculate the value of (r0) where the value of $\tau_{rz} = \tau_0$. For r<r0, the velocity gradient is zero and the

fluid flows with a constant velocity. The detail calculation for two different regions r<r0 and r>r0 are given below.

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

At $r = r_0, \tau_{rz} = \tau_0$. Thus,

$$\tau_0 = \frac{(P_0 - P_L)}{2L} r_0$$

or

$$r_0 = \frac{2\tau_0 L}{(P_0 - P_L)}$$

For r<r0, we equate Equations (21.12) and (22.4), that is

$$-\mu \frac{dv_z}{dr} + \tau_0 = \tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Finally, we obtain,

$$v_z = \frac{\tau_0}{\mu} r - \left(\frac{(P_0 - P_L)}{4\mu L} r^2 \right) + c_1$$

No slip Boundary condition at r=R , $v_z = 0$ may be used to calculate c1 as shown below Substituting this value in Equation (22.11), we get

$$c_1 = \left(\frac{(P_0 - P_L)}{4\mu L} R^2 \right) - \frac{\tau_0}{\mu} R$$

Finally, the velocity profile is given by

$$v_z = \frac{\tau_0}{\mu}(r-R) + \left(\frac{(P_0-P_L)}{4\mu L}R^2\right)\left(1-\left(\frac{r}{R}\right)^2\right)$$

Equation (22.12) gives the velocity profile is region $r_0 \le r \le R$. as shown in Fig. 22.2. Equation (22.9) shows that as we keep increasing the pressure difference (P_0-P_L), the value of r0 keep on decreasing and the velocity profile changes as shown in Fig. 22.2.

The value of r0 also depends on τ_0, and reduces with it. If we substitute $\tau_0 = 0$ in Equation (22.12), we obtain the same expression for velocity profile as we had earlier obtain for Newtonian fluids. This result implies that if the value of pressure difference (P_0-P_L) is significantly high then the Bingham fluid may show behaviour similar to Newtonian fluids.

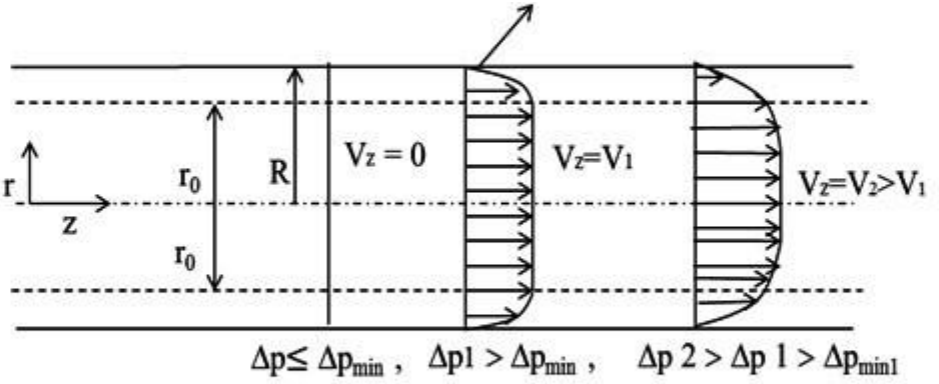


Fig 22.2 Effect of differential pressure flow through pipe for Bingham fluid

Now, we may determine the velocity profile in the plug flow region (r>r0) by substituting r= r0 in Equation (22.12)

$$v_0 = \frac{\tau_0}{\mu}(r_0-R) + \left(\frac{(P_0-P_L)}{4\mu L}R^2\right)\left(1-\left(\frac{r_0}{R}\right)^2\right)$$

Falling film problem for Bingham fluid

$$\eta = \mu_0 \pm \frac{\tau_0}{\left|\frac{dv_z}{dx}\right|}$$

$$\frac{dv_z}{dx} = 0$$ or

$$v_z = v_0$$

For region (2) where $\tau \geq \tau_0$ and $\delta \geq \delta_0$, we have

$$\eta = \left\{ \mu \pm \frac{\tau_0}{\left|\frac{dv_z}{dx}\right|} \right\}$$

Here $\frac{dv_z}{dx}$ is negative. Therefore, after removing the modulus sign and substituting the value of η in Generalized Newton's law of viscosity. we obtain,

$$v_0 = \frac{\rho g \cos\beta \delta^2}{2\mu}\left[1 - \frac{\delta_0^2}{\delta^2}\right] - \frac{\tau_0 \delta}{\mu}\left[1 - \frac{\delta_0}{\delta}\right]$$

$$\tau_{xz} = -\left\{\mu \frac{dv_z}{dx} - \tau_0\right\} = \rho g \cos\beta x$$

or

$$-\mu \frac{dv_z}{dx} = \rho g \cos\beta x - \tau_0$$

or

$$\frac{dv_z}{dx}=\frac{-\rho g\cos\beta x}{\mu}+\frac{\tau_0}{\mu}$$

Finally, we obtain the velocity profile, as given below

$$v_z=\frac{-\rho g\cos\beta}{\mu}\frac{x^2}{2}+\frac{\tau_0}{\mu}x+c_2$$

where c2 is an integral constant. By using no slip boundary condition at $x=\delta$, $v_z=0$, we obtain

$$c_2=\frac{\rho g\cos\beta}{\mu}\frac{\delta^2}{2}-\frac{\tau_0}{\mu}\delta$$

Therefore,

$$v_z=\frac{\rho g\cos\beta\delta^2}{2\mu}\left[1-\frac{x^2}{\delta^2}\right]-\frac{\tau_0\delta}{\mu}\left[1-\frac{x}{\delta}\right]$$

Equation shows the velocity profile in region $\delta_0\leq x\leq\delta$. . From Equation, we may also calculate the velocity of plug flow region by substituting the value $x=\delta_0$

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SCHOOL OF BIO & CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – II – Modelling and transport of bioprocesses – SCH1312

2. Introduction to Heat Transport

Practically all the operations that are carried out by the chemical engineers involve the production or absorption of energy in the form of heat. The study of temperature distribution and heat transfer is of great importance to engineers because of its almost universal occurrence in many branches of science and engineering. The first step in the optimal design of heat exchangers such as boilers, heaters, refrigerators and radiators is a detailed analysis of heat transfer. This is essential to determine the feasibility and cost of the undertaking, as well as the size of equipment required to transfer a specified amount of heat in a given time. Difference between thermodynamics and heat transfer Thermodynamic tells us (i) How much heat is transferred (ii) How much work is done (iii) Final state of the system.

Heat transfer tells us: (i) How much heat is transferred (with what modes) (ii) At what rate heat is transferred (iii) Temperature distribution inside the body.

The various modes of heat transfer are (i) conduction (ii) convection (iii) radiation. Conduction Heat transfer by the actual but invisible movement of molecules within the continuous substance due to temperature gradient is known as conduction. When a current or macroscopic particle of fluid crosses a specific surface, it carries with it a definite quantity of enthalpy. Such a flow of enthalpy is called convection. Convection is the mode of heat transfer in which the heat flow is associated with the movement of fluid. Transfer of energy through space by electromagnetic waves is known as radiation.

2.1 Applications of heat transfer

Energy production and conversion -steam power plant, solar energy conversion etc. Refrigeration and air-conditioning Domestic applications -ovens, stoves, toaster Cooling of electronic equipment Manufacturing / materials processing -welding, casting, soldering, laser machining Automobiles / aircraft design

2.1.1 Conduction: It is the transfer of internal energy by microscopic diffusion and collisions of particles or quasi-particles within a body due to a temperature gradient. The microscopically diffusing and colliding objects include molecules, electrons, atoms, and phonons. They transfer disorganized microscopic kinetic and potential energy, which are jointly known as internal energy. Conduction can only take place within an object or material, or between two objects that are in direct or indirect contact with each other. On a microscopic scale, heat conduction occurs as hot, rapidly moving or vibrating atoms and molecules interact with neighboring atoms and molecules, transferring some of their energy (heat) to these neighboring particles. In other words, heat is transferred by conduction when adjacent atoms vibrate against one another, or as electrons move from one atom to another.

Fourier’s law of heat Conduction

The rate of heat transfer due to conduction is governed by Fourier’s Law, as shown

$$q = kA\left(\frac{\Delta T}{\Delta x}\right)$$

The terms in Eqn. 1 are:

q – rate of heat transfer (W)

k – thermal conductivity (W/m·K)

A – surface area across which heat is transferred (m^2)

ΔT – difference in temperature over which heat is transferred (K)

Δx – distance over which heat is transferred (m)

Thermal conductivity indicates the ease of heat transfer through a material and is a material dependent property. The ΔT term is the driving force for heat transfer.

2.1.2 Convection

The rate of heat transfer due to convection is described by

$$q = hA(\Delta T)$$

In Eqn. 2 the new term is:

h –heat transfer coefficient ($W/m^2 \cdot K$)

In Eqn. 2, the heat transfer coefficient replaces the k/Δx term in Eqn. 1. The reason this happens is because convection has a mobile phase, and thickness is no longer an effective way of describing how the heat is transferred. The heat transfer coefficient can be thought of as the inverse of the resistance to heat transfer. Also, because temperature is a function of distance from a surface, the ΔT term is calculated between the surface and the bulk temperature of the mobile phase.

2.1.3 Radiation

The concept for radiation is that all materials are constantly emitting infrared radiation that is absorbed by other materials. For this module, we will assume that radiation is emitted directly outward from the surface of objects. While conduction and convection are driven by a temperature gradient, radiation is only based on the temperature of the object emitting radiation.

The rate of heat transfer due to radiation can be described by

Radiation Emitted: $q_{out} = \varepsilon A\sigma({T_s}^4)$

Radiation Absorbed: $q_{in} = \alpha A\sigma({T_o}^4)$

Heat transfer is the exchange of thermal energy between physical systems. The rate of heat transfer is dependent on the temperatures of the systems and the properties of the intervening medium through which the heat is transferred. The three fundamental modes of heat transfer are *conduction*, *convection* and *radiation*.

Heat transfer, the flow of energy in the form of heat, is a process by which a system changes its internal energy, hence is of vital use in applications of the First Law of Thermodynamics. Conduction is also known as diffusion, not to be confused with diffusion related to the mixing of constituents of a fluid. The direction of heat transfer is from a region of high temperature to another region of lower temperature, and is governed by the Second Law of Thermodynamics. Heat transfer changes the internal energy of the systems from which and to which the energy is transferred. Heat transfer will occur in a direction that increases the entropy of the collection of systems. Thermal equilibrium is reached when all involved bodies and the surroundings reach the same temperature. Thermal expansion is the tendency of matter to change in volume in response to a change in temperature.

Newton's law of cooling states that *the rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.* As such, it is equivalent to a statement that the heat transfer coefficient, which mediates between heat losses and temperature differences, is a constant. This condition is generally true in thermal conduction (where it is guaranteed by Fourier's law), but it is often only approximately true in conditions of convective heat transfer, where a number of physical processes make effective heat transfer coefficients somewhat dependent on temperature differences. Finally, in the case of heat transfer by thermal radiation, Newton's law of cooling is not true.

Thermal conductivity (often denoted *k*, *λ*, or *κ*) is the property of a material to conduct heat. It is evaluated primarily in terms of Fourier's Law for heat conduction. Heat transfer occurs at a lower rate across materials of low thermal conductivity than across materials of high thermal conductivity. Correspondingly, materials of high thermal conductivity are widely used in heat sink applications and materials of low thermal conductivity are used as thermal insulation. The thermal conductivity of a material may depend on temperature. The reciprocal of thermal conductivity is called thermal resistivity. Thermal conductivity is actually a tensor, which means it is possible to have different values in different directions.

Table 1: Thermal conductivity at room temperature for some metals and non-metals

Metals	Ag	Cu	Al	Fe	Steel
k [W/m-K]	420	390	200	70	50

Non-metals	H_20	Air	Engine oil	H_2	Brick	Wood	Cork
k [W/m-K]	0.6	0.026	0.15	0.18	0.4 -0 .5	0.2	0.04

2.2 Effect of temperature on thermal conductivity

Thermal conductivity is the physical property of the substance. It depends upon temperature gradient. For pure metals, thermal conductivity decreases with an increase in temperature. For gases and insulators, thermal conductivity increases with an increase in temperature. For small ranges of temperature, k may be considered constant. For larger temperature ranges, thermal conductivity can be approximated by an equation of the form $k = a + bT$, where a and b are empirical constants.

Steady-State Conduction It is the form of conduction which happens when the temperature difference driving the conduction is constant so that after an equilibrium time, the spatial distribution of temperatures (temperature field) in the conducting object does not change any further. In steady state conduction, the amount of heat entering a section is equal to amount of heat coming out. Unsteady state conduction It is the form of conduction which happens when the temperature difference driving the conduction is not constant so that after an equilibrium time, the spatial distribution of temperatures (temperature field) in the conducting object changes as a function of time. Heat flux is denoted as q/A and it is defined as the rate of heat flow passing through a material per cross-sectional area and its unit is w/m^2.

Silver is the material which possess highest thermal conductivity and being a solid it is composed of closed packing arrangement and due to this more molecular interactions within the molecules and hence the thermal conductivity is high.

Thermal conductivity is ability of a material to transport heat energy through it from high temperature region to low temperature region. The heat energy, Q, transported across a plane of area A in presence of a temperature gradient $\Delta T/\Delta l$ is given where k is the thermal conductivity of the material. It has units as W/m.K. It is a microstructure sensitive property. • Its value range o for metals 20-400 for ceramics 2-50 for polymers order of 0.3 Mechanisms - Thermal conductivity

Heat is transported in two ways – electronic contribution, vibrational (phonon) contribution. In metals, electronic contribution is very high. Thus metals have higher thermal conductivities. It is same as electrical conduction. Both conductivities are related through Wiedemann-Franz law where L – Lorentz constant ($5.5\text{x}10^{-9}$ cal.ohm/sec.K^2) • As different contributions to conduction vary with temperature, the above relation is valid to a limited extension for many metals. • With increase in temperature, both number of carrier electrons and contribution of lattice vibrations increase. Thus thermal conductivity of a metal is expected to increase. However, because of greater lattice vibrations, electron mobility decreases. The combined effect of these factors leads to very different behavior for different metals. Eg.: thermal conductivity of iron initially decreases then increases slightly; thermal conductivity decreases with increase in temperature for aluminium; while it increases for platinum.

2.3.1 HEAT TRANSFER THROUGH A PLANE WALL

Let us consider a plane wall of thickness L, thermal conductivity k, inside surface temperature Ti, outside surface temperature To. Let Q be the rate of heat transferred through the plane wall.

By Fourier's law of heat conduction

$Q = -kA\, dT/dx$

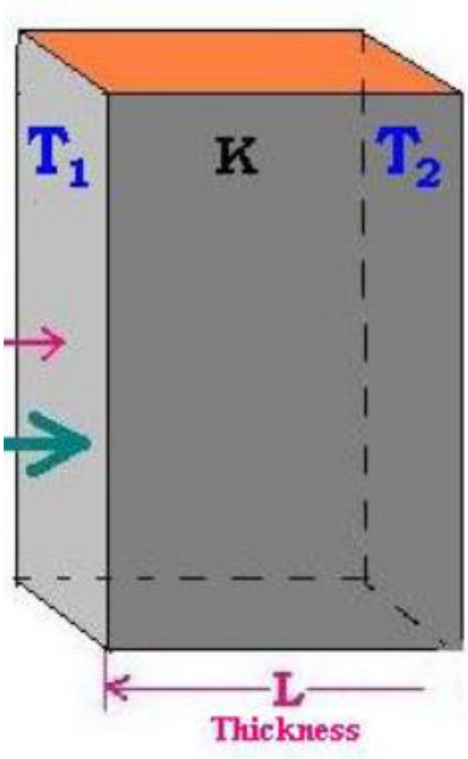


Fig. 1: Heat flow through a metal wall

$$Q \int_{0}^{L} dx = -k\,A \int_{Ti}^{To} dT$$

On integrating the above eqn, $Q = KA(T_i - T_o) / L$

Q = KA ΔT / L i.e Q = ΔT / (KA/L) where KA/L = R_{th} where R_{th} – thermal resistance measured in °C / Watts or K / Watts. Hence Q = ΔT / R_{th}.

2.3.2 HEAT TRANSFER THROUGH A HOLLOW CYLINDER

Let us consider a hollow cylinder. The inside radius of the cylinder is r_1, the outside radius is r_2, and the length of the cylinder is L. The thermal conductivity of the material of which the cylinder is made is k. The temperature of the outside surface is T_2, and that of the inside surface is T_1.

By Fourier's law of heat conduction,

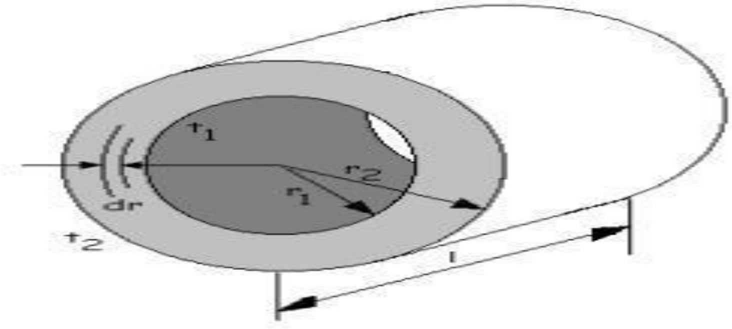


Fig. 2 Heat flow through a cylinder

Q = -kA dT/dr

$$Q \int_{r1}^{r2} \frac{dr}{r} = -k\, 2\pi L \int_{T1}^{T2} dT$$

integrating the above eqn,

$$Q = \frac{2\pi k L\ (T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

2.3.2.1 LOGARITHMIC MEAN RADIUS AND ARITHMETIC MEAN RADIUS

Logarithmic mean radius is the radius that when applied to the integrated equation for a flat wall, will give the correct rate of heat flow through a thick walled cylinder. It is given by the expression

$$\overline{r_L} = \frac{r_o - r_i}{\ln\left(\frac{r_o}{r_i}\right)}$$

where $\overline{r_L}$ is the logarithmic mean radius of the cylinder

r_o is the outer radius of the pipe and

r_i is the inner radius of the pipe.

Hence using the above expression Q = 2∏ k L (Ti-To) * (ro-ri) / ln (ro/ri) *(ro-ri)

Using the Logarithmic mean radius expression in above, we get

Q = 2∏ k L ΔT * r_{lm} / (ro-ri) hence, Q = ΔT / (ro-ri) / A_{lm} * k

Where A_{lm} = 2∏ L * r_{lm} , A_{lm} is the logarithmic mean area which is used for thin cylinders.

1.3.3 COMPOUND RESISTANCES IN SERIES

(I) Heat Transfer Through A Composite Plane Wall

Let us consider a flat wall constructed of a series of 3 layers as shown. Let the thickness of the layers be L_1, L_2, L_3 and the average thermal conductivities of the materials of which the layers are made be k_1, k_2, k_3 respectively. Let us consider a hot fluid at a temperature T_a and heat transfer coefficient h_a inside the wall and cold fluid at a temperature T_b and heat transfer coefficient h_b outside the wall. Let T_1, T_2, T_3 and T_4 be the interface temperatures. It is desired to derive an equation for calculating the rate of heat flow through the series of resistances.

