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SCHOOL OF BIO AND CHEMICAL ENGINEERING

DEPARTMENT OF BIOMEDICAL ENGINEERING

UNIT – I – Classification of Signals and Systems – SBMA1304

CLASSIFICATION OF SIGNALS AND SYSTEMS

1.INTRODUCTION

1.1.SIGNAL: Any physical quantity if its varies with respect to time it is termed as a signal.

Eg: time, pressure, velocity, mass, temperature

1.2.SYSTEM: Any physical device which performs an operation on the signal it is termed as system.

Eg: amplifier system, filter system, rectifier systems

1.3.CONTINUOUS TIME SIGNAL: If the amplitude of the signal varies continuously with respect to time then it is termed as continuous time signal. It can be represented by the symbol x(t).

1.4DISCRETE TIME SIGNAL: It has got the discrete set of values. It has the specific amplitude for the specific time intervals. It can be represented by the symbol x(n).

1.5.CT & DT WAVEFORMS

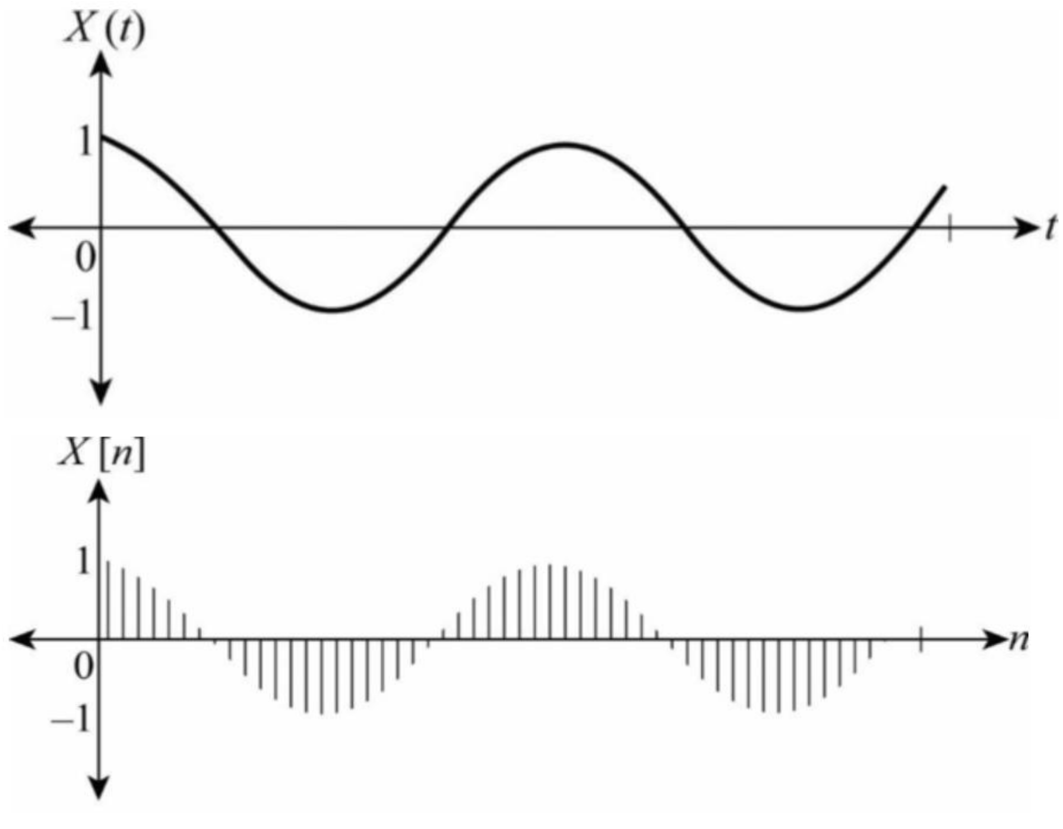


Fig. 1: CT and DT waveforms

1.6.CLASSIFICATION OF CT AND DT SIGNALS

- Periodic and non periodic signal
- Even and odd signal
- Energy and power signal
- Deterministic and random signal

1.6.1.PERIODIC AND NON PERIODIC SIGNAL

A signal is said to be periodic if it repeats at regular time interval. Non periodic signals do not repeat at regular intervals

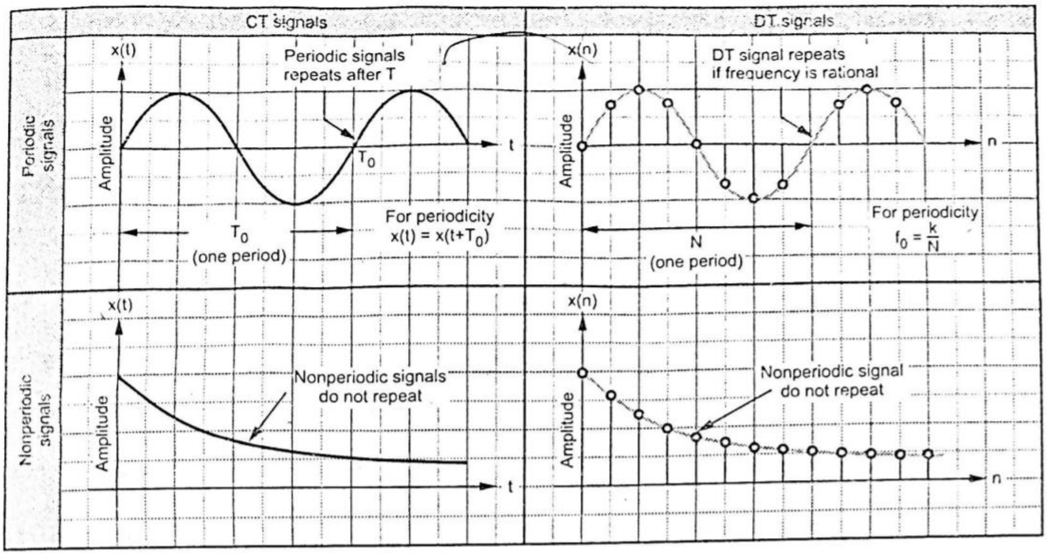


Fig. 2: Periodic and nonperiodic waveforms

- **Condition for periodicity of CT signal**

The CT signal repeat after certain period T_o i.e.,

$$x(t) = x(t+T_0)$$

- **Condition for periodicity of DT signal**

Consider DT cosine wave, $x(n) = \cos(2\pi f_0 n)$

$$x(n+N) = \cos[2\pi f_0 (n+N)]$$

$$= \cos(2\pi f_0 n + 2\pi f_0 N)$$

For periodicity, $x(n) = x(n+N)$

$$\cos(2\pi f_0 n) = \cos(2\pi f_0 n + 2\pi f_0 N)$$

Above equation is satisfied only if $2\pi f_0 N$ is integer multiple of 2π i.e.,

$$2\pi f_0 N = 2\pi k, \text{ Where k is integer}$$

$$\boxed{f_0 = \frac{k}{N}}$$

- **Periodicity of signal $x_1(t)+x_2(t)$**

Let us consider that the signal $x(t)=x_1(t)+x_2(t)$

Then $x_1(t)$ will be periodic if,

$$x_1(t) = x_1(t+T_1) = x_1(t+2T_1) = \dots$$

$$\text{or} \quad x_1(t) = x_1(t+mT_1), \quad \text{Here 'm' is an integer.}$$

Similarly $x_2(t)$ will be periodic if,

$$x_2(t) = x_2(t+T_2) = x_2(t+2T_2) = \dots$$

$$\text{or} \quad x_2(t) = x_2(t+nT_2), \quad \text{Here 'n' is an integer}$$

Then x(t) will be periodic if,

$$mT_1 = nT_2 = T_0, \quad \text{here } T_0 \text{ is period of } x(t)$$

This means 'T_0' is integer multiple of periods of $x_1(t)$ and $x_2(t)$. For above equation we have,

$$\boxed{\frac{T_1}{T_2} = \frac{n}{m}, \text{ i.e. ratio of two integers}}$$

- Periodicity of $x_1(n)+x_2(n)$

Here $x(n) = x_1(n)+x_2(n)$ is periodic if,

$$\boxed{\frac{N_1}{N_2} = \frac{n}{m} \text{ i.e. ratio of two integers}}$$

The period of $x(n)$ will be least common multiple of N_1 and N_2.

1.6.2.EVEN AND ODD SIGNAL

- A signal is said to be even signal if the inversion of time axis does not change the amplitude. It is also called as symmetric signal.
- x(t) = x(-t)
- x(n) = x(-n)
- A signal is said to be odd signal if the inversion of time axis also inverts the amplitude of the signal. It is also called as antisymmetric signal.

  x(t) = -x(-t)

  x(n) = -x(-n)

  Examples of even and odd signal

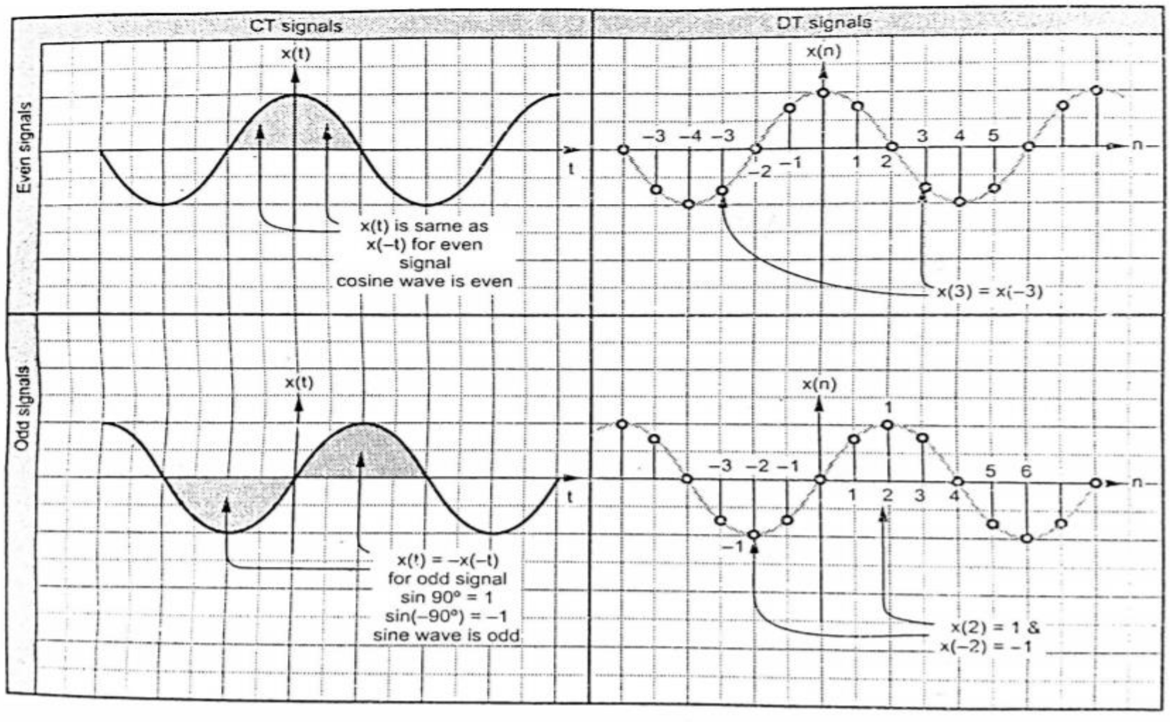


Fig. 3: Even and odd waveforms

Representation of signal in even and odd parts

i) Let the signal be represented into its even and odd parts as,

$$x(t) = x_e(t) + x_o(t)$$

Here $x_e(t)$ is even part of $x(t)$ and

$x_o(t)$ is odd part of $x(t)$

ii) Substitute −t for t in above equation,

$$x(-t) = x_e(-t) + x_o(-t)$$

Now by definition of even signal, $x_e(-t) = x_e(t)$ and by definition of odd signal $x_o(-t) = -x_o(t)$. Hence above equation will be,

$$x(-t) = x_e(t) - x_o(t)$$

iii) Adding equation (1.2.7) and equation (1.2.8),

$$x(t) + x(-t) = 2x_e(t) \Rightarrow x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$$

Subtracting equation (1.2.8) from equation (1.2.7).

$$x(t) - x(-t) = 2x_o(t) \Rightarrow x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$$

Even part : $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$

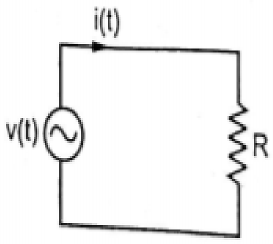
Odd part : $x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$

Similarly for DT signals we can write,

Even part : $x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$

Odd part : $x_o(n) = \frac{1}{2}\{x(n) - x(-n)\}$

1.6.3.ENERGY AND POWER SIGNAL



The instantaneous power dissipated in the load resistance R is given as

$$p(t) = \frac{v^2(t)}{R} = i^2(t)R$$

Normalized power

It is the power dissipated in R = 1Ω load.

$$p(t) = v^2(t) = i^2(t)$$

Let v(t) and i(t) denoted by x(t),then the normalized power is

$$p(t) = x^2(t)$$

Energy

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \quad \text{for CT signal} \qquad E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad \text{for DT signal}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \quad \text{for CT signal}$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \quad \text{for DT signal}$$

Power

A signal is said to be power signal, if its normalized power is non zero and finite

For power signal, $0 < P < \infty$

A signal is said to be energy signal, if its total energy is finite and nonzero

For Energy signal, $0 < E < \infty$

1.7.COMPARISON

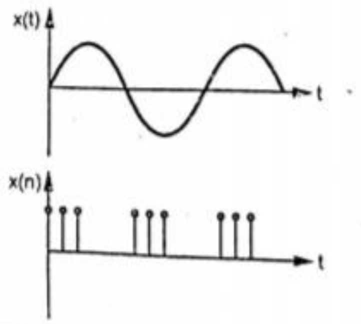
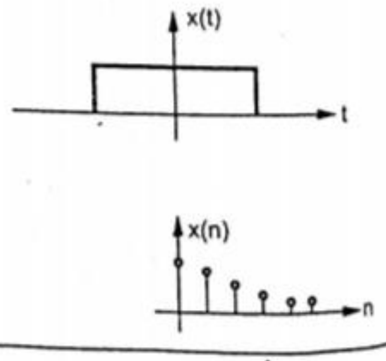
Sr. No.	Parameter	Power signal	Energy signal
1.	Definition	$0 < P < \infty$	$0 < E < \infty$
2.	Equation	$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$ $\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x(n)\rvert^2$	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$ $= \sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2$
3.	Periodicity	Most of periodic signals are Power signals	Most of the nonperiodic signals are Energy signals.
4.	Energy and power	Energy of the power signal is infinite.	Power of the energy signal is zero.
5.	Examples		

Table. 1: Comparison

1.8.DETERMINISTIC AND RANDOM SIGNAL

- Deterministic: If any signal can be represented by proper mathematical equation then it is termed as deterministic signal
- Eg: sine wave, cosine wave, exponential

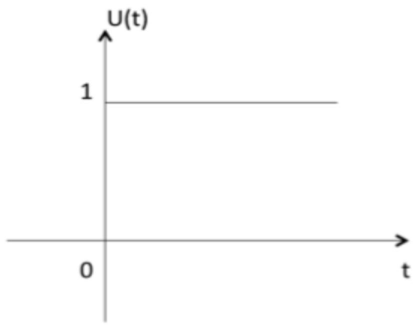
- Random: It can not be represented by any mathematical equation
- Eg: all kind of noises during electronic experiments

1.9.STANDARD ELEMETRY SIGNALS

- Unit step signal
- Unit Ramp signal
- Unit impulse signal
- Exponential signal

1.9.1.UNIT STEP SIGNAL

Unit step function is denoted by u(t). It is defined as u(t) = $\begin{cases} 1 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



- It is used as best test signal.
- Area under unit step function is unity.

Fig. 4: Unit step signal

1.9.2.UNIT RAMP SIGNAL

area under unit ramp is unity

Ramp signal is denoted by r(t), and it is defined as r(t) = $\begin{cases} t & t \geqslant 0 \\ 0 & t < 0 \end{cases}$

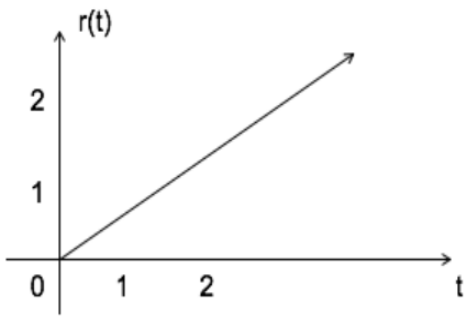


Fig. 5: Unit ramp signal

1.9.3.UNIT IMPULSE SIGNAL

Impulse function is denoted by δ(t). and it is defined as δ(t) = $\begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$

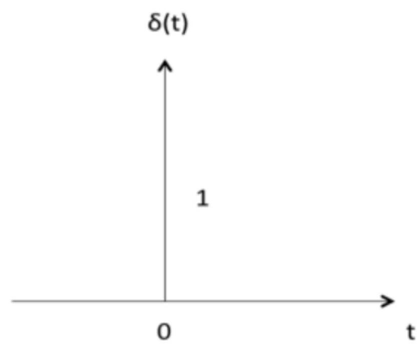


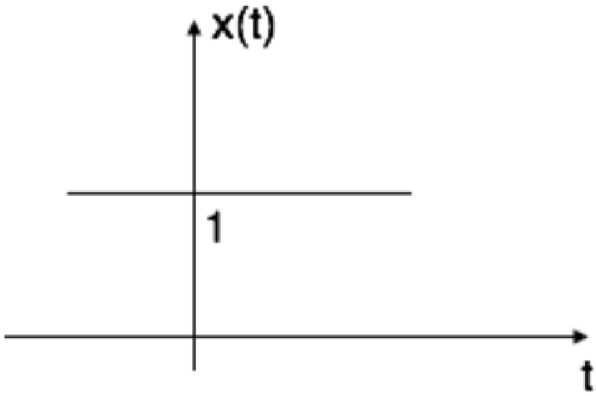
Fig. 6: Unit impulse signal

1.9.4.EXPONENTIAL SIGNAL

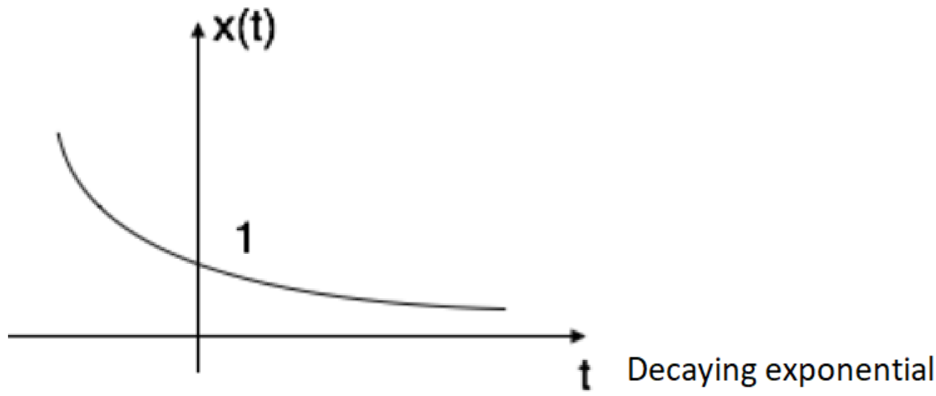
Exponential signal is in the form of x(t) = $e^{\alpha t}$

The shape of exponential can be defined by α .

Case i: if α = 0 $\rightarrow$ x(t) = e^0 = 1



Case ii: if α < 0 i.e. -ve then x(t) = $e^{-\alpha t}$



Case iii: if α > 0 i.e. +ve then x(t) = $e^{\alpha t}$

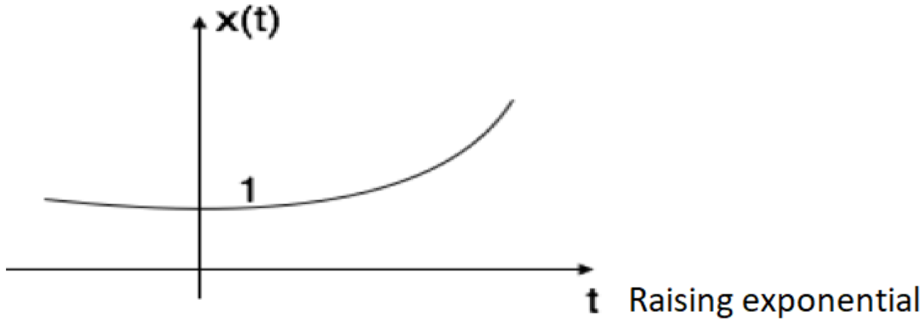


Fig. 7: Exponential waveforms

1.10.STANDARD ELEMETRY SIGNALS IN DISCRETE TIME

- Unit step signal
- Unit ramp signal
- Unit impulse signal
- Exponential signal
- Sinusoidal signal
- Complex exponential signal

1.10.1.UNIT STEP SIGNAL

The unit step sequence is defined as

$$u(n) = 1 \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$

The graphical representation of $u(n)$ is shown in Fig

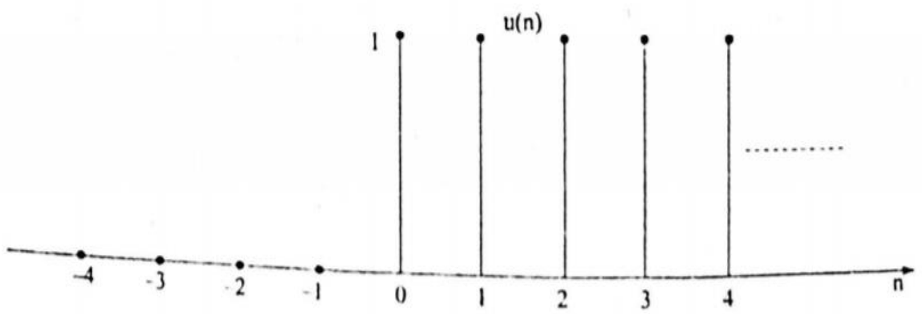


Fig. 8: Unit step signal

1.10.2.UNIT RAMP SIGNAL

The unit ramp sequence is defined as

$$r(n) = n \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$

The graphical representation of $r(n)$ is shown in Fig.

