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SCHOOL OF BIO AND CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – I – TRANSPORT PHENOMENA – SCH1309

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1.1 Introduction to Momentum Transport

Momentum transport deals with the transport of momentum which is responsible for flow in fluids. Momentum transport describes the science of fluid flow also called fluid dynamics. A few basic assumptions are involved in fluid flow and these are discussed below.

No slip boundary condition

This is the first basic assumption used in momentum transport. It deals with the fluid flowing over a solid surface, and states that whenever a fluid comes in contact with any solid boundary, the adjacent layer of the fluid in contact with the solid surface has the same velocity as the solid surface. Hence, we assumed that there is no slip between the solid surface and the fluid or the relative velocity is zero at the fluid–solid interface. For example, consider a fluid flowing inside a stationary tube of radius *R* as shown in Fig 7.1. Since the wall of the tube at *r*=*R* is stationary, according to the no-slip condition implies that the fluid velocity at *r*=*R* is also zero.

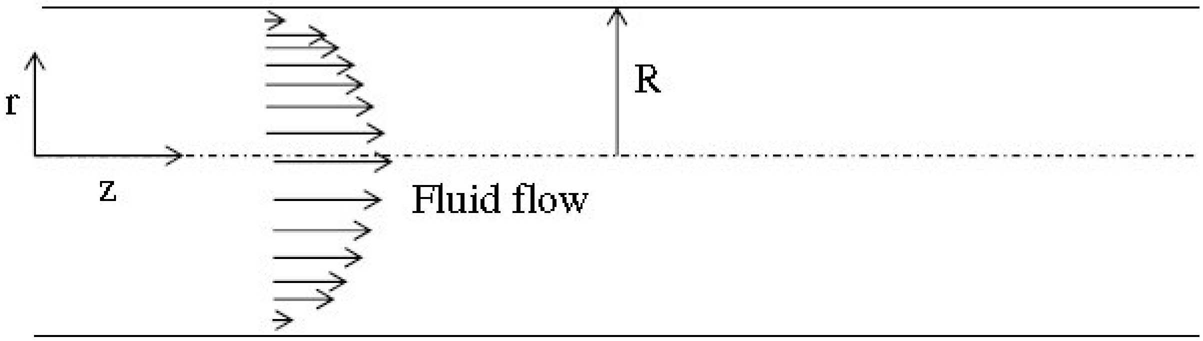

Fig 1 Fluid flow in a circular tube of radius R

In the second example as shown in Fig. 7.2, there are two plates which are separated by a distance *h*, and some fluid is present between these plates. If the lower plate is forced to move with a velocity *V* in x direction and the upper plate is held stationary, no-slip boundary conditions may be written as follows

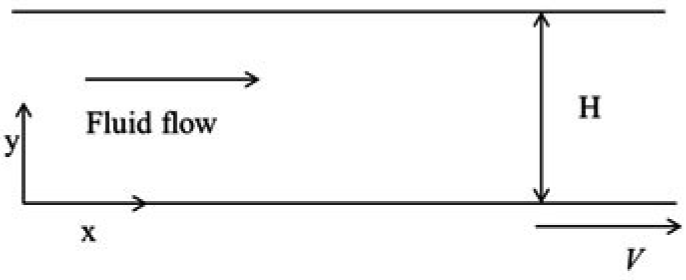

Fig 2 Two parallel plates at stationary condition

$$y = 0,\ v_x = v$$

$$y = H,\ v_x = 0$$

Thus, every layer of fluid is moving at a different velocity. This leads to shear forces which are described in the next section.

1.2 Newton's Law of Viscosity

Newton's law of viscosity may be used for solving problem for Newtonian fluids. For many fluids in chemical engineering the assumption of Newtonian fluid is reasonably acceptable. To understand Newtonian fluid, let us consider a hypothetical experiment, in which there are two infinitely large plates situated parallel to each other, separated by a distance h. A fluid is present between these two plates and the contact area between the fluid and the plates is *A*.
A constant force *F1* is now applied on the lower plate while the upper plate is held stationary. After steady state has reached, the velocity achieved by the lower plate is measured as V_1. The force is then changed, and the new velocity of the plate associated with this force is measured.

If the *F/A* is plotted against *V/h*, we may observe that they lie on a straight line passing through the origin.

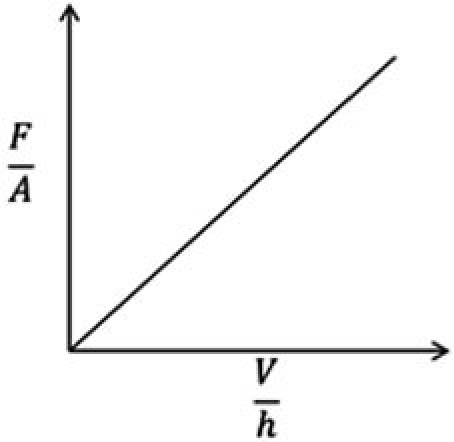

Fig 3 Shear stress vs. shear stain

Thus, it may be said that *F/A* is proportional to *v/h* for a Newtonian fluid.

$$\frac{F}{A} \propto \frac{v}{h}$$

It may be noted that it is the velocity gradient which leads to the development of shear forces. The above equation may be re-written as

$$\Rightarrow \frac{F_1}{A} \propto \frac{v_1 - 0}{h - 0} = \frac{\Delta v_x}{\Delta y}$$

In the limiting case, as *h* → *0*, we have

$$\frac{F}{A} \propto \frac{dv_x}{dy}$$

$$\frac{F}{A} = \pm \mu \frac{dv_x}{dy}$$

where, μ is a constant of proportionality, and is called as the viscosity of the fluid. The quantity *F/A* represents the shear forces/stress. It may be represented as τ_{yx}., where the subscript *x* indicates the direction of force and subscript *y* indicates the direction of outward normal of the surface on which this force is acting. The quantity $\frac{dv_x}{dy}$ or the velocity gradient is also called the shear rate. μ is a property of the fluid and is measured the resistance offered by the fluid to flow. Viscosity may be constant for many Newtonian fluids and may change only with temperature.

Thus, the Newton’s law of viscosity, in its most basic form is given as

$$\tau_{yx} = \pm \mu \frac{dv_x}{dy}$$

Here, both ‘+’ or ‘–’ sign are valid. The positive sign is used in many fluid mechanics books whereas the negative sign may be found in transport phenomena books. If the positive sign is used then τ_{yx}. may be called the shear force while if the negative sign is used τ_{yx}. may be referred to as the momentum flux which flows from a higher value to a lower value.

$$\tau_{yx} = +\mu \frac{dv_x}{dy} \quad \rightarrow \text{Shear force}$$

$$\tau_{yx} = -\mu \frac{dv_x}{dy} \quad \rightarrow \text{Momentum flux}$$

The reason for having a negative sign for momentum flux in the transport phenomena is to have similarities with Fourier's law of heat conduction in heat transport and Ficks law of diffusion in mass transport. For example, in heat transport, heat flows from higher temperature to lower temperature indicating that heat flux is positive when the temperature gradient is negative. Thus, a minus sign is required in the Fourier's law of heat conduction. The interpretation of τ_{yx}. as the momentum flux is that *x* directed momentum flows from higher value to lower value in *y* direction.

The dimensions of viscosity are as follows:

$$\mu = \frac{Force/Area}{\left[\dfrac{dv_x}{dy}\right]} = \frac{MLT^{-1}L^{-2}}{LT^{-1}L^{-1}} = ML^{-1}T^{-1}$$

The SI unit of viscosity is *kg/m.s* or *Pa.s*. In CGS unit is *g/cm.s* and is commonly known as poise (*P*). where *1 P = 0.1 kg/m.s*. The unit poise is also used with the prefix *centi-*, which refers to one-hundredth of a poise, i.e. *1 cP = 0.01 P*. The viscosity of air at 25^oC is *0.018 cP,* water at 25^oC is *1 cP* and for many polymer melts it may range from *1000* to *100,000 cP*, thus showing a long range of viscosity.

1.3 Laminar and turbulent flow

Fluid flow can broadly be categorized into two kinds: laminar and turbulent. In laminar flow, the fluid layers do not inter-mix, and flow separately. This is the flow encountered when a tap is just opened and water is allowed to flow very slowly. As the flow increases, it becomes much more irregular and the different fluid layers start mixing with each other leading to turbulent flow. Osborne Reynolds tried to distinguish between the two kinds of flow using an ingenious experiment and known as the Reynolds’s experiment. The basic idea behind this experiment is described below.

1.4 Reynolds’s experiment

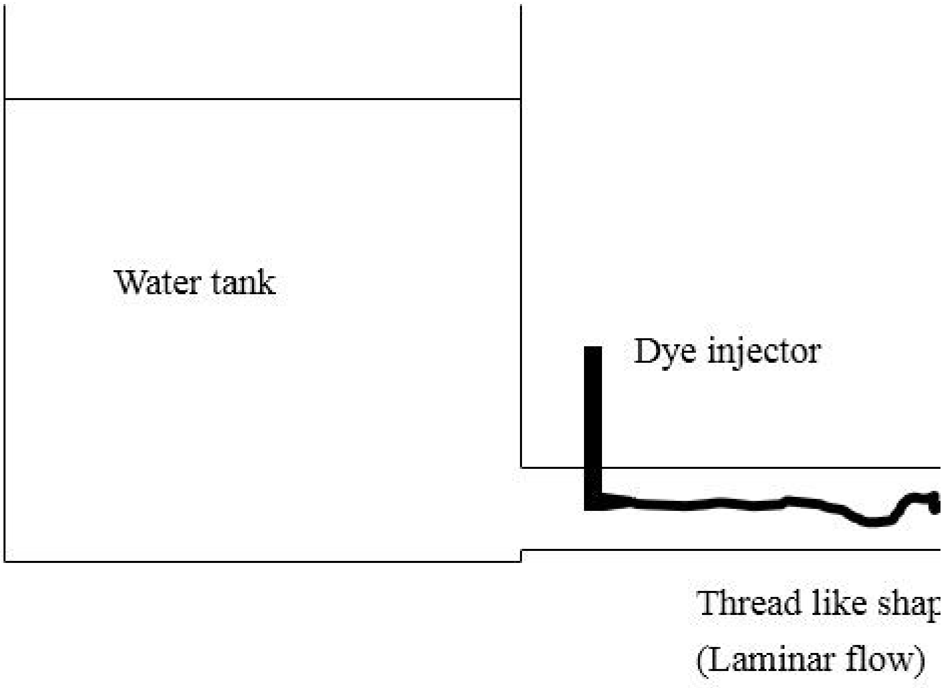

Fig 5 Reynolds's experiments

The experiment setup used for performing the Reynolds's experiment is shown in Fig. 5. The average velocity of fluid flow through the pipe diameter can be varied. Also, there is an arrangement to inject a colored dye at the center of the pipe. The profile of the dye is observed along the length of the pipe for different velocities for different fluids. If this experiment is performed, it may be seen that for certain cases the dye shows a regular thread type profile, which is seen at low fluid velocity and flow is called laminar flow. when the fluid velocity is increased the dye starts to mixed with the fluid and for larger velocities simply disappears. At this point fluid flow becomes turbulent.

For the variables average velocity of fluid $v_{z\ avg}$, pipe diameter D, fluid density ρ, and the fluid viscosity μ, Reynolds found a dimensionless group which could be used to characterize the type of fluid flow in the tube. This dimensionless quantity is known as the Reynolds number. From the experiment, It was observed that if $Re > 2100$, the dye simply disappeared and the flow has changed to laminar to turbulent flow.

$$\mathrm{Re} = \frac{\rho v_{z,avg} D}{\mu}$$

Thus, for Re <2100, we have laminar flow, i.e., no mixing in the radial direction leading to a thread like flow and for *Re >2100,* we have the turbulent flow, i.e., mixing in the radial direction between layers of fluid.

In laminar flow, the fluid flows as a stream line flow with no mixing between layers. In turbulent flow, the fluid is unstable and mixes rapidly due to fluctuations and disturbances in the flow. The disturbance might be present due to pumps, friction of the solid surface or any type of noise present in the system. This makes solving fluid flow problem much more difficult. To understand the difference in the velocity profile in two kinds of fluid flows, we consider a fluid flowing to a horizontal tube in *z* direction under steady state condition. Then, we can intuitively see the velocity profile may be shown below

For laminar flow, it is observed that fluid flows as smooth stream line and all other components of velocity are zero. Thus

$$v_z = v_z\left(r\right)$$

$$v_r = 0$$

$$v_\theta = 0$$

For turbulent flow, if we observe the fluid flows at a local point. It is observed that fluid flows in very random manner in all directions where these local velocities may be the function of any dimensions.

$$v_z = v_z(r, z, \theta, t)$$

$$v_r = v_r(r, z, \theta, t)$$

$$v_\theta = v_\theta(r, z, \theta, t)$$

Thus, we see that for laminar flow there is only one component of velocity present and it depends only on one coordinate whereas the solution of turbulent flow may be vary complex. For turbulent flow, one can ask the question that if the fluid is flowing in the *z* direction then why are the velocity components in *r* and *θ* direction non-zero? The mathematical answer for this question can be deciphered from the equation of motion. The equation of motion is a non-linear partial differential equation. This non-linear nature of the equation causes instability in the system which produces flow in other directions. The instability in the system may occur due to any disturbances or noise present in the environment. On the other hand, if the velocity of fluid is very low the deviation due to disturbances may decay with time, and becomes negligible after that. Thus the flow remains in laminar region. Consider a practical example in which some cars are moving on the highway in the same direction but in the different lanes at different speeds. If suddenly, some obstacle comes on the road, then if the car's speed is sufficiently low, it can move on to other lane smoothly and come back to its original lane after the obstacle is crossed. This is the regular laminar case. On the other hand, if the car is moving at a high speed and suddenly encounters an obstacle, then the driver may lose control, and this car may move haphazardly and hit other cars and after that traffic may never return to normal traffic conditions. This is the turbulent case.

1.4.1 Internal and external flows

Depending on how the fluid and the solid boundaries contact each other, the flow may be classified as internal flow or external flow. In internal flows, the fluid moves between solid boundaries. As is the case when fluid flows in a pipe or a duct. In external flows, however, the fluid is flowing over an external solid surface, the example may be sited is the flow of fluid over a sphere as shown in Fig. 8.1.

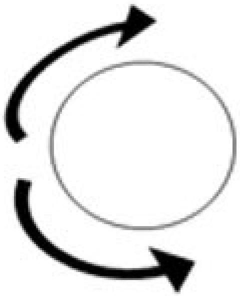

Fig 8.1 External flow around a sphere

Boundary layers and fully developed regions

Let us now consider the example of fluid flowing over a horizontal flat plate as shown in Fig. 8.2. The velocity of the fluid is V_∞ before it encounters the plate. As the fluid touches the plate, the velocity of the fluid layer just adjacent to the plate surface becomes zero due to the no slip boundary condition. This layer of fluid tries to drag the next fluid layer above it and reduces its velocity. As the fluid proceeds along the length of the plate (in x-direction), each layer starts to drag adjacent fluid layer but the effect of drag reduces as we go further away from the plate in y-direction. Finally, at some distance from the plate this drag effect disappears or becomes insignificant. This region where the velocity is changing or where the velocity gradients exists, is called the boundary layer region. The region beyond boundary layer where the velocity gradients are insignificant is called the potential flow region.

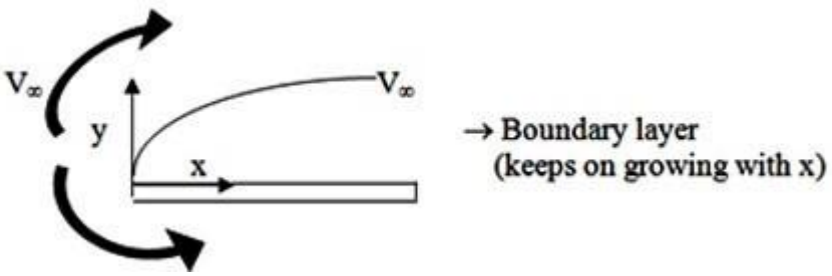

Fig 8.2 External flow over a flat plate

As depicted in Fig. 8.2, the boundary layer keeps growing along the x-direction, and may be referred to as the developing flow region. In internal flows (e.g. fluid flow through a pipe), the boundary layers finally merge after flow over a distance as shown in Fig. 8.3 below.

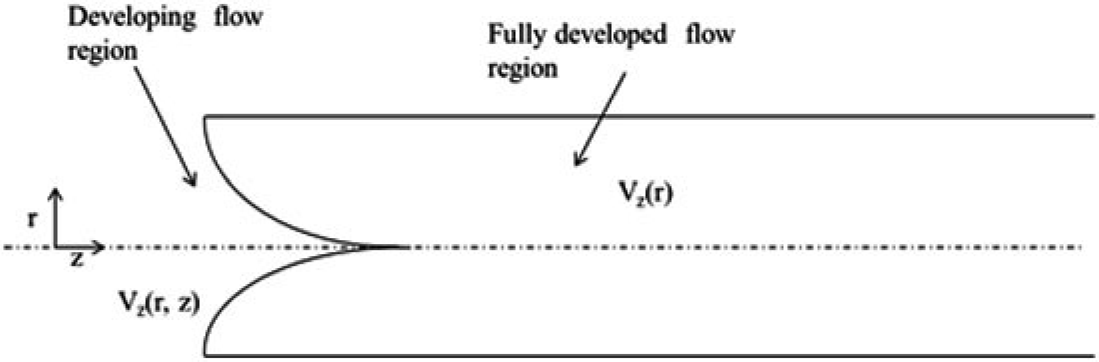

Fig 8.3 Developing flow and fully developed flow region

The region after the point at which the layers merge is called the fully developed flow region and before this it is called the developing flow region. In fact, fully developed flow is another important assumption which is taken for finding solution for varity of fluid flow problem. In the fully developed flow region (as shown in Figure 8.3), the velocity vz is a function of r direction only. However, the developing flow region, velocity vz is also changing in the z direction.

Main axioms of transport phenomena

The basic equations of transport phenomena are derived based on following five axioms.

- Mass is conserved, which leads to the equation of continuity.
- Momentum is conserved, which leads to the equation of motion.
- Moment of momentum is conserved leads to an important result that the 2nd order stress tensor $\underset{\sim}{\tau}$ is symmetric.

- Energy is conserved, which leads to equation of thermal energy.
- Mass of component i in a multi-component system is conserved, which leads to the convective diffusion equation.

The solution of equations, resulting from axiom 2, 4 and 5 leads to the solution of velocity, temperature and concentration profiles. Ones these profiles are known, all other important information needed can be determined. We first take the axiom -1. Other axioms will be taken up one by one letter on.
There are three types of control volumes (CV) which may be chosen for deriving the equations based these axioms.

- Rectangular shaped control volume fixed in space

In this case, the control volume is rectangular volume element and is fixed in space. This method is the easiest to understand but requires more number of steps.

- Irregular shaped control volume element fixed in space

In this case, the control volume can be of any shape, but it is again fixed in space. This method is somewhat more difficult than the previous method as it requires little better understanding vector analysis and surface and volume integrals.

- Material volume approach

In this case, the control volume can be of any shape but moves with the velocity of the flowing fluid. This method is most difficult in terms of mathematics, but requires least number of steps for deriving the equations.

All three approaches when applied to above axiom, lead to the same equations. In this web course, we follow the first approach. Other approaches may be found elsewhere.
Axioms-1

Mass is conserved

Consider a fluid of density ρ flowing with velocity $\underset{\sim}{v}$ as shown in Fig. 8.4. Here, ρ and $\underset{\sim}{v}$ are functions of space (x,y,z) and time (t). For conversion of mass, the rate of mass entering and leaving from the control volume (net rate of inflow) has to be evaluated and this should be equal to the rate of accumulation of mass in the control volume (CV). Thus, conservation of mass may be written in words as given below

$$\begin{pmatrix}\text{rate of accumulation of}\\ \text{mass in control volume}\end{pmatrix}=\begin{pmatrix}\text{rate of inflow of mass}\\ \text{into control volume}\end{pmatrix}-\left(\begin{matrix}\text{rate}\\ \text{out}\end{matrix}\right.$$

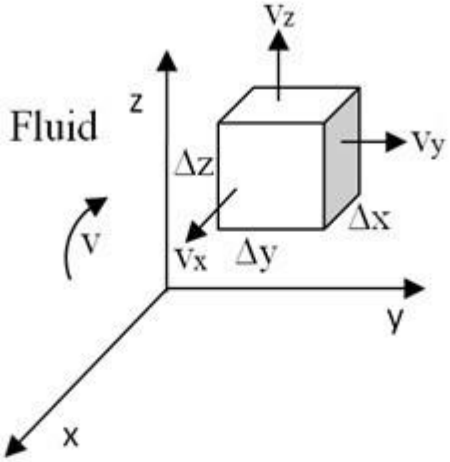

Fig 8.4 Fixed rectangular volume element through which fluid is flowing

The equation is then divided by the volume of the CV and converted into a partial differential equation by taking the limit as all dimensions go to zero. This limit effectively means that CV collapses to a point, thereby making the equation valid at every point in the system.

Let m and m+Δm be the mass of the control volume at time t and t+Δt respectively. Then, the rate of accumulation,

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

In order to evaluate the rate of inflow of mass into the control volume, we need to inspect how mass enters the control volume. Since the fluid velocity has three components vx, vy and vz, we need to identify the components which cause the inflow or the outflow at each of the six faces of the rectangular CV. For example, it is the component vx which forces the fluid to flow in the x direction, and thus it makes the fluid enter or exit through the faces having area ΔyΔz at x = x and x = x+Δx respectively. The component vy forces the fluid in y direction, and thus it makes the fluid enter or exit through the faces having area ΔxΔz at y = y and y = y+Δy respectively. Similarly, the component vz forces the fluid to flow in z direction, and thus it makes fluid enter or exit through the faces having area ΔxΔy at z = z and z = z+Δ z respectively.

The rate mass entering in x direction through the surface ΔyΔz is (ρvxΔyΔz|x), the rate of mass entering in y direction through the surface ΔxΔz is (ρvyΔxΔz|y) and the rate of mass entering from z direction through the surface ΔxΔy is (ρvzΔxΔy|z). In a similar manner, expressions for the rate of mass leaving from the control volume may be written.

Thus, the conservation of mass leads to the following expressions

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z)=\Big[\rho v_x\Delta y\Delta z\big|_x - \rho v_x\Delta y\Delta z\big|_{x+\Delta x}+\rho v_y\Delta x\Delta z\big|_y - \rho v_y\Delta x\Delta z\big|_{y+\Delta y}+\rho v_z$$

Dividing the Equation (8.1) by the volume ΔxΔyΔz, we obtain

$$\frac{\partial\rho}{\partial t}=\left[\frac{(\rho v_x)|_x-(\rho v_x)|_{x+\Delta x}}{\Delta x}\right]+\left[\frac{(\rho v_y)|_y-(\rho v_y)|_{y+\Delta y}}{\Delta y}\right]+\Big[\frac{(\rho v_z)|_z-(\rho}{\Delta z}$$

Note that each term in Equation (8.2) has the unit of mass per unit volume per unit time. Now, taking the limits Δx→0, Δy→0 and Δz→0, we get

$$\frac{\partial\rho}{\partial t}=\lim_{\Delta x\to 0}\left[\frac{(\rho v_x)|_x-(\rho v_x)|_{x+\Delta x}}{\Delta x}\right]+\lim_{\Delta y\to 0}\left[\frac{(\rho v_y)|_y-(\rho v_y)|_{y+\Delta y}}{\Delta y}\right]+\lim_{\Delta z\to 0}\Big[$$

and using the definition of derivative, we finally obtain

$$\frac{\partial\rho}{\partial t}=-\left[\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}\right]$$

Equation (8.4) is applicable to each point of the fluid. Rearranging the terms, we get the equation of continuity, may be written as given below.

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)=0$$

We need not to derive the equation of continuity again and again in other coordinate system (that is, spherical or cylindrical). The idea is to rewrite Equation (8.5) in vector and tensor form. Once it is written in this form, the same equation may be applied to other coordinate system as well. Thus, the Equation (8.5) may be rewritten in vector and tensor form as shown below.

$$\frac{\partial\rho}{\partial t}+\underset{\sim}{\nabla}.(\rho\underset{\sim}{v})=0$$

Vector and tensor analysis of cylindrical and spherical coordinate systems is not done here, and can be looked up elsewhere. Thus, the final expressions in cylindrical and spherical coordinates are given as below.

cylindrical coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Spherical coordinates (r, θ, φ)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho v_\varphi)=0$$

Equation of continuity in terms of substantial derivative

The second term in Equation (8.6) may be broken into two parts as shown below. Partial derivative present in the Equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{\partial \rho}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In the above equation, the first two terms may be combined using the definition of substantial derivative to obtain the following equation.

$$\frac{D\rho}{Dt}+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In some cases, the fluid may be incompressible, i.e. density ρ is a constant with time as well as space coordinates. For example, water may be assumed as an incompressible fluid under isothermal conditions. In fact, all liquids may be assumed as incompressible fluids under isothermal conditions. For this special case, the equation of continuity may be further simplified as shown below

$$\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \quad (\rho \text{ is constant})$$

The above equation for an incompressible fluid does not mean that the system is under steady state conditions. The velocity of the fluid may still be a function of time. It only implies that if the velocity of the fluid changes in a particular direction (x, y or z) then it should also change in the other directions such that mass is conserved without changing its density. The equation of continuity provides additional information about the velocity profile and helps in solution of

1.5 EQUATION OF MOTION.

Internal and external flows

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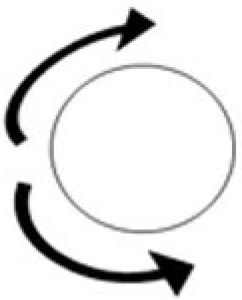

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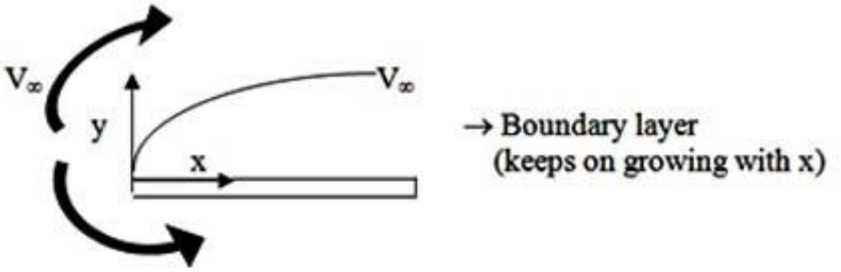

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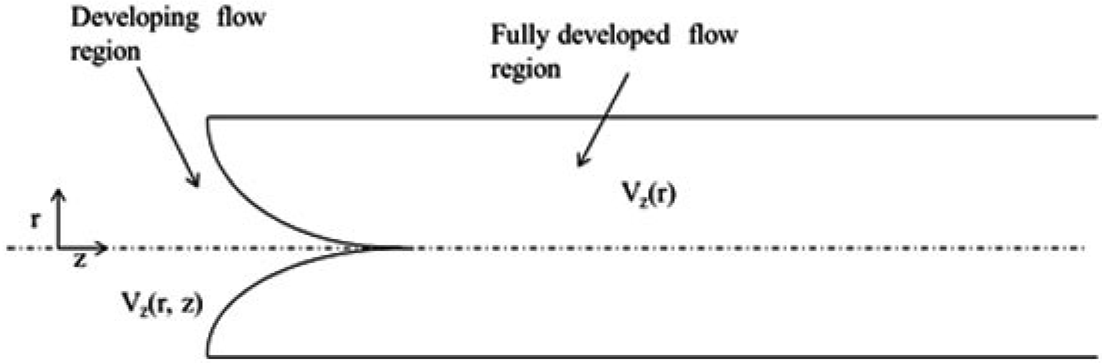

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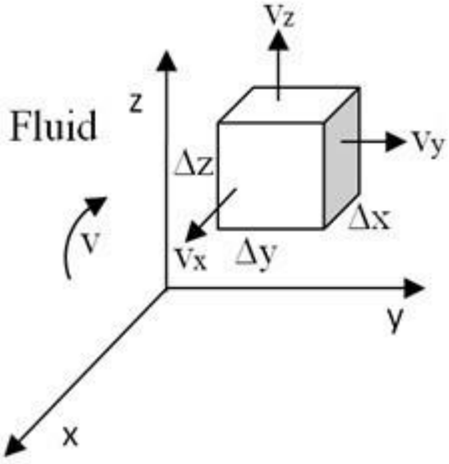

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Let m and m+Δm be the mass of the control volume at time t and t+Δt respectively. Then, the rate of accumulation,

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

In order to evaluate the rate of inflow of mass into the control volume, we need to inspect how mass enters the control volume. Since the fluid velocity has three components vx, vy and vz, we need to identify the components which cause the inflow or the outflow at each of the six faces of the rectangular CV. For example, it is the component vx which forces the fluid to flow in the x direction, and thus it makes the fluid enter or exit through the faces having area ΔyΔz at x = x and x = x+Δx respectively. The component vy forces the fluid in y direction, and thus it makes the fluid enter or exit through the faces having area ΔxΔz at y = y and y = y+Δy respectively. Similarly, the component vz forces the fluid to flow in z direction, and thus it makes fluid enter or exit through the faces having area ΔxΔy at z = z and z = z+Δ z respectively.

The rate mass entering in x direction through the surface ΔyΔz is (ρvxΔyΔz|x), the rate of mass entering in y direction through the surface ΔxΔz is (ρvyΔxΔz|y) and the rate of mass entering from z direction through the surface ΔxΔy is (ρvzΔxΔy|z). In a similar manner, expressions for the rate of mass leaving from the control volume may be written.

Thus, the conservation of mass leads to the following expressions

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z) = \Big[\rho v_x\Delta y\Delta z\big|_x - \rho v_x\Delta y\Delta z\big|_{x+\Delta x} + \rho v_y\Delta x\Delta z\big|_y - \rho v_y\Delta x\Delta z\big|_{y+\Delta y} + \rho v_z$$

Dividing the Equation (8.1) by the volume ΔxΔyΔz, we obtain

$$\frac{\partial\rho}{\partial t} = \left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \Big[\frac{(\rho v_z)|_z - (\rho}{\Delta z}$$

Note that each term in Equation (8.2) has the unit of mass per unit volume per unit time. Now, taking the limits Δx→0, Δy→0 and Δz→0, we get

$$\frac{\partial\rho}{\partial t} = \lim_{\Delta x\to 0}\left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \lim_{\Delta y\to 0}\left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \lim_{\Delta z\to 0}\Big[$$

and using the definition of derivative, we finally obtain

$$\frac{\partial\rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right]$$

Equation (8.4) is applicable to each point of the fluid. Rearranging the terms, we get the equation of continuity, may be written as given below.

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

We need not to derive the equation of continuity again and again in other coordinate system (that is, spherical or cylindrical). The idea is to rewrite Equation (8.5) in vector and tensor form. Once it is written in this form, the same equation may be applied to other coordinate system as well. Thus, the Equation (8.5) may be rewritten in vector and tensor form as shown below.

$$\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}.(\rho\underset{\sim}{v}) = 0$$

Vector and tensor analysis of cylindrical and spherical coordinate systems is not done here, and can be looked up elsewhere. Thus, the final expressions in cylindrical and spherical coordinates are given as below.

cylindrical coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Spherical coordinates (r, θ, φ)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho v_\varphi)=0$$

1.6 Equation of continuity in terms of substantial derivative

The second term in Equation (8.6) may be broken into two parts as shown below. Partial derivative present in the Equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{\partial \rho}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In the above equation, the first two terms may be combined using the definition of substantial derivative to obtain the following equation.

$$\frac{D\rho}{Dt}+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In some cases, the fluid may be incompressible, i.e. density ρ is a constant with time as well as space coordinates. For example, water may be assumed as an incompressible fluid under isothermal conditions. In fact, all liquids may be assumed as incompressible fluids under isothermal conditions. For this special case, the equation of continuity may be further simplified as shown below

$$\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \quad (\rho \text{ is constant})$$

The above equation for an incompressible fluid does not mean that the system is under steady state conditions. The velocity of the fluid may still be a function of time. It only implies that if the velocity of the fluid changes in a particular direction (x, y or z) then it should also change in the other directions such that mass is conserved without changing its density. The equation of continuity provides additional information about the velocity profile and helps in solution of equation of motion.

Solution of momentum transport problem by shell momentum balances

Here, we solve few simple problems of fluid mechanics with simple geometries by using the shell momentum balance approach. This will lead to greater understanding of various terms involved in the application of conservation of momentum in fluid given in Equation (9.1)

Flow through circular tube

Flow of fluids through a circular tube is a common problem, encountered frequently in different fields of engineering. Consider an incompressible, Newtonian fluid, flowing through a horizontal circular tube as shown in Fig. (10.1). Assume that the fluid flow is laminar and under steady state conditions. Determine the velocity profile and average velocity of the fluid using shell momentum balance approach.

solution procedure

Assumptions

- Fluid density and viscosity are constants.
- System is in steady state.
- Laminar flow (simple shear flow).
- Newton's law of viscosity is applicable.
- Fully developed flow.

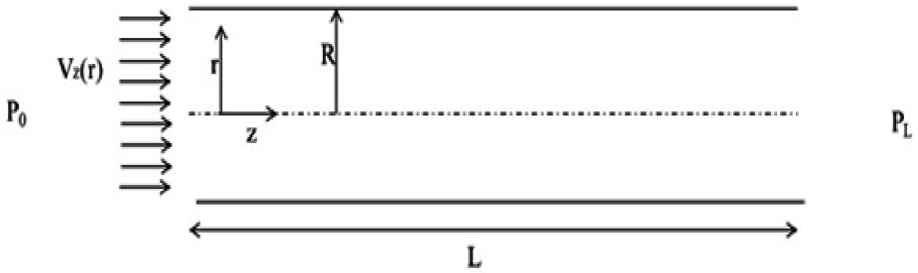


Fig Laminar flow in a horizontal pipe

Intuitively guess the velocity profile

Since the flow is steady and laminar, we may intuitively say that the velocities in r direction and θ direction are zero. Due to steady state conditions, the fluid velocity in z direction, vz, is not dependent on time t. Furthermore, due to the axisymmetric geometry fluid flow the velocity vz is independent of θ. Thus,

$$v_r = 0,\ v_\theta = 0,\ v_z = v_z(r, z, \cancel{\theta}, \cancel{t})$$

$$v_z = v_z(r, z)$$

By applying the equation of continuity in cylindrical coordinates

$$\frac{\partial v_z}{\partial z} = 0$$

Hence,

$$v_z = v(r)$$

Since the fluid is flowing in z direction, we may conclude the following.

- Since vr=0, r directional momentum balance is not important.
- Since vθ=0 , θ directional momentum balance is again not important.
- Since vz≠0, z directional momentum balance is most important.

1.7 Equation for circular pipe:

The control volume should be decided very carefully. The geometry and size of the control volume should be taken according to the geometry of the system and based on the conditions given in the problem. In this case, the geometry of the pipe is cylindrical, hence we use the cylindrical control volume. The fluid is flowing in the z direction but velocity is changing only in r direction. Therefore, the control volume is taken in such a way that the variable thickness of the control volume is in the r direction. As the flow is not dependent on z and θ coordinates, we may choose any dimension in z or θ directions. This means that z may be any length. It may be L/4, L/2 or L. In a similar manner, any value of θ may be taken. It may be 2π or π or $\pi/2$ or $\pi/4$. However, in the r direction, we need to take the differential thickness dr. These arguments leads us to a control volume as shown in Fig. (10.2). The length of the cylindrical shell is L which is equal to length of pipe and thickness is dr.

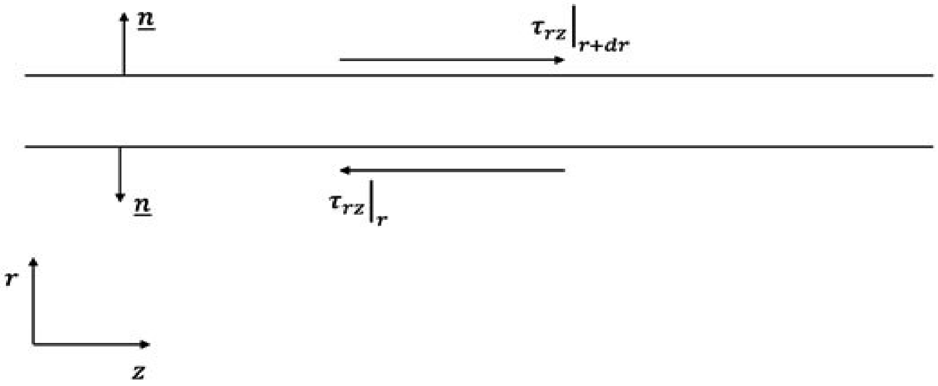


Fig 10.2 Control volume for flow through pipe

Momentum balance

As discussed earlier, the shear stress/forces may be written in two ways:

- Taking shear stress as actual shear forces.
- Taking shear stress as momentum flux.

Here, we show that both methods lead to the same final results for velocity profile.

Momentum balance using shear stress as shear force

Momentum flux entering the control volume by convection
=

$$(\rho v_z^2 2\pi r dr)\Big|_{z=0}$$

Momentum flux leaving the control volume by convection
=

$$(\rho v_z^2 2\pi r dr)\Big|_{z=L}$$

Since the pipe is horizontal, the force due to gravity is zero. No other body forces are acting on the control volume.

Surface forces

- Pressure force: Fluid is flowing in z direction only. So pressure forces which are working on the surface normal to z direction are

Pressure force at z=0 is

$$P_0 2\pi r dr\Big|_{z=0} \quad \textbf{(10.5)}$$

Pressure force at z=L is

$$-\ P_L 2\pi r dr\Big|_{z=L} \quad \textbf{(10.6)}$$

- Shear forces: The shear stress tensor in cylindrical coordinate is given below.

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

Among all 9 components the first column of stresses are important for r directional flow, the second column of stresses are important for θ directional flow, and the third column are important for z directional flow. Since the fluid is flowing in the z direction, only the third column needs to be considered. Since the Velocity gradient is present only in the r-direction, only τ_{rz} needs to be considered, the remaining two terms are not significant. Now, we need to decide the direction in which the shear forces are acting. Recall

$$\underset{\sim}{T}_n = \underset{\sim}{\delta}_n . \underset{\sim}{\tau}$$

Where the unit vector $\underset{\sim}{\delta}_n$ is the outer normal of a surface and if it is in positive direction then $\underset{\sim}{T}_n$ is also positive while if it is in negative direction then $\underset{\sim}{T}_n$ is shown as negative direction. Therefore, τ_{rz} (as a force) is positive at r+dr and negative at r as shown in Fig. 10.2.(Note: the first index, z, in τ_{rz} from right to left indicates the direction of force and second index, r, indicates the surface on which it acts).

Accumulation term: Due to steady state system, the rate of accumulation of momentum equals to zero .

General momentum balance is given below

$$\begin{pmatrix} \text{rate of accummulation} \\ \text{of momentum in CV} \end{pmatrix} = \begin{pmatrix} \text{rate of momentum} \\ \text{entering CV} \\ \text{by convection} \end{pmatrix} - \begin{pmatrix} \text{rate of momentum} \\ \text{leaving CV} \\ \text{by convection} \end{pmatrix} + \left(\sum \text{applied forces} \right)$$

or in this case

$$0 = (rv_z^2 2\pi r dr)|_{z=0} - (rv_z^2 2\pi r dr)|_{z=L} + 0 + P_0 2\pi r dr$$
$$- P_L 2\pi r dr + (\tau_{rz} 2\pi r L)|_{r+dr} - (\tau_{rz} 2\pi r L)|_r$$

Since the velocity is constant along the axial direction as shown in Equation (10.2), the first two terms in Equation (10.8) are cancel out and we are left with following Equation.

$$0 = 2\pi L\left[\left(\tau_{rz} r\right)|_{r+dr} - \left(\tau_{rz} r\right)|_r\right] + P_0 2\pi r dr - P_L 2\pi r dr$$

Dividing by $2\pi r dr$, we have

$$0 = \frac{P_0 - P_L}{L} + \frac{(\tau_{rz} r)|_{r+dr} - (\tau_{rz} r)|_r}{r dr}$$

As dr→0, the Equation (10.10) may be rewritten as given below.

(Note that, τ_{rz} is a function of r only which means we get the total derivative instead of the partial derivative.)

$$\frac{d(\tau_{rz} \mathrm{r})}{dr} = \frac{-\mathrm{r}\,(\mathrm{P}_0 - \mathrm{P}_L)}{L}$$

Further integrating the Equation (10.11) once with respect to the variable r, we obtain

$$\tau_{rz}\mathrm{r} = -\frac{r^2}{2}\frac{(\mathrm{P}_0 - \mathrm{P}_L)}{L} + c_1$$

or

$$\tau_{rz} = -\frac{r}{2}\frac{(\mathrm{P}_0 - \mathrm{P}_L)}{L} + \frac{c_1}{r}$$

Here, c1 is a constant of integration. Equation (10.12) shows that if r=0, the value of τ_{rz} will be infinite, which is physically not possible. Therefore, c1 must be zero. Hence,

$$\tau_{rz} = -\frac{r}{2}\frac{(P_0 - P_L)}{L}$$

Now, by applying Newton’s law of viscosity, and taking τ_{rz} as force, we obtain

$$\tau_{rz} = +\mu\frac{dv_z}{dr} = -\frac{r}{2}\frac{(P_0 - P_L)}{L}$$

Momentum balance using shear stress as momentum flux

Now, we will employ the second method where shear force are considered as momentum flux. To indicate the direction of momentum flux, we draw the arrow in r direction and find where this arrow enters the control volume and also leaves the control volume as shown in Fig (10.3). Thus, the momentum flux enters the control volume through the surface 2π rL at r=r and leaves through the surface 2π rL at r=r+dr.

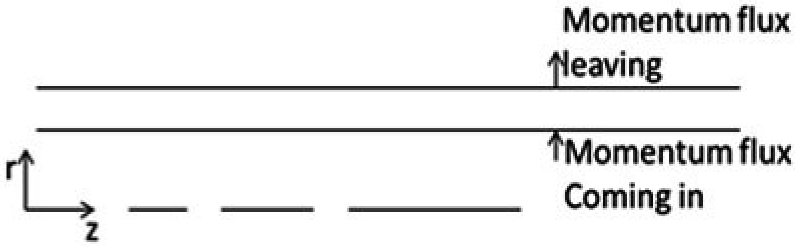


Fig 10.3 Momentum flux applied on control volume

Thus,
Momentum flux at r = r is

$$(\tau_{rz}2\pi rL)\big|_{r} \tag{10.15}$$

Momentum flux at r = r +Δr is

$$(\tau_{rz}2\pi rL)\big|_{r+dr} \tag{10.16}$$

(Note: when we consider τ_{rz} as the momentum flux, first index, z, indicates the direction of momentum flux, while the second index, r, indicates the direction of flow of momentum flux from higher to lower value. Subsequently, it will become clear that if we follow the coordinate system’s directions and assume momentum is flowing in this direction, the sign convention for momentum flux is automatically taken place.)

In this case, momentum balance in Equation (9.2) may be modified as shown below

$$\begin{pmatrix}\text{rate of accummulation}\\ \text{of momentum in CV}\end{pmatrix} = \begin{pmatrix}\text{rate of momentum}\\ \text{entering CV}\\ \text{by convection}\end{pmatrix} - \begin{pmatrix}\text{rate of momentum}\\ \text{leaving CV}\\ \text{by convection}\end{pmatrix} + (\sum$$

$$+ \begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{entering CV}\end{pmatrix} - \begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{leaving CV}\end{pmatrix}$$

Here, the shear stress are taken into account as momentum flux. The pressure and gravity are the only applied forces.