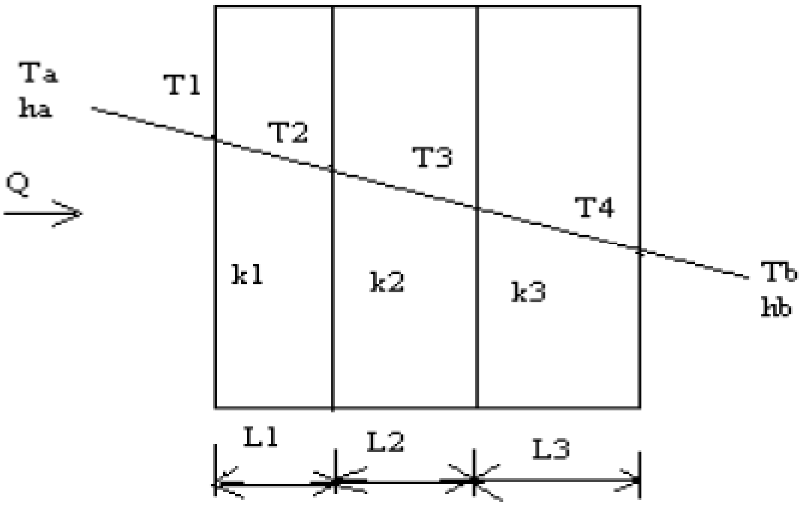


Fig. 3 Heat flow through a composite wall

Rate of heat flow from the hot fluid to the inner surface of the wall

By Newton's law of cooling

$Q = h_a A (T_a - T_b)$

By rearranging the above eqn, we get

$$Q = \frac{(T_a - T_1)}{\frac{1}{h_a A}}$$

Rate of heat flow through the I layer, by Fourier's law of heat conduction,

$Q = KA(T_1-T_2) / L$ On rearranging this,

$Q = (T_1-T_2) / (L_1/K_1.A)$

Rate of heat flow through the II layer,

$Q = (T_2-T_3) / (L_2/K_2.A)$

Rate of heat flow through the III layer,

$Q = (T_3-T_4) / (L_3/K_3.A)$

Rate of heat flow from outer surface of the wall to the cold fluid By Newton's law of cooling

$Q = h_b A (T_4 - T_b)$ and $Q = (T_4 - T_b) / 1/ h_b A$

Overall rate of heat flow =*overall thermal resistance / overall temperature drop*

Overall rate of heat flow

$$Q = \frac{(T_a - T_b)}{\frac{1}{h_a A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A} + \frac{1}{h_b A}}$$

In steady state heat conduction through a composite wall, this can be written as

$Q = (T_1 - T_4) / (L_1/K_1.A + L_2/K_2.A + L_3/K_3.A)$

i.e $Q = \Delta T / R_{th1} + R_{th2} + R_{th3}$

Hence, $Q = \Delta T / \sum R_{th}$

$$Rate\ of\ heat\ flow = \frac{Temperaure\ difference}{Resistance}$$

$$R_1 = \frac{x_1}{k_1 A} \qquad R_2 = \frac{x_2}{k_2 A} \qquad R_3 = \frac{x_3}{k_3 A}$$

Table 2 Various flows and their driving forces

Flow	Driving force
Electricity flow	Electric potential gradient
Fluid flow	Pressure gradient
Heat flow	Temperature gradient

It was observed in the previous discussion that for the given plane wall the area for heat transfer was constant along the heat flow direction. The plane solid wall was one of the geometries but if we take some other geometry (tapered plane, cylindrical body, spherical body etc.) in which the area changes in the direction of heat flow. Now we will consider geometrical configuration which will be mathematically simple and also of great engineering importance like hollow cylinder and hollow sphere. In these cases the heat transfer area varies in the radial direction of heat conduction. We will take up both the cases one by one in the following sections.

1.4) Heat transfer through co-axial cylinder Provided with one layer of insulation

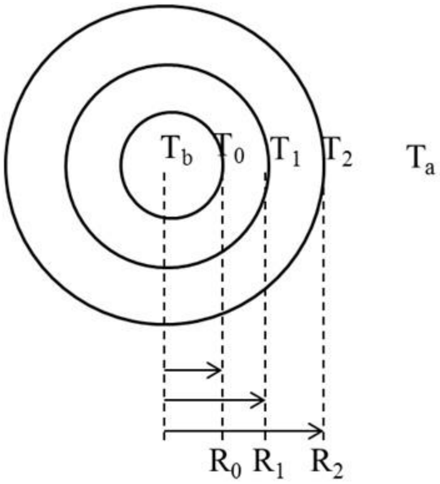


Fig. 4 Heat flow through a composite cylinder

Let us consider coaxial cylinders constructed of a series of 3 layers as shown in fig. Let R_o R_1 and R_2 be the radii of the cylinders and the average thermal conductivities of the materials of which the layers are made be k_1 and k_2 respectively. Let us consider a hot fluid at a temperature T_b and heat transfer coefficient h_b inside the cylinder and cold fluid at a temperature T_a and heat transfer coefficient h_a outside the cylinder. Let T_o, T_1 and T_2 be the interface temperatures. It is desired to derive an equation for calculating the rate of heat flow through the composite cylinder provided with series of resistances.

Consider a very thin hollow cylinder of thickness*d_r* in the main geometry (fig.2.9a) at a radial distance *r*. If *d_r* is small enough with respect to *r*, then the area of the inner and outer surface of the thin cylinder may be considered to be of same area. In other words, for very small *d_r* with respect to *r*, the lines of heat flow may be considered parallel through the differential element in radial outward direction.

We may ignore the heat flow through the ends if the cylinder is sufficiently large. We may thus eliminate any dependence of the temperature on the axial coordinate and for one dimensional steady state heat conduction, the rate of heat transfer for the thin cylinder,

$$\dot{q} = -k\frac{dT}{dr}\ 2\pi rL$$

Where *dT* is the temperature difference between the inner and outer surface of the thin cylinder considered above and *k* is the thermal conductivity of the cylinder.
On rearranging,

$$\frac{dr}{r} = -\frac{2\pi Lk}{\dot{q}}dT$$

To get the heat flow through the thick wall cylinder, the above equation can be integrated between the limits,

$$\int_{r_i}^{r_o}\frac{dr}{r} = -\frac{2\pi Lk}{\dot{q}}\int_{T_i}^{T_o}dT$$

On solving,

$$\dot{q} = k(2\pi L)\frac{(T_i - T_o)}{ln(r_o/r_i)}$$

$$\dot{q} = kA_{LM}\frac{(T_i - T_o)}{r_o - r_i}$$

Rate of heat flow from the hot fluid to the inner surface of the wall

By Newton’s law of cooling,

$Q = hb\ A\ (T_b – T_0)$

$Q = hb\ 2\ \prod RL\ (T_b – To)$

By rearranging the above,

Q = (Tb – To) / 1/ 2п R_o L

Rate of the heat flow through the cylinder By Fourier’s law of heat conduction

$$Q = \frac{2\pi kL\ (T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

using the above conditions from the fig,

$Q = 2\prod\ k_1 L\ (T_o – T_1) / \ln (R_1/R_o)$

By rearranging the above,

$Q = (T_o – T_1) / \ln (R_1/R_o) / 2\prod\ k_1 L$

Rate of the heat flow through the II layer By Fourier’s law of heat conduction

$Q = (T_1 – T_2) / \ln (R_2/R_1) / 2\prod\ k_2 L$

Rate of heat flow from the outer surface of the wall to the cold fluid

By Newton’s law of cooling

Q = ha A (T2 – Ta)

Q = ha 2пR_2L (T2– Ta) on rearranging, Q = (T2– Ta) / 1 / ha 2$\prod$ R_2L

Overall rate of heat flow =*overall thermal resistance* / *overall temperature drop*

Hence overall rate of heat flow is given by

$Q = (T_b\text{-}T_a) / (1/ hb\ 2\square\ R_o) + (\ln (R_1/R_o) / 2\prod k_1 L) + (\ln (R_2/R_1) / 2\prod k_2 L) +(1/ ha\ 2\prod R_2L)$

For steady state conduction , heat transfer coefficients can be neglected and the heat flow is given by

$Q = (T_b\text{-}T_a) / (\ln (R_1/R_o) / 2\prod k_1 L) + (\ln (R_2/R_1) / 2\prod k_2 L)$

Hence $Q = \Delta T / R_{th1} + R_{th2}$

Where $R_{th1} = (\ln (R_1/R_o) / 2\prod k_1 L)$ and $R_{th2} = (\ln (R_2/R_1) / 2\prod k_2 L)$

$Q = \Delta T / \sum R_{th}$

1.5 Insulation

The addition of insulation material on a surface reduces the amount of heat flow to the ambient. There are certain instances in which the addition of insulation to the outside surface of cylindrical of spherical walls does not reduce the heat loss. Under certain circumstances it actually increases the heat loss up to a certain thickness of insulation. It is well known fact that the rate of heat transfer will approach zero if an infinite amount of insulation are added. This means that there must be a value of radius for which rate of heat transfer is maximum. This value is known as the critical radius of insulation, rc.

Variation of thermal conductivity with respect to solids, liquids and gases

In solids, solids, heat conduction conduction is due to two effects effects: the lattice lattice vibrational vibrational waves induced by the vibrational motions of the molecules positioned at relatively fixed positions in a periodic manner called a lattice, and the energy transported via the free flow of electrons in the solid. • The thermal conductivity of a solid is obtained by adding the lattice and electronic components. The relatively high thermal conductivities of pure metals are primarily due to the electronic component. The lattice component of th l erma cond ii uct v ty strongly d d epen s on the way the molecules are arranged. For example, diamond, which is a highly ordered crystalline solid has the highest known thermal conductivity at room. Conduction crystalline solid, has the highest known thermal conductivity at room temperature.

The thermal thermal conductivities conductivities of materials materials vary with temperature temperature. The variation of thermal conductivity over certain temperature ranges is negligible for some materials, but significant for others. The thermal conductivities of certain solids exhibit dramatic increases at temperatures near absolute zero, when these solids become superconductors.

2. 4 CRITICAL RADIUS OF INSULATION IN PIPES

Let us consider an insulating layer in the form of a hollow cylinder of length L. Let ri and ro be the inner and outer radii of insulation. The thermal conductivity of the material of which the layer is made be k. Let the inside surface of insulation be at a temperature Ti, and the outside surface at a temperature To be dissipating heat by convection to the surroundings at a temperature Tb with a heat transfer coefficient h. Then the rate of heat transfer Q through this insulation layer is

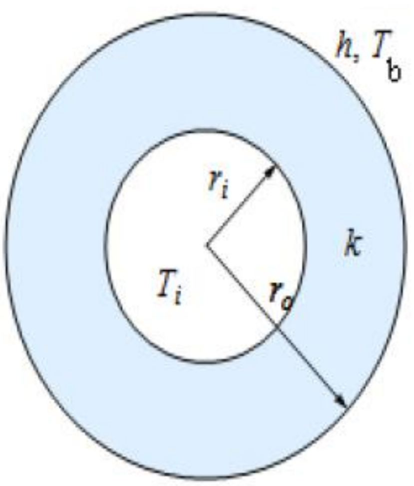


Fig. 5 Heat flow through a cylindrical pipe for critical radius

$$Q = \frac{2\pi L\left(T_i - T_b\right)}{\dfrac{\ln\left(\dfrac{r_o}{r_i}\right)}{k} + \dfrac{1}{hr_o}} \qquad (1)$$

The value of critical radius r_c, that is r_o for which Q is a maximum may be obtained by equating dQ/dr_o to zero.

$$\frac{dQ}{dr_o} = \frac{0 - \left(T_i - T_b\right)\left[\dfrac{1}{2\pi kLr_o} - \dfrac{1}{2\pi hLr_o^2}\right]}{\left[\dfrac{\ln\left(\dfrac{r_o}{r_i}\right)}{2\pi kL} + \dfrac{1}{2\pi hLr_o}\right]^2} \qquad (2)$$

$\left(T_i - T_b\right) \neq 0$(Since it is the driving force)

$$\therefore \quad \frac{1}{2\pi kLr_o} - \frac{1}{2\pi hLr_o^2} = 0 \qquad (3)$$

$$r_0 = \frac{k}{h} = r_c$$

The radius at which the rate of heat transfer is maximum is known as the critical radius of insulation.

2.6 VARIABLE THERMAL CONDUCTIVITY

Let us a hollow cylinder. The inside radius of the cylinder is ri, the outside radius is ro, and the length of the cylinder is L. The thermal conductivity of the material of which the cylinder is varies with temperature as k= ko(1 + αT). The temperature of the outside surface is To and that of the inside surface is Ti. This can be used with many equations such as

$k = ko\ (\alpha + \beta T)$

$k = ko\ (\alpha + \beta T + \gamma T^2)$

$k = ko\ (a + bT)$

$k = ko\ (a + bT + cT^2)$

By Fourier’s law of heat conduction

$$Q = -kA \frac{dT}{dr} \qquad (1)$$

$$Q = -k_o (1 + \beta T) A \frac{dT}{dr}$$

$$Q \int_{ri}^{ro} \frac{dr}{r} = -k_o 2\pi L \int_{Ti}^{To} (1 + \beta T)\, dT \qquad (2)$$

$$Q \ln \frac{r_o}{r_i} = k_0 2\pi L \left(1 + \beta \frac{[T_i + T_o]}{2} \right) (T_i - T_o)$$

$$Q = \frac{k_0 2\pi L \left(1 + \beta \frac{[T_i + T_o]}{2} \right) (T_i - T_o)}{\ln \frac{r_o}{}} \qquad (3)$$

2.7 Introduction to Unsteady state heat transfer

A solid body is said to be in a steady state if its temperature does not vary with time. If however there is an abrupt change in its surface temperature or environment it takes some time before the body to attain an equilibrium temperature or steady state. During this interim period the temperature varies with time and the body is said to be in an unsteady or transient state. The analysis of unsteady state heat transfer is of great interest to engineers because of its widespread occurrence such as in boiler tubes, rocket nozzles, automobile engines, cooling of IC engines, cooling and freezing of food, heat treatment of metals by quenching, etc. For practical purposes it is necessary to know the time taken to attain a certain temperature when the environment suddenly changes. The solution of an unsteady sate problem will be more complex than that of steady state one because of the presence of another variable time, t.

Transient heat conduction problems can be divided into periodic heat flow and non periodic heat flow problems. Periodic heat flow problems are those in which the temperature varies on a regular basis, eg., the variation of temperature of the surface of the earth during a twenty four hour period..

In the non periodic type, the temperature at any point within the system varies non linearly with time.

Introduction to this point, we have considered conductive heat transfer problems in which the temperatures are independent of time. In many applications, however, the temperatures are varying with time, and we require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed.

$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\overset{\Lambda}{q}}{k}$$

Where $\alpha=\frac{k}{\rho c}$ is the thermal diffusivity. Without any heat generation and considering spatial variation of temperature only in x-direction, the above equation reduces to:

2.8 Systems with negligible internal resistance – Lumped Heat Analysis

Heat transfer in heating and cooling of a body is dependent upon both the internal and surface resistances. The simplest unsteady state problem is one in which the internal resistance is negligible, that is, the convective resistance at the surface boundary is very large when compared to the internal resistance due to conduction. In other words, the solid has an infinite thermal conductivity so that there is no variation of temperature inside the solid and temperature is a function of time only. This situation cannot exist in reality because all the solids have a finite thermal conductivity and there will always be a temperature gradient inside whenever heat is added or removed. Problems such as heat treatment of metals by quenching, time response of thermocouples and thermometers, etc can be analyzed by this idealization of negligible internal resistance. The process in which the internal resistance is ignored being negligible in comparison with its surface resistance is called the Newtonian heating and cooling process. In Newtonian heating and cooling process the temperature throughout the solid is considered to be uniform at a given time. Such an analysis is called the lumped heat capacity analysis.

2.9 Systems with negligible surface resistance

Another class of transient problems met with in practice is one in which the surface resistance is negligible compared to the overall resistance. This amounts to saying that the convective heat transfer coefficient at the surface is infinity. For such a process the surface temperature remains constant for all the time and its value is equal to that of ambient temperature.

Dimensionless parameters:

$$\frac{T_i - T_s}{T_s - T_\infty} = \frac{\bar{h}L}{k} = \text{Biot number}$$

$$\mathbf{Bi} = \frac{\textbf{resistance to internal heat flow}}{\textbf{resistance to external heat flow}}$$

The Biot number is dimensionless, and it can be thought of as the ratio to the internal and external heat flows. Whenever the Biot number is small, the internal temperature gradients are also small and a transient problem can be treated by the “lumped thermal capacity” approach. The lumped capacity assumption implies that the object for analysis is considered to have a single mass-averaged temperature.

In general, a characteristic length scale may be obtained by dividing the volume of the solid by its surface area: $L = V/A_s$

Using this method to determine the characteristic length scale, the corresponding Biot number may be evaluated for objects of any shape, for example a plate, a cylinder, or a sphere. As a thumb rule, if the Biot number turns out to be less than 0.1, lumped capacity assumption is applied. In this context, *a dimensionless time,* known as the Fourier number, can be obtained by multiplying the dimensional time by the thermal diffusivity and dividing by the square of the characteristic length:

$$\textbf{dimensionless time} = \frac{\alpha t}{L^2} = \mathbf{Fo}$$

Lumped heat capacity analysis: temperature distribution inside or outside the solid is neglected.

The cases considered so far have been those in which the heat conducting solid is free of internal heat generation. However, the situations where the internal heat is generated are very common cases in chemical industries for example, the exothermic reaction in the solid pallet of a catalyst.

We have learnt that how the Fourier equation is used for the steady-state heat conduction through the composites of three different geometries that were not having any heat source in it. However, the heat generation term would come into the picture for these geometries. It would not be always easier to remember and develop heat conduction relations for different standard and non-standard geometries. Therefore, at this point we should learn how to develop a general relation for the heat conduction that should be applicable to the entire situation such as steady-state, unsteady state, heat source, different geometry, heat conduction in different direction, etc. Again here we will consider that the solid is isotropic in nature, which means the thermal conductivity of the material is same in all the direction of heat flow.

To get such a general equation the differential form of the heat conduction equation is most important. For simplicity, we would consider an infinitesimal volume element in a Cartesian coordinate system. The dimensions of the infinitesimal volume element are d_x, d_y, and d_z in the respective direction as shown in the fig

The simplest situation in an unsteady heat transfer process is to use the lump assumption, wherein we neglect the temperature distribution inside the solid a with the heat transfer between the solid and the ambient fluids. In other w assuming that the temperature inside the solid is constant and is equal to temperature.

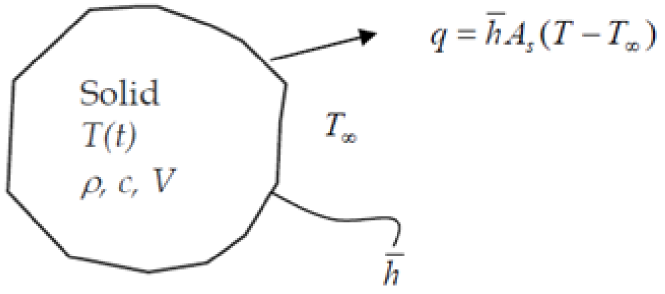


Fig. 6

The solid object shown in figure 5.2 is a metal piece which is being cooled in forming. Thermal energy is leaving the object from all elements of the surface shown for simplicity by a single arrow. The first law of thermodynamics ap problem is

$$\begin{pmatrix}\text{heat out of object}\\ \text{during time } dt\end{pmatrix} = \begin{pmatrix}\text{decrease of internal thermal}\\ \text{energy of object during time } dt\end{pmatrix}$$

Now, if Biot number is small and temperature of the object can be considered to this equation can be written as

$$\bar{h}A_s\left[T(t) - T_\infty\right]dt = -\rho c V dT$$

or,
$$\frac{dT}{(T - T_\infty)} = -\frac{\bar{h}A_s}{\rho c V}dt$$

Integrating and applying the initial condition $T(0) = T_i$,

$$\ln\frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{\bar{h}A_s}{\rho c V}t$$

Taking the exponents of both sides and rearranging,

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

where

$$b = \frac{\bar{h}A_s}{\rho c V} \qquad (1/\text{s})$$

$$b = \frac{\overline{h} A_s}{\rho c V} \qquad (1/\mathrm{s})$$

b is a positive quantity having dimension (time)-1. The reciprocal of *b* is usually called *time constant*, which has the dimension of time.