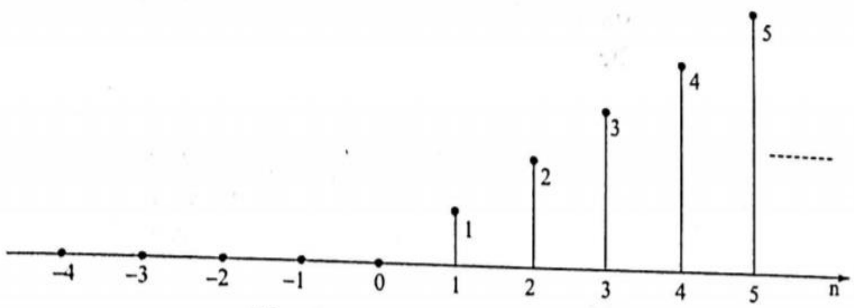


Fig. 9: Unit ramp signal

1.10.3.UNIT IMPULSE SIGNAL

The unit-sample sequence is defined as

$$\delta(n) = 1 \text{ for } n = 0$$
$$= 0 \text{ for } n \neq 0$$

The graphical representation of $\delta(n)$ is shown in Fig

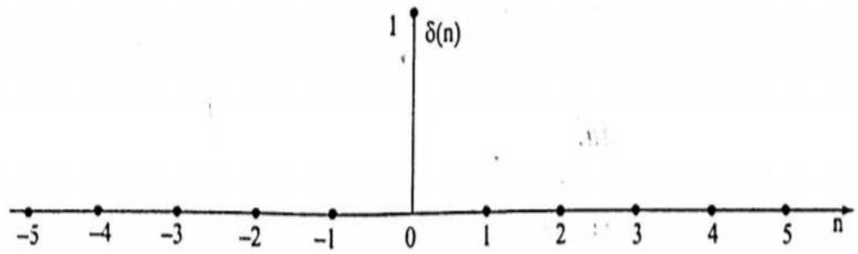


Fig. 10: Unit impulse signal

1.10.4.EXPONENTIALSIGNAL

The exponential signal is of the form $x(n) = a^n$ for all n

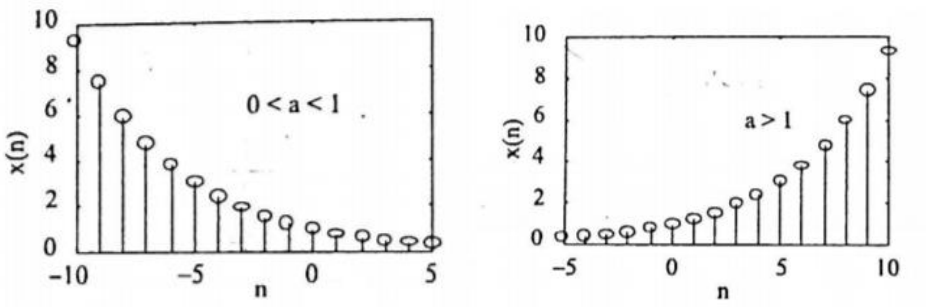


Fig. 11: Exponential waveforms

1.10.5.SINUSOIDAL SIGNAL

The discrete-time sinusoidal signal is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$

where ω_0 is the frequency (in radians per sample) and ϕ is the phase (in radians).

Using Euler's identity, we can write

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

Complex Exponential

The discrete-time complex exponential signal is given by

$$x(n) = a^n e^{j(\omega_0 n+\phi)}$$
$$= a^n \cos(w_0 n + \phi) + j a^n \sin(\omega_0 n + \phi)$$

1.11.PERIODICITY PROBLEMS

(i) $x(n) = \cos 0.01\,\pi\,n$

Compare with, $x(n) = \cos 2\pi f n$

$\therefore\ 2\pi f n = 0.01\,\pi n \Rightarrow f = \frac{0.01}{2} = \frac{1}{200} = \frac{k}{N}$

Here f is expressed as ratio of two integers with k = 1 and N = 200. Hence the signal is, periodic with N = 200.

ii) $x(n) = \cos 3\pi n$

Compare with $x(n) = \cos 2\pi f n$

$\therefore\ 2\pi f n = 3\pi n \Rightarrow f = \frac{k}{N} = \frac{3}{2}$ i.e. ratio of two integers.

Hence this signal is, periodic with N = 2

iii) $x(n) = \sin 3n$

Compare with $x(n) = \cos 2\pi f n$

$2\pi f n = 3n \Rightarrow f = \frac{k}{N} = \frac{3}{2\pi}$ which is not ratio of two integers.

Hence the signal is not periodic

iv) $x(n) = \cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

Compare with, $x(n) = \cos 2\pi f_1 n + \cos 2\pi f_2 n$

$$2\pi f_1 n = \frac{2\pi n}{5} \Rightarrow f_1 = \frac{1}{5} = \frac{k_1}{N_1}, \quad \therefore N_1 = 5$$

and $$2\pi f_2 n = \frac{2\pi n}{7} \Rightarrow f_2 = \frac{1}{7} = \frac{k_2}{N_2}, \quad \therefore N_2 = 7$$

Here since $\frac{N_1}{N_2} = \frac{5}{7}$ is the ratio of two integers, the sequence is periodic. The period of $x(n)$ is least common multiple of N_1 and N_2. Here least common multiple of $N_1 = 5$ and $N_2 = 7$ is 35. Therefore this sequence is, **periodic with N = 35.**

v) $x(n) = \cos\left(\frac{n}{8}\right)\cos\frac{n\pi}{8}$:

Here $$2\pi f_1 n = \frac{n}{8} \Rightarrow f_1 = \frac{1}{16\pi}, \quad \text{which is not rational}$$

and $$2\pi f_2 n = \frac{n\pi}{8} \Rightarrow f_1 = \frac{1}{16}, \quad \text{which is rational}$$

Thus $\cos\left(\frac{n}{8}\right)$ is not periodic and $\cos\left(\frac{n\pi}{8}\right)$ is periodic. $x(n)$ is non periodic since it is the product of periodic and nonperiodic signal.

vi) $x(n) = \sin(\pi + 0.2n)$

Compare with, $x(n) = \sin(2\pi f n + \theta)$

$\therefore \theta = \pi$ i.e. phase shift and

$2\pi f n = 0.2n \Rightarrow f = \frac{0.2}{2\pi} = \frac{1}{10\pi}$ *which is not rational. Hence this signal is* **not periodic.**

vii) $x(n) = e^{\left(j\frac{\pi}{4}\right)n}$

$$= \cos\frac{\pi}{4}n + j\sin\frac{\pi}{4}n$$

Compare with, $x(n) = \cos 2\pi f n + j\sin 2\pi f n$

Here $2\pi f n = \frac{\pi}{4}n \Rightarrow f = \frac{1}{8} = \frac{k}{N}$, which is rational.

Hence this signal is, **periodic with N = 8**

$x(t) = \cos t + \sin\sqrt{2}\,t$

Compare with, $x(t) = \cos 2\pi f_1 t + \sin 2\pi f_2 t$

$$2\pi f_1 t = t \quad \Rightarrow \quad f_1 = \frac{1}{2\pi} = \frac{1}{T_1}, \text{ Hence } T_1 = 2\pi$$

and.

$$2\pi f_2 t = \sqrt{2}t \quad \Rightarrow \quad f_2 = \frac{\sqrt{2}}{2\pi} = \frac{1}{T_2}, \text{ Hence } T_2 = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$$

The ratio of two periods is, $\frac{T_1}{T_2} = \frac{2\pi}{\sqrt{2}\pi} = \sqrt{2}$. Since the ratio $\frac{T_1}{T_2}$ is not ratio of two integers (i.e. not rational number), the signal is **not periodic.**

$x(t) = 2\cos 100\pi t + 5\sin 50t$

Compare with, $x(t) = A_1\cos 2\pi f_1 t + A_2 \sin 2\pi f_2 t$

$$2\pi f_1 t = 100\pi t \Rightarrow f_1 = \frac{100}{2} = 50 = \frac{1}{T_1}, \text{ Hence } T_1 = \frac{1}{50}$$

and

$$2\pi f_2 t = 50\,t \Rightarrow f_2 = \frac{50}{2\pi} = \frac{25}{\pi} = \frac{1}{T_2}, \text{ Hence } T_2 = \frac{\pi}{25}$$

The ratio of two peeriods is, $\frac{T_1}{T_2} = \frac{1/50}{\pi/25} = \frac{1}{2\pi}$, which is not rational. Therefore this signal is **not periodic.**

$$x(t) = 2\cos t + 3\cos\frac{t}{3}$$

Compare with, $x(t) = A\cos 2\pi f_1 t + B\cos 2\pi f_2 t$

$$2\pi f_1 t = t \quad \Rightarrow \quad f_1 = \frac{1}{2\pi} = \frac{1}{T_1}, \text{ Hence } T_1 = 2\pi$$

and $$2\pi f_2 t = \frac{t}{3} \quad \Rightarrow \quad f_2 = \frac{1}{6\pi} = \frac{1}{T_2}, \text{ Hence } T_2 = 6\pi$$

The ratio of two peeriods is, $\frac{T_1}{T_2} = \frac{2\pi}{6\pi} = \frac{1}{3}$. Which is rational.

Hence the signal is **periodic.** The fundamental period is least common multiple of $T_1 = 2\pi$ and $T_2 = 6\pi$. It will be $T = 6\pi$.

$$x(t) = \cos^2(2\pi t)$$

$$= \frac{1+\cos 4\pi t}{2} \qquad \text{since } \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\cos 4\pi t$$

Here $\frac{1}{2}$ is the DC shift added to the signal $x_1(t) = \frac{1}{2}\cos 4\pi t$.

But $x_1(t)$ is periodic. It will remain periodic after adding the DC shift. Hence the signal is **periodic.**

And, $2\pi f t = 4\pi t \Rightarrow f = 2 = \frac{1}{T}$, hence period $T = \frac{1}{2}$.

1.11.2.EVEN AND ODD COMPONENTS PROBLEMS

$x(n) = e^{-(n/4)} \quad u(n)$

Even and odd parts of the sequence $x(n)$ are given by equation

Even part, $x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$ and

Odd part, $x_o(n) = \frac{1}{2}\{x(n) - x(-n)\}$

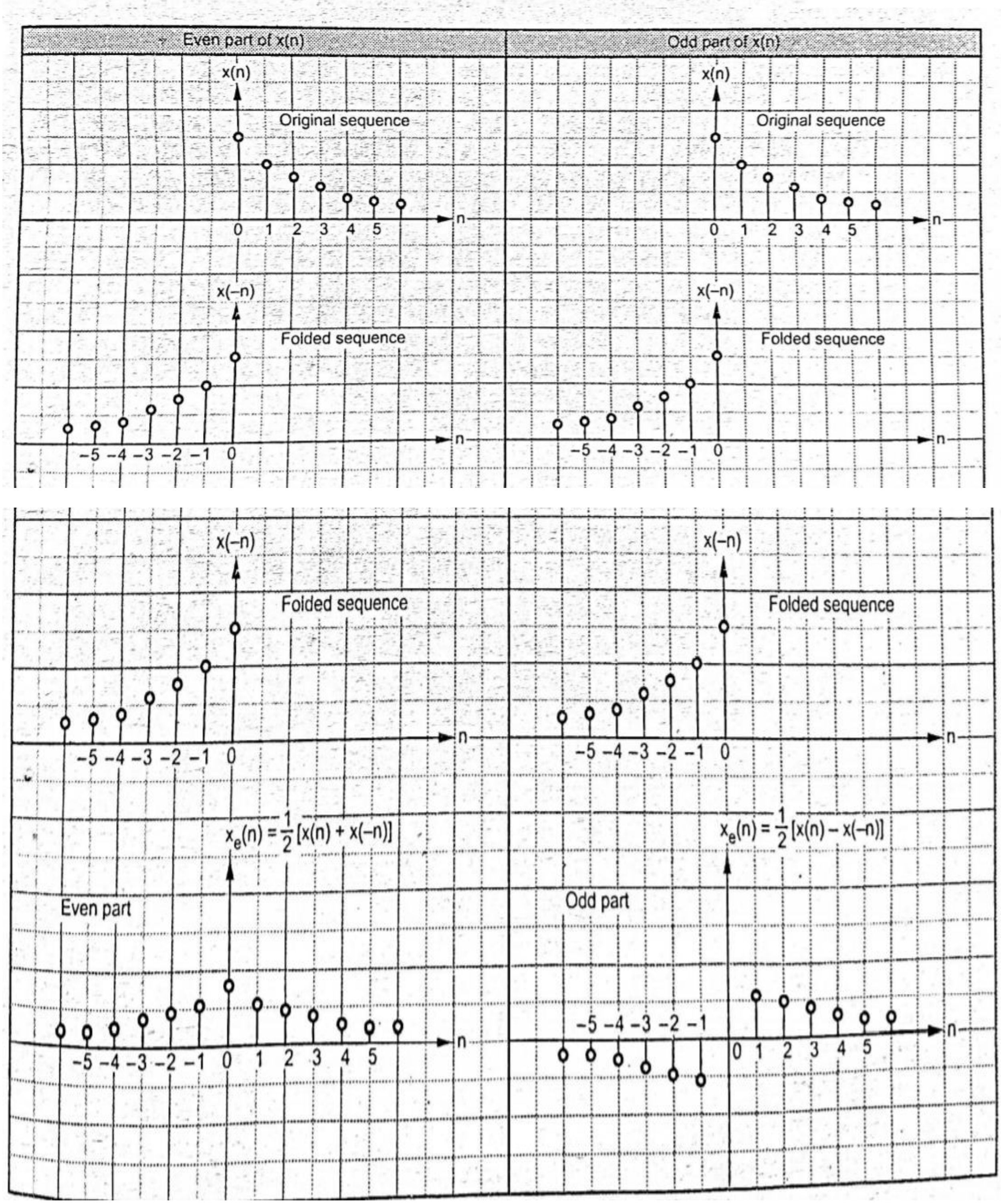


Fig. 12: Even and odd signal

$$x(n) = \text{Im}[e^{jn\pi/4}]$$

$$= \text{Im}\left[\cos\frac{n\pi}{4} + j\sin\frac{n\pi}{4}\right], \quad \sin e^{j0} = \cos\theta + j\sin\theta$$

$$= \sin\frac{n\pi}{4}$$

Compare this equation with $x(n) = \sin 2\pi f n$, hence $2\pi f n = \frac{n\pi}{4} \Rightarrow f = \frac{1}{8}$ cycles/sample. Since $f = \frac{k}{N} = \frac{1}{8}$. There will be 8 samples in one period of DT sine wave.

Fig. 1.2.10 shows the waveform of $x(n) = \sin\frac{n\pi}{4}$ and its even and odd parts are also shown. Even and odd parts are given as,

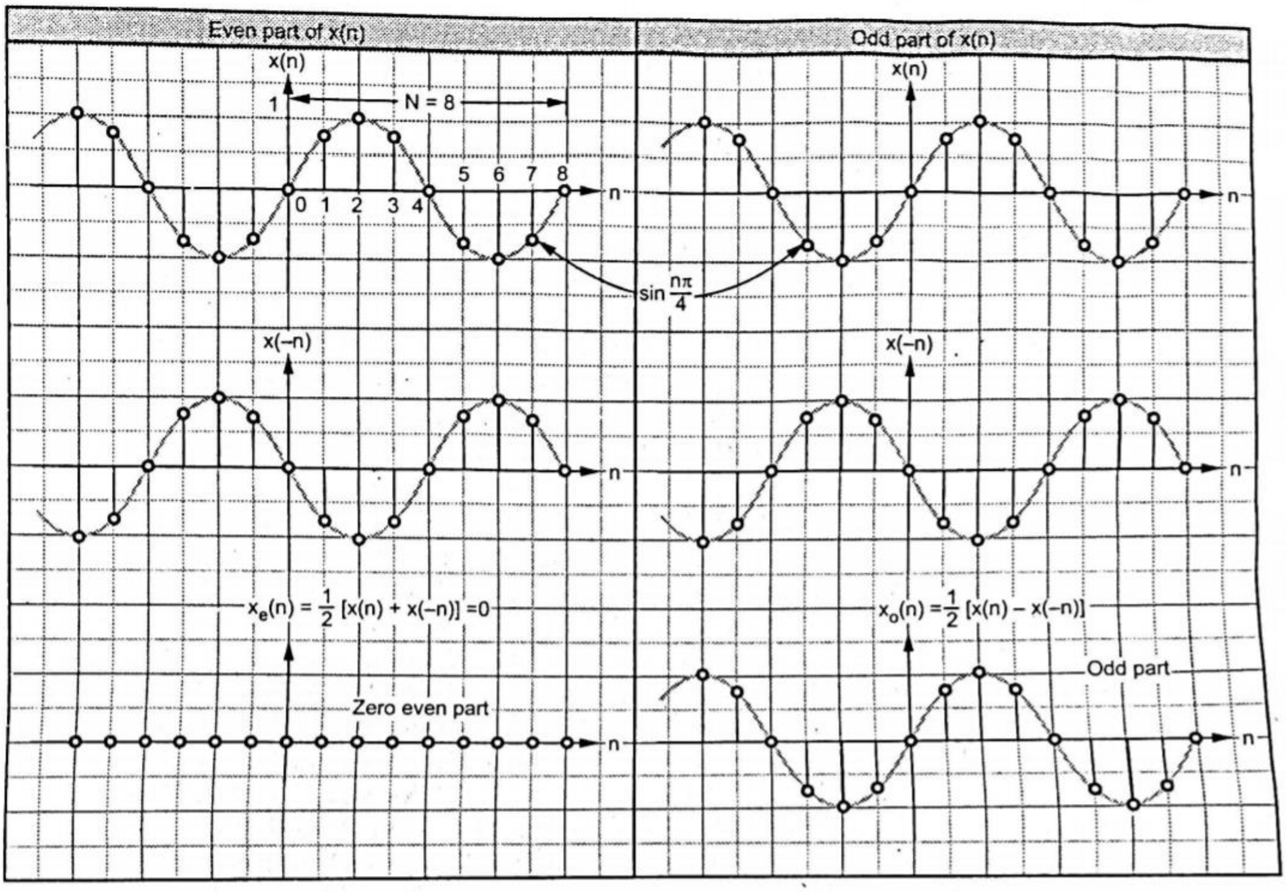


Fig. 13: Even and odd signal

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases}$$

Even part $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$ and

Odd part $x_0(t) = \frac{1}{2}\{x(t) - x(-t)\}$

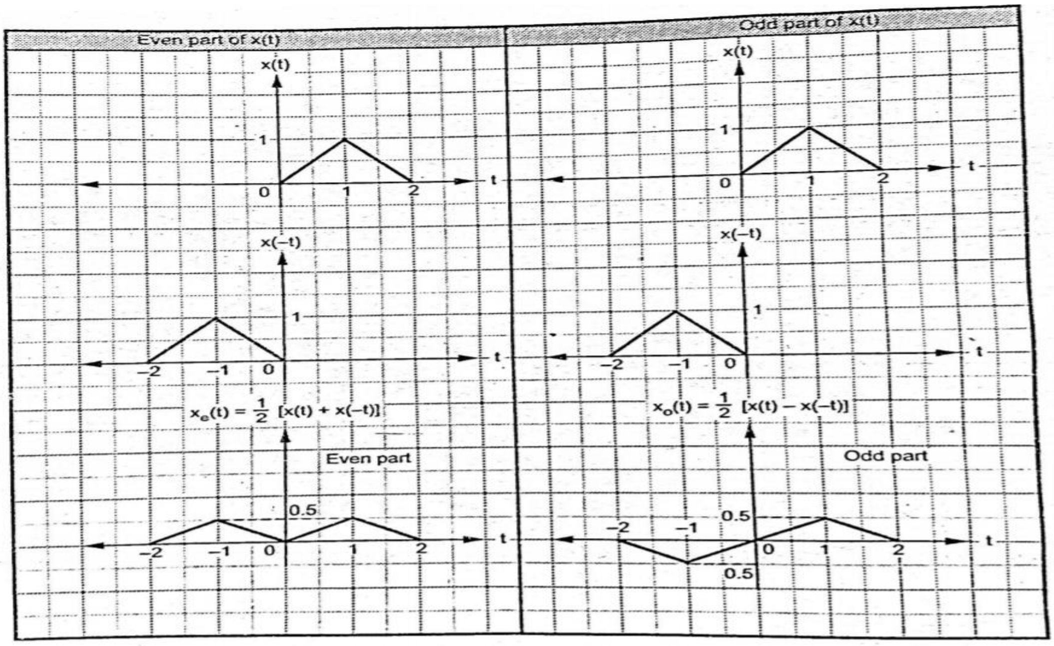


Fig. 14: Even and odd signal

$$x(t) = \cos^2\left(\frac{\pi}{2}t\right)$$

$$= \frac{1+\cos\pi t}{2} \text{ by } \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\cos\pi t$$

Compare this equation with $x(t) = A + A\cos 2\pi f t$. Hence $2\pi f t = \pi t \Rightarrow f = \frac{1}{2}$ or $T = 2$.