Substituting various terms in above equation, we obtain

$$0 = 0 + (P_0 - P_L)\, 2\pi r dr + (\tau_{rz} 2\pi r L)\big|_r - (\tau_{rz} 2\pi r L)\big|_{r+dr}$$

Dividing by $2\pi r dr$, we obtain

$$0 = (\tau_{rz} r)\big|_r - (\tau_{rz} r)\big|_{r+dr} + \frac{(P_0 - P_L)}{L}$$

Again as dr→0 Equation (10.17) leads to

$$0 = -\frac{d(\tau_{rz} r)}{r dr} + \frac{(P_0 - P_L)}{L}$$

or

$$\frac{d(\tau_{rz} r)}{dr} = \frac{(P_0 - P_L)}{L} r$$

By integrating the Equation (10.18), we have

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r + \frac{c_1}{r}$$

As we discussed earlier, c1 should be zero. Therefore,

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Now applying Newton’s law of viscosity where shear stress is taken as momentum flux, we obtain

$$\mu \frac{dv_z}{dr} = -\frac{(P_0 - P_L)}{2L} r$$

Equation (10.14) and (10.20) are identical and hence show that both methods finally lead to the same result.
To obtain velocity profile we further integrating the Equation (10.21)

$$v_z = -\frac{(P_0 - P_L)}{2\mu L} \frac{r^2}{2} + c_2$$

Here c2 is the second constant of integration which may be determined by using appropriate boundary condition.

Boundary condition

By no-slip boundary condition
vz=0 at r=R

$$0 = -\frac{(P_0 - P_L)}{4\mu L} R^2 + c_2$$

$$c_2 = \frac{(P_0 - P_L)}{4\mu L} R^2$$

Substituting the value of c2 in Equation (10.22), we finally get

$$v_z = \frac{(P_0 - P_L)}{4\mu L}(R^2 - r^2)$$

Note: c1 can also be calculated by using the boundary condition in terms of velocity vz: i.e., vz is finite at r=0

or $\left.\frac{dv_z}{dr}\right|_{r=0} = 0$ (since the velocity profile is symmetric about r=0).

Thus, the velocity profile for flow through pipe is given by the following expression

$$v_z = \frac{(P_0 - P_L)}{4\mu L}R^2\left(1 - \frac{r^2}{R^2}\right)$$

The maximum velocity of the fluid will be exhibited at the centre of the pipe and is given by

$$v_{z\max} = v_z\,|_{r=0} = \frac{(P_0 - P_L)}{4\mu L}R^2$$

Alternatively, the velocity profile may also be expressed in terms of the maximum velocity as

$$v_z = v_{z,\max}\left(1 - \frac{r^2}{R^2}\right)$$

The average velocity of the fluid in the pipe is the average of all local velocities. Thus, this may be calculated by estimating the volumetric flow rate through the pipe and then dividing it by the cross sectional area of the pipe. The total volumetric flow in the system is

$$Q = \int dQ$$

where, dQ is the volumetric flow rate from small cylindrical strip of thickness dr.

$$= \int v_z 2\pi r dr$$

By substituting the value of v z from equation (10.27), we have

$$= \int v_{z\,max}\left[1-\frac{r^2}{R^2}\right]2\pi r dr$$

By integrating the equation (10.30) from r=0 to r=R, we obtain

$$= \int_0^R 2\pi\, v_{z\,max}\left[r-\frac{r^3}{R^2}\right]dr$$

or

$$= 2\pi v_{z\,max}\left[\frac{r^2}{2}-\frac{r^4}{4R^2}\right]_0^R$$

Thus,

$$Q = \pi v_{z\,max}\frac{R^2}{2}$$

and average velocity is

$$v_{z,avg} = \frac{Q}{A_c}$$

$$= \frac{\pi\, v_{z,max}\frac{R^2}{2}}{\pi R^2}$$

or

$$v_{z,avg} = \frac{v_{z,max}}{2}$$

The velocity profile for laminar flow in a circular tube is shown in Fig. 10.5.

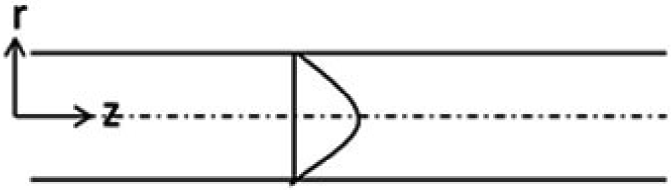


Fig 10.5 Velocity profile in horizontal pipe

We can also find the radial distance at which the local velocity of fluid flow equals the average velocity. For this, substitute $v_z = v_{z,avg} = \frac{v_{z,\max}}{2}$ into Equation (10.26), we obtain

$$1 - \frac{r^2}{R^2} = \frac{1}{2}$$

$$\frac{r^2}{R^2} = \frac{1}{2}$$

$$\text{or, } r = \frac{R}{\sqrt{2}} \qquad \mathbf{(10.34)}$$

Finally , the volumetric flow rate in terms of pressure drop is as follows

$$Q = v_{zavg} \pi R^2 = \frac{\pi(\mathrm{P}_0 - \mathrm{P}_\mathrm{L})}{8\mu L} R^4 \qquad \mathbf{(10.35)}$$

$$Q = \frac{\pi(\mathrm{P}_0 - \mathrm{P}_\mathrm{L})}{128 u L} D^4 \qquad \mathbf{(10.36)}$$

Equation (10.36) is known as the Hagen – Poiseuille equation. Thus, if the pressure drop is given, we can calculate the volumetric flow rate in the pipe and vice-versa. This equation can also be used for the calculation of viscosity in capillary flow viscometer. However, it may be noted that Hagen – Poiseuille equation is valid only for fully developed laminar flow. Therefore, when this equation is used for various calculations there may be some errors due to developing and exiting flow at both ends of the pipe. Hence, this equation has to be modified for real situations.

Friction factor

The friction factor is a dimensionless number, which provides an idea about the magnitude of shear stress produced by a solid boundary as fluid flows. This is defined as the ratio of shear stress at the wall and the kinetic energy head of the fluid, $\frac{1}{2} \rho v^2_{zavg}$. Here, ρ is the density and v_{zavg} is the average velocity of fluid. The friction factor is thereby defined as

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2_{zavg}}$$

where, τ_w is the shear force per unit area on the wall of the tube. This may be calculated as shown below

$$\tau_w = -\left(-\tau_{rz}\big|_{r=R}\right)$$

Here, first minus sign is used as the inside surface of the tube wall has outer normal in the negative r direction and second minus sign is used because τ_{rz} is treated here as momentum flux. If τ_{rz} is treated as actual shear force then positive sign would have to be taken. For fully developed laminar flow, the velocity profile is parabolic and is given by

$$v_z = v_{zmax}\left[1 - \frac{r^2}{R^2}\right]$$

Evaluating the velocity gradient at the wall (r=R), we have

$$\left.\frac{dv_z}{dr}\right|_{r=R} = -\frac{2v_{z\,max}}{R}$$

Thus, the shear stress considered as momentum flux is given by

$$\tau_{rz}\big|_{r=R} = -\mu \left.\frac{dv_z}{dr}\right|_{r=R} = +\frac{2\mu v_{zmax}}{R}$$

or

$$\tau_w = \tau_{rz}\big|_{r=R} = \frac{2\mu v_{zmax}}{R} = \frac{2\mu(P_0 - P_L)}{4\mu L}\frac{R^2}{R} = \frac{(P_0 - P_L)}{2L}R$$

The friction factor may now be calculated as shown below

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2_{zavg}} = \frac{\frac{(P_0 - P_L)}{2L}R}{\frac{1}{2}\frac{\rho(P_0 - P_L)}{8\mu L}R^2} = \frac{8\mu}{\rho R v_{zavg}}$$

or

$$f = \frac{16\mu}{\rho D v_{zavg}} = \frac{16}{Re}$$

Equation (11.7) shows that the friction factor in laminar flow region depends only on the Reynolds number. Clearly, the friction factor is also a dimensionless number.

Friction factor in turbulent flow

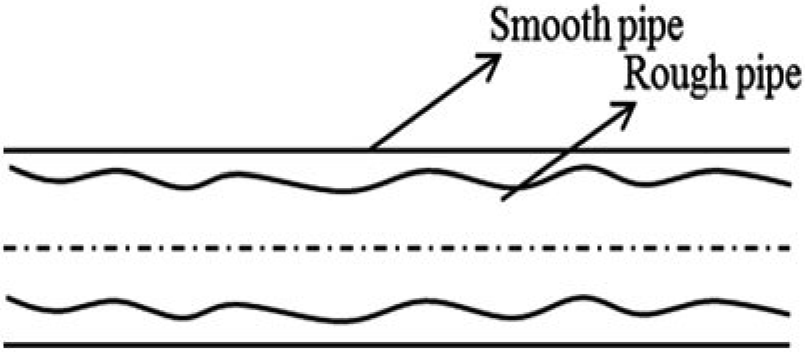
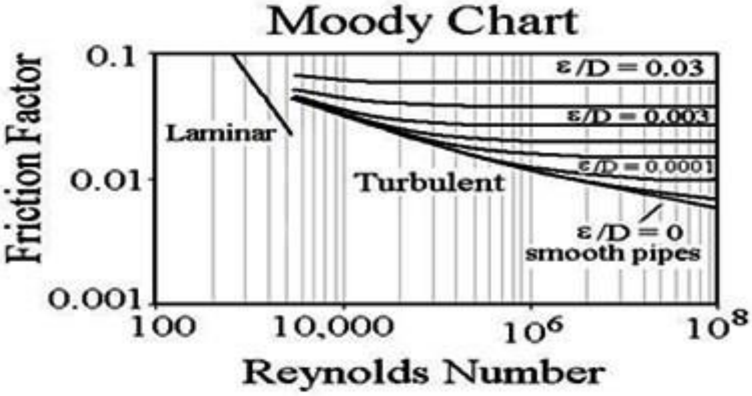


Fig 11.1 Smooth and rough surface of pipe

In turbulent flow, the friction factor also depends on the surface of the pipe. A rough pipe leads to higher turbulence than a smoother pipe, so that the friction factor for smoother pipes is less than that for rougher pipes. The ratio of surface roughness height (∈) to pipe diameter (D) is used to quantify the “roughness” of the pipe surface. In practice, the shear stress on the wall may be calculated by measuring the pressure drop across the pipe for a given flow rate. Thus, friction factor may be calculated as the function of Reynolds number and plotted on a log-log plot for a given surface roughness. The curves are different for different surface roughness as shown in figure. (11.2). The collection of these f-Re plots is called Moody Chart as shown in figure below, and can be used for estimating the friction factor for given flow parameters.



Ref: http://www.brighthub.com/engineering/civil/articles

Solution of some more fluid flow problems by shell momentum balance approach

In this section, we solve a few more fluid mechanics problems in simple geometries using the shell momentum balance approach. The detail procedure, which was also used in previous example, is outlined below.

1) Make a diagram of the flow geometry with the appropriate coordinate system

2) Specify all necessary assumptions

3) Intuitively assume the velocity profile

This is an important step for solving these problems. In laminar flow, the fluid flows in parallel layer without mixing. Thus, it is easy to guess the non-zero components of velocities by intuition.

4) Apply of the equation of continuity to modify the velocity profile

5) Determine the non-zero shear stress component(s)

Since the shear stress components depend on the velocity profile, the non-zero shear stress components may now be determine.

6) Determine control volume and make shell momentum balance for the control volume

Draw control volume in system diagram according to system shape, size and problem statement. The selection of proper control volume is very important to solve problem correctly. The control volume should be select in such way that it can be easily integrated for whole system. The differential length of control volume should be taken in direction of changing velocity.
Write momentum balance equation for the control volume. The shear stress may be considered as shear force or as momentum flux, both provide the same results as shown in previous example . Write down all surface and body force acting on the fluid carefully. Finally obtain a appropriate

differential equation and integrate.

7) Boundary conditions

Use appropriate boundary conditions which help us to determine the constant of integration in above step.

1.8 Falling film on an inclined flat surface

An inclined surface of length L and width W is situated at an angle B to the vertical direction as shown in Fig. (11.3). A Newtonian fluid is freely falling on the surface as a film of thickness δ. Assuming the flow to be laminar, determine the velocity profile, flow rate and shear force on the surface by the fluid.

Solution

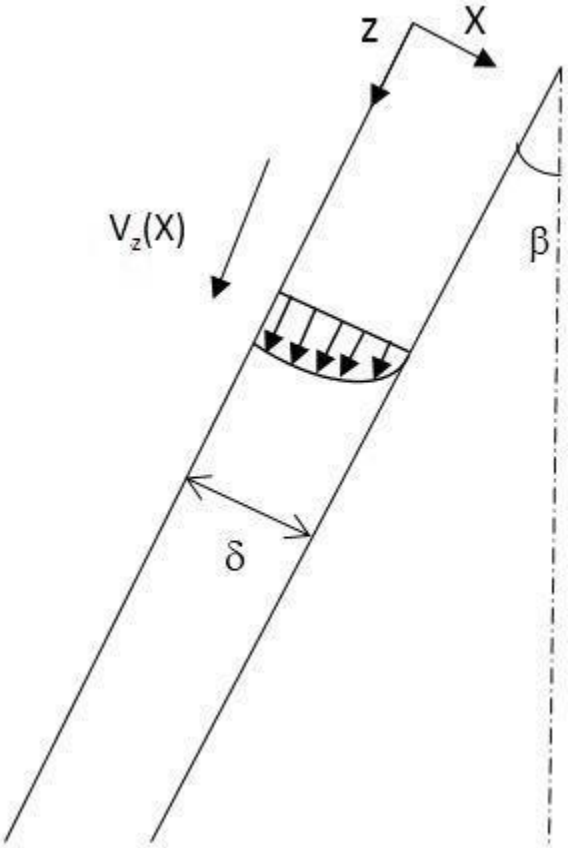


Fig 11.3 Laminar flow on an inclined surface

Assumptions

- Constant density, viscosity
- Steady state
- Laminar flow (simple shear flow)
- Fully developed flow
- Newton's law of viscosity is applicable

Assume velocity profile
The fluid is flowing in the z direction, hence only the z component of velocity is non-zero. Thus, we may assume

$$v_x = 0 \quad , \quad v_y = 0 \quad \text{an}$$

$$v_z = v_z(x, y, z, t)$$

We may further assume that vz does not depends upon y coordinate. Since the flow is steady, vz does not depend on time. Thus,

$$v_z = v_z(x, z)$$

Using the equation of continuity in the cartesian coordinates for constant fluid density, we have

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

which reduces to

$$\frac{\partial v_z}{\partial z} = 0$$

Equation (11.11) indicates that vz does not depend on the z coordinate. Thus,

$$v_z = v_z(x)$$

There are nine components of the shear stress as shear force or momentum flux, namely

$\tau_{xx}\,\tau_{xy}\,\tau_{xz} \rightarrow$ acting on x surface

$\tau_{yx}\,\tau_{yy}\,\tau_{yz} \rightarrow$ acting on y surface

$\tau_{zx}\,\tau_{zy}\,\tau_{zz} \rightarrow$ acting on z surface

Since vz is only the non-zero velocity, and also it is the function of x coordinate , τ_{zx} is the only significant component of shear stress and we need to write momentum balance only in z direction. Because the pressure is same at both ends of the inclined plane, there is no pressure force on the fluid. Now, we can solve this problem by assuming shear stress as a shear force or shear stress as momentum flux.

Assuming τ_{xz} as momentum flux
Draw a control volume of length L, width W and differential thickness dx.

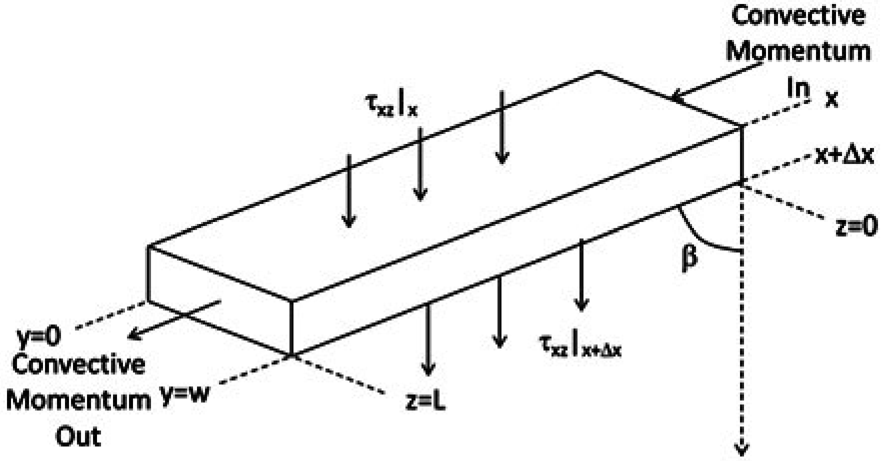


Fig 11.3 Control volume for falling film problem

Momentum balance in x direction

Rate of momentum flux entering CV due to viscous transport at

$$x \;=\; LW\,\tau_{xz}\big|_x$$

Rate of momentum flux leaving CV due to viscous transport at

$$x+\Delta x = LW\tau_{xz}\big|_{x+\Delta x}$$

Gravity force acting on fluid in z direction

$$= (LW\Delta x)(\rho g \ cos\beta)$$

Rate of momentum flux entering in CV due to convective transport

$$= (\rho v_z W dx \ v_z)\Big|_{z=0}$$

Rate of momentum flux leaving from CV due to convective transport

$$= (\rho v_z W dx \ v_z)\Big|_{z=L}$$

Now, when above terms are substituted for z-momentum balance, we obtain

$$(\rho v_z w dx v_z)\Big|_{z=0} - (\rho v_z w dx \, v_z)\Big|_{z=L} + LW(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Since the velocity vz does not depends on z coordinate, the first two terms cancel out and we obtain

$$LW(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Dividing Equation (11.19) by volume of the control volume (LWΔx), we have

$$\frac{(\tau_{xz}\,|_x - \tau_{xz}\,|_{x+dx})}{\Delta x} + (\rho g \cos\beta) = 0$$

As Δx→0 , The Equation (11.20)simplified to

$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta$$

The Newton’s law of viscosity (here, shear stress is defined as momentum flux) is given by

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$

$$\frac{d}{dx}\left(-\mu \frac{dv_z}{dx}\right) = \rho g \cos \beta$$

or

$$-\mu \frac{d^2 v_z}{dx^2} = \rho g \cos \beta$$

or

$$\frac{d^2 v_z}{dx^2} = -\left(\frac{\rho g \cos \beta}{\mu}\right)$$

By integrating the Equation (11.25), we have

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos \beta}{\mu}\right)x + c_1$$

or

$$v_z = -\left(\frac{\rho g \cos \beta}{\mu}\right)\frac{x^2}{2} + c_1 x + c_2$$

The above equation requires two boundary conditions for determining c1 and c2.

Boundary conditions

1 At x=0 the liquid surface is in contact with air where the shear stresses at both gas liquid phases should be equal. Thus,

$$\tau_{xz\,(air)}\Big|_{x=0} = \tau_{xz\,(liquid)}\Big|_{x=0}$$

Since both may be assumed Newtonian fluids, we have

$$\mu_g \rho_g \left.\frac{dv_{z(air)}}{dx}\right|_{x=0} = \mu\rho \left.\frac{dv_z}{dx}\right|_{x=0}$$

where ρg is the density and µg is the viscosity of air. Thus

$$\left.\frac{dv_z}{dx}\right|_{x=0} = \frac{\mu_g \rho_g}{\mu\rho} \left.\frac{dv_{z(air)}}{dx}\right|_{x=0}$$

Since, µg and ρg is much smaller than µ and ρ, and Equation (11.30) may be approximately written as

$$\left.\frac{dv_z}{dx}\right|_{x=0} = 0$$

Substituting above boundary condition in Equation (11.26), we obtain

$$c_1 = 0$$

2. At x=δ no slip boundary condition may be applied, i.e.,

at

$$x = \delta \ , v_z = 0$$

Thus, from Equation (11.27), we get

$$0 = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2} x + c_2$$

or

$$c_2 = \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2}$$

Finally the velocity profile is obtained as

$$v_z = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{x^2}{2} + \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2}$$

or

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left(1 - \left(\frac{x}{\delta}\right)^2\right)$$

Falling film "Assuming τ_{xz} as shear force"

Now, we again solve the same problem (falling film over an inclined plane) by treating shear stress as a shear force. For this purpose, we take the same control volume as before.
For momentum balance in z direction, all terms are same as before except the terms for shear forces. Here, τ_{xz} represents the force in z direction acting on the surfaces which have normal in x direction. Shear force is positive if the outward normal is in positive direction and negative if normal is in negative direction. Thus,

shear force at x=x is
$-LW\tau_{xz}\big|_x$

Shear force at x=x+Δx is
$+LW\tau_{xz}\big|_{x+\Delta x}$
;

The z momentum balance for this case is as follows

$$LW(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Dividing Equation (12.3) by the volume of control volume WLΔx, we have

$$\frac{(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x)}{\Delta x} + (\rho g \cos\beta) = 0$$

As Δx→0 Equation (12.4) leads to

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos\beta$$

Now, substituting the Newton's law of viscosity for shear stress as a force

$$\tau_{xz} = \mu \frac{dv_z}{dx}$$

Therefore,

$$\mu \frac{d^2 v_z}{dx^2} = -\rho g \cos\beta$$

Equations (11.24) and (12.7) are the same, which show that both approaches provide the same answer.

Maximum velocity

It is clear from Equation (11.37) that the maximum velocity is given by

$$v_{z,max} = \frac{\rho g \delta^2 \cos\beta}{2\mu}$$

Average velocity and volumetric flow rate of falling film

vz is the linear velocity in z direction. Hence, the volumetric flow rate can be determined by integrating it over the cross section of flow (Wδ).Thus ,

$$Q = \int_0^W \int_0^\delta v_z \, dx\, dy$$

From Equation (11.37), we get

$$Q = \int_0^W \int_0^\delta \frac{\rho g \delta^2 \cos\beta}{2\mu} \left(1 - \left(\frac{x}{\delta} \right)^2 \right) dx dy$$

By integrating Equation (12.10), we find

$$Q = \frac{W \rho g \delta^3 \cos\beta}{3\mu}$$

To obtain the average velocity, we divide the volumetric flow rate by the cross sectional area.

$$\langle v_z \rangle_{avg} = \frac{Q}{\int_0^W \int_0^\delta dx\,dy}$$

or

$$\langle v_z \rangle_{avg} = \frac{\rho g \delta^2 \cos\beta}{3\mu}$$

Equation (12.12) may also be written as

$$\langle v_z \rangle_{avg} = \frac{2}{3} v_{z,max}$$

Force acting on solid surface due to the fluid

$$F = \int_0^L \int_0^W +\left(+\tau_{xz}\big|_{x=\delta}\right) dy\,dz$$

(Note: in Equation (12.14), first '+' sign shows the direction of the normal of the inclined surface and second '+' sign is taken since shear stress is defined as shear force). Thus,

$$F = \rho g \delta L W \cos\beta$$

In this lecture, we have once again seen that the shear stress tensor may be assumed as a shear force or as a momentum flux. In either case, we finally obtain the same expression for the velocity profile. The only difference is that when we treat shear stress as a shear force, it is included in the summation of all forces term in the momentum balance equation, while when we treat shear stress as momentum flux, it is written as momentum entering and leaving by the viscous transport. From now onwards, we will treat shear stress as momentum flux as it is more consistent with what we see in heat transfer as Fourier's law of heat conduction and in mass transfer as Fick's law of diffusion. Thus, in transport phenomena (Momentum transport, Heat transport, and Mass transport) for the basic transport laws we have minus sign in front the relevant gradient implying fluxes flow from higher values to lower values.

Falling film on the outside of a circular shell

In an experiment, a fluid flows upward through a small circular shell and then flows downward out side the tube under laminar conditions as shown in Fig. 12.2. We need to set up a relevant momentum balance and determine the velocity profile, mass flow rate and the force acting on outer surface of the tube.

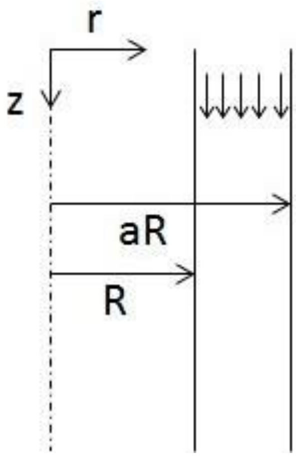


Fig 12.2 Falling film outside the circular tube

Assumptions

- Density and viscosity are constants.
- Steady state.
- Fully developed laminar flow.
- Newton's law of viscosity is applicable.

Non-zero velocities
Fluid is flowing in the z direction due to gravity. There is no driving force in the θ direction and a solid surface is present in the r direction. Therefore, we may intuitively assume that

$$v_z = v_z(r,z)$$

$$v_\theta = 0$$

$$v_r = 0$$

Now, using the equation of continuity in cylindrical coordinate system, we have

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

or

$$\rho \frac{\partial}{\partial z}(v_z) = 0$$

From Equation (12.18), we obtain that vz is independent to z. Therefore,

$$v_z = v_z(r)$$

Choose a control volume in the film of differential thickness dr and length L (it is a cylindrical shell).

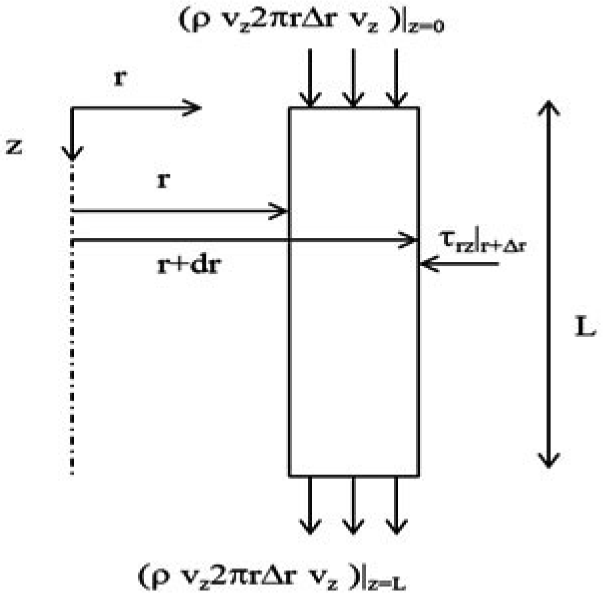


Fig 12.3 Control volume for falling film outside the circular tube

There are nine components of shear stress tensor. Since the fluid is flowing in z direction and it is a function of r only, we may argue as before τ_{rz} is the only important component of the shear stress tensor. The other components are insignificant for momentum balance in z direction. The momentum balance in z-direction is given below.

Momentum balance for control volume

Convective momentum entering the control volume at z=0 is

$$(\rho v_z 2\pi r\Delta r v_z)\big|_{z=0} \qquad \textbf{(12.20)}$$

Convective momentum leaving the control volume at z=L is

$$(\rho v_z 2\pi r\Delta r v_z)\big|_{z=L} \qquad \textbf{(12.21)}$$

Shear stress as momentum flux entering the control volume at r= r is

$$(2\pi r L\tau_{rz})\big|_{r=r} \qquad \textbf{(12.22)}$$

Shear stress as momentum flux entering the control volume at r= r + Δr is

$$(2\pi r L\tau_{rz})\big|_{r=r+\Delta r} \qquad \textbf{(12.23)}$$

{Note: If you consider shear stress as momentum flux, then it always flows in the positive direction of axes}

Fluid is flowing only due to gravity and may be written as

$$(2\pi r\Delta r L\rho g) \qquad \textbf{(12.24)}$$

Substituting above terms, we obtain

$$(\rho v_z 2\pi r D r v_z)\big|_{z=0} - (\rho v_z 2\pi r\Delta r v_z)\big|_{z=L} + (2\pi r L\tau_{rz})\big|_{r=r} - (2\pi r L\tau_{rz})\big|_{r=r+\Delta r} + 2\pi r\Delta r L\rho g = 0$$

Since velocity, vz, is not dependent on the z, the first two terms in above equation are equal and cancel out, leaving the following equation for momentum balance.

$$(2\pi r L\tau_{rz})\big|_{r=r} - (2\pi r L\tau_{rz})\big|_{r=r+\Delta r} + 2\pi r\Delta r L\rho g = 0$$

Dividing Equation (12.26) by volume of control volume $2\pi r\Delta r L$, weobtain

$$\frac{(r\tau_{rz}\big|_{r} - r\tau_{rz}\big|_{r+\Delta r})}{r\Delta r} = -\rho g$$

As dr→0, Equation (12.27) reduces to

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = -\rho g$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-\mu\frac{dv_z}{dr}\right)\right) = \rho g$$

After integration we obtain

$$\frac{dv_z}{dr} = -\frac{\rho g r}{2\mu} + \frac{c_1}{r}$$

and

$$v_z = -\frac{\rho g r^2}{4\mu} + c_1 ln r + c_2$$

Here, c_1 and c_2 are the constants of integration.

Boundary conditions

1. r=aR we have the air water interface where we may assume that

$$\frac{d\,v_z}{d\,r} = 0$$

(The explanation is given earlier in Lecture 11.)
Substituting the above boundary condition in Equation (12.31), we obtain

$$c_1 = \frac{\rho g a^2 R^2}{2\mu}$$

2. At r = R, no slip boundary condition is applicable. Thus,

$$v_z\big|_{r=R} = 0$$

Using this boundary condition, we obtain

$$c_2 = \frac{\rho g R^2}{4\mu} - c_1 ln R$$

or,

$$c_2 = \frac{\rho g R^2}{4\mu} - \frac{\rho g a^2 R^2}{2\mu} ln R$$

Therefore, the velocity profile is given by

$$v_z = -\frac{\rho g r^2}{4\mu} + \frac{\rho g a^2 R^2}{2\mu} ln r + \frac{\rho g a^2 R^2}{4\mu} - \frac{\rho g a^2 R^2}{2\mu} ln R$$

or

$$v_z = \frac{\rho g R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right]$$

Maximum velocity

At r = aR, the velocity is maximum. Thus,

$$v_{z,max} = \frac{\rho g R^2}{4\mu}\left[1-a^2+2a^2 \ln a\right]$$

1.9 Flow through Annulus

A Newtonian fluid is flowing in a narrow slit (B<<W<<L), formed by two parallel plates as shown in Fig. (13.1), due to the combined effect of both gravity and pressure. Determine the velocity profile, average velocity, and mass flow rate for laminar and steady flow.

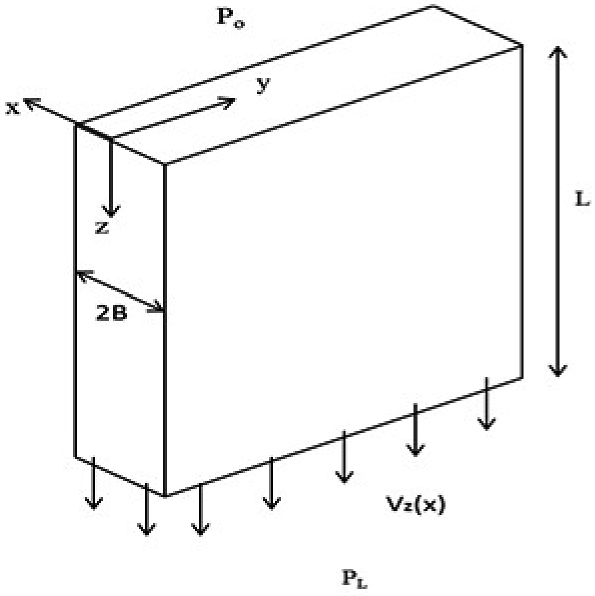


Fig 13.1 Laminar flow in narrow slit

Assumptions

- Density and viscosity are constant.
- Steady state.
- Laminar Flow(simple shear flow).

- Newton's law of viscosity is applicable.

Fluid is flowing in the z direction due to both gravity and pressure difference. Therefore, vz is the only important velocity component. As the slit is very narrow (B<<W<<L), we may assume that end effects are negligible in y direction and vz is not a function of y.

Thus, intuitively we assume the velocity profile as,

$$v_z = v_z(x,z)$$

$$v_x = 0$$

$$v_y = 0$$

Now, using the equation of continuity in cartesian coordinate system

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

or

$$\rho \frac{\partial v_z}{\partial z} = 0$$

Therefore,

$$v_z = v_z(x)$$

From above velocity profile, we may conclude that τ_{xz} is the only important shear stress component. We now select a cuboidal control volume of dimensions L, W, Δx, as shown in Fig. 13.2 (Note: differential thickness is chosen in x direction)

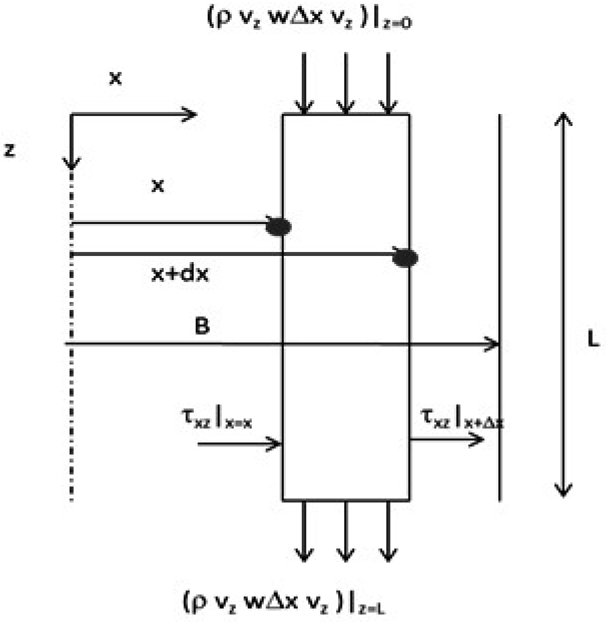


Fig 13.2 Control volume for laminar flow in narrow slit.

Momentum balance in z direction

Convective momentum entering the CV at z=0 is

$$(\rho v_z W \Delta x v_z)\big|_{z=0} \qquad \textbf{(13.5)}$$

Convective momentum leaving the CV at z=L is

$$(\rho v_z W \Delta x v_z)\big|_{z=L} \qquad \textbf{(13.6)}$$

Momentum entering CV by viscous transport at x=x is

$$(LW\tau_{xz})\big|_{x=x} \qquad \textbf{(13.7)}$$

Momentum leaving the CV by viscous transport at x=x+Δx is

$$(LW\tau_{xz})\big|_{x=x+\Delta x} \qquad \textbf{(13.8)}$$

Pressure force at z=0 is

$$P_0 W\Delta x \qquad \textbf{(13.9)}$$

Pressure force at z=L is

$$-P_L W\Delta x \qquad \textbf{(13.10)}$$

Gravity force on CV is

$$\rho\ g\ \Delta x L W \qquad \textbf{(13.11)}$$

Substituting these terms into the momentum balance in z direction, we get

$$(\rho v_z w\Delta x v_z)\big|_{z=0} - (\rho v_z w\Delta x v_z)\big|_{z=L} + (LW\tau_{rz})\big|_{r=r} - (LW\tau_{rz})\big|_{r=r+\Delta r}$$
$$+P_0 w\Delta x - P_L w\Delta x + \rho g\Delta x LW = 0$$

Since, vz is not a function of z, the first two convective momentum terms represented by Equations (13.5) and (13.6) are equal and hence cancel out from the above equation and we get

$$(LW\tau_{xz})\Big|_{x=x} - (LW\tau_{xz})\Big|_{x=x+\Delta x} + P_0 w\Delta x - P_L w\Delta x + \rho g\Delta x LW = 0$$

Dividing Equation (13.13) by the volume of the control volume ΔxLW , we obtain

$$\frac{(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x})}{\Delta x} = \frac{P_L - P_0}{L} - \rho g$$

Combining the pressure force with gravity, and taking the limit as Δx→0, we have

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{(P_0 - \rho g z(0)) - (P_L + \rho g z(L))}{L}\right)$$

or

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{P_{c0} - P_{cL}}{L}\right)$$

where, $P_c = P - \rho g z$

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{P_{c0} - P_{cL}}{L}\right)x + c_1$$

Substituting Newton's law of viscosity, we have

$$\tau_{xz} = -\mu\frac{dv_z}{dx}$$

or

$$-\mu\frac{dv_z}{dx}=\left(\frac{P_{cL}-P_{c0}}{L}\right)x+c_1$$

and finally after integration, we get

$$v_z=-\left(\frac{P_{c0}-P_{cL}}{\mu L}\right)\frac{x^2}{2}-\frac{c_1}{\mu}x+c_2$$

Boundary conditions are

1. At x=0 , the velocity profile must be symmetric. Therefore, $\left.\frac{dv_z}{dx}\right|_{x=0}=0$
or

$$c_1=0$$

2. At x=B , no slip boundary condition is applicable. Thus, $v_z=0$

or

$$c_2=\left(\frac{P_{c0}-P_{cL}}{\mu L}\right)\frac{B^2}{2}$$

Thus, velocity profile may be written as

$$v_z=\left(\frac{P_{c0}-P_{cL}}{\mu L}\right)\frac{B^2}{2}\left[1-\left(\frac{x}{B}\right)^2\right]$$

Equation (13.23) describes the velocity profile in the narrow slit.

Mass flow rate and average velocity

Mass flow rate = Volumetric flow rate × Density

$$= \rho \int_0^W \int_{-B}^{B} v_z \, dx dy$$

By substituting the value of velocity from Equation (13.23), we have

$$= \int_0^W \int_{-B}^{B} \left[\left(\frac{P_{c0} - P_{cL}}{\mu L} \right) \frac{B^2}{2} \left[1 - \left(\frac{x}{B} \right)^2 \right] dx dy \right] \rho$$

or

$$m = \frac{2}{3} \frac{\rho W \left(P_{c0} - P_{cL} \right) B^3}{\mu L}$$

Average velocity = Volumetric flow rate/ Area of cross section

$$= \frac{\frac{2}{3} \frac{W \left(P_{c0} - P_{cL} \right) B^3}{\mu L}}{2 B W}$$

or

$$\langle v_z \rangle = \frac{1}{3} \frac{\left(P_{c0} - P_{cL} \right) B^2}{\mu L}$$

Annular flow with inner cylinder moving axially

In a wire coating machine, a wire of radius kR is moving into a cylindrical hollow die. The radius of the die is R , and the wire is moving with a velocity v0 along the axis. The die is filled with a Newtonian fluid, a coating material. The pressure at both ends of the die is same. Find the velocity distribution in the narrow annular region. Obtain the viscous force acting on the wire of length L . Also, find the mass flow rate through the annular region.

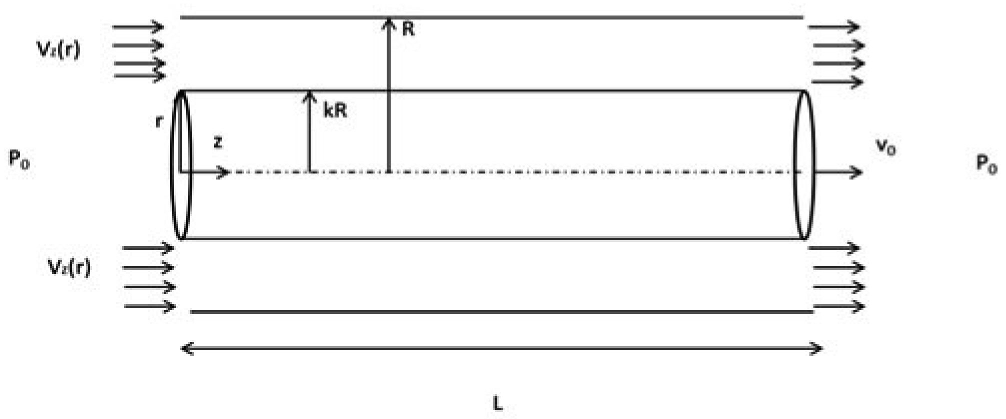


Fig 13.3 Annular flow with the inner cylinder moving axially

Assumptions

- density and viscosity are constant
- steady state.
- laminar (simple shear flow).
- Newton's law of viscosity is applicable.

Velocity components

The fluid is moving due to the motion of the wire in z direction so vz is the only important velocity component. There is no solid boundary in θ direction, and the flow is steady, therefore vz will not depend on θ and t. Hence,

$$v_z = v_z\left(r,\, z\right)$$

Now, applying the equation of continuity in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

or

$$\rho\,\frac{dv_z}{dz} = 0$$

Thus,

$$v_z = v_z(r)$$

This result indicates that τ_{rz} is the only significant shear stress among the 9 components for momentum balance in z direction. Now, consider a control volume of differential thickness dr and length L at a distance r away from the center. We may write the momentum balance in z direction.

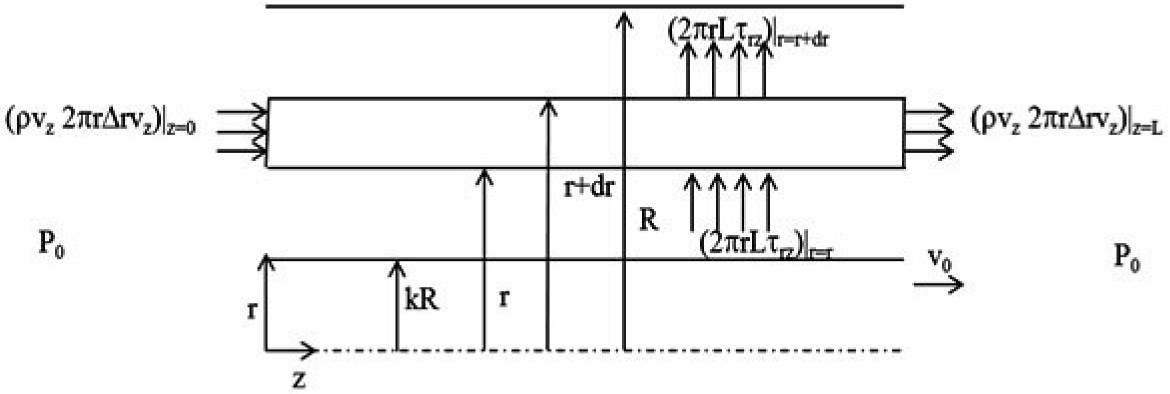


Fig 13.4 Control volume for annular flow with the inner cylinder moving axially

Convective momentum entering at z=0 is

$$(\rho v_z 2\pi r\Delta r v_z)|_{z=0}$$

Convective momentum leaving at z=L is

$$(\rho v_z 2\pi r\Delta r v_z)|_{z=L}$$

Momentum entering control volume by viscous transport at r = r is

$$(2\pi rL\tau_{rz})|_{r=r}$$

Momentum leaving control volume by viscous transport at at r = r +Δr is

$$(2\pi rL\tau_{rz})|_{r=r+\Delta r}$$

Now, the momentum balance over the control volume is below

$$(\rho v_z 2\pi r\Delta r v_z)\big|_{z=0} - (\rho v_z 2\pi r\Delta r v_z)\big|_{z=L} + (2\pi r L\tau_{rz})\big|_{r=r} - (2\pi r L\tau_{rz})\big|_{r=r+\Delta r} = 0$$

Since velocity vz is not dependent on z coordinate therefore the convective terms represented by equations (13.29) and (13.30) are equal and hence cancelled out. Leaving with the following equation,

$$(2\pi r L\tau_{rz})\big|_{r=r} - (2\pi r L\tau_{rz})\big|_{r=r+\Delta r} = 0$$

Dividing equation (13.34) by volume of the control volume, $2\pi r\Delta r L$

$$\frac{(r\tau_{rz}\big|_{r} - r\tau_{rz}\big|_{r+\Delta r})}{r\Delta r} = 0$$

Taking the limit as dr→0, we have

$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = 0$$

and after integration

$$\tau_{rz} = \frac{c_1}{r}$$

where c_1 is an integration constant. Now, using Newton's law of viscosity, we get

$$-\mu\frac{dv_z}{dr} = \frac{c_1}{r}$$

or

$$v_z = -\frac{c_1}{\mu}\ln r + c_2$$

where c_2 is another integration constant.