2.10 Heat conduction through spherical surface

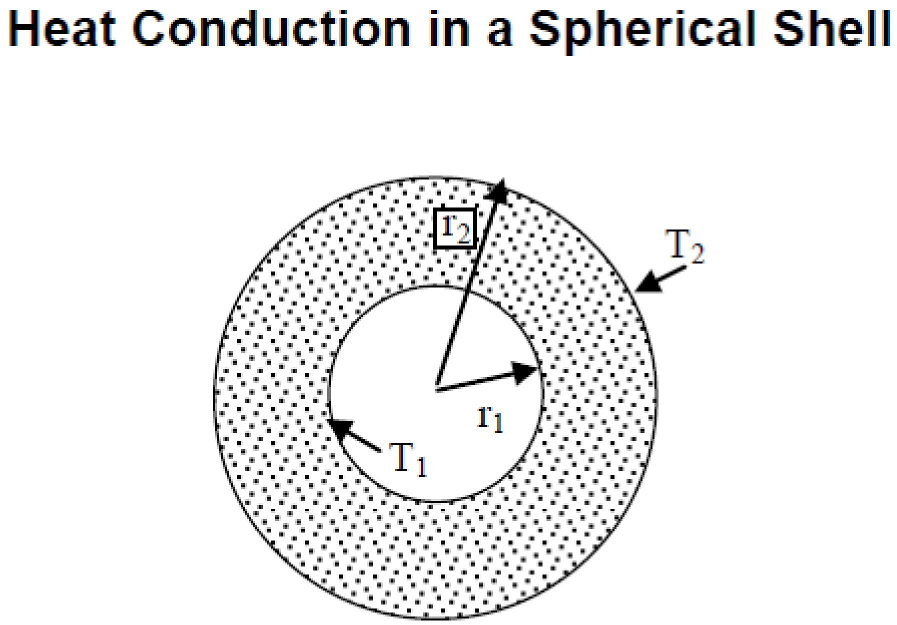


Fig. 7 Heat flow through a spherical surface

Consider a spherical shell with inside radius r_1 and outside radius r_2. Let T_1 be the inside temperature and T_2 be the outside temperature. K be the thermal conductivity of the material. Q be the heat flow through the spherical shell with a cross sectional area to be A. based on Fourier's law of heat conduction and following the assumptions,

(i) Heat flow is uniform
(ii) Heat flow is normal to the surface
(iii) The material is uniform and possess constant thermal conductivity
(iv) Heat flow is uni-directional

Let us consider a differential element of thickness dr which is lying between inside and outside radius. For such an element the heat flow is given by

Q = -kA dT/dr where area of the sphere is $A = 4\pi r^2$

Substituting the area of a sphere

Integrating, between r = r_1 and r_2, and T_1 and T_2,

$$\frac{q_r}{4\pi}\left|-\frac{1}{r}\right|_{r_1}^{r_2} = -k\left|T\right|_{T_1}^{T_2}$$

$$\frac{q_r(r_2 - r_1)}{4\pi r_1 r_2} = -k(T_2 - T_1)$$

$$q_r = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{(r_2 - r_1)}$$

The thermal resistance is expressed as

Q = ΔT / ($r_2 - r_1$) / 4πk r_1r_2 where the resistance is given by ($r_2 - r_1$) / 4πk r_1r_2

Geometric mean radius is given by $r_m^2 = r_1.r_2$ and hence

R_{th} = (r_2-r_1) / A_m . k where $A_m = 4\pi\ r_m^2$ and A_m is the geometric mean area.

Similarly, rate of heat flow for a composite spherical shell with one layer of insulation is given by $Q = \Delta T / [(r_2 – r_1) / 4\pi k_1\ r_1 r_2 + (r_3 – r_2) / 4\pi k_2\ r_2 r_3]$

In a plane wall the area perpendicular to the direction of heat flow adding more insulation to a wall always decreases heat transfer. If thicker is insulation, the lower is the heat transfer rate. This is due to the fact the outer surface have always the same area.

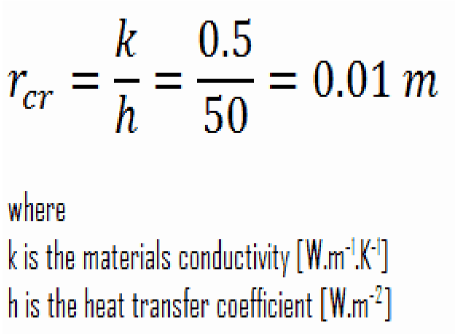
But in cylindrical and spherical coordinates, the addition of insulation also increases the outer surface, which decreases the convection resistance at the outer surface. Moreover, in some cases, a decrease in the convection resistance due to the increase in surface area can be more important than an increase in conduction resistance due to thicker insulation. As a result the total resistance may actually decrease resulting in increased heat flow.

The thickness upto which heat flow increases and after which heat flow decreases is termed as critical thickness. In the case of cylinders and spheres it is called critical radius. It can be derived the critical radius of insulation depends on the thermal conductivity of the insulation k and the external convection heat transfer coefficient h.

Optimum thickness of insulation

The radius of insulation at which the total annual cost is minimum corresponds to optimum thickness of insulation.

Assume a steel pipe of r_1 = 10 mm, which is exposed to natural convection at $h = 50\ W/m^2.K$. This pipe is insulated by material of thermal conductivity $k = 0.5\ W/m.K$. Determine the critical thickness of this combination:



$$r_{cr} = \frac{k}{h} = \frac{0.5}{50} = 0.01\ m$$

where

k is the materials conductivity $[W.m^{-1}.K^{-1}]$

h is the heat transfer coefficient $[W.m^{-2}]$

Hence $r_{cr} > r_1$ and heat transfer will increase with the addition of insulation up to a thickness of $r_{cr} – r_1 = (0.010 – 0.005)m = 0.005\ m$

P.No. 1. A furnace wall consists of two layers, 22.5cm of fire brick(k=1.2kcal/hr m °C)and 12.5cm of insulating brick (k=0.15kcal/hr m °C) . The temperature inside the furnace is 1650 °C and the inside heat transfer coefficient is 60kcal/hr m °C . The temperature of the surrounding atmosphere is 27°C and the outside heat transfer coefficient is 10kcal/hr m^2 °C . Determine the rate of heat of loss per square meter of the wall.

Solution:

L_1 - 22.5 x 10^{-2} m

L_2 - 12.5 x 10^{-2} m

k_1 - 1.2 kcal/hr m °C

k_2 - 0.15 kcal/hr m °C

h_a - 60 kcal/hr m^2 °C

h_b - 10kcal/hr m^2 °C

T_a – 1650 °C

T_b – 27 °C

$$Q = \frac{(T_a - T_b)}{\frac{1}{h_a A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{1}{h_b A}}$$

Q / A = 1426.8 W / m^2

P.No.2. A pipe carrying steam at 220 °C has an I.D. of 15cm. The convection coefficient on the inside wall is 60W/m^2K . The pipe wall thickness is 15mm and the thermal conductivity is 35W/mK . The outside is exposed to a chemical at 130°C with a convection coefficient of 15W/m^2K . If the pipe wall is covered with two insulation layers, the first 3cm thickness with k=0.12W/mK and the second 4cm thickness with k= 0.35W/m K. Determine the rate of heat transfer.

Solution :

r_1 - 75 x 10^{-3} m

r_2 - 90 x 10^{-3} m

r_3 - 120 x 10^{-3} m

r_4 - 160 x 10^{-3} m

k_1 - 35 W / m K

k_2 - 0.12 W / m K

k_2 - 0.35 W / m K

h_a - 60 W / m^2 K

h_b – 15 W / m^2 K

$$Q = \frac{2\pi L\,(T_a - T_b)}{\frac{1}{h_a r_1} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{k_1} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{k_2} + \frac{\ln\left(\frac{r_4}{r_3}\right)}{k_3} + \frac{1}{h_b r_4}}$$

Q = 146.32 W

2.11 Analogy between heat flow and electricity:

Heat flow is represented by Fourier's law of heat conduction whereas electrical flow is represented by ohm's law.

Heat flow = Temperature gradient / thermal resistance

Ohm's law is given by

Electrical flow = voltage drop / electrical resistance

The various parameters analogous to each other in both the laws are heat flow and electrical flow, voltage drop with temperature gradient, thermal and electrical resistance.

2.12 Relation between individual and overall heat transfer coefficient

Film temperature: It is the average between the temperature of the surface and the fluid. indeed if the entire Overall Heat Transfer Coefficient Let us consider a plane wall of thickness x_w and thermal conductivity kw. The warm fluid at a mean temperature of T_h is flowing through the inside surface of the wall. The cold fluid at a mean temperature of T_c is flowing through the outside surface of the wall. The inside surface temperature is T_{wh} and outside surface temperature is T_{wc}. The overall heat transfer coefficient is constructed from the individual coefficients and the resistances of the wall in the following manner. (i) Overall Heat transfer Coefficient based on outside surface area. The rate of heat transfer from the warm fluid to the inner surface of the wall in differential form:

$$\frac{dq}{dA_i} = h_i(T_h - T_{wh})$$

By rearranging the above,

$$dq = \frac{(T_h - T_{wh})}{\frac{1}{h_i dA_i}}$$

. The rate of heat transfer through the wall in differential form is given by

$$\frac{dq}{d\bar{A}_L} = \frac{k_w(T_{wh} - T_{wc})}{x_w}$$

on rearranging,

$$dq = \frac{(T_{wh} - T_{wc})}{\frac{x_w}{k_w d\bar{A}_L}}$$

the rate of heat transfer from the outer surface of the wall to the cold fluid in differential form is given by

$$\frac{dq}{dA_o} = h_o(T_{wc} - T_c)$$

on rearranging,

$$dq = \frac{(T_{wc} - T_c)}{\dfrac{1}{h_o dA_o}}$$

If the eqns are solved for the temperature differences and the temperature differences added, the result is

$$(T_h - T_{wh}) + (T_{wc} - T_c) + (T_{wc} - T_c) = T_h - T_c = \Delta T = dQ\left(\frac{1}{h_i dA_i} + \frac{x_w}{k_w d\bar{A}_L} + \frac{1}{h_o dA_o}\right)$$

Assume that the heat transfer rate is arbitrarily based on the outside area. If the eqn is solved for dQ, and if both sides of the resulting equations are divided by dAo, the result is

$$\frac{dQ}{dA_o} = \frac{T_h - T_c}{\dfrac{1}{h_i}\left(\dfrac{dA_o}{dA_i}\right) + \dfrac{x_w}{k_w}\left(\dfrac{dA_o}{d\bar{A}_L}\right) + \dfrac{1}{h_o}\left(\dfrac{dA_o}{dA_o}\right)}$$

$$\frac{dA_o}{dA_i} = \frac{r_o}{r_i} \quad ; \qquad \frac{dA_o}{d\bar{A}_L} = \frac{r_o}{\bar{r}_L}$$

Then the above Eqn becomes

$$\frac{dQ}{dA_o} = \frac{T_h - T_c}{\dfrac{1}{h_i}\left(\dfrac{r_o}{r_i}\right) + \dfrac{r_o}{k_w}\ln\left(\dfrac{r_o}{r_i}\right) + \dfrac{1}{h_o}}$$

2.3 Overall heat transfer coefficient based on inside surface area The rate of heat transfer from the warm fluid to the inner surface of the wall in differential form:

$$\frac{dq}{dA_i} = h_i(T_h - T_{wh})$$

On rearranging,

$$dq = \frac{(T_h - T_{wh})}{\dfrac{1}{h_i dA_i}}$$

The rate of heat transfer through the wall in differential form

$$\frac{dq}{d\bar{A}_L} = \frac{k_w(T_{wh} - T_{wc})}{x_w}$$

By rearranging the above,

$$dq = \frac{(T_{wh} - T_{wc})}{\dfrac{x_w}{k_w d\bar{A}_L}}$$

The rate of heat transfer from the outer surface of the wall to the cold fluid in differential form:

$$\frac{dq}{dA_o} = h_o(T_{wc} - T_c)$$

On rearranging

$$dq = \frac{(T_{wc} - T_c)}{\dfrac{1}{h_o dA_o}}$$

If the eqns to are solved for the temperature differences and the temperature differences added, the result is

$$(T_h - T_{wh}) + (T_{wc} - T_c) + (T_{wc} - T_c) = T_h - T_c = \Delta T = dQ\left(\frac{1}{h_i dA_i} + \frac{x_w}{k_w d\overline{A}_L} + \frac{1}{h_o dA_o}\right)$$

Assume that the heat transfer rate is arbitrarily based on the inside area. If the Eqn is solved for dQ, and if both sides of the resulting equations are divided by dAi, the result is

$$\frac{dQ}{dA_i} = \frac{T_h - T_c}{\dfrac{1}{h_i}\left(\dfrac{dA_i}{dA_i}\right) + \dfrac{x_w}{k_w}\left(\dfrac{dA_i}{d\overline{A}_L}\right) + \dfrac{1}{h_o}\left(\dfrac{dA_i}{dA_o}\right)}$$

$$\frac{dA_i}{dA_o} = \frac{r_i}{r_o} \quad ; \qquad \frac{dA_i}{d\overline{A}_L} = \frac{r_i}{\overline{r}_L}$$

The above eqn becomes

$$\frac{dQ}{dA_i} = \frac{T_h - T_c}{\dfrac{1}{h_i}\left(\dfrac{r_i}{r_i}\right) + \dfrac{r_i}{k_w}\ln\left(\dfrac{r_o}{r_i}\right) + \dfrac{1}{h_o}\left(\dfrac{r_i}{r_o}\right)}$$

the overall heat transfer coefficient based on outside surface area is,

$$U_o = \frac{1}{\dfrac{1}{h_i}\left(\dfrac{r_o}{r_i}\right) + \dfrac{r_o}{k_w}\ln\left(\dfrac{r_o}{r_i}\right) + \dfrac{1}{h_o}}$$

Overall heat transfer coefficient based on inside surface area

The rate of heat transfer from the warm fluid to the inner surface of the wall in differential form

$$\frac{dq}{dA_i} = h_i\left(T_h - T_{wh}\right)$$

On rearranging,

$$dq = \frac{\left(T_h - T_{wh}\right)}{\dfrac{1}{h_i dA_i}}$$

The rate of heat transfer from the outer surface of the wall to the cold fluid in differential form:

$$\frac{dq}{dA_o} = h_o\left(T_{wc} - T_c\right)$$

on rearranging,

$$dq = \frac{\left(T_{wc} - T_c\right)}{\dfrac{1}{h_o dA_o}}$$

substituting the above equations,

$$\left(T_h - T_{wh}\right) + \left(T_{wc} - T_c\right) + \left(T_{wc} - T_c\right) = T_h - T_c = \Delta T = dQ\left(\frac{1}{h_i dA_i} + \frac{x_w}{k_w d\bar{A}_L} + \frac{1}{h_o dA_o}\right)$$

Assume that the heat transfer rate is arbitrarily based on the inside area. If the eqn is solved for dQ, and if both sides of the resulting equations are divided by dAi, the result is

$$\frac{dQ}{dA_i} = \frac{T_h - T_c}{\frac{1}{h_i}\left(\frac{dA_i}{dA_i}\right) + \frac{x_w}{k_w}\left(\frac{dA_i}{d\bar{A}_L}\right) + \frac{1}{h_o}\left(\frac{dA_i}{dA_o}\right)}$$

$$\frac{dA_i}{dA_o} = \frac{r_i}{r_o} \quad ; \qquad \frac{dA_i}{d\bar{A}_L} = \frac{r_i}{\bar{r}_L}$$

$$\frac{dQ}{dA_i} = \frac{T_h - T_c}{\frac{1}{h_i}\left(\frac{r_i}{r_i}\right) + \frac{r_i}{k_w}\ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_o}\left(\frac{r_i}{r_o}\right)}$$

Hence, the overall heat transfer coefficient based on inside area is

$$U_i = \frac{1}{\frac{1}{h_i} + \frac{r_i}{k_w}\ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_o}\left(\frac{r_i}{r_o}\right)}$$

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SCHOOL OF BIO & CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – III – Modeling and transport of bioprocesses – SCH1312

3.1 Introduction of Mass Transfer

When a system contains two or more components whose concentrations vary from point to point, there is a natural tendency for mass to be transferred, minimizing the concentration differences within a system. The transport of one constituent from a region of higher concentration to that of a lower concentration is called mass transfer.

The transfer of mass within a fluid mixture or across a phase boundary is a process that plays a major role in many industrial processes. Examples of such processes are:

(i) Dispersion of gases from stacks
(ii) Removal of pollutants from plant discharge streams by absorption
(iii) Stripping of gases from waste water
(iv) Neutron diffusion within nuclear reactors
(v) Air conditioning

Many of air day-by-day experiences also involve mass transfer, for example:

(i) A lump of sugar added to a cup of coffee eventually dissolves and then eventually diffuses to make the concentration uniform.
(ii) Water evaporates from ponds to increase the humidity of passing-air-stream
(iii) Perfumes present a pleasant fragrance which is imparted throughout the surrounding atmosphere.

The mechanism of mass transfer involves both molecular diffusion and convection.

3.2 Properties of Mixtures

Mass transfer always involves mixtures. Consequently, we must account for the variation of physical properties which normally exist in a given system. When a system contains three or more components, as many industrial fluid streams do, the problem becomes unwidely very quickly. The conventional engineering approach to problems of multicomponent system is to attempt to reduce them to representative binary (i.e., two component) systems.

In order to understand the future discussions, let us first consider definitions and relations which are often used to explain the role of components within a mixture.

3.2.1 Concentration of Species:

Concentration of species in multicomponent mixture can be expressed in many ways. For species A, mass concentration denoted by ρ_A is defined as the mass of A,m_A per unit volume of the mixture.

$$\rho_A = \frac{m_A}{V} \quad \text{----------------------------------} \quad \textbf{(1)}$$

The total mass concentration density ρ is the sum of the total mass of the mixture in unit volume:

$$\rho = \sum_i \rho_i$$

where ρ_i is the concentration of species i in the mixture.

Molar concentration of, A, C_A is defined as the number of moles of A present per unit volume of the mixture.

By definition,

$$Number\,of\ moles = \frac{mass\,of\ A}{molecular\,weight\,of\ A}$$

$$n_A = \frac{m_A}{M_A} \quad \text{----------------------------- } \mathbf{(2)}$$

Therefore from (1) & (2)

$$C_A = \frac{n_A}{V} = \frac{\rho_A}{M_A}$$

For ideal gas mixtures,

$$n_A = \frac{p_A V}{RT} \quad \textbf{[from Ideal gas law PV = nRT]}$$

$$C_A = \frac{n_A}{V} = \frac{p_A}{RT}$$

where p_A is the partial pressure of species A in the mixture. V is the volume of gas, T is the absolute temperature, and R is the universal gas constant.

The total molar concentration or molar density of the mixture is given by

$$C = \sum_i C_i$$

3.2.2 Velocities

In a multi-component system the various species will normally move at different velocities; and evaluation of velocity of mixture require the averaging of the velocities of each species present.

If ν $_I$ is the velocity of species i with respect to stationary fixed coordinates, then mass-average velocity for a multicomponent mixture defined in terms of mass concentration is,

$$\nu = \frac{\sum_i \rho_i \nu_i}{\sum_i \rho_i} = \frac{\sum_i \rho_i \nu_i}{\rho}$$

By similar way, molar-average velocity of the mixture ν * is

$$\nu^* = \frac{\sum_i C_i V_i}{C}$$

For most engineering problems, there will be title difference in ν * and ν and so the mass average velocity, ν, will be used in all further discussions.

The velocity of a particular species relative to the mass-average or molar average velocity is termed as diffusion velocity

(i.e) Diffusion velocity = ν $_i$ - ν

The mole fraction for liquid and solid mixture, x $_A$,and for gaseous mixtures, y $_A$, are the molar concentration of species A divided by the molar density of the mixtures.

$$x_A = \frac{C_A}{C}$$ **(liquids and solids)**

$$y_A = \frac{C_A}{C}$$ **(gases).**

The sum of the mole fractions, by definition must equal 1;

(i.e.) $$\sum_i x_i = 1$$

$$\sum_i y_i = 1$$

by similar way, mass fraction of A in mixture is;

$$w_A = \frac{\rho_A}{\rho}$$

10. The molar composition of a gas mixture at 273 K and $1.5 * 10^5$ Pa is:

O_2	7%
CO	10%
CO_2	15%
N_2	68%

Determine

a) the composition in weight percent
b) average molecular weight of the gas mixture
c) density of gas mixture

d) partial pressure of O_2.