Fig. shows the waveforem of x(t). Even and odd parts are given as,

Even part, $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$ *and*

Odd part, $x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$

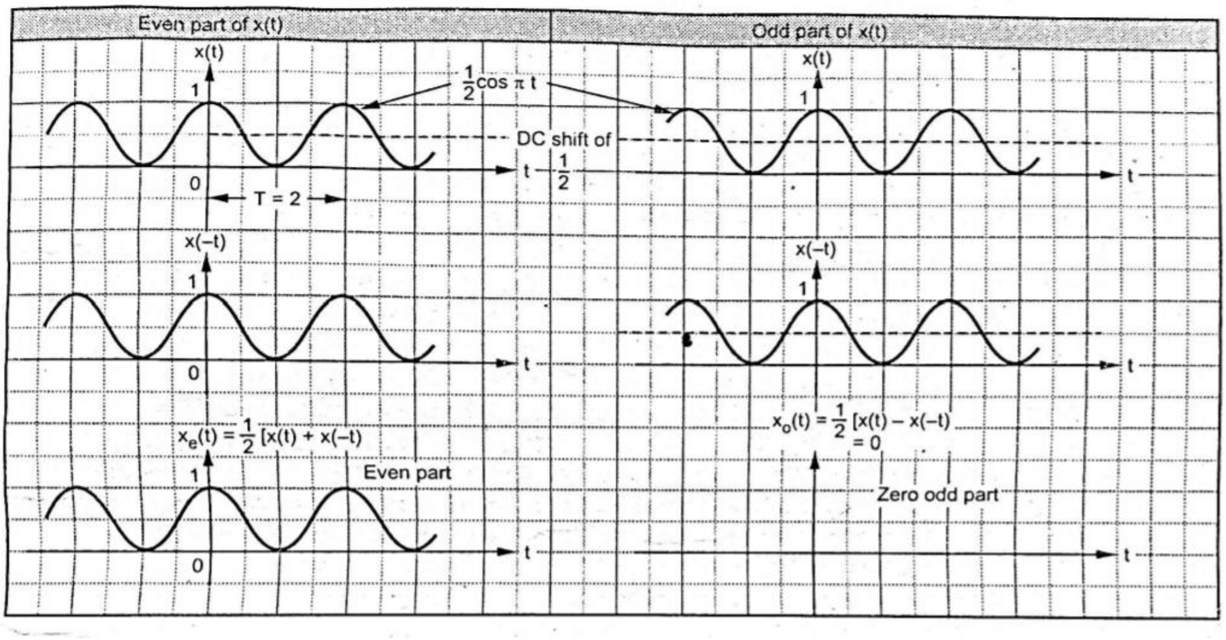


Fig. 15: Even and odd signal

1.11.3.POWER OF THE ENERGY SIGNAL

Here limits are changed to $-\infty$, ∞ as $T \to \infty$

Let *x(t)* be an energy signal. The power is given by equation

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[\lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \right]$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[\int_{-\infty}^{\infty} |x(t)|^2 \, dt \right]$$

$$= \lim_{T \to \infty} \frac{1}{T} \cdot E, \quad \text{Since quantity inside brackets is E.}$$

$$= 0 \times E = 0, \quad \text{since } \lim_{T \to \infty} \frac{1}{T} = 0$$

Thus, **Power of energy signal is zero over infinite time.**

1.11.4.ENERGY OF THE POWER SIGNAL

Energy of the signal is given by equation

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Let us change the limits of integration as $-\frac{T}{2}, \frac{T}{2}$ and take $\lim_{T \to \infty}$. This will not change meaning of above equation.

i.e.,

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \left[T \cdot \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \right] \quad \text{by rearranging}$$

$$= \lim_{T \to \infty} T \cdot \left[\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \right]$$

$$= \lim_{T \to \infty} T \cdot P \text{ since quantity inside brackets is P.}$$

$$= \infty \text{ By takint limits as } T \to \infty$$

Thus, **Energy of the power signal is infinite over infinite time**

1.11.5.PROBLEMS OF ENERGY AND POWER SIGNALS

Step 1 : Observe the signal carefully. If it is periodic and infinite duration then it can be power signal. Hence calculate its power directly.

Step 2 : If the signal is periodic but of finite duration, then it can be energy signal. Hence calculate its energy directly.

Step 3 : If the signal is not periodic, then it can be energy signal. Hence calculate its energy directly.

PROBLEMS

This signal is not periodic. Hence as per step 3, calculate its energy directly.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad \text{By definition}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \qquad \text{since u(n) = 1 for n = 0 to } \infty$$

Here use, $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$. The above equation will be

$$E = \frac{1}{1-\frac{1}{2}} = 2$$

Since energy is finite and non zero, it is energy signal with E = 2.

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

This signal is periodic (since u(n) repeats after every sample) and of inifinite duration. Hence it may be power signal. Therefore let calculate power directly,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^2 \qquad \text{Since u(n) = 1 for } 0 \le n \le \infty$$

Here $\sum_{n=0}^{N} (1)^2$ means 1 + 1 + 1 + 1...... for n = 0 to N. In other words, 1 + 1 + 1 + 1 ...... (N + 1) times = (N + 1). Therefore above equation will be,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot (N+1)$$

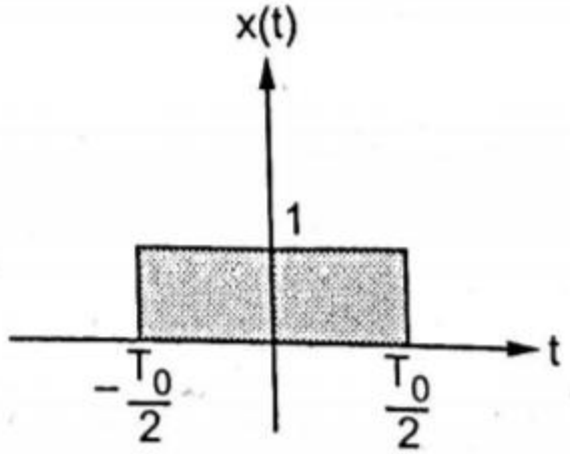
$$= \lim_{N \to \infty} \frac{N+1}{2N+1} = \lim_{N \to \infty} \frac{1+\frac{1}{N}}{2+\frac{1}{N}} = \frac{1}{2}$$

The power is finite and non zero, hence unit step function is power signal with $P = \frac{1}{2}$.

$$x(n) = u(n)$$

$$x(t) = rect\left(\frac{t}{T_o}\right)$$

$$rect\left(\frac{t}{T_o}\right) = \begin{cases} 1 & \text{for } -\frac{T_o}{2} \le t \le \frac{T_o}{2} \\ 0 & \text{elsewhere} \end{cases}$$



It is non periodic. Hence it can be energy signal as per step 3. Hence calculate energy directly

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{\frac{T_o}{2}}^{\frac{T_o}{2}} (1)^2 \, dt$$

$$= [t]_{-T_o/2}^{T_o/2} = T_o$$

The energy is finite and nonzero. It is **energy signal with $E = T_o$.**

$x(t) = \cos^2 \omega_o t$

This is squared cosine wave, hence it is periodic. Therefore this can be periodic signal. As per step 1, calculate power of this signal directly,

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \qquad \text{By definition}$$

The given signal $x(t) = \cos^2 \omega_0 t$ has some period T_o and it is real signal. There above equation will be,

$$P = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} [\cos^2 \omega_o t]^2 \, dt$$

Here $[\cos^2 \omega_o t]^2 = \cos^4 \omega_0 t$. It can be expanded by standard trigonometric relations as follows :

$$\therefore \qquad P = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{8}[3 + 4\cos 2\omega_0 t + \cos 4\,\omega_o t]\, dt$$

$$P = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} [\cos^2 \omega_0 t]^2 \, dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left[\frac{1 + \cos 2\omega_0 t}{2}\right]^2 dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{4}[1 + \cos 2\omega_0 t]^2 \, dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{4}[1 + \cos^2 2\omega_0 t + 2\cos 2\omega_0 t] \, dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{4}\left[1 + \left(\frac{1 + \cos 4\omega_0 t}{2}\right) + 2\cos 2\omega_0 t\right] dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{8}[2 + 1 + \cos 4\omega_0 t + 4\cos 2\omega_0 t] \, dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{8}[3 + 4\cos 2\omega_0 t + \cos 4\omega_0 t] \, dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{3}{8}\, dt + \underbrace{\lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} 4\cos 2\omega_o t\, dt}_{\text{This term will be zero since it is integration of cosine wave over "full cycles"}} + \underbrace{\lim_{T_0 \to \infty} \int_{-T_0/2}^{T_0/2} \cos 4\,\omega_o t\, dt}_{\text{This term will be also zero since it is integration of cosine wave over "full cycles"}}$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} \cdot [t]_{-T_0/2}^{T_0/2} + 0 + 0$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} \cdot T_0 = \frac{3}{8}$$

The power of the signal is finite and nonzero, hence it is **power signal with $P = \frac{3}{8}$**

$$x(t) = rect\left(\frac{t}{T_o}\right)\cos \omega_o t$$

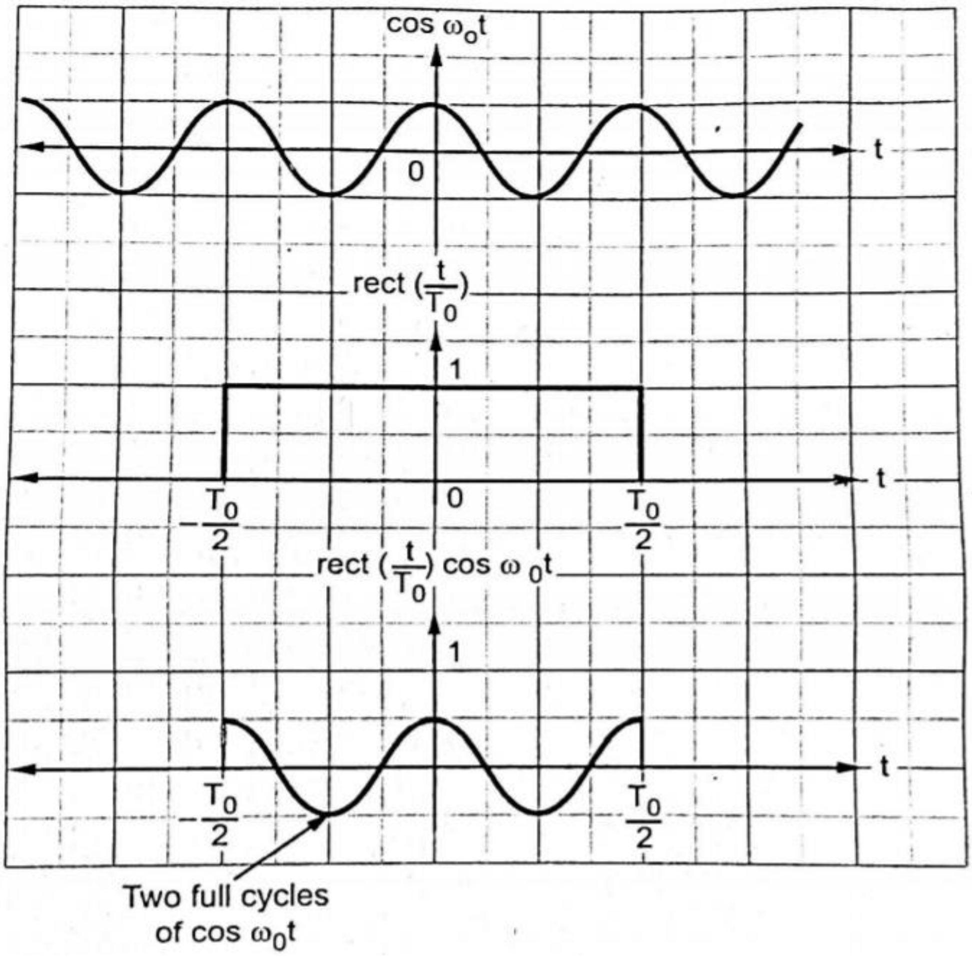


Fig. 16: Rectangular and sine wave

The given function is the product of cosine wave and *rect* function Fig. show how $x(t)$ is derived.

- Cos $\omega_0 t$ is periodic and infinite duration signal.
- Basically it is power signal.
- $\cos\omega_0 t$ is multiplied with the rectangular pulse. Hence the resultant signal is a cosine wave of duration $-\frac{T_0}{2} \le t \le \frac{T_0}{2}$.
- It is assumed that there are multiple number of cycles of cosine wave in $-\frac{T_0}{2} \le t \le \frac{T_0}{2}$.

The final signal is periodic but of finite duration. Hence it can be energy signal. Therefore calculate as per step2, energy directly.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad \text{By definition}$$

$$= \int_{-T_0/2}^{T_0/2} [\cos \omega_0 t]^2 dt = \int_{-T_0/2}^{T_0/2} \left(\frac{1+\cos 2\omega_0 t}{2}\right) dt$$

$$= \frac{1}{2}\int_{-T_0/2}^{T_0/2} dt + \underbrace{\frac{1}{2}\int_{-T_0/2}^{T_0/2} \cos 2\omega_0 t}_{\text{This term will be zero since it is integration of cosine wave over "full cycles"}} = \frac{T_0}{2}$$

Here energy is finite and non zero, hence it is **Energy signal with $E = \frac{T_0}{2}$.**

1.12.CLASSIFICATION OF CT & DT SYSTEMS

- Static and dynamic
- Linear and nonlinear
- Time invariant and time variant

- Causal and noncausal
- Stable and unstable

1.12.1.STATIC AND DYNAMIC SYSTEM

When the output of the system depends only upon the present input sample, then it is called *static* system. For example,

$$y(n) = 10 \cdot x(n)$$

or

$$y(n) = 15 \cdot x^2(n) + 10\,x(n)$$

are the static systems. Here the $y(n)$ depends only upon n^{th} input sample. Hence such systems do not need memory for its operation. A system is said to be *dynamic* if the output depends upon the past values of input also. For example,

$$y(n) = x(n) + x(n-1)$$

This is the dynamic system. In this system the n^{th} output sample value depends upon n^{th} input sample and just previous i.e. $(n-1)^{th}$ input sample. This systems needs to store the previous sample value. Consider the following equation of a system.

1.12.2.TIME INVARIANT AND TIMEVARIANT SYSTEM

To test for shift invariance, excite the system by the input $x(n)$ and get output $y(n)$. Then delay (or shift) the input by 'k' samples and calculate the output. Let this output be,

$$y(n,k) = T\left[x(n-k)\right]$$

Thus $y(n,k) \Rightarrow$ response due to delayed/shifted input.

Now we have $y(n)$ computed earlier. Hence obtain $y(n-k)$ from $y(n)$ by delaying by 'k' samples. i.e., $y(n-k) \Rightarrow$ output delayed/shifted directly. Then the system is shift invariant or time invariant if,

$$y(n,k) = y(n-k) \quad \textit{for all values of 'k'}$$

And the system is time variant or shift variant if,

$$y(n,k) \neq y(n-k) \quad \textit{even for single value of 'k'}$$

1.12.3.LINEAR AND NONLINEAR SYSTEM

A system is said to be linear if it satisfies the superposition principle. Let $x_1(n)$ and $x_2(n)$ be the two input sequences. Then the system is said to be linear if and only if

$$T\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

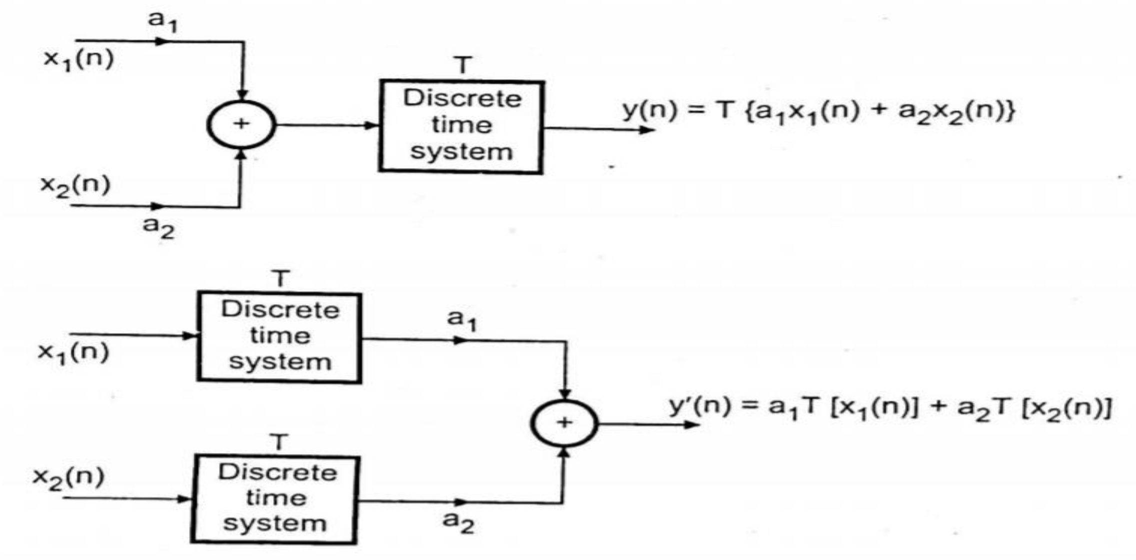


Fig. 17: Linear system

1.12.4.CAUSAL AND NONCAUSAL SYSTEM

In the causal system the output depends upon past and present inputs only. That is the output is the function of $x(n), x(n-1), x(n-2), x(n-3)\ldots$ and so on. The system is noncausal if its output depends upon the future inputs also. i.e. $x(n+1), x(n+2)\ldots.$ and so on. Thus the noncausal systems are physically unrealizable. The following example illustrates causal and noncausal systems.

STABLE AND UNSTABLE SYSTEM

When the every bounded input produces a bounded output, then the system is called Bounded Input Bounded Output (BIBO) stable. The input $x(n)$ is said to be bounded if there exists some finite number M_x such that,

$$|x(n)| \leq M_x < \infty$$

Similarly output $y(n)$ is bounded if there exists some finite number M_y such that,

$$|y(n)| \leq M_y < \infty$$

If the output is unbounded for any bounded input, then the system is unstable. The unstable systems produce erratic outputs.

1.13.PROBLEMS OF DISCRETE TIME SYSTEMS

$$y(n) = \cos[x(n)]$$

A system is static if its output depends only upon the present input sample. Here since $y(n)$ depends upon cosine of $x(n)$, i.e. present input sample, the system is static.

For two separate inputs the system produces the response of,

$$y_1(n) = T\{x_1(n)\} = \cos[x_1(n)]$$

$$y_2(n) = T\{x_2(n)\} = \cos[x_2(n)]$$

The response of the system to linear combination of two inputs will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\} = \cos[a_1 x_1(n) + a_2 x_2(n)]$$

The linear combination of two outputs will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n) = a_1 \cos[x_1(n)] + a_2 \cos[x_2(n)]$$

Clearly $y_3(n) \neq y'_3(n)$. Hence system is nonlinear.

The system is said to be shift invariant or time invariant if its characteristics do not change with shift of time origin. The given system is,

$$y(n) = T\{x(n)\} = \cos[x(n)]$$

Let us delay the input by k samples. Then output will be,

$$y(n, k) = T\{x(n-k)\} = \cos[x(n-k)]$$

Now let us delay the output $y(n)$ given by equation (4.2.25) by 'k' samples, i.e. $y(n-k)$. This is equivalent to replacing n by $n-k$ in equation (4.2.25). i.e.,

$$y(n-k) = \cos[x(n-k)]$$

Comparing above equation with equation (4.2.26) we observe that,

$$y(n, k) = y(n-k)$$

This shows that the system is shift invariant.

The system is said to be causal if output depends upon past and present inputs only. The output is given as,

$$y(n) = \cos[x(n)]$$

Here observe that n^{th} sample of output depends upon n^{th} sample of input $x(n)$. Hence the system is a causal system.

For any bounded value of $x(n)$ the cosine function has bounded value. Hence $y(n)$ has bounded value. Therefore the system is said to be BIBO stable.

Thus the given system is,

Static, nonlinear, shift invariant, causal and stable.

$$y(n) = x(n)\cos(\omega_0 n)$$

This is a static system since, the output of the system depends only upon the present input sample. i.e., n^{th} output sample depends upon n^{th} input sample. Hence this is a static system.

We know that the given system is,

$$y(n) = T\{x(n)\} = x(n)\cos(\omega_0 n)$$

When the two inputs $x_1(n)$ and $x_2(n)$ are applied separately, the responses $y_1(n)$ and $y_2(n)$ will be,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(n)\cos(\omega_0 n) \\ y_2(n) &= T\{x_2(n)\} = x_2(n)\cos(\omega_0 n) \end{aligned}\right\}$$

The response of the system due to linear combination of inputs will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\} = [a_1 x_1(n) + a_2 x_2(n)]\cos(\omega_0 n)$$

$$= a_1 x_1(n)\cos(\omega_0 n) + a_2 x_2(n)\cos(\omega_0 n)$$

Now the linear combination of the two outputs will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1 x_1(n)\cos(\omega_0 n) + a_2 x_2(n)\cos(\omega_0 n)$$

From above equation and equation (4.2.28),

$y_3(n) = y'_3(n)$. Hence the system is linear.

The system equation is,

$$y(n) = x(n)\cos(\omega_0 n)$$

The response of the system to delayed input will be,

$$y(n,k) = T\{x(n-k)\}$$

$$= x(n-k)\cos(\omega_0 n)$$

Now let us delay or shift the output $y(n)$ by 'k' samples. i.e.,

$$y(n-k) = x(n-k)\cos[\omega_0(n-k)]$$

Here every 'n' is replaced by $(n-k)$. On comparing above equation with equatio (4.2.29) we find that,

$y(n,k) \neq y(n-k)$ *Hence the system is shift variant.*

In the given system, $y(n)$ depends upon $x(n)$, i.e. present output depends upon present input. Hence the system is causal.

The given system equation is,

$$y(n) = x(n)\cos(\omega_0 n)$$

Here value of $\cos(\omega_0 n)$ is always bounded. Hence as long as $x(n)$ is bounded, $y(n)$ is also bounded. Hence the system is stable.

This system is,

Static, linear, shift variant, causal and stable.

$y(n) = x(-n+2)$

It is clear from above equation that n^{th} sample of output is equal to $(-n+2)^{th}$ sample of input. Hence the system needs memory storage. Therefore the system is Dynamic.

It is very easy to prove that this system is linear.

The output $y(n)$ for delayed input will be,

$$\begin{aligned} y(n,k) &= T\{x(n-k)\} \\ &= x(-n+2-k) \end{aligned}$$

Now the delayed output will be obtained by replacing n by $(n-k)$ in the system equation i.e.,

$$\begin{aligned} y(n-k) &= x[-(n-k)+2] \\ &= x(-n+2+k) \end{aligned}$$

On comparing above equation with equation (4.2.30) we find that,

$y(n,k) \neq y(n-k)$ *Hence the system is shift variant*

In the given system equation, when we put $n=0$ we get,

$$y(0) = x(2)$$

Thus the output depends upon future inputs. Hence the system is noncausal.