Boundary conditions are

at r = kR , $v_z = v_0$ **(13.40)**

or

$$v_0 = -\frac{c_1}{\mu} \ln kR + c_2$$

and at r = R, $v_z = 0$ **(13.42)**

or

$$c_2 = \frac{c_1}{\mu} \ln R$$

From Equation (13.41)

$$v_0 = -\frac{c_1}{\mu} \ln kR + \frac{c_1}{\mu} \ln R$$

or

$$v_0 = \frac{c_1}{\mu} \ln \frac{R}{kR}$$

or

$$c_1 = \frac{v_0 \mu}{\ln(1/k)}$$

By substituting the value of c1 into Equation (13.39), the velocity profile may be obtained as

$$v_z = -\frac{v_0 \ln r}{\ln(1/k)} + \frac{v_0 \ln R}{\ln(1/k)}$$

or

$$\frac{v_z}{v_0} = \frac{ln(r / R)}{ln(1 / k)}$$

Mass flow rate in the annular region

$$w = \rho \int_{0}^{2\pi} \int_{kR}^{R} v_z r dr d\theta$$

or

$$w = \frac{2\pi\rho v_0}{ln \quad k} \int_{kR}^{R} r \, ln\left(\frac{r}{R}\right) dr$$

or

$$w = \frac{\pi R^2 \rho v_0}{2} \left[\frac{(1-k^2)}{ln(1/k)} - 2k^2 \right]$$

Drag force acting on the wire may be calculated as

$$F = 2\pi kRL + \left(-\tau_{rz}\right)\Big|_{r=kR}$$

or

$$F = 2\pi kRL \left(\mu \frac{dv_z}{dr}\bigg|_{r=kR} \right)$$

By substituting the value of velocity vz, we obtain

$$F = 2\pi kRL\mu \frac{d}{dr}\left[v_0 \frac{ln(r / R)}{ln(1 / k)} \right]\Bigg|_{r=kR}$$

Finally, we obtain the expression for drag force as

$$F = \frac{2\pi L \mu v_0}{ln(1/k)}$$

1.10 Flow of two immiscible fluids between two parallel plates

Two immiscible liquids are flowing in between two adjacent, parallel plates. Solve the problem for velocity profile and mass flow rate.

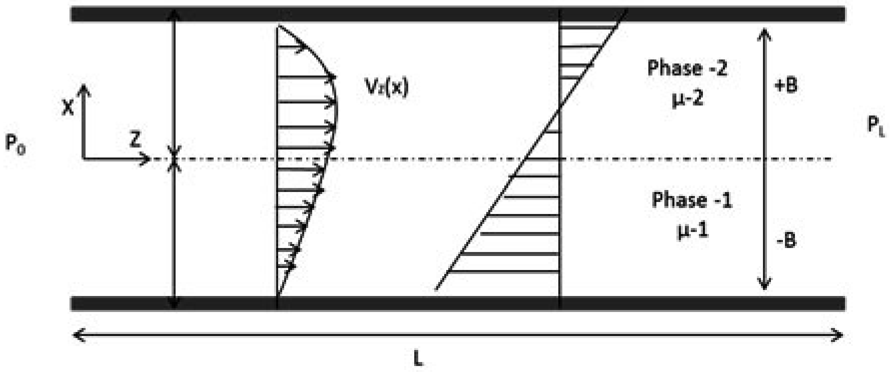


Fig 14.1 Flow of two immiscible fluids between a pair of horizontal plates

Assumptions

- Density and viscosity are constants.
- Steady state.
- Laminar (simple shear flow) fully developed.
- Newton’s law of viscosity is applicable.

Since fluid is flowing in z direction only, therefore vz is the only non-zero velocity component. We can assume that end effects are negligible in y direction and hence, vz is not a function of y. thus,

$$v_z = v_z(x, z)$$ (1

$$v_x = 0$$
$$v_y = 0$$ (

Now using equation of continuity for Cartesian coordinate system

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)=0$$

$$\rho\,\frac{dv_z}{dz}=0$$

which implies that

$$v_z = v_z(x)$$

As before, we may concluded that τ_{xz} is only important shear stress component for momentum balance in z direction. Choosing a differential strip of thickness dx and length L as a control volume, we have

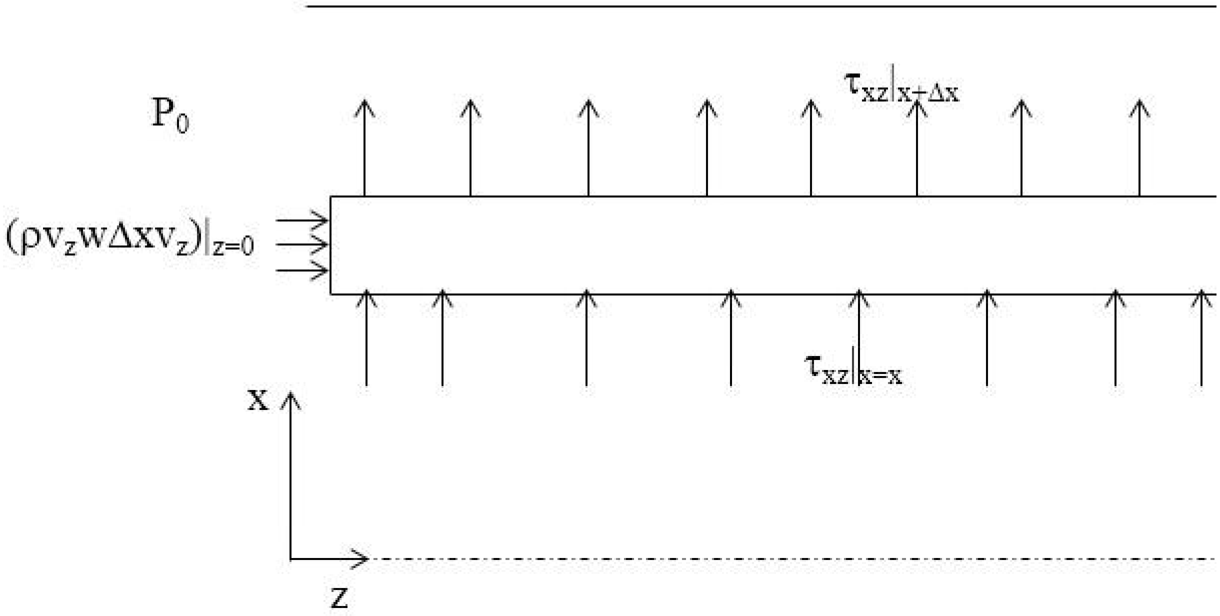


Fig 14.2 Control volume for Flow of two immiscible fluid between a pair of horizontal plates

Momentum balance in control volume

Convective momentum entering CV at z=0 is

$$(\rho v_z w\Delta x v_z)|_{z=0} \qquad \textbf{(14.5)}$$

Convective momentum leaving CV at z=0 is

$(\rho v_z w\Delta x v_z)\big|_{z=L}$ **(14.6)**

Momentum entering CV by viscous transport at x=x is $(LW\tau_{xz})\big|_{x=x}$ **(14.7)**

Momentum leaving CV by viscous transport at x=x+Δx is $(LW\tau_{xz})\big|_{x=x+\Delta x}$ **(14.8)**

Pressure force at z=0 is

$P_0\Delta x w$ **(14.9)**

Pressure force at z=L is

$-P_L\Delta x w$ **(14.10)**

The equation for momentum balance can be written as

$$(\rho v_z w\Delta x v_z)\big|_{z=0} - (\rho v_z w\Delta x v_z)\big|_{z=L} + (LW\tau_{rz})\big|_{x=x} - (LW\tau_{rz})\big|_{x=x+\Delta x} + P_0 w\Delta x - P_L$$

As before, convective terms cancel out and Equation (14.11) reduces to the following equation.

$$(LW\tau_{rz})\big|_{x=x} - (LW\tau_{rz})\big|_{x=x+\Delta x} + P_0 w\Delta x - P_L w\Delta x = 0$$

Dividing Equation (14.12) by volume of control volume ΔxLW, we obtain

$$\frac{(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x})}{\Delta x} = \frac{P_L - P_0}{L}$$

Now, as Δx →0 Equation (14.13) becomes

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{P_0 - P_L}{L}\right)$$

After substituting Newton’s law of viscosity in Equation (14.14) and integrating it, we obtain

$$v_z = -\left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1}{\mu}x + c_2$$

This equation is valid for both regions. Therefore,

$$v_z^{1} = -\left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1^{1}}{\mu}x + c_2^{1}$$

$$v_z^{2} = -\left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1^{2}}{\mu}x + c_2^{2}$$

Here, superscript (1) represents the phase-1 and superscript (2) represents the phase-2.

Boundary conditions

There are four boundary conditions needed to solve the problem and given below

1. $x=0,\ v_z^{1} = v_z^{2}$
2. $x=-b,\ v_z^{1} = 0$
3. $x=+b,\ v_z^{2} = 0$
4. $x=0,\ \frac{dv_z^{1}}{dx}\Big|_{x=0} = \frac{dv_z^{2}}{dx}\Big|_{x=0}$

This leads to the solution

$$c_1^{\,1} = c_1^{\,2}$$

$$c_2^{\,1} = c_2^{\,2}$$

$$c_1^{\,2} = \left(\frac{P_0 - P_L}{2\mu^1 L}\right) b^2 \left(\frac{2\mu^1}{\mu^1 + \mu^2}\right)$$

$$\tau_{xz} = \left(\frac{P_0 - P_L}{L}\right)\left[\frac{x}{b} - \frac{1}{2}\left(\frac{\mu^1 - \mu^2}{\mu^1 + \mu^2}\right)\right]$$

and

$$v_z^{\,1} = \left(\frac{P_0 - P_L}{2\mu^1 L}\right) b^2 \left[\left(\frac{2\mu^1}{\mu^1 + \mu^2}\right) + \left(\frac{\mu^1 - \mu^2}{\mu^1 + \mu^2}\right)\frac{x}{b} - \left(\frac{x}{b}\right)^2\right]$$

$$v_z^{\,2} = \left(\frac{P_0 - P_L}{2\mu^1 L}\right) b^2 \left[\left(\frac{2\mu^2}{\mu^1 + \mu^2}\right) + \left(\frac{\mu^1 - \mu^2}{\mu^1 + \mu^2}\right)\frac{x}{b} - \left(\frac{x}{b}\right)^2\right]$$

1.11 Derivation of equation of motion

In this section, we derive the equation of motion, which may be used for solving any fluid mechanics problem. This equation is based on axiom 2, i.e., the momentum is conserved. We consider a control volume having volume Δx,Δy,Δz fixed in space.

According to the momentum conservation equation,

$$\begin{aligned}\begin{pmatrix}\text{Rate of accumulation of}\\ \text{momentum in control volume}\end{pmatrix} &= \begin{pmatrix}\text{Net rate of inflow of}\\ \text{momentum by convectio}\end{pmatrix}\\ &+ \begin{pmatrix}\text{Net rate of inflow of}\\ \text{by viscous transpor}\end{pmatrix}\\ &+ (\text{Pressure forces}) + (\text{For}\end{aligned}$$

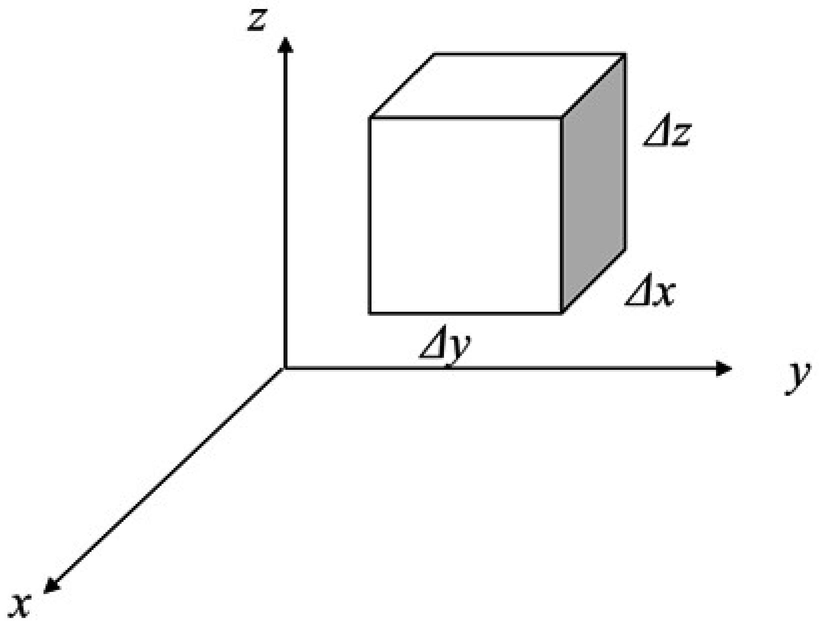


Fig 15.1 Cubical control volume fixed in space
Momentum balance in x direction

Rate of accumulation of x directed momentum in control volume

$$= \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z v_x)$$

Net rate of inflow of x directed momentum into CV by convection from x-phases

$$\left[(\rho v_x \Delta z \Delta y) v_x\right]_x - \left[(\rho v_x \Delta z \Delta y) v_x\right]_{x+\Delta x}$$

Net rate of inflow of x-momentum into CV from y-phases

$$= \left[(\rho v_y \Delta x \Delta z) v_x\right]_y - \left[(\rho v_y \Delta x \Delta z) v_x\right]_{y+\Delta y}$$

Net rate of inflow of x-momentum into CV from z-phases

$$= \left[(\rho v_z \Delta x \Delta y) v_x\right]_z - \left[(\rho v_z \Delta x \Delta y) v_x\right]_{z+\Delta z}$$

Net rate of inflow of momentum into CV due to viscous transport

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

In all shear stress component, the second index shows the direction of momentum flux and first index shows the direction in which the momentum is flowing. For example, τ_{yx} denotes the x directed momentum flowing in y direction. Therefore, the x directed momentum fluxes are τ_{xx}, τ_{yx} and τ_{zx}. Thus,

Net rate of inflow of x directed momentum by viscous transport from x phase are

$$= (\tau_{xx} \Delta z \Delta y)|_x - (\tau_{xx} \Delta z \Delta y)|_{x+\Delta x}$$

Net rate of inflow of x directed momentum by viscous transport from y phase are

$$= (\tau_{yx} \Delta x \Delta z)|_y - (\tau_{yx} \Delta x \Delta z)|_{y+\Delta y}$$

Net rate of inflow of x directed momentum by viscous transport from z phase are

$$= (\tau_{zx} \Delta x \Delta y)|_z - (\tau_{zx} \Delta x \Delta y)|_{z+\Delta z}$$

Net pressure force in x direction =

$$(P \Delta y \Delta z)|_x - (P \Delta y \Delta z)|_{x+\Delta x} \qquad \textbf{(15.9)}$$

Gravity force in x direction =

$$(\rho \Delta x \Delta y \Delta z)\, g_x \qquad \textbf{(15.10)}$$

Adding all the above terms and dividing by the volume of control volume Δx,Δy,Δz and finally taking the limits,
Δx→0, Δy→0,and Δz→0, we obtain

$$\frac{\partial(\rho v_x)}{\partial t} = -\left[\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right] - \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] - \frac{\partial p}{\partial x} + \rho g$$

The above equation is the x component of equation of motion in cartesian coordinate system. Similarly, for y-direction

$$\frac{\partial(\rho v_y)}{\partial t} = -\left[\frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z}\right] - \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] - \frac{\partial p}{\partial y}$$

and for z-direction

$$\frac{\partial(\rho v_z)}{\partial t} = -\left[\frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z}\right] - \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] - \frac{\partial}{\partial}$$

The above three equation may be combined in vector tensor form as

$$\frac{\partial(\rho \underset{\sim}{v})}{\partial t} = -\underset{\sim}{\nabla}.(\rho \underset{\sim}{v}\underset{\sim}{v}) - \underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

In above form, the equation of motion may be used in any coordinate system.

Equation (15.14) may be written in substantial derivative form as shown below

$$\frac{\partial(\rho \underset{\sim}{v})}{\partial t} + \underset{\sim}{\nabla}.(\rho \underset{\sim}{v}\underset{\sim}{v}) = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

if $\underset{\sim}{x}$ and $\underset{\sim}{z}$ are the two vectors. We may use the following vector identity.

$$\nabla.(\underset{\sim}{x}\underset{\sim}{z}) = \underset{\sim}{x}.\underset{\sim}{\nabla}\underset{\sim}{z} + \underset{\sim}{z}(\underset{\sim}{\nabla}.\underset{\sim}{x})$$

Now, replace $\underset{\sim}{x}$ by $\rho \underset{\sim}{v}$ and $\underset{\sim}{z}$ by $\underset{\sim}{v}$, then we have

$$\nabla.(\underset{\sim}{x}\underset{\sim}{z}) = \nabla.(\rho \underset{\sim}{v}\underset{\sim}{v}) = \rho \underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v} + \underset{\sim}{v}(\underset{\sim}{\nabla}.\rho \underset{\sim}{v})$$

also,

$$\frac{\partial(\rho\underset{\sim}{v})}{\partial t} = \rho\frac{\partial\underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\frac{\partial\rho}{\partial t}$$

(15.

After substituting Equations (15.17) and (15.18) in Equation (15.14), the equation of motion reduces to

$$\underset{\sim}{v}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\underset{\sim}{v}}{\partial t} + \rho\underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v} + \underset{\sim}{v}(\underset{\sim}{\nabla}.\rho\underset{\sim}{v}) = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla}p + \rho\underset{\sim}{g}$$

Rearranging the terms on the left hand side, we have

$$\rho\left[\frac{\partial\underset{\sim}{v}}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v}\right] + \underset{\sim}{v}\left[\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}.\rho\underset{\sim}{v}\right] = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla}p + \rho\underset{\sim}{g}$$

But from the equation of continuity

$$\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}.\rho\underset{\sim}{v} = 0$$

Hence,

$$\rho\left[\frac{\partial\underset{\sim}{v}}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v}\right] = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla}p + \rho\underset{\sim}{g}$$

or

$$\rho\frac{D\underset{\sim}{v}}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla}p + \rho\underset{\sim}{g}$$

Equations (15.20) and (15.21) are the generalized form of equation of motion without any assumption and may be applied to any coordinate system. The detailed form of this equation in cartesian, cylindrical and spherical coordinate system is given in Appendix-3.

Navier Stokes Equation for incompressible Newtonian fluid

The equation of motion may be further simplified by substituting the Newton's law of viscosity for the momentum flux term appearing in the equation of motion.

For a one-dimensional system where

vy=0,vz=0.. (16.1)

we have seen that the Newton's law of viscosity may be written as,

$$\tau_{yx} = -\mu \frac{\partial v_x}{\partial y}$$

where, τ_{yx} represents x directed momentum flowing in the y direction However, in general, for a three dimensional flow, all 9 components of shear stress may be important. Thus,

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Here, τ_{xx} , τ_{yy} and τ_{zz} are the normal stresses and the remaining are shear stress.

Axiom 3: Moment of momentum is conserved

This axiom 3 leads to a very simple conclusion that the shear stress tensor is symmetric in nature. The derivative itself is lengthy and is not reproduced here. $\underset{\sim}{\tau}$ is symmetric implies that

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

Newton’s law of viscosity may now be generalized as given below. Again, the basis for this representation is not shown here, but it may be found in any standard books in fluid mechanics.

$$\underset{\sim}{\tau} = -\mu\underset{\sim}{\Delta} + \frac{2}{3}\mu(\underset{\sim}{\nabla}.\underset{\sim}{v})\delta_{ij}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_j$$

where,

$$\underset{\sim}{\Delta} = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j$$

Hence, we have the nine components of shrear stress as

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3} \mu \underset{\sim}{\nabla} . \underset{\sim}{v}$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3} \mu \underset{\sim}{\nabla} . \underset{\sim}{v}$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3} \mu \underset{\sim}{\nabla} . \underset{\sim}{v}$$

$$\tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = -\mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)$$

$$\tau_{xz} = \tau_{zx} = -\mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

The detail form of Newtons law of viscosity in all coordinate system is given in Appendix- 01. Now, consider the situation when an incompressible fluid is flowing only in x direction and depends on y coordinate only. In such a case, we have $v_x = v_x(y)$, $v_y = 0$ and $v_z = 0$. We can easily see that for this case,

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = 0$$

$$\tau_{yz} = \tau_{zy} = 0$$

$$\tau_{xz} = \tau_{zx} = 0$$

and only significant components of stress are τ_{xy} and τ_{yx}. Also, the expression for τ_{yx} is the same as given earlier as Newton's law of viscosity. For rectangular coordinate system, substituting the value of $-\underset{\sim}{\nabla}.\underset{\sim}{\tau}$ in the x component of equation of motion, we obtain

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_x = -\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$$

Assuming that ρ and μ are constant, we obtain

$$-\underset{\sim}{\nabla}.\underset{\sim}{\tau} = -\left[\frac{\partial}{\partial x}\left(-2\mu\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(-\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right)\right]$$

or

$$= \mu\left[\frac{\partial}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial}{\partial y}\frac{\partial v_x}{\partial y} + \frac{\partial}{\partial y}\frac{\partial v_y}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial}{\partial z}\frac{\partial v_z}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_x}{\partial x}\right]$$

or

$$= \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial}{\partial x}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]\right]$$

But from equation of continuity for an incompressible fluid, we have

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Therefore,

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_x = \mu\left[\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right]$$

$$= \mu\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]v_x$$

or

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_x = \mu\underset{\sim}{\nabla}^2 v_x$$

or

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_y = \mu\underset{\sim}{\nabla}^2 v_y$$

similarly

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)_z = \mu\underset{\sim}{\nabla}^2 v_z$$

Therefore, in vector and tensor form

$$-\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right) = \mu\underset{\sim}{\nabla}^2 \underset{\sim}{v}$$

Thus, the equation of motion reduce to

$$\rho\frac{D\underset{\sim}{v}}{Dt} = -\underset{\sim}{\nabla}p + \mu\underset{\sim}{\nabla}^2\underset{\sim}{v} + \rho\underset{\sim}{g}$$

Equation (16.25) is known as the Navier Stokes equation and is used for solving problems involving Newtonian fluids of constant density and viscosity. For non-Newtonian and compressible fluids, the generalized form of equation of motion given earlier must be used. The detailed forms of the equations of motion along with Navier Stokes equations in cartesian, cylindrical and spherical coordinates are given in the Appendix-03.

Solution of momentum transport problems using Navier Stokes equation

In this section, transport problems involving Newtonian fluids are solved by making use of the equation of motion or Navier Stokes equation. We will firstly solve the falling film problem and flow through a circular tube for comparing the solutions obtained earlier by using the shell momentum balance method. We will then proceed to solve some more fluid mechanics problems.

1.12 Falling film on an inclined surface

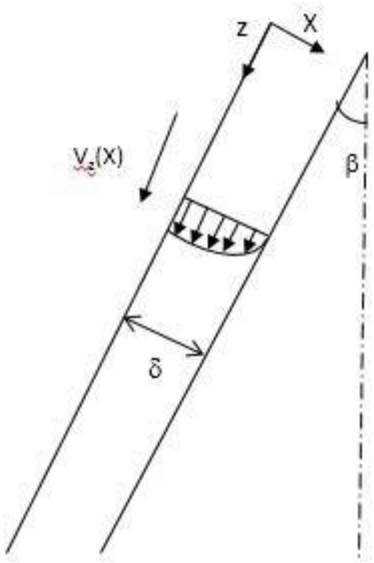


Fig 17.1 Falling film on inclined surface

This problem was solved earlier by the shell momentum balance technique. We will now try to solve this problem by using the Navier Stokes equations.

We are again required to make the same necessary assumptions as done earlier using the shell momentum balance technique. We postulate the non- zero components of the velocity and from there, determine the non-zero components of the shear stress tensor. These steps are the same as earlier and lead us to conclude that $v_z = v_z(r)$ and τ_{rz} is the only important component of shear stress. We now use the Navier Stokes equation in cartesian coordinates as given in Appendix-03.

x component is

$$\rho g_x = 0 \qquad \textbf{(17.1)}$$

y component is

$$\rho g_y = 0 \qquad \textbf{(17.2)}$$

z component is

$$\mu \frac{d^2 v_z}{dx^2} + \rho g_z = 0$$

where

$$g_z = g\,cos\beta$$

Integrating Equation (17.3), we have

$$\frac{dv_z}{dx} = -(g\cos\beta)x + c_1$$

and

$$v_z = -(g\cos\beta)x + c_1 x + c_2$$

The boundary conditions are also the same as used earlier,

at

$$x = 0,\ \left.\frac{dv_z}{dx}\right|_{x=0} = 0$$

and

at

$$x = \delta,\ v_z = 0$$

This leads to the solution for velocity profile, as

$$v_z = -\frac{g\delta^2\cos\beta}{2\mu}\left[1-\left(\frac{x}{\delta}\right)^2\right]$$

which is same as obtain earlier using shell momentum balance approach.

Fluid flow through a vertical tube

A Newtonian fluid is flowing inside a vertical tube having circular cross section due to pressure difference and gravity. Solve the problem using the Navier Stokes equations.

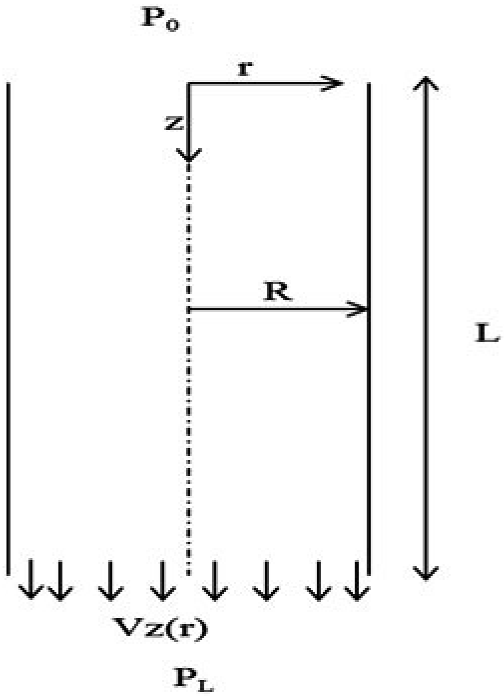


Fig 17.2 Flow through a vertical circular tube

A similar type of problem (for a horizontal pipe) was solved earlier using the shell momentum balance technique. Therefore, the initial steps are the same and include making appropriate assumptions and postulating the non- zero velocity components. As shown earlier, it leads to the conclusion that $v_z = v_z(r)$

Now using the Navier Stokes equation for cylindrical co-ordinates, after eliminating all zero terms, we have r- component of Navier Stokes equation

$$\frac{\partial P}{\partial r} = 0$$

θ- component

$$\frac{\partial P}{\partial \theta} = 0$$

z - component

$$-\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z = 0$$

We can combine gravity and pressure forces as to rewrite Equation (17.11) as,

$$-\frac{\partial P_c}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=0$$

where, Pc is the effective pressure including the gravity, and is defined as,

$$P_c = P - \rho g_z z$$

Note that since pressure changes in only z direction and vz is a function of r only the partial derivative may be converted to total derivative. Furthermore, in Equation (17.12), the first term is only a function of z and the second term is only a function of r, i.e.,

$$F_1(z) + F_2(r) = 0$$

This leads to result that F1 and F2 both are constants as Equation (17.13) is true for all values of z and r.

$$F_1(z) = c_1; F_2(r) = -c_1$$

Therefore,

$$\frac{dP_c}{dz} = c_1$$

By integrating the Equation (17.15)

$$P_c = c_1 z + c_2$$

Boundary conditions are

at

$$z=0, P_c = P_{c0}$$

and

at

$$z=L, P_c = P_{cL}$$

This leads to the following solution

$$\frac{P_{c0} - P_{cL}}{L} = c_1$$

By substituting in Equation (17.12)

$$-\left(\frac{P_{c0} - P_{cL}}{L}\right) + \mu \frac{1}{r} \frac{d}{dr}\left(r \frac{dv_z}{dr}\right) = 0$$

or

$$v_z = -\left(\frac{P_{c0} - P_{cL}}{4\mu L}\right) r^2 + c_3 \ln r + c_4$$

Boundary conditions are

at r= 0, v_z is finite

and

at r = R,
$v_z = 0$

This leads to

$$v_z = \left(\frac{P_{c0} - P_{cL}}{4\mu L}\right) R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

which is again similar to what we have seen for a horizontal tube except for pressure difference term. In fact, it can be shown that the velocity profile given in Equation (17.22) is valid for any configuration, horizontal, vertical, or inclined, with effective pressure is defined as

$$P_c = P - \rho g_z z$$

$$T_z = \int_0^{2\pi} \int_0^R (-\tau_{\theta\phi})\big|_{\theta=\pi/2} r dr r d\phi = 2\pi \left(\frac{\mu\Omega}{\psi}\right) \int_0^R r^2 dr$$

Radial flow between two parallel discs

A part of a lubrication system consists of two circular discs and the lubricant flows in the radial direction. The flow takes place because of modified pressure *(p_1- p_2)* between the inner and outer radii r_1 and r_2 respectively. Formulate the problem for velocity profile and mass flow rate through the system.

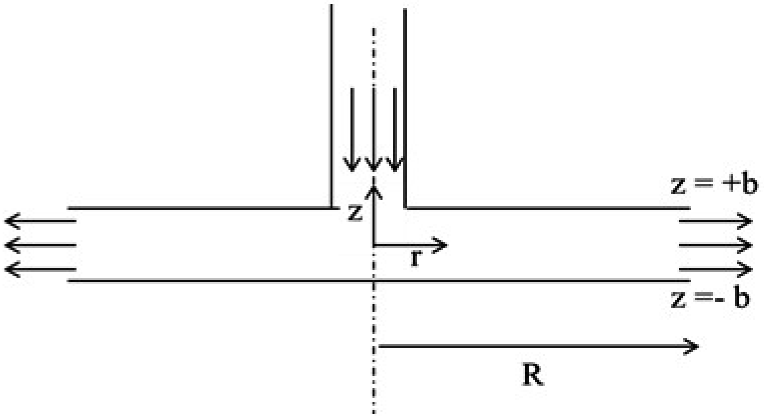


Fig 19.1 Radial flow in space between two parallel circular discs

Assumptions

- Density and viscosity are constant
- Steady state.
- Laminar flow (simple shear flow).
- Newton's law of viscosity is applicable.

Velocity profile

The fluid is flowing in the *r* direction. Hence, the only non-zero component of velocity is v_r and it depends on the both *r* and *z*. It will not depend on the *θ* coordinate due to cylindrical symmetry. i.e.,

$$v_r = v_r(r, z) \tag{19.1}$$

Applying the equation of continuity in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{19.2}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r\right) = 0 \qquad \textbf{(19.3)}$$

or

$$\frac{\rho}{r}\frac{\partial}{\partial r}\left(r v_r\right) = 0 \qquad \textbf{(19.4)}$$

Thus, $r v_r$ is a constant and which may be a function of the z,

$$r v_r = F\left(z\right) \qquad \textbf{(19.5)}$$

Using the *r*-component of the Navier–Stokes equation in cylindrical co-ordinate systems, we have

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)\right) + \frac{\partial^2 v_r}{\partial z^2}\right] \qquad \textbf{(19.6)}$$

By substituting Equation (19.5), we get

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu\frac{\partial^2 v_r}{\partial z^2} \qquad \textbf{(19.7)}$$

Equation (19.7) is a second order partial differential equation and may not solve analytically. However, we may obtain an analytical solution for the limiting case when the flow is very slow (also called a creeping flow). In such a scenario, we may neglect the convective term (on the left hand side) in Equation (19.7) and thus, we have

$$\frac{\partial P}{\partial r} = \mu\frac{\partial^2 v_r}{\partial z^2} \qquad \textbf{(19.8]}$$

Multiplying *r* on both sides, we have

$$r\frac{\partial P}{\partial r} = r\mu\frac{\partial^2 v_r}{\partial z^2}$$

or

$$r\frac{\partial P}{\partial r} = \mu\frac{\partial^2 F}{\partial z^2} \qquad \textbf{(19.9)}$$

In Equation (19.9), the left hand side is a function of *r* only, while the right hand side is a function of *z* only. Since this equation is valid for all possible values of *r* and *z*, both the terms should be equal to each other, and in turn equal to a constant, c_2 , independent of *r* and *z*. Therefore,

$$r\,\frac{dP}{dr} = c_2 \qquad \textbf{(19.10)}$$

or

$$\mu\,\frac{d^2F}{dz^2} = c_2 \qquad \textbf{(19.11)}$$

From Equation (19.10), we get

$$\int_{P_1}^{P_2} dP = c_2\int_{r_1}^{r_2}\frac{dr}{r} \qquad \textbf{(19.12)}$$

or

$$c_2 = \frac{P_2 - P_1}{ln\left(\dfrac{r_2}{r_1}\right)} \qquad \textbf{(19.13)}$$

Substituting Equation (19.13) into Equation (19.11), we find

$$\mu \frac{d^2 F}{dz^2} = -\frac{P_1 - P_2}{\mu \, ln\left(\frac{r_2}{r_1}\right)} \tag{19.14}$$

or

$$F = \frac{P_1 - P_2}{2\mu ln\left(\frac{r_2}{r_1}\right)} z^2 + c'_1 z + c'_2 \tag{19.15}$$

Boundary conditions

No-slip is valid at both the plates. Thus,

at

$$z = \pm b \, , v_r = 0$$

(1

By substituting these boundary condition*s* in Equation (19.15), we have

$$c'_2 = \frac{P_1 - P_2}{2\mu \, ln\left(\frac{r_2}{r_1}\right)} b^2 \tag{19.17}$$

At *z=0*, the velocity profile is symmetric. Therefore, this is the second required boundary condition for the problem

$$\left.\frac{\partial F}{\partial z}\right|_{z=0} = 0$$

(19.

This leads to the solution

$$c'_1 = 0$$

and

$$F = \frac{P_1 - P_2}{2\mu \ln\left(\frac{r_2}{r_1}\right)} b^2 \left[1 - \left(\frac{z}{b}\right)^2\right]$$ **(19.**

or

$$rv_r = \frac{P_1 - P_2}{2\mu \ln\left(\frac{r_2}{r_1}\right)} b^2 \left[1 - \left(\frac{z}{b}\right)^2\right]$$ **(19.**

Finally, we obtain the velocity profile

$$v_r = \frac{P_1 - P_2}{2r\mu \ln\left(\frac{r_2}{r_1}\right)} b^2 \left[1 - \left(\frac{z}{b}\right)^2\right]$$ **(19.**

The mass flow rate of at any *r* in the system must be the same (in fact that was the reason, why we got rv_r constant for a given in the first place). Select the surface at $r = r_2$ to obtain mass flow rate

$$w = \int_0^{2\pi} \int_{-b}^{+b} \rho v_r \big|_{r=r_2} \, r dz d\theta$$

or

$$w = \frac{4\pi (P_1 - P_2) b^2 \rho}{3\mu \ln\left(\frac{r_2}{r_1}\right)}$$ **(19.2**

Parallel – disc viscometer

A fluid is placed in a gap (of thickness B) between two parallel discs of radius *R*. The lower disc is kept stationary while the upper disc is made to rotate at a constant angular velocity Ω. Formulate the problem for determining the viscosity at low shear rates.

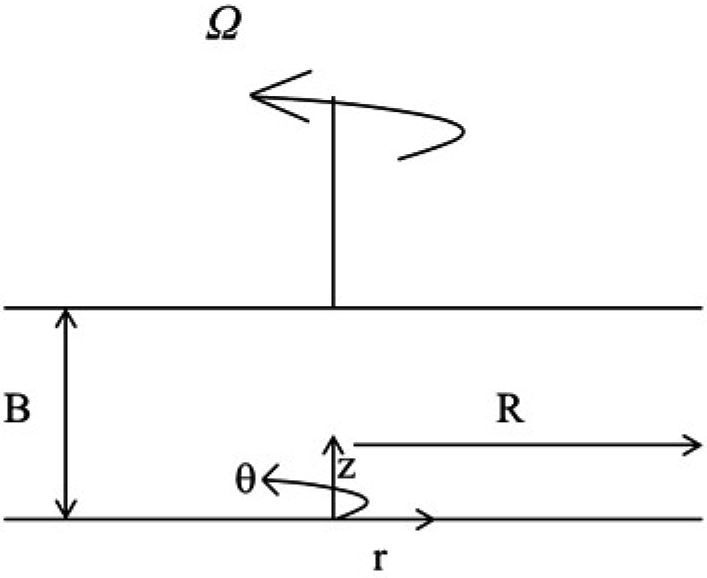


Fig 19.2 Front view of two-plate viscometer

Assumptions

- Density and viscosity are constant.
- Steady state.
- Laminar flow (simple shear flow).
- Newton's law of viscosity is applicable.

Velocity profile

The fluid is sheared in the θ direction; hence, v_θ is the non-zero component of velocity. Applying the equation of continuity in cylindrical coordinate, we obtain

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\rho r v_\theta)=0$$

(19.

Thus, v_θ, does not depend on the θ coordinate, or

$$v_\theta = v_\theta(r, z)$$ (19

For simplifying the problem further, we may assume that for low shear rates

$$v_\theta = r f(z)$$ (19

Using the θ component of the Navier – Stokes equation for cylindrical co-ordinate systems

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2} = 0$$ (19

By substituting Equation (19.25), we get

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 f(z)\right)\right) + \frac{r\partial^2 f(z)}{\partial z^2} = 0$$ (19.

or

$$\frac{\partial^2 f(z)}{\partial z^2} = 0$$

After integration, we finally obtain

$$f(z) = c_1 z + c_2$$ (1

where c_1 and c_2 are the integral constant.

Boundary conditions are

at $z = 0$, $v_\theta = 0$ or $f(z) = 0$

thus,

$$c_2 = 0$$

and at $z = B, v_\theta = v$ or

$$f(z) = v/r = \Omega$$ **(19**

This boundary condition leads to the solution

$$c_1 = \frac{\Omega}{B}$$ **(1**

and

$$f(z) = \frac{\Omega z}{B}$$ **(1**

Finally, we obtain the velocity profile

$$v_\theta = \frac{\Omega\, zr}{B}$$ **(1**

Now, the *z*-component of the torque exerted on the fluid by the upper rotating disc, may be calculated as

$$T_z = \int_0^{2\pi}\int_0^{R} (-\tau_{z\theta})\, r \big|_{z=B}\, r\,dr\,d\theta$$ **(19**

or

$$= 2\,\pi\mu\,\frac{\Omega}{B}\int_0^{R} r^3\,dr$$

Finally, we obtain the value of torque.

$$T_z = \pi\mu\frac{\Omega}{2B}R^2$$ **(**

Thus, by plotting the angular velocity Ω vs torque T_z, the viscosity may be

Non-Newtonian fluids

Non-Newtonian fluids are the fluids which do not obey Newton’s law of viscosity. For describing Non-Newtonian fluids, let’s recall the Newton's law of viscosity experiment. There are two long parallel plate situated at distance h to each other. Top plate is stationary and bottom plate is moving with velocity $\underset{\sim}{v}$ as shown in Fig. (20.1).

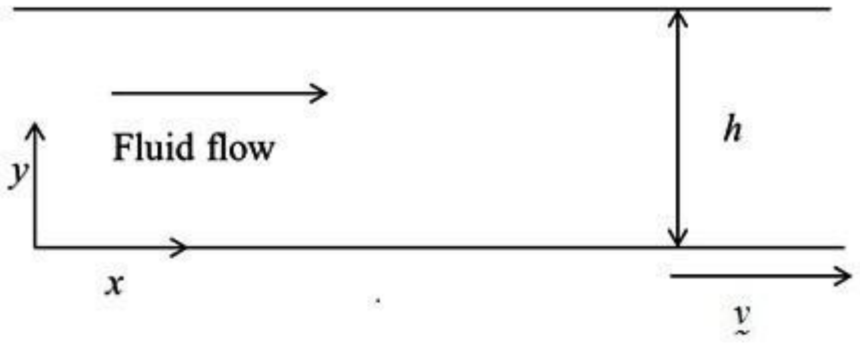


Fig 20.1 Non-Newtonian flow between two parallel plates

If a force, F, is applied to move plate, then (τ_{xy})

$$\tau_{xy} = \frac{F}{A}$$

and under steady state conditions when h is small and when

$$\frac{d v_x}{d y} = \frac{\underset{\sim}{v}}{h}$$

Now, we calculate τ_{xy} by repeating experiments for different applied forces and velocity achieved by the bottom plate and plotting a graph as shown in Fig. (20.2). Depending on the nature of fluid, different types of curves may be obtained.

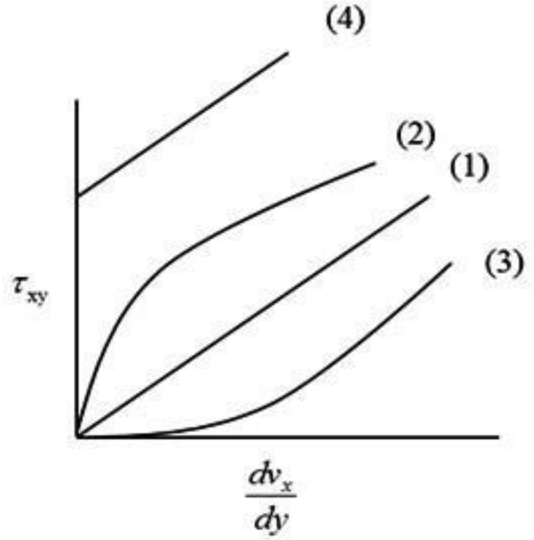


Fig 20.2 Shear stress vs. shear strain diagram for Newtonian and non-Newtonian fluids

1.13 Rheological behaviour of fluids

If fluid shows the behaviour like curve (1) then it is a Newtonian fluid. Other fluids are non-Newtonian fluids. Curve (2) represents a Pseudo-plastic fluid, curve (3) represents a Dilatant fluid, and curve (4) represents a Bingham plastic fluid. There are several Theoretical and empirical models available to describe the rheological behaviour of non-Newtonian fluids. Here, we discuss some of them, which come under the group of generalized Newtonian models. Basic equation for a generalized non-Newtonian fluid is given below

$$\tau_{yx} = -\eta \frac{dv_x}{dy}$$

Here, η is the apparent viscosity, which is clearly a function of shear rate as may be seen from Fig. (20.2). Therefore,

$$\eta = f\left(\frac{dv_x}{dy}\right) \qquad (20.3)$$

If the apparent viscosity increases with increase in shear rate, $\frac{dv_x}{dy}$, then the fluid is called Dilatant fluid and if it decreases with increase in shear rate, $\frac{dv_x}{dy}$ then fluid is called Pseudo-plastic fluid. Some fluids require a critical shear stress to initiate the flow. These fluids are called Bingham fluids. Some important rheological models for non-Newtonian fluids are given below.

1 Power Law or Ostwald De Waele model

Power law or Ostwald De Waele model is the most generalized model for non-Newtonian fluids. The expression of this model is given in Equation (20.3)

$$\tau_{yx} = -m\left|\left(\frac{dv_x}{dy}\right)\right|^{n-1}\frac{dv_x}{dy}$$

Here, apparent viscosity η is defined as,

$$\eta = m\left|\left(\frac{dv_x}{dy}\right)\right|^{n-1}$$

This is a two-parameter model where m and n are the two parameters.

If n = l then η =m
where m is similar to the viscosity of the fluid and model shows the Newtonian behaviour .
If n>1, then η increases with increasing shear rate and the model shows the Dilatant behaviour.

If n<1, then η decreases with increasing shear rate and the model shows the Pseudo-plastic behaviour.

Modulus sign

In power law model, modulus sign can be removed according to the value of shear rate.

1. If $\frac{dv_x}{dy}$ is positive, then

$$\eta = m\left(\frac{dv_x}{dy}\right)^{n-1}$$

2. If $\frac{dv_x}{dy}$ is negative, then

$$\eta = m\left(-\frac{dv_x}{dy}\right)^{n-1}$$

Several fluids do not show single type of rheological behaviour. They show Newtonian behaviour for a range of shear stress and Non-Newtonian behaviour for some other ranges of shear stresses. Several models have been suggested for these types of fluids. Some popular models like Eyring model, Ellis model, Reiner Philipp off model and Bingham Fluid model are discussed here.

2. Eyring model

Eyring model is a two-parameter model. The equation of Eyring model is as follow

$$\sinh\left(\frac{\tau_{yx}}{A}\right) = -\frac{1}{B}\frac{dv_x}{dy}$$

where A, B are the two parameters.