Calculations:

Let the gas mixture constitutes 1 mole. Then

O_2 = 0.07 mol

CO = 0.10 mol

CO_2 = 0.15 mol

N_2 = 0.68 mol

Molecular weight of the constituents are:

O_2 = 2 * 16 = 32 g/mol

CO = 12 + 16 = 28 g/mol

CO_2 = 12 + 2 * 16 = 44 g/mol

N_2 = 2 * 14 = 28 g/mol

Weight of the constituents are: (1 mol of gas mixture)

O_2 = 0.07 * 32 = 2.24 g

CO = 0.10 * 28 = 2.80 g

CO_2 = 0.15 * 44 = 6.60 g

N_2 = 0.68 * 28 = 19.04 g

Total weight of gas mixture = 2.24 + 2.80 + 6.60 + 19.04

= 30.68 g

Composition in weight percent:

$$O_2 = \frac{2.24}{30.68} * 100 = 7.30\%$$

$$CO = \frac{2.80}{30.68} * 100 = 9.13\%$$

$$CO_2 = \frac{6.60}{30.68} * 100 = 21.51\%$$

$$N_2 = \frac{19.04}{30.68} * 100 = 62.06\%$$

Average molecular weight of the gas mixture $M = \frac{\text{Weight of gas mixture}}{\text{Number of moles}}$

$$M = \frac{30.68}{1} = 30.68\ g/mol$$

Assuming that the gas obeys ideal gas law,

PV = nRT

$$\frac{n}{V} = \frac{P}{RT}$$

$$\frac{n}{V} = \text{molar density} = \rho_m$$

Therefore, density (or mass density) = $\rho_m M$

Where M is the molecular weight of the gas.

$$Density = \rho_m M = \frac{PM}{RT} = \frac{1.5 * 10^5 * 30.68}{8314 * 273}\ kg/m^3$$

$= 2.03\ kg/m^3$

Partial pressure of O_2 = [mole fraction of O_2] * total pressure

$$= \frac{7}{100} * \left(1.5 * 10^5\right)$$

$= 0.07 * 1.5 * 10^5$

$= 0.105 * 10^5$ Pa

3.3 Diffusion flux

Just as momentum and energy (heat) transfer have two mechanisms for transport-molecular and convective, so does mass transfer. However, there are convective fluxes in mass transfer, even on a molecular level. The reason for this is that in mass transfer, whenever there is a driving force, there is always a net movement of the mass of a particular species which results in a bulk motion of molecules. Of course, there can also be convective mass transport due to macroscopic fluid motion. In this chapter the focus is on molecular mass transfer.

The mass (or molar) flux of a given species is a vector quantity denoting the amount of the particular species, in either mass or molar units, that passes per given increment of time through a unit area normal to the vector. The flux of species defined with reference to fixed spatial coordinates, N_A is

$$N_A = C_A \nu_A \quad \text{----------------------} \; (1)$$

This could be written interms of diffusion velocity of A, (i.e., $\nu_A - \nu$) and average velocity of mixture, ν, as

$$N_A = C_A (\nu_A - \nu) + C_A \nu \quad \text{--------------} \; (2)$$

By definition

$$\nu = \nu^* = \frac{\sum_i C_i \nu_i}{C}$$

Therefore, equation (2) becomes

$$N_A = C_A (\nu_A - \nu) + \frac{C_A}{C} \sum_i C_i \nu_i$$

$$= C_A (\nu_A - \nu) + y_A \sum_i C_i \nu_i$$

For systems containing two components A and B,

$$N_A = C_A (v_A - v) + y_A (C_A v_A + C_B v_B)$$

$$= C_A (v_A - v) + y_A (N_A + N_B)$$

$$N_A = C_A (v_A - v) + y_A N \quad \text{-----------} (3)$$

The first term on the right hand side of this equation is diffusional molar flux of A, and the second term is flux due to bulk motion.

3.3.1 Fick's law:

An empirical relation for the diffusional molar flux, first postulated by Fick and, accordingly, often referred to as Fick's first law, defines the diffusion of component A in an isothermal, isobaric system. For diffusion in only the Z direction, the Fick's rate equation is

$$J_A = -D_{AB} \frac{dC_A}{dZ}$$

where D_{AB} is diffusivity or diffusion coefficient for component A diffusing through component B, and dC_A / dZ is the concentration gradient in the Z-direction.

A more general flux relation which is not restricted to isothermal, isobasic system could be written as

$$J_A = -C D_{AB} \frac{d y_A}{d Z} \qquad \text{-----------------} (4)$$

using this expression, Equation (3) could be written as

$$N_A = -C D_{AB} \frac{d y_A}{d Z} + y_A N \qquad \text{---------------} (5)$$

3.3.2 Relation among molar fluxes:

For a binary system containing A and B, from Equation (5),

$$N_A = J_A + y_A N$$

or $$J_A = N_A + y_A N \qquad \text{-----------------------} \mathbf{(6)}$$

Similarly,

$$J_B = N_B + y_B N \qquad \text{--------------------} \mathbf{(7)}$$

Addition of Equation (6) & (7) gives,

$$J_A + J_B = N_A + N_B - (y_A + y_B) N \qquad \text{----------} \mathbf{(8)}$$

By definition N = N $_A$ + N $_B$ and y $_A$ + y $_B$ = 1.
Therefore equation (8) becomes,

$$J_A + J_B = 0$$

$$J_A = -J_B$$

$$C D_{AB} \frac{d y_A}{d z} = - C D_{BA} \frac{d y_B}{d Z} \quad \text{--------------} \ (9)$$

From $y_A + y_B = 1$

$dy_A = - dy_B$

Therefore Equation (9) becomes,

$$D_{AB} = D_{BA} \quad \text{----------------------------------} \quad (10)$$

This leads to the conclusion that diffusivity of A in B is equal to diffusivity of B in A.

3.4 Diffusivity

Fick's law proportionality, D_{AB}, is known as mass diffusivity (simply as diffusivity) or as the diffusion coefficient. D_{AB} has the dimension of L^2 / t, identical to the fundamental dimensions of the other transport properties: Kinematic viscosity, $\nu\eta = (\mu / \rho)$ in momentum transfer, and thermal diffusivity, α (= $k / \rho C_\rho$) in heat transfer.

Diffusivity is normally reported in cm^2 / sec; the SI unit being m^2 / sec.

Diffusivity depends on pressure, temperature, and composition of the system.

In table, some values of D_{AB} are given for a few gas, liquid, and solid systems.

Diffusivities of gases at low density are almost composition independent, incease with the temperature and vary inversely with pressure. Liquid and solid diffusivities are strongly

concentration dependent and increase with temperature.

General range of values of diffusivity:

Gases :	**5×10^{-6}**	**------------**	**1×10^{-5}**	**m^2 / sec.**
Liquids :	**10^{-6}**	**------------**	**10^{-9}**	**m^2 / sec.**
Solids :	**5×10^{-14}**	**------------**	**1×10^{-10}**	**m^2 / sec.**

In the absence of experimental data, semitheoretical expressions have been developed which give approximation, sometimes as valid as experimental values, due to the difficulties encountered in experimental measurements.

3.4.1 Diffusivity in Gases:

Pressure dependence of diffusivity is given by

$$D_{AB} \propto \frac{1}{p}$$ **(for moderate ranges of pressures, upto 25 atm).**

And temperature dependency is according to

$$D_{AB} \propto T^{3/2}$$

Diffusivity of a component in a mixture of components can be calculated using the diffusivities for the various binary pairs involved in the mixture. The relation given by Wilke is

$$D_{1-mixture} = \frac{1}{\dfrac{y'_2}{D_{1-2}} + \dfrac{y'_3}{D_{1-3}} + \ldots\ldots\ldots. + \dfrac{y'_n}{D_{1-n}}}$$

Where $D_{1\text{-mixture}}$ is the diffusivity for component 1 in the gas mixture; $D_{1\text{-n}}$ is the

diffusivity for the binary pair, component 1 diffusing through component n; and y'_n is the mole fraction of component n in the gas mixture evaluated on a component –1 – free basis, that is

$$y'_2 = \frac{y_2}{y_2 + y_3 + \ldots\ldots y_n}$$

$$D_2m = \frac{1.847}{1.2 * 10^5} = 1.539 * 10^{-5}\ m^2/\sec$$

By Similar calculations Diffusivity of N_2 in the mixture can be calculated, and is found to be, $D_{3m} = 1.588 * 10^{-5}$ m^2/sec.

3.4.2 Diffusivity in liquids:

Diffusivity in liquid are exemplified by the values given in table … Most of these values are nearer to 10^{-5} cm^2 / sec, and about ten thousand times shower than those in dilute gases. This characteristic of liquid diffusion often limits the overall rate of processes accruing in liquids (such as reaction between two components in liquids).

In chemistry, diffusivity limits the rate of acid-base reactions; in the chemical industry, diffusion is responsible for the rates of liquid-liquid extraction. Diffusion in liquids is important because it is slow.

Certain molecules diffuse as molecules, while others which are designated as electrolytes ionize in solutions and diffuse as ions. For example, sodium chloride (NaCl), diffuses in water as ions Na^+ and Cl^-. Though each ions has a different mobility, the electrical neutrality of the solution indicates the ions must diffuse at the same rate; accordingly it is possible to speak of a diffusion coefficient for molecular electrolytes such as NaCl. However, if several ions are present, the diffusion rates of the individual cations and

anions must be considered, and molecular diffusion coefficients have no meaning.

Diffusivity varies inversely with viscosity when the ratio of solute to solvent ratio exceeds five. In extremely high viscosity materials, diffusion becomes independent of viscosity.

3.4.3 Diffusivity in solids:

Typical values for diffusivity in solids are shown in table. One outstanding characteristic of these values is their small size, usually thousands of time less than those in a liquid, which are inturn 10,000 times less than those in a gas.

Diffusion plays a major role in catalysis and is important to the chemical engineer. For metallurgists, diffusion of atoms within the solids is of more importance.

3.5 Steady State Diffusion

In this section, steady-state molecular mass transfer through simple systems in which the concentration and molar flux are functions of a single space coordinate will be considered.

In a binary system, containing A and B, this molar flux in the direction of z, as given by Eqn (5) is [section 3.3.1]

$$N_A = -C D_{AB} \frac{d y_A}{d z} + y_A \left(N_A + N_B\right) \quad \text{--- (1)}$$

3.5.1 Diffusion through a stagnant gas film

The diffusivity or diffusion coefficient for a gas can be measured, experimentally using Arnold diffusion cell. This cell is illustrated schematically in figure.

The narrow tube of uniform cross section which is partially filled with pure liquid A, is maintained at a constant temperature and pressure. Gas B which flows across the open

end of the tub, has a negligible solubility in liquid A, and is also chemically inert to A. (i.e. no reaction between A & B).

Component A vaporizes and diffuses into the gas phase; the rate of vaporization may be physically measured and may also be mathematically expressed interms of the molar flux.

Consider the control volume S Δ z, where S is the cross sectional area of the tube. Mass balance on A over this control volume for a steady-state operation yields

[Moles of A leaving at z + Δz] – [Moles of A entering at z] = 0.

(i.e.) $$S N_A \Big|_{z+\Delta z} - S N_A \Big|_{z} = 0. \qquad \text{-------------- (1)}$$

Dividing through by the volume, SΔZ, and evaluating in the limit as ΔZ approaches zero, we obtain the differential equation

$$\frac{d N_A}{d z} = 0 \qquad \text{------------------------- (2)}$$

This relation stipulates a constant molar flux of A throughout the gas phase from Z_1 to Z_2.

A similar differential equation could also be written for component B as,

$$\frac{d N_B}{d Z} = 0,$$

and accordingly, the molar flux of B is also constant over the entire diffusion path from z_1 and z_2.

Considering only at plane z_1, and since the gas B is insoluble is liquid A, we realize that N_B, the net flux of B, is zero throughout the diffusion path; accordingly B is a stagnant

gas.

From equation (1) (of section 3.5)

$$N_A = -CD_{AB}\frac{dy_A}{dz} + y_A(N_A + N_B)$$

Since N $_B$ = 0,

$$N_A = -CD_{AB}\frac{dy_A}{dz} + y_A N_A$$

Rearranging,

$$N_A = \frac{-CD_{AB}}{1-y_A}\frac{dy_A}{dz} \qquad \text{------------ (3)}$$

This equation may be integrated between the two boundary conditions:

at z = z $_1$ Y $_A$ = Y $_{A1}$

And at z = z $_2$ Y $_A$ = y $_{A2}$

Assuming the diffusivity is to be independent of concentration, and realizing that N $_A$ is constant along the diffusion path, by integrating equation (3) we obtain

$$N_A \int_{Z_1}^{Z_2} dz = CD_{AB} \int_{y_{A1}}^{y_{A2}} \frac{-dy_A}{1-y_A}$$

$$N_A = \frac{C D_{AB}}{Z_2 - Z_1} \ln\left(\frac{1 - y_{A2}}{1 - y_{A1}}\right)$$ **--------------(4)**

The log mean average concentration of component B is defined as

$$y_{B,lm} = \frac{y_{B2} - y_{B1}}{\ln\left(y_{B2}/y_{B1}\right)}$$

Since $y_B = 1 - y_A$,

$$y_{B,lm} = \frac{(1 - y_{A2}) - (1 - y_{A1})}{\ln\left(y_{A2}/y_{A1}\right)} = \frac{y_{A1} - y_{A2}}{\ln\left(y_{A2}/y_{A1}\right)}$$ **------- (5)**

Substituting from Eqn (5) in Eqn (4),

$$N_A = \frac{C D_{AB}}{Z_2 - z_1} \frac{(y_{A1} - y_{A2})}{y_{B,lm}}$$ **-------------------- (6)**

For an ideal gas $C = \frac{n}{V} = \frac{p}{RT}$ **, and**

for mixture of ideal gases $y_A = \frac{p_A}{P}$

Therefore, for an ideal gas mixture equation. (6) becomes

$$N_A = \frac{D_{AB}}{RT(z_2 - z_1)} \frac{(p_{A1} - p_{A2})}{p_{B,lm}}$$

This is the equation of molar flux for steady state diffusion of one gas through a second stagnant gas.

Many mass-transfer operations involve the diffusion of one gas component through another non-diffusing component; absorption and humidification are typical operations defined by these equation.

The concentration profile (y_A vs. z) for this type of diffusion is shown in figure:

Oxygen is diffusing in a mixture of oxygen-nitrogen at 1 std atm, 25°C. Concentration of oxygen at planes 2 mm apart are 10 and 20 volume % respectively. Nitrogen is non-diffusing.

(a) Derive the appropriate expression to calculate the flux oxygen. Define units of each term clearly.
(b) Calculate the flux of oxygen. Diffusivity of oxygen in nitrogen = 1.89 * 10^{-5} m^2/sec.

Solution:

Let us denote oxygen as A and nitrogen as B. Flux of A (i.e.) N_A is made up of two components, namely that resulting from the bulk motion of A (i.e.), Nx_A and that resulting from molecular diffusion J_A:

$$N_A = Nx_A + J_A \text{ ---------------------------------- (1)}$$

From Fick's law of diffusion,

$$J_A = -D_{AB} \frac{dC_A}{dz} \text{ --- (2)}$$

Substituting this equation (1)

$$N_A = Nx_A - D_{AB} \frac{dC_A}{dz} \text{ ----------------------------- (3)}$$

Since N = $N_A + N_B$ and $x_A = C_A / C$ equation (3) becomes

$$N_A = \left(N_A + N_B\right) \frac{C_A}{C} - D_{AB} \frac{dC_A}{dz}$$

Rearranging the terms and integrating between the planes between 1 and 2,

$$\int \frac{dz}{cD_{AB}} = -\int_{C_{A1}}^{C_{A2}} \frac{dC_A}{N_A C - C_A \left(N_A + N_B\right)} \text{ -------------- (4)}$$

Since B is non diffusing $N_B = 0$. Also, the total concentration C remains constant. Therefore, equation (4) becomes

$$\frac{z}{CD_{AB}} = -\int_{C_{A1}}^{C_{A2}} \frac{dC_A}{N_A C - N_A C_A}$$

$$= \frac{1}{N_A} \ln \frac{C - C_{A2}}{C - C_{A1}}$$

Therefore,

$$N_A = \frac{CD_{AB}}{z} \ln \frac{C - C_{A2}}{C - C_{A1}} \text{ ---------------------------- (5)}$$

Replacing concentration in terms of pressures using Ideal gas law, equation (5) becomes

$$N_A = \frac{D_{AB} P_t}{RTz} \ln \frac{P_t - P_{A2}}{P_t - P_{A1}} \text{ --------------------------- (6)}$$

where

D_{AB} = molecular diffusivity of A in B

P_T = total pressure of system

R = universal gas constant

T = temperature of system in absolute scale

z = distance between two planes across the direction of diffusion

P_{A1} = partial pressure of A at plane 1, and

P_{A2} = partial pressure of A at plane 2

3.5.2 Psuedo steady state diffusion through a stagnant film:

In many mass transfer operations, one of the boundaries may move with time. If the length of the diffusion path changes a small amount over a long period of time, a pseudo steady state diffusion model may be used. When this condition exists, the equation of steady state diffusion through stagnant gas' can be used to find the flux.

If the difference in the level of liquid A over the time interval considered is only a small fraction of the total diffusion path, and $t_0 - t$ is relatively long period of time, at any given instant in that period, the molar flux in the gas phase may be evaluated by

$$N_A = \frac{C D_{AB} (y_{A1} - y_{A2})}{z y_{B,lm}} \quad \text{------------------ (1)}$$

where z equals $z_2 - z_1$, the length of the diffusion path at time t.

The molar flux N_A is related to the amount of A leaving the liquid by

$$N_A = \frac{\rho_{A,L}}{M_A} \frac{dz}{dt} \quad \text{------------------------- (2)}$$

where $\frac{\rho_{A,L}}{M_A}$ **is the molar density of A in the liquid phase**

under Psuedo steady state conditions, equations (1) & (2) can be equated to give

$$\frac{\rho_{A,L}}{M_A}\frac{dz}{dt} = \frac{C D_{AB}(y_{A1}-y_{A2})}{z\, y_{B,lm}} \quad \text{--------------- (3)}$$

Equation. (3) may be integrated from t = 0 to t and from z = z $_{t0}$ to z = z$_t$ as:

$$\int_{t=0}^{t} dt = \frac{\rho_{A,L}\, y_{B,lm} / M_A}{C D_{AB}(y_{A1}-y_{A2})} \int_{Z_{t0}}^{Z_t} z\, dz$$

yielding

$$t = \frac{\rho_{A,L}\, y_{B,lm} / M_A}{C D_{AB}(y_{A1}-y_{A2})} \left(\frac{z_t^2 - z_{t0}^2}{2}\right) \quad \text{-------- (4)}$$

This shall be rearranged to evaluate diffusivity D$_{AB}$ as,

$$D_{AB} = \frac{\rho_{A,L}\, y_{B,lm}}{M_A C\,(y_{A1}-y_{A2})t} \left(\frac{z_t^2 - z_{t0}^2}{2}\right)$$

3.5.3 Equimolar counter diffusion:

A physical situation which is encountered in the distillation of two constituents whose molar latent heats of vaporization are essentially equal, stipulates that the flux of one gaseous component is equal to but acting in the opposite direction from the other gaseous component; that is, N$_A$ = - N$_B$.

The molar flux N$_A$, for a binary system at constant temperature and pressure is described by

$$N_A = -C D_{AB}\frac{d y_A}{dz} + y_A\,(N_A + N_B)$$

or $$N_A = -D_{AB}\frac{dC_A}{dz} + y_A (N_A + N_B) \qquad \text{------- (1)}$$

with the substitution of N_B = - N_A, Equation (1) becomes,

$$N_A = -D_{AB}\frac{dC_A}{dz} \qquad \text{---------------- (2)}$$

For steady state diffusion Equation. (2) may be integrated, using the boundary conditions:

at $z = z_1$ $C_A = C_{A1}$
and $z = z_2$ $C_A = C_{A2}$

Giving,

$$N_A \int_{Z_1}^{Z_2} dz = -D_{AB} \int_{C_{A1}}^{C_{A2}} dC_A$$

from which

$$N_A = \frac{D_{AB}}{z_2 - z_1}(C_{A1} - C_{A2}) \qquad \text{------------------- (3)}$$

For ideal gases, $C_A = \frac{n_A}{V} = \frac{p_A}{RT}$ **. Therefore Equation. (3) becomes**

$$N_A = \frac{D_{AB}}{RT(z_2 - z_1)}(P_{A1} - P_{A2}) \qquad \text{---------- (4)}$$

This is the equation of molar flux for steady-state equimolar counter diffusion.