It is clear from the given system equation that, as long as input is bounded, the output is bounded. Hence the system is a stable system.

Thus the given system is,

Dynamic, linear, shift variant, noncausal and stable.

$y(n) = |x(n)|$

The output is equal to magnitude of present input sample. Hence the system does not need memory storage. Therefore the system is static.

The given system equation is,

$$y(n) = T\{x(n)\} = |x(n)|$$

For two separate inputs $x_1(n)$ and $x_2(n)$ the system has the response of,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = |x_1(n)| \\ y_2(n) &= T\{x_2(n)\} = |x_2(n)| \end{aligned}\right\}$$

The response of the system to linear combination of two inputs $x_1(n)$ and $x_2(n)$ will be,

$$\begin{aligned} y_3(n) &= T\{a_1 x_1(n) + a_2 x_2(n)\} \\ &= |a_1 x_1(n) + a_2 x_2(n)| \end{aligned}$$

Now the linear combination of two outputs will be,

$$\begin{aligned} y'_3(n) &= a_1 y_1(n) + a_2 y_2(n) \\ &= a_1 |x_1(n)| + a_2 |x_2(n)| \end{aligned}$$

Here observe that $y_3(n) \neq y'_3(n)$. Hence the system is nonlinear.

Delaying the input by 'k' samples, output will be,

$$y(n, k) = T\{x(n-k)\}$$
$$= |x(n-k)|$$

And the delayed output will be,

$$y(n-k) = |x(n-k)|$$

Since $y(n, k) = y(n-k)$, the system is shift invariant.

The system equation is $y(n) = |x(n)|$. The output depends upon present input. Hence the system is causal.

From the given equation it is clear that as long as $x(n)$ is bounded, $y(n)$ will be bounded. Hence the system is stable.

Thus the given system is,

Static, nonlinear, shift invariant, causal and stable.

$y(n) = x(n)\, u(n)$

The output depends upon present input only. Hence the system is static.

The given system equation is,

$$y(n) = T\{x(n)\} = x(n)\, u(n)$$

The response of this system to the two inputs $x_1(n)$ and $x_2(n)$ when applied separately will be,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(n)\, u(n) \\ y_2(n) &= T\{x_2(n)\} = x_2(n)\, u(n) \end{aligned}\right\}$$

The response of the system to linear combination of inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$
$$= [a_1 x_1(n) + a_2 x_2(n)]\, u(n)$$
$$= a_1 x_1(n)\, u(n) + a_2 x_2(n)\, u(n)$$

The linear combination of two outputs of equation

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$
$$= a_1 x_1(n)\, u(n) + a_2 x_2(n)\, u(n)$$

From above equation

$y_3(n) = y'_3(n)$, hence the system is linear.

The response of the system to delayed input will be,

$$y(n,k) = T\{x(n-k)\}$$
$$= x(n-k)\,u(n)$$

The delayed output will be obtained by replacing 'n' by $n-k$. i.e.,

$$y(n-k) = x(n-k)\,u(n-k)$$

Here, on comparing above equation and equation (4.2.35), we find that,

$y(n,k) \neq y(n-k)$ *Hence the system is shift variant.*

The system equation is, $y(n) = x(n)\,u(n)$. The output depends upon present input only. Hence the system is causal.

We know that $u(n) = 1$ for $n \geq 0$ and $u(n) = 0$ for $n < 0$. This means $u(n)$ is a bounded sequence. Hence as long as $x(n)$ is bounded, $y(n)$ is also bounded. Hence this system is stable.

Thus, the given system is,

Static, linear, shift variant, causal and stable.

$y(n) = x(n) + n\,x(n+1)$

From the given equation it is clear that, the output depends upon the present input and next input. Hence system is dynamic.

The given system equation is,

$$y(n) = T\{x(n)\} = x(n) + n\,x(n+1)$$

If we apply two inputs $x_1(n)$ and $x_2(n)$ separately, then the outputs become,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(n) + n\,x_1(n+1) \\ \text{and}\quad y_2(n) &= T\{x_2(n)\} = x_2(n) + n\,x_2(n+1) \end{aligned}\right\}$$

Response of the system to linear combination of inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1\,x_1(n) + a_2\,x_2(n)\}$$
$$= a_1\left[x_1(n) + n\,x_1(n+1)\right] + a_2\left[x_2(n) + n\,x_2(n+1)\right]$$

The linear combination of two outputs given by equation (4.2.37) (a) will be,

$$y'_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$$
$$= a_1\left[x_1(n) + n\,x_1(n+1)\right] + a_2\left[x_2(n) + n\,x_2(n+1)\right]$$

On comparing above equation with equation (4.2.38) we observe that,

$y_3(n) = y'_3(n)$, Hence the system is linear.

The given system equation is,

$$y(n) = T\{x(n)\} = x(n) + n\,x(n+1)$$

Response of the system to delayed input will be,

$$y(n, k) = T\{x(n-k)\}$$

$$= x(n-k) + n\,x(n-k+1)$$

Now let us delay the output of equation (4.2.39) by 'k' samples. i.e.,

$$y(n-k) = x(n-k) + (n-k)\,x(n-k+1)$$

Here we have replaced 'n' by '$n-k$'. On comparing above equation with equation (4.2.40) we observe that,

$y(n, k) \neq y(n-k)$, Hence the system is shift variant.

The given system equation is,

$$y(n) = x(n) + n\,x(n+1)$$

Here observe that n^{th} output sample depends upon $(n+1)^{th}$ i.e. next input sample. That is the output depends upon future input. Hence the system is noncausal.

In the given system equation observe that as $n \to \infty$, $y(n) \to \infty$ even if $x(n)$ is bounded. Hence the system unstable.

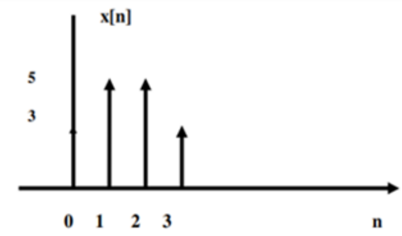
Thus the given system is,

Dynamic, linear, shift variant, noncausal and unstable.

1.14. REPRESENTATION OF DISCRETE TIME SIGNALS

- Graphical Representation
- Functional Representation
- Tabular Representation
- Sequence Representation

Graphical Representation



Functional Representation

$$x(n) = \begin{cases} 3 \text{ for } n = 0 \\ 5 \text{ for } n = 1 \\ 5 \text{ for } n = 2 \\ 3 \text{ for } n = 3 \end{cases}$$

Tabular Representation

n	0	1	2	3
x(n)	3	5	5	3

Sequence Representation

x(n) = {3,5,5,3}

Fig. 18: Representation of signals

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SCHOOL OF BIO AND CHEMICAL ENGINEERING

DEPARTMENT OF BIOMEDICAL ENGINEERING

UNIT – II – Analysis of Continuous Time Signals – SBMA1304

ANALYSIS OF CONTINUOUS TIME SIGNALS

Analysis of CT signals and systems can be performed using following tools :

i) Fourier transform

ii) Laplace transform

iii) Various properties of systems

iv) Characterization of LTI systems

In this chapter we will study following topics :

i) Properties and examples of Fourier transform

ii) Properties and examples on Laplace transform

iii) Properties of systems such as causality, linearity, time invariance stability etc.

iv) Convolution integral for LTI systems.

v) Use of Fourier and Laplace transforms for systems analysis.

2.1. FOURIER TRANSFORM

Periodic signals which extend over the interval $(-\infty, \infty)$ can be effectively represented with the help of fourier series. A periodic signals which are strictly time limited can also be represented by fourier series. A time limited signal means it has zero value outside the specified interval. And asymptotically time limited means as time approaches to infinity (∞), the value of signal becomes zero $[i.e. \, x(t) \to 0 \; as \; |t| \to \infty]$. Such time limited signals can be more conveniently represented by *fourier transform* in frequency domain. These signals are aperiodic because their period $T_0 \to \infty$.

Fourier transform can also be found for periodic signals. It provides effective reversible transformation link between frequency domain and time domain representation of the signal. We have seen previously that for nonperiodic signals $T_0 \to \infty$. As the period of the signal $T_0 \to \infty$, $f_0 = 0$. Therefore the spacing between the spectral components becomes infinitesimal and hence the frequency spectrum appears to be *continuous*. Whereas periodic signals has fixed period T_0. Therefore their frequency spectrum is discontinuous as we have seen in the examples in the last section.

2.2. DEFINITION

Let $x(t)$ be the signal which is function of time t. The fourier transform of $x(t)$ is given

Fourier Transform : $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ or

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \text{since} \quad \omega = 2\pi f$$

Similarly $x(t)$ can be recovered from its fourier transform $X(f)$ by using inverse fourier transform.

Inverse Fourier Transform : $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

The functions $x(t)$ and $X(f)$ form a fourier transform pair is written by a shorthand symbol as shown below,

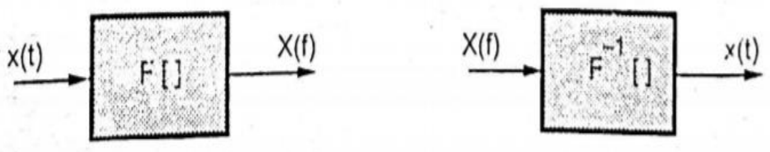
$$x(t) \leftrightarrow X(f)$$

Other shorthand notation for fourier transform is as shown below,

$$X(f) = F[x(t)]$$

and $$x(t) = F^{-1}[X(f)]$$

Fourier transform thus can be considered as a linear operator as shown in Fig.



(a) Fourier transformation and (b) Inverse fourier transformation

2.3. PROPERTIES OF FOURIER TRANSFORM

Linearity (Superposition)

Let $x_1(t) \leftrightarrow X_1(f)$ represent a fourier transform pair and $x_2(t) \leftrightarrow X_2(f)$ represent another fourier transform pair. Then for all constants like C_1 and C_2 we have,

$$C_1 x_1(t) + C_2 x_2(t) \leftrightarrow C_1 X_1(f) + C_2 X_2(f)$$

Proof : *By definition of Fourier transform,*

$$F[C_1 x_1(t) + C_2 x_2(t)] = \int_{-\infty}^{\infty} [C_1 x_1(t) + C_2 x_2(t)] e^{-j2\pi f t} dt$$

$$= C_1 \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t} dt + C_2 \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t} dt$$

$$= C_1 X_1(f) + C_2 X_2(f) \qquad \text{By definition of FT}$$

Linearity : $C_1 x_1(t) + C_2 x_2(t) \leftrightarrow C_1 X_1(f) + C_2 X_2(f)$

Time Scaling

Let $x(t)$ and $X(f)$ be a fourier transform pair and 'a' is some constant.

Then by time scaling property,

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Solution : By definition of fourier transform

$$F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t} dt$$

Let $\tau = at$ *then*

$$dt = \frac{1}{a} d\tau$$

Here two cases are possible ; $a > 0$ and $a < 0$;

For $a > 0$;

$$F[x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{\frac{-j\pi f \tau}{a}}\, d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi \left(\frac{f}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{f}{a}\right)$$

For $a < 0$

$$F[x(at)] = -\frac{1}{a} X\left(\frac{f}{a}\right)$$

Combining equation above two equations,

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Thus the time scaling property is proved.

Duality or Symmetry Property

Duality property of fourier transform states that if

$$x(t) \leftrightarrow X(-f)$$

then $$X(t) \leftrightarrow x(-f)$$

Proof : By definition of inverse fourier transform in equation

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

$$\text{for } t = -t$$

$$x(-t) = \int_{-\infty}^{\infty} X(f)\, e^{-j2\pi ft}\, df$$

Interchanging t and f we get,

$$x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi ft}\, dt = F[X(t)]$$

$$\therefore \quad x(-f) \leftrightarrow X(t)$$

Time Shifting

If $$x(t) \leftrightarrow X(f), \text{ then}$$

Time shifting : $x(t-t_0) \leftrightarrow X(f)\, e^{-j2\pi f t_0}$

Proof : By definition of fourier transform

$$f\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi ft}\, dt$$

Let $$t - t_0 = \tau$$

Then $t = t_0 + \tau \rightarrow dt = d\tau$

$$F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f(t_0+\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f t_0}\, e^{-j2\pi f\tau}\, d\tau$$

$$= e^{-j2\pi f t_0}\, X(f) \qquad \textit{By definition of FT.}$$

Thus a time shift 't_0' has no change on the amplitude spectrum but there is a phase shift of $-2\pi f t_0$.

Frequency Shifting

If $x(t) \leftrightarrow X(f)$ then

Frequency shifting : $e^{j2\pi f_c t}\, x(t) \leftrightarrow X(f - f_c)$

Here f_c is real constant. This property is also called modulation theorem.

Proof : By definition of FT.

$$F\left[e^{j2\pi f_c}\, x(t)\right] = \int_{-\infty}^{\infty} e^{j2\pi f_c t}\, x(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f-f_c)t}\, dt$$

$$= X(f - f_c)$$

Multiplication of the function $x(t)$ by $e^{j2\pi f_c t}$ results in shifting of fourier spectrum $X(f)$ in positive side by f_c.

Area Under $x(t)$

If $x(t) \leftrightarrow X(f)$, then

$$\textit{Area under } x(t) : \int_{-\infty}^{\infty} x(t)\, dt = X(0)$$

That is area under $x(t)$ is equal to its fourier transform at zero frequency.

Proof : By definition of FT,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

Let $$f = 0$$

$\therefore$ $$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

Area under $X(f)$

If $x(t) \leftrightarrow X(f)$, then

$$\textit{Area under } X(f) : \int_{-\infty}^{\infty} X(f)\, df = x(0)$$

That is the area under fourier spectrum of a signal is equal to its value at $t = 0$

Proof : By definition of IFT,

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

Let $$t = 0,$$

$$x(0) = \int_{-\infty}^{\infty} X(f)\, df$$

Differentiation in Time Domain

If $x(t) \leftrightarrow X(f)$ and first derivative of $x(t)$ is fourier transformable, then

$$\frac{d}{dt} x(t) \leftrightarrow (j 2 \pi f) X(f)$$

Differentiation of function $x(t)$ in time domain is equivalent to multiplying its fourier transform by $(j 2 \pi f)$.

Proof : By definition of FT,

$$F\left[\frac{d}{dt} x(t)\right] = \int_{-\infty}^{\infty} \frac{d}{dt} x(t)\, e^{-j2\pi ft}\, dt$$

Integrating by parts,

$$F\left[\frac{d}{dt} x(t)\right] = e^{-j2\pi ft}\left[x(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-j 2 \pi f)\, e^{-j\pi ft}\, dt$$

$$= j 2 \pi f \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= j 2 \pi f X(f)$$

Similarly it can be easily shown that

$$F\left\{\frac{d^2 x(t)}{dt^2}\right\} = (j2\pi f)^2 X(f)$$

$$= (j\omega)^2 X(f) = -\omega^2 X(f)$$

Hence $F\left\{\frac{d^n x(t)}{dt^n}\right\} = (j2\pi f)^n X(f)$

$$= (j\omega)^n X(f)$$

Integration in Time Domain

If $x(t) \leftrightarrow X(f)$, and provided that $X(0) = 0$, then,

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j2\pi f} X(f)$$

Assuming that $X(0) = 0$, the integration of $x(t)$ in time domain has the effect of dividing its fourier transform by $(j2\pi f)$.

Proof : Let $x(t)$ be expressed as,

$$x(t) = \frac{d}{dt}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}$$

We know that, $x(t) \leftrightarrow X(f)$

$$\therefore \quad F[x(t)] = F\left[\frac{d}{dt}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right] = j2\pi f\left[F\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right]$$

By differentiation property

i.e.
$$X(f) = j2\pi f\left[F\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right]$$

$$\therefore \quad F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j2\pi f} X(f)$$

Conjugate Functions

If $x(t) \leftrightarrow X(f)$, then for complex valued time function $x(t)$ we have

$$x^*(t) \leftrightarrow X^*(-f)$$

Proof : By definition of IFT,

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

By taking complex conjugates of both sides

$$x^*(t) = \int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi ft}\, df$$

Now by replacing f with $-f$ gives,

$$x^*(t) = \int_{-\infty}^{\infty} X^*(-f)\, e^{j2\pi ft}\, df$$

$$= F^{-1}[X^*(-f)]$$

$$x^*(t) \leftrightarrow X^*(-f)$$

Multiplication in Time Domain (Multiplication Theorem)

Let the two fourier transform pairs be $x_1(t) \leftrightarrow X_1(f)$ and $x_2(t) \leftrightarrow X_2(f)$, then

$$x_1(t)\,x_2(t) \leftrightarrow \int_{-\infty}^{\infty} X_1(\lambda)\,X_2(f-\lambda)\,d\lambda$$

That is multiplication of two signals in time domain is transformed into convolution of their fourier transforms in frequency domain.

The short hand notation for this property is,

$$\boxed{x_1(t)\,x_2(t) \leftrightarrow X_1(f) * X_2(f)}$$

Proof : Let us write the RHS of equation 2.2.49 as follows,

$$x_1(t)\,x_2(t) \leftrightarrow X_{12}(f)$$

i.e. $F[x_1(t)\,x_2(t)] = X_{12}(f)$

$$= \int_{-\infty}^{\infty} x_1(t)\,x_2(t)\,e^{-j2\pi ft}\,dt \quad \text{by definition of FT}$$

Let $\lambda = f - f'$, then by arranging above equation,

$$X_{12}(f) = \int_{-\infty}^{\infty} X_2(f-\lambda)\,d\lambda \int_{-\infty}^{\infty} x_1(t)\,e^{-j2\pi(f-f')t}\,dt$$

$$= \int_{-\infty}^{\infty} X_2(f-\lambda)\,d\lambda \int_{-\infty}^{\infty} x_1(t)\,e^{-j2\pi\lambda t}\,dt$$

The second integral above is $X_1(\lambda)$ from definition of FT

$$\therefore \quad X_{12}(f) = \int_{-\infty}^{\infty} X_2(f-\lambda)\,d\lambda\,X_1(\lambda)$$

or

$$= \int_{-\infty}^{\infty} X_1(\lambda)\,X_2(f-\lambda)\,d\lambda$$

$$\therefore \quad x_1(t)\,x_2(t) \leftrightarrow \int_{-\infty}^{\infty} X_1(\lambda)\,X_2(f-\lambda)\,d\lambda$$

This property is some times called as multiplication theorem.

Since convolution is commutative, equation (2.2.50) can also be written as,

$$x_1(t)\,x_2(t) \leftrightarrow X_1(f) * X_2(f).$$

Convolution in Time Domain (Convolution Theorem)

If $x_1(t) \leftrightarrow X_1(f)$ and $x_2(t) \leftrightarrow X_2(f)$

then, $$\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \leftrightarrow X_1(f)\, X_2(f)$$

This property states that convolution of two signals in time domain is transformed into multiplication of their individual fourier transforms in frequency domain.

The short hand notation of convolution can be used to represent this property as follows,

i..e. $$\boxed{x_1(t) * x_2(t) \leftrightarrow X_1(f)\, X_2(f)}$$

Proof : Convolution of $x_1(t)$ *and* $x_2(t)$ as given equation 2.2.52 is,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right) e^{-j2\pi ft}\, dt \qquad \text{By definition of FT}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f\tau}\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{j2\pi f\tau}\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f\tau}\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt$$

Let $t-\tau = \alpha$ in the second integral.

$$\therefore\ F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f\tau}\, d\tau \int_{-\infty}^{\infty} x_2(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha$$

From definition of FT applied to RHS

$$F[x_1(t) * x_2(t)] = X_1(f)\, X_2(f)$$

i.e. $$\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \leftrightarrow X_1(f)\, X_2(f)$$

Find the fourier transform of the decaying exponential as shown in

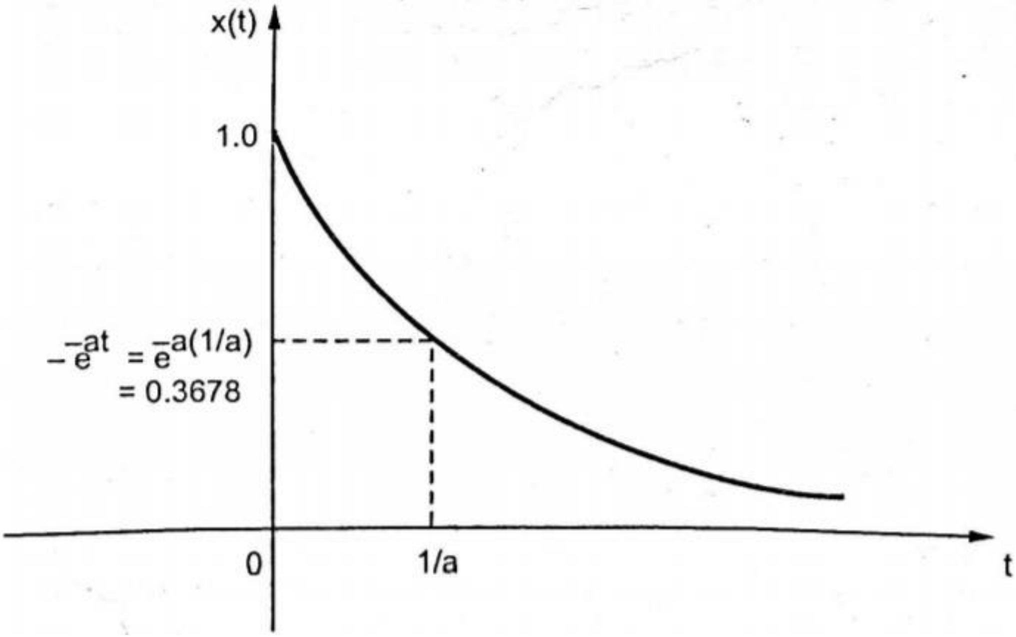


Fig. 1: Decaying Exponential

Normally to show time delays in the function and sign of time, use of unit step function $u[(t)]$ is made. The value of unit step function is always unity i.e.

$$u(t) = 1 \qquad \text{for } t \geq 0$$

∴ The exponential pulse in Fig. 2.2.2 is represented as,

$$x(t) = e^{-at}\, u(t) \qquad \text{Here } u(t) = 1$$

By definition of fourier transform we have,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, u(t) \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt$$

The lower limit is taken `0' since $x(t) = 0$, for $t < 0$. And $u(t) = 1$ for $t \geq 0$

$$\therefore \quad X(f) = \frac{1}{-(a+j2\pi f)} \left[e^{-(a+j2\pi f)t}\right]_0^{\infty}$$

$$= \frac{1}{a + j2\pi f}$$

Thus the fourier transform pair becomes,

$$\textit{Decaying exponential pulse}: e^{-at}\ u(t) \leftrightarrow \frac{1}{a + j2\pi f}$$

To calculate magnitude and phase spectrum :

The function $X(f)$ is expressed as,

$$X(f) = A(f) + jB(f)$$

Here $A(f)$ is real part of $X(f)$ and $B(f)$ is imaginary part of $X(f)$.