In Eyring model, if $\tau_{xy} \rightarrow 0$ which means very low shear forces, we have

$$\sinh\left\{\frac{\tau_{yx}}{A}\right\} \rightarrow \frac{\tau_{yx}}{A}$$

Therefore, as $\tau_{yx} \rightarrow 0$, the model shows Newtonian behaviour

$$\tau_{yx} = -\frac{A}{B}\frac{dv_x}{dy}$$

Here, viscosity = $\left(\frac{A}{B}\right)$

If τ_{yx} is very large, the model shows Non-Newtonian behaviour as shown Fig. 20.3

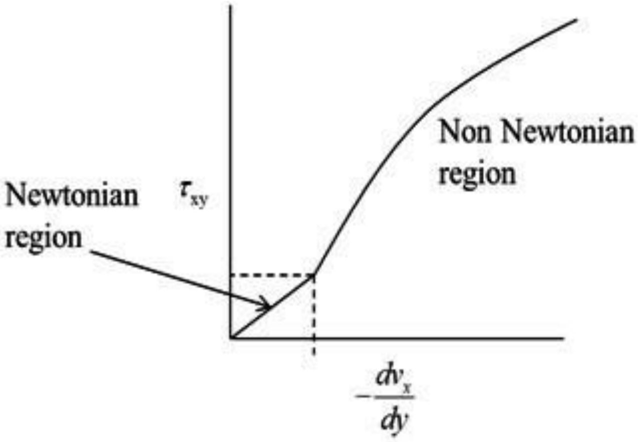


Fig 20.3 Shear stress vs. shear strain diagram for Eyring model

Therefore, Eyring model may be used for a fluid which shows Newtonian behaviour at low shear rates and non- Newtonian behaviour at high shear rates.

3. Ellis model

Ellis model is a three-parameter model. The equation of this model is as follows

$$-\frac{dv_x}{dy} = \left\{\varphi_o + \varphi_1 \left|\tau_{yx}\right|^{\alpha-1}\right\}\tau_{yx}$$

Here, φ_0, φ_1 and α are the three parameters .

Here, we consider some special cases,

1. If $\varphi_1 = 0$ then Equation (20.11) reduce to

$$\frac{dv_x}{dy} = -\varphi_0 \tau_{yx}$$

or

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

which is same as Newton's law of viscosity with $\left(\frac{1}{\varphi_0}\right)$ as the viscosity of the fluid.

2. If $\varphi_0 = 0$, then

$$-\frac{dv_x}{dy} = -\varphi_1 \, | \tau_{yx} |^{a-1} \, \tau_{yx}$$

which is similar to a Power law model

3. If α >1 and τ_{yx} is small then the second term is approximately zero and equation reduces to

$$\tau_{yx} = -\frac{1}{\varphi_0} \frac{dv_x}{dy}$$

which is similar to Newton's law of viscosity.

4. If α <1 and τ_{yx} is very large, then again, second term is negligible and we have

$$\tau_{yx} = -\frac{1}{\varphi_0} \frac{dv_x}{dy}$$

Which again shows Newtonian behaviour. Therefore, Ellis model may be used for fluids which show Newtonian behaviour at very low and very high shear stresses, but non-Newtonian behaviour at intermediate value of shear stresses.

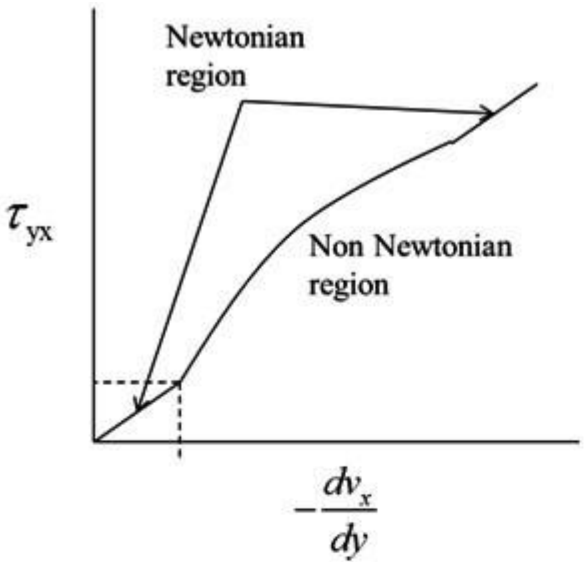


Fig 20.4 Shear stress vs. shear strain diagram for Ellis model

This type of behaviour may be shown by some polymer melts

4. Reiner Philipp off model

This is also a three-parameter model. The equation of Reiner Philipp off model is as follows,

$$-\frac{dv_x}{dy} = \left[\frac{1}{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1+\left(\frac{\tau_{yx}}{\tau_s}\right)^2}}\right]\tau_{yx}$$

where, μ_0 , μ_∞ and τ_s are the three parameters.

In Reiner Philipp off model, if τ_{yx} is very large, the equation reduces to,

$$-\frac{dv_x}{dy} = \frac{1}{\mu_\infty}\tau_{yx}$$

or

$$\tau_{yx} = -\mu_\infty \frac{dv_x}{dy}$$

which is same as the Newton’s law of viscosity,

If τ_{yx} is very small then equation reduces to

$$-\frac{dv_x}{dy} = \frac{1}{\mu_0}\tau_{yx}$$

or

$$\tau_{yx} = -\mu_0 \frac{dv_x}{dy}$$

which is also same as the Newton’s law of viscosity. Therefore, Reiner Philipp off model may be used for a fluid which shows Newtonian behaviour at very low and very high shear stresses but non-Newtonian behaviour for intermediate values of shear stress. Here, μ_0 and μ_∞ represent the viscosity of fluid at very low and very high shear stress conditions respectively.

5. Bingham Fluid model

Bingham fluid is special type of fluid which require a critical shear stress to start the flow.
The equation of Bingham fluid model are given below

$$\tau_{yx} = -\left[\mu + \frac{\tau_0}{\left|\frac{dv_x}{dy}\right|}\right]\frac{dv_x}{dy}$$

if

$$|\tau_{yx}| > \tau_0 \qquad \textbf{(20.19)}$$

$\frac{dv_x}{dy}=0$ if $\left|\tau_{yx}\right| \leq \tau_0$ or

$\eta = 0$ **(20.2**

A typical shear stress vs. shear rate diagram for a Binghum model is shown below

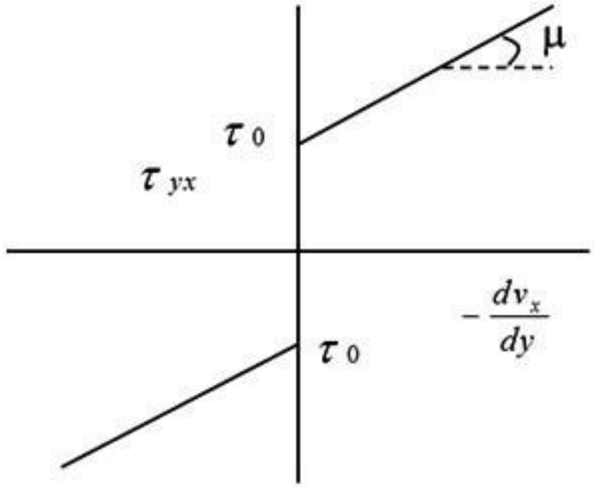


Fig 20.5 Shear stress vs. shear strain diagram for Bingham model

Momentum transport problems involving Power law and Bingham fluids:

In this section, we will solve fluid mechanics problem for Power law and Bingham plastic fluids. These problems have been earlier solved for Newtonian fluids. We have chosen the same problems here for better understanding.

Falling film on inclined plane

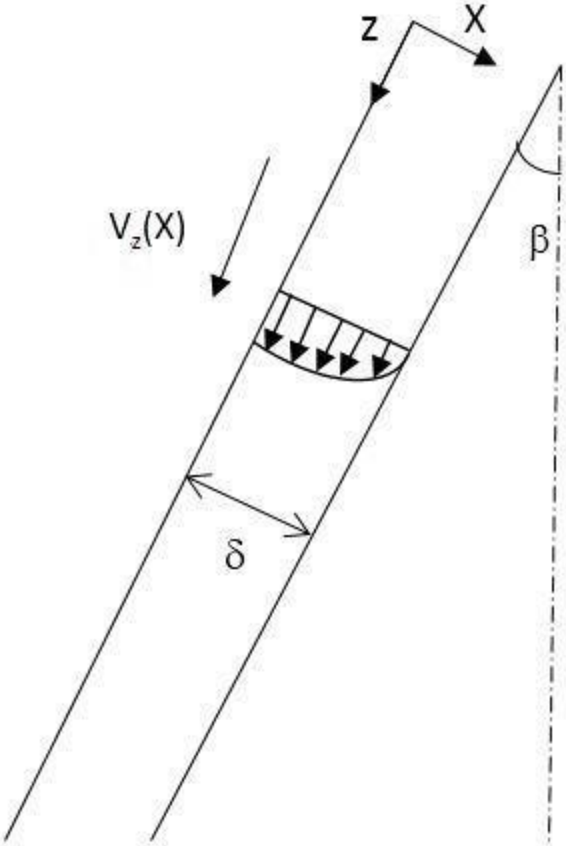


Fig 21.1 Falling film problem for non-Newtonian fluid

Initial steps, such as making appropriate assumptions, finding important velocity components, applying equation of continuity, and determining important shear stress components are similar as steps seen for Newtonian fluid in lecture 11 and 17. As before $v_z = v_z(x)$ and is the only non-zero velocity component and τ_{xz} is the only important shear stress component.

(Note: Since the forms of shear stress $\underset{\sim}{\tau}$ for Newtonian and non-Newtonian fluids are same, the only difference is the viscosity μ for Newtonian fluids and apparent viscosity η for non-Newtonian fluids and furthermore as non-zero components of velocities are also same, the same components of shear stress $\underset{\sim}{\tau}$ are significant for both Newtonian and non- Newtonian fluids.)

To solve the problem, we start with the generalized equation of motion in terms of $\underset{\sim}{\tau}$. Since the fluid is moving in z direction, discarding all terms which are zero, z-component of the equation of motion reduces to

$$-\frac{d\tau_{xz}}{dx}+\rho g_z=0$$

where

$$g_z = g\ cos\beta$$

therefore,

$$\tau_{xz}=\rho g\cos\beta x+c_1$$

For Power law fluids

$$\tau_{xz}=-\eta\frac{dv_z}{dx}$$

$$\eta=m\left|\frac{dv_z}{dx}\right|^{n-1}$$

Since vz is decreasing with increasing value of x , the negative sign should be used for removing the modulus sign, i.e. ,

$$\tau_{xz}=-m\left(-\frac{dv_z}{dx}\right)^{n-1}\left(\frac{dv_z}{dx}\right)$$

or

$$\tau_{xz}=m\left(-\frac{dv_z}{dx}\right)^{n}$$

By substituting Equation (21.7) in Equation. (21.1), we obtain

$$\left(-\frac{dv_z}{dx}\right)^n = \frac{\rho g \cos\beta}{m}x + c_1$$

By applying the boundary condition, at $x = 0, \quad \tau_{xz}\big|_{air} = \tau_{xz}\big|_{fluid}$ which simplifies to

$$\left.\frac{dv_z}{dx}\right|_{x=0} = 0$$

as disused in lecture 11

By substituting this boundary condition in Equation (21.8), we get $c_1 = 0$. Therefore,

$$\left(-\frac{dv_z}{dx}\right)^n = \frac{\rho g \cos\beta}{m}x$$

or

$$v_z = -\sqrt[n]{\frac{\rho g \cos\beta}{m}}\,\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c_2$$

Here, c_2 is another integral constant.

Now, using the second boundary condition, at $x = \delta \;, v_z = 0$, we finally obtain

$$v_z = \frac{n}{n+1}\sqrt[n]{\frac{\rho g \cos\beta}{m}}\,\delta^{\frac{n+1}{n}}\left[1-\left(\frac{x}{\delta}\right)^{\frac{n+1}{n}}\right]$$

Tube flow problem for Power law fluid

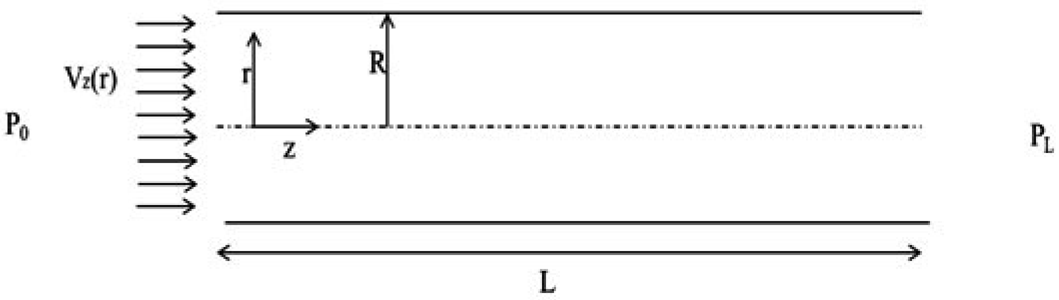


Fig 21.1 Flow through pipe for non-Newtonian fluid

As we discussed in lecture 10, the only non-zero component of velocity is vz, which depends on r only. The important component of shear stress is τ_{rz}.

By applying general equation of motion in cylindrical co-ordinate, we get

$$\frac{\partial P}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})=0$$

Equation (21.11) may be further simplified as before

$$\frac{P_0-P_L}{L}-\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})=0$$

or

$$\tau_{rz}=\frac{(P_0-P_L)}{2L}r+\frac{c_1}{r}$$

By applying the boundary condition, at r=0, velocity is finite, we obtain

$$c_1 = 0$$

and for power law fluids

$$\tau_{rz}=m\left(-\frac{dv_z}{dr}\right)^n$$

(Note: Since vz is decreasing with increasing value of r, the negative sign should be used for removing the modulus sign.)

By substituting Equation (21.13) to Equation (21.12), we get

$$m\left(-\frac{dv_z}{dr}\right)^n = \frac{(P_0-P_L)}{2L}r$$

Integrating above equation, we obtain

$$v_z = -\sqrt[n]{\frac{(P_0-P_L)}{2Lm}}\,\frac{r^{\frac{n+1}{n}}}{\frac{n+1}{n}} + c_2$$

Now, by applying the no-slip boundary condition at, $r = R, v_z = 0$, we obtain

$$c_2 = -\sqrt[n]{\frac{(P_0-P_L)}{2Lm}}\,\frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}}$$

Thus,

$$v_z = \sqrt[n]{\frac{(P_0-P_L)}{2Lm}}\,\frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}}\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

Equation (21.15) represents the velocity profile of freely falling film on an inclined surface for a Power law non-Newtonian fluid. If we substitute the n=1 and m=μ in this expression, we get Equation (10.25) which was derived earlier for a Newtonian fluid.

Tube Flow Problem for a Bingham Fluid

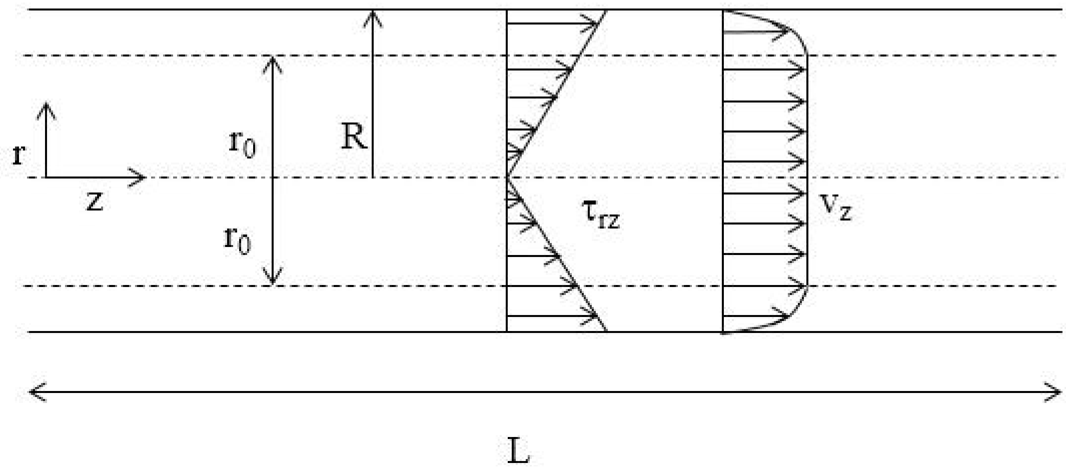


Fig 22.1 Flow through pipe for Bingham fluid

As mentioned in the previous lecture, the forms of shear stress $\underset{\sim}{\tau}$ for Newtonian and non-Newtonian fluids are the same. Therefore, Equation (21.12) is applicable for a Bingham fluids also, i.e.,

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Equations (20.19) and (20.20) may be written for this system

1. For $\tau_{rz} < \tau_0$ ($r < r_0$), where r_0 is to be determine latter,

$$\frac{dv_z}{dr} = 0$$

, or

$$v_z = v_0 \text{ (constant)}$$

2. For $\tau_{rz} \geq \tau_0$ ($r \geq r_0$)

$$\eta = \mu_0 + \frac{\tau_0}{\left|\frac{dv_z}{dr}\right|}$$

In Equation (22.2), $\frac{dv_z}{dr}$ is negative. Therefore, after removing the modulus sign, we obtain

$$\eta = \mu_0 - \frac{\tau_0}{\left(\frac{dv_z}{dr}\right)}$$

Thus,

$$\tau_{rz} = -\left\{ \mu - \frac{\tau_0}{\frac{dv_z}{dr}} \right\} \frac{dv_z}{dr}$$

or

$$\tau_{rz} = -\mu \frac{dv_z}{dr} + \tau_0$$

Condition for movement of fluid

As we start to pressurize the fluid by imposing pressure difference , fluid does not move initially. As we continue to increase the pressure difference the fluid may start to move at some critical pressure difference ($P_{c0} - P_{cL}$). This critical value may be determined by setting $\tau_{rz}\big|_{r=R} = \tau_0$. Thus,

$$\tau_0 = \frac{(P_{c0} - P_{cL})}{2L} R$$

Thus, the fluid will flow if

$$\tau_0 \le \frac{(P_{c0} - P_{cL})}{2L} R$$

Suppose the pressure difference across the tube exceeds this critical value of pressure ($P_{c0} - P_{cL}$) then the fluid will start to flow. Now, under this condition we may calculate the value of (r0) where the value of $\tau_{rz} = \tau_0$. For r<r0, the velocity gradient is zero and the

fluid flows with a constant velocity. The detail calculation for two different regions r<r0 and r>r0 are given below.

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

At $r = r_0, \tau_{rz} = \tau_0$ .Thus,

$$\tau_0 = \frac{(P_0 - P_L)}{2L} r_0$$

or

$$r_0 = \frac{2\tau_0 L}{(P_0 - P_L)}$$

For r<r0, we equate Equations (21.12) and (22.4), that is

$$-\mu \frac{dv_z}{dr} + \tau_0 = \tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Finally, we obtain,

$$v_z = \frac{\tau_0}{\mu} r - \left(\frac{(P_0 - P_L)}{4\mu L} r^2 \right) + c_1$$

No slip Boundary condition at r=R , $v_z = 0$ may be used to calculate c1 as shown below

Substituting this value in Equation (22.11), we get

$$c_1 = \left(\frac{(P_0 - P_L)}{4\mu L} R^2 \right) - \frac{\tau_0}{\mu} R$$

Finally, the velocity profile is given by

$$v_z = \frac{\tau_0}{\mu}(r-R) + \left(\frac{(P_0-P_L)}{4\mu L}R^2\right)\left(1-\left(\frac{r}{R}\right)^2\right)$$

Equation (22.12) gives the velocity profile is region $r_0 \le r \le R$. as shown in Fig. 22.2. Equation (22.9) shows that as we keep increasing the pressure difference (P_0-P_L), the value of r0 keep on decreasing and the velocity profile changes as shown in Fig. 22.2.

The value of r0 also depends on τ_0 , and reduces with it. If we substitute $\tau_0 = 0$ in Equation (22.12), we obtain the same expression for velocity profile as we had earlier obtain for Newtonian fluids. This result implies that if the value of pressure difference (P_0-P_L) is significantly high then the Bingham fluid may show behaviour similar to Newtonian fluids.

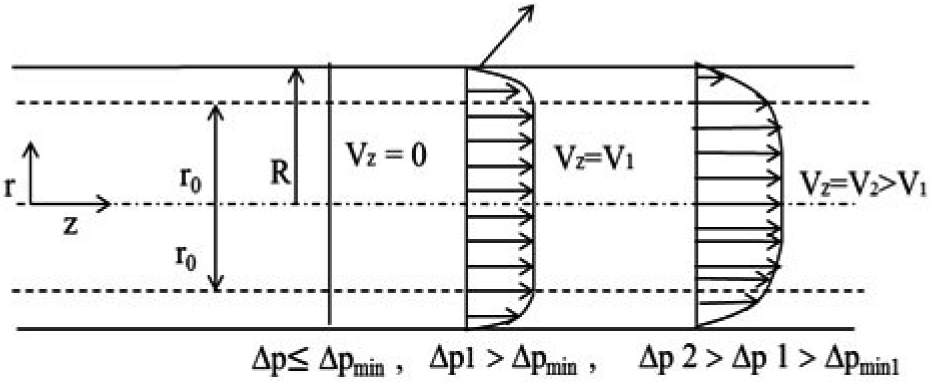


Fig 22.2 Effect of differential pressure flow through pipe for Bingham fluid

Now, we may determine the velocity profile in the plug flow region (r>r0) by substituting r= r0 in Equation (22.12)

$$v_0 = \frac{\tau_0}{\mu}(r_0-R) + \left(\frac{(P_0-P_L)}{4\mu L}R^2\right)\left(1-\left(\frac{r_0}{R}\right)^2\right)$$

Falling film problem for Bingham fluid

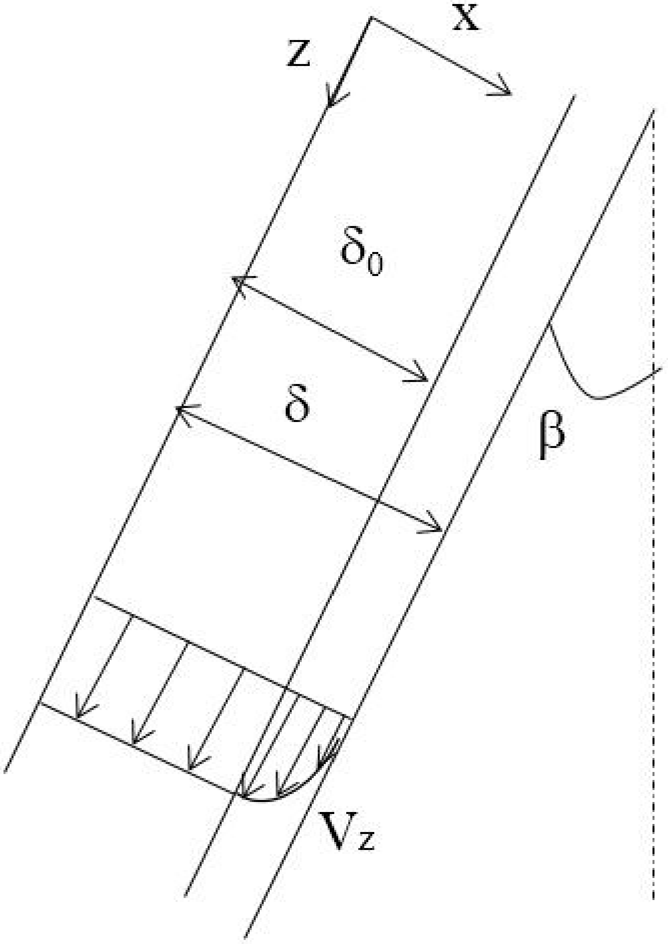


Fig 22.3 Flow on inclined surface for Bingham fluid

As we discussed earlier, the expression of shear stress is same, as we had derived for Newtonian fluids and Power law fluids in lecture 11 and lecture 21. Therefore, from Equation (21.3)

$$\tau_{xz} = \rho g x \cos \beta$$

For this system, Bingham fluid model may be written as,

1. For $\tau < \tau_0$, $x < \delta_0$

$$\frac{dv_z}{dx} = 0$$

2. For $\tau \geq \tau_0$, $x \geq \delta_0$

$$\eta = \mu_0 \pm \frac{\tau_0}{\left|\frac{dv_z}{dx}\right|}$$

As the critical thickness of film δ_0 is unknown, (the fluid flows only when $\delta \geq \delta_0$) we may calculated from Equation (22.17), i.e.,

$\tau_{xz} = \tau_0$ at $x = \delta_0$ or,

$$\tau_0 = \rho g \delta_0 \cos \beta$$

or

$$\delta_0 = \frac{\tau_0}{\rho g \cos \beta}$$

From region (1) where $\tau < \tau_0$ and $x < \delta_0$, we have

$$\frac{dv_z}{dx} = 0$$

or

$$v_z = v_0$$

For region (2) where $\tau \geq \tau_0$ and $\delta \geq \delta_0$, we have

$$\eta = \left\{ \mu \pm \frac{\tau_0}{\left|\frac{dv_z}{dx}\right|} \right\}$$

Here $\frac{dv_z}{dx}$ is negative. Therefore, after removing the modulus sign and substituting the value of η in Generalized Newton’s law of viscosity. we obtain,

$$\tau_{xz} = -\left\{\mu \frac{dv_z}{dx} - \tau_0\right\} = \rho g \cos \beta x$$

or

$$-\mu \frac{dv_z}{dx} = \rho g \cos \beta x - \tau_0$$

or

$$\frac{dv_z}{dx} = \frac{-\rho g \cos \beta x}{\mu} + \frac{\tau_0}{\mu}$$

Finally, we obtain the velocity profile, as given below

$$v_z = \frac{-\rho g \cos \beta}{\mu} \frac{x^2}{2} + \frac{\tau_0}{\mu} x + c_2$$

where c2 is an integral constant. By using no slip boundary condition at $x = \delta$, $v_z = 0$, we obtain

$$c_2 = \frac{\rho g \cos \beta}{\mu} \frac{\delta^2}{2} - \frac{\tau_0}{\mu} \delta$$

Therefore,

$$v_z = \frac{\rho g \cos \beta \delta^2}{2\mu}\left[1 - \frac{x^2}{\delta^2}\right] - \frac{\tau_0 \delta}{\mu}\left[1 - \frac{x}{\delta}\right]$$

Equation (22.22) shows the velocity profile in region $\delta_0 \le x \le \delta$. From Equation(22.22), we may also calculate the velocity of plug flow region by substituting the value $x = \delta_0$. Thus,

$$v_0 = \frac{\rho g \cos\beta \delta^2}{2\mu}\left[1 - \frac{\delta_0^2}{\delta^2}\right] - \frac{\tau_0 \delta}{\mu}\left[1 - \frac{\delta_0}{\delta}\right]$$

$$\tau_{r\theta} = \frac{c_1}{r^2}$$

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SCHOOL OF BIO AND CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – II – TRANSPORT PHENOMENA – SCH1309

I. Introduction

CONTENTS

2.1 Introduction to Heat Transfer

In heat transfer, we deal with transfer of thermal energy or heat which takes place between different bodies/fluids. Here, we start with Axiom-4 of transport phenomena.

This axiom is similar to the first law of thermodynamics. It states that "energy is conserved" which means energy can not be created or destroyed. Energy may be transferred from one form to another or one place to another. Transfer of energy in a system depends how it interacts with the surrounding. Here, the system is defined as the region of an equipment / unit which is under investigation. The remainder of everything else is called the surrounding which is outside the boundaries of the system. The system may be classified in three types, based on how the system is interacting with the surrounding in terms of heat, work, and mass exchange.

(1) Isolated system

Here, the system can not exchange either heat, work or mass with the surrounding. Therefore, the total energy of an isolated system does not be change or $\Delta E = E_1 - E_2 = 0$ where ΔE is the change in total energy of the system at two different states 1 and 2.

(2) Closed system

Here, the system can not exchange mass with the surrounding but heat and work may be exchanged. Therefore, the change in total energy of a closed system within two different states can be calculated as $\Delta E = \Delta Q + \Delta W$ where, ΔE is the change in energy of the system, ΔQ is the heat added to the system, and ΔW is the work done on the system by the surrounding. The change in total energy of a system, ΔE equals to the summation of changes in potential, kinetic, and internal energies of the system. However, the change in potential and kinetic energies of the system are usually negligible and thus, the total energy E changes only due to the change in internal energy, U. Therefore, for a closed system, we may write,

$$\Delta U = \Delta Q + \Delta W$$

(3) Open system

In an open system all three mass, heat, and work may be exchanged with the surrounding.

Therefore, the change in total energy of an open system may be calculated as,

$$\Delta E = \Delta Q + \Delta W + \begin{pmatrix} \text{Addition or removal of} \\ \text{energy due to net inflow} \\ \text{of mass in the system} \end{pmatrix}$$

the following manner.

$$\begin{pmatrix}\text{Rate of accumulation}\\ \text{of energy in the system}\end{pmatrix} = \begin{pmatrix}\text{Net rate of inflow}\\ \text{of energy by convection}\end{pmatrix} + \begin{pmatrix}\text{Net rate of heat}\\ \text{addtion by conduction}\end{pmatrix}$$

$$\begin{pmatrix}\text{Net rate of work}\\ \text{done on the system}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation/ consumption}\\ \text{by the heat source or sink}\end{pmatrix}$$

$$+\begin{pmatrix}\text{Heat loss or gain}\\ \text{by radiation}\end{pmatrix}$$

The above equation takes into account the heat transfer by three different modes as shown below.

- **Conduction:** In heat transfer, heat conduction is the transfer of heat from higher temperature region to lower temperature region due to temperature gradients.
- **Convection:** The energy transfer may also occur due to the transport of material from the boundaries of the system.
- **Radiation:** This term implies transfer of heat energy due to electromagnetic waves under certain range of wavelength. Radiation does not require a material medium for energy transport like in conduction and convection. Unless the temperature is high, the heat addition by radiation may be neglected.

While studying the subject of heat transfer, the main objective is to find the rate of heat transfer from a body or a system. Fourier's law of heat conduction provides the relation between the rate of heat transfer and temperature gradients.

Fourier's law of heat conduction

When a temperature gradient exists in a body, experience has shown that the heat is transferred from higher temperatures to lower temperatures. Consider a solid block of surface area A, which is located between two parallel planes, set a distance H apart as shown in Fig. (24.1). Initially, for $t<0$, the solid block is maintained at a homogeneous temperature T_1 throughout. After some time $t=0$, lower plane is suddenly brought to a higher temperature T_2 and maintained at that temperature for $t>0$. Once the steady state is achieved, it is found that a constant heat flux in x direction is required to maintain the temperature difference, Q / A is required to maintain the constant temperature difference (T_2-T_1) across the solid block.

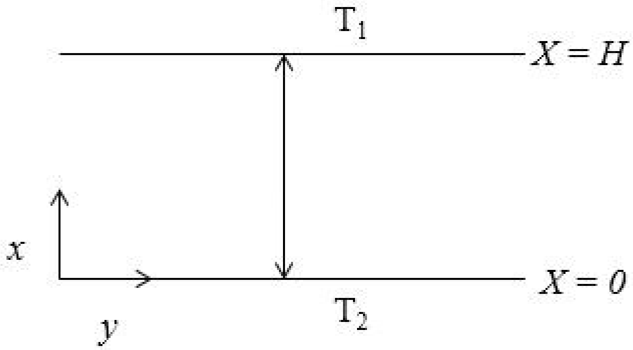

Fig 24.1 Flow of heat between two parallel plates

Repetition of the above experiment with different temperature differences $\Delta T=(T_2-T_1)$ shows that the heat flux is proportional to ΔT/H as shown in Fig. (24.2).

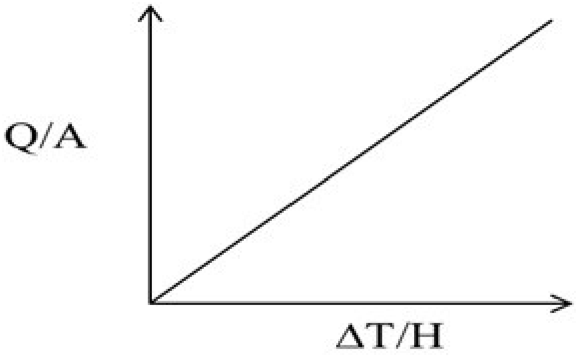

Fig 24.2 Heat flus vs. temperature gradient

This implies that

$$q_x = \frac{Q}{A} \propto \frac{\Delta T}{H}$$

or

$$q_x = -k\frac{dT}{dx}$$

where *k* is called thermal conductivity. Negative sign indicates that the heat flows from higher temperatures to lower temperatures.

Unit of thermal conductivity is

$$k = \frac{[q_x]}{\left[\frac{dT}{dx}\right]} = \frac{{cal}/{cm^2 \sec}}{{°c}/{cm}} = \frac{cal}{cm - \sec °c}$$

The Equation (24.2) is called the Fourier’s law of heat conduction. By extending this equation in three dimensions, we obtain

$$q_x = -k \frac{dT}{dx}$$

ector form as

$$q_y = -k \frac{dT}{dy}$$

the molecular heat transport or conduction for an isotropic body and ystem. The detail forms of Fourier's law in all coordinate systems are

$$q_z = -k \frac{dT}{dz}$$

ome simple heat transfer problems due to conduction by using shell energy balance.

2.2 Heat conduction through a composite wall

Consider a composite wall of height L , width W and thickness $\delta_1 + \delta_2$. The wall contains two layers of different materials which have the thermal conductivity K0 and K1, and different thickness δ_1 and δ_2 respectively. At x=0, the composite wall is maintained at a constant temperature T0, while at $x=\delta_1+\delta_2$, it has a constant temperature T2 as shown in Fig. (25.1).

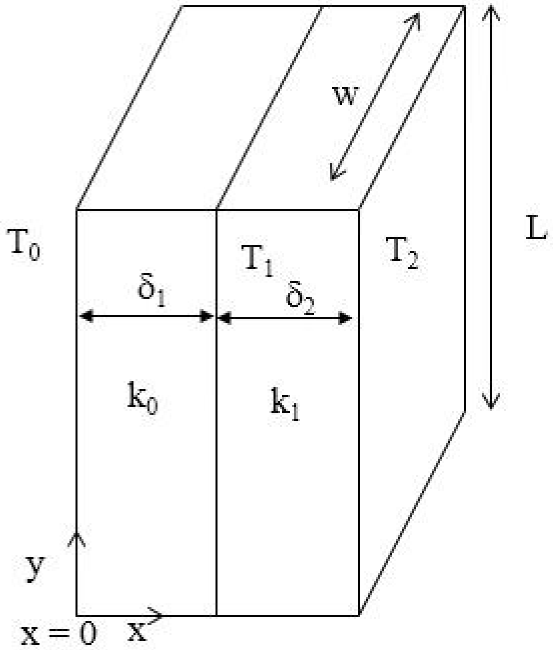

Fig 25.1 Heat conduction through a composite wall

Assumptions

- System is in steady state.
- Thermal conductivity for both walls, K0 , and K1 are constants.
- System follows Fourier’s law of heat conduction.
- Heat loss from side walls in direction of y and z are negligible.

Non-zero components of heat flux and the control volume

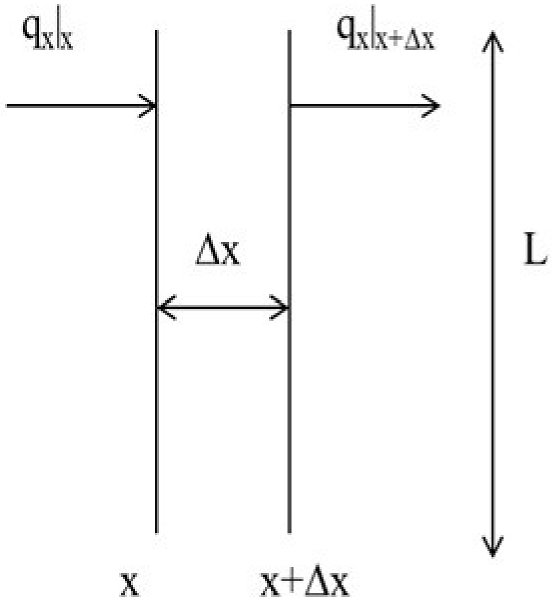

Fig 2 Control volume for heat flow through a composite wall

Since temperature is changing in x direction only, the control volume is chosen such that it has differential thickness in x direction as shown in Fig. (25.2) and qx is the only component of heat flux.

Energy balance

Heat flux entering into the control volume at x = x is

$$q_x HW \big|_x \qquad ($$

Heat flux leaving from the control volume at x = x + dx is p

$$q_x HW \big|_{x+\Delta x} \qquad ($$

Any source or sink of heat is not present in the control volume and work done on the system is zero. The thermal energy balance for this control volume may be written as

$$0 = \left(q_x HW\right)\Big|_{x=x} - \left(q_x HW\right)\Big|_{x+\Delta x}$$

After dividing by LWΔx and taking the limit Δx to zero, we get

$$\frac{dq_x}{dx} = 0$$

Integration of this equation gives

$$q_x = c_1$$

Here C1 is the integration constant. Equation (25.5) implies that heat flux is constant throughout the composite wall.
By applying the Fourier’s law of heat conduction, we get

$$q_x = \frac{-kdT}{dx} = c_1$$

Equation (25.6) may also be written as

$$\frac{dT}{dx} = \frac{-c_1}{k}$$

Now, the problem may be solved for both layers of composite wall separately.
Layer 1: $0 < x < \delta_1$
Here, thermal conductivity is k0 Therefore, Equation (25.7) may be changed to

$$\frac{dT}{dx} = \frac{-c_1}{k_0}$$

At x=0, the temperature of the composite wall is given as T=T0. Also the temperature of the first at can be assumed as T=T11. By integration of Equation (25.8) and substituting the boundary conditions, we obtain

$$T_{11} - T_0 = -\frac{c_1\delta_1}{k_0}$$

Layer 2: $\delta_1 < x < \delta_2$

Similar to solution for layer one, solution for layer 2 may also be found and given below

$$T_2 - T_{12} = \frac{-c_1(\delta_2 - \delta_1)}{k_1}$$

Here, T12 is the assumed temperature of second layer of

$$x = \delta_1 \qquad \textbf{(25.1}$$

It may be noted that at the interface of the two layers at X=δ1, the heat fluxes are same for both layers. Thus, equating the heat fluxes we find the integration constant c1 is same for both layers. Also, thermal equilibrium may assume at interface and therefore,

$$T_{11} = T_{12}$$

Using above boundary condition in Equation (25.9) and (25.10) and adding, we obtain

$$T_2 - T_0 = -c_1 \left\{ \frac{\delta_1}{k_0} + \frac{\delta_2 - \delta_1}{k_1} \right\}$$

or

$$c_1 = q_x = \frac{T_0 - T_2}{\left\{ \frac{\delta_1}{k_0} + \frac{\delta_2 - \delta_1}{k_1} \right\}}$$

The above equation (25.14) provides resulting heat flow per unit area of the composite wall.
In the last example, we had solved a heat transfer problem which involved cartesian coordinates. To understand the formulation of problem in other coordinate systems two more example are considered here. One in cylindrical coordinate system and second in spherical coordinate system.

2.3 Heat transfer in a cylindrical shell

Consider a long cylindrical shell of inner radius R1, outer radius R2, and length L shown in Fig. 26.1. The inner wall of cylindrical shell is maintained at constant temperature T1 and outer wall is maintained at constant temperature T2. Calculate the heat transfer rate in radial direction from the cylindrical shell.

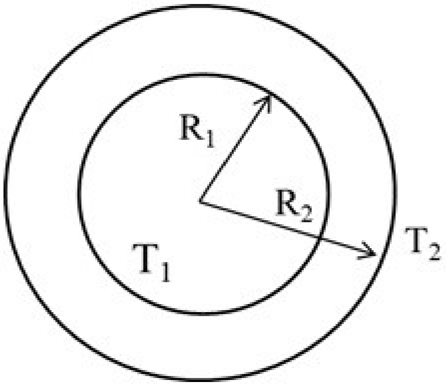

Fig 26.1 Heat transfer in cylindrical shell

Assumptions

- System is in steady state.
- Thermal conductivity, k, is constant.
- System follows Fourier's law of heat conduction.
- Heat loss in axial direction is negligible.

Non-zero heat flux component

Since temperature is changing in r direction only, qr is present. Now, consider a control volume of differential thickness Δr as shown below

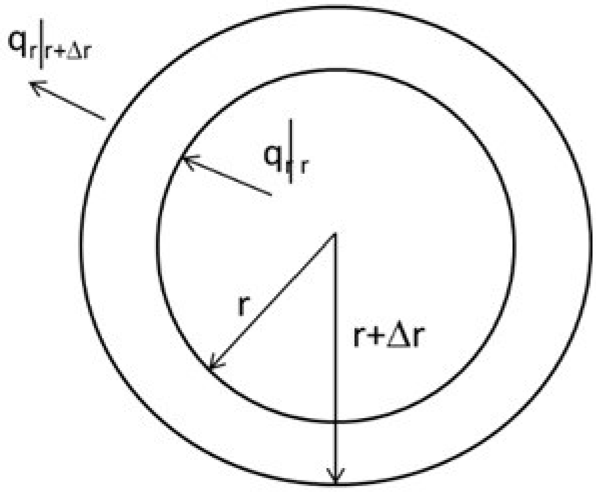

Fig 26.2 Control volume for heat transfer in cylindrical shell

Shell energy balance across the control volume

Heat flux entering the control volume at r=r is

$$2\pi rLq_r\big|_r \qquad (26.1)$$

Heat flux leaving from the control volume at r= r+ Δr is

$$2\pi rLq_r\big|_{r+\Delta r} \qquad (26.2)$$

No source or sink of heat is present in the control volume and work done on the system is zero. Thus, the thermal energy balance is reduced to

$$0 = q_r 2\pi rL\big|_r - q_r 2\pi rL\big|_{r+\Delta r}$$

By dividing Equation (26.3) to the volume of control volume 2πrLΔr and taking the limit Δr going to zero, we obtain

$$\frac{d(rq_r)}{dr} = 0$$

By integrating Equation (26.4), we get

$$q_r = \frac{c_1}{r}$$

where c1 is a integration constant.
Substituting Fourier's law of heat conduction in Equation (26.5) , we obtain

$$\frac{dT}{dr} = -\frac{c_1}{-kr}$$

or

$$T = -c_1 lnr + c_2$$

Here, c2 is the constant of integrations.
Boundary conditions are

at $r = R_1$, $T = T_1$

(26.7)

and

at $r = R_2$, $T = T_2$

$$(26.8)$$

This leads to the solution

$$T_2 - T_1 = -c_1 \frac{ln(R_2 / R_1)}{k}$$

$$c_1 = \frac{k\left(T_1 - T_2\right)}{ln\left(R_2 / R_1\right)}$$

Substituting the value of c1 in Equation (26.5), we finally obtain

$$q_r = \frac{k\left(T_1 - T_2\right)}{r\,ln\left(R_2 / R_1\right)}$$

The rate of heat transfer through cylindrical shell may be calculated as shown below,

$$Q_0 = \left(2\pi L r \times q_r\right)\Big|_{r=R_1\ or\ r=R_2} = \frac{2\pi L k\left(T_1 - T_2\right)}{ln\left(R_2 / R_1\right)}$$

2.4 Heat transfer in a spherical shell

Consider a spherical shell of inner radius R1 and outer radius R2, whose inside and outside surfaces are maintained at the constant temperatures T1 and T2 respectively as shown in Fig. 26.3. Calculate the heat flux from the spherical shell

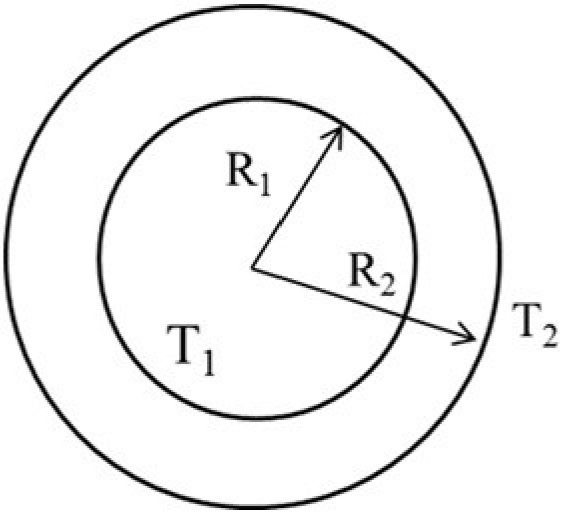

Fig Heat transfer in hollow sphere

Assumptions

- System is in steady state.
- Thermal conductivity, k, is constant.
- System follows Fourier’s law of heat conduction.