Concentration profile in these equimolar counter diffusion may be obtained from,

$$\frac{d}{dz}(N_A) = 0$$ **(Since N_A is constant over the diffusion path).**

And from equation. (2)

$$N_A = -D_{AB}\frac{dC_A}{dz}.$$

Therefore

$$\frac{d}{dz}\left(-D_{AB}\frac{dC_A}{dz}\right) = 0 .$$

or

$$\frac{d^2 C_A}{dz^2} = 0.$$

This equation may be solved using the boundary conditions to give

$$\frac{C_A - C_{A1}}{C_{A1} - C_{A2}} = \frac{z - z_1}{z_1 - z_2} \quad \text{-------------- (5)}$$

Equation, (5) indicates a linear concentration profile for equimolar counter diffusion.

For equimolar counter current diffusion:

$$N_A = \frac{D_{AB}}{RTz}\left(p_{A1} - p_{A2}\right) \text{------------------------- (1)}$$

Diffusion in Liquids:

Equation derived for diffusion in gases equally applies to diffusion in liquids with some modifications. Mole fraction in liquid phases is normally written as 'x' (in gases as y). The concentration term 'C' is replaced by average molar density, $\left(\frac{\rho}{M}\right)_{av}$.

a) For steady – state diffusion of A through non diffusivity B:

N_A = constant , $N_B = 0$

$$N_A = \frac{D_{AB}}{z\, x_{BM}}\left(\frac{\rho}{M}\right)_{av}\left(x_{A1} - x_{A2}\right)$$

where $Z = Z_2 - Z_1$, the length of diffusion path; and

$$X_{BM} = \frac{X_{B2} - X_{B1}}{\ln\left(X_{B2} / X_{B1}\right)}$$

b) For steady – state equimolar counter diffusion :

$N_A = - N_B$ = const

$$N_A = \frac{D_{AB}}{Z}\left(C_{A1} - C_{A2}\right) = \frac{D_{AB}}{Z}\left(\frac{\rho}{M}\right)_{av}\left(x_{A1} - x_{A2}\right)$$

Diffusion in solids

In certain unit operation of chemical engineering such as in drying or in absorption,

mass transfer takes place between a solid and a fluid phase. If the transferred species is distributed uniformly in the solid phase and forms a homogeneous medium, the diffusion of the species in the solid phase is said to be structure independent. In this cases diffusivity or diffusion coefficient is direction – independent.

At steady state, and for mass diffusion which is independent of the solid matrix structure, the molar flux in the z direction is :

$$N_A = -D_{AB}\frac{dC_A}{dz} = \text{constant}$$, as given by Fick's law.

Integrating the above equation,

$$N_A = \frac{D_{AB}\left(C_{A1} - C_{A2}\right)}{z}$$

which is similar to the expression obtained for diffusion in a stagnant fluid with no bulk motion (i.e. N = 0).

Diffusion in process solids:

In some chemical operations, such as heterogeneous catalysis, an important factor, affecting the rate of reaction is the diffusions of the gaseous component through a porous solid. The effective diffusivity in the solid is reduced below what it could be in a free fluid, for two reasons. First, the tortuous nature of the path increases the distance, which a molecule must travel to advance a given distance in the solid. Second, the free cross – sectional area is restricted. For many catalyst pellets, the effective diffusivity of a gaseous component is of the order of one tenth of its value in a free gas.

If the pressure is low enough and the pores are small enough, the gas molecules will collide with the walls more frequently than with each other. This is known as Knudsen flow or Knudsen diffusion. Upon hitting the wall, the molecules are momentarily absorbed and then given off in random directions. The gas flux is reduced by the wall collisions.

By use of the kinetic flux is the concentration gradient is independent of pressure ; whereas the proportionality constant for molecular diffusion in gases (i.e. Diffusivity) is inversely proportional to pressure.

Knudsen diffusion occurs when the size of the pore is of the order of the mean free path of the diffusing molecule.

3.6 Transient Diffusion/Unsteady state diffusion

Transient processes, in which the concentration at a given point varies with time, are referred to as unsteady state processes or time – dependent processes. This variation in concentration is associated with a variation in the mass flux.

These generally fall into two categories:

i) the process which is in an unsteady state only during its initial startup, and

ii) the process which is in a batch operation throughout its operation.

In unsteady state processes there are three variables-concentration, time,

and position. Therefore the diffusion process must be described by partial rather than ordinary differential equations.

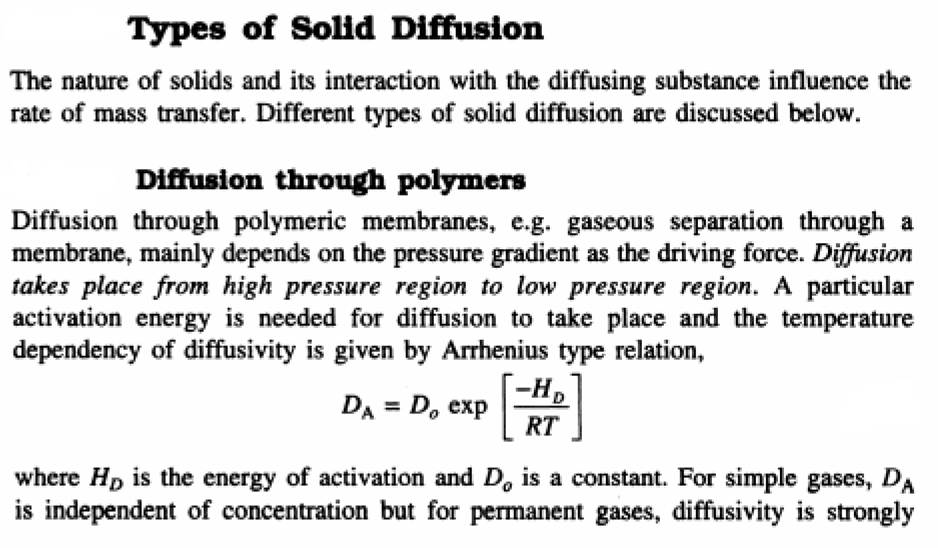
Although the differential equations for unsteady state diffusion are easy to establish, most solutions to these equations have been limited to situations involving simple geometries and boundary conditions, and a constant diffusion coefficient.

Many solutions are for one-directional mass transfer as defined by Fick's second law of diffusion :

$$\frac{\partial C_A}{\partial t} = D_{AB} \frac{\partial^2 C_A}{\partial z^2} \qquad \text{\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

This partial differential equation describes a physical situation in which there is no bulk–motion contribution, and there is no chemical reaction. This situation is encountered when the diffusion takes place in solids, in stationary liquids, or in system having equimolar counter diffusion.

The solution to Fick's second law usually has one of the two standard forms. It may appear in the form of a trigonometric series which converges for large values of time, or it may involve series of error functions or related integrals which are most suitable for numerical evaluation at small values of time. These solutions are commonly obtained by using the mathematical techniques of separation of variables or Laplace transforms.



Types of Solid Diffusion

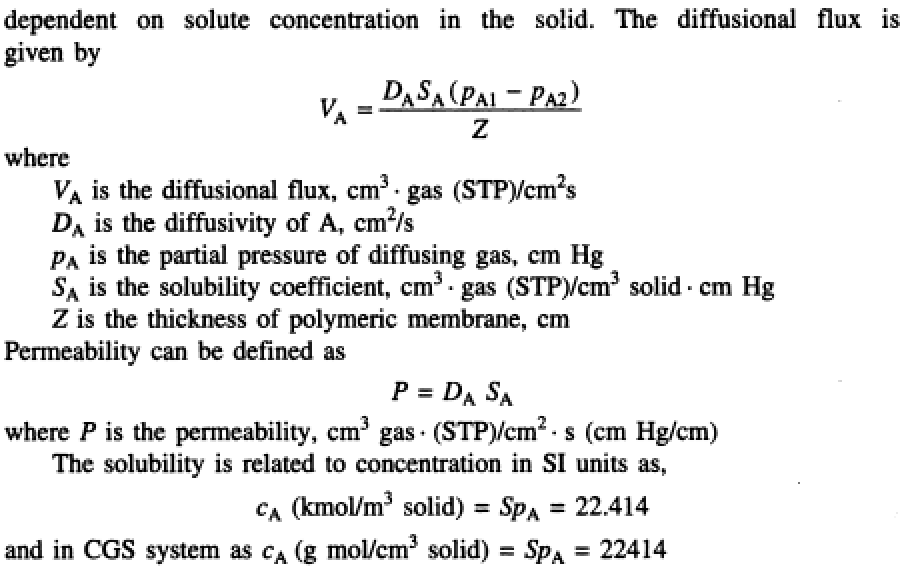
The nature of solids and its interaction with the diffusing substance influence the rate of mass transfer. Different types of solid diffusion are discussed below.

Diffusion through polymers

Diffusion through polymeric membranes, e.g. gaseous separation through a membrane, mainly depends on the pressure gradient as the driving force. *Diffusion takes place from high pressure region to low pressure region.* A particular activation energy is needed for diffusion to take place and the temperature dependency of diffusivity is given by Arrhenius type relation,

$$D_A = D_o \exp\left[\frac{-H_D}{RT}\right]$$

where H_D is the energy of activation and D_o is a constant. For simple gases, D_A is independent of concentration but for permanent gases, diffusivity is strongly



dependent on solute concentration in the solid. The diffusional flux is given by

$$V_A = \frac{D_A S_A (p_{A1} - p_{A2})}{Z}$$

where

V_A is the diffusional flux, $\text{cm}^3 \cdot \text{gas (STP)/cm}^2\text{s}$

D_A is the diffusivity of A, cm^2/s

p_A is the partial pressure of diffusing gas, cm Hg

S_A is the solubility coefficient, $\text{cm}^3 \cdot \text{gas (STP)/cm}^3 \text{ solid} \cdot \text{cm Hg}$

Z is the thickness of polymeric membrane, cm

Permeability can be defined as

$$P = D_A S_A$$

where P is the permeability, $\text{cm}^3 \text{ gas} \cdot \text{(STP)/cm}^2 \cdot \text{s (cm Hg/cm)}$

The solubility is related to concentration in SI units as,

$$c_A \text{ (kmol/m}^3\text{ solid)} = Sp_A = 22.414$$

and in CGS system as c_A (g mol/cm^3 solid) $= Sp_A = 22414$

Diffusion through crystalline solids

Solute nature and crystalline structure are the important parameters in this type of diffusion. Some of the mechanisms followed for diffusion through crystal geometry are given below:

1. Interstitial mechanism—Solute atoms diffuse from one interstitial site to the next in the crystal lattice.
2. Vacancy mechanism—If lattice sites are vacant, an atom in an adjacent site may jump into the vacant site.
3. Interstitialcy mechanism—A large atom occupying in an interstitial site pushes the neighbouring lattice into an interstitial position and moves into the vacancy produced.
4. Crowded-ion mechanism—An extra atom in a chain of close-packed atoms can displace several atoms in the line from their equilibrium position.
5. Diffusion along grain boundaries—Diffusion takes place in crystal interfaces and dislocatic

Diffusion in porous solids

The solid may be porous in nature such as adsorbents or membrane and the diffusion takes place either by virtue of concentration gradient or by hydrodynamic flow behaviour because of pressure difference. In steady state diffusion of gases, there are two types of diffusive movement, depending on the ratio of pore diameter d, to the mean free path of the gas molecules, λ.

If the ratio $d/\lambda > 20$, molecular diffusion predominates

$$N_A = \left(\frac{N_A}{N_A + N_B}\right)\left(\frac{D_{AB,eff} P_t}{RTZ}\right) \ln \left[\frac{\dfrac{N_A}{N_A + N_B} - y_{A2}}{\dfrac{N_A}{N_A + N_B} - y_{A1}}\right] \qquad (2.62)$$

If $d/\lambda < 0.2$, the rate of diffusion is governed by the collisions of the gas molecules within the pore walls and follows Knudsen's law.

$$N_A = \frac{D_{K,A}\,(p_{A1} - p_{A2})}{RTl}$$

where

$D_{K,A}$ is the Knudsen diffusivity, cm^2/s
l is the length of the pore, cm
p_A is the partial pressure of diffusing substance, cmHg

Knudsen diffusivity can be determined by using an empirical relation,

$$D_{K,A} = \left(\frac{d}{3}\right)\left(\frac{8\, g_c\, RT}{\pi M_A}\right)^{1/2}$$

2.5.2 Unsteady State Diffusion

Since solids are not readily transported, as fluids, unsteady state diffusional conditions arise more frequently in solids than in fluids. For unsteady state diffusion, Fick's second law is applied,

$$\frac{\partial C}{\partial t} = D_{AB}\left[\frac{\partial^2 C_A}{\partial x^2} + \frac{\partial^2 C_A}{\partial y^2} + \frac{\partial^2 C_A}{\partial z^2}\right] \tag{2.67}$$

Theories of mass transfer

Various theories have been used as models for explaining the turbulent mass transfer. These models can be used for predicting the mass transfer coefficients and they can be correlated with experimental data to obtain the design parameters of process equipments.

Film Theory

Consider turbulent flow of liquid over a solid surface and a simultaneous mass transfer is taking place. The film theory postulates that there is a stagnant film of thickness, Z_f adjacent to the interface, where the concentration difference is attributed to molecular diffusion as shown in Fig. 3.3. As the molecular diffusion is occurring only in Z_f, the flux equation can be written as

$$N_A = k_C (C_{A1} - C_{A2}) = \left(\frac{D_{AB}}{Z_f}\right)(C_{A1} - C_{A2}) \tag{3.21}$$

where $(C_{A1} - C_{A2})$ is the concentration difference. Hence, $k_C = (D_{AB}/Z_f)$, the mass transfer coefficient is proportional to $D_{AB}^{1.0}$. However, the J_D factor is given by,

$$J_D = \left(\frac{k_C}{V}\right)(N_{Sc})^{2/3} = \left(\frac{k_C}{V}\right)\left(\frac{\mu}{\rho D_{AB}}\right)^{2/3} \tag{3.22}$$

Hence, the film theory deviates from the actual turbulent mass transfer.

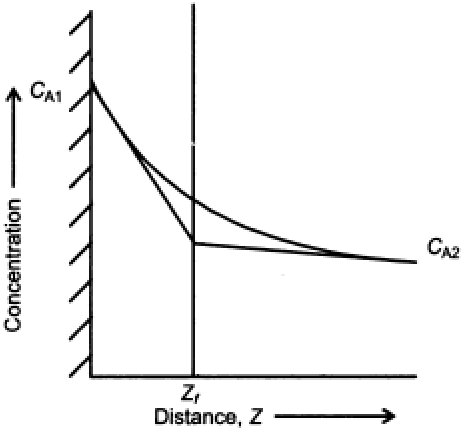


Fig. Concentration distribution in film theory.

Penetration Theory

This theory explains the mass transfer at fluid surface and was proposed by Higbie. In many situations, the time of exposure for mass transfer is too short and hence, there may not be sufficient time for the steady state concentration gradient of film theory to develop. This theory has been described in Fig. 3.4. An eddy *b*, rising

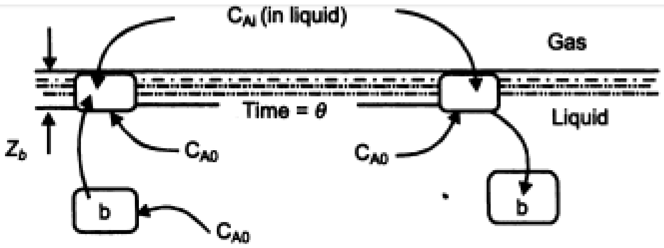


Fig. 3.4 Higbie's theory.

from the turbulent liquid is exposed for a short time, θ at the interface for absorption. In this situation, the exposure time is assumed to be constant for all the eddies or particles of liquid.

Initially the eddy concentration is C_{A0} and when it comes to the surface, the interfacial concentration is C_{Ai}. Since the exposure time is less, molecules of solute from gas never reach the depth Z_b, which is nothing but the thickness of eddy. The liquid particle is subjected to unsteady state diffusion and hence Fick's second law is applicable, i.e.

$$\frac{\partial C_A}{\partial \theta} = D_{AB}\left[\frac{\partial^2 C_A}{\partial Z^2}\right]$$

From the solute point of view, the depth Z_b is considered to be infinite.
The boundary conditions are as follows:

$$C_A = C_{A0} \text{ at } \theta = 0 \text{ for all } Z$$
$$C_A = C_{Ai} \text{ at } Z = 0 \;\; \theta > 0$$
$$C_A = C_{A0} \text{ at } Z = \infty \text{ for all } \theta$$

By solving the above Eq. (3.23), the average flux can be obtained as described in falling film

Hence,

$$N_{A,av} = 2(C_{Ai} - C_{A0})\sqrt{\frac{D_{AB}}{\pi\theta}}$$

$$k_{L,av} = \sqrt{\frac{D_{AB}}{\pi\theta}}$$

Thus, in penetration theory k_L is proportional to $D_{AB}^{0.5}$. However, the exponent on D_{AB} varies from zero to 0.8 or 0.9.

Surface Renewal Theory

In reality, the time of exposure of all eddies as proposed in penetration theory is not constant. Hence, Danckwerts modified the penetration theory to account for varying lengths of time of exposure.

If S is the fractional rate of replacement of elements,

Then, $$N_{A,av} = (C_{Ai} - C_{A0})\sqrt{D_{AB}S}$$

Hence, $k_{L,av}$ is proportional to $D_{AB}^{0.5}$ in this theory.

Combination of Film–Surface Renewal Theory

Film theory is meant for steady state diffusion where $k_L \propto D_{AB}$ and in surface renewal theory, $k_L \propto D_{AB}^{0.5}$. So k_L is proportional to D_{AB}^n with '*n*' dependent upon circumstances. In this theory Dobbins replaced the third boundary condition of Eq. (3.23) by $C_A = C_{A0}$ at $Z = Z_b$, where Z_b is of finite depth. Finally he obtained

$$k_{L,av} = \sqrt{D_{AB}S}\ \coth\sqrt{\frac{SZ_b^2}{D_{AB}}}$$

Surface–Stretch Theory

Lightfoot and his coworkers explained this theory where they found that the mass transfer at the interface varies with time periodically. When mass transfer is proceeding for a particular system the central portion of the drop is thoroughly turbulent and resistance to mass transfer resides in a surface layer with varying thickness and the drop is elongated as shown in Fig. 3.5. According to this theory,

$$k_{L,av} = \frac{\left[(A/A_r)\sqrt{\dfrac{D_{AB}}{\pi\theta_r}}\right]}{\sqrt{\displaystyle\int_0^{\theta/\theta r}(A/A_r)^2\,d\theta}}$$

where *A* is time dependent interfacial surface, A_r is reference value of A, defined for every situation and θ_r is constant with dimensions of time or drop formation time.

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SCHOOL OF BIO & CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – IV – Modeling and transport of bioprocesses-SCH1312

I Michaelis-Menten kinetics

The goal of this chapter is to develop the mathematical techniques to quantitatively model biochemical reactions. Biochemical reactions in living cells are often catalyzed by enzymes. These enzymes are proteins that bind and subsequently react specifically with other molecules (other proteins, DNA, RNA, or small molecules) defined as substrates. A few examples:

1. The conversion of glucose (substrate) into glucose-6-phosphate (product) by the protein hexokinase (enzyme).
2. Transcription: binding of the RNA polymerase (enzyme) to the promoter region of the DNA (substrate) results in transcription of the mRNA (product).
3. The phosphorylation of a protein: the unphosphorylated protein CheY (substrate, regulating the direction of rotation of the bacterial flagella) is phosphorylated by a phosphate CheZ (enzyme) resulting in CheY-p (product).

All these reactions involve a substrate S reacting with an enzyme E to form a complex ES which then in turn is converted into product P and the enzyme:

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} ES \xrightarrow{k_2} E + P \qquad \textbf{[I.1]}$$

In this scheme there are two fundamental different reactions. The first reaction depicted with the double arrow is a reversible reaction reflecting the reversible binding and unbinding of the enzyme and the substrate. The second reaction is an irreversible reaction in which the enzyme-substrate complex is irreversibly converted into product and enzyme symbolized by the single arrow. The rate of a reaction is proportional to the product of the concentrations of the reactants. The kinetics of the chemical equations above is described by the following set of coupled differential equations:

$$\begin{aligned}
\frac{d[S]}{dt} &= -k_1[E][S] + k_{-1}[ES] \\
\frac{d[E]}{dt} &= -k_1[E][S] + (k_{-1} + k_2)[ES] \\
\frac{d[ES]}{dt} &= k_1[E][S] - (k_{-1} + k_2)[ES] \\
\frac{d[P]}{dt} &= k_2[ES] \equiv v
\end{aligned} \qquad \textbf{[I.2]}$$

Note that k_1 and k_{-1} have different units, 1/(Ms) and 1/s respectively. The turnover rate v is defined as the increase (or decrease) in product over time, which is directly proportional to the concentration of enzyme-substrate complex [ES]. For the analysis below we will assume initial conditions: $[S]_{t=0} = S_o$; $[E]_{t=0} = E_o$; $[ES]_{t=0} = 0$; $[P]_{t=0} = 0$.