Therefore magnitude spectrum of $X(f)$ is given as,

$$|X(f)| = \sqrt{A^2(f) + B^2(f)}$$

And phase spectrum is given as,

$$\theta(f) = tan^{-1}\ \frac{B(f)}{A(f)}$$

Consider the equation

$$X(f) = \frac{1}{a + j2\pi f}$$

Multiply and divide RHS by $a - j2\pi f$,

$$X(f) = \frac{1}{a + j2\pi f} \times \frac{a - j2\pi f}{a - j2\pi f}$$

$$= \frac{a - j2\pi f}{a^2 + (2\pi f)^2}$$

$$= \frac{a}{a^2 + (2\pi f)^2} + j\ \frac{-2\pi f}{a^2 + (2\pi f)^2}$$

$$\left.\begin{aligned} \text{Here real part } A(f) &= \frac{a}{a^2 + (2\pi f)^2} \\ \text{and imaginary part } B(f) &= \frac{-2\pi f}{a^2 + (2\pi f)^2} \end{aligned}\right\}$$

From equation magnitude of $X(f)$ will be,

$$|X(f)| = \sqrt{\frac{a^2}{\left[a^2+(2\pi f)^2\right]^2} + \frac{(-2\pi f)^2}{\left[a^2+(2\pi f)^2\right]^2}}$$

$$= \sqrt{\frac{1}{a^2+(2\pi f)^2}}$$

From equation 2.2.18 phase spectrum will be,

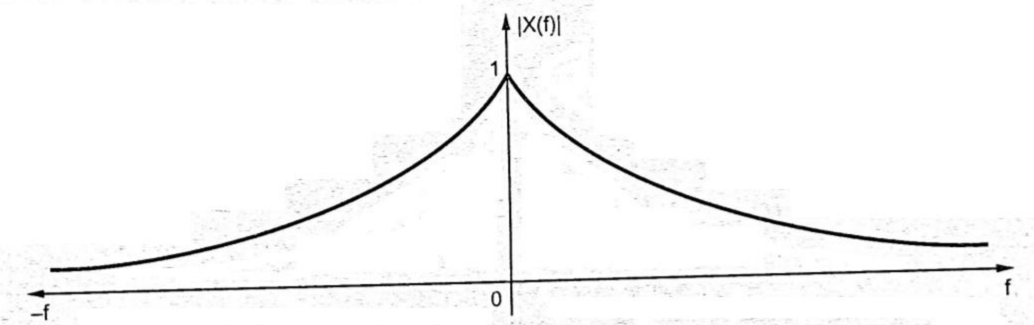
$$\theta(f) = \tan^{-1}\left\{\frac{-2\pi f / \left[a^2+(2\pi f)^2\right]}{a / \left[a^2+(2\pi f)^2\right]}\right\}$$

$$= \tan^{-1}\left(\frac{-2\pi f}{a}\right)$$

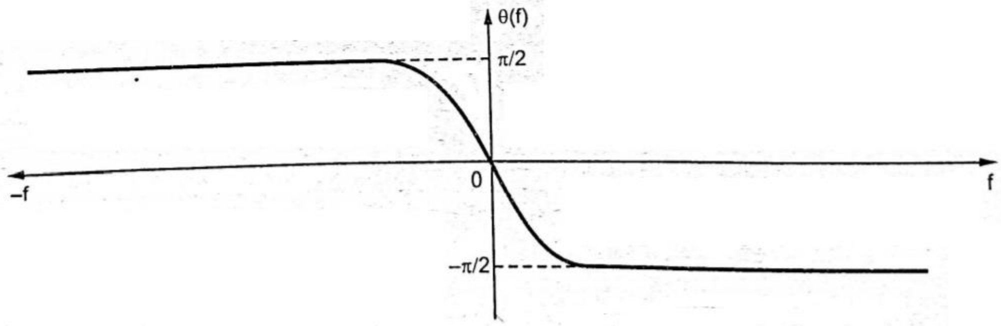
$$= -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

f	$\lvert X(f) \rvert$	$\theta(f)$
f	$\lvert X(f)\rvert = \frac{1}{\sqrt{a^2 + (2\pi f)^2}} = \frac{1}{\sqrt{1+(2\pi f)^2}}$ since $a = 1$	$\theta(f) = -\tan^{-1}\left(\frac{2\pi f}{a}\right) = -\tan^{-1}(2\pi f)$ since $a = 1$
0	$\lvert X(f)\rvert = \frac{1}{a} = 1$	$\theta(f) = 0$
0.1 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(0.2\pi)^2}} = 0.846$	$\theta(f) = -\tan^{-1}(0.2\pi) = -32.14°$
0.5 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(\pi)^2}} = 0.3033$	$\theta(f) = -\tan^{-1}(\pi) = -72.34°$
1.0 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(2\pi)^2}} = 0.157$	$\theta(f) = -\tan^{-1}(2\pi) = -80.95°$
10.0 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(20\pi)^2}} = 0.0159$	$\theta(f) = -\tan^{-1}(20\pi) = -89.08$
⋮	⋮	⋮

− 0.1 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(-0.2\pi)^2}} = 0.846$	$\theta(f) = -\tan^{-1}(-0.2\pi) = 32.14°$
− 0.5 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(-\pi)^2}} = 0.3033$	$\theta(f) = -\tan^{-1}(-\pi) = 72.34°$
− 1.0 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(-2\pi)^2}} = 0.157$	$\theta(f) = -\tan^{-1}(-2\pi) = 80.95°$
− 10.00 Hz	$\lvert X(f)\rvert = \frac{1}{\sqrt{1+(-20\pi)^2}} = 0.0159$	$\theta(f) = -\tan^{-1}(-20\pi) = 89.08°$
⋮	⋮	⋮

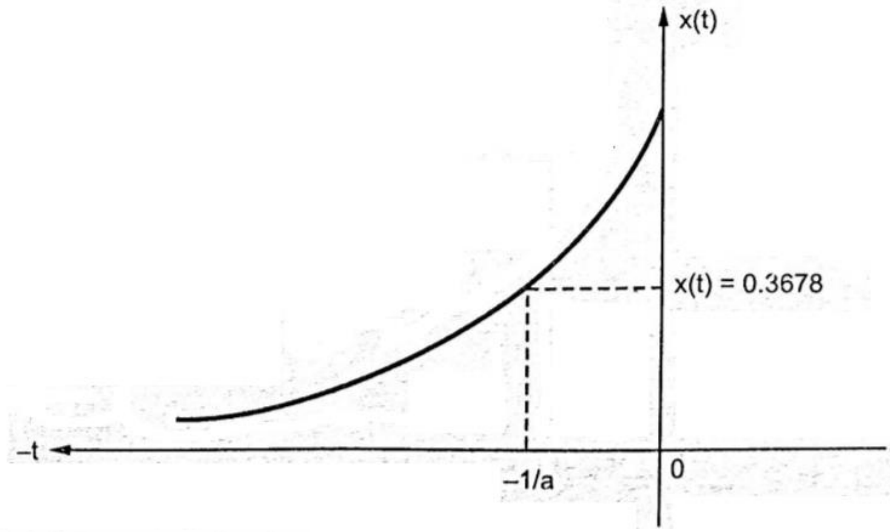


Amplitude spectrum of decaying exponential pulse of Fig. Here a = 1 (assumed). It is even function of frequency.



Phase spectrum. It is odd function of frequency.

Obtain the Fourier transform of Raising Exponential as shown in the figure



Rising

$$x(t) = e^{at}\, u(-t)$$

Sign of t in $u(-t)$ is negative and it represents negative time of the pulse. It is zero for positive time $(t>0)$. Therefore integration can be performed from $-\infty$ to 0 instead of $-\infty$ to $+\infty$.

$\therefore$ Fourier transform of $x(t)$ becomes

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

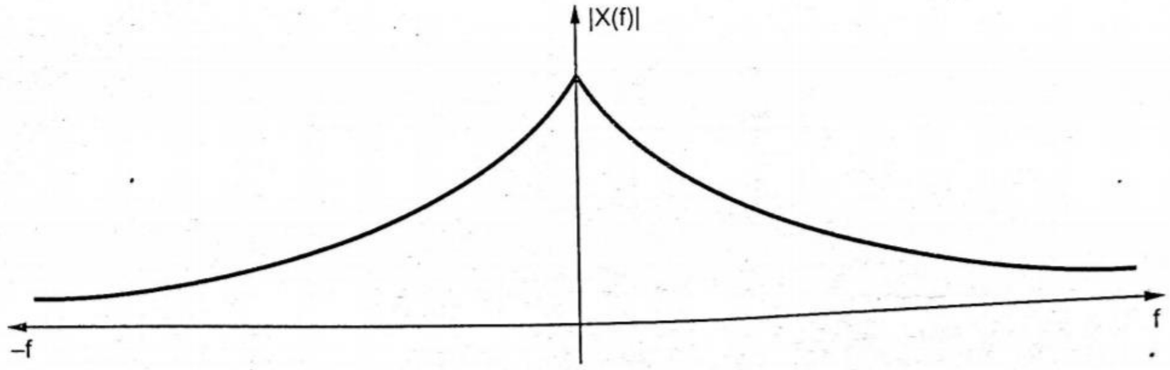
$$= \int_{-\infty}^{0} e^{at}\, u(-t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{0} e^{[t(a-j2\pi f)]}\, dt$$

$$= \frac{1}{a-j2\pi f}\left[e^{0}-e^{\infty}\right] = \frac{1}{a-j2\pi f}$$

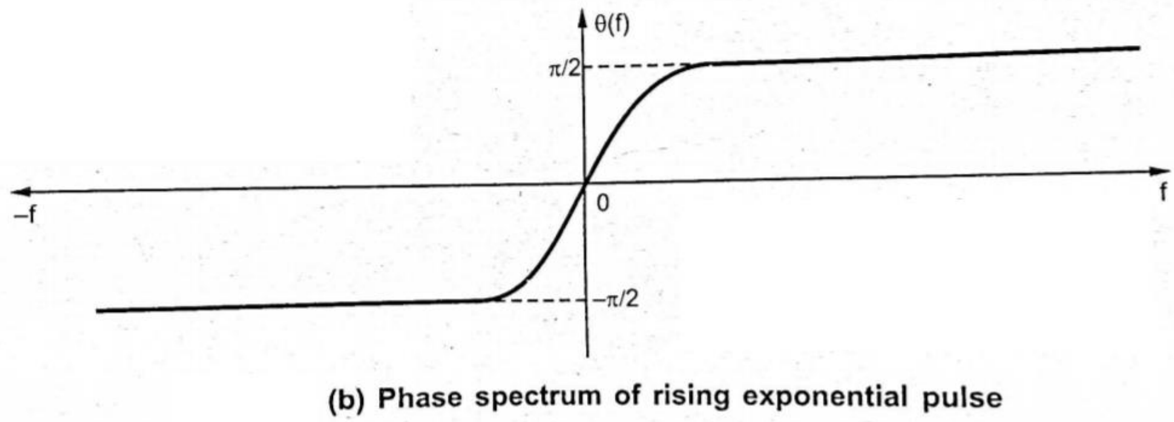
This fourier transform pair for rising exponential pulse can be represented by following equation,

Rising exponential pulse : $e^{at}\, u(-t) \leftrightarrow \dfrac{1}{a-j2\pi f}$

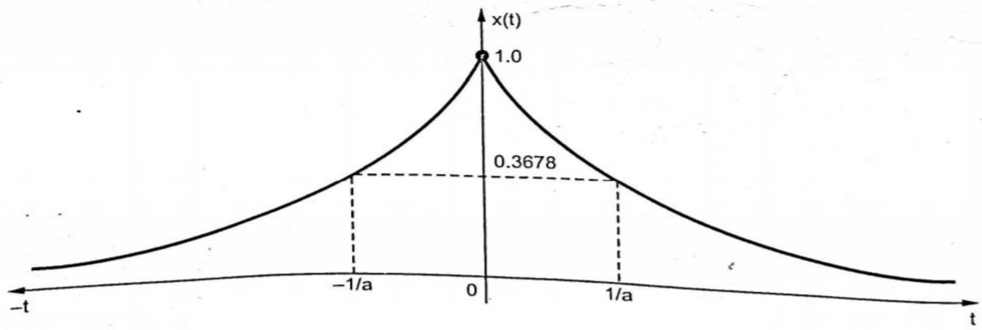
$$|X(f)| = \frac{1}{\sqrt{a^2+(2\pi f)^2}}$$



(a) Amplitude spectrum and



(b) Phase spectrum of rising exponential pulse



obtain the Fourier transform of double exponential signal as shown in the figure

Solution : The double exponential pulse of above figure can be represented as,

$$x(t) = e^{-at} \; ; \qquad t > 0$$
$$= 1 \; ; \qquad t = 0$$
$$= e^{at} \; ; \qquad t < 0$$

Fourier transform : $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$ by equation

$$= \int_{-\infty}^{0-} e^{at}\, e^{-j2\pi ft}\, dt + \int_{0-}^{0+} 1 \cdot e^{-j2\pi ft}\, dt + \int_{0+}^{\infty} e^{-at}\, e^{-j2\pi ft}\, dt$$

$$\int_{0-}^{0+} 1 \cdot e^{-j2\pi ft}\, dt = 1 \int_{0-}^{0+} e^{-j2\pi ft}\, dt = 1\left[e^{0+} - e^{0-}\right] = 1\,[1-1] = 0$$

Or in other words integration at single point with upper and lower limits same is zero only.

$$X(f) = \frac{1}{a - j2\pi f} + 0 + \frac{1}{a + j2\pi f}$$

$$= \frac{2a}{a^2 + (2\pi f)^2}$$

Equation can be written in short hand as $e^{-a|t|}u(t)$. [Here when $t < 0$; $e^{-a(-t)}u(-t) = e^{at}u(-t)$].

$\therefore$ Fourier transform pair is represented as shown below,

Symmetric double exponential pulse : $e^{-a\|t\|} \leftrightarrow \dfrac{2a}{a^2 + (2\pi f)^2}$

Here $\quad |X(f)| = \dfrac{2a}{a^2 + (2\pi f)^2}$

and $\quad \theta(f) = 0$

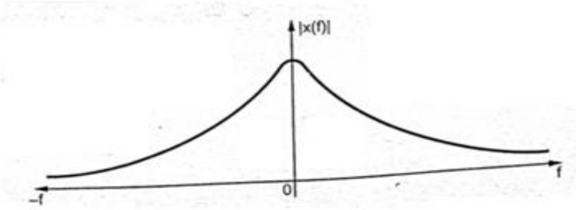
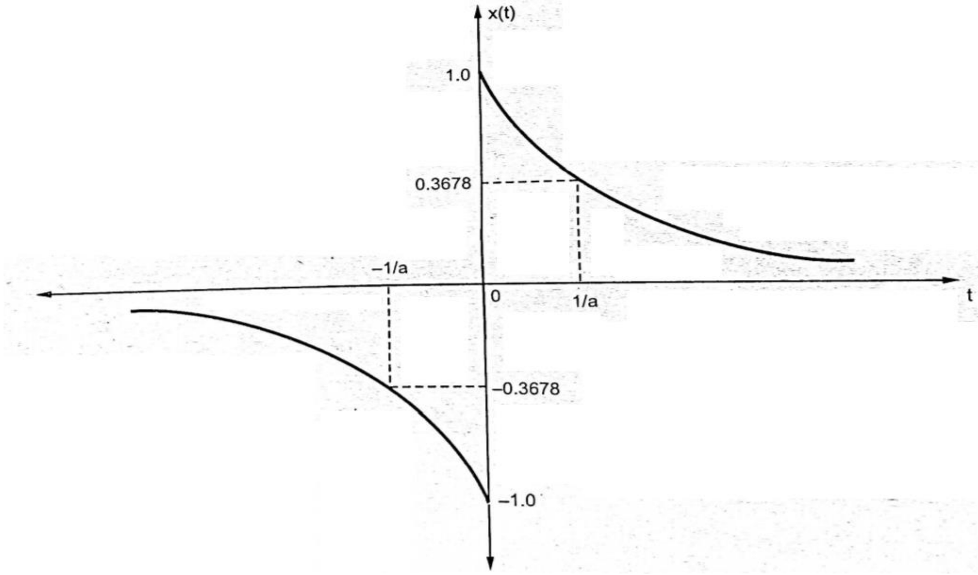


Fig. 2: Magnitude response



Obtain the Fourier transform of antisymmetric exponential pulse as shown in the figure

Solution : This pulse can be very easily represented with the help of equation We have used in last problem. Here overall value of the pulse is negative for negative time.

$$x(t) = e^{-at}; \quad t > 0$$

$$= |1|; \quad t = 0$$

$$= -e^{at}; \quad t < 0$$

∴ Fourier transform will be,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \text{from equation}$$

$$= \int_{-\infty}^{0-} -e^{at}\, e^{-j2\pi ft}\, dt + \int_{0-}^{0+} |1|\, e^{-j2\pi ft}\, dt + \int_{0+}^{\infty} e^{-at}\, e^{-j2\pi ft}\, dt$$

$$= -\frac{1}{a - j2\pi f} + \frac{1}{a + j2\pi f}$$

$$= -\frac{j4\pi f}{a^2 + (2\pi f)^2}$$

Signum function can be used to represent equation compactly, A signum function is defined as,

$$sgn(t) = 1; \quad t > 0$$

$$= -1; \quad t < 0$$

∴ Equation can be written as,

$$x(t) = e^{-a|t|} sgn(t)$$

∴ Fourier transform pair becomes,

$$\boxed{e^{-a|t|}\; sgn(t) \leftrightarrow \frac{-j4\pi f}{a^2 + (2\pi f)^2}}$$

$$f(t) = e^{-0.5t}\, u(t)$$

Using scaling property find the Fourier transform of

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Now consider the given example. FT can be obtained directly as,

$$X(f) = \frac{1}{0.5 + j\,2\pi f}$$

We know that $F[e^{-t}] = \frac{1}{1 + j\,2\pi f}$

Now by scaling property and $a = 0.5$

$$F\left[e^{-0.5t}\right] = \frac{1}{0.5} \frac{1}{1 + j\,2\pi\left(\frac{f}{0.5}\right)}$$

$$= \frac{1}{0.5 + j\,2\pi f}$$

Obtain the fourier transform of rectangular pulse of duration T and amplitude `A' as shown in Fig. below.

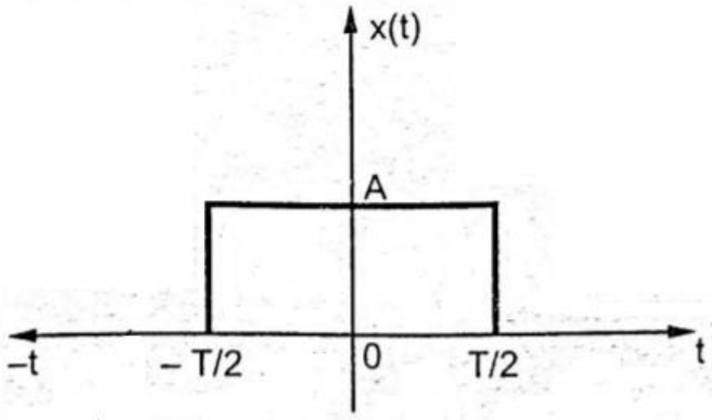


Fig. 3: Rectangular pulse

Solution : This rectangular pulse is defined as,

$$rect\left(\frac{t}{T}\right) = \begin{cases} A & for \; -\frac{T}{2} < t < \frac{T}{2} \\ 0 & elsewhere \end{cases}$$

$$x(t) = A\; rect\left(\frac{t}{T}\right)$$

FT of $x(t)$ $\quad X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$

$$= \int_{-T/2}^{T/2} A\, e^{-j2\pi ft}\, dt$$

$$= \frac{A}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-T/2}^{T/2}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right]$$

$$= \frac{A}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right]$$

$$= \frac{A}{\pi f}\, sin\, (\pi f T)$$ By Euler's theorem.

$$= AT\, \frac{sin(\pi f T)}{\pi f T}$$ By rearranging the equation.

$$= AT\, sinc\, (fT)$$ Since $sinc\ x = \frac{\sin(\pi x)}{\pi x}$

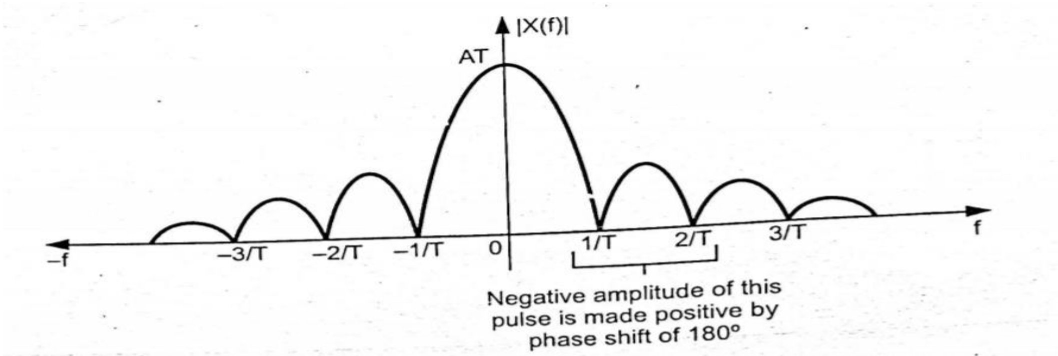


Fig. 4: Amplitude spectrum

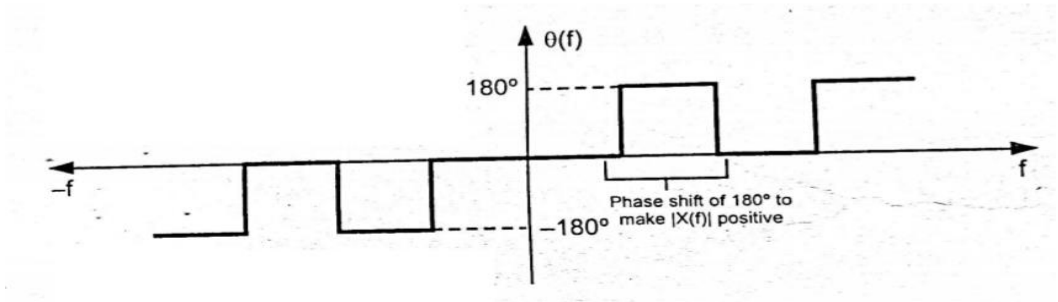


Fig. 5: Phase spectrum spectrum

The fourier transform pair of sinc and rectangular function is,

$$A\,rect\left(\frac{t}{T}\right) \leftrightarrow AT\,sinc(fT)$$

i.e. *Rectangular pulse ↔ sinc pulse.*

Obtain the fourier transform of the impulse function ,shown in below.