Non-zero heat flux component

Since temperature changing in r direction, only qr is present. The control volume may be drawn of differential thickness δr as shown in Fig (26.4).

Shell energy balance across the control volume

Heat flux entering control volume at r= r is $4\pi r^2 q_r |_r$ (26.12)

Heat flux leaving control volume at r= r+ Δr is $4\pi r^2 q_r |_{r+\Delta r}$ (26.13)

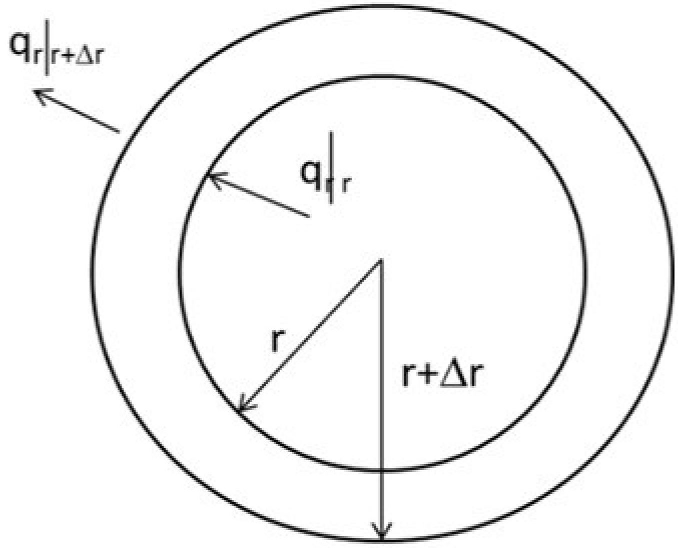

Fig 26.4 Control volume for heat transfer in hollow sphere

Any heat source or sink is not present in the control volume and work done on the system is zero. Thus, the thermal energy balance is reduced to

$$0=\left(q_r 4\pi r^2\right)\big|_r-\left(q_r 4\pi r^2\right)\big|_{r+dr}$$

Dividing Equation (26.14) by volume of control volume $4\pi r^2\Delta r$ and taking the limit Δr→0 , we get

$$\frac{d}{dr}(r^2 q_r)=0$$

and integrating Equation (26.15), we find

$$q_r=\frac{c_1}{r^2}$$

where, c1 is an integration constant.

By substituting Fourier’s law of heat conduction in Equation (26.16) and integrating, we obtain

$$T=\frac{c_1}{kr}+c_2$$

where c2 is another integration constant.

The Equation (26.17) is subjected to the boundary conditions,

at $r = R_1$, $T = T_1$

(26.18)

and

at $r = R_2$, $T = T_2$

(26.19)

Using above boundary conditions and evaluating the constants of integration c1 and c2, we finally obtain heat flux through spherical shell as given below,

$$q_r = \frac{1}{r^2}\frac{k\left(T_1 - T_2\right)}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

2.5 Heat transfer from a cylindrical composite wall : Use of heat transfer coefficients

Consider a cylindrical composite wall whose inner surface is exposed to a fluid at constant temperature Tb and the outer surface is exposed to atmosphere at a temperature Ta. Ro is the inner radius of cylinder while outer radius is R1. The cylinder is insulated and radius of insulation changes from R1 to R2 as shown in Fig. (27.1). The inside and out side heat transfer coefficients are hb and ha respectively. K01 and K12 are the thermal conductivity of cylinder material and insulation respectively. Calculate the overall heat loss through the cylindrical wall.

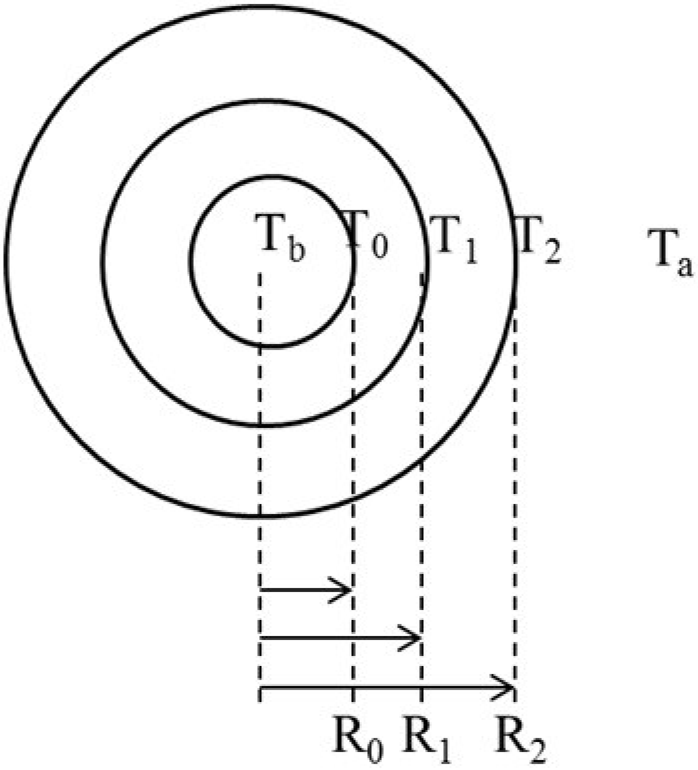

Fig 1 Heat transfer in cylindrical composite wall

Assumptions

- System is in steady state.
- Thermal conductivities K01and K12 are constants.
- System follows Fourier's law of heat conduction.
- Heat loss in axial directions are negligible.

Heat flux component

Here, the temperature is changing in the radial direction, and therefore, T=T(r) and qr is the only non-zero heat flux. As we have seen in the previous problems, the energy balance for cylinder material and insulation may be written as

$$rq_r = c_1 = constant$$

where, c1= R0q0 = R1 q1= R2q2 where q0, q1, and q2 are the heat fluxes at R0, R1 and R2 respectively. Substituting Fourier's law of heat conduction, we obtain,

$$-kr\frac{dT}{dr} = c_1$$

Assume the unknown temperatures are T0, T1 and T2 respectively as shown in Fig. (27.1). To solve Equation (27.2) for metal wall as well as for insulation material, it may be noted that the metal wall subjected to the following boundary conditions.

at $r = R_0$, $T = T_0$ (27.3)

and

at $r = R_1$, $T = T_1$ (27.4)

Similarly for insulation, we have the boundary condition (27.4) and second boundary condition is given below

$$r = R_2, T = T_2$$

The final solution in for Equation (27.2) is given below

$$q_0 R_0 = \frac{k_{01}(T_0 - T_1)}{ln(R_1 / R_0)}$$

or

$$(T_0 - T_1) = q_0 R_0 \frac{ln(R_1 / R_0)}{k_{01}}$$

or

and similarly

$$(T_1 - T_2) = q_0 R_0 \frac{ln(R_2 / R_1)}{k_{12}}$$

Heat transfer coefficient

For complete solution of this problem, we need to solve problems of heat transfer in fluid inside the cylindrical tube as well as in the atmosphere outside the insulation. However, if we know the heat transfer coefficients, we may avoid finding these solutions. Recall the Newton's law of cooling which states the rate of heat transfer from a body is proportional to the difference in temperature between the body and its surrounding where q= AδT where A is the constant of proportionality. The heat transfer coefficients are similar to the coefficient A in the Newton's law of cooling, using these coefficients

$$q_0 = h_a\left(T_a - T_0\right)$$

or

$$\left(T_a - T_0\right) = \frac{q_0}{h_a} = \frac{R_0 q_0}{h_a R_0}$$

and for insulation, we have

$$q_2 = h_b\left(T_2 - T_b\right)$$

or

$$\left(T_2 - T_b\right) = \frac{q_2}{h_b} = \frac{R_2 q_2}{R_2 h_b}$$

By adding Equations (27.7), (27.8),(27.10) and (27.11) and, noting Equation (27.1), we get

$$\left(T_a - T_b\right) = q_0 R_0 \left\{ \frac{1}{h_b R_2} + \frac{ln(R_1/R_0)}{k_{01}} + \frac{ln(R_2/R_1)}{k_{12}} + \frac{1}{R_0 h_a} \right\}$$

or

$$q_0 = \frac{\left(T_a - T_b\right)}{R_0 \left\{ \frac{1}{h_b R_2} + \frac{ln(R_1/R_0)}{k_{01}} + \frac{ln(R_2/R_1)}{k_{12}} + \frac{1}{h_a R_0} \right\}}$$

Heat conduction with a heat source

A cylindrical rod of radius R0 and length L is producing heat which is equal to Sc per unit time per unit volume, may be due to conversion of electrical energy into heat. The surface of the cylinder is maintained at temperature T0. Determine the temperature profile.

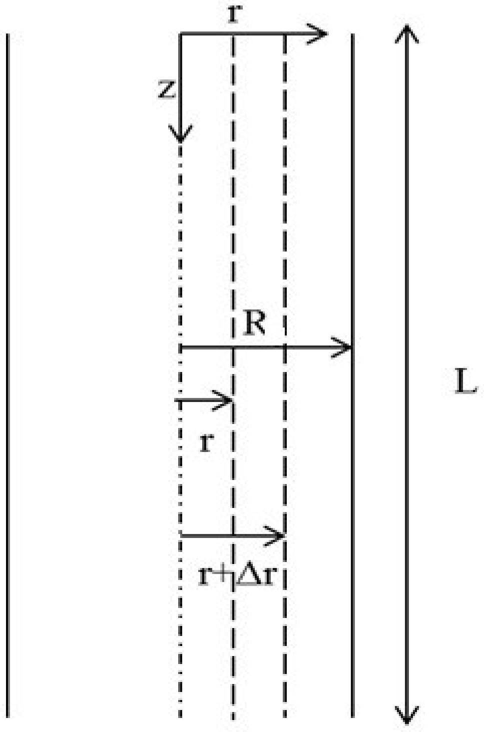

Fig 27.2 Heat transfer in cylindrical shell

Assumptions

- System is in steady state.
- Thermal conductivity of metal is constant.
- System follow Fourier's law of thermal heat conduction.
- Heat loss in axial directions are negligible.

Once again, temperature is changing in r direction only and qr is the only non-zero component of heat flux. Writing the following terms required for energy balance for a control volume shown in Fig. 27.2, we obtain the following terms

Heat entering the control volume is $2\pi r L q_r |_r$

(27.13)

Heat leaving control volume is $2\pi rLq_r\big|_{r+\Delta r}$

(27.14)

the heat produced in metal rod is given by

$$Sc\times 2\pi r\Delta r\, L$$

Writing the energy balance, we find

$$2\pi rLq_r\big|_r - 2\pi rLq_r\big|_{r+\Delta r} + Sc2\pi r\Delta r\, L = 0$$

which leads to the following differential equation

$$\frac{1}{r}\frac{d}{dr}\left(rq_r\right) = Sc$$

Integrating Equation (27.17), we finally obtain

$$q_r = \frac{rSc}{2} + \frac{c_1}{r}$$

To evaluate the constant of integration c1, we may apply the boundary condition that at r=0, qr is finite; Therefore,
c1 =0 (27.19)

or

$$q_r = \frac{rSc}{2}$$

Applying Fourier's law of heat conduction, we have

$$-k\,\frac{dT}{dr}=\frac{rSc}{2}$$

Thus, after integration, we obtain

$$T=-\frac{Scr^2}{4k}+c_2$$

The second boundary condition for this problem is that

at

$$r=R,T=T_0$$

or

$$c_2=\ T_0+\frac{ScR^2}{4k}$$

Thus, the temperature profile may be determined as below

$$\left(T-T_0\right)=\frac{ScR^2}{4k}\left\{1-\left(\frac{r}{R}\right)^2\right\}$$

2.6 Critical radius of insulation

Consider a cylindrical rod which is insulated by an insulation material as shown in Fig. (27.3). The radius of the rod is R0 and rod is maintained at temperature T0. The insulated rod is surrounded by a medium at temperature Ta. The out-side heat transfer coefficient is ha. Determine the critical radius of insulation at which the heat loss is maximum.

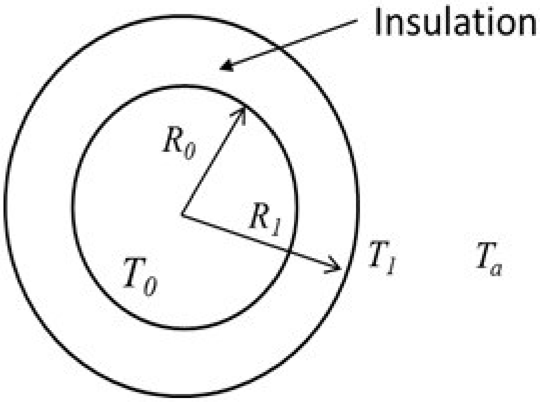

Fig 27.3Insulated cylindrical pipe

Once again, as shown earlier, the energy balance for cylindrical shell in insulation material leads to the following differential equation.

$$\frac{1}{r}\frac{d}{dr}\left(rq_r\right)=0$$

Integrating Equation (27.25) and applying Fourier's law of heat conduction, we obtain

$$q_r=\frac{c_1}{r}=-k_I\frac{dT}{dr}$$

The above equation may be integrated subject to the following boundary conditions

at $r=R_0$, $T=T_0$

(27.27)

and

at $r=R_1$, $T=T_1$

(27.28)

Also, since the out side heat transfer coefficient is given, we have

$$q_r\big|_{r=R_1}=h_a\left(T_1-T_a\right)$$

(27.

The solution of this problem is similar to the problem solved earlier for composite cylindrical shell. Thus, we finally obtain

$$q_0 = \frac{(T_0 - T_a)}{R_0\left\{\dfrac{1}{h_a R_1} + \dfrac{ln(R_1 / R_0)}{k_1}\right\}}$$

In Equation (27.28), we may define the overall heat transfer resistance as

$$U = R_0\left\{\frac{1}{h_a R_1} + \frac{ln(R_1 / R_0)}{k_1}\right\}$$

In Equation (27.30), the first term in right hand side represents the convective heat transfer resistance and the second term represents the conductive heat transfer resistance. It can be easily observed that convective resistance decreases and conductive resistance increases as we increase the thickness of the insulation, i.e., R1 thus, over all heat loss may initially increase and then decreases as shown in Fig. 27.8. The value of R1 where the heat loss is maximum or the overall heat transfer resistance is minimum, is called the critical radius of insulation.

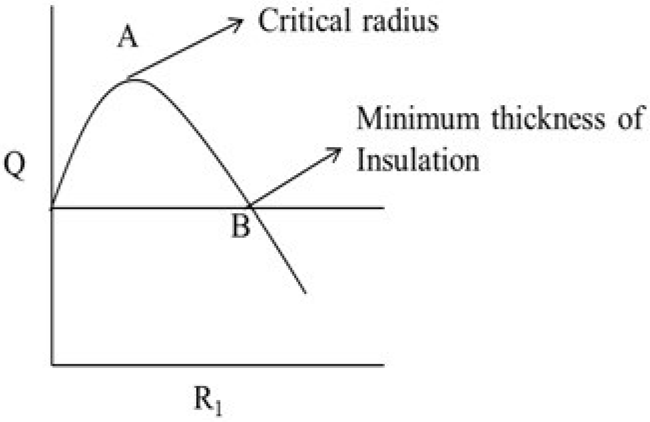

Fig 27.3 Thickness of insulation Vs. heat flux

Since, the overall heat transfer resistance should be minimum at critical radius, we have

$$\left.\frac{dU}{dR_1}\right|_{Critical\ radius} = 0 \qquad ($$

or

$$\frac{1}{k_I R_I} - \frac{1}{R_I^{\,2} h_a} = 0 \tag{27.33}$$

which provides the critical radius of insulation

$$R_I\Big|_{\text{Critical radius}} = R_c = \frac{k_I}{h_a}$$ (27

If the overall resistance U is differentiated twice with respect to the radius of insulation R1, we obtain

$$\frac{d^2U}{dR^2{}_I} = -\frac{1}{k_I R_I^{\,2}} + \frac{2}{R_I^{\,3} h_a}$$

At critical radius of insulation the value of the right hand side in Equation of (27.35) is positive as shown below

$$\frac{d^2U}{dR_I^2} = \frac{-h_a^{\,2}}{k^2} + \frac{2h_a^{\,2}}{h_a k^3} = \frac{h_b^{\,2}}{k^3} > 0$$

Since, the value of second derivative is always positive, verifies that U is minimum at critical radius of insulation. Thus, the heat loss is maximum at critical radius of insulation given by Equation (27.33).

2.7 Derivation of equation of energy

In this section, we derive the equation of energy by using Axiom-4, which states that energy is conserved. The equation of total energy may be further divided into two parts. First is the equation of mechanical energy and second is the equation of thermal energy. The equation of mechanical energy is derived from equation of motion. The equation of thermal energy is derived by subtracting the equation of mechanical energy from the equation of total energy. Later, the equation of thermal energy is modified in temperature explicit form, which may be used for obtaining temperature profile

Consider a stationary control volume of dimension Δx, Δy and Δz. The fluid is flowing with a velocity $\underset{\sim}{v}$, which has components vx, vy and vz in x, y and z directions respectively, as shown in the Fig.(28.1).

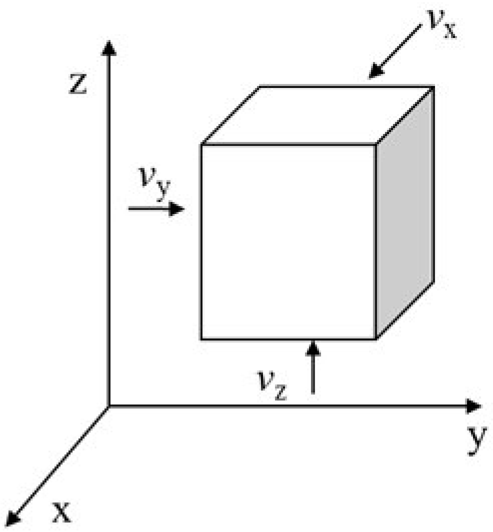

Fig 28.1 Cubical control volume

The total energy consists of potential energy $\hat{G}$, Internal energy $\hat{U}$ and kinetic energy $\frac{1}{2}v^2$. Since the control volume is fixed in the space, the change in potential energy is negligible. Therefore, the energy balance may be written as .

$$\begin{pmatrix}\text{Rate of accumulation of internal}\\ \text{\& kinetic energy in control volume}\end{pmatrix} = \begin{pmatrix}\text{Rate of net change of internal}\\ \text{\& kinetic energy by convection}\end{pmatrix} + \left(\begin{matrix}\text{N}\\ \text{ad}\end{matrix}\right.$$

$$+\left(\begin{matrix}\text{Work done on the system against vari}\\ \text{forces}(\text{Pressure, Gravity and Shear})\end{matrix}\right.$$

$$+\begin{pmatrix}\text{Rate of heat addition}\\ \text{by some heat sources}\end{pmatrix}$$

Now, we take each term in equation (28.1) and write them separately as given below.

(1) Rate of accumulation of internal and kinetic energy

The rate of accumulation of internal and kinetic energy in the control may be written as

$$\frac{\partial}{\partial t}\left[(\rho\Delta x\Delta y\Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]$$

where, $\hat{U}$ is the internal energy per unit mass of the system.

(2) Rate of net change of kinetic and internal energy by convection

Net inflow of kinetic and internal energy by convection may be written as

$$\left[\left(\rho v_x \Delta y \Delta z\right)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_x-\left[\left(\rho v_x \Delta y \Delta z\right)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_{x+\Delta x}$$

$$+\left[\left(\rho v_y \Delta x \Delta z\right)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_y-\left[\left(\rho v_y \Delta x \Delta z\right)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_{y+\Delta y}$$

$$+\left[\left(\rho v_z \Delta y \Delta x\right)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_z-\left[\left(\rho v_z \Delta x \Delta y\right)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_{z+\Delta z}$$

(3) Rate of heat addition by conduction

As shown in Fig. (28.2), heat flux, $\underset{\sim}{q}$, has three components qx, qy and qz respectively. Therefore, the net heat addition by conduction is given below

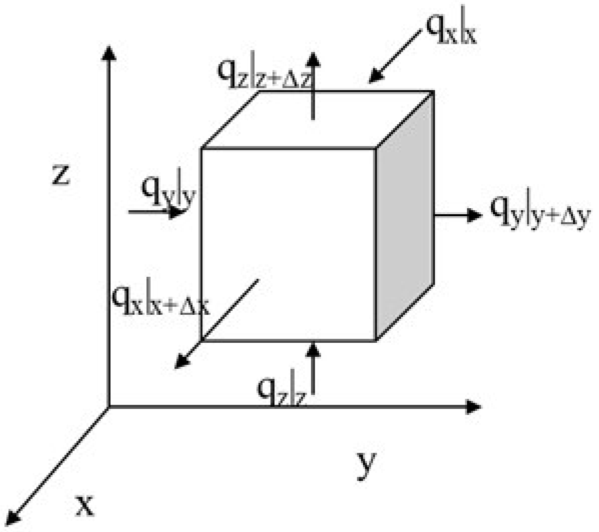

Fig 28.2 Cubical control volume with heat transfer

$$(q_x\Delta y\Delta z)\big|_x - (q_x\Delta y\Delta z)\big|_{x+\Delta x} + (q_y\Delta x\Delta z)\big|_y - (q_y\Delta x\Delta z)\big|_{y+\Delta y} + (q_z\Delta x\Delta y)\big|_z - (q_z\Delta x\Delta y)\big|_{z+\Delta z}$$

(4) Rate work done on the system

Work done on the system is defined as a scalar product of force and displacement vectors. If $\underset{\sim}{F}$ is a force and $\partial \underset{\sim}{l}$ is the displacement then work done delta w,is

$$\partial w = \underset{\sim}{F} . \partial \underset{\sim}{l}$$

or rate of work done is given by

$$\frac{\partial w}{\partial t} = \underset{\sim}{F} \cdot \frac{\partial \underset{\sim}{l}}{\partial t}$$

or

$$= \underset{\sim}{F} . \underset{\sim}{v}$$

or

$$= F_x v_x + F_y v_y + F_z v_z$$

Now, consider the forces acting on a fluid element. These forces are

- Gravity force
- Pressure force
- Shear force

Therefore, we need to consider work done by these force on the control volume separately.

1. Rate of work done against Gravity forces

By following Equation (28.6), the work done against gravity force may be written as

$$= \rho \Delta x \Delta y \Delta z g_x v_x + \rho \Delta x \Delta y \Delta z g_y v_y + \rho \Delta x \Delta y \Delta z g_z v_z$$

2. Rate of work done by pressure forces

The pressure forces always act in the opposite direction to the outer normal of a plane. It is a compressible force. Therefore, the work done on the control volume by pressure forces may be calculated as follows
Rate of work done by x directed pressure forces:

$$= \left(v_x P\right)\Big|_x \Delta y \Delta z - \left(v_x P\right)\Big|_{x+\Delta x} \Delta y \Delta z$$

Rate of work done by y directed pressure forces:

$$= \left(v_y P\right)\Big|_y \Delta x \Delta z - \left(v_y P\right)\Big|_{y+\Delta y} \Delta x \Delta z$$

Rate of work done by z directed pressure forces:

$$= \left(v_z P\right)\Big|_z \Delta x \Delta y - \left(v_z P\right)\Big|_{z+\Delta z} \Delta x \Delta y$$

Derivation of equation of energy

3. Rate of Work done by shear forces

There are nine components of shear stress tensor. Three of these act on x directed face, similarly the net three acts on the y directed face and remaining three act on z directed faces (shown in

Fig. 29.1) as discussed earlier.

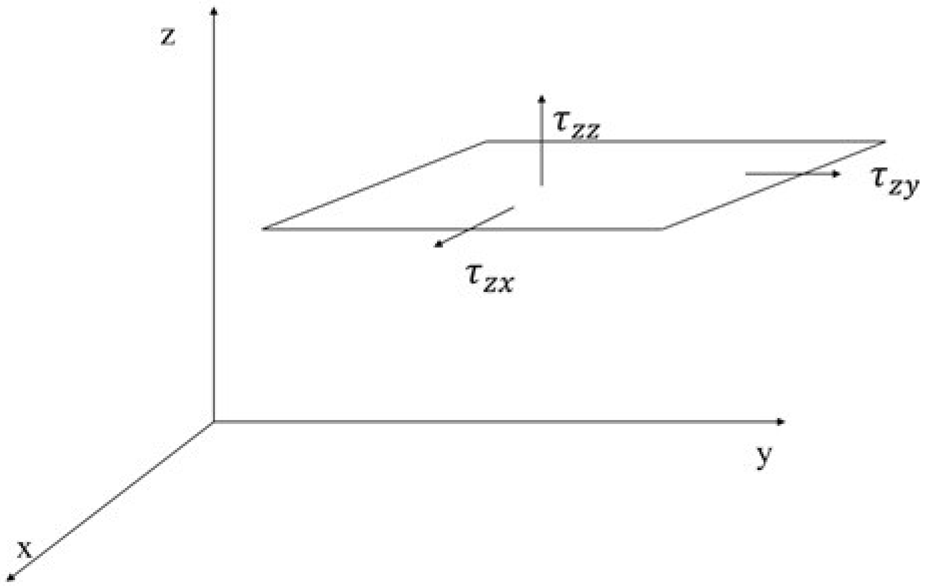

Fig.29.1 Shear stresses, acting on z directed plane

Therefore, the work done by shear forces on the control volume, acting on x directed plane may be calculated as

$$\left(\tau_{xx}v_x+\tau_{xy}v_y+\tau_{xz}v_z\right)\Big|_{x+\Delta x}\Delta y\Delta z-\left(\tau_{xx}v_x+\tau_{xy}v_y+\tau_{xz}v_z\right)\Big|_{x}\Delta y\Delta z$$

Similarly, the work done by shear forces on the control volume, acting on y directed plane may be calculated as

$$\left(\tau_{yx}v_x+\tau_{yy}v_y+\tau_{yz}v_z\right)\Big|_{y+\Delta y}\Delta x\Delta z-\left(\tau_{yx}v_x+\tau_{yy}v_y+\tau_{yz}v_z\right)\Big|_{y}\Delta x\Delta z$$

and the work done by shear force on the control volume, acting on z directed plane may be calculated as

$$\left(\tau_{zx}v_x+\tau_{zy}v_y+\tau_{zz}v_z\right)\Big|_{z+\Delta z}\Delta x\Delta y-\left(\tau_{zx}v_x+\tau_{zy}v_y+\tau_{zz}v_z\right)\Big|_{z}\Delta x\Delta y$$

5. Rate of Heat addition by Heat source or sink

If any heat source/sink is present in the control volume which generates the heat as Sc per unit volume then heat generated in the control volume may be written as.

$$= Sc\Delta x\Delta y\Delta z$$

We now substitute all terms given above in Eqution (28.1) and then divide by ΔxΔyΔz. After taking the limits Δx,Δy and Δz going to zero, we obtain the following equation

$$\frac{\partial}{\partial t}\left[\rho\left(\hat{U}+\frac{v^2}{2}\right)\right]=-\left[\frac{\partial}{\partial x}\left\{\rho v_x\left(\hat{U}+\frac{v^2}{2}\right)\right\}+\frac{\partial}{\partial y}\left\{\rho v_y\left(\hat{U}+\frac{v^2}{2}\right)\right\}+\frac{\partial}{\partial z}\left\{\rho v_z\left(\hat{U}+\frac{v^2}{2}\right)\right\}\right]-\left[\frac{\partial}{\partial x}\right.$$

$$\rho\left(v_x g_x+v_y g_y+v_z g_z\right)-\left[\frac{\partial}{\partial x}(Pv_x)+\frac{\partial}{\partial y}(Pv_y)+\frac{\partial}{\partial z}(Pv_z)\right]$$

$$+\left[\frac{\partial}{\partial x}\left(\tau_{xx}v_x+\tau_{xy}v_y+\tau_{xz}v_z\right)+\frac{\partial}{\partial y}\left(\tau_{yx}v_x+\tau_{yy}v_y+\tau_{yz}v_z\right)+\frac{\partial}{\partial z}\left(\tau_{zx}v_x+\tau_{zy}v_y+\right.\right.$$

In Equation (29.5), the stress tensor $\underset{\sim}{\tau}$ was taken as shear forces. To change it into momentum flux, we replace all components of $\underset{\sim}{\tau}$ with a minus sign. In addition, if we rewrite the Equation (29.8) in vector and tensor form , we obtain the following result for equation of energy.

$$\frac{\partial}{\partial t}\left[\rho\left(\hat{U}+\frac{v^2}{2}\right)\right]=-\underset{\sim}{\nabla}.\left[\rho\left(\hat{U}+\frac{v^2}{2}\right)\right]\underset{\sim}{v}-\underset{\sim}{\nabla}.\underset{\sim}{q}+\rho\underset{\sim}{v}.\underset{\sim}{g}-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)+Sc$$

We may further simplify Equation (29.6) by combining the internal energy and kinetic energy terms as shown below. If

$$s=\hat{U}+\frac{v^2}{2}$$

Then, Equation (29.6) may be written as

$$\frac{\partial(\rho s)}{\partial t}+\underset{\sim}{\nabla}.\rho s\underset{\sim}{v}=-\underset{\sim}{\nabla}.\underset{\sim}{q}+\rho\underset{\sim}{v}.\underset{\sim}{g}-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)+Sc$$

The left hand side of Equation (29.8) may be modified as shown below

$$\frac{\partial(\rho s)}{\partial t} + \underset{\sim}{\nabla}.(\rho s\underset{\sim}{v})$$

or

$$\frac{\rho\partial s}{\partial t} + \frac{s\partial\rho}{\partial t} + s\underset{\sim}{\nabla}.(\rho\underset{\sim}{v}) + (\rho\underset{\sim}{v})\underset{\sim}{\nabla}s$$

or

$$\rho\left[\frac{\partial s}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}s\right] + s\left[\frac{\partial\rho}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}\rho\underset{\sim}{v}\right]$$

But,

$$\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}.\rho\underset{\sim}{v} = 0$$

Therefor, Equation (29.10) simplifies to,

$$= \rho\left[\frac{\partial s}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}s\right]$$

or

$$= \rho\,\frac{Ds}{Dt}$$

Therefore, Equation (29.08) simplifies to

$$\frac{\rho Ds}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - \underset{\sim}{\nabla}.(p\underset{\sim}{v}) + \rho\underset{\sim}{v}.\underset{\sim}{g} - \underset{\sim}{\nabla}.(\underset{\sim}{\tau}.\underset{\sim}{v}) + Sc$$

or

$$\frac{\rho D\left(\hat{U}+{v^2}/{2}\right)}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + \rho\underset{\sim}{v}.\underset{\sim}{g} - \underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + S_c$$

Above equation represents the equation of total energy in terms of substantial derivative. Since observer is moving with fluid in the case of substantial derivative, the convective terms are not presents in the above equation.

Derivation of equation of energy

In previous lecture, we had derived the equation of energy which may be further divided into two parts

1. Equation of mechanical energy
2. Equation of thermal

energy Equation of mechanical

energy

For understanding the nature of mechanical energy, consider a simple case of a single particle moving in one direction as shown in Fig. 30.1. Assume the particle has mass m and is located at height h from a reference plane and moving upward with velocity $\underset{\sim}{v}$. Gravity is the only force working on the particle.

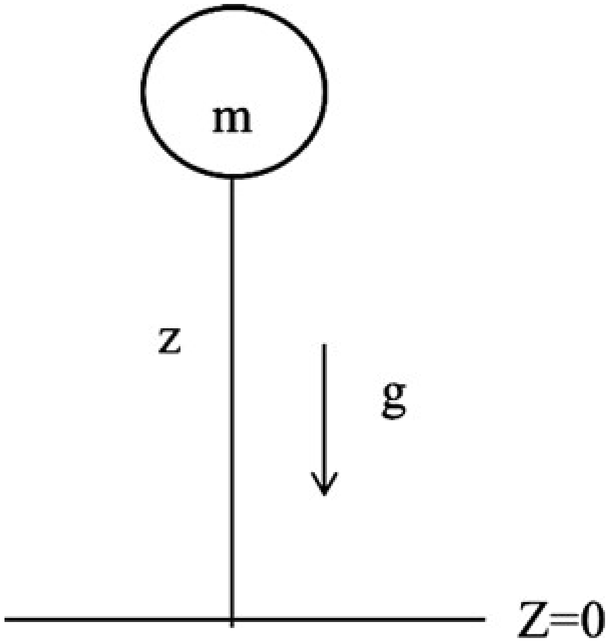

Fig 30.1 A particle of mass m situated at height z

Starting with Newton's second law of motion, we have

Force = mass x acceleration

where

$$\underset{\sim}{F} = m\underset{\sim}{a}$$

(30.1

or

$$\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt}$$

(3

By taking dot product of equation (30.2) with velocity , we find that

$$\underset{\sim}{v}\cdot\left[\underset{\sim}{F} = m\,\frac{d\underset{\sim}{v}}{dt}\right]$$

or

$$\underset{\sim}{v}\cdot\underset{\sim}{F} = m\underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt}$$

Using vector identity, we have

$$\frac{d\left(\underset{\sim}{v}\cdot\underset{\sim}{v}\right)}{dt} = \underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt} + \frac{d\underset{\sim}{v}}{dt}\cdot\underset{\sim}{v} = 2\underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt}$$

or

$$\underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt} = \frac{1}{2}\frac{d\left(v^2\right)}{dt}$$

where, v is the magnitude of the velocity vector $\underset{\sim}{v}$.
By substituting Equation (30.6) in Equation (30.4), we obtain

$$\underset{\sim}{v}\cdot\underset{\sim}{F} = m\frac{d\left(\dfrac{v^2}{2}\right)}{dt}$$

or,

$$v_1F_1 + v_2F_2 + v_3F_3 = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

For the example given above, we have ,
F1 = 0, F2 = 0, F3=-mg .... (30.8)

and

ν1 = 0, ν2 = 0, ν3 = ν .... (30.9)

Thus, Equation (30.7), reduces to

$$-vmg = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Substitute ν = (dz/dt). Thus we obtain,

$$-mg\frac{dz}{dt} = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Since, m and g are constants. We may rewrite above equation as,

$$-\frac{d(mgz)}{dt} = \frac{d\left(m\frac{v^2}{2}\right)}{dt}$$

or

$$\frac{d}{dt}\left(mgz + m\frac{v^2}{2}\right) = 0$$

Thus $mgz + m\frac{v^2}{2} = constant$

......

First term in above equation is the potential energy and second term represents the kinetic energy. Therefore, the above equation states that the sum of kinetic and potential energy remains constant. This is the equation of mechanical energy for a particle and similar equation may be derived for fluids as shown below.
Equation of mechanical energy for fluids
The equation of motion for a fluid is equivalent to Newton's second law of motion for solid bodies. Therefore, to derive the equation of mechanical energy for fluids we take the dot product of velocity with equation of motion for fluids. i.e.,

$$\underset{\sim}{v}.\left[\rho\frac{D\underset{\sim}{v}}{Dt} = \rho\underset{\sim}{g} - \underset{\sim}{\nabla}P - \underset{\sim}{\nabla}.\underset{\sim}{\tau}\right]$$

As before,

$$\underset{\sim}{v}.\frac{D\underset{\sim}{v}}{Dt} = \frac{D\left(\frac{v^2}{2}\right)}{Dt}$$

(Note: substantial derivatives behave like normal derivatives.). Thus,

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \rho\underset{\sim}{g}.\underset{\sim}{v} - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}P\right) - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)$$

The following vector and tensor identities may be used for simplifying Equation (30.14)

$$\underset{\sim}{\nabla}.\left(P\,\underset{\sim}{v}\right) \;=\; P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) \;+\; \underset{\sim}{v}.\left(\underset{\sim}{\nabla}P\right)$$

and if $\underset{\sim}{\tau}$ is a second order symmetric tensor then we also have

$$\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) \;=\; \underset{\sim}{v}.\underset{\sim}{\nabla}.\underset{\sim}{\tau} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Thus, we obtain

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right) - \left[\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + \left(-P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\right)\right] - \left[\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)\right]$$

or

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right) - \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Equation (30.18) is called the equation of mechanical energy for fluids. Significance of each term is given below.

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right)\left\{\begin{matrix}\text{Rate of change of}\\ \text{kinetic energy}\\ \text{per unit volume}\end{matrix}\right\} = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right)\left\{\begin{matrix}\text{work done by gravity}\\ \text{force on the system}\end{matrix}\right\}$$

$$-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)\left\{\begin{matrix}\text{work done by pressure}\\ \text{force on the system}\end{matrix}\right\} + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\left\{\begin{matrix}\text{reversible conversion of}\\ \text{kinetic energy into}\\ \text{the internal energy}\end{matrix}\right.$$

$$-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)\left\{\begin{matrix}\text{work done by viscous}\\ \text{forces on system}\end{matrix}\right\} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\left\{\begin{matrix}\text{irreversible conversion of}\\ \text{kinetic energy in to the hea}\end{matrix}\right.$$

As discussed earlier, the equation of thermal energy can be derived by subtracting the equation of mechanical energy (Equation (30.18)) from the equation of total energy (Equation (29.15)), i.e.,

$$\left(\begin{matrix}\text{Equation of thermal}\\ \text{energy}\end{matrix}\right) = \left(\begin{matrix}\text{Equation of}\\ \text{energy}\end{matrix}\right) - \left(\begin{matrix}\text{Equation of mechanical}\\ \text{energy}\end{matrix}\right.$$

thus

$$\rho \frac{D\left(\hat{U}\right)}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + S_c + \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) - P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \left(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)$$

or

$$\rho \frac{D\left(\hat{U}\right)}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} + S_c - P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \left(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)$$

The significance of each term in equation of thermal energy, Equation (31.21) is given below

$$\rho \frac{D\left(\hat{U}\right)}{Dt}\left\{\begin{matrix}\text{Rate of change of}\\ \text{internal energy}\\ \text{per unit volume}\end{matrix}\right\} = -\underset{\sim}{\nabla}.\underset{\sim}{q}\left\{\begin{matrix}\text{Heat transferred}\\ \text{by conduction}\end{matrix}\right\} + S_c\left\{\begin{matrix}\text{Heat gener}\\ \text{by source}\end{matrix}\right.$$

$$-P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\left\{\begin{matrix}\text{reversible conversion of}\\ \text{kinetic energy into}\\ \text{the internal energy}\end{matrix}\right\} - \left(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)\left\{\begin{matrix}\text{irreversible conversion of}\\ \text{kinetic energy in to the he}\end{matrix}\right.$$

Here $(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v})$, is known as the viscous heat dissipation and the significance of this will be discussed later.

Equation of mechanical energy of fluids and its interpretation

If we consider a special case of non-viscous fluid, where the shear stress is zero, Equation (30.14) simplifies as shown below

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla}P + \rho\left(\underset{\sim}{v}.\underset{\sim}{g}\right)$$

Here, the gravity may be represented by gradient of a scalar quantity Φ, or

$$\underset{\sim}{g} = -\underset{\sim}{\nabla}\hat{\phi}$$

Then, Equation (30.23) may be rewritten as

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla} P + \rho\left(-\underset{\sim}{v}.\underset{\sim}{\nabla}\hat{\phi}\right)$$

Further, if we assume that pressure and gravity do not depend on time. Thus, we have

$$\frac{\partial P\left(t\right)}{\partial t} = 0$$

and

$$\rho \frac{\partial \phi}{\partial t} = 0$$

After substituting these values in Equation (30.25), we obtain

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla} P - \frac{\partial P}{\partial t} + \rho\left(-\underset{\sim}{v}.\underset{\sim}{\nabla}\hat{\phi}\right) - \rho \frac{\partial \hat{\phi}}{\partial t}$$

which may be further simplified as

$$\frac{D\left(\frac{v^2}{2}\right)}{Dt} = \frac{D\left(\frac{P}{\rho}\right)}{Dt} - \frac{D\hat{\phi}}{Dt}$$

or

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2} + \frac{p}{\rho} + \hat{\phi}\right) = 0$$

which leads to

$$\left(\frac{v^2}{2} + \frac{p}{\rho} + \hat{\phi}\right) = constant$$

The above equation is called Bernoulli’s equation. This states that the sum of the kinetic energy, pressure and potential energy heads is constant for a non-viscous fluid.

Comparisons of mechanical and thermal energy

Equations (30.18) and (30.20) show that the heat flux by conduction, $\underset{\sim}{q}$, is not present in the equation of mechanical energy and its contribution shows up only in the equation of thermal energy. Similarly, the heat addition by a heat so appears only in the equation of thermal energy.

Also, as is shown letter the term $\left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{v}\right)$ is always positive and appears with a minus sign in the equation of mechanical energy while it appears with a positive sign in the equation of thermal energy. Indicating the mechanical energy always degrades and converts into thermal energy. All these facts indicate that theses equations are consistence with the second law of thermodynamics which specifies that for an isolated system work can be converted into heat but not vice versa .

Temperature explicit form of the equation of thermal energy

Internal energy, $\hat{U}$, is a state function which may be written in the terms of two intensive variables. If the intensive variables are temperature and volume, we may write $\hat{U}(T,\hat{V})$ and

$$d\hat{U} = \left(\frac{\partial \hat{U}}{\partial T}\right)_{\hat{V}} dT + \left(\frac{\partial \hat{U}}{\partial \hat{V}}\right)_{T} d\hat{V} \tag{31.1}$$

where the symbol “^” represents the value of the property per unit mass or per unit mole.

From thermodynamics for fluids

$$d\hat{U} = C_v dT + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right] d\hat{V} \tag{31.2}$$

For example, if fluid is an ideal gas then substituting the ideal gas law in the right hand side of Equation (31.2), we obtain

$$P\hat{V} = RT$$

or

$$\left(\frac{dP}{dT}\right)_{\hat{V}} = \frac{R}{\hat{V}}$$

Thus,

$$d\hat{U} = C_v dT + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{v}}\right] = C_v dT + \left[-P + \frac{TR}{\hat{V}}\right] = 0 \qquad \textbf{(31.3)}$$

For real fluids, Equation (31.2) may be written in the form of the substantial derivative,i.e.,

$$\frac{D\hat{U}}{Dt} = C_v \frac{DT}{Dt} + \left[-P + T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}\right]\frac{D\hat{V}}{Dt} \qquad \textbf{(31.4)}$$

Furthermore, the specific volume $\hat{V}$ is the inverse of the density ρ, or

$$\hat{V} = \left(\frac{1}{\rho}\right) \qquad \textbf{(31.5)}$$

Therefore,

$$\frac{D\hat{V}}{Dt} = \frac{D\left(\frac{1}{\rho}\right)}{Dt} = \frac{-1}{\rho^2}\frac{D\rho}{Dt} \qquad \textbf{(31.6)}$$

Thus, Equation (31.4) reduces to

$$\frac{D\hat{U}}{Dt} = C_v \frac{DT}{Dt} + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} \right]\left(\frac{-1}{\rho^2}\frac{D\rho}{Dt} \right)$$

or

$$\rho\frac{D\hat{U}}{Dt} = \rho C_v \frac{DT}{Dt} + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} \right]\left(\frac{-1}{\rho}\frac{D\rho}{Dt} \right) \qquad \textbf{(31.7)}$$

However, the equation of thermal energy is

$$\rho\frac{D\left(\hat{U}\right)}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} + Sc - P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \left(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) \qquad \textbf{(31.8}$$

From Equations (31.7) and (31.8), we obtain the temperature explicit form of equation of thermal energy as

$$\rho C_v \frac{DT}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - p\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \frac{1}{\rho^2}\left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} \right]\frac{D\rho}{Dt} + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) + Sc \qquad \textbf{(31.9}$$

Using the equation of continuity $\left(\frac{D\rho}{Dt} + \rho\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) = 0 \right)$, in above equation, we finally obtain

$$\rho C_v \frac{DT}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \frac{1}{\rho}\left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} \right]\left[\rho\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\right] + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) + Sc \qquad \textbf{(31.}$$

which simplifies the equation (31.10) leads to

$$\rho C_v \frac{DT}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} \left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) + Sc$$

or

$$\rho C_v \left[\frac{\partial T}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} T\right] = -\underset{\sim}{\nabla}.\underset{\sim}{q} - T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}} \left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) + Sc$$

(3

Equation (31.11) represents the equation of thermal energy in terms of temperatures for a real fluid. Expended form of Equation (31.11) in cartesian, cylindrical and spherical coordinate system is given in Appendix-5. Some limiting cases of Equation (31.11) are discussed below.