Since the enzyme is a catalyst that facilitates the reaction but does not react itself, the total concentration of enzyme (free + bound) should be constant:

$$E_o = [E] + [ES] \qquad \textbf{[I.3]}$$

Using this conservation law the four differential equations [I.2] reduce to three coupled ordinary differential equations:

$$\begin{aligned}
\frac{d[S]}{dt} &= -k_1E_o[S] + (k_1[S] + k_{-1})[ES] \\
\frac{d[ES]}{dt} &= k_1E_o[S] - (k_1[S] + k_{-1} + k_2)[ES] \\
\frac{d[P]}{dt} &= k_2[ES] \equiv v
\end{aligned} \qquad \textbf{[I.4]}$$

with the initial conditions $[S]_{t=0} = S_o$, $[ES]_{t=0} = 0$, and $[P]_{t=0} = 0$. Matlab code 1 solves these equations and calculates the time dependence of the concentrations [S], [ES] and [P] as a function of the initial concentrations $[S_o]$ and $[E_o]$ and the rate constants k_1, k_{-1}, and k_2. In this case the systems can also be solved analytically. Figure 1 shows an example of the time dependence of the chemical components for $k_1[S_o] \approx k_{-1} >> k_2$. This is often the regime of biological relevance since the substrate-enzyme binding occurs at much faster time scales than the turnover into product. The thermodynamic equilibrium or steady state (t→∞) of this system would be [S] = [ES] = 0; [E] = [E_o]; [P] = [S_o]. However the relevant time-scale to consider is the time range in which [ES] and [E] are relatively constant. This state is often called the quasi-equilibrium or pseudo-steady state. Under these circumstances one expects that after an initial short transient period there will be a balance between the formation of the enzyme-substrate complex and the breaking apart of complex (either to enzyme and substrate, or to enzyme and product). In the pseudo-steady state (d[ES]/dt = d[E]/dt = 0) (I.4) reduces to:

$$[ES] = \frac{k_1[S]E_o}{k_1[S] + k_{-1} + k_2}$$

$$v = \frac{dP}{dt} = \frac{k_2[S]E_o}{\frac{k_{-1}+k_2}{k_1} + [S]} \qquad \textbf{[I.5]}$$

In the case of many more substrate than enzyme molecules ($S_o >> E_o$), this pseudo-steady state will be achieved before there is perceptible transformation of substrate into product. In this case the equation [I.5] leads to the traditional Michaelis-Menten equation, which predicts the initial turnover rate of the enzymatic reaction v_o as a function of initial substrate concentration S_o:

$$v_o = \frac{v_{max}S_o}{K_m + S_o} \qquad \textbf{[I.6]}$$

where the constant $K_m = (k_{-1}+k_2)/k_l$ is called the Michaelis constant and $v_{max} = k_2E_o$ is the maximum turn-over rate. The Michaelis constant has units of concentration and reflects the affinity of the reaction. Strong affinity means small K_m. At a concentration K_m the turn-over rate is $0.5v_{max}$ (Fig. 2).

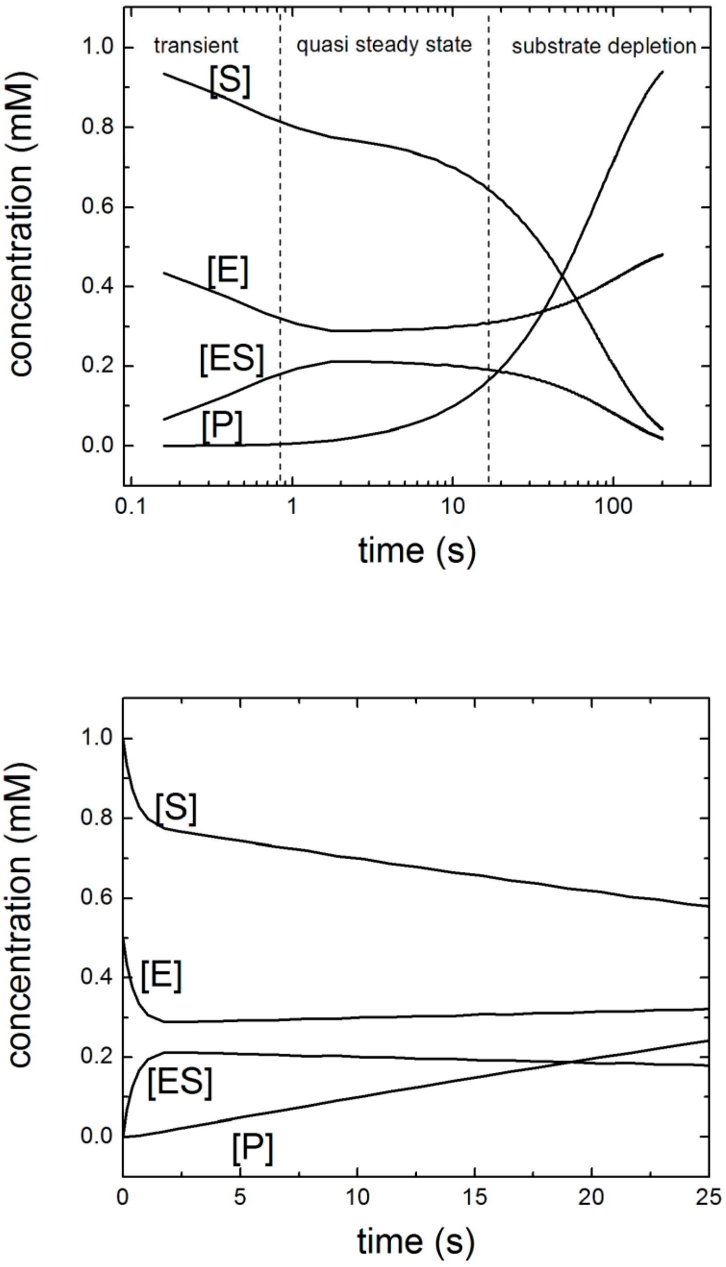

Figure 1. The time dependence of the substrate, enzyme, enzyme-substrate complex, and product concentration. This graph was generated by using Matlab code 1. The upper panel uses a logarithmic x-axis whereas the lower panel uses a linear scale.

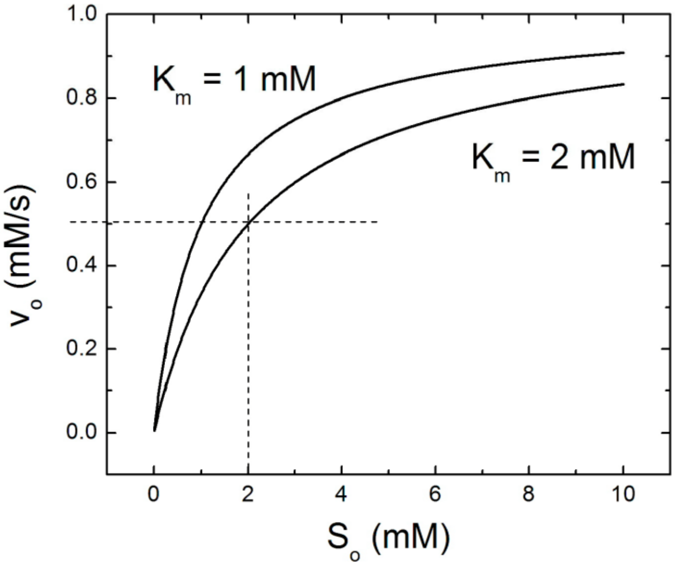

Figure 2. The initial turnover rate as given by the Michaelis-Menten formula [I.6].

Matlab code 1: Michaelis-Menten kinetics

```
% filename: mm.m

k1=1e3;       % units 1/(Ms)

k_1=1;        % units 1/s

k2=0.05;      % units 1/s
E0=0.5e-3;    % units M
options=[];

[t y]=ode23('mmfunc',[0 100],[1e-3 0 0],options,k1,k_1,k2,E0);
S=y(:,1);

ES=y(:,2);

E=E0-ES;
P=y(:,3);

plot(t,S,'r',t,E,'b',t,ES,'g',t,P,'c');


% filename: mmfunc.m
```

```
function dydt = f(t,y,flag,k1,k_1,k2,E0)
% [S] = y(1), [ES] = y(2), [P] = y(3)
```

II Equilibrium binding and cooperativity

In the previous Section we considered Michaelis-Menten kinetics. We found that the traditional form of the Michaelis-Menten equation [I.6] is derived by assuming a quasi-steady state in which the concentration of enzyme-substrate complex is fairly constant over time. Additionally we had to assume that initially the substrate is in excess. In this Section, we first will take a step back and focus on the steady state behavior of reversible reactions and introduce the concept of multiple binding sites. Initially we will consider multiple binding sites that are independently binding substrates. However for most protein complexes the binding of substrates is not independent. For example, after binding the first substrate molecule the binding probability of the second substrate is affected. This phenomenon is called cooperativity.

In the previous section it was assumed that one substrate molecule binds to one enzyme molecule. In biological reactions however proteins often bind multiple substrates. Assume a protein has n binding sites for a substrate. P_j denotes the protein bound to j substrate molecules S. The reactions describing this process are:

$$S + P_{j-1} \leftrightarrow P_j \qquad \textbf{[II.1]}$$

where j = 1, 2, …, n.

The time-evolution of the concentration of unbound protein P_o is (j=1):

$$\frac{d[P_0]}{dt} = -k_{+1}[P_o][S] + k_{-1}[P_1] \qquad \textbf{[II.2]}$$

where k_{+1} and k_{-1} are the forward and backward rate constants of [II.1] for j=1. The association and dissociation constants are defined as:

$$K_a = \frac{k_{+1}}{k_{-1}}$$

$$K_d = \frac{k_{-1}}{k_{+1}} = \frac{1}{K_a} \qquad \textbf{[II.3]}$$

In steady state, d[P$_o$]/dt = 0:

$$K_a = \frac{[P_1]}{[P_o][S]} \qquad \textbf{[II.4]}$$

To characterize all n reactions, we introduce the n association constants K_j, j = 1, 2, … ,n.

$$K_j = \frac{[P_j]}{[P_{j-1}][S]} \qquad \textbf{[II.5]}$$

It is experimentally difficult to measure $[P_j]$, a more convenient quantity is the average number r ($0 < r < n$) of substrates bound to the protein. Because there are j substrates bound to P_j, r is given by:

$$r = \frac{[P_1] + 2[P_2] + 3[P_3] + \ldots + n[P_n]}{[P_o] + [P_1] + [P_2] + \ldots + [P_n]} \qquad \textbf{[II.6]}$$

combining [II.5] and [II.6] gives Adair's equation:

$$r = \frac{K_1[S] + 2K_1K_2[S]^2 + 3K_1K_2K_3[S]^3 + \ldots + nK_1K_2\ldots K_n[S]^n}{1 + K_1[S] + K_1K_2[S]^2 + \ldots + K_1K_2\ldots K_n[S]^n} \qquad \textbf{[II.7]}$$

Note that $0 < r < n$, one often uses the normalized form, called the saturation function Y = r/n ($0 < Y < 1$).

Identical and independent binding sites

For now let's assume we have n identical binding sites and that binding at a given site is independent of the state of binding of all other sites. The rate constants k_+ and k_- characterize the binding and unbinding rates respectively. In steady state, [II.2] can now be written as:

$$0 = -nk_+[P_o][S] + k_-[P_1] \qquad \textbf{[II.8]}$$

The factor n takes into account that there are n possible binding sites available for binding the first substrate. On the other hand there is only one possibility to loose a substrate going from state P_1 to P_o. Similarly for j=2 we can deduce:

$$0 = -(n-1)k_+[P_1][S] + 2k_-[P_2] \qquad \textbf{[II.9]}$$

because there are (n-1) possibilities to add a substrate and only 2 possibilities to remove a substrate. If the intrinsic association constant K is defined as:

$$K \equiv \frac{k_+}{k_-} \qquad \textbf{[II.10]}$$

$$K_j = \frac{(n - j + 1)K}{j} \qquad \textbf{[II.11]}$$

for j = 1, 2, … , n. By substituting [II.11] in [II.7] an explicit equation for r as a function of K, n, and [S] is found. We will not go through the details of the derivation. If you are interested, see for example Bisswanger (2002, p. 11-16). The final result is elegantly simple:

$$r = \frac{nK[S]}{1 + K[S]} \qquad \textbf{[II.12]}$$

Note that the mathematical form of this equation is very similar to Michaelis-Menten kinetics. However this result is a steady-state (equilibrium) property while Michaelis-Menten equation is not. Equation [II.12] can also be derived in a more hand waving manner. As the n binding sites are identical and independent, it is not important to view them as clustered in one protein. If [F] is the concentration of free binding site and [B] the concentration of bound sites in steady state, then the association constant for this equilibrium is given by:

$$K = \frac{[B]}{[F][S]} \qquad \textbf{[II.13]}$$

The total number of sites is: n[P]=[F]+[B], this combined with [II.13] gives:

$$r = \frac{[B]}{[P]} = \frac{nK[S]}{1 + K[S]} \qquad \textbf{[II.14]}$$

Non-identical and independent binding sites

Now consider the case in which the binding sites are non-identical. Each binding site family (with n_j binding sites) is characterized by its own association constant K_j. At low concentrations first the binding sites with the high affinities will be occupied, the lower affinity binding site will only be occupied at larger [S]. As the binding site are independent the binding equation (18) holds for each binding site family and r is just the sum of the different individual processes:

$$r = \frac{n_1K_1[S]}{1 + K_1[S]} + \frac{n_2K_2[S]}{1 + K_2[S]} + \dots + \frac{n_mK_m[S]}{1 + K_m[S]} \qquad \textbf{[II.15]}$$

Identical and interacting binding sites

In the following discussion we will confine ourselves to two binding sites (n=2). First, let us assume that both binding sites are identical. In this case we only have to consider three states for the protein-substrate complex: no substrate bound, one substrate molecule bound, and two substrate molecules bound. The rate constants k_+ and k_- characterize the transitions between the unbound and single-bound state, and k^*_+ and k^*_- the transitions between single-bound and double-bound states. The intrinsic association constants are defined by: $K = k_+/k_-$ and $K^* = k^*_+/k^*_-$. Analogous to [II.10] and [II.11] we find:

$$K_1 = 2K$$

$$K_2 = \frac{1}{2}K^* \qquad \textbf{[II.16]}$$

By using Adair's equation [II.7] we find:

$$r = \frac{2K[S] + 2KK^*[S]^2}{1 + 2K[S] + KK^*[S]^2} \qquad \textbf{[II.17]}$$

The saturation function Y = r/n is:

$$Y = \frac{K[S] + KK^*[S]^2}{1 + 2K[S] + KK^*[S]^2} \qquad \textbf{[II.18]}$$

For K=K* we recover the hyperbolic (Michaelis-Menten like) equation [II.12]:

$$\tilde{Y} = \frac{K[S]}{1 + K[S]} \qquad \textbf{[II.19]}$$

Let's compare the functional forms of [II.18] and [II.19] in more detail. The difference between the two functions is:

$$Y - \tilde{Y} = \frac{(K^* - K)K[S]^2}{(1+K[S])(1+2K[S]+KK^*[S]^2)} \qquad \textbf{[II.20]}$$

Positive cooperativity is often defined as $Y - \tilde{Y} > 0$, and negative cooperativity as $Y - \tilde{Y} < 0$. In other words, positive cooperativity occurs when the affinity of binding a second ligand is larger than binding the first ligand ($K^* > K$). For negative cooperativity the binding affinity for the second ligand is smaller than for the first ($K^* < K$).

Another, often used, definition for cooperativity is sigmoidality (from ‘S shaped’). For a sigmoidal curve the second derivative should change sign. Let’s introduce the dimensionless variables $\beta = K^*/K$ and $x = K[S]$:

$$Y = \frac{x(1+\beta x)}{1+2x+\beta x^2}$$

$$\frac{dY}{dx} = \frac{1+2x\beta+\beta x^2}{(1+2x+\beta x^2)^2} \qquad \textbf{[II.21]}$$

$$\frac{d^2Y}{dx^2} = 2\frac{\beta-2-\beta x\left[3+3x\beta+\beta x^2\right]}{(1+2x+\beta x^2)^3}$$

The second derivative can only change sign if $\beta > 2$. Note that this definition yields a different criterion for cooperativity. According to the first definition a reaction is cooperative for $\beta > 1$, whereas according to the second definition $\beta > 2$. During the rest of the course we will use the first definition.

Now consider the limit for which intermediate states can be neglected. In this example, that would mean that single-bound states are very unlikely. The effective reaction would be:

$$P_o + 2S \leftrightarrow P_2 \qquad \textbf{[II.22]}$$

The saturation function is now:

$$Y = \frac{K[S]^2}{1+K[S]^2} \qquad \textbf{[II.23]}$$

where $K = [P_2]/([P_o][S]^2)$ is the association constant of reaction [II.22]. Note that is this case the units of K are $(M)^{-2}$. This limit was first consider by Hill who proposed a graphical way to represent equations such as [II.23]. In a Hill plot one plots $\ln\left[Y/(1-Y)\right]$ versus $\ln[S]$. The slope of this graph is called the Hill number which is in this case equals 2. The Hill number is often used as an estimation of the number of binding sites of a protein. However one should be very careful as [II.23] involves a major assumption (no intermediate states). Let’s calculate the Hill number n_H for the case [II.21] in which intermediate states are allowed:

$$n_H = \frac{d}{d(\ln[S])}\ln\left\lceil \frac{Y}{\quad} \right\rceil = x\frac{d}{\quad}\ln\left\lceil \frac{Y}{\quad} \right\rceil = 1 + \frac{(\beta-1)x}{\quad} \qquad \textbf{[II.24]}$$

$\lfloor 1 - Y \rfloor$ dx $\lfloor 1 - Y \rfloor$

$(1+$
$x)($
$1+$
$\beta x)$

The Hill number is plotted in Fig. 3 as a function of x at different values of β. The Hill number only approaches 2 for very large β and small x.

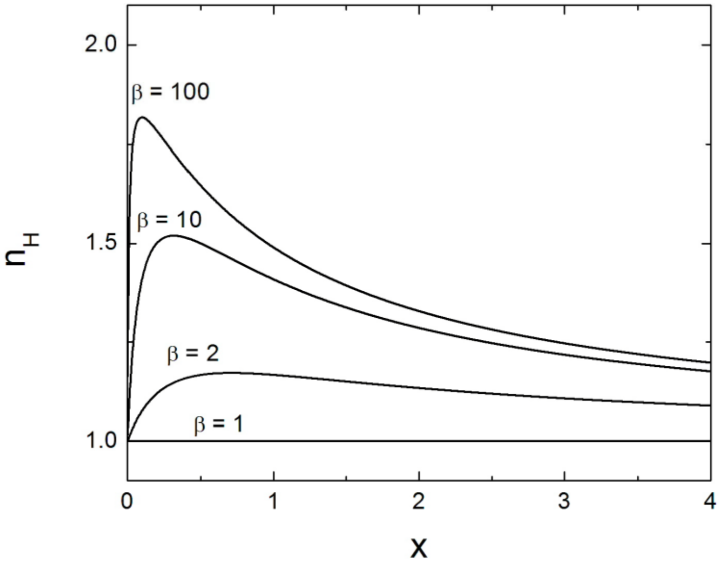

Figure 3. The Hill number as a function of the dimensionless concentration at different values of β for a protein with two identical interacting binding sites. The mathematical form is given by equation [II.24].