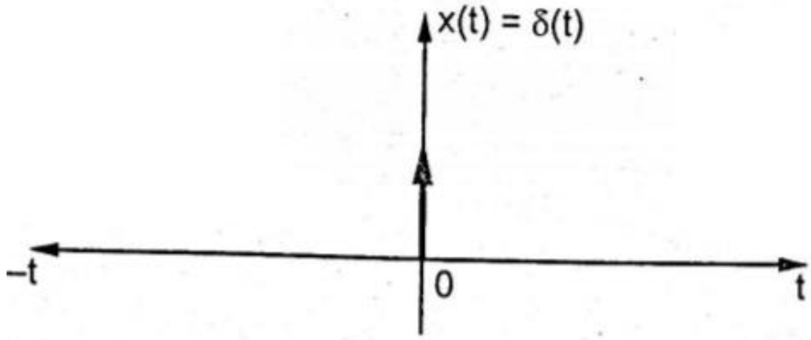


Fig. 6: Impulse Function

Solution : By definition of FT,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt$$

The sifting property of impulse function is given as,

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-t_0) = f(t_0)$$

Here $f(t) = e^{-j2\pi ft}$ and $t_0 = 0$

$$\therefore \quad X(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft}\, \delta(t-0)\, dt$$

By rearranging equation

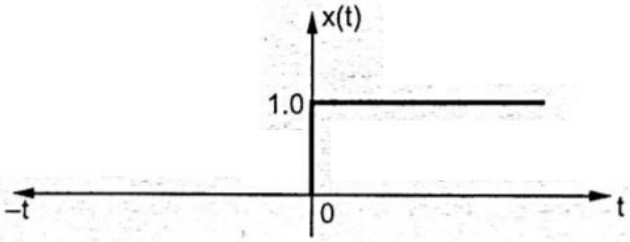
$$= e^{-2\pi f \cdot 0}$$ By applying sifting property.

$$= 1$$

Delta Function : $\delta(t) \leftrightarrow 1$

Fig shows the amplitude spectrum of delta function. It shows that delta function or unit impulse contains all the frequencies with same amplitude in its spectrum.

Obtain the fourier transform of the unit step function shown in Fig.



Solution : The unit step signal is defined as,

$$u(t) = 1 \quad t \geq 0$$
$$= 0 \quad t < 0$$

By definition of FT, $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$

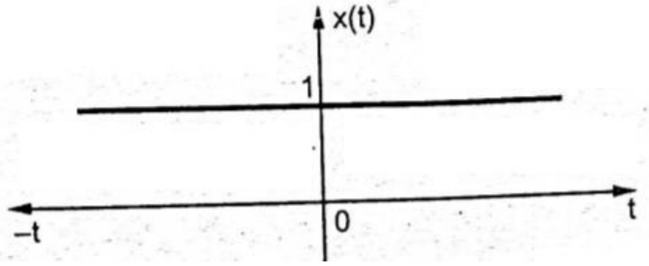
$$= \int_{0}^{\infty} 1\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{-j2\pi f}\left[e^{-j2\pi f t}\right]_0^{\infty}$$

$$= \frac{1}{j2\pi f}$$

Thus $\quad u(t) \leftrightarrow \dfrac{1}{j2\pi f}$

Obtain the fourier transform of DC signal shown in Fig.



$$\delta(t) \leftrightarrow 1$$

Here $X(f) = 1$

i.e. $\delta(t) \leftrightarrow X(f)$

By applying duality property of equation 2.2.39

$$X(t) \leftrightarrow x(-f)$$

Here $X(t) = 1$ and $\delta(-f) = \delta(f)$ since $\delta(f)$ is even function.

Therefore, $\boxed{1 \leftrightarrow \delta(f)}$

Above equation states that dc signal $[x(t) = 1]$ is transformed into delta function $[\delta(f)]$ in frequency domain. Thus dc signal and delta function in frequency domain form a fourier transform pair.

i.e., $$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Here if $x(t) = 1$, then $X(f) = \delta(f)$ from equation

$$\int_{-\infty}^{\infty} 1\, e^{-j2\pi f t}\, dt = \delta(f)$$

$$\boxed{\int_{-\infty}^{\infty} e^{-j2\pi f t}\, dt = \delta(f)}$$

This is the important relation and gives result of integration of exponential function.

If $1 \leftrightarrow \delta(t)$ form a fourier transform pair of dc signal in time domain and delta function in frequency domain, then

$$\int_{-\infty}^{\infty} \cos(2\pi f t)\, dt = \delta(f)$$

Solution : By definition of FT,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Now if $x(t) = 1, X(f) = \delta(f)$ from the given condition in example.

$$\therefore \int_{-\infty}^{\infty} 1\, e^{-j2\pi f t}\, dt = \delta(f)$$

$$\text{or} \quad \int_{-\infty}^{\infty} e^{-j2\pi f t}\, dt = \delta(f)$$

$$\cos(2\pi f t) = \frac{e^{j2\pi f t} + e^{-j2\pi f t}}{2} \quad \text{and}$$

$$\sin(2\pi f t) = \frac{e^{j2\pi f t} - e^{-j2\pi f t}}{2j}$$

From above two equations we have,

$$\cos(2\pi f t) - j\sin(2\pi f t) = \frac{e^{2\pi f t} + e^{-j2\pi f t}}{2} - \frac{e^{j2\pi f t} - e^{-j2\pi f t}}{2}$$

$$= e^{-j2\pi f t}$$

Now substitute value of $e^{-j2\pi f t}$ from above equation in equation

$$\int_{-\infty}^{\infty} [\cos(2\pi f t) - j\sin(2\pi f t)]\, dt = \delta(f)$$

Now since $\delta(f)$ is real valued, complex function will have zero value in above expression,

$$\therefore \int_{-\infty}^{\infty} \cos(2\pi f t) = \delta(f)$$

Hence the result is proved.

Obtain the fourier transform of $x(t) = e^{j2\pi f_c t}$

Solution : By definition of FT,

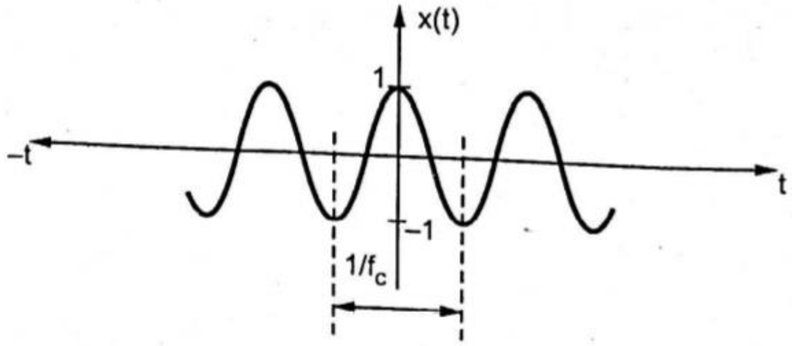
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{j2\pi f_c t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(f-f_c)t}\, dt$$

$$= \delta\,(f - f_c)$$

Find out the fourier transform of cosine wave shown in Fig. below.



$$x(t) = \cos(2\pi f_c t) = \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2} \qquad \text{By Euler's theorem.}$$

By definition of FT,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right] e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left\{e^{-j2\pi(f-f_c)t} + e^{-j2\pi(f+f_c)t}\right\} dt$$

$$= \frac{1}{2} \left\{ \int_{-\infty}^{\infty} e^{-j2\pi(f-f_c)t}\, dt + \int_{-\infty}^{\infty} e^{-j2\pi(f+f_c)t}\, dt \right\}$$

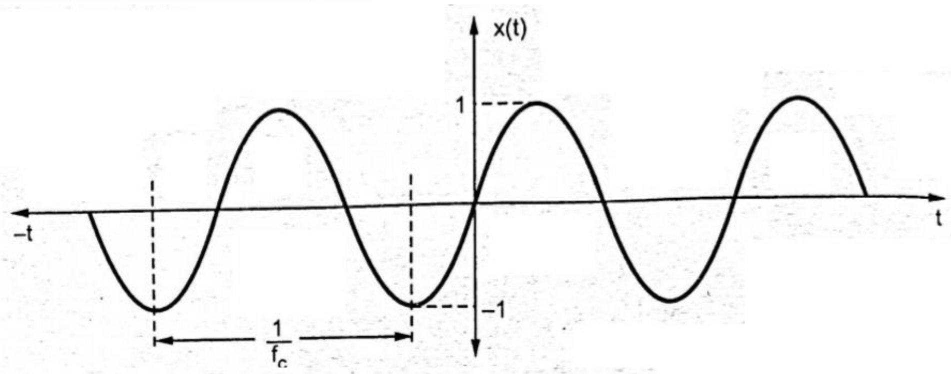
$$= \frac{1}{2} \left[\delta(f-f_c) + \delta(f+f_c)\right] \quad \textit{from equation 2.2.61}$$

Thus,

Cosine wave : $\cos(2\pi f_c t) \leftrightarrow \frac{1}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right]$

Find out the fourier transform of sine function $\sin(2\pi f_c t)$ shown in Fig.

$$x(t) = \sin(2\pi f_c t)$$



Solution : Here since $V_m = 1$;

$$x(t) = \sin(2\pi f_c t)$$

$$\sin(2\pi f_c t) = \frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j} \qquad \textit{By Euler's theorem}$$

By definition of FT, $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$

$$= \int_{-\infty}^{\infty} \sin(2\pi f_c t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j}\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2j} \int_{-\infty}^{\infty} \left[e^{-j2\pi(f-f_c)t} - e^{-j2\pi(f+f_c)t}\right] dt$$

$$= \frac{1}{2j}\left[\delta(f-f_c) - \delta(f+f_c)\right]$$

Thus,

$$\textit{Sine wave}: \sin(2\pi f_c t) \leftrightarrow \frac{1}{2j}\left[\delta(f-f_c) - \delta(f+f_c)\right]$$

2.4 LAPLACE TRANSFORM

Fourier transform represents continuous time signal in terms of complex sinusoidals, i.e. $e^{j\omega t}$. The laplace transform provides broader characterization compared to fourier transform. Laplace transform represents continuous time signals in terms of complex exponentials, i.e. e^{-st}. Hence laplace transform can be used to analyze the signals or functions which are not absolutely integrable. Continuous time systems are also analyzed more effectively using laplace transforms. Laplace transform can be applied to the analysis of unstable systems also. Laplace transform of the impulse response is called system function or transfer function.

i) Bilateral or two sided laplace transform and

ii) Unilateral or one sided laplace transform.

Consider the continuous time signal $x(t)$. It's laplace transform is denoted by $X(s)$. Then the laplace transform is given as,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Here the independent variable 's' is complex in nature and it is given as,

$$s = \sigma + j\omega$$

Here σ is the real part of 's'. It is called attenuation constant. And $j\omega$ is the imaginary part of 's' and it is called complex frequency. In equation 2.3.1 observe that the integration is taken from $-\infty$ to $+\infty$. Hence it is called bilateral or double sided laplace transform.

The laplace transform pair $x(t)$ and $X(s)$ is represented as,

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

The unilateral laplace transform is given as,

$$\boxed{X(s) = \int_{0-}^{\infty} x(t)\, e^{-st}\, dt}$$

2.5. INVERSE LAPLACE TRANSFORM

This laplace transform is mainly used for causal signals. The lower limit is taken from $0-$. This is to include the time just before zero. Thus for the continuous function, integration is effectively taken from 0 to ∞. Unilateral laplace transform is useful in the analysis of networks and solving differential equations.

The inverse laplace transform is given as,

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds}$$

This formula involves complex integration. Inverse laplace transform can also be obtained using partial fraction expansion.

We know that fourier transform is given as,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier transform can be calculated only if $x(t)$ is absolutely integrable. i.e.,

Relationship between Fourier Transform and Laplace Transform

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

We know that $s = \sigma + j\omega$. Hence equation can be written as,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left\{x(t) e^{-\sigma t}\right\} e^{-j\omega t}\, dt$$

Comparing above equation with equation 2.3.6 we find that, laplace transform of $x(t)$ is basically the fourier transform of $x(t)\, e^{-\sigma t}$. If $s = j\omega$, i.e. $\sigma = 0$, then above equation becomes,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= X(j\omega)$$

Thus $\quad X(s) = X(j\omega)$ *when* $s = j\omega$

This means laplace transform is same as fourier transform when $s = j\omega$. Above equation shows that fourier transform is special case of laplace transform. Thus laplace transform provides broader characterization compared to fourier transform. $s = j\omega$ indicates imaginary axis in complex s-plane. Thus laplace transform is basically fourier transform on imaginary $(j\omega)$ axis in the s-plane.

2.6.CONVERGENCE

From equation we know that laplace transform is basically the fourier transform of $x(t)e^{-\sigma t}$. Hence if fourier transform of $x(t)e^{-\sigma t}$ exists, then laplace transform of $x(t)$ exists. For fourier transform to exist, $x(t)e^{-\sigma t}$ must be absolutely integrable. i.e.,

$$\int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt < \infty$$

If this condition is satisfied, then fourier transform of $x(t)e^{-\sigma t}$ will exist. In other words we can say that laplace transform of $x(t)$ will exist, if above condition is satisfied. Thus above equation gives necessary condition for laplace transform to exist. The range of values of 'σ' for which laplace transform converges is called *region of convergence* or ROC.

Calculate the laplace transform of $x(t) = e^{at}\, u(t)$ and plot the ROC.

Solution : We know that laplace transform is given as,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{at}\, u(t)\, e^{-st}\, dt$$

$u(t) = 1$ for $t \geq 0$. Hence above equation will be,

$$X(s) = \int_{0}^{\infty} e^{at}\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

Above equation can be written as,

$$X(s) = \lim_{t\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right] - \lim_{t\to 0}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]$$

We know that e^x converges if x is negative. Hence the first term in above equation will converge if $(s-a)>0$. i.e.,

$$\lim_{t\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right] = 0 \text{ if } (s-a)>0$$

Thus the first term will be zero as $t\to\infty$.

Therefore equation 2.3.11 can be written as,

$$X(s) = 0-\left[\frac{e^0}{-(s-a)}\right] \quad for \quad (s-a)>0$$

$$= \frac{1}{s-a} \quad for \quad (s-a)>0 \quad or \quad s>a$$

For $s<a$, the laplace transform cannot be calculated since the integral is unbounded. Therefore the region of convergence is $s>a$. This is shown in Fig. The shaded area shows the ROC. Thus the laplace transform pair is,

$$e^{at}\,u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}, \quad ROC \quad s>a$$

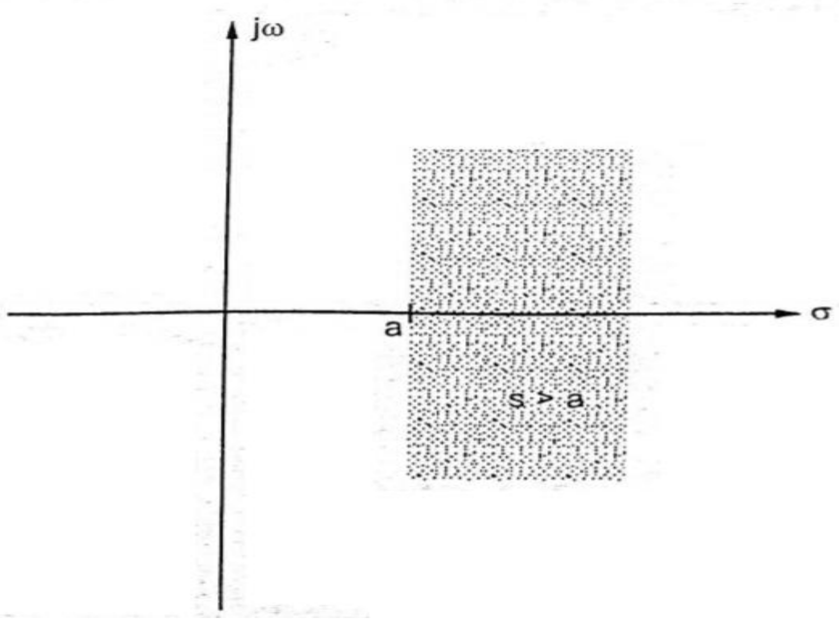


Fig. 7. Region of Convergence

Determine the laplace transform and ROC for the signal

$x(t) = -e^{at}\, u(-t)$

Solution : Laplace transform is given as,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} -e^{at}\, u(-t)\, e^{-st}\, dt$$

We know that,

$$u(-t) = \begin{cases} 0 & \text{for} \quad t \geq 0 \\ 1 & \text{for} \quad t < 0 \end{cases}$$

Hence the integration limits of laplace transform will be changed as follows :

$$X(s) = \int_{-\infty}^{0} -e^{at}\, e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{s-a}\right]_{-\infty}^{0}$$

$$= \lim_{t\to 0}\left[\frac{e^{-(s-a)t}}{s-a}\right] - \lim_{t\to -\infty}\left[\frac{e^{-(s-a)t}}{s-a}\right]$$

The second term will converge if power of exponent is negative. Note that 't' tends to $-\infty$. Hence $(s-a)$ must be negative to make overall exponent negative. Therefore we can write,

$$X(s) = \frac{e^{-(s-a)0}}{s-a} - \frac{e^{-(s-a)(-\infty)}}{s-a}$$

$$= \frac{1}{s-a} - \frac{0}{s-a} \quad for \quad (s-a)<0$$

$$= \frac{1}{s-a} \quad for \quad s-a<0 \quad or \quad s<a$$

Thus the laplace transform will converge if $s<a$. For $s>a$, the integration will be unbounded. Fig. shows the ROC of $X(s)$.

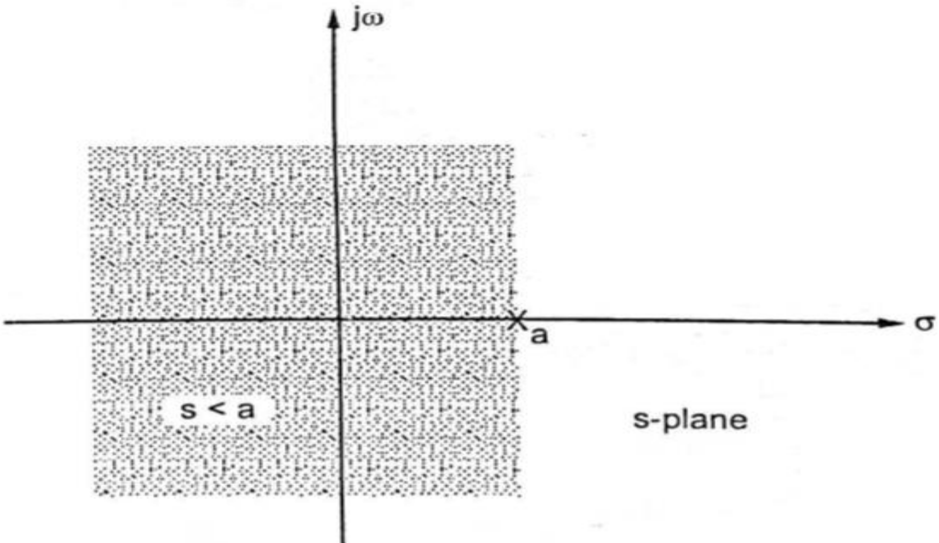


Fig. 8. Region of Convergence

$$x(t) = e^{-at}\, u(-t)$$

We have obtained the laplace transform of $e^{at}\, u(t)$ in example From equation it is given as,

$$e^{at}\, u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}\ , \ ROC\ \ s > a$$

Hence laplace transform of $e^{-at}\, u(t)$ will be,

$$\boxed{e^{-at}\, u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a},\ \ ROC\ \ s > -a}$$

$$x(t) = -e^{-at}\, u(-t)$$

We have obtained laplace transform of $-e^{at}\, u(-t)$ in example From equation it is given as,

$$-e^{at}\, u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}\ , \ ROC\ \ s < a$$

Hence laplace transform of $-e^{-at}\, u(-t)$ will be,

$$\boxed{-e^{-at}\, u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a},\ \ ROC\ \ s < -a}$$

2.7.PROPERTIES OF LAPLACE TRANSFORM

Linearity

Let $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$, ROC : R_1

and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$, ROC : R_2

Then linearity property states that,

$$\boxed{\mathcal{L}\left[a_1\, x_1(t) + a_2\, x_2(t)\right] = a_1\, X_1(s) + a_2\, X_2(s)\ , \quad ROC\ : R_1 \cap R_2}$$

Proof : By definition of laplace transform we can write,

$$\mathcal{L}\left[a_1 x_1(t)+a_2 x_2(t)\right] = \int_{-\infty}^{\infty}\left[a_1 x_1(t)+a_2 x_2(t)\right]e^{-st}\,dt$$

$$= a_1\int_{-\infty}^{\infty} x_1(t)e^{-st}\,dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-st}\,dt$$

$$= a_1 X_1(s)+a_2 X_2(s) \quad ROC\ : R_1 \cap R_2$$

Here ROC : $R_1 \cap R_2$ indicates the intersection of R_1 and R_2.