Case 1: heat conduction in solids

In solids, all velocities are zero and Equation (31.11) simplifies to

$$\rho C_v \frac{\partial T}{\partial t} = -\underset{\sim}{\nabla}.\underset{\sim}{q} + Sc$$

(3

where, the heat flux $\underset{\sim}{q}$ may be estimated by the Fourier’s law of heat conduction. If *k* is a constant then

$$\underset{\sim}{q} = -k\underset{\sim}{\nabla} T$$

or,

$$\rho C_v \frac{\partial T}{\partial t} = +k\nabla^2 T + Sc$$

Equation (31.13) is also known as the Fourier's second law of heat conduction.

Case 2 Heat transfer in fluids with constant ρ and k

For constant density, equation of continuity reduces to

$$\underset{\sim}{\nabla}.\underset{\sim}{v} = 0$$

In addition, the heat capacity at constant volume and constant pressures are the same. Thus,

$$C_v = C_p$$

For this case, the equation of thermal energy may be simplified to

$$\rho C_v \left[\frac{\partial T}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} T \right] = k\nabla^2 T + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v} \right) + Sc$$

or

$$\frac{\partial T}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} T = \frac{k}{\rho C_p}\nabla^2 T + \frac{\left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v} \right)}{\rho C_p} + \frac{Sc}{\rho C_p}$$

Equation (31.18) represents the equation of thermal energy for constant density and heat conductivity. It may be noted that Equation (31.18) has a similar form as the equation of motion for constant density and viscosity (Navior Stoke equation), i.e.,

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \rho.\underset{\sim}{\nabla}\underset{\sim}{v} = \frac{\mu}{\rho}\nabla^2 \underset{\sim}{v} - \frac{\underset{\sim}{\nabla} p}{\rho} + \underset{\sim}{g}$$

Both equations show the similarities between momentum and heat transport. The detail form of Equation (31.18) for different coordinate system is given in the Appendix -5

2.8 Viscous heat dissipation / viscous heating

The viscous heat dissipation term $\left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v} \right)$, in the equation of thermal energy represents the conversion of mechanical energy into thermal energy due to viscous dissipation. This term is always positive. For Newtonian fluids, $\underset{\sim}{\tau}$ may be calculated by using the Newton's law viscosity. The details are not shown here but may be found elsewhere, i.e.,

$$\underset{\sim}{\tau} = -\mu \underset{\sim}{\nabla} \underset{\sim}{v}$$

or

$$-\underset{\sim}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{v} = \mu \phi_v$$

where, Φ_v is a scalar quantity and the value of Φ_v for different coordinate system is given in the Appendix -6. As may be seen from this table that all terms present in the expression from Φ_v are positive. Thus, this viscous dissipation leads to increase the value of thermal energy and raises the temperature of fluid.

Significance of Viscous dissipation / heating

As discussed in previous lecture, the viscous dissipation leads to rise in the temperature of fluid. Here, we solve a problem which help us to understand the significance of this term in terms of rise in temperature.

Consider a fluid flowing under laminar conditions between two parallel plates which are kept at same temperature T0 as shown in Fig. (32.1). The incoming fluid, at z = 0, have the same temperature T0. Assuming steady state, determine the increase in temperature of fluid due to viscous heat dissipation.

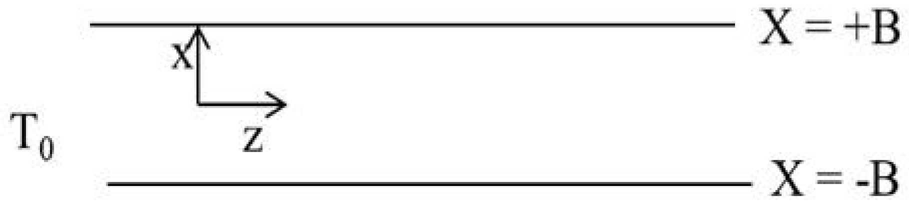

Fig 32.1 Viscous flow between two parallel plates

Assumptions

- Density ρ, viscosity μ and thermal conductivity k are constants.
- System is in steady state.

- Flow is laminar (simple shear flow) and fully developed.
- Newton’s law of viscosity is applicable.

Fluid is flowing in z direction only and the Non-zero velocity components are
vx = 0, vy = 0, vz = vz(x)
the velocity profile vz may be easily obtained for a Newtonian fluid as given below

$$v_z = v_{z,max}\left[1-\frac{x^2}{B^2}\right] \tag{32.1}$$

where $v_{z,max}$ is the maximum velocity of fluid.

At steady state, the heat produced by viscous dissipation is removed from both plates to keep the temperature of the plates at T_0. Thus, temperature of the fluid increases until the heat generation by viscous dissipation matches the heat remove from the plates and the temperature is no longer a function of z coordinate. For this fully developed region, the temperature is a function of *x* coordinate only, i.e.,

$$T = T(x) \tag{32.2}$$

For this case, the equation of thermal energy may be simplified as follows

$$0 = k\frac{d^2T}{dx^2} + \mu\varphi_v \tag{32.3}$$

Substituting the value of viscous dissipation in the equation of thermal energy, we have

$$0 = k\frac{d^2T}{dx^2} + \mu\left(\frac{dv_z}{dx}\right)^2 \tag{32.4}$$

The velocity gradient in equation (32.4) may be estimated from Equation (32.1) as

$$\frac{dv_z}{dx} = v_{z\,max}\left(\frac{-2x}{b^2}\right) \tag{32.5}$$

Therefore, the Equation (32.4) finally simplifies as

$$0 = k\frac{d^2T}{dx^2} + \mu\frac{4x^2}{B^4}v_{z,max}^2$$

or

$$\frac{d^2T}{dx^2} = -\left(\frac{4\mu}{k}\frac{v_{z,max}^2}{B^4}\right)x^2 = Ax^2 \tag{32.6}$$

where, A is a constant.

Integrating Equation (32.6), we finally obtain the temperature profile as

$$T = \frac{Ax^4}{12} + c_1x + c_2 \tag{32.7}$$

where c_1 and c_2 are the constants of integrations. These constants may be determined by using the following boundary conditions,

$$x = \pm B, T = T_0 \tag{32.8}$$

or

$$T_0 = \frac{AB^4}{12} \pm c_1B + c_2 \tag{32.9}$$

which leads to

$$c_1 = 0 \tag{32.10}$$

and

$$c_2 = T_0 - \frac{AB^4}{12} \tag{32.11}$$

Thus, the rise in temperature of fluid is given by the following equation by substituting the value of c_1 and c_2 in Equation (32.7), we finally obtain,

$$T - T_0 = \frac{-AB^4}{12}\left[1-\left(\frac{x}{B}\right)^4\right]$$

or

$$T - T_0 = \frac{\mu v_{z,max}^2}{3k}\left[1-\left(\frac{x}{B}\right)^4\right] \tag{32.12}$$

Equation (32.12) shows that the rise in temperature is maximum at $x = 0$ and is given by

$$\left(T - T_0\right)_{max} = \frac{\mu v_{z,max}^2}{3k} \tag{32.13}$$

For example, if water is flowing at a maximum velocity *100 ft/sec*, then we find,

$$\left(T - T_0\right)_{max} = \frac{\mu v_{z,max}^2}{3k} = \frac{100^2}{3}\,\frac{\mu_{water\ 25^0 C}}{k_{water\ 25^0 C}} \approx 1^\circ F \tag{32.14}$$

Thus, even for such large velocities, the rise in temperature is only about 1^0F. For highly viscous fluids, such as polymer solutions with viscosity *1000 Cp*, the temperature rise may be

$$(T - T_0)_{max} \approx 1000^\circ F \tag{32.15}$$

Therefore, we may conclude that the viscous dissipation may be important for highly viscous fluids, for low viscosity fluid like air, the viscous dissipation term may be safely neglected.

2.9 Transpiration Cooling/ heating

Transpiration cooling / heating is used to reduce or enhance the heat transfer rates by a convective flow of fluid in or opposite direction of the actual heat transfer. Additional convective flow provides forced convection to produce the desired effect. A classic example of the transpiration cooling is the design of a storage tank for liquefied gas nitrogen or oxygen. Both liquefied gas have cryogenic boiling points. The required thickness of insulation material may be quite high in order to reduce heat gained by conduction from surrounding at atmospheric temperature to very low temperature in the tank. Here, we may use transpiration cooling to reduce the heat transfer. Design of a cryogenic storage tank is shown below

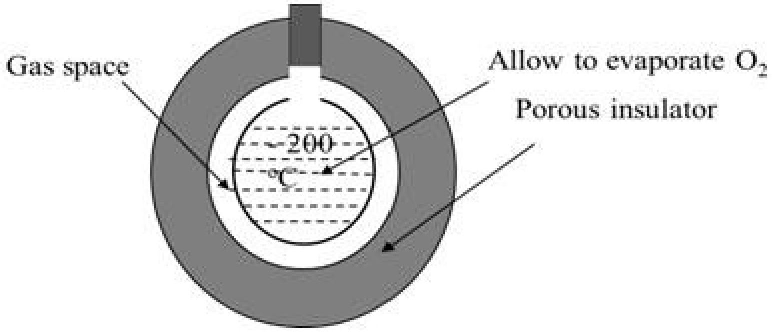

Fig 3 Use of transpiration cooling in liquefied gas container

Here, the cryogenic liquid is stored in a spherical container, which is surrounded by a porous insulation, as shown in Fig. (32.2). Some amount of stored liquid is allowed to vaporize. This vaporized gas fills the space between liquid gas container and the porous insulation. It provides the cold jacket to the container and reduces the heat transfer. As this gas starts to diffuse through pores, it provides a convective transpiration cooling and prevents the heat transferred from outside to the liquid container.

Here, we solve a simpler example of transpiration cooling. Assume the radius of two concentric porous spheres are *kR* and *R* as shown in Fig. (32.3). The temperature of inner sphere is T_k and outer sphere is T_0. The inner sphere is refrigerated. An air stream at temperature T_k with the mass flow rate w is forced from the inner sphere to outer sphere for maintaining the internal sphere temperature at T_k. Determine the heat flux with and without transpiration cooling(w = 0).

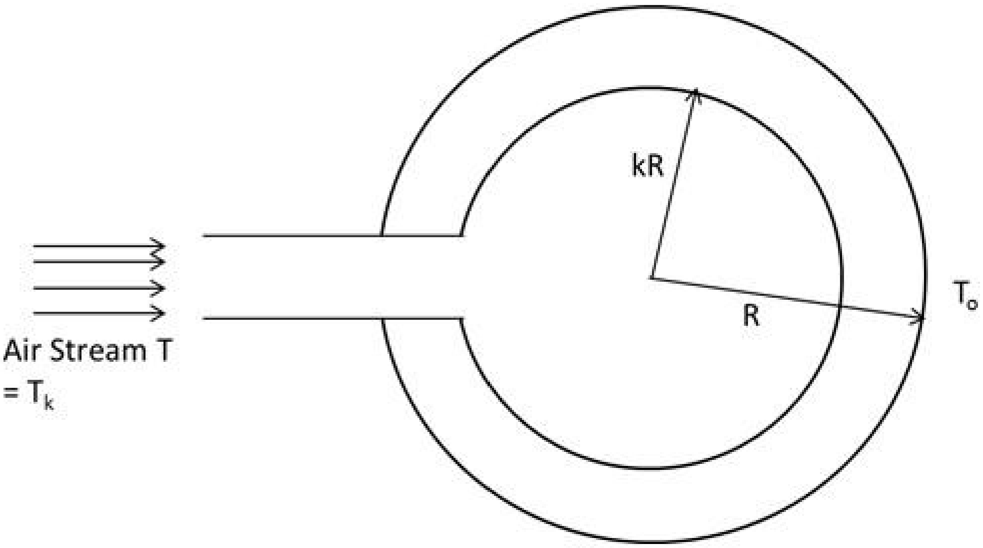

Fig 32.3 Transpiration cooling between two concentric spherical shell

Assumptions

- Density *ρ*, viscosity *μ* & thermal conductivity *k* are constant.
- System is in steady state.
- Flow is laminar (simple shear flow) and fully developed.

Air flows only in the radial direction. Therefore, the non-zero velocity components are

$$v_\theta = v_\phi = 0 \;\&\; v_r = v_r\left(r\right) \qquad \textbf{(32.16)}$$

Assume, the temperature is changing only in radial direction, or,

$$T = T(r) \quad (32.17)$$

The equation of continuity in spherical coordinates provides,

$$\frac{1}{r}\frac{d}{dr}(\rho r^2 v_r) = 0$$

By integrating above equation, we obtain

$$\rho r^2 v_r = constant \quad (32.18)$$

Here, the mass flow rate may be calculated as

$$w = 4\pi r^2 \rho v_r \quad (32.19)$$

Thus,

$$v_r = \frac{w}{4\pi\rho r^2} \quad (32.20)$$

Since, the viscosity of air is very low, we may neglect the viscous dissipation term from the equation of thermal energy, or

$$\rho C_p v_r \frac{dT}{dr} = k\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)\right] \quad (32.21$$

Substituting the value of v_r from Equation (32.20) in above equation, we obtain

$$\rho C_p \frac{w}{4\pi\rho r^2}\frac{dT}{dr} = \frac{k}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr} \right)$$

or

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr} \right) = \frac{C_p w}{4\pi k}\frac{dT}{dr} = R_0 \frac{dT}{dr}$$

(32.22)

where,

$$R_0 = \frac{C_p w}{4\pi k}$$

(32.23)

Equation (32.22) may be solved as follows

$$\frac{dT}{dr} = Y$$

(32.24

Thus, Equation (32.22) may be rewritten as,

$$\frac{d}{dr}\left(r^2 Y \right) = R_0 Y$$

(32.2

or

$$r^2 \frac{dY}{dr} + 2rY = R_0 Y$$

or

$$\frac{dY}{Y} = \frac{(R_0 - 2r)dr}{r^2}$$

which may be integrated to gives the following expression,

$$ln\,Y + ln\,r^2 = \frac{-R_0}{r} + c_1 = ln\left(Yr^2\right)$$ (32

Here, c_1 is the integral constant. Thus, we have

$$Y = \frac{c_2}{r^2}\,e^{-(R_0/r)} = \frac{dT}{dr}$$ (3

where, $c_2 = e^{c_1}$

By integrating Equation (32.27), we finally obtain

$$T = c_2\int\frac{e^{-(R_0/r)}}{r^2}dr + c_3$$ (

Here, c_3 is another integration constant.

Boundary conditions for this problem are

at
$r = R,\ T = T_0$

and at
$r = KR, T = T_k$ (

Finally evaluating c_2 and c_3 , we obtain the temperature profile between $r = KR$ to R,

$$\frac{T - T_0}{T_k - T_0} = \frac{e^{-R_0/r} - e^{-R_0/R}}{e^{-R_0/KR} - e^{-R_0/R}}$$

Now, we may calculate heat loss from inner sphere by first calculating the temperature gradient from above equation then multiplying it by the surface area of the sphere. Thus,

$$Q = \left(area\ of\ inner\ sphere \right) \times \left(heat\ flux \right)\Big|_{r=kR}$$

or

$$Q = -\left(4\pi r^2 q_r \right)\Big|_{r=kR}$$

where, $q_r = -k\dfrac{dT}{dr}$

Here, '−' sign is used as the outer normal of the inner sphere is in negative r direction. Thus,

$$Q = \frac{4\pi k R_0 (T_0 - T_k)}{e^{(R_0/KR)(1-K)} - 1}$$

where

$$R_0 = \frac{w_r C_p}{4\pi k}$$

Now, consider the situation when transpiration cooling in not used. Therefore, the velocity of air stream v_r is zero. In this case, the equation of thermal energy may be written as

$$\left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) \right] = 0$$

By integrating Equation (32.35) and using the above boundary conditions, we obtain

$$\frac{T-T_0}{T_k-T_0}=\frac{K}{1-K}\left(\frac{1}{r}-\frac{1}{R}\right)$$

We may once again calculate the heat loss from inner sphere by following the same procedure as earlier, thus,

$$Q_0=\frac{4\pi KR(T_0-T_k)}{1-K}$$

Here, Q_0 is the heat loss without transpiration cooling. The transpiration cooling efficiency may now be defined as

$$\varepsilon=\frac{Q_0-Q}{Q_0}=1-\frac{\varphi}{e^{\varphi}-1}$$

where,

$$\varphi=\frac{R_0(1-K)}{KR}$$

Relation between the transpiration efficiency and Φ is shown in Fig. (32.4). As may be seen from this figure, the transpiration efficiency increases with increase in the value of Φ or mass flow rate, *w*. For very large values of Φ, transpiration efficiency approaches to one, which implies that heat transfer from outer sphere to inner sphere, Q, is negligible.

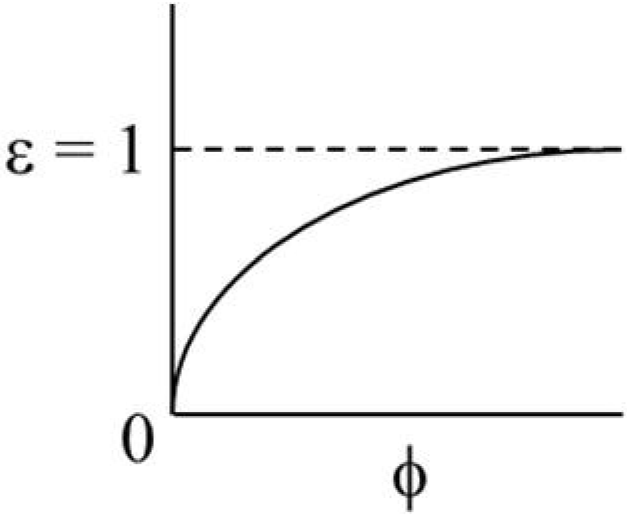

Fig 32.4 Efficiency of transpiration cooling

2.10 Heat conduction in a rectangular fin

Fins are used to enhance the heat transfer from any surface by increasing the heat transfer surface area. Due to heat conduction, this additional surface is at lower and lower temperatures as more and more surface area is added. Thus, the rate of heat transfer decreases as we move away from the original surface and efficiency of the fin is reduced. The effectiveness of a fin is defined as the ratio of actual heat transfer through the fin and heat transfer when the whole fin surface is available at the same temperature as that of the original surface. A simple rectangular fin is shown in Fig. (33.1). The wall temperature is T_w and the ambient temperature is T_a. Dimensions of fin are as shown in Fig. (33.1). Formulate the problem and determine the temperature profile. Finally also calculate the efficiency of the fin.

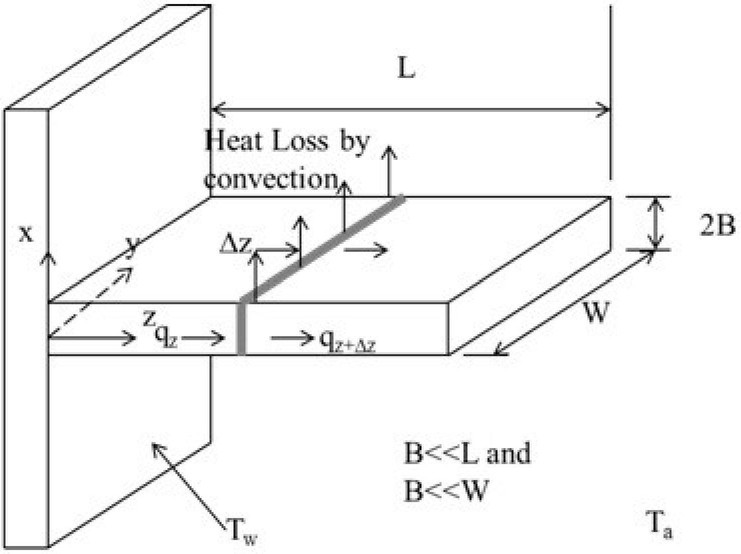

Fig 33.1 Rectangular fin

Assumption

- System is at steady state.
- Fin has a constant heat conductivity.
- The outside heat transfer coefficient is h_a.

We may solve this problem for two different cases.

Case 1: when L, W and B are of the same order of magnitude

In this case the temperature is a function of *x*, *y* and *z* coordinates, i.e.,

$$T = T\left(x,\, y,\, z\right)$$

and the equation of thermal energy may be written as

$$-\underset{\sim}{\nabla}.\underset{\sim}{q} = 0$$

After substituting the Fourier's law of conduction, we obtain

$$k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]=0$$

The above equation may needs to be solved subject to the following boundary conditions,

$z = 0$, $T = T_w$

$z = L$, $q_z\big|_{z=L} = h_a(T\big|_{z=L} - T_a)$

$x = +B$, $q_z\big|_{x=B} = h_a(T\big|_{x=B} - T_a)$

$x = -B$, $q_z\big|_{x=-B} = h_a(T\big|_{x=-B} - T_a)$

$y = 0$, $q_y\big|_{y=w} = h_a(T\big|_{y=w} - T_a)$

$y = w$, $q_y\big|_{y=w} = h_a(T\big|_{y=0} - T_a)$

and it is obvious that Equation (33.3) can not be solved analytically and requires computational methods.

Case 2: Now, consider a case when thickness of the fin is negligible in comparison with height and width, i.e., L & W >> 2B

In this case, the heat flux in y direction is small and we may write temperature as function of *x* and *z* coordinates only,

$$T = T(x, z)$$

In this case, the equation of thermal energy may be simplified as

$$k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right] = 0$$

This may also require numerical solutions. To find simple analytical solution, we may further assume that the average temperature, T_{avg} , at any cross section of fin is a function of *z* coordinate only, i.e.,

$$T_{avg} = T_{avg}(z)$$

Thus, we are reducing the original two dimensional into a one-dimensional problem to obtain an approximate solution. This assumption is reasonably valid since the thickness of the fin is very small and the variation of temperature in *x* direction may be averaged out. Since, we have reduced a two dimensional problem to a one-dimensional problem by averaging the temperature, the equation of thermal energy is no longer applicable. We may use the shell energy balance approach to find a appropriate solution. For the sake of convenience,T_{avg} is now replaced by T . The control volume is a strip of the fin with thickness *2B*, length Δz and width *W* as shown in Fig. (33.2).

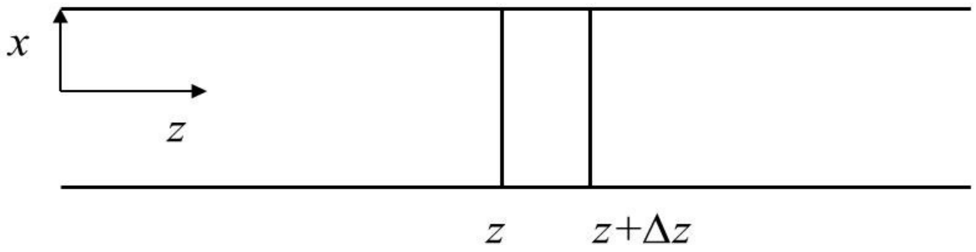

Fig 33.2 Control volume for a rectangular fin

Heat entering the control volume by conduction

$$= q_z \big|_z \times 2BW$$

Heat leaving the control volume by conduction

$$= q_z \big|_{z+\Delta z} \times 2BW$$

Heat loss from upper and lower surfaces of the control volume

$$= 2h_a\left(T - T_a\right) W\Delta z$$

Thus, energy balance may be written as

$$(q_z\big|_z - q_z\big|_{z+\Delta z})\ 2BW - 2h_a\left(T - T_a\right) W\Delta z = 0$$

After dividing the Equation (33.17) by *2BWΔz* and taking the limit Δz→0 , we obtain

$$\frac{dq_z}{dz} = -\frac{h_a}{B}(T - T_a)$$

Fourier’s law may be applied here as

$$q_z = -k\frac{dT}{dz}$$

and finally, we obtain

$$\frac{d^2T}{dz^2} = \frac{h_a}{kB}(T - T_a)$$

The Boundary conditions may be written as

at

$$z=0,\ T=T_w$$

and at

$$z=L, \quad -k\left.\frac{dT}{dz}\right|_{z=L}=h_a\left(T|_{z=L}-T_a\right)$$

However, for a sufficiently long fin $\left(T|_{z=L}-T_a\right)$ is small and we may assume that at

$$z=L, \left.\frac{dT}{dz}\right|_{z=L}=0$$

By applying the above boundary conditions in Equation (33.20), we obtain the following solution

$$\frac{(T-T_a)}{(T_w-T_a)}=\cosh N\varepsilon-(\tanh N)\sinh N\varepsilon$$

where

$$N=\frac{hL^2}{KB}$$ and

$$\varepsilon=\frac{z}{L}$$

This equation may be arranged as

$$\theta=\frac{\cosh N(1-\varepsilon)}{\cosh N}$$

with

$$\theta=\frac{T-T_a}{T_w-T_a}$$

Going back to the original two dimensional problem given in Equation (33.11), even without assuming anything further, we may still obtain the same results as given by Equation (33.26), provided *B* is very small. Thus, the fin temperature is the function of both *x* and *z* coordinates, i.e., *T*(*x*,*z*), and the equation of thermal energy may be written as

$$\frac{\partial q_x}{\partial x}+\frac{\partial q_z}{\partial z}=0$$

If *2B* is small (from the definition of a differential), we may approximate the first term as

$$\frac{\partial q_x}{\partial x}=\lim_{B\to 0}\frac{q_x\big|_{x=B}-q_x\big|_{x=-B}}{2B}$$

By applying the boundary conditions given in the Equations (33.6) and (33.7), we get,

$$\frac{\partial q}{\partial x}\cong\frac{h_a(T-T_a)-\left(-h_a(T-T_a)\right)}{2B}$$

or

$$\frac{\partial q}{\partial x}\cong\frac{h_a}{B}(T-T_a)$$

where the partial derivative is replaced by the total derivative. By substituting the Equation (33.31) in (33.28), we again obtain

$$\frac{dq_z}{dz}+\frac{h_a(T-T_a)}{B}=0$$

where the partial derivative is replaced by the total derivative. After using the Fourier's law of heat conduction, we obtain the following differential equation

$$-k\frac{d^2T}{dz^2}+\frac{h_a}{B}(T-T_a)=0$$

Equation (33.33) has the same form as Equation (33.20) and therefore the solutions is also the same

Efficiency of the rectangular fin

As state above , the efficiency of fin may be defined as,

$$\eta = \frac{Actual\ rate\ of\ heat\ loss\ from\ fin}{Rate\ of\ heat\ loss\ from\ an\ isothermal\ fin\ at\ T_w}$$

Substituting the values, we get

$$\eta = \frac{\int_0^W \int_0^L h_a\ (T - T_a) dz\ dy}{\int_0^W \int_0^L h_a\ (T_W - T_a) dz\ dy}$$

which may be simplified by using the above dimensionless numbers, defined in Equation (33.25) and (33.27), as

$$\eta = \frac{\int_0^W \theta d\varepsilon}{\int_0^W d\varepsilon}$$

The Equation (33.35) leads to the following simple solution

$$\eta = \frac{1}{coshN} \left(-\frac{1}{N} sinhN(1-\varepsilon) \right) \Bigg|_0^1$$

which may be further simplified as

$$\eta = \frac{tan h\, N}{N}$$

As may be noted from above equation η depends only on N. A simple plot for η vs N is shown below

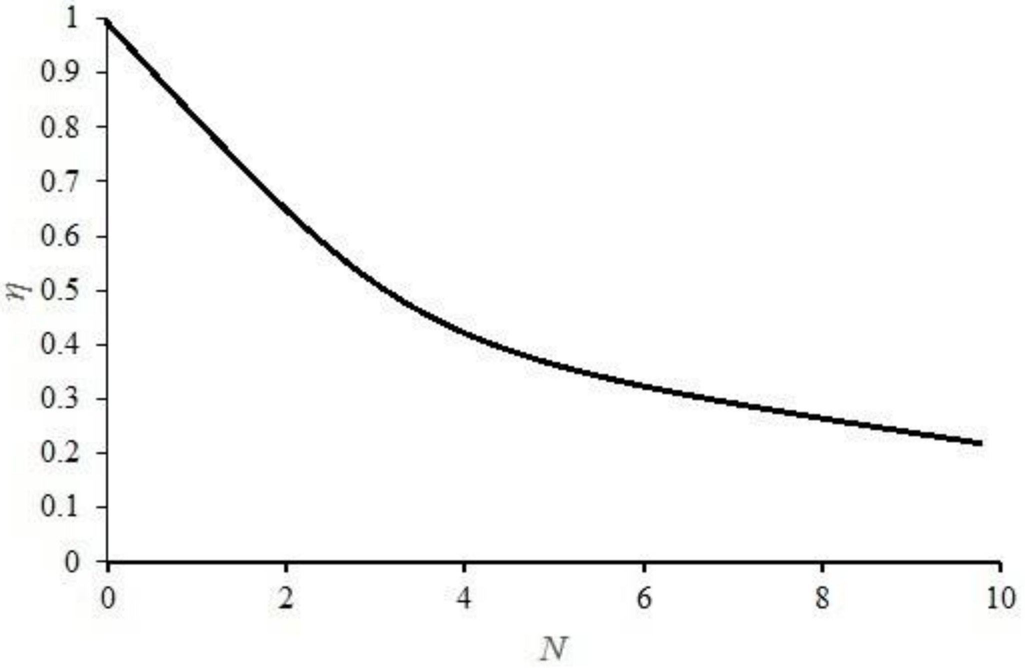

Fig 33.3 Efficiency of fin vs N

Fig. 4 shows that long fins have lesser efficiency. The optimum fin length may be found by optimizing the enhancement in heat transfer and the cost of this additional surface area. In the real life, the fins need not to be rectangular in shape and the different shaped fins may be designed for enhanced fin efficiency.

Fourier's Law of Heat Conduction

$$\underset{\sim}{q} = -k \underset{\sim}{\nabla} T$$

Cartesian coordinates(x,y,z)

$$q_x = -k\frac{\partial T}{\partial x}$$

$$q_y = -k\frac{\partial T}{\partial y}$$

$$q_z = -k\frac{\partial T}{\partial z}$$

Cylindrical coordinates (r,θ,z)

$$q_r = -k\frac{\partial T}{\partial r}$$

$$q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$q_z = -k\frac{\partial T}{\partial z}$$

Spherical coordinate (r, θ,Φ)

$$q_r = -k\frac{\partial T}{\partial r}$$

$$q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$q_\phi = -k\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

Equation of Thermal Energy in Terms of heat flux

$$\rho\hat{c}_{\hat{V}}\left[\frac{\partial T}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}T\right]=-\underset{\sim}{\nabla}.\underset{\sim}{q}-T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)+\left(-\underset{\sim}{\tau}:\underset{\sim}{\nabla}\underset{\sim}{v}\right)+Sc$$

Cartesian coordinates(x,y,z)

$$\rho\hat{c}_{\hat{V}}\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=-\left[\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right]-T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)-(\underset{\sim}{\tau}:\underset{\sim}{\nabla}$$

Cylindrical coordinates (r,θ,z)

$$\rho\hat{c}_{\hat{V}}\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=-\left[\frac{1}{r}\frac{\partial}{\partial r}(rq_r)+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}\right]-T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}(\underset{\sim}{\nabla}.\underset{\sim}{v})$$

Spherical coordinate (r, θ,Φ)

$$\rho\hat{c}_{\hat{V}}\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)=-\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2q_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(q_0\sin\theta)+\right.$$

$$-T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}(\underset{\sim}{\nabla}.\underset{\sim}{v})-(\underset{\sim}{\tau}:\underset{\sim}{\nabla}\underset{\sim}{v})+Sc$$

Equation of Energy for Pure Newtonian Fluid with Constant ρ and k

$$\frac{\partial T}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}T=\frac{k}{\rho\hat{c}_p}\nabla^2T+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Cartesian coordinates(x,y,z)

$$\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=\frac{k}{\rho\hat{c}_p}\left[\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}+\frac{\partial^2T}{\partial z^2}\right]+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Cylindrical coordinates (r,θ,z)

$$\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=\frac{k}{\rho\hat{c}_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2T}{\partial \theta^2}+\frac{\partial^2T}{\partial z^2}\right]+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Spherical coordinate (r, θ,Φ)

$$\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)=\frac{k}{\rho\hat{c}_p}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right.$$

$$+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Appendix - 6

Dissipation Function for

Newtonian Fluids Cartesian

$$\phi_v = 2\left(\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right) + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2$$

coordinates(x,y,z)

Cylindrical coordinates (r,θ,z)

$$\phi_v = 2\left(\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right) + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]^2$$

$$-\frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right]^2$$

Spherical coordinate (r, θ,Φ)

$$\phi_v = 2\left(\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot\theta}{r}\right)^2\right) + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial}{\partial}\right.$$

$$+\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2$$

$$-\frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]^2$$

SCHOOL OF BIO AND CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – III – TRANSPORT PHENOMENA – SCH1309

Contents

3 MASS TRANSFER

3.1 Introduction of Mass Transfer

When a system contains two or more components whose concentrations vary from point to point, there is a natural tendency for mass to be transferred, minimizing the concentration differences within a system. The transport of one constituent from a region of higher concentration to that of a lower concentration is called mass transfer.

The transfer of mass within a fluid mixture or across a phase boundary is a process that plays a major role in many industrial processes. Examples of such processes are:

(i) Dispersion of gases from stacks
(ii) Removal of pollutants from plant discharge streams by absorption
(iii) Stripping of gases from waste water
(iv) Neutron diffusion within nuclear reactors
(v) Air conditioning

Many of air day-by-day experiences also involve mass transfer, for example:

(i) A lump of sugar added to a cup of coffee eventually dissolves and then eventually diffuses to make the concentration uniform.
(ii) Water evaporates from ponds to increase the humidity of passing-air-stream
(iii) Perfumes presents a pleasant fragrance which is imparted throughout the surrounding atmosphere.

The mechanism of mass transfer involves both molecular diffusion and convection.

The mole fraction for liquid and solid mixture, x_A ,and for gaseous mixtures, y_A, are the molar concentration of species A divided by the molar density of the mixtures.

$$x_A = \frac{C_A}{C} \quad \text{(liquids and solids)}$$

$$y_A = \frac{C_A}{C} \quad \text{(gases).}$$

The sum of the mole fractions, by definition must equal 1;

(i.e.) $$\sum_i x_i = 1$$

$$\sum_i y_i = 1$$

by similar way, mass fraction of A in mixture is;

$$w_A = \frac{\rho_A}{\rho}$$

1. The molar composition of a gas mixture at 273 K and $1.5 * 10^5$ Pa is:

$$\frac{n}{V} = \text{molar density} = \rho_m$$

Therefore, density (or mass density) = $\rho_m M$

Where M is the molecular weight of the gas.

$$Density = \rho_m M = \frac{PM}{RT} = \frac{1.5 * 10^5 * 30.68}{8314 * 273}\ kg/m^3$$

$$= 2.03\ kg/m^3$$

Partial pressure of O_2 = [mole fraction of O_2] * total pressure

$$= \left[\frac{7}{100} * 1.5\right] * 10^5$$

$$= 0.07 * 1.5 * 10^5$$

$$= 0.105 * 10^5\ Pa$$

3.2 Diffusion flux

Just as momentum and energy (heat) transfer have two mechanisms for transport-molecular and convective, so does mass transfer. However, there are convective fluxes in mass transfer, even on a molecular level. The reason for this is that in mass transfer, whenever there is a driving force, there is always a net movement of the mass of a particular species which results in a bulk motion of molecules. Of course, there can also be convective mass transport due to macroscopic fluid motion. In this chapter the focus is on molecular mass transfer.

The mass (or molar) flux of a given species is a vector quantity denoting the amount of the particular species, in either mass or molar units, that passes per given increment of time through a unit area normal to the vector. The flux of species defined with reference to fixed spatial coordinates, N_A is

$$N_A = C_A \nu_A \qquad \text{---------------------- (1)}$$

This could be written interms of diffusion velocity of A, (i.e., $\nu_A - \nu$) and average velocity of mixture, ν, as

$$N_A = C_A (\nu_A - \nu) + C_A \nu \qquad \text{--------------- (2)}$$

By definition

$$u^* = \frac{\sum_i C_i u_i}{C}$$

Therefore, equation (2) becomes

$$N_A = C_A (u_A - u^*) + \frac{C_A}{C} \sum_i C_i u_i$$

$$= C_A (u_A - u^*) + y_A \sum_i C_i u_i$$

For systems containing two components A and B,

$$N_A = C_A (u_A - u^*) + y_A (C_A u_A + C_B u_B)$$
$$= C_A (u_A - u^*) + y_A (N_A + N_B)$$
$$N_A = C_A (u_A - u^*) + y_A N \qquad \text{----------- (3)}$$

The first term on the right hand side of this equation is diffusional molar flux of A, and the second term is flux due to bulk motion.

3.2.1 Fick’s law:

An empirical relation for the diffusional molar flux, first postulated by Fick and, accordingly, often referred to as Fick’s first law, defines the diffusion of component A in an isothermal, isobaric system. For diffusion in only the Z direction, the Fick’s rate equation is

$$J_A = -D_{AB} \frac{dC_A}{dZ}$$

where D_{AB} is diffusivity or diffusion coefficient for component A diffusing through component B, and dC_A / dZ is the concentration gradient in the Z-direction.

A more general flux relation which is not restricted to isothermal, isobasic system could be written as

$$J_A = -C D_{AB} \frac{dy_A}{dZ} \qquad \text{----------------- (4)}$$

using this expression, Equation (3) could be written as

$$N_A = -C D_{AB} \frac{dy_A}{dZ} + y_A N \qquad \text{--------------- (5)}$$

Relation among molar fluxes:

For a binary system containing A and B, from Equation (5),

$$N_A = J_A + y_A N$$

or $$J_A = N_A - y_A N \quad \text{-----------------------} \ (6)$$

Similarly,

$$J_B = N_B - y_B N \quad \text{--------------------} \ (7)$$

Addition of Equation (6) & (7) gives,

$$J_A + J_B = N_A + N_B - (y_A + y_B) N \quad \text{----------} \ (8)$$

By definition N = $N_A + N_B$ and $y_A + y_B = 1$.
Therefore equation (8) becomes,

$$J_A + J_B = 0$$
$$J_A = -J_B$$

$$C D_{AB} \frac{d y_A}{d z} = - C D_{BA} \frac{d y_B}{d Z} \quad \text{---------------} \ (9)$$

From $y_A + y_B = 1$
$dy_A = - dy_B$

Therefore Equation (9) becomes,

$$D_{AB} = D_{BA} \quad \text{----------------------} \quad (10)$$

This leads to the conclusion that diffusivity of A in B is equal to diffusivity of B in A.

Diffusivity

Fick's law proportionality, D_{AB}, is known as mass diffusivity (simply as diffusivity) or as the diffusion coefficient. D_{AB} has the dimension of L^2 / t, identical to the fundamental dimensions of the other transport properties: Kinematic viscosity, $\nu = (\mu / \rho)$ in momentum transfer, and thermal diffusivity, α ($= k / \rho C_p$) in heat transfer.

Diffusivity is normally reported in cm^2 / sec; the SI unit being m^2 / sec.

Diffusivity depends on pressure, temperature, and composition of the system.

In table, some values of D_{AB} are given for a few gas, liquid, and solid systems.

Diffusivities of gases at low density are almost composition independent, incease with the temperature and vary inversely with pressure. Liquid and solid diffusivities are strongly concentration dependent and increase with temperature.

General range of values of diffusivity:

Gases :	5×10^{-6}	-------------	1×10^{-5}	m^2 / sec.
Liquids :	10^{-6}	-------------	10^{-9}	m^2 / sec.
Solids :	5×10^{-14}	-------------	1×10^{-10}	m^2 / sec.

In the absence of experimental data, semitheoretical expressions have been developed which give approximation, sometimes as valid as experimental values, due to the difficulties encountered in experimental measurements.

Diffusivity in Gases:

Pressure dependence of diffusivity is given by

$$D_{AB} \propto \frac{1}{p}$$ (for moderate ranges of pressures, upto 25 atm).

And temperature dependency is according to

$$D_{AB} \propto T^{3/2}$$

Diffusivity of a component in a mixture of components can be calculated using the diffusivities for the various binary pairs involved in the mixture. The relation given by Wilke is

$$D_{1-mixture} = \frac{1}{\frac{y'_2}{D_{1-2}} + \frac{y'_3}{D_{1-3}} + \dots\dots\dots\dots + \frac{y'_n}{D_{1-n}}}$$

Where $D_{1-mixture}$ is the diffusivity for component 1 in the gas mixture; D_{1-n} is the diffusivity for the binary pair, component 1 diffusing through component n; and y'_n is the mole fraction of component n in the gas mixture evaluated on a component –1 – free basis, that is

$$y'_2 = \frac{y_2}{y_2 + y_3 + \dots\dots\dots\dots y_n}$$

Diffusivity in liquids:

Diffusivity in liquid are exemplified by the values given in table … Most of these values are nearer to 10^{-5} cm^2 / sec, and about ten thousand times shower than those in dilute gases. This characteristic of liquid diffusion often limits the overall rate of processes accruing in liquids (such as reaction between two components in liquids).

In chemistry, diffusivity limits the rate of acid-base reactions; in the chemical industry, diffusion is responsible for the rates of liquid-liquid extraction. Diffusion in liquids is important because it is slow.

Certain molecules diffuse as molecules, while others which are designated as electrolytes ionize in solutions and diffuse as ions. For example, sodium chloride (NaCl), diffuses in water as ions Na^+ and Cl^-. Though each ions has a different mobility, the electrical neutrality of the solution indicates the ions must diffuse at the same rate; accordingly it is possible to speak of a diffusion coefficient for molecular electrolytes such as NaCl. However, if several ions are present, the diffusion rates of the individual cations and anions must be considered, and molecular diffusion coefficients have no meaning.

Diffusivity varies inversely with viscosity when the ratio of solute to solvent ratio exceeds five. In extremely high viscosity materials, diffusion becomes independent of viscosity.

Diffusivity in solids:

Typical values for diffusivity in solids are shown in table. One outstanding characteristic of these values is their small size, usually thousands of time less than those in a liquid, which are inturn 10,000 times less than those in a gas.

Diffusion plays a major role in catalysis and is important to the chemical engineer. For metallurgists, diffusion of atoms within the solids is of more importance.

3.2.2 Steady State Diffusion

In this section, steady-state molecular mass transfer through simple systems in which the concentration and molar flux are functions of a single space coordinate will be considered.

In a binary system, containing A and B, this molar flux in the direction of z, as given by Eqn (5) is [section 3.3.1]

$$N_A = -C D_{AB} \frac{dy_A}{dz} + y_A (N_A + N_B) \quad \text{--- (1)}$$

Problem. Oxygen is diffusing in a mixture of oxygen-nitrogen at 1 std atm, 25°C. Concentration of oxygen at planes 2 mm apart are 10 and 20 volume % respectively. Nitrogen is non-diffusing.

(a) Derive the appropriate expression to calculate the flux oxygen. Define units of each term clearly.

(b) Calculate the flux of oxygen. Diffusivity of oxygen in nitrogen = 1.89 * 10 $^{-5}$ m ^{2}/sec.