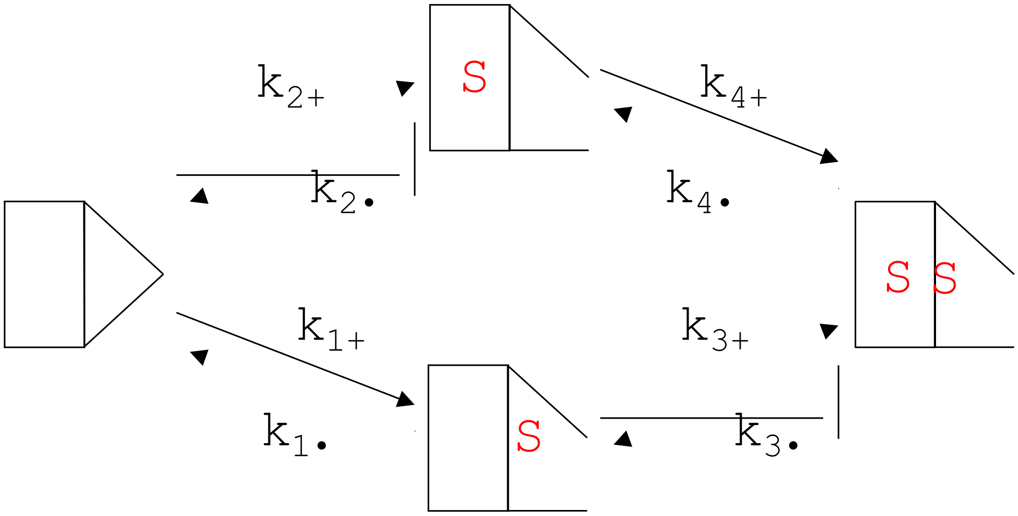

Figure 4. Two independent interacting binding sites.

Non-identical and interacting binding sites

How would the analysis above change if the two binding sites are non-identical? The ligand binding to the two binding sites is now characterized by the rate constants $k_{\pm1}$, $k_{\pm2}$, $k_{\pm3}$, and $k_{\pm4}$ (Fig. 4) and the four intrinsic association constants $K_j=k_{+j}/k_{-j}$ ($j=1,2,3,4$). In this case there are four states of the protein-ligand complex: nothing bound, site 1 bound, site 2 bound, and two sites bound. The principal of detailed balance (thermodynamic equilibrium) does not allow any net fluxes between states.

Microbial Growth Kinetics

Introduction-Studying growth of a microorganism is the basis of biotechnological exploitation of microflora for production of desired product. Optimization of growth of microorganism in a particular media is desirable due to economical and availability of particular growth constituent in a region. Despite this, some microorganisms have specific requirement and they grow in a particular growth media. Common media for growth of different microorganism, yeast and animal cells is discussed in future lecture. In today’s lecture we will discuss bacterial cell division, methods of measuring growth, different phase in bacterial growth and growth kinectics.

Modes of Bacterial Cell Division-

1. Binary division-binary division is the most common mode of cell division in bacteria (Figure 9.1). In this mode of cell division, a single bacteria cell grows transversely with the synthesis of chromosomal DNA. A transverse septum appears in the middle of the cell body that divides the bacterial cell into the two with a distribution of chromosomal DNA, ribosome and other cellular machinery.

Binary Division **Budding**

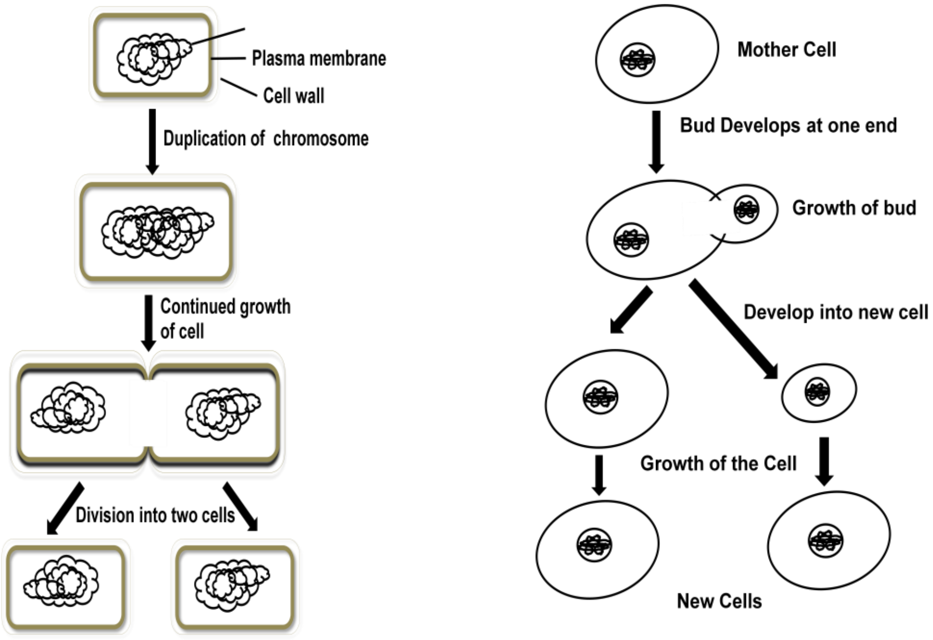

Figure 9.1: Different modes of cell division in bacteria.

2. Budding-In this mode of cell division, chromosomal DNA divides to form two copies. Sister chromosomal DNA moves to the one side of the cell and this portion of the cells protrude from main body to form bud. Eventually bud grows in size and get separated from main cell to develop a new cell.

3. Fragmentation-This mode of asexual division is more common in filamentous bacteria. In this mode, filament of the growing cell gets fragmented into small bacillary or coccoid cells, these cellular fragments eventually develop into new cell.

Measuring Bacterial growth- A number of methds have been developed to measure bacterial growth in liquid media and in solid support media. A few are discussed below:

Microscopic count-bacterial cells can be counted easily on a "petroff-hausser counting chamber" (Figure 9.2). The chamber has a ruling to make square (1/400 mm^2) of equivalent volume. A glass slide is placed (~1/50mm height) to make a chamber filled with bacterial cell suspension. Volume of each chamber is 1/20,000 mm^3. This chamber can be used to observe bacteria with phase contrast microscope. For example, if each chamber has 8 bacteria then there are 8x20,000,000 or 1.6×10^8 bacteria/ml. A very high or low concentration of bacterial sample can not be counted accurately.

Plate count method-In this method, a defined amount of bacterial culture suspension is introduced onto solid support media to grow and give colonies. If number of colonies on solid media is too high, then serial dilution of original stock can be plated on solid media and number of colony can be counted with a colony counter. A manual colony counter has lamp at the bottom, a grid to divide the bacterial culture plate and a magnifying glass to visualize and count single colony. A plate with colony count of 30-300 can be used to determine the number of bacteria present in original stock.

Number of bacteria per ml= Number of colonies counted on plate X dilution of sample

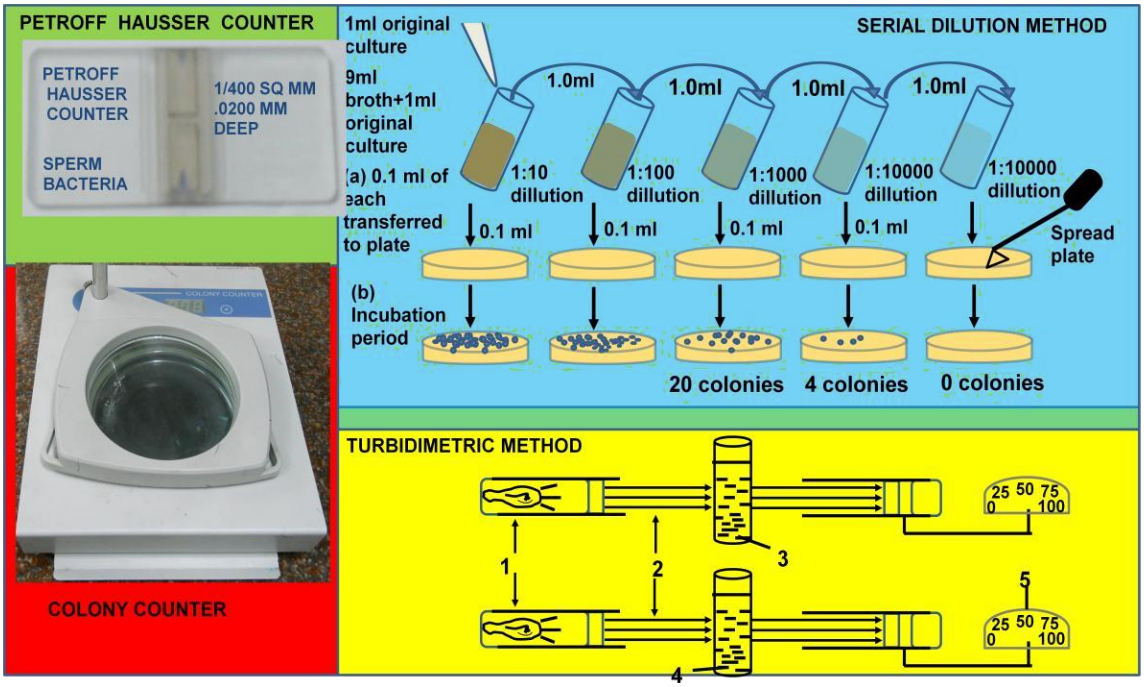

Figure 9.2: Different methods of bacterial counting.

Turbidimetric methods-This method is based on light scattering principles of particulate matter such as bacteria. A bacteria cell suspension is placed in test cuvette and corresponding media in reference cuvette. The optical density or absorbance of the bacterial suspension is used to measure the number of bacteria number. This method can not distinguish between live or dead bacteria as both form contribute to the turbidity.

Nitrogen content and Dry weight- A bacterial cell mass can be measured by direct measuremenof dry weight of culture or nitrogen content.

Growth cycle of bacteria- As discussed earlier, the most common method of bacteria division is binary fission and by this method, one bacteria cell gives two daughter cells. The time a bacteria takes to complete one division is called as generation time and it depends on bacteria species and media properties.

Hence, if we start from one bacteria, it divides after every generation time as follows-

Generation (n)	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**n**
No. of bacteria	**1**	**2**	**4**	**8**	**16**	**32**	**64**	
No. of bacteria	**1**	**2^1**	**2^2**	**2^3**	**2^4**	**2^5**	**2^6**	**2^n**

Hence, After n generations, no of bacteria will be

$N=1$ x 2^n ....Eq 9.1

But assume if number of bacteria at time 0 is N_o, then

$N=No$ X 2^n... Eq 9.2

$\text{Log } N=\text{Log } N_o+n \log_{10} 2$..Eq 9.3

$n= 3.3 (\text{Log}_{10} N-\text{Log}_{10} N_o)$ .... Eq 9.4

Eq 9.2 can be used to determine number of bacteria, if initial number of bacteria and number of generation is known where as Eq 9.4 can directly been used to calculate number of generations.

Bacterial growth in a liquid media is given in Figure 9.3 and it has 4 distinct phases:

1. Lag Phase-The single cell inoculation into the liquid media doesn't start dividing as per its generation time. During this phase bacteria gets adjusted to the new media and grow in size instead of dividing into daughter cells. In this phase, bacteria synthesize the most crucial enzymes or co-enzyme present in traces and required for optimal growth and multiplication. In addition, cell is metabolically active and be busy in synthesizing large amount of protoplasm. At the end of this phase, each bacterial cell divides and enter into the next phase of active multiplication.

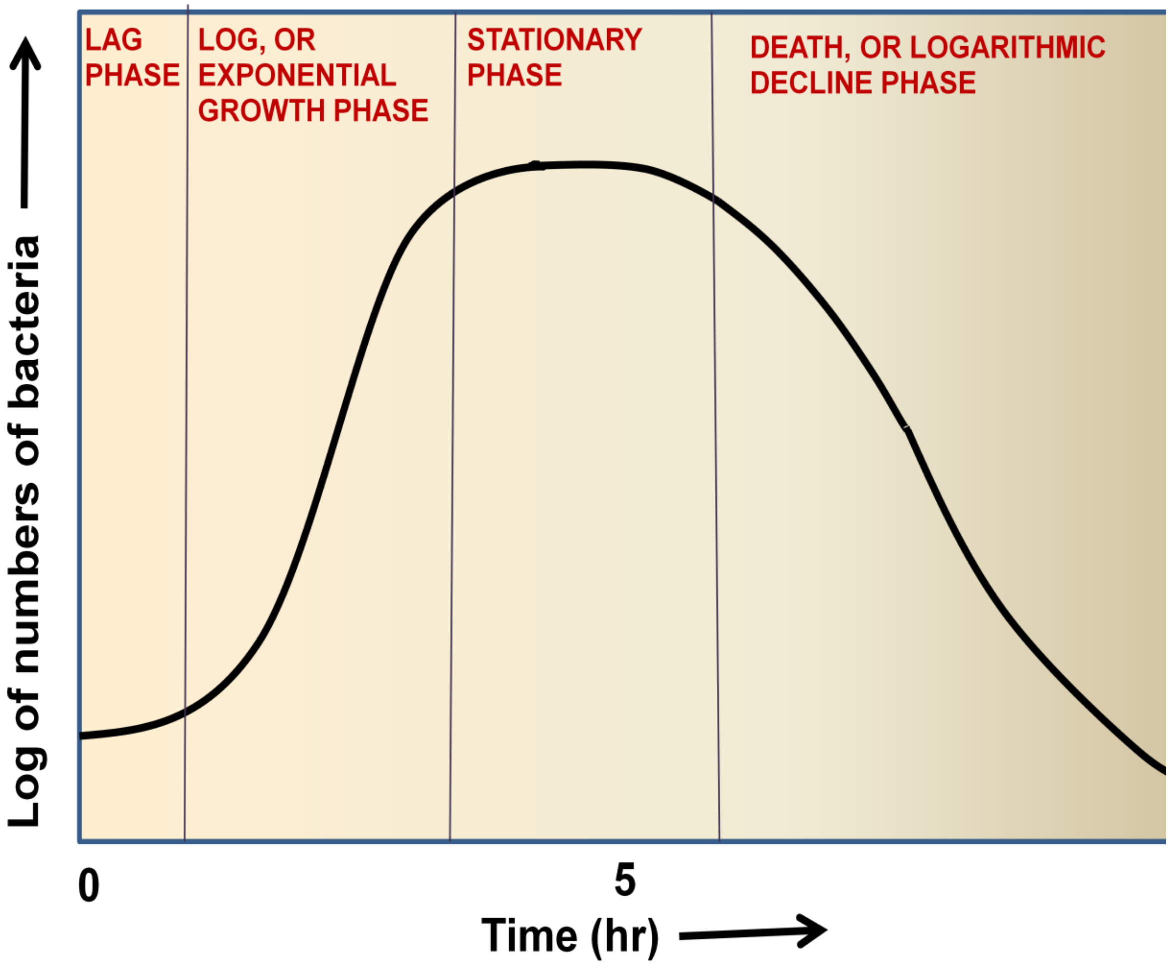

Figure 9.3: kinetics of Growth of bacteria.

2. Log Phase-In this phase, bacterial cell population is involved in active division and whole cell population is more or less homogenous in terms of chemical composition, physiology and metabolic activity. A plot of number of cell (in log scale) against time gives straight line. The growth of bacterial cell population is increasing at a constant rate and continues until substrate concentration is not limiting.

3. Stationary Phase-Once substrate is limiting, the logarithmic phase of growth begins to decline gradually with a constant number of cells to give a staright line. The population remains constant because number of divisions are equal to the number of death events. As substrate is limiting, death of old cell provides enough nutrient for remaining cells to grow and multiply to maintain the constant number.

4. Death Phase-When substrate is not sufficient from dying cells, death rate of bacteria superseed rate of growth and as a result number of bacteria declines sharply.

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UNIT – V – Modelling and transport of bioprocesses – SCH1312

Centrifuge – Introduction

A centrifuge is a device for separating particles from a solution according to their size, shape, density, viscosity of the medium and rotor speed. In biology, the particles are usually cells, sub cellular organelles, viruses, large molecules such as proteins and nucleic acids.

Ultracentrifuge

Ultracentrifuges are available with a wide variety of rotors suitable for a great range of experiments. Most rotors are designed to hold tubes that contain the samples. Swinging bucket rotors allow the tubes to hang on hinges so the tubes reorient to the horizontal as the rotor initially accelerate. Fixed angle rotors are made of a single block of material and hold the tubes in cavities bored at a predetermined angle. Zonal rotors are designed to contain a large volume of sample in a single central cavity rather than in tubes. Some zonal rotors are capable of dynamic loading and unloading of samples while the rotor is spinning at high speed.

Preparative rotors are used in biology for pelleting of fine particulate fractions, such as cellular organelles (mitochondria, microsomes, ribosomes) and viruses. They can also be used for gradient separations, in which the tubes are filled from top to bottom with an increasing concentration of a dense substance in solution. Sucrose gradients are typically used for separation of cellular organelles. Gradients of caesium salts are used for separation of nucleic acids. After the sample has spun at high speed for sufficient time to produce the separation, the rotor is allowed to come to a smooth stop and the gradient is gently pumped out of each tube to isolate the separated components.

Hazards

The tremendous rotational kinetic energy of the rotor in an operating ultracentrifuge makes the catastrophic failure of a spinning rotor a serious concern. and it can explode. Rotors conventionally have been made from high strength-to-weight metals such as aluminum or titanium. The stresses of routine use and harsh chemical solutions eventually cause rotors to deteriorate. Proper use of the instrument and rotors within recommended limits and careful maintenance of rotors to prevent corrosion and to detect deterioration is necessary to mitigate this risk.

More recently some rotors have been made of lightweight carbon fiber composite material, which are up to 60% lighter, resulting in faster acceleration/deceleration rates. Carbon fiber composite rotors also are corrosion-resistant, eliminating a major cause of rotor failure.

Ultracentrifugation is a specialized technique used to spin samples at exceptionally high speeds. Current ultracentrifuges can spin to as much as 150 000 rotations per minute (rpm) (equivalent to 1 000 000 g) (Biocompare, 2019b). However, extreme centrifugal forces may cause overheating, so to avoid sample damage, ultracentrifuges are equipped with vacuum systems that keep a constant temperature in the centrifuge's rotor (Biocompare, 2019b).

Centrifugation, and ultracentrifugation, is nowadays, at the core of the laboratory routine. Benchtop centrifuges are essential devices in any biology or chemistry laboratory, and they are used on a day-to-day basis in a wide range of experimental protocols, from concentrating solutions to isolating cells and subcellular components. Ultracentrifugation widened the applications of benchtop centrifugation, allowing the isolation of smaller sized particles, and the study of purified molecules and molecular complexes (Ohlendieck & Harding, 2017). In biology, the development of ultracentrifugation in the early 1900s, widened the possibilities of scientific research to the subcellular level, allowing for the differential separation of cellular components, such as organelles, lipid membranes, and even to purify proteins and ribonucleic acids (DNA and RNA).

The Principle of Ultracentrifugation

The basis of ultracentrifugation is the same as normal centrifugation: to separate the components of a solution based on their size and density, and the density (viscosity) of the medium (solvent) As a general principle, (ultra)centrifugation abides by the following rules

the denser a biological structure is, the faster it sediments in a centrifugal field

the more massive a biological particle is, the faster it moves in a centrifugal field

the denser the biological buffer system is, the slower the particle moves in a centrifugal field

the greater the frictional coefficient (i.e., the friction between the component and the neighbouring environment) is, the slower a particle moves

the greater the centrifugal force is, the faster the particle sediments

the sedimentation rate of a given particle will be zero when the density of the particle and the surrounding medium is equal.

Centrifugation Versus Ultracentrifugation

The use of ultracentrifugation over centrifugation (and vice-versa) denotes basic differences between the two techniques. Fundamental differences between centrifugation and ultracentrifugation include:

Spinning velocity, and therefore, the centrifugal force applied to the samples. The rotor of an ultracentrifuge can spin as high as 1 000 000 x g, while most common benchtop centrifuges are limited to 65 000 x g. This brings up the second basic difference:

Refrigeration and vacuum systems, which are mandatory in ultracentrifuges. Because of the extremely high spinning speed, ultracentrifuges are always equipped with vacuum and refrigeration systems, to avoid sample and/or device damage due to frictional force and overheating. In benchtop centrifuges, these two systems are optional, with the most simple centrifuges, like mini micro-centrifuges, displaying none.