Time Shifting

Let $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, ROC : R

then

$$\boxed{\mathcal{L}\left[x(t-t_0)\right] = e^{-st_0}X(s) \quad ROC : R}$$

Proof : By definition, laplace transform of $x(t-t_0)$ will be,

$$\mathcal{L}\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\,dt$$

Let $\tau = t-t_0$

$\therefore$ $d\tau = dt$ *and* $t = \tau + t_0$

And when $t = -\infty$, $\tau = -\infty - t_0 = -\infty$ *and*

when $t = \infty$, $\tau = \infty - t_0 = \infty$

Hence laplace transform becomes,

$$\mathcal{L}\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} x(\tau)\, e^{-s(\tau+t_0)}\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, e^{-st_0}\, d\tau$$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau$$

$$= e^{-st_0} X(s)\ ,\ ROC : R$$

Thus delay of t_0 in time domain is equivalent to multiplication of laplace transform by e^{-st_0}.

Shifting in s-domain

Let $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, ROC : R

then $$\boxed{\mathcal{L}\left[e^{s_0 t}\, x(t)\right] = X(s-s_0)\ ,\ ROC : R + Re(s_0)}$$

Proof : By definition of laplace transform,

$$\mathcal{L}\left[e^{s_0 t}\, x(t)\right] = \int_{-\infty}^{\infty} e^{s_0 t}\, x(t) e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(s-s_0)t} dt$$

$$= X(s-s_0)\ \text{with}\ ROC : R + Re(s_0)$$

Thus the frequency shift of 's_0' is equivalent to multiplying $x(t)$ by $e^{s_0 t}$. The ROC is also shifted by $Re(s_0)$.

Time Scaling

Let $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, with ROC : R

then $$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad ROC : \frac{R}{a}$$

Proof : By definition of laplace transform,

$$\mathcal{L}[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt$$

Let $$at = \tau \qquad \therefore \quad t = \frac{\tau}{a}$$

and $$dt = \frac{1}{a}\, d\tau$$

Limits of integration will remain same. Hence laplace transform becomes,

$$\mathcal{L}[x(at)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-s\frac{\tau}{a}}\, \frac{1}{a}\, d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right) \quad ROC : \frac{R}{a}$$

Now let us consider negative value of a. i.e.,

$$\mathcal{L}\left[x(-at)\right] = \int_{-\infty}^{\infty} x(-at)\, e^{-st}\, dt$$

Let $\qquad -at = \tau, \qquad \therefore\ t = -\frac{\tau}{a}$

and $\qquad dt = -\frac{1}{a}\, d\tau$

Limits of integration will interchange. i.e.,

$$\mathcal{L}\left[x(-at)\right] = \int_{\infty}^{-\infty} x(\tau)\, e^{-s\left(-\frac{\tau}{a}\right)} \left(-\frac{1}{a}\right) d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-\left(\frac{s}{-a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{-a}\right), \quad ROC : \frac{R}{-a}$$

From above equation and equation 2.4.5 we can write,

$$\mathcal{L}\left[x(at)\right] = \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad ROC : \frac{R}{a}$$

Thus the time scaling property is proved. This property shows that expanding time axis is equivalent to compression in frequency domain. The ROC is also compressed or expanded depending upon value of a.

As a special case with $a = -1$ we have,

$$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s), \quad ROC : R$$

This result shows that inverting the time axis inverts frequency axis as well as ROC.

Differentiation in Time Domain

Let $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $ROC : R$

then

$$\frac{d}{dt} x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \quad ROC : R$$

Proof : Consider the inverse laplace transform given by equation

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

Differentiate both sides of above equation with respect to 't' i.e.,

$$\frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, s\, e^{st}\, ds = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} [sX(s)]\, e^{st}\, ds$$

This equation shows that inverse laplace transform of $sX(s)$ is $\frac{d}{dt}x(t)$. This proves

then

$$\frac{d^n}{dt^n} x(t) \overset{\mathcal{L}}{\longleftrightarrow} s^n X(s) \quad , \quad ROC\ containing\ R$$

Differentiation in s-domain

Let $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $ROC : R$

then

$$-t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds} X(s), \quad ROC : R$$

Proof : By definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

differentiating above equation with respect to 's',

$$\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t)\,(-t)\, e^{-st} dt = \int_{-\infty}^{\infty} [-t\ x(t)]\, e^{-st}\, dt$$

This equation shows that the laplace transform of $-t\,x(t)$ is $\frac{d}{ds}X(s)$. Hence equation 2.4.10 is proved. ROC is unchanged. Above result can be extended easily for multiple s-domain differentiations i.e.,

$$(-t)^n\, x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d^n}{ds^n} X(s) \quad , \quad ROC : R$$

Convolution in Time Domain

Let $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$, $ROC : R_1$

and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$, $ROC : R_2$

then $x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)\, X_2(s)$, $ROC :$ *containing* $R_1 \cap R_2$

That is, the laplace transform of convolution of two functions is equivalent to multiplication of their laplace transforms.

Proof : By definition of convolution,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau)\, d\tau$$

Taking laplace transform of both the sides,

$$\mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau)\, d\tau \right\} e^{-st}\, dt$$

changing the order of integration,

$$\mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt$$

The second integration in above equation represents laplace transform of $x_2(t)$ with delay of 'τ'. Hence applying the time delay property we can write second integration term as,

$$\mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \quad e^{-s\tau} X_2(s)$$

$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau)\, e^{-s\tau}\, d\tau$$

Above integration represents laplace transform of $x_1(\tau)$,

$$\mathcal{L}[x_1(t) * x_2(t)] = X_2(s)\, X_1(s)$$

$$= X_1(s)\, X_2(s) \ , \ ROC : R_1 \cap R_2$$

Thus convolution property can be proved. The ROC is at least intersection of R_1 and R_2.

Integration in Time Domain

Let $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, $ROC : R$

then $$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\mathcal{L}} \frac{X(s)}{s} \quad , \quad ROC : R \cap [Re(s) > 0]$$

Proof : Let us consider the convolution of $x(t)$ with $u(t)$.

$$x(t) * u(t) = \int_{-\infty}^{\infty} u(t-\tau)\, x(\tau)\, d\tau$$

We know that $u(t-\tau) = \begin{cases} 1 & for\ t \geq \tau \quad i.e.\ \tau \leq t \\ 0 & elsewhere \end{cases}$

Hence convolution becomes,

$$x(t) * u(t) = \int_{-\infty}^{t} 1\ x(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau$$

i.e. $$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

Taking laplace transform of both sides,

$$\mathcal{L}\left\{ \int_{-\infty}^{t} x(\tau)\, d\tau \right\} = \mathcal{L}\{x(t) * u(t)\}$$

Using convolution property we can write RHS as,

$$\mathcal{L}\left\{ \int_{-\infty}^{t} x(\tau)\, d\tau \right\} = \mathcal{L}\{x(t)\}\, \mathcal{L}\{u(t)\}$$

$$= X(s)\frac{1}{s}$$

$$= \frac{X(s)}{s} \quad , \quad ROC : R \cap [Re(s) > 0]$$

Thus integration property is proved. The ROC is intersection of ROC of $X(s)$ and ROC of $u(t)$ i.e. $Re(s) > 0$.

The relation of equation 2.4.15 can be generalized further for multiple integrations. i.e.,

$$\mathcal{L}\left\{ \int_{-\infty}^{t} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} x(t)\, dt_1\, dt_2 \ldots dt_n \right\} = \frac{X(s)}{s^n}$$

Integration in s-domain

Let $$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC : R$$

then $$\frac{x(t)}{t} \xleftrightarrow{\mathcal{L}} \int_s^{\infty} X(s)ds, \quad ROC : R$$

Proof : Consider the RHS of above equation,

$$\int_s^{\infty} X(s)ds = \int_s^{\infty} \left[\int_{-\infty}^{\infty} x(t)e^{-st}\,dt \right] ds$$

Changing the order of integration and rearranging the terms,

$$\int_s^{\infty} X(s)ds = \int_{-\infty}^{\infty} x(t) \left[\int_s^{\infty} e^{-st} ds \right] dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{e^{-st}}{-t} \right]_s^{\infty} dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\lim_{s \to \infty} \frac{e^{-st}}{-t} - \frac{e^{-st}}{-t} \right] dt$$

In above equation $\lim_{s \to \infty} \frac{e^{-st}}{-t}$ becomes zero in specified ROC.

Hence,

$$\int_s^{\infty} X(s)ds = \int_{-\infty}^{\infty} x(t) \frac{e^{-st}}{t}\,dt \quad , \quad ROC : R$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{t} e^{-st}\,dt \quad , \quad ROC : R$$

$$= \mathcal{L}\left[\frac{x(t)}{t} \right]$$

2.8 LAPLACE TRANSFORM OF STANDARD ELEMETRY SIGNALS

Determine the laplace transform of unit step function. The unit step

$$u(t)=\begin{cases}1 & \text{for } t\geq 0\\ 0 & \text{otherwise}\end{cases}$$

Solution : By definition of laplace transform,

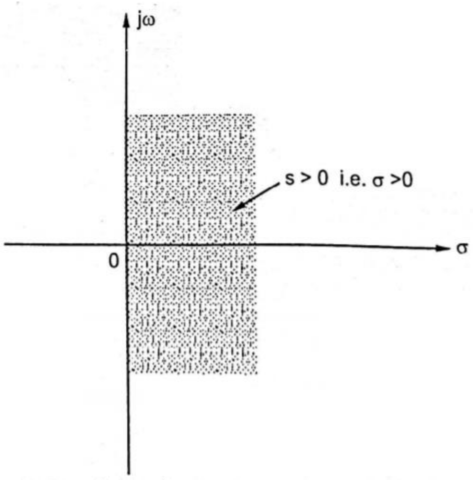
$$\mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

changing the limits from 0 to ∞, since $u(t)=1$ for $t\geq 0$ above equation will be,

$$\mathcal{L}[u(t)] = \int_{0}^{\infty} 1\, e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \lim_{t\to\infty}\left[\frac{e^{-st}}{-s}\right]-\lim_{t\to 0}\left[\frac{e^{-st}}{-s}\right]$$

First term in above equation will be zero as $t\to\infty$ if sign of exponent is negative. This is possible if $s>0$. Hence above equation becomes,

$$\mathcal{L}[u(t)] = 0-\frac{e^0}{-s} \quad for \quad s>0 = \frac{1}{s} \quad for \quad s>0$$



Here $s>0$ indicates ROC of laplace transform. Fig. 2.5.1 indicates this ROC. Here observe that $s>0$ means $\sigma>0$. Hence sometimes ROC is also written as $Re(s)>0$ or $\sigma>0$. Thus we obtained the laplace transform pair as,

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}, \quad ROC: s>0 \;\; or \;\; \sigma>0$$

Determine the laplace transform of impulse function, $\delta(t)$.

Solution : The impulse function is defined as,

$$\delta(t) = 0 \quad for \quad t \neq 0$$

And at $t = 0$, area under $\delta(t)$ is equal to '1'.

By definition of laplace transform,

$$\mathcal{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

From definition of laplace transform,

$$\mathcal{L}[\delta(t)] = \int_{t=0} \delta(t)\, e^{-s\,0}\, dt$$

Above integral is evaluated only at $t = 0$ since $\delta(t)$ is non zero only at $t = 0$. And $\delta(t) = 0$ for $t \neq 0$. Hence above equation becomes,

$$\mathcal{L}[\delta(t)] = \int_{t=0} \delta(t)\, 1\, dt$$

$$= \int_{t=0} \delta(t)\, dt$$

Above integral indicates area under the unit impulse as $t \to 0$. The area under the unit impulse is equal to '1' at $t = 0$. Hence above equation becomes,

$$\mathcal{L}[\delta(t)] = 1$$

Here note that ROC will be entire s-plane, since there is no condition for evaluation of integration. Thus the laplace transform pair is,

$$\boxed{\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1, \quad ROC : entire\ s\text{-}plane}$$

Determine the laplace transform of the ramp function. The unit ramp function

$$r(t) = \begin{cases} t & for\ t \geq 0 \\ 0 & otherwise \end{cases}$$

$$or\ \ r(t) = t\,u(t)$$

Solution : By definition of the laplace transform,

$$\mathcal{L}[r(t)] = \int_{-\infty}^{\infty} r(t)\, e^{-st}\, dt$$

Since $r(t) = t$ for $t \geq 0$, the limits of above integration will be changed as 0 to ∞. Hence,

$$\mathcal{L}[r(t)] = \int_{0}^{\infty} t\, e^{-st}\, dt$$

Integrating above equation by parts we get,

$$\mathcal{L}[r(t)] = \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1\,\frac{e^{-st}}{-s}\, dt$$

Consider the first term in above equation,

$$\left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} = \lim_{t\to\infty}\left[t\,\frac{e^{-st}}{-s}\right] - \lim_{t\to 0}\left[t\,\frac{e^{-st}}{-s}\right]$$

The first term in above equation will be zero if $s>0$. For negative values of 's', first term will be unbounded. And as $t\to 0$, second term will be zero. Hence above equation becomes,

$$\left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} = 0 \quad for \quad s>0$$

Therefore equation 2.5.3 becomes,

$$\mathcal{L}[r(t)] = 0 - \int_0^{\infty} \frac{e^{-st}}{-s}\,dt \quad for \;\; s>0$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{1}{s}\lim_{t\to\infty}\left[\frac{e^{-st}}{-s}\right] - \frac{1}{s}\lim_{t\to 0}\left[\frac{e^{-st}}{-s}\right]$$

The first term in above equation will be zero if $s>0$. For negative values of 's', first term will be unbounded. Hence above equation becomes,

$$\mathcal{L}[r(t)] = \frac{1}{s}\times 0 - \frac{1}{s}\left[\frac{e^{0}}{-s}\right] \quad for \;\; s>0$$

$$= \frac{1}{s^2} \quad for \;\; s>0 \;\; or \;\; ROC: s>0$$

Above laplace transform exists only for $s>0$. Hence ROC is $s>0$. It is also written as $\sigma>0$. Thus the laplace transform pair is,

$$\boxed{r(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2}\;,\;\; ROC: s>0 \;\; or \;\; \sigma>0}$$

(i) $x(t) = A\, sin\, \omega_0 t \quad u(t)$

By definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} A\, sin\, \omega_0 t \;\; u(t)\, e^{-st}\, dt$$

We know that $sin\, \omega_0\, t = \frac{1}{2j}\left[e^{j\omega 0 t} - e^{-j\omega 0 t}\right]$. Hence above equation becomes,

$$X(s) = \int_{-\infty}^{\infty} A \frac{1}{2j}\left[e^{j\omega 0 t} - e^{-j\omega 0 t}\right] u(t) e^{-st}\, dt$$

$$= \frac{A}{2j}\left\{\int_{-\infty}^{\infty} e^{j\omega 0 t}\, u(t) e^{-st}\, dt - \int_{-\infty}^{\infty} e^{-j\omega 0 t}\, u(t) e^{-st}\, dt\right\}$$

$$= \frac{A}{2j}\left\{\mathcal{L}\left[e^{j\omega 0 t}\, u(t)\right] - \mathcal{L}\left[e^{-j\omega 0 t}\, u(t)\right]\right\}$$

We know that,

$$\mathcal{L}\left[e^{at}\,u(t)\right] = \frac{1}{s-a}, \quad ROC \;\; s>a \;\; or \; \sigma>a$$

$$\mathcal{L}\left[e^{j\omega_0 t}\,u(t)\right] = \frac{1}{s-j\omega_0}, \quad ROC : s>j\omega_0$$

and $$\mathcal{L}\left[e^{-j\omega_0 t}\,u(t)\right] = \frac{1}{s+j\omega_0}, \quad ROC : s>-j\omega_0$$

Here note that ROC : $s>j\omega_0$ means,

ROC : $\sigma+j\omega > 0+j\omega_0$

Normally we consider only real part for ROC.

Hence above equation can be written as,

ROC : $\sigma>0$ i.e. $Re(s)>0$

Similarly for the second term $s>-j\omega_0$ we can write,

ROC : $\sigma+j\omega>0-j\omega_0$

ROC : $\sigma>0$ or $Re(s)>0$

Thus the ROC for both the terms is $Re(s)>0$.

Hence equation 2.5.5 becomes,

$$X(s) = \frac{A}{2j}\left\{\frac{1}{s-j\omega_0}-\frac{1}{s+j\omega_0}\right\} \quad ROC : Re(s)>0$$

$$= \frac{A}{2j}\,\frac{2j\omega_0}{s^2+\omega_0^2}$$

$$= \frac{A\omega_0}{s^2+\omega_0^2}$$

Thus we obtained the laplace transform pair as,

$$\boxed{A\sin\omega_0 t \;\; u(t) \xleftrightarrow{\mathcal{L}} \frac{A\omega_0}{s^2+\omega_0^2}, \quad ROC \; : Re(s)>0}$$

(ii) $x(t) = A \cos \omega_0 t\, u(t)$

By definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} A \cos \omega_0 t\, u(t)\, e^{-st}\, dt$$

We know that $\cos \omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$. Hence above equation becomes,

$$X(s) = \int_{-\infty}^{\infty} A \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] u(t)\, e^{-st}\, dt$$

$$= \frac{A}{2}\left\{\int_{-\infty}^{\infty} e^{j\omega_0 t}\, u(t) e^{-st}\, dt + \int_{-\infty}^{\infty} e^{-j\omega_0 t}\, u(t) e^{-st}\, dt\right\}$$

$$= \frac{A}{2}\left\{\mathcal{L}\left[e^{j\omega_0 t}\, u(t)\right] + \mathcal{L}\left[e^{-j\omega_0 t}\, u(t)\right]\right\}$$

We know that,

$$\mathcal{L}\left[e^{at}\, u(t)\right] = \frac{1}{s-a}\ , \quad ROC\ :\ s > a \text{ or } Re(s) > a$$

$$\therefore \quad \mathcal{L}\left[e^{j\omega_0 t}\, u(t)\right] = \frac{1}{s - j\omega_0}\ , \quad ROC\ :\ Re(s) > 0$$

$$\text{and} \quad \mathcal{L}\left[e^{-j\omega_0 t}\, u(t)\right] = \frac{1}{s + j\omega_0}\ , \quad ROC\ :\ Re(s) > 0$$

Hence equation 2.5.7 becomes,

$$X(s) = \frac{A}{2}\left\{\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right\}\ , \quad ROC\ :\ Re(s) > 0$$

$$= \frac{A}{2} \cdot \frac{2s}{s^2 + \omega_0^2} = \frac{A \cdot s}{s^2 + \omega_0^2}$$

Thus we obtained the laplace transform pair as,

$$\boxed{A \cos \omega_0 t\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{A \cdot s}{s^2 + \omega_0^2}\ , \quad ROC\ :\ Re(s) > 0}$$

Find out the laplace transform of impulse function using differentiation property.

The differentiation of unit step function gives unit impulse function.

$$\delta(t) = \frac{d}{dt} u(t)$$

Taking laplace transform of both sides,

$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{d}{dt} u(t)\right]$$

We know that $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$. Using differentiation property of laplace transform above equation can be written as,

$$\mathcal{L}[\delta(t)] = s \cdot \frac{1}{s}$$

$$\mathcal{L}[\delta(t)] = 1. \quad ROC : \text{entire s-plane}$$

A damped sine wave is given as,

$$f(t) = e^{-at} \sin \omega t$$

Find laplace transform of this signal.

Solution : With the help of Euler's identity,

$$f(t) = e^{-at}\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right] = \frac{1}{2j}\left\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right\}$$

By taking laplace transform of both sides,

$$\mathcal{L}\, f(t) = \frac{1}{2j}\, \mathcal{L}\left\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2j}\left\{\frac{1}{s+(a-j\omega)} - \frac{1}{s+(a+j\omega)}\right\}$$

$$= \frac{1}{2j} \cdot \frac{2j\omega}{(s+a)^2+\omega^2}\ ,\quad ROC : s > -a \ \text{ or } \ Re(s) > -a$$

$$= \frac{\omega}{(s+a)^2+\omega^2}$$

Thus, $$L\left[e^{-at} \sin \omega t\right] = \frac{\omega}{(s+a)^2+\omega^2}\ ,\quad ROC : Re(s) > -a$$

A damped cosine wave is given as, $f(t) = e^{-at} \cos \omega t$

find out laplace transform of this signal.

Solution : With the help of Euler's identity,

$$f(t) = e^{-at}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right]$$

$$= \frac{1}{2}\cdot\left[e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right]$$

By taking laplace transform of both sides,

$$\mathcal{L}\ f(t) = \frac{1}{2}\ \mathcal{L}\ \left\{e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2}\cdot\left\{\frac{1}{s+(a-j\,\omega)} + \frac{1}{s+(a+j\,\omega)}\right\}$$

$$= \frac{1}{2}\cdot\frac{2\,(s+a)}{(s+a)^2+\omega^2}\ ,\quad ROC : s > -a \ \text{ or } \ Re(s) > -a$$

$$= \frac{s+a}{(s+a)^2+\omega^2}$$

$$\left[e^{-at}\cos\omega t\right] = \frac{s+a}{(s+a)^2+\omega^2}\ ,\quad ROC : Re(s) > -a$$

Determine the laplace transform of $x(t) = t\, e^{-at}\, u(t)$

Solution : We know that,

$$e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} ROC : Re(s) > -a$$

From differentiation in s-domain

$$-t\, x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds} X(s)$$

$$-t\, e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}\left[\frac{1}{s+a}\right]$$

$$t\, e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^2} ROC : Re(s) > -a$$

This is the laplace transform of given equation.