Solution:

Let us denote oxygen as A and nitrogen as B. Flux of A (i.e.) N $_A$ is made up of two components, namely that resulting from the bulk motion of A (i.e.), Nx $_A$ and that resulting from molecular diffusion J $_A$:

$$N_A = N x_A + J_A \qquad (1)$$

From Fick's law of diffusion,

$$J_A = -D_{AB}\frac{dC_A}{dz} \text{------------------------------} (2)$$

Substituting this equation (1)

$$N_A = N x_A - D_{AB}\frac{dC_A}{dz} \text{------------------------------} (3)$$

Since N = N $_A$ + N $_B$ and x $_A$ = C $_A$ / C equation (3) becomes

$$N_A = (N_A + N_B)\frac{C_A}{C} - D_{AB}\frac{dC_A}{dz}$$

Rearranging the terms and integrating between the planes between 1 and 2,

$$-\int\frac{dz}{cD_{AB}} = \int_{C_{A1}}^{C_{A2}}\frac{dC_A}{N_A C - C_A(N_A + N_B)} \text{--------------} (4)$$

Since B is non diffusing N $_B$ = 0. Also, the total concentration C remains constant. Therefore, equation (4) becomes

$$\frac{z}{CD_{AB}} = -\int_{C_{A1}}^{C_{A2}}\frac{dC_A}{N_A C - N_A C_A}$$
$$= \frac{1}{N_A}\ln\frac{C - C_{A2}}{C - C_{A1}}$$

Therefore,

$$N_A = \frac{C D_{AB}}{z} \ln \frac{C - C_{A2}}{C - C_{A1}} \quad \text{---------------------------- (5)}$$

Replacing concentration in terms of pressures using Ideal gas law, equation (5) becomes

$$N_A = \frac{D_{AB} P_t}{RTz} \ln \frac{P_t - P_{A2}}{P_t - P_{A1}} \quad \text{--------------------------- (6)}$$

where

D_{AB} = molecular diffusivity of A in B
P_T = total pressure of system
R = universal gas constant
T = temperature of system in absolute scale
z = distance between two planes across the direction of diffusion
P_{A1} = partial pressure of A at plane 1, and
P_{A2} = partial pressure of A at plane 2

Given:

$D_{AB} = 1.89 * 10^{-5}$ m^2/sec
$P_t = 1$ atm $= 1.01325 * 10^5$ N/m^2
T = 25°C = 273 + 25 = 298 K
z = 2 mm = 0.002 m
$P_{A1} = 0.2 * 1 = 0.2$ atm (From Ideal gas law and additive pressure rule)
$P_{A2} = 0.1 * 1 = 0.1$ atm

Substituting these in equation (6)

$$N_A = \frac{\left(1.89 * 10^{-5}\right)\left(1.01325 * 10^5\right)}{(8314)(298)(0.002)} \ln\left(\frac{1 - 0.1}{1 - 0.2}\right)$$

$$= 4.55 * 10^{-5} \text{ kmol/m}^2\text{.sec}$$

3.3 Psuedo steady state diffusion through a stagnant film:

In many mass transfer operations, one of the boundaries may move with time. If the length of the diffusion path changes a small amount over a long period of time, a pseudo steady state diffusion model may be used. When this condition exists, the equation of steady state diffusion through stagnant gas' can be used to find the flux.

If the difference in the level of liquid A over the time interval considered is only a small fraction of the total diffusion path, and $t_0 - t$ is relatively long period of time, at any given instant in that period, the molar flux in the gas phase may be evaluated by

$$N_A = \frac{C D_{AB} (y_{A1} - y_{A2})}{z y_{B,lm}} \quad \text{------------------ (1)}$$

where z equals $z_2 - z_1$, the length of the diffusion path at time t.

The molar flux N_A is related to the amount of A leaving the liquid by

$$N_A = \frac{\rho_{A,L}}{M_A}\frac{dz}{dt} \qquad \text{-------------------------- (2)}$$

where $\frac{\rho_{A,L}}{M_A}$ is the molar density of A in the liquid phase

under Psuedo steady state conditions, equations (1) & (2) can be equated to give

$$\frac{\rho_{A,L}}{M_A}\frac{dz}{dt} = \frac{C D_{AB}(y_{A1} - y_{A2})}{z\, y_{B,lm}} \qquad \text{--------------- (3)}$$

Equation. (3) may be integrated from t = 0 to t and from $z = z_{t0}$ to $z = z_t$ as:

$$\int_{t=0}^{t} dt = \frac{\rho_{A,L}\, y_{B,lm} / M_A}{C D_{AB}(y_{A1} - y_{A2})}\int_{Z_{t0}}^{Z_t} z\, dz$$

yielding

$$t = \frac{\rho_{A,L}\, y_{B,lm} / M_A}{C D_{AB}(y_{A1} - y_{A2})}\left(\frac{z_t^2 - z_{t0}^2}{2}\right) \qquad \text{-------- (4)}$$

This shall be rearranged to evaluate diffusivity D_{AB} as,

$$D_{AB} = \frac{\rho_{A,L}\, y_{B,lm}}{M_A C\, (y_{A1} - y_{A2})\, t}\left(\frac{z_t^2 - z_{t0}^2}{2}\right)$$

Equimolar counter diffusion:

A physical situation which is encountered in the distillation of two constituents whose molar latent heats of vaporization are essentially equal, stipulates that the flux of one gaseous component is equal to but acting in the opposite direction from the other gaseous component; that is, $N_A = - N_B$.

The molar flux N_A, for a binary system at constant temperature and pressure is described by

$$N_A = - C D_{AB}\frac{d y_A}{d z} + y_A\,(N_A + N_B)$$

or $$N_A = -D_{AB}\frac{dC_A}{dz} + y_A\,(N_A + N_B) \qquad \text{------- (1)}$$

with the substitution of $N_B = -N_A$, Equation (1) becomes,

$$N_A = -D_{AB}\frac{dC_A}{dz} \qquad \text{----------------- (2)}$$

For steady state diffusion Equation. (2) may be integrated, using the boundary conditions:

at $z = z_1$ $C_A = C_{A1}$
and $z = z_2$ $C_A = C_{A2}$

Giving,

$$N_A \int_{z_1}^{z_2} dz = -D_{AB}\int_{C_{A1}}^{C_{A2}} dC_A$$

from which

$$N_A = \frac{D_{AB}}{z_2 - z_1}\,(C_{A1} - C_{A2}) \qquad \text{------------------- (3)}$$

For ideal gases, $C_A = \frac{n_A}{V} = \frac{p_A}{RT}$. Therefore Equation. (3) becomes

$$N_A = \frac{D_{AB}}{RT\,(z_2 - z_1)}\,(P_{A1} - P_{A2}) \qquad \text{---------- (4)}$$

This is the equation of molar flux for steady-state equimolar counter diffusion.

Concentration profile in these equimolar counter diffusion may be obtained from,

$$\frac{d}{dz}(N_A) = 0$$ (Since N_A is constant over the diffusion path).

And from equation. (2)

$$N_A = -D_{AB}\frac{dC_A}{dz}.$$

Therefore

$$\frac{d}{dz}\left[-D_{AB}\frac{dC_A}{dz}\right] = 0\,.$$

or $$\frac{d^2 C_A}{d z^2} = 0.$$

This equation may be solved using the boundary conditions to give

$$\frac{C_A - C_{A1}}{C_{A1} - C_{A2}} = \frac{z - z_1}{z_1 - z_2} \text{ -------------- (5)}$$

Problem. Methane diffuses at steady state through a tube containing helium. At point 1 the partial pressure of methane is p_{A1} = 55 kPa and at point 2, 0.03 m apart P_{A2} = 15 KPa. The total pressure is 101.32 kPa, and the temperature is 298 K. At this pressure and temperature, the value of diffusivity is 6.75 * 10^{-5} m^2/sec.

i) calculate the flux of CH_4 at steady state for equimolar counter diffusion.
ii) Calculate the partial pressure at a point 0.02 m apart from point 1.

Calculation:

For steady state equimolar counter diffusion, molar flux is given by

$$N_A = \frac{D_{AB}}{R\,T\,z}\left(p_{A1} - p_{A2}\right) \text{ -------------------------- (1)}$$

Therefore;

$$N_A = \frac{6.75 * 10^{-5}}{8.314 * 298 * 0.03}\left(55 - 15\right) \frac{kmol}{m^2 .\sec}$$

$$= 3.633 * 10^{-5} \frac{kmol}{m^2 \sec}$$

And from (1), partial pressure at 0.02 m from point 1 is:

$$3.633 * 10^{-5} = \frac{6.75 * 10^{-5}}{8.314 * 298 * 0.02}\left(55 - p_A\right)$$

p_A = 28.33 kPa

Problem. In a gas mixture of hydrogen and oxygen, steady state equimolar counter diffusion is occurring at a total pressure of 100 kPa and temperature of 20°C. If the partial pressures of oxygen at two planes 0.01 m apart, and perpendicular to the direction of diffusion are 15 kPa and 5 kPa, respectively and the mass diffusion flux of oxygen in the mixture is 1.6 * 10^{-5} kmol/m^2.sec, calculate the molecular diffusivity for the system.

Solution:

For equimolar counter current diffusion:

$$N_A = \frac{D_{AB}}{RTz}(p_{A1} - p_{A2}) \text{ ---------------- (1)}$$

where

N $_A$ = molar flux of A (1.6 * 10 $^{-5}$ kmol/m ^{2}.sec):
D $_{AB}$ = molecular diffusivity of A in B
R = Universal gas constant (8.314 kJ/kmol.k)
T = Temperature in absolute scale (273 + 20 = 293 K)
z = distance between two measurement planes 1 and 2 (0.01 m)
P $_{A1}$ = partial pressure of A at plane 1 (15 kPa); and
P $_{A2}$ = partial pressure of A at plane 2 (5 kPa)

Substituting these in equation (1)

$$1.6 * 10^{-5} = \frac{D_{AB}}{(8.314)(293)(0.01)}(15 - 5)$$

Therefore, D $_{AB}$ = 3.898 * 10 $^{-5}$ m ^{2}/sec

Problem. A tube 1 cm in inside diameter that is 20 cm long is filled with Co_2 and H_2 at a total pressure of 2 atm at 0°C. The diffusion coefficient of the Co_2 – H_2 system under these conditions is 0.275 cm^2/sec. If the partial pressure of Co_2 is 1.5 atm at one end of the tube and 0.5 atm at the other end, find the rate of diffusion for:

i) steady state equimolar counter diffusion ($N_A = - N_B$)
ii) steady state counter diffusion where $N_B = -0.75\, N_A$, and
iii) steady state diffusion of Co_2 through stagnant H_2 ($N_B = 0$)

$$i)\ N_A = -C D_{AB} \frac{d\,y_A}{d\,z} + y_A (N_A + N_B)$$

Given

N $_B$ = - N $_A$

Therefore $N_A = -C D_{AB} \dfrac{d\,y_A}{d\,z} = -D_{AB} \dfrac{d\,C_A}{d\,z}$

(For ideal gas mixture $C_A = \dfrac{p_A}{R\,T}$ where p_A is the partial pressure of A; such that p $_A$ + p $_B$ = P)

Therefore $N_A = -D_{AB} \dfrac{d\,(p_A / RT)}{d\,z}$

For isothermal system, T is constant

Therefore $N_A = -\frac{D_{AB}}{RT}\frac{dp_A}{dz}$

(i.e.) $N_A \int_{Z_1}^{Z_2} dz = -\frac{D_{AB}}{RT}\int_{P_{A1}}^{P_{A2}} dp_A$

$N_A = \frac{D_{AB}}{RT\,z}\left[p_{A1} - p_{A2}\right]$ ---------------------------------- (1)

where $Z = Z_2 - Z_1$

Given: $D_{AB} = 0.275\ cm^2/sec = 0.275 * 10^{-4}\ m^2/sec$; T = 0°C = 273 k

$$N_A = \frac{0.275*10^{-4}}{8314*273*0.2}\left[1.5*1.01325*10^5 - 0.5*1.01325*10^5\right]$$

$$= 6.138*10^{-6}\ \frac{k\,mol}{m^2\ sec}$$

Rate of diffusion = $N_A S$

Where S is surface area

Therefore rate of diffusion = $6.138 * 10^{-6} * \pi r^2$

$= 6.138 * 10^{-6} * \pi (0.5 * 10^{-2})^2$

$= 4.821 * 10^{-10}$ k mol/sec

$= 1.735 * 10^{-3}$ mol/hr.

ii) $N_A = -CD_{AB}\frac{dy_A}{dz} + y_A\left(N_A + N_B\right)$

given: $N_B = -0.75\ N_A$

Therefore $N_A = -CD_{AB}\frac{dy_A}{dz} + y_A\left(N_A - 0.75\ N_A\right)$

$$= -C D_{AB}\frac{dy_A}{dz} + 0.25 y_A N_A$$

$$N_A - 0.25\, y_A N_A = -C D_{AB}\frac{dy_A}{dz}$$

$$N_A\, dz = -C D_{AB}\frac{dy_A}{1 - 0.25\, y_A}$$

for constant N_A and C

$$N_A \int_{Z_1}^{Z_2} dz = -CD_{AB} \int_{y_{A1}}^{y_{A2}} \frac{dy_A}{1-0.25\,y_A}$$

$$\left[\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)\right]$$

$$N_A z = -C D_{AB}\left(\frac{1}{-0.25}\right)\left[\ln(1-0.25\,y_A)\right]_{y_{A1}}^{y_{A2}}$$

$$N_A = \frac{4CD_{AB}}{z}\ln\left[\frac{1-0.25\,y_{A2}}{1-0.25\,y_{A1}}\right] \text{ ---------------------------------- (2)}$$

Given:

$$C = \frac{p}{RT} = \frac{2 * 1.01325 * 10^5}{8314 * 273} = 0.0893\ Kmol/m^3$$

$$y_{A1} = \frac{p_{A1}}{P} = \frac{1.5}{2} = 0.75$$

$$y_{A2} = \frac{p_{A2}}{P} = \frac{0.5}{2} = 0.25$$

Substituting these in equation (2),

$$N_A = \frac{4 * 0.0893 * 0.275 * 10^{-4}}{0.2}\ln\left[\frac{1-0.25 * 0.25}{1-0.25 * 0.75}\right]$$

$$= 7.028 * 10^{-6}\ \frac{kmol}{m^2 \sec}$$

Rate of diffusion = N_A S = $7.028 * 10^{-6} * \pi * (0.5 * 10^{-2})^2$

= $5.52 * 10^{-10}$ kmol/sec

= $1.987 * 10^{-3}$ mol/hr.

iii) $N_A = -CD_{AB}\frac{dy_A}{dz} + y_A\left(N_A + N_B\right)$

Given: $N_B = 0$

Therefore $N_A = -CD_{AB}\frac{dy_A}{dz} + y_A N_A$

$$N_A \int_{Z_1}^{Z_2} dz = -CD_{AB} \int_{y_{A1}}^{y_{A2}} \frac{dy_A}{1-y_A}$$

$$= \frac{CD_{AB}}{Z}\ln\left[\frac{1-y_{A2}}{1-y_{A1}}\right]$$

$$= \frac{0.0893 * 0.275 * 10^{-4}}{0.2}\left[\ln\left(\frac{1-0.25}{1-0.75}\right)\right]$$

$$= 1.349 * 10^{-5} \frac{kmol}{m^2 . \sec}$$

Rate of diffusion = 1.349 8 10 $^{-5}$ * π * (0.5 * 10 $^{-2}$) ^{2}

= 1.059 Kmol / sec
= 3.814 mol/hr

3.4 Diffusion in Liquids:

Equation derived for diffusion in gases equally applies to diffusion in liquids with some modifications. Mole fraction in liquid phases is normally written as ‘x’ (in gases as y). The concentration term ‘C’ is replaced by average molar density, $\left(\frac{\rho}{M}\right)_{av}$.

a) For steady – state diffusion of A through non diffusivity B:

N_A = constant , N_B = 0

$$N_A = \frac{D_{AB}}{Z\,x_{BM}} \left(\frac{\rho}{M}\right)_{av} (x_{A1} - x_{A2})$$

where Z = Z $_2$ – Z $_1$, the length of diffusion path; and

$$X_{BM} = \frac{X_{B2} - X_{B1}}{\ln\left(X_{B2}/X_{B1}\right)}$$

b) For steady – state equimolar counter diffusion :

N_A = – N_B = const

$$N_A = \frac{D_{AB}}{Z}(C_{A1} - C_{A2}) = \frac{D_{AB}}{Z}\left(\frac{\rho}{M}\right)_{av}(x_{A1} - x_{A2})$$

Problem. Calculate the rate of diffusion of butanol at 20°C under unidirectional steady state conditions through a 0.1 cm thick film of water when the concentrations of butanol at the opposite sides of the film are, respectively 10% and 4% butanol by weight. The diffusivity of butanol in water solution is 5.9 * 10 $^{-6}$ cm ^{2}/sec. The densities of 10% and 4% butanol solutions at 20°C may be taken as 0.971 and 0.992 g/cc respectively. Molecular weight of Butanol (C_4H_9OH) is 74, and that of water 18.

Calculations

For steady state unidirectional diffusion,

$$N_A = \frac{D_{AB}}{Z} C \frac{(x_{A1} - x_{A2})}{x_{B,lm}}$$

where C is the average molar density.

$$= \left(\frac{\rho}{M}\right)_{avg}$$

Conversion from weight fraction the Mole fraction:

$$x_{A1} = \frac{(0.1/74)}{\left[(0.1/74) + (0.9/18)\right]} = 0.026$$

$$x_{A2} = \frac{(0.04/74)}{\left[(0.04/74) + (0.96/18)\right]} = 0.010$$

Average molecular weight at 1 & 2:

$$M_1 = \frac{1}{(0.1/74) + (0.9/18)} = 19.47\, kg/Kmol$$

$$M_2 = \frac{1}{(0.04/74) + 0.96/18} = 18.56\, kg/Kmol$$

$$\left(\frac{\rho}{M}\right)_{avg} = \frac{\left(\rho_1/M_1 + \rho_2/M_2\right)}{2}$$

$$= \frac{0.971/19.47 + 0.992/18.56}{2}$$

= 0.0517 gmol / cm ^{3}
= 51.7 kmol/m ^{3}

$$x_{B,lm} = \frac{x_{B2} - x_{B1}}{\ln(x_{B2}/x_{B1})} = \frac{(1 - x_{A2}) - (1 - x_{A1})}{\ln\left[\frac{(1 - x_{A2})}{(1 - x_{A1})}\right]}$$

(i.e.) $$x_{B,lm} = \frac{(1 - 0.01) - (1 - 0.026)}{\ln\left[\frac{1 - 0.01}{1 - 0.026}\right]}$$

$$= \frac{0.016}{0.0163} = 0.982$$

Therefore $$N_A = \frac{D_{AB}}{2}\left(\frac{\rho}{M}\right)_{avg}\frac{(x_{A1} - x_{A2})}{x_{B,lm}}$$

$$= \frac{5.9 * 10^{-6} * 10^{-4} * 51.7}{0.1 * 10^{-2}} * \frac{(0.026 - 0.010)}{0.982}$$

$$= 4.97 * 10^{-7}\, \frac{kmol}{m^2 \sec}$$

$$= 1.789\, \frac{gmol}{m^2.hr.}$$

$$= 1.789 * 74\, \frac{g}{m^2.hr.}$$

$$= 132.4\, \frac{g}{m^2.hr.}$$

3.5 Diffusion in solids

In certain unit operation of chemical engineering such as in drying or in absorption, mass transfer takes place between a solid and a fluid phase. If the transferred species is distributed uniformly in the solid phase and forms a homogeneous medium, the diffusion of the species in the solid phase is said to be structure independent. In this cases diffusivity or diffusion coefficient is direction – independent**.**

At steady state, and for mass diffusion which is independent of the solid matrix structure, the molar flux in the z direction is :

$$N_A = -D_{AB}\frac{dC_A}{dz} = \text{constant}$$, as given by Fick's law.

Integrating the above equation,

$$N_A = \frac{D_{AB}\left(C_{A1} - C_{A2}\right)}{z}$$

which is similar to the expression obtained for diffusion in a stagnant fluid with no bulk motion (i.e. N = 0).

In some chemical operations, such as heterogeneous catalysis, an important factor, affecting the rate of reaction is the diffusions of the gaseous component through a porous solid. The effective diffusivity in the solid is reduced below what it could be in a free fluid, for two reasons. First, the tortuous nature of the path increases the distance, which a molecule must travel to advance a given distance in the solid. Second, the free cross – sectional area is restricted. For many catalyst pellets, the effective diffusivity of a gaseous component is of the order of one tenth of its value in a free gas.

If the pressure is low enough and the pores are small enough, the gas molecules will collide with the walls more frequently than with each other. This is known as Knudsen flow or Knudsen diffusion. Upon hitting the wall, the molecules are momentarily absorbed and then given off in random directions. The gas flux is reduced by the wall collisions.

By use of the kinetic flux is the concentration gradient is independent of pressure ; whereas the proportionality constant for molecular diffusion in gases (i.e. Diffusivity) is inversely proportional to pressure.

Knudsen diffusion occurs when the size of the pore is of the order of the mean free path of the diffusing molecule.

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DEPARTMENT OF CHEMICAL ENGINEERING

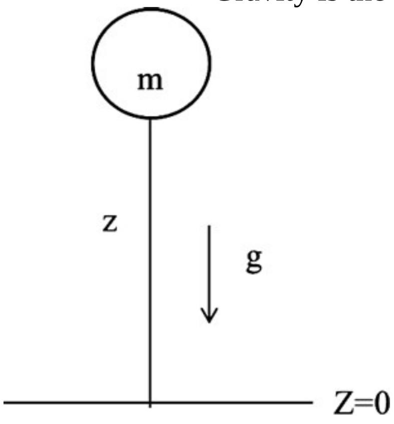
UNIT – IV – TRANSPORT PHENOMENA – SCH1309

4.1 Derivation of equation of energy

1. Equation of mechanical energy
2. Equation of thermal energy

Equation of mechanical energy

For understanding the nature of mechanical energy, consider a simple case of a single particle moving in one direction as shown in Fig. 30.1. Assume the particle has mass m and is located at height h from a reference plane and moving upward with velocity $\underset{\sim}{v}$. Gravity is the only force working on the particle.



Starting with Newton’s second law of motion,

we have Force = mass x acceleration

$$\underset{\sim}{v}.\underset{\sim}{F} = m\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt}$$

$$\underset{\sim}{F} = m\underset{\sim}{a}$$

$$\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt} \qquad \underset{\sim}{v}.\left[\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt}\right]$$

Using vector identity, we have

$$\frac{d\left(\underset{\sim}{v}.\underset{\sim}{v}\right)}{dt} = \underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt} + \frac{d\underset{\sim}{v}}{dt}.\underset{\sim}{v} = 2\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt}$$

or

$$\underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt}=\frac{1}{2}\frac{d\left(v^2\right)}{dt}$$

where, ν is the magnitude of the velocity vector $\underset{\sim}{v}$. By substituting

$$\underset{\sim}{v}.\underset{\sim}{F}=m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

$$v_1F_1+v_2F_2+v_3F_3=m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

For the example given above, we have ,

F1 = 0, F2 = 0, F3=-mg

and

ν1 = 0, ν2 = 0, ν3 = ν

$$-vmg=m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Substitute ν = (dz/dt). Thus we obtain,

$$-mg\frac{dz}{dt} = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Since, m and g are constants. We may rewrite above equation as,

$$-\frac{d(mgz)}{dt} = \frac{d\left(m\frac{v^2}{2}\right)}{dt}$$

or

$$\frac{d}{dt}\left(mgz + m\frac{v^2}{2}\right) = 0$$

Thus $$mgz + m\frac{v^2}{2} = constant$$

$$\underset{\sim}{v}.\left[\rho\frac{D\underset{\sim}{v}}{Dt} = \rho\underset{\sim}{g} - \underset{\sim}{\nabla}P - \underset{\sim}{\nabla}.\underset{\sim}{\tau}\right]$$

As before,

$$\underset{\sim}{v}.\frac{D\underset{\sim}{v}}{Dt} = \frac{D\left(\frac{v^2}{2}\right)}{Dt}$$

(Note: substantial derivatives behave like normal derivatives.). Thus,

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \rho\underset{\sim}{g}.\underset{\sim}{v} - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}P\right) - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)$$

The following vector and tensor identities may be used for simplifying Equation (30.14)

$$\underset{\sim}{\nabla}.(P\underset{\sim}{v}) = P(\underset{\sim}{\nabla}.\underset{\sim}{v}) + \underset{\sim}{v}.(\underset{\sim}{\nabla}P)$$

and if $\underset{\sim}{\tau}$ is a second order symmetric tensor then we also have

$$\underset{\sim}{\nabla}.(\underset{\sim}{\tau}.\underset{\sim}{v}) = \underset{\sim}{v}.\underset{\sim}{\nabla}.\underset{\sim}{\tau} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Thus, we obtain

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho\underset{\sim}{g}\right) - \left[\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + \left(-P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\right)\right] - \left[\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)\right]$$

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho\underset{\sim}{g}\right) - \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Equation is called the equation of mechanical energy for fluids.

Significance of each term is given below.

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right)\left\{\begin{matrix}\text{Rate of change of}\\ \text{kinetic energy}\\ \text{per unit volume}\end{matrix}\right\} = \underset{\sim}{v}.\left(\rho\underset{\sim}{g}\right)\left\{\begin{matrix}\text{work done by gravity}\\ \text{force on the system}\end{matrix}\right\}$$

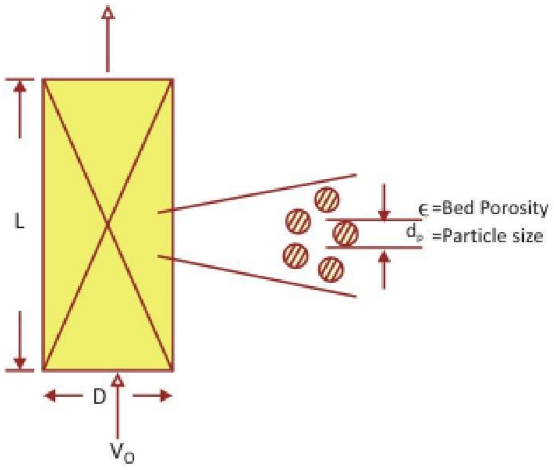
$$-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)\left\{\begin{matrix}\text{work done by pressure}\\ \text{force on the system}\end{matrix}\right\} + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\left\{\begin{matrix}\text{reversible conversion of}\\ \text{kinetic energy into}\\ \text{the internal energy}\end{matrix}\right\}$$

$$-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)\left\{\begin{matrix}\text{work done by viscous}\\ \text{forces on system}\end{matrix}\right\} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\left\{\begin{matrix}\text{irreversible conversion of}\\ \text{kinetic energy in to the heat}\end{matrix}\right\}$$

$$\rho \frac{D\left(\hat{U}\right)}{Dt}\begin{Bmatrix}\textit{Rate of change of}\\ \textit{internal energy}\\ \textit{per unit volume}\end{Bmatrix} = -\underset{\sim}{\nabla}.\underset{\sim}{q}\begin{Bmatrix}\textit{Heat transferred}\\ \textit{by conduction}\end{Bmatrix} + S_e\begin{Bmatrix}\textit{Heat generated / removed}\\ \textit{by source or sink}\end{Bmatrix}$$

$$-P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\begin{Bmatrix}\textit{reversible conversion of}\\ \textit{kinetic energy into}\\ \textit{the internal energy}\end{Bmatrix} - \left(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)\begin{Bmatrix}\textit{irreversible conversion of}\\ \textit{kinetic energy in to the heat}\end{Bmatrix}$$

4.2 FLOW THROUGH PACKED BED



For the theoretical analysis to calculate pressure–drop, actual flow channels are replaced with parallel cylindrical conduits of constant cross–section. Particles are assumed to be of the same size and shape having constant sphericity, Φ_s.

Pressure–drop occurs due to inertial and viscous effects. At high Reynolds number, inertial effects prevail, whereas the viscous effects are important at low Reynolds number. In general,

$$(\Delta p)_{total} = (\Delta p)_{viscous} + (\Delta p)_{inertial}$$

$$\frac{F_D}{As} = \frac{\text{Drag over the channel} - \text{walls consiting of packed bed particles}}{\text{Total surface area of particles}}$$

$$\boxed{= K_1 \left(\frac{\mu_f V}{r_h}\right) + K_2 \left(\rho_f V^2\right)}$$

$$\tau = \frac{4\mu_f V}{r} \text{ or } \tau \,\alpha\, \frac{\mu_f V}{r_h}$$

$$r_h = \text{hydraulic radius} = \frac{\text{Total cross} - \text{section of conduits}}{\text{Wetted parameter}}$$

$$As = \underset{\substack{\downarrow \\ Total\ \#\ of\ particles}}{N_p} \times \underset{\substack{\downarrow \\ Surface\ area\ of\ one\ particle}}{S_p}$$

$$= \frac{S_o L(1-\epsilon)}{v_p} \times s_p, \text{ where } v_p = \text{volume of one partide}$$

$$\frac{v_p}{s_p} = \frac{6}{\phi_p\ d_p}; v = \frac{v_o}{\epsilon}, \text{where } v_o = \text{superficial velocity}$$

Similarly, pressure–drop at high Reynolds number, $\Delta p \,\alpha\, \rho_f V^2$**. Therefore, Pressure-drop in packed beds is related to pressure–drop due to viscous and inertial effects via two empirical constants,** K_1 and K_2**.**

$= \frac{\text{Total volume of voids}}{\text{Total surface area of particles}}$ **(multiply both numerator and denominator by L)** $= \frac{(S_o L)\epsilon}{As}$ **, S_0 = cross sectional area of packed–bed**

$$\frac{F_D}{As} = \frac{F_D \phi_P\, d_P}{S_o L(1-\epsilon)\times 6} = \left[K_1\,\frac{\mu_f\, v_o}{\epsilon^2\; S_o L} \times \frac{S_o L(1-\epsilon)6}{\phi_P\; d_P} + K_2 \rho_f \frac{V_o^2}{\epsilon^2}\right]$$

$$= \frac{\rho_f\, v_o^2}{\epsilon^2}\left[6K_1 \frac{\mu_f\,(1-\epsilon)}{v_o \phi_s\; d_p\;\; \rho_f} + K_2\right]$$

$$F_D = \text{drag} - \text{force} = (\Delta p)_{\text{total}} \times S_o \epsilon$$

$$\frac{(\Delta p)_{\text{total}}\;(S_o\epsilon)\phi_s\; d_P}{S_o L(1-\epsilon)\times 6} = \rho_f \frac{V_o^2}{\epsilon^2}\left[\frac{6K_1\mu_f\,(1-\epsilon)}{\phi_s\; d_P v_o\; \rho_f} + K_2\right]$$

$$\frac{\Delta p}{L}\left(\frac{\epsilon^3}{1-\epsilon}\right)\left(\frac{\phi_s\; d_P}{\rho_f\; V_o^2}\right) = \frac{36\; K_1\; \mu_f\,(1-\epsilon)}{\phi_s\; d_P v_o\;\rho_f} + 6K_2$$

$$\boxed{\frac{\Delta p}{L}\left(\frac{\epsilon^3}{1-\epsilon}\right)\left(\frac{\phi_s\; d_P}{\rho_f\; V_o^2}\right) = \frac{150(1-\epsilon)\mu_f}{\phi_s\; d_P v_o\;\rho_f} + \;1.75}$$

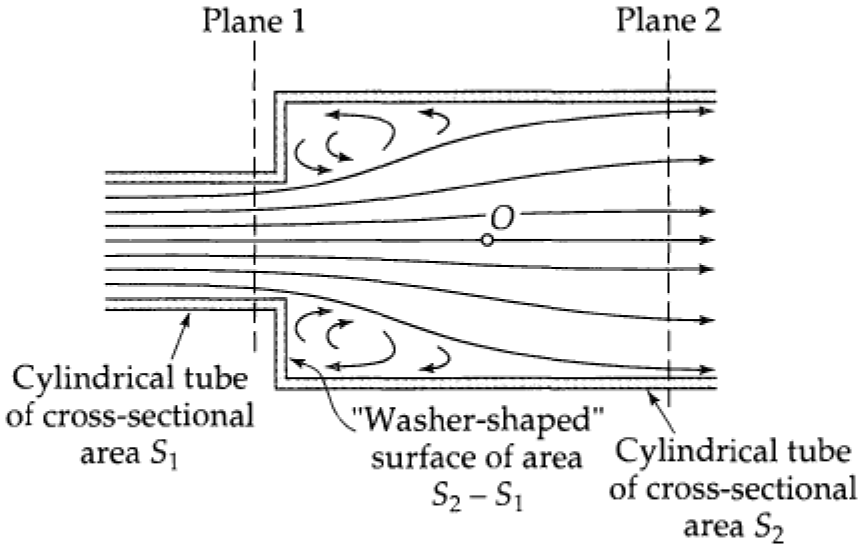
$$f_\rho = \frac{\Delta p}{L\;\rho_f\; V_o^2}\left(\frac{\phi_s\; d_P \epsilon^3}{(1-\epsilon)}\right)$$

$$= \left(\frac{\Delta p}{\rho}\right)_{\text{Friction}}$$

$$= \frac{150(1-\epsilon)^2\,\mu_f\,v_o L}{\epsilon^3\,d_p{}^2\,\phi_s{}^2\,\rho_f} + \frac{1.75\,(1-\epsilon)\,V_o^2 L}{\epsilon^3\,\phi_s\,d_p}$$

4.3 SUDDEN ENLARGEMENT

An incompressible fluid flows from a small circular tube into a large tube in turbulent flow, as shown in Fig. 7.6-1. The cross-sectional areas of the tubes are S_1 and S_2. Obtain an expression for the pressure change between planes 1 and 2 and for the friction loss associated with the sudden enlargement in cross section. Let $\beta = S_1/S_2$, which is less than unity.



(a) ***Mass balance.*** For steady flow the mass balance gives

$$w_1 = w_2 \quad \text{or} \quad \rho_1 v_1 S_1 = \rho_2 v_2 S_2$$

For a fluid of constant density, this gives

$$\frac{v_1}{v_2} = \frac{1}{\beta}$$

(b) ***Momentum balance.*** The downstream component of the momentum balance is

$$\mathbf{F}_{f \to s} = (v_1 w_1 - v_2 w_2) + (p_1 S_1 - p_2 S_2)$$

The force $\mathbf{F}_{f \to s}$ is composed of two parts: the viscous force on the cylindrical surfaces parallel to the direction of flow, and the pressure force on the washer-shaped surface just to the right of plane 1 and perpendicular to the flow axis. The former contribution we neglect (by intuition) and the latter we take to be $p_1(S_2 - S_1)$ by assuming that the pressure on the washer-shaped surface is the same as that at plane 1.

$$-p_1(S_2 - S_1) = \rho v_2 S_2 (v_1 - v_2) + (p_1 S_1 - p_2 S_2)$$

Solving for the pressure difference gives

$$p_2 - p_1 = \rho v_2 (v_1 - v_2)$$

or, in terms of the downstream velocity,

$$p_2 - p_1 = \rho v_2^2 \left(\frac{1}{\beta} - 1 \right)$$

(c) ***Angular momentum balance.*** This balance is not needed. If we take the origin of coordinates on the axis of the system at the center of gravity of the fluid located between planes 1 and 2, then $[\mathbf{r}_1 \times \mathbf{u}_1]$ and $[\mathbf{r}_2 \times \mathbf{u}_2]$ are both zero, and there are no torques on the fluid system.

(d) ***Mechanical energy balance.*** There is no compressive loss, no work done via moving parts, and no elevation change, so that

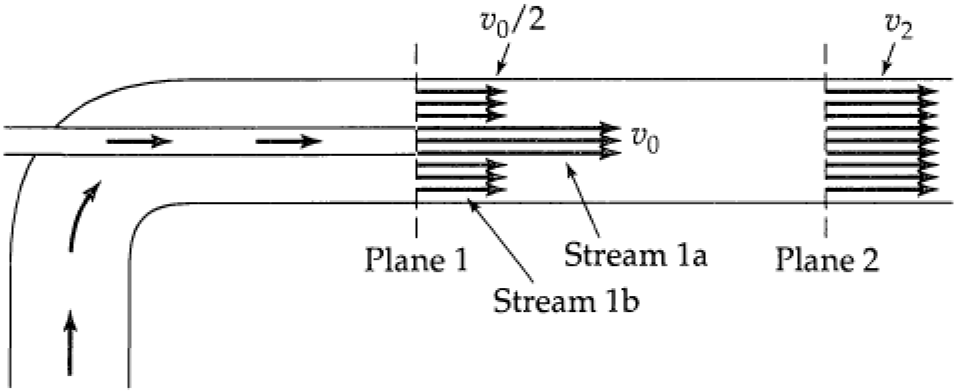
$$\hat{E}_v = \tfrac{1}{2}(v_1^2 - v_2^2) + \frac{1}{\rho}(p_1 - p_2)$$

Insertion of Eq. 7.6-6 for the pressure rise then gives, after some rearrangement,

$$\hat{E}_v = \tfrac{1}{2} v_2^2 \left(\frac{1}{\beta} - 1 \right)^2$$

4.4 LIQUID - LIQUID EJECTOR

A diagram of a liquid–liquid ejector is shown in Fig. It is desired to analyze the mixing of the two streams, both of the same fluid, by means of the macroscopic balances. At plane 1 the two fluid streams merge. Stream 1a has a velocity v_0 and a cross-sectional area $\frac{1}{3}S_1$, and stream 1b has a velocity $\frac{1}{2}v_0$ and a cross-sectional area $\frac{2}{3}S_1$. Plane 2 is chosen far enough downstream that the two streams have mixed and the velocity is almost uniform at v_2. The flow is



(a) ***Mass balance.*** At steady state, Eq. (A) of Table 7.6-1 gives

$$w_{1a} + w_{1b} = w_2$$

or

$$\rho v_0(\tfrac{1}{3}S_1) + \rho(\tfrac{1}{2}v_0)(\tfrac{2}{3}S_1) = \rho v_2 S_2$$

Hence, since $S_1 = S_2$, this equation gives

$$v_2 = \tfrac{2}{3}v_0$$

for the velocity of the exit stream. We also note, for later use, that $w_{1a} = w_{1b} = \frac{1}{2}w_2$.

(b) ***Momentum balance.*** From Eq. (B) of Table 7.6-1 the component of the momentum balance in the flow direction is

$$(v_{1a}w_{1a} + v_{1b}w_{1b} + p_1S_1) - (v_2w_2 + p_2S_2) = 0$$

or using the relation at the end of (a)

$$\begin{aligned}(p_2 - p_1)S_2 &= (\tfrac{1}{2}(v_{1a} + v_{1b}) - v_2)w_2 \\ &= (\tfrac{1}{2}(v_0 + \tfrac{1}{2}v_0) - \tfrac{2}{3}v_0)(\rho(\tfrac{2}{3}v_0)S_2)\end{aligned}$$

from which

$$p_2 - p_1 = \tfrac{1}{18}\rho v_0^2$$

This is the expression for the pressure rise resulting from the mixing of the two streams.

(c) ***Angular momentum balance.*** This balance is not needed.

(d) ***Mechanical energy balance.*** Equation (D) of Table 7.6-1 gives

$$(\tfrac{1}{2}v_{1a}^2 w_{1a} + \tfrac{1}{2}v_{1b}^2 w_{1b}) - \left(\tfrac{1}{2}v_2^2 + \frac{p_2 - p_1}{\rho}\right)w_2 = E_v$$

or, using the relation at the end of (a), we get

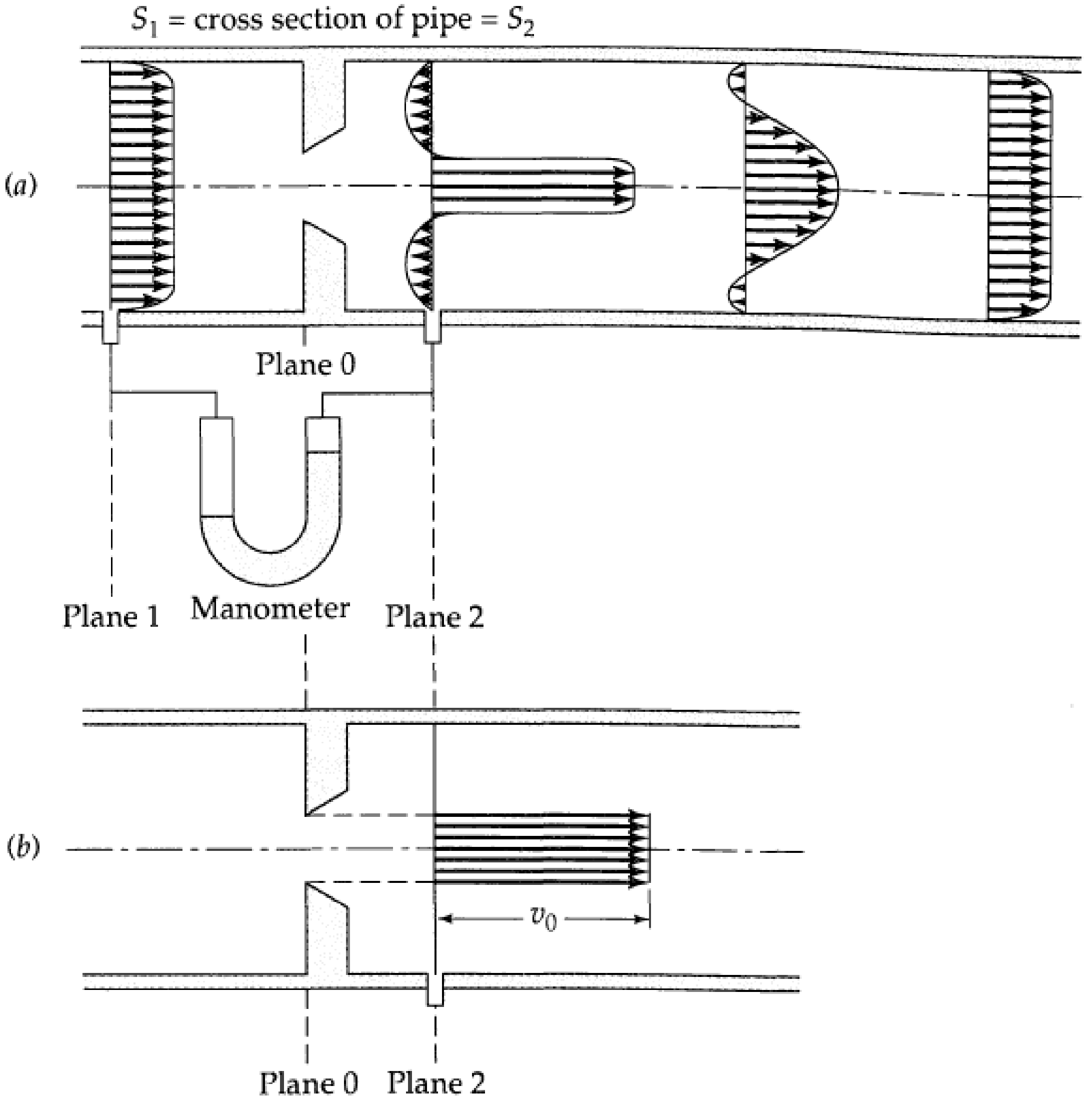
$$(\tfrac{1}{2}v_{1a}^2(\tfrac{1}{2}w_2) + \tfrac{1}{2}(\tfrac{1}{2}v_0)^2(\tfrac{1}{2}w_2)) - (\tfrac{1}{2}(\tfrac{2}{3}v_0)^2 + \tfrac{1}{18}v_0^2)w_2 = E_v$$

Hence

$$\hat{E}_v = \frac{E_v}{w_2} = \frac{5}{144}v_0^2$$

4.5 ISOTHERMAL FLOW OF AN LIQUID THROUGH AN ORIFICE

A common method for determining the mass rate of flow through a pipe is to measure the pressure drop across some "obstacle" in the pipe. An example of this is the orifice, which is a thin plate with a hole in the middle. There are pressure taps at planes 1 and 2, upstream and downstream of the orifice plate.