Type of pellet that is produced from sample fractionation: because ultracentrifuges can achieve much higher spinning speeds, the type of sediment (pellet) that results from one or the other is also different, with ultracentrifugation allowing for the isolation of smaller particles than benchtop centrifugation. In biology labs, subcellular fractionation to separate cytosolic contents (such as whole cytosol, mitochondria or chloroplasts) from cell nuclei can be attained by benchtop centrifuges. However, to isolate smaller components such as ribosomes and small vesicles, higher centrifugal forces, only possible with ultracentrifuges, are necessary.

Types of Ultracentrifugation: Analytical Versus Preparative

There is, currently on the market, a wide variety of ultracentrifuges. The choice among different brands and models must consider the type of experimental applications to be performed, the availability of different rotors (making it possible to adapt the ultracentrifuge to different experimental settings) and the temperature range. Some ultracentrifuges further offer remote monitoring and control, and password-protection. It is important to ask: For which experimental setting do I need an ultracentrifuge? In that sense, two types of ultracentrifuges are available: analytical and preparative. Analytical ultracentrifugation is used in the study of purified macromolecules or supramolecular assemblies, while preparative ultracentrifugation is used in the actual separation of tissues, cells, subcellular components and other biochemically interesting particles.

Analytical ultracentrifuges are equipped with optical detection systems that allow the researcher to follow the centrifugation process in real-time. These systems may use ultraviolet (UV) light absorption or refracting index interference (RII) optical detection systems (ultracentrifuges may be equipped with one or both types of optical systems). While UV detection directly measures the absorbance (abs) of a substance at a specific wavelength, RII measures changes in the refraction index (radiation direction) of a given substance, compared to the solvent it is dissolved in. The purpose of analytical centrifugation is different from other types of centrifugation. Although component isolation is possible with analytical centrifugation, the goal of this technique is to obtain data to characterize the sample that is spun (sedimentation velocity, viscosity, concentration, etc.). With analytical centrifugation, it is possible to follow the variations in sample concentration as a function of the applied centrifugal force. This technique is used in two main experimental settings: sedimentation velocity and sedimentation equilibrium studies, which are key in macromolecular characterization. Results from sedimentation velocity experiments provide data that are used to calculate the molecular size (molecular weight), shape, and molar masses of new chemical molecules, ribonucleic acids, proteins, and others.

Preparative ultracentrifuges are mostly used to process biological samples for further analysis. The most common application of preparative ultracentrifugation is in tissue and subcellular fractionation, to isolate increasingly smaller components of the biological. For that, two main centrifugation methods are used: differential and density-gradient centrifugation.

Differential centrifugation is used to separate the components of a solution based on differences in the sedimentation rate of the different components of the mixture. As explained above (see section 2: The Principle of Ultracentrifugation), the sedimentation properties of a substance depend on its size and density but also on the density of the solvent. In medical and biology labs, crude tissue homogenates containing organelles, membrane vesicles, and other structural fragments are divided into different fractions by the stepwise increase of the applied centrifugal field. Furthermore, differential centrifugation is also routinely used in the isolation of non-living substances, like nanoparticles, colloids, and viruses.

Density gradient centrifugation goes further in particle separation than differential centrifugation. It is ideal when the goal is to isolate particles of similar sizes, but different densities. In this case, it is possible to establish density gradient solutions with increasing concentrations of specific materials, in the spinning tube. Cesium salt gradients are used in the separation of DNA, and sucrose gradients are used in subcellular fractionation to isolate organelles and multiprotein complexes, like ribosomes. Today, there are several commercial gradient solution kits to isolate specific particles. Roughly, there are two types of density gradient centrifugation: rote-zonal centrifugation and isopycnic centrifugation (also called equilibrium centrifugation), which differ in the way particles are separated across the gradient.

Rote-zonal centrifugation – particle separation depends mostly on particle mass. Zones, or bands, are generated, each containing a particle fraction of a specific mass. However, care must be taken when performing rote-zonal centrifugation. Because the mass of the particles is higher than the density of the solvent, if they are centrifuged for too long, all particles will eventually deposit in the bottom of the tube.

Isopycnic (equilibrium) centrifugation – particle separation depends solely on their density. In isopycnic separation, particles are mixed with the gradient solution, and during centrifugation, they will move until they reach the gradient phase which equals their density (isopycnic or equilibrium point). Because the density of the gradient medium is always higher than the density of particles, these will never sediment, independently of the centrifugation time. Continuous gradients may be used in isopycnic centrifugation, however, discontinuous gradients in which particles form bands at the interface between the density gradient layers are more suitable for the separation of some biological samples, like the separation of lymphocytes from whole blood.

Applications of Analytical and Preparative Ultracentrifugation

Due to their intrinsic differences, analytical and preparative ultracentrifugation are used for different purposes:

Analytical ultracentrifugation

determination of the purity (including the presence of aggregates) and oligomeric state of macromolecules, by recording sedimentation velocity data

determination of the average molecular mass of solutes in their native state
Study of changes in the molecular mass of supramolecular complexes,
using either sedimentation velocity, sedimentation equilibrium (or both)
the detection of conformation and conformational changes
Preparative ultracentrifugation

subcellular fractionation

affinity purification of membrane vesicles

separation of DNA components

colloid separation

virus purification

The Ultracentrifuge: How to Use and How to Care

Modern ultracentrifuges are heavy, sturdy equipment that requires certain know-how for proper usage and care.

Rotor balance. As in all centrifuges, sample spinning requires a proper balance of the weight inside the rotor. Given the extremely high spinning speed inside the ultracentrifuge's rotor, the impact of subtle imbalances may be shockingly strong. Modern ultracentrifuges have some buffer capacity, to absorb slight weight imbalances, and when there is too much imbalance, an automatic system shuts off the device. Moreover, in all ultracentrifuges, the rotor is encapsulated in a strong heavy metallic cage, to avoid vibrations and projections that could damage the sample and endanger the operator. Yet, it is of vital importance that the ultracentrifuge is properly loaded, according to the manufacturer's instructions.

Sample position in rotor. All rotor positions must be filled. Even when there are only a few tubes, the rest of the positions must be occupied with blank samples of equivalent weight. To avoid both rotor and sample damage, it is important to set the ultracentrifuge to slow acceleration and deceleration modes. This is especially important in density gradients, as the sudden stop of the spinning may affect the separation of the gradient layers (Ohlendieck & Harding, 2017). Ultracentrifuges are expensive devices, which are required to accurately separate particles in solution. To ensure the proper function of the ultracentrifuge, care measures must be undertaken regularly. Apart from safety, proper loading of the rotor avoids excessive vibration, which can cause damage to the device.

Centrifuge cleaning. Maintenance and cleaning of the rotor must be done with non-abrasive detergents to avoid corrosion. Rotor cleaning is especially important to ensure that there are no remnants of the samples that were centrifuged, and therefore, should always be performed after spinning.

Storage. Whenever the device is not used, or simply for overnight storage, rotors must be kept in a dry room, properly cleaned, and left to dry in an inverted position, to avoid the accumulation of water in the sample cells.

Regular maintenance. This should be done by certified operators to ensure the proper long-term function of the ultracentrifuge.

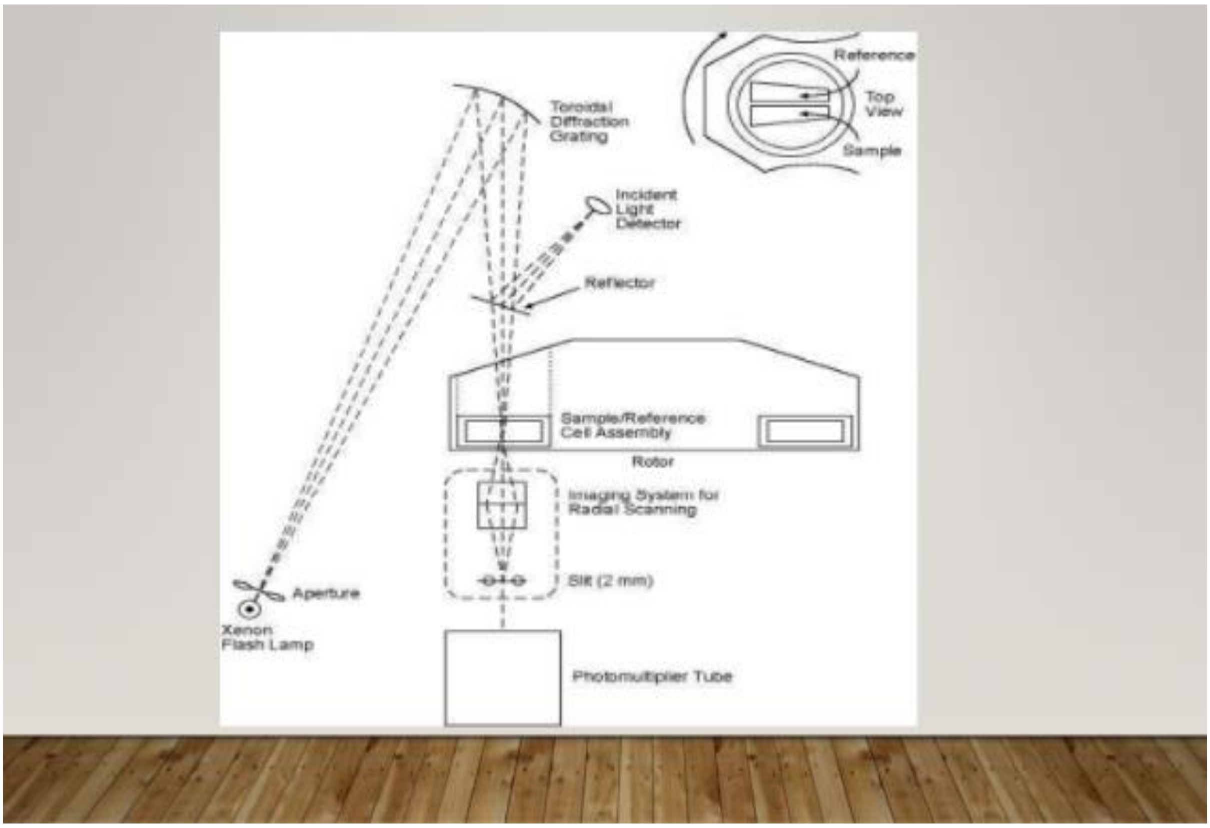
Advantages and Limitations of Ultracentrifugation

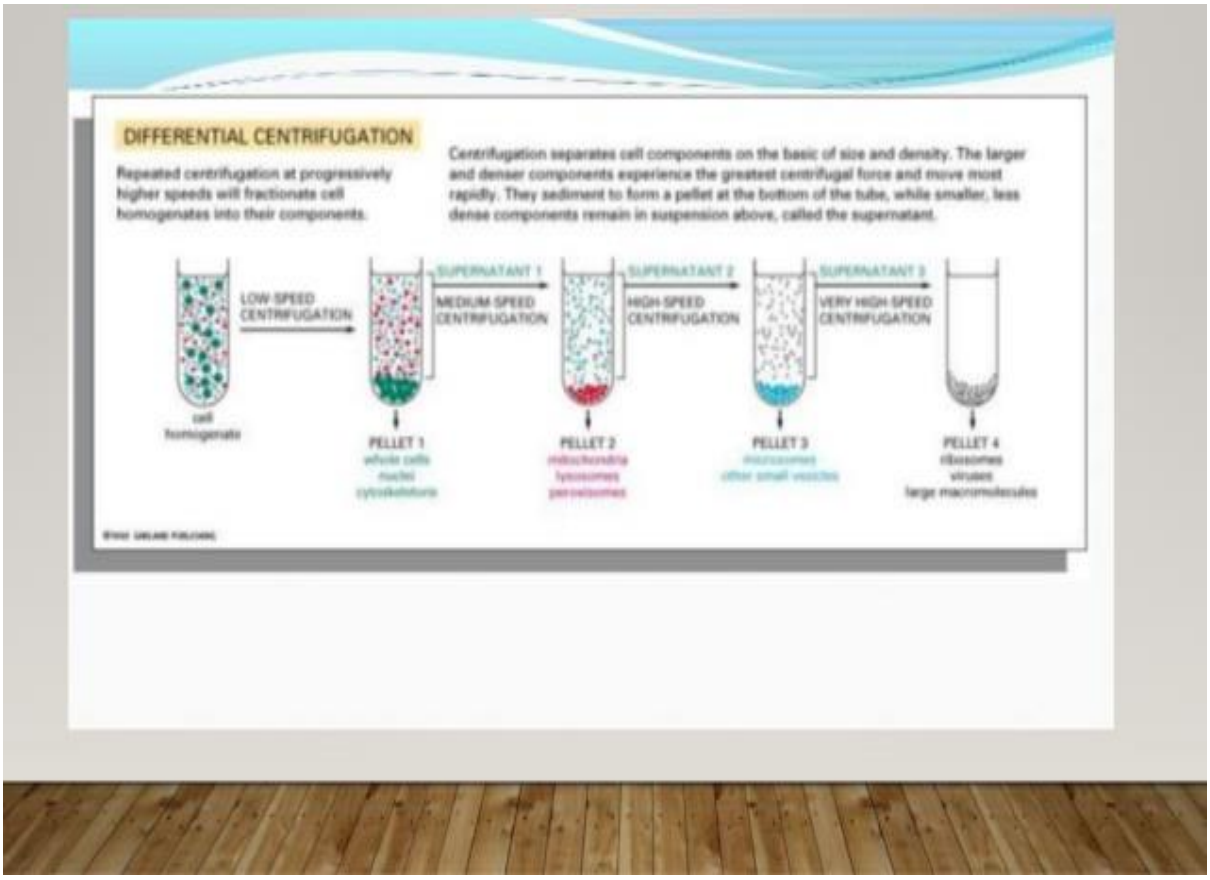
From the development of the first ultracentrifuge in the 1920s by Svedberg, up to today, the scientific advances that resulted from the application of ultracentrifugation to biology, chemistry, material science, and others, are countless. In its most obvious approaches, ultracentrifugation extended the limits of biology research to the subcellular level, by allowing the isolation of particles as small as ribosomes, subcellular organelles, membranes, and ribonucleic acids. With the advent of analytical ultracentrifugation, research took another step further towards the understanding of the submicroscopic world, with the ability to further characterize molecular size, shape, and structure. However, ultracentrifugation has its own limitations, like any other laboratory technique. These include:

Low sample yield – In preparative ultracentrifugation, samples must be washed several times after spinning, to ensure that there is no cross-contamination between fractions. Samples for preparative centrifugation are usually limited in size (e.g., tissues) or volume (e.g., cell suspensions or blood). In every wash step that a sample is subjected, there is loss of material, and thus, after an ultracentrifugation protocol, the yield can be very low.

Ultracentrifugation is still a time-consuming process, and it can take up to several hours to fractionate all the components of a single mixture.

Ultracentrifuges are extremely expensive devices, which require constant maintenance.





DIFFERENTIAL CENTRIFUGATION

Repeated centrifugation at progressively higher speeds will fractionate cell homogenates into their components.

Centrifugation separates cell components on the basic of size and density. The larger and denser components experience the greatest centrifugal force and move most rapidly. They sediment to form a pellet at the bottom of the tube, while smaller, less dense components remain in suspension above, called the supernatant.

Electro-osmosis

Electroosmotic flow (or electro-osmotic flow, often abbreviated EOF; synonymous with electroosmosis or electroendosmosis) is the motion of liquid induced by an applied potential across a porous material, capillary tube, membrane, microchannel, or any other fluid conduit. Because electroosmotic velocities are independent of conduit size, as long as the electrical double layer is much smaller than the characteristic length scale of the channel, electroosmotic flow will have little effect. Electroosmotic flow is most significant when in small channels. Electroosmotic flow is an essential component in chemical separation techniques, notably capillary electrophoresis. Electroosmotic flow can occur in natural unfiltered water, as well as buffered solutions.

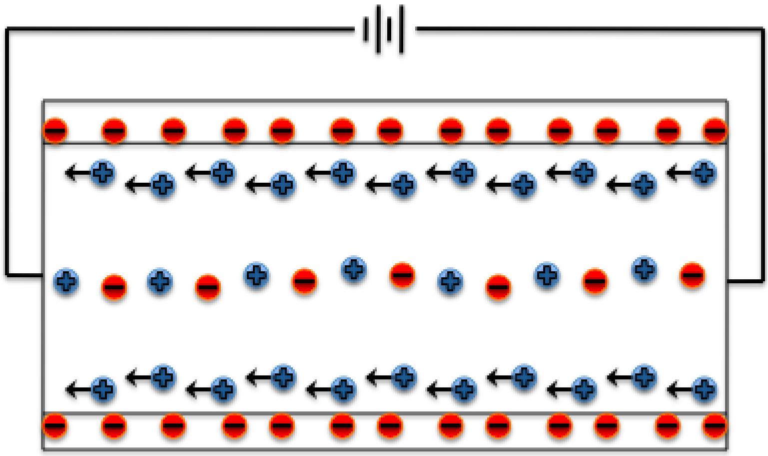
Electroosmotic flow is caused by the Coulomb force induced by an electric field on net mobile electric charge in a solution. Because the chemical equilibrium between a solid surface and an electrolyte solution typically leads to the interface acquiring a net fixed electrical charge, a layer of mobile ions, known as an electrical double layer or Debye layer, forms in the region near the interface. When an electric field is applied to the fluid (usually via electrodes placed at inlets and outlets), the net charge in the electrical double layer is induced to move by the resulting Coulomb force. The resulting flow is termed electroosmotic flow.

The resulting flow from applying a voltage is a plug flow. Unlike a parabolic profile flow generated from a pressure differential, a plug flow’s velocity profile is approximately planar, with slight variation near the electric double layer. This offers significantly less deleterious dispersive effects and can be controlled without valves, offering a high-performance method for fluid separation, although many complex factors prove this control to be difficult. Because of difficulties measuring and monitoring flow in microfluidic channels, primarily disrupting the flow pattern, most analysis is done through numerical methods and simulation.

Electroosmotic flow through microchannels can be modeled after the Navier-Stokes equation with the driving force deriving from the electric field and the pressure differential. Thus it is governed by the continuity equation

Electro-osmotic flow is commonly used in microfluidic devices,[8][9] soil analysis and processing,[10] and chemical analysis,[11] all of which routinely involve systems with highly charged surfaces, often of oxides. One example is capillary electrophoresis,[9][11] in which electric fields are used to separate chemicals according to their electrophoretic mobility by applying an electric field to a narrow capillary, usually made of silica. In electrophoretic separations, the electroosmotic flow affects the elution time of the analytes.

Electro-osmotic flow is actuated in a FlowFET to electronically control fluid flow through a junction. It is projected that micro fluidic devices utilizing electroosmotic flow will have applications in medical research. Once controlling this flow is better understood and implemented, the ability to separate fluids on the atomic level will be a vital component for drug dischargers. Mixing fluids at the micro scale is currently troublesome. It is believed that electrically controlling fluids will be the method in which small fluids are mixed. A controversial use of electro-osmotic systems is the control rising damp in the walls of buildings While there is little evidence to suggest that these systems can be useful in moving salts in walls, such systems are claimed to be especially effective in structures with very thick walls. However some claim that there is no scientific base for those systems, and cite several examples for their failure.



Electro-osmosis is the flow of liquid that is in contact with a charged solid surface when an electric field is applied, and it becomes an important consideration with the increased surface area-to-volume ratio associated with small diameter capillaries.

Electro-osmosis refers to the movement of liquid in a porous material due to an applied electric field. Electro-osmosis is a very effective instrument when treating heterogeneous, silt and clay-rich soil. The phenomenon of electro-osmosis is very useful in chemical separation techniques and buffered solutions.

Electro-osmosis can be used for organics removal. It reduces the need for specialized electrodes. Electro-osmosis has achieved two benefits when properly applied:

Provides uniform pore water movement in most types of soil. The size of the pore is not important. Electro-osmotic flow rate is primarily a function of applied voltage.

The electricity applied to the soil directly results in the heating of the soil. The soil warming not only increases the mobilization of volatile organics, but also increases the electro-osmotic permeability by lowering the viscosity of the pore water.

Electro-osmosis techniques are commonly used in chemical analysis, soil analysis and processing, and microfluidic devices. All of these uses involve highly charged surfaces, often of oxides. In capillary electrophoresis, which is another use of electro-osmosis, chemicals are separated according to their electrophoretic mobility by applying an electric field