S_1 = cross section of pipe = S_2
(a)
Plane 0
Plane 1
Manometer
Plane 2
(b)
v_0
Plane 0
Plane 2

(a) ***Mass balance.*** For a fluid of constant density with a system for which $S_1 = S_2 = S$, the mass balance in Eq. 7.1-1 gives

$$\langle v_1 \rangle = \langle v_2 \rangle$$

With the assumed velocity profiles this becomes

$$v_1 = \frac{S_0}{S} v_0$$

and the volume rate of flow is $w = \rho v_1 S$.

(b) ***Mechanical energy balance.*** For a constant-density fluid in a flow system with no elevation change and no moving parts, Eq. 7.4-5 gives

$$\frac{1}{2}\frac{\langle v_2^3 \rangle}{\langle v_2 \rangle} - \frac{1}{2}\frac{\langle v_1^3 \rangle}{\langle v_1 \rangle} + \frac{p_2 - p_1}{\rho} + \hat{E}_v = 0$$

The viscous loss $\hat{E}_v$ is neglected, even though it is certainly not equal to zero. With the assumed velocity profiles, Eq.

$$\tfrac{1}{2}(v_0^2 - v_1^2) + \frac{p_2 - p_1}{\rho} = 0$$

$$v_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho} \frac{1}{(S/S_0)^2 - 1}}$$

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SCHOOL OF BIO AND CHEMICAL ENGINEERING

DEPARTMENT OF CHEMICAL ENGINEERING

UNIT – V – TRANSPORT PHENOMENA – SCH1309

5.1 Analogy of Mass Transfer

Mass transfer by convection involves the transport of material between a boundary surface (such as solid or liquid surface) and a moving fluid or between two relatively immiscible, moving fluids.

There are two different cases of convective mass transfer:

1. Mass transfer takes place only in a single phase either to or from a phase boundary, as in sublimation of naphthalene (solid form) into the moving air.
2. Mass transfer takes place in the two contacting phases as in extraction and absorption.

Convective Mass Transfer Coefficient

In the study of convective heat transfer, the heat flux is connected to heat transfer coefficient as

$$Q/A = q = h(t_s - t_m) \quad \text{--------------------} \ (1.1)$$

The analogous situation in mass transfer is handled by an equation of the form

$$N_A = k_c(C_{As} - C_A) \quad \text{--------------------} \ (1.2)$$

The molar flux N_A is measured relative to a set of axes fixed in space. The driving force is the difference between the concentration at the phase boundary, C_{AS} (a solid surface or a fluid interface) and the concentration at some arbitrarily defined point in the fluid medium, C_A. The convective mass transfer coefficient k_C is a function of geometry of the system and the velocity and properties of the fluid similar to the heat transfer coefficient, h.

Significant Parameters in Convective Mass Transfer

Dimensionless parameters are often used to correlate convective transfer data. In momentum transfer Reynolds number and friction factor play a major role. In the correlation of convective heat transfer data, Prandtl and Nusselt numbers are important. Some of the same parameters, along with some newly defined dimensionless numbers, will be useful in the correlation of convective mass-transfer data.

The molecular diffusivities of the three transport process (momentum, heat and mass) have been defined as:

$$\text{Momentum diffusivity} \quad \nu = \frac{\mu}{\rho} \quad \text{-----------------------------} \ (1.3)$$

$$\text{Thermal diffusivity} \quad \alpha = \frac{k}{\rho C_p} \quad \text{---------------------------} \ (1.4)$$

and

$$\text{Mass diffusivity } D_{AB} \quad \text{---------------------------} \ (1.5)$$

It can be shown that each of the diffusivities has the dimensions of L^2 / t, hence, a ratio of any of the two of these must be dimensionless.

The ratio of the molecular diffusivity of momentum to the molecular diffusivity of heat (thermal diffusivity) is designated as the Prandtl Number

$$\frac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}} = \Pr = \frac{\nu}{\alpha} = \frac{C_p \mu}{K} \quad \text{------------------------} \ (1.6)$$

The analogous number in mass transfer is Schmidt number given as

$$\frac{\text{Momentum diffusivity}}{\text{Mass diffusivity}} = Sc = \frac{\nu}{D_{AB}} = \frac{\mu}{\rho D_{AB}} \quad \text{-------------- (1.7)}$$

The ratio of the molecular diffusivity of heat to the molecular diffusivity of mass is designated the Lewis Number, and is given by

$$\frac{\text{Thermal diffusivity}}{\text{Mass diffusivity}} = Le = \frac{\alpha}{D_{AB}} = \frac{k}{\rho C_p D_{AB}} \quad \text{------------- (1.8)}$$

Lewis number is encountered in processes involving simultaneous convective transfer of mass and energy.

Let us consider the mass transfer of solute A from a solid to a fluid flowing past the surface of the solid. The concentration and velocity profile is depicted .For such a case, the mass transfer between the solid surface and the fluid may be written as

$$N_A = k_c \left(C_{As} - C_{A\infty}\right) \quad \text{---------------------- (1 a)}$$

Since the mass transfer at the surface is by molecular diffusion, the mass transfer may also described by

$$N_A = - D_{AB} \left.\frac{d\,C_A}{d\,y}\right|_{y=0} \quad \text{------------------------- (1.9)}$$

When the boundary concentration, C_{As} is constant, equation (9) may be written as

$$N_A = - D_{AB}$$

$$\left.\frac{d\left(C_A - C_{As}\right)}{dy}\right|_{y=0} \text{------------------------} (1.10)$$

Equation (4.1a) and (4.10) may be equated, since they define the same flux of component A leaving the surface and entering the fluid

$$k_c\left(C_{As} - C_{A\infty}\right) = -D_{AB}\frac{d}{dy}\left(C_A - C_{As}\right)\Big|_{y=0} \quad \text{---------------} \ (1.11)$$

This relation may be rearranged into the following form:

$$\frac{k_c}{D_{AB}} = -\frac{d\left(C_A - C_{As}\right)/dy}{\left(C_A - C_{A\infty}\right)}\Bigg|_{y=0} \quad \text{--------------------} \ (1..12)$$

Multiplying both sides of equation(4.12) by a characteristic length, L we obtain the following dimensionless expression:

$$\frac{k_c L}{D_{AB}} = -\frac{d\left(C_A - C_{As}\right)/dy\,\big|_{y=0}}{\left(C_{AS} - C_{A\infty}\right)/L} \quad \text{--------------------} \ (1.13)$$

The right hand side of equation (4.13) is the ratio of the concentration gradient at the surface to an overall or reference concentration gradient; accordingly, it may be considered as the ratio of molecular mass-transport resistance to the convective mass-transport resistance of the fluid. This ratio is generally known as the Sherwood number, Sh and analogous to the Nusselt number Nu, in heat transfer.

5.2 Application of Dimensional Analysis to Mass Transfer

One of the method of obtaining equations for predicting mass-transfer coefficients is the use of dimensionless analysis. Dimensional analysis predicts the various dimensionless parameters which are helpful in correlating experimental data.

There are two important mass transfer processes, which we shall consider, the transfer of mass into a steam flowing under forced convection and the transfer of mass into a phase which is moving as the result of natural convection associated with density gradients.

Transfer into a stream flowing under forced convection

Consider the transfer of mass from the walls of a circular conduit to a fluid flowing through the conduit. The mass transfer is due to the concentration driving force $C_{As} - C_A$.

These variables include terms descriptive of the system geometry, the flow and fluid properties and the quantity of importance, k_c.

By the Buckingham method of grouping the variables, the number of dimensionless ☐ groups is equal to the number of variables minus the number of fundamental dimensions. Hence the number of dimensionless group for this problem will be three.
With D_{AB}, ☐ and D as the core variables, the three ☐ groups to be formed are

$$\square_1 \square D_{AB}^{a} \square^{b} D^{c} k_{c} \quad \text{----------------------------} \ (1.14)$$

$$\square_2 \square D_{AB}^{d} \square^{e} D^{f} \square \quad \text{----------------------------} \ (1.15)$$

and $$\square_3 \square D_{AB}^{g} \square^{h} D^{i} \square \quad \text{----------------------------} \ (1.16)$$

Substituting the dimensions for ☐ ,

$$\square_1 \square D_{AB}^{a} \square^{b} D^{c} k_{c} \quad \text{----------------------------} \ (1.17)$$

$$1 = \left(\frac{L^2}{t}\right)^a \left(\frac{M}{L^3}\right)^b (L)^c \left(\frac{L}{t}\right) \text{ ----------------------------- } (1.18)$$

Equating the exponents of the fundamental dimensions on both sides of the equation, we have

L : $0 = 2a - 3b + c + 1$

t : $0 = -a - 1$

M : $0 = b$

Solving these equations,

$a = -1$, $b = 0$ and $c = 1$

Thus $\pi_1 = \frac{k_c D}{D_{AB}}$ which is the Sherwood number.

The other two π groups could be determined in the same manner, yielding

$$\pi_2 = \frac{Dv}{D_{AB}} \text{ ---------- } (1.19)$$

and $$\pi_3 = \frac{\mu}{\rho D_{AB}} = S_c \text{ ---------- } (1.20)$$

which is termed as Schmidt Number

Dividing π_2 by π_3, we get

$$\frac{\pi_2}{\pi_3} = \frac{\left(\frac{Dv}{D_{AB}}\right)}{\left(\frac{\mu}{\rho D_{AB}}\right)} = \frac{Dv\rho}{\mu} = Re \text{ ---------- } (1.21)$$

which is the Reynolds Number

The result of the dimensional analysis of mass transfer by forced convection in a circular conduit indicates that a correlating relation could be of the form,

$$Sh = \phi(\mathrm{Re}, Sc) \text{ -------------------------- } (1.22)$$

Which is analogous to the heat transfer correlation

$$Nu = \phi(\mathrm{Re}, \mathrm{Pr}) \text{ --------------------------- } (1.23)$$

Transfer into a phase whose motion is due to Natural Convection

Natural convection currents develop if there exists any variation in density within the fluid phase. The density variation may be due to temperature differences or to relatively large concentration differences.

According to Buckingham theorem, there will be three dimensionless groups. Choosing D $_{AB}$, L and μ as the core variables, the π groups to be formed are

$$\pi_1 = D_{AB}^{a} L^{b} \mu^{c} k_c \text{ ---------------------------- } (4.24)$$

$$\pi_2 = D_{AB}^{d} L^{e} \mu^{f} \rho \text{ ---------------------------- } (4.25)$$

and $$\pi_3 = D_{AB}^{g} L^{h} \mu^{i} g \Delta\rho_A \text{ ---------------------------- } (4.26)$$

Solving for the dimensionless groups, we obtain

$$\pi_1 = \frac{k_c L}{D_{AB}} = Nu\text{, the Nusselt number} \text{ ---------------------- } (4.27)$$

$$\pi_2 = \frac{\rho D_{AB}}{\mu} = \frac{1}{Sc}$$, the reciprocal of Schmidt number -------- (4.28)

and $$\pi_3 = \frac{L^3 g \Delta\rho_A}{\mu D_{AB}}$$ ------------------------------- (4.29)

With the multiplication of π_2 and π_3, we obtain a dimensionless parameter analogous to the Grashof number in heat transfer by natural convection

$$\pi_2 \pi_3 = \left(\frac{\rho D_{AB}}{\mu}\right)\left(\frac{L^3 g \Delta\rho_A}{\mu D_{AB}}\right)$$

$$= \frac{L^3 \rho g \Delta\rho_A}{\mu^2} = Gr_{AB}$$ -------------------------- (4.30)

The result of the dimensional analysis of mass transfer by natural convection indicates that a correlating relation could be of the form,

$$Sh = f\left(Gr_{AB}, Sc\right)$$ ---------------------------- (4.31)

5.3 Analogy among Mass, Heat and Momentum Transfer

Analogies among mass, heat and momentum transfer have their origin either in the mathematical description of the effects or in the physical parameters used for quantitative description.

To explore those analogies, it could be understood that the diffusion of mass and conduction of heat obey very similar equations. In particular, diffusion in one dimension is described by the Fick's Law as

$$J_A = -D_{AB}\frac{d\,C_A}{d\,z} \quad \text{--------------------------------} \ (4.32)$$

Similarly, heat conduction is described by Fourier’s law as

$$q = -k\frac{dT}{dz} \text{ - } (4.33)$$

Where k is the thermal conductivity.

The similar equation describing momentum transfer as given by Newton’s law is

$$\square = -\square\frac{d\square}{dz} \text{ .... } (4.34)$$

Where ☐ is the momentum flux (or shear stress) and ☐ is the viscosity of fluid.

At this point it has become conventional to draw an analogy among mass, heat and momentum transfer. Each process uses a simple law combined with a mass or energy or momentum balance.

In this section, we shall consider several analogies among transfer phenomenon which has been proposed because of the similarity in their mechanisms. The analogies are useful in understanding the transfer phenomena and as a satisfactory means for predicting behaviour of systems for which limited quantitative data are available.

The similarity among the transfer phenomena and accordingly the existence of the analogies require that the following five conditions exist within the system

1. The physical properties are constant

2. There is no mass or energy produced within the system. This implies that there is no chemical reaction within the system

3. There is no emission or absorption of radiant energy.

4. There is no viscous dissipation of energy.

5. The velocity profile is not affected by the mass transfer. This implies there should be a low rate of mass transfer.

5.4 Reynolds Analogy

The first recognition of the analogous behaviour of mass, heat and momentum transfer was reported by Osborne Reynolds in 1874. Although his analogy is limited in application, it served as the base for seeking better analogies.

Reynolds postulated that the mechanisms for transfer of momentum, energy and mass are identical. Accordingly,

$$\frac{k_c}{v_\infty} = \frac{h}{\rho v_\infty C_p} = \frac{f}{2} \quad \text{-------------------------------- (4.35)}$$

Here h is heat transfer coefficient

f is friction factor

v_∞ is velocity of free stream

The Reynolds analogy is interesting because it suggests a very simple relation between different transport phenomena. This relation is found to be accurate when Prandtl and Schmidt numbers are equal to one. This is applicable for mass transfer by means of turbulent eddies in gases. In this situation, we can estimate mass transfer coefficients from heat transfer coefficients or from friction factors.

5.5 Chilton – Colburn Analogy

Because the Reynold's analogy was practically useful, many authors tried to extend it to liquids. Chilton and Colburn, using experimental data, sought modifications to the Reynold's analogy that would not have the restrictions that Prandtl and Schmidt numbers must be equal to one. They defined for the j factor for mass transfer as

$$j_D = \frac{k_c}{v}\left(Sc\right)^{2/3} \text{ ---------------------------- } (4.36)$$

The analogous j factor for heat transfer is

$$j_H = St \Pr^{2/3} \text{ .... } (4.37)$$

where St is Stanton number = $\frac{Nu}{\text{Re Pr}} = \frac{h}{\rho v C_p}$

Based on data collected in both laminar and turbulent flow regimes, they found

$$j_D = j_H = \frac{f}{2} \text{ - } (4.38)$$

This analogy is valid for gases and liquids within the range of 0.6 < Sc < 2500 and 0.6 < Pr < 100.

The Chilton-Colburn analogy has been observed to hold for many different geometries for example, flow over flat plates, flow in pipes, and flow around cylinders.

5.6 The Prandtl analogy

In the turbulent core the transport is mainly by eddies and near the wall, that is laminar sub-layer, the transport is by molecular diffusion. Therefore, Prandtl modified the above two analogies using universal velocity profile while driving the analogy

$$St = \frac{\frac{f}{2}}{1+5\sqrt{\frac{f}{2}}(Pr-1)} \qquad (4.36)$$

5.7 The Van Karman analogy

Though Prandtl considered the laminar and turbulent laminar sublayers but did not consider the buffer zone. Thus, Van Karman included the buffer zone into the Prandtl analogy to further improve the analogy.

$$St = \frac{\frac{f}{2}}{1+5\sqrt{\frac{f}{2}}\left[(Pr-1)+\ln\left\{1+\frac{5}{6}(Pr-1)\right\}\right]} \qquad (4.37)$$

Problem. A stream of air at 100 kPa pressure and 300 K is flowing on the top surface of a thin flat sheet of solid naphthalene of length 0.2 m with a velocity of 20 m/sec. The other data are:

Mass diffusivity of naphthalene vapor in air = 6 * 10 $^{-6}$ m ^{2}/sec
Kinematic viscosity of air = 1.5 * 10 $^{-5}$ m ^{2}.sc
Concentration of naphthalene at the air-solid naphthalene interface = 1 * 10 $^{-5}$ kmol/m^{3}

Calculate:

(a) the overage mass transfer coefficient over the flat plate
(b) the rate of loss of naphthalene from the surface per unit width

Note: For heat transfer over a flat plate, convective heat transfer coefficient for laminar flow can be calculated by the equation.

$$Nu = 0.664 \, Re_L^{1/2} \, Pr^{1/3}$$

you may use analogy between mass and heat transfer.

Solution:

Given: Correlation for heat transfer

$$Nu = 0.664 \, Re_L^{1/2} \, Pr^{1/3}$$

The analogous relation for mass transfer is

$$Sh = 0.664 \, Re_L^{1/2} \, Sc^{1/3} \text{ ---}(1)$$

where

Sh = Sherwood number = kL/D $_{AB}$

Re$_L$ = Reynolds number = L□□/□

Sc = Schmidt number = □ / (□ D$_{AB}$)

k = overall mass transfer coefficient

L = length of sheet

D $_{AB}$ = diffusivity of A in B

□ = velocity of air

□ = viscosity of air

□ = density of air, and

□/□ = kinematic viscosity of air.

Substituting for the known quantities in equation (1)

$$\frac{k(0.2)}{6 * 10^{-6}} = 0.664 \left(\frac{(0.2)(20)}{1.5 * 10^{-5}}\right)^{1/2} \left(\frac{1.5 * 10^{-5}}{6 * 10^{-6}}\right)^{1/3}$$

k = 0.014 m/sec

Rate of loss of naphthalene = k (C $_{Ai}$ – C $_{A□}$)

= 0.014 (1 * 10 $^{-5}$ – 0) = 1.4024 * 10 $^{-7}$ kmol/m ^{2} sec

Rate of loss per meter width = (1.4024 * 10 $^{-7}$) (0.2) = 2.8048 * 10 $^{-8}$ kmol/m.sec

= 0.101 gmol/m.hr.

5.8 Convective Mass Transfer Correlations

Extensive data have been obtained for the transfer of mass between a moving fluid and certain shapes, such as flat plates, spheres and cylinders. The techniques include sublimation of a solid, vapourization of a liquid into a moving stream of air and the dissolution of a solid into water.

These data have been correlated in terms of dimensionless parameters and the equations obtained are used to estimate the mass transfer coefficients in other moving fluids and geometrically similar surfaces.

Flat Plate

From the experimental measurements of rate of evaporation from a liquid surface or from the sublimation rate of a volatile solid surface into a controlled air-stream, several correlations are available. These correlation have been found to satisfy the equations obtained by theoretical analysis on boundary layers,

$$Sh = 0.664\,\mathrm{Re}_L^{1/2}\,Sc^{1/3} \quad (\text{laminar})\ \mathrm{Re}_L < 3*10^5 \qquad \text{-------------} \ (4.39)$$

$$Sh = 0.036\,\mathrm{Re}_L^{0.8}\,Sc^{1/3} \quad (\text{turbulent})\ \mathrm{Re}_L > 3*10^5 \qquad \text{-----------} \ (4.40)$$

Using the definition of j factor for mass transfer on equation (4.39) and (4.40) we obtain

$$j_D = 0.664\,\mathrm{Re}_L^{-1/2} \quad (\text{laminar})\ \mathrm{Re}_L < 3*10^5 \qquad \text{-------------} \ (4.41)$$

$$J_D = 0.037\,\mathrm{Re}_L^{-0.2} \ (\text{turbulent})\ \mathrm{Re}_L > 3*10^5 \qquad \text{-----------} \ (4.42)$$

These equations may be used if the Schmidt number in the range 0.6 < Sc < 2500.

7. If the local Nusselt number for the laminar boundary layer that is formed over a flat plate is

$$Nu_x = 0.332\,\mathrm{Re}_x^{1/2}\,Sc^{1/3}$$

Obtain an expression for the average film-transfer coefficient $\bar{k}_c$, when the Reynolds number for the plate is

a) $Re_L = 100\,000$
b) $Re_L = 1500\,000$

The transition from laminar to turbulent flow occurs at $Re_x = 3 * 10^5$.

Derivation:

By definition : $$\bar{k}_c = \frac{\int_o^L k_c\,dx}{\int_o^L dx}$$

and $$Nu_x = \frac{k_c x}{D_{AB}}; \qquad Re_x = \frac{x v \rho}{\mu}; \qquad Sc = \frac{\mu}{\rho D_{AB}}$$

For $Re_L = 100\,000$; (which is less than the Reynolds number corresponding to Transition value of $3 * 10^5$)

$$\bar{k}_c = \frac{\int_o^L 0.332 \left(\frac{x v \rho}{\mu}\right)^{1/2} (Sc)^{1/3} \frac{D_{AB}}{x}\,dx}{L}$$

$$= \frac{0.332\,(Sc)^{1/3} \left(\frac{v \rho}{\mu}\right)^{1/2}}{L} D_{AB} \int_o^L \frac{dx}{x^{1/2}}$$

$$= \frac{0.332}{\frac{1}{2} L} Sc^{1/3} \left(\frac{v \rho}{\mu}\right)^{1/2} D_{AB} \left[x^{1/2}\right]_o^L$$

(i.e.) $$\frac{\bar{k}_c L}{D_{AB}} = 0.664\, Re_L^{1/2}\, Sc^{1/3}$$ [answer (a)]

For Re $_L$ = 1500 000 (> $3 * 10^5$)

$$\bar{k}_c = \frac{D_{AB}}{L}\left[\int_0^{L_t} 0.332\,Re_x^{1/2}\,Sc^{1/3}\,\frac{dx}{x} + \int_{L_t}^{L} 0.0292\,Re_x^{4/5}\,Sc^{1/3}\,\frac{dx}{x}\right]$$

where L_t is the distance from the leading edge of the plane to the transition point where $Re_x = 3 * 10^5$.

$$\bar{k}_c = D_{AB}\,\frac{\left[0.332\,Sc^{1/3}\left(\frac{V\rho}{\mu}\right)^{1/2}\int_0^{L_t}\frac{dx}{x^{1/2}} + 0.0292\,Sc^{1/3}\left(\frac{V\rho}{\mu}\right)^{4/5}\int_{L_t}^{L}\frac{dx}{x^{1/5}}\right]}{L}$$

$$\frac{\bar{k}_c L}{D_{AB}} = 0.664\,Re_t^{1/2}\,Sx^{1/3} + \frac{0.0292}{4/5}\,Sc^{1/3}\left(\frac{V\rho}{\mu}\right)^{4/5}\left(x^{4/5}\right)_{L_t}^{L}$$

$$= 0.664\,Re_t^{1/2}\,Sc^{1/3} + 0.0365\,Sc^{1/3}\left(Re_L^{4/5} - Re_t^{4/5}\right)$$

$$\frac{\bar{k}_c L}{D_{AB}} = 0.664\,Re_t^{1/2}\,Sc^{1/3} + 0.0365\,Re_L^{4/5}\,Sc^{1/3} - 0.0365\,Re_t^{4/5}\,Sc^{1/3}$$ where $Re_t = 3 * 10^5$

Single Sphere

Correlations for mass transfer from single spheres are represented as addition of terms representing transfer by purely molecular diffusion and transfer by forced convection, in the form

$Sh = Sh_o + C\,Re^m\,Sc^n$ ---------------------- (4.43)

Where C, m and n are constants, the value of n is normally taken as 1/3

For very low Reynold’s number, the Sherwood number should approach a value of 2. This value has been derived in earlier sections by theoretical consideration of molecular diffusion from a sphere into a large volume of stagnant fluid. Therefore the generalized equation becomes

$$Sh = 2 + C\,\mathrm{Re}^{m}\,Sc^{1/3} \qquad (4.44)$$

For mass transfer into liquid streams, the equation given by Brain and Hales

$$Sh = \left[4 + 1.21\, Pe_{AB}^{2/3}\right]^{1/2} \tag{4.45}$$

correlates the data that are obtained when the mass transfer Peclet number, Pe $_{AB}$ is less than 10,000. This Peclet number is equal to the product of Reynolds and Schmidt numbers (i.e.)

$$Pe_{AB} = Re\, Sc \tag{4.46}$$

For Peclet numbers greater than 10,000, the relation given by Levich is useful

$$Sh = 1.01\, Pe_{AB}^{1/3} \tag{4.47}$$

The relation given by Froessling

$$Sh = 2 + 0.552\, Re^{1/2}\, Sc^{1/3} \tag{4.48}$$

correlates the data for mass transfer into gases for at Reynold’s numbers ranging from 2 to 800 and Schmidt number ranging 0.6 to 2.7.

For natural convection mass transfer the relation given by Schutz

$$Sh = 2 + 0.59\left(Gr_{AB}\, Sc\right)^{1/4} \tag{4.49}$$

is useful over the range

$2 * 10^{8} < Gr_{AB}\, Sc < 1.5 * 10^{10}$

Problem. The mass flux from a 5 cm diameter naphthalene ball placed in stagnant air at 40°C and atmospheric pressure, is 1.47 * 10 $^{-3}$ mol/m ^{2}. sec. Assume the vapor pressure of naphthalene to be 0.15 atm at 40°C and negligible bulk concentration of naphthalene in air. If air starts blowing across the surface of naphthalene ball at 3 m/s by what factor will the mass transfer rate increase, all other conditions remaining the same?

For spheres : Sh = 2.0 + 0.6 (Re) $^{0.5}$ (Sc)$^{0.33}$

Where Sh is the Sherwood number and Sc is the Schmids number. The viscosity and density of air are 1.8 * 10 $^{-5}$ kg/m.s and 1.123 kg/m ^{3}, respectively and the gas constant is 82.06 cm ^{3} atm/mol.K.

Calculations:

$Sh = \frac{k_c L}{D_{AB}}$ where L is the characteristic dimension for sphere L = Diameter.

$$Sc = \frac{\mu}{\rho D_{AB}}$$

$$R_c = \frac{D v \rho}{\mu}$$

$$Mass\ flux,\ N_A = K_c \Delta c \quad \text{------------------------------(1)}$$

Sh = 2.0 + 0.6 (Re) $^{0.5}$ (Sc) $^{0.33}$

$$\frac{k_c D}{D_{AB}} = 2.0 + 0.6 \left(\frac{DV\rho}{\mu}\right)^{0.5} \left(\frac{\mu}{\rho D_{AB}}\right)^{0.33} \quad \text{----------------------- (2)}$$

also $N = K_G \Delta p_A$

Therefore $\frac{k_c}{RT} = K_G$

Given:

$$N = 1.47 * 10^{-3} \frac{mol}{m^2.sec} = \frac{K_c}{RT} \left(p_A \right)$$

$$\frac{k_c}{RT}\left(\frac{0.15}{1} - 0\right) = 1.47 * 10^{-3} * 10^{-4} \frac{mol}{cm^2.sec}$$

$$k_c = \frac{1.47 * 10^{-7}}{0.15} * 82.06 * (273 + 40)$$

= 0.0252 cm/sec

$k_c = 2.517 * 10^{-4}$ m/sec------------------------------(3)

Estimation of D_{AB}:

From (2),

$$\frac{2.517 * 10^{-4} * 5 * 10^{-2}}{D_{AB}} = 2 \quad \text{(since v = 0)}$$

Therefore $D_{AB} = 6.2925 * 10^{-6}$ m²/sec.

And

$$\frac{k_c * 5 * 10^{-2}}{6.2925 * 10^{-6}} = 2 + 0.6 \left(\frac{5 * 10^{-2} * 3 * 1.123}{1.8 * 10^{-5}}\right)^{0.5} \left(\frac{1.8 * 10^{-5}}{1.123 * 6.2925 * 10^{-6}}\right)^{0.33}$$

$7946\, k_c = 2 + 0.6 * (96.74) * (1.361)$

$k_c = 0.0102$ m/sec...(4)

$$\frac{(4)}{(3)} \Rightarrow \frac{N_{A2}}{N_{A1}} = \frac{0.0102}{2.517 * 10^{-4}} = 40.5$$

Therefore, rate of mass transfer increases by 40.5 times the initial conditions.

Single Cylinder

Several investigators have studied the rate of sublimation from a solid cylinder into air flowing normal to its axis. Bedingfield and Drew correlated the available data in the form

$$\frac{k_G P Sc^{0.56}}{G_m} = 0.281 \left(Re^{/}\right)^{-0.4} \text{ ----------------------- (4.50)}$$

which is valid for $400 < Re^{/} < 25000$

and $0.6 < Sc < 2.6$

Where $Re^{/}$ is the Reynold's number in terms of the diameter of the cylinder, G_m is the molar mass velocity of gas and P is the pressure.

5.9 Flow Through Pipes

Mass transfer from the inner wall of a tube to a moving fluid has been studied extensively. Gilliland and Sherwood, based on the study of rate of vapourization of nine different liquids into air given the correlation

$$Sh \frac{p_{B,lm}}{P} = 0.023 \, Re^{0.83} \, Sc^{0.44} \text{ -------------- (4.51)}$$

Where $p_{B,\, lm}$ is the log mean composition of the carrier gas, evaluated between the surface and bulk stream composition. P is the total pressure. This expression has been found to be valid over the range

$2000 < Re < 35000$

$0.6 < Sc < 2.5$

Linton and Sherwood modified the above relation making it suitable for large ranges of Schmidt number. Their relation is given as

$$Sh = 0.023\ Re^{0.83}\ Sc^{1/3} \text{ --------------------- } (4.52)$$

and found to be valid for

$2000 < Re < 70000$

and $1000 < Sc < 2260$

Problem. A solid disc of benzoic acid 3 cm in diameter is spin at 20 rpm and 25°C. Calculate the rate of dissolution in a large volume of water. Diffusivity of benzoic acid in water is $1.0 * 10^{-5}$ cm^2/sec, and solubility is 0.003 g/cc. The following mass transfer correlation is applicable:

$$Sh = 0.62\ Re^{1/2}\ Sc^{1/3}$$

Where $Re = \frac{D^2 \omega \rho}{\mu}$ and ω is the angular speed in radians/time.

Calculations:

$$\text{Dissolution rate} = N_A\ S \text{ --------------------------------- } (1)$$

Where N_A = mass flux, and

S = surface area for mass transfer

$$N_A = k_c (C_{As} - C_{A\infty}) \text{ .... } (2)$$

Where C_{As} is the concentration of benzoic and at in water at the surface of the dose.

$C_{A\infty}$ is the concentration benzoic acid in wate for an from the surface of the disc.

Given:

Sh = 0.62 Re $^{1/2}$ Sc $^{1/3}$

(i.e.) $$\frac{k_c D}{D_{AB}} = 0.62\left(\frac{D^2\omega\rho}{\mu}\right)^{\frac{1}{2}}\left(\frac{\mu}{\rho D_{AB}}\right)^{\frac{1}{3}}$$ ------------------- (3)

1 rotation = 2π radian

Therefore 20 rotation per minute = 20 * 2π radian/min

$$= \frac{20}{60} * 2\pi \quad radian/sec$$

For water ρ = 1 g/cm ^{3} μ = 1 centipoise = 0.01 g/cm.sec.

From (3),

$$k_c = 0.62\ D_{AB}\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\left(\frac{\mu}{\rho D_{AB}}\right)^{\frac{1}{3}}$$

$$= 0.62 * 1.0 * 10^{-5} * \left(\frac{(40\pi/60) * 1}{0.01}\right)^{\frac{1}{2}}\left(\frac{0.01}{1 * 1.0 * 10^{-5}}\right)^{\frac{1}{3}}$$

= 8.973 * 10 $^{-4}$ cm/sec.

From (2),

N $_A$ = 8.973 * 10 $^{-4}$ (0.003 – 0)

= 2.692 * 10 $^{-6}$ g/cm ^{2}.sec

From (1),

N $_A$ S = N $_A$ * (2πr ^{2})

= 2.692 * 10 $^{-6}$ * (2π * 1.5 ^{2})

= 3.805 * 10 $^{-5}$ g/sec

= 0.137 g/hr.

5.10 Hot Wire Anemometer

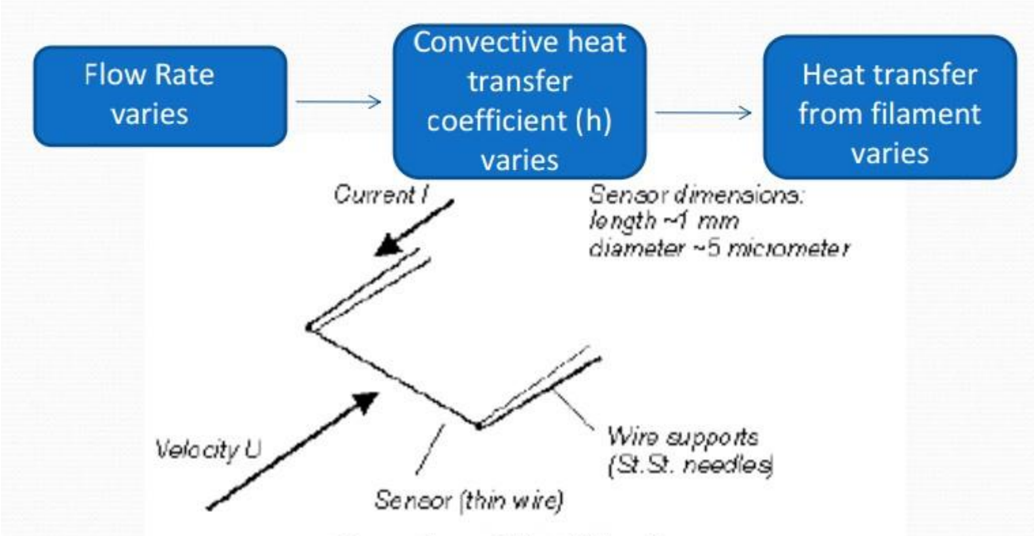
A hot-wire anemometer measures local instantaneous velocity based on principles of heat transfer. However, it requires that the fluid itself be at a uniform temperature. It can be used to measure three components of velocity and velocity fluctuations arising in turbulent flow. This is possible because of the high speed of response of the hot-wire probe and the associated feedback circuit. A hot-wire probe is used in gas flows, while a hot-film is used for liquid flow. The hot- wire has a limitation that it is insensitive to the flow direction. Further, it has a non-linear input- output relationship which makes its sensitivity non-uniform over any velocity range. In particular, the sensitivity decreases with increasing velocity. The hot-wire probe is a platinum- coated tungsten wire, typically of 5 μm diameter and about mm length, supported between highly conducting prongs. Tungsten has high temperature coefficient of resistance (i.e., resistance increases rapidly with temperature) and the platinum coating affords strength as well as protection against corrosion of the thin wire.

Characteristics of Hot Wire Anemometry

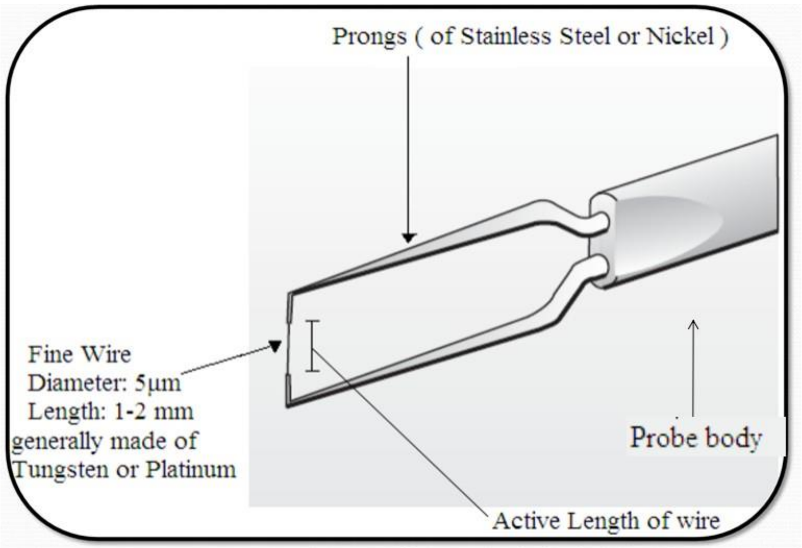
- Intrusive Technique
- Measurement of instantaneous velocities and temperature at a point in a flow.
- Hot wire anemometry is an ideal tool for measurement of velocity fluctuations in time domain in turbulent flows.
- Principal tool for basic studies of physics of turbulent flows.

Advantages of HWA

- Good Frequency response
- Measurements to several hundred kHz possible, 1 MHz also feasible
- Velocity Measurement: measures magnitude and direction of velocity and velocity fluctuations, Wide velocity range Temperature Measurements
- Two Phase Flow: Measurements in flows containing continuous turbulent phase and distributed bubbles



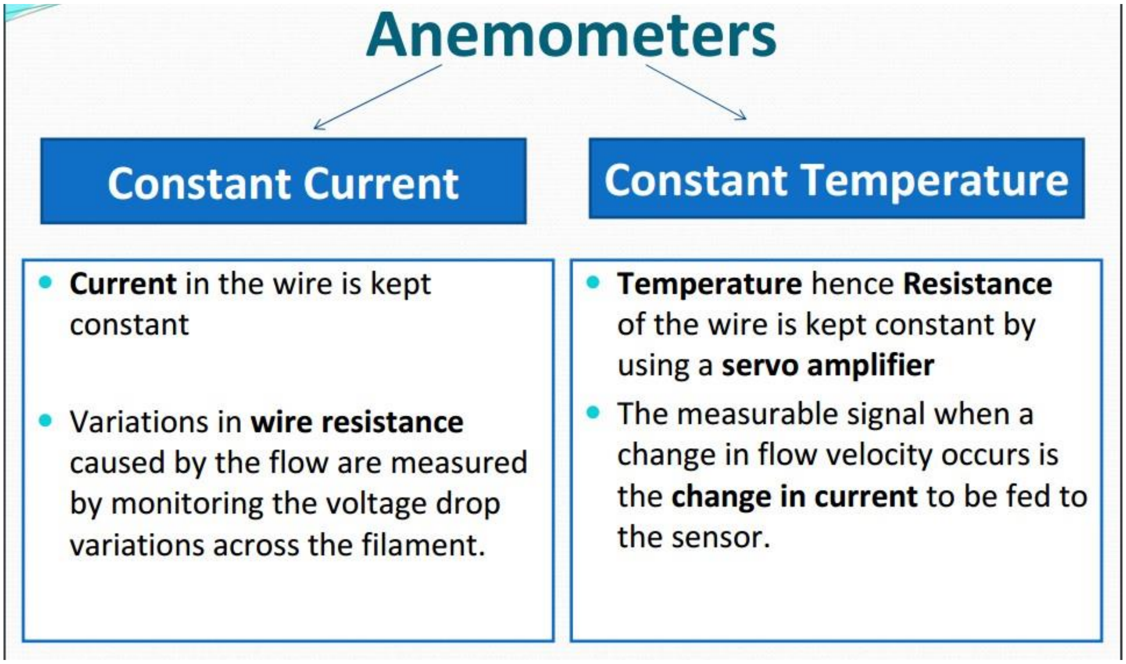
Operation of Hot Wire Sensor



Structure of hot wire probe

Mode of Operation

1. **Constant Current Anemometer**
2. **Constant Temperature Anemometer**



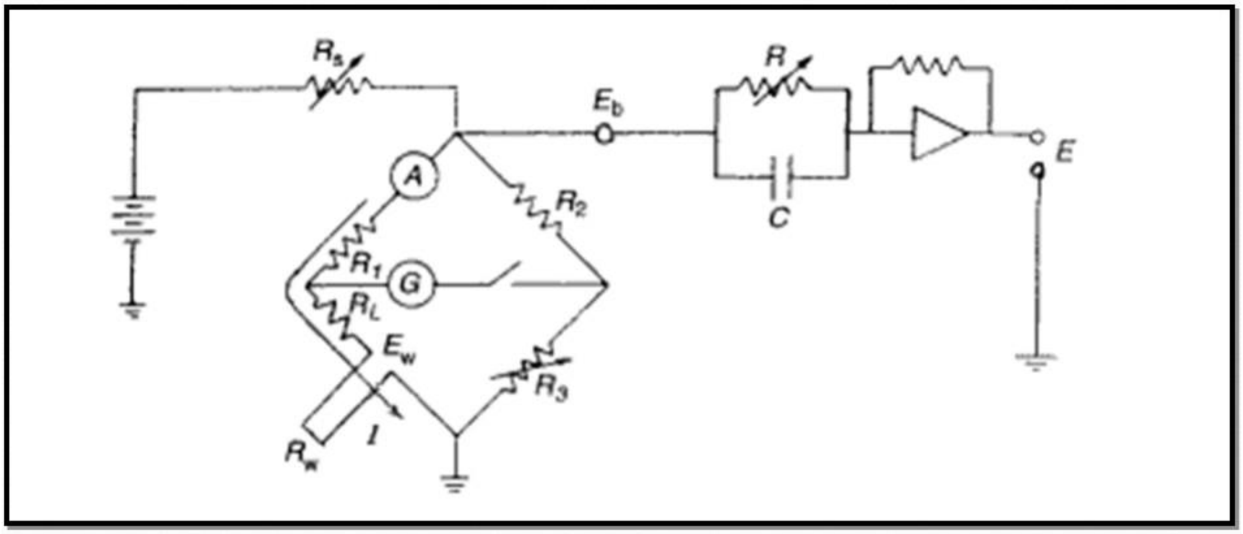
Anemometers

Constant Current

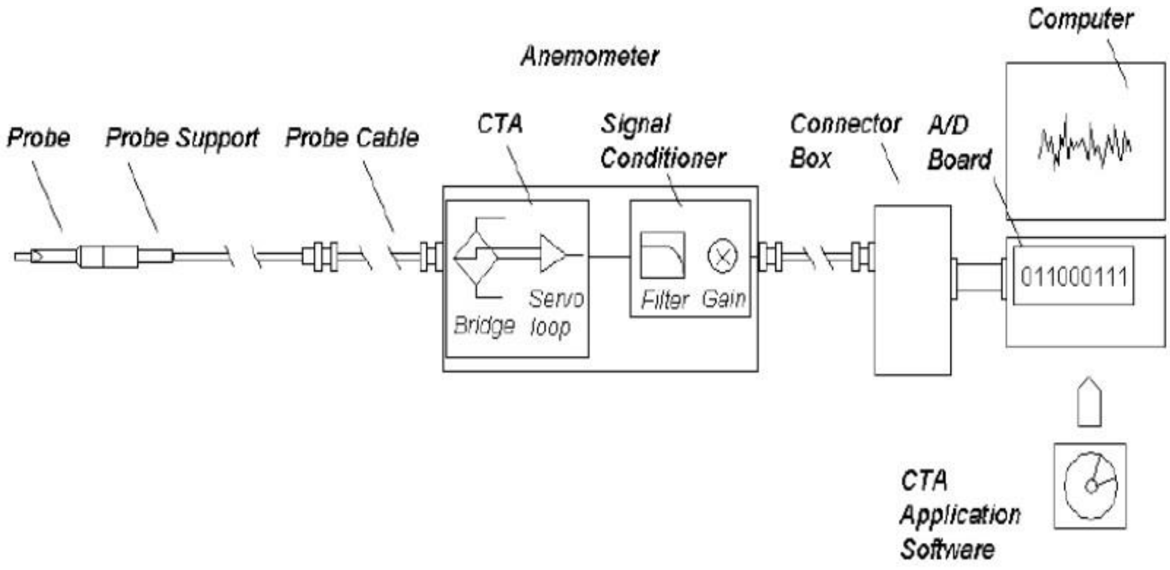
- **Current** in the wire is kept constant
- Variations in **wire resistance** caused by the flow are measured by monitoring the voltage drop variations across the filament.

Constant Temperature

- **Temperature** hence **Resistance** of the wire is kept constant by using a **servo amplifier**
- The measurable signal when a change in flow velocity occurs is the **change in current** to be fed to the sensor.



Circuit Diagram of Constant Current Anemometer



Basic CTA Measuring Chain