The same result can be extended as,

$$-t\left[t\, e^{-at}\, u(t)\right] \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}\left[\frac{1}{(s+a)^2}\right]$$

$$\frac{t^2}{2}\, e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^3}$$

Hence the general equation becomes,

$$\frac{t^{n-1}}{(n-1)!}\, e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}\ , \quad ROC : Re(s) > -a$$

Table. 1: Laplace Transform Formula

Sr. No.	Time domain signal $x(t)$	Laplace transform $X(s)$	ROC
1.	$\delta(t)$	1	entire s-plane
2.	$u(t)$	$\frac{1}{s}$	$Re(s)>0$
3.	$r(t)$	$\frac{1}{s^2}$	$Re(s)>0$
4.	$e^{at}u(t)$	$\frac{1}{s-a}$	$Re(s)>a$
5.	$-e^{at}u(-t)$	$\frac{1}{s-a}$	$Re(s)<a$
6.	$A\sin\omega_0 t\, u(t)$	$\frac{A\omega_0}{s^2+\omega_0^2}$	$Re(s)>0$
7.	$A\cos\omega_0 t\, u(t)$	$\frac{As}{s^2+\omega_0^2}$	$Re(s)>0$
8.	$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re(s)>-a$
9.	$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s)>-a$
10.	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$Re(s)>0$
11.	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$Re(s)<0$
12.	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$Re(s)>-a$
13.	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$Re(s)<-a$

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SCHOOL OF BIO AND CHEMICAL ENGINEERING

DEPARTMENT OF BIOMEDICAL ENGINEERING

UNIT – III – Linear Time Invarient Continuous Time Systems – SBMA1304

LINEAR TIME INVARIENT CONTINUOUS TIME SYSTEMS

3.1. FOURIER METHOD FOR ANALYSIS

consider the convolution,

$$y(t) = x(t) * h(t)$$

By convolution property of fourier transform we can write above equation as,

$$\left.\begin{aligned} Y(f) &= X(f)\, H(f) \\ \text{or} \quad Y(\omega) &= X(\omega)\; H(\omega) \end{aligned}\right\}$$

$$\left.\begin{aligned} H(\omega) &= \frac{Y(\omega)}{X(\omega)} \\ \text{or} \quad H(f) &= \frac{Y(f)}{X(f)} \end{aligned}\right\}$$

Here $H(\omega)$ or $H(f)$ represent the frequency response of the LTI-CT system. These functions are also called as system transfer functions.

The impulse response of the continuous time system is given as,

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Determine the frequency response and plot the magnitude phase plots.

Take fourier transform of the given impulse response. i.e.,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} e^{-t/RC}\, u(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t\left(j\omega + \frac{1}{RC}\right)}\, dt$$

$$= \frac{1}{RC} \left(-\frac{1}{j\omega + \frac{1}{RC}}\right) \left[e^{-t\left(j\omega + \frac{1}{RC}\right)}\right]_0^{\infty}$$

$$= \frac{1/RC}{j\omega + 1/RC} = \frac{1}{1 + j\omega RC}$$

Now let us determine the magnitude and phase of $H(\omega)$. Let us rearrange above equation as,

$$H(\omega) = \frac{1}{1+j\omega RC} \times \frac{1-j\omega RC}{1-j\omega RC} = \frac{1-j\omega RC}{1+(\omega RC)^2}$$

$$= \frac{1}{1+(\omega RC)^2} + j\,\frac{-\omega RC}{1+(\omega RC)^2}$$

Thus $H(\omega)$ is expressed into its real and imaginary parts. Now magnitude can be obtained as,

$$|H(\omega)| = \left\{\frac{1}{\left[1+(\omega RC)^2\right]^2} + \frac{(\omega RC)^2}{\left[1+(\omega RC)^2\right]^2}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

This is the magnitude response of the given system. And the phase response will be,

$$\angle H(\omega) = tan^{-1}\left\{\frac{(-\omega RC)/\left[1+(\omega RC)^2\right]}{1/\left[1+(\omega RC)^2\right]}\right\}$$

$$= -tan^{-1}(\omega RC)$$

Let $RC = 1$, then magnitude and phase response will be,

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

and $$\angle H(\omega) = -tan^{-1}(\omega)$$

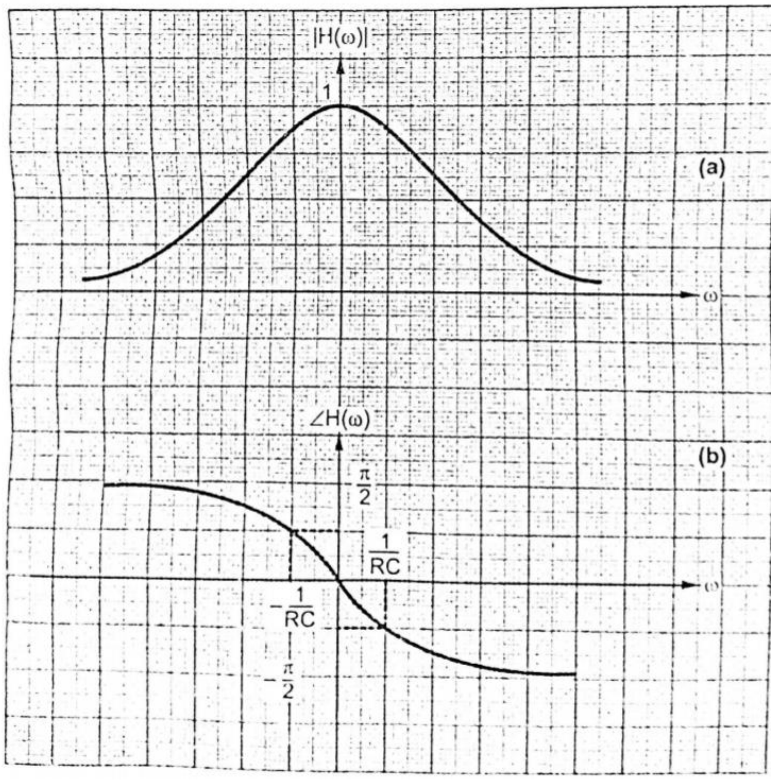


Fig. 1: Magnitude and Phase Response

The system produces the output of $y(t) = e^{-t}u(t)$ for an input of $x(t) = e^{-2t}u(t)$. Determine the impulse response and frequency response of the system.

Solution : Here $y(t) = e^{-t}\ u(t)$

and $x(t) = e^{-2t}\ u(t)$

Consider the standard fourier transform pair,

$$e^{-at}\ u(t) \xleftrightarrow{FT} \frac{1}{a + j2\pi f}$$

Hence fourier transforms of $y(t)$ and $x(t)$ will be,

$$Y(t) = \frac{1}{1 + j2\pi f}$$

and $$X(f) = \frac{1}{2 + j2\pi f}$$

From equation 2.11.3 we can obtain the transfer function as,

$$H(f) = \frac{Y(f)}{X(f)}$$

Putting the values of $X(f)$ and $Y(f)$,

$$H(f) = \frac{1/(1 + j2\pi f)}{1/(2 + j2\pi f)}$$

Let us multiply the numerator and denominator by $1-j2\pi f$. i.e.,

$$H(f) = \frac{2+j2\pi f}{1+j2\pi f} \times \frac{1-j2\pi f}{1-j2\pi f}$$

$$= \frac{2+(2\pi f)^2}{1+(2\pi f)^2} + j\,\frac{-2\pi f}{1+(2\pi f)^2}$$

Hence magnitude of $H(f)$ will be,

$$|H(f)| = \left\{\left[\frac{2+(2\pi f)^2}{1+(2\pi f)^2}\right]^2 + \left[\frac{(2\pi f)}{1+(2\pi f)^2}\right]^2\right\}^{\frac{1}{2}}$$

Simplifying the above equation we get,

$$|H(f)| = \sqrt{\frac{4+(2\pi f)^2}{1+(2\pi f)^2}}$$

This is the magnitude response of the system. And the phase response will be,

$$\angle H(f) = tan^{-1}\,\frac{(-2\pi f)/\left[1+(2\pi f)^2\right]}{\left[2+(2\pi f)^2\right]/\left[1+(2\pi f)^2\right]}$$

$$= -tan^{-1}\left\{\frac{2\pi f}{2+(2\pi f)^2}\right\}$$

Now consider the transfer function of equation 2.11.8. i.e.,

$$H(f) = \frac{2+j2\pi f}{1+j2\pi f}$$

Let us rearrange the above equation as,

$$H(f) = \frac{1+j2\pi f+1}{1+j2\pi f}$$

$$= 1 + \frac{1}{1+j2\pi f}$$

Using the fourier transform pair of equation we get,

$$h(t) = IFT\{H(f)\}$$

$$= \delta(t) + e^{-t}\, u(t)$$

This is the impulse response of the given system.

3.2. DIFFERENTIAL EQUATIONS

Now let us see how the differential equation can be represented in fourier (frequency domain). This representation is useful in obtaining frequency response and impulse response of the system.

Consider the differential equation,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

The differentiation property of fourier transform is,

$$\left.\begin{aligned} &\frac{d}{dt} x(t) \xleftrightarrow{FT} j2\pi f\, X(\omega) \\ \text{or} \quad &\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega\, X(\omega) \end{aligned}\right\}$$

Let us apply this property to the differential equation. i.e.,

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

This is the system transfer function. i.e.,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The impulse response as well as frequency response can be obtained from above equation.

The differential equation of the system is given as,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6\,y(t) = -\frac{dx(t)}{dt}$$

Determine the frequency response and impulse response of this system.

The differential equation is,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6\,y(t) = -\frac{dx(t)}{dt}$$

Taking fourier transform of above equation as per equation

$$(j\omega)^2\,Y(\omega) + 5\,(j\omega)\,Y(\omega) + 6\,Y(\omega) = -j\omega\,X(\omega)$$

$$\therefore \qquad Y(\omega)\left\{(j\omega)^2 + 5\,j\omega + 6\right\} = -j\omega\,X(\omega)$$

Hence system transfer function is,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6}$$

Frequency response can be obtained from above by separating the real and imaginary parts. Magnitude and phase are then calculated.

Let us expand above equation in partial fractions,

$$H(\omega) = \frac{-j\omega}{(j\omega+2)(j\omega+3)}$$

$$= \frac{2}{j\omega+2} - \frac{3}{j\omega+3}$$

$$= 2\cdot\frac{1}{2+j\omega} - 3\cdot\frac{1}{3+j\omega}$$

Here we will use the fourier transform pair,

$$e^{-at}\, u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

$$\therefore \quad h(t) = IFT\,\{H(\omega)\}$$

$$= \left[2\, e^{-2t} - 3\, e^{-3t}\right] u(t)$$

This is the impulse response of the system

Impulse response h (t) is given by

$h(t) = e^{-t}\, u(t) + e^{2t}\, u(-t)$ *The system is excited by*

$x(t) = e^{-2t}\, u(t)$

Solution : Output $\quad y(t) = x(t) * h(t)$

& $\quad Y(f) = X(f)\, H(f)$

Here $\quad y(t) = F^{-1}\{Y(f)\}$

By definition of FT,

$$H(f) = F\{h(t)\} = \int_{-\infty}^{\infty} \left[e^{2t}\, u(-t) + e^{-t}\, u(t)\right] e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{0} e^{2t}\, e^{-j2\pi f t}\, dt + \int_{0}^{\infty} e^{-t}\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2-j2\pi f} e^{(2-j2\pi f)t} \Big|_{-\infty}^{0} + \frac{-1}{1+j2\pi f} e^{-(1+j2\pi f)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{2-j2\pi f} + 0 + 0 - \frac{-1}{1+j2\pi f}$$

$$= \frac{1}{2-j2\pi f} + \frac{1}{1+j2\pi f} = \frac{3}{(1+j2\pi f)(2-j2\pi f)}$$

& $X(f) = F\{x(t)\}$

$$= \int_0^{\infty} e^{-2t}\, e^{-j2\pi f t}\, dt = \int_0^{\infty} e^{-(2+j2\pi f)t}\, dt$$

$$= \frac{1}{2+j2\pi f}$$

Now, $Y(f) = H(f)\, X(f)$

$$= \frac{3}{(1+j2\pi f)(2-j2\pi f)(2+j2\pi f)}$$

$$= \frac{a}{1+j2\pi f} + \frac{b}{2-j2\pi f} + \frac{c}{2+j2\pi f}$$

Here, $a = \left. \frac{3}{(2-j2\pi f)(2+j2\pi f)} \right|_{j2\pi f = -1} = 1$

$$b = \left. \frac{3}{(1+j2\pi f)(2-j2\pi f)} \right|_{j2\pi f = 2} = 1/4$$

$$c = \left. \frac{3}{(1+j2\pi f)(2-j2\pi f)} \right|_{j2\pi f = -2} = -3/4$$

$\therefore$ $Y(f) = \frac{1}{1+j2\pi f} + \frac{1/4}{2-j2\pi f} + \frac{-3/4}{2+j2\pi f}$

The inverse fourier transform of

$$F^{-1}\left\{\frac{1/4}{2-j2\pi f}\right\} = F^{-1}\left\{\frac{1/4}{-(-2+j2\pi f)}\right\} = -\frac{1}{4} e^{2t} u(t)$$

$$y(t) = e^{-t} u(t) - \frac{3}{4} e^{-2t} u(t) + \frac{1}{4} e^{2t} u(-t).$$

Solve the differential equation,

$$\frac{dy(t)}{dt} + 5y(t) = x(t)$$

with initial condition $y(0^+) = -2$ *and input* $x(t) = 3\,e^{-2t}\,u(t)$.

Solution : The given differential equation is,

$$\frac{dy(t)}{dt} + 5\,y(t) = x(t)$$

Taking unilateral laplace transform of above equation,

$$sY(s) - y(0^-) + 5\,Y(s) = X(s)$$

The initial condition is given as $y(0^+) = y(0^-) = -2$

Input $x(t) = 3\,e^{-2t}\,u(t)$. The laplace transform of exponential function is,

$$e^{at} \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}$$

$$\therefore \qquad X(s) = \mathcal{L}\{3\,e^{-2t}\} = 3\,\frac{1}{s+2}$$

Putting for $y(0^-)$ and $X(s)$ in equation

$$sY(s) + 2 + 5\,Y(s) = \frac{3}{s+2}$$

$$Y(s)\,\{s+5\} = \frac{3}{s+2} - 2$$

$$\therefore \qquad Y(s) = \frac{3}{(s+2)(s+5)} - \frac{2}{s+5}$$

Expanding $Y(s)$ into partial fractions,

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+5} - \frac{2}{s+5}$$

$$= \frac{1}{s+2} - \frac{3}{s+5}$$

Taking inverse laplace transform of above equation

$$y(t) = e^{-2t}\,u(t) - 3\,e^{-5t}\,u(t)$$

Solve the following differential equation,

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5\,y(t) = 5\,x(t)$$

$$\text{with } y(0^-) = 1 \text{ and } \left.\frac{dy(t)}{dt}\right|_{0^-} = 2$$

And input $x(t) = u(t)$

Solution : The given differential equation is,

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5\,y(t) = 5\,x(t)$$

Taking laplace transform of both sides,

$$\left[s^2\,Y(s) - s\,y(0^-) - \left.\frac{dy(t)}{dt}\right|_{0^-}\right] + 4\left[s\,Y(s) - y(0^-)\right] + 5\,Y(s) = 5\,X(s)$$

Here $X(s) = \frac{1}{s}$ and putting initial conditions,

$$\left[s^2\,Y(s) - s - 2\right] + 4\left[s\,Y(s) - 1\right] + 5\,Y(s) = \frac{5}{s}$$

$$\therefore \quad Y(s) = \frac{s^2+6s+5}{s(s^2+4s+5)}$$

$$= \frac{s^2+6s+5}{s(s+2+j)(s+2-j)}$$

Expanding above equation in partial fractions,

$$Y(s) = \frac{1}{s} + \frac{-j}{s+2+j} + \frac{j}{s+2-j}$$

$$= \frac{1}{s} + \frac{2}{(s+2)^2+1}$$

$$e^{-\alpha t} \sin \omega t \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega}{(s+\alpha)^2+\omega^2}$$

and

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$$

Taking inverse laplace transform of equation

$$y(t) = u(t) + 2e^{-2t} \sin t$$

This is the required solution.

3.3. SYSTEM TRANSFER FUNCTION

We know that the output of the LTI CT system is given as,

$$y(t) = h(t) * x(t)$$

Taking the laplace transform of above equation,

$$Y(s) = H(s)\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

Here $H(s)$ is called the system transfer function. The impulse response of the system can be obtained by taking inverse laplace transform of $H(s)$.

The system transfer function can also be obtained from the differential equation. Consider the differential equation,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

At the beginning of this section we have seen that e^{st} is the eigen function of the system and eigenvalue is $H(s)$. Hence if $x(t) = e^{st}$, then $y(t) = e^{st}\,H(s)$. Putting these values in the differential equation,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt_k} e^{st}\,H(s) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} e^{st}$$

Here use $\frac{d^k}{dt^k} e^{st} = s^k\, e^{st}$. *Then we get,*

$$\sum_{k=0}^{N} a_k\, s^k\, e^{st}\, H(s) = \sum_{k=0}^{M} b_k\, s^k\, e^{st}$$

$$\therefore \qquad H(s) = \frac{\sum_{k=0}^{M} b_k\, s^k}{\sum_{k=0}^{N} a_k\, s^k}$$

This is the rational form of the system transfer function. Impulse response can be obtained by taking inverse laplace transform.

Find the impulse response of the system given by,

$$\tau_0 \frac{dy(t)}{dt} + y(t) = x(t)$$

$$or\ RC \frac{dy(t)}{dt} + y(t) = x(t)$$

Solution : The given difference equation is,

$$\tau_0 \frac{dy(t)}{dt} + y(t) = x(t)$$

Taking laplace transform of above equation,

$$\tau_0 s\, Y(s) + Y(s) = X(s) \quad \text{assuming zero initial conditions}$$

$$Y(s)\,(1+\tau_0 s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1+\tau_0 s}$$

$$= \frac{1}{\tau_0 \left(s + \frac{1}{\tau_0}\right)}$$

Taking inverse laplace transform of above equation,

$$h(t) = \frac{1}{\tau_0} e^{-t/\tau_0}\, u(t)$$

This is the impulse response of given system.

For the second differential equation, $\tau = RC$, hence impulse response becomes,

$$h(t) = \frac{1}{RC} e^{-t/RC}\, u(t)$$

The input-output relation of a system at initial rest is given by,

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Find the system transfer function, frequency response and impulse response.

Solution : **(i) To obtain system transfer function**

Since the system is at rest initially, we can assume zero initial conditions. Taking laplace transform of the differential equation (with zero initial conditions),

$$s^2 Y(s) + 4s\,Y(s) + 3\,Y(s) = s\,X(s) + 2\,X(s)$$

$$\therefore\ Y(s)\left[s^2 + 4s + 3\right] = X(s)\,[s+2]$$

$$\therefore\quad H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2+4s+3}$$

This is the rational form of system transfer function. It can be directly obtained using equation 2.12.25 also.

(ii) To obtain frequency response

We know that $s = \sigma + j\omega$ Hence frequency response can be obtained by putting $s = j\omega$ in equation 2.12.26. i.e.,

$$H(\omega) = H(s)\big|_{s=j\omega} = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

(iii) To obtain impulse response

Consider the system transfer function,

$$H(s) = \frac{s+2}{s^2+4s+3} = \frac{s+2}{(s+3)\,(s+1)}$$

Expressing this equation into partial fractions,

$$H(s) = \frac{1/2}{s+3} + \frac{1/2}{s+1}$$

Taking inverse laplace transform of above,

$$h(t) = \left[\frac{1}{2}\,e^{-3t} + \frac{1}{2}\,e^{-t}\right] u(t)$$

Determine the transfer function of the system described by the differential equation,

$$\frac{d^2y(t)}{dt^2} + 6\,\frac{dy(t)}{dt} + 9\,y(t) = \frac{d^2x(t)}{dt^2} + 3\,\frac{dx(t)}{dt} + 2x(t)$$

Solution : Assuming zero initial conditions and taking laplace transform of the given differential equation,

$$s^2 Y(s)+6s\, Y(s)+9\, Y(s) = s^2 X(s)+3\, sX(s)+2\, X(s)$$

$$\therefore\ Y(s)\left[s^2+6s+9\right] = X(s)\left[s^2+3s+2\right]$$

$$\therefore \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{s^2+3s+2}{s^2+6s+9}$$

This is the required system transfer function.

The system transfer function is given as,

$$H(s) = \frac{s}{s^2+5\,s+6}$$

The input to the system is $x(t) = e^{-t}\, u(t)$. Determine the output assuming zero initial conditions.

Solution : The input is $x(t) = e^{-t}\, u(t)$

Hence laplace transform of the input will be,

$$X(s) = \frac{1}{s+1}$$

The output of the system is given as,

$$y(t) = h(t) * x(t)$$

Taking laplace transform of the above equation,

$$Y(s) = H(s)\, X(s)$$

$$= \frac{s}{s^2+5\,s+6} \times \frac{1}{s+1}$$

$$= \frac{s}{(s+2)(s+3)} \times \frac{1}{s+1}$$

Expressing the above equation into partial fractions,

$$Y(s) = \frac{2}{s+2} - \frac{3/2}{s+3} - \frac{1/2}{s+1}$$

Taking inverse laplace transform of above equation,

$$y(t) = \left[2\,e^{-2t} - \frac{3}{2}\,e^{-3t} - \frac{1}{2}\,e^{-t}\right] u(t)$$

3.4. PROPERTIES OF CONVOLUTION

1. Commutative property of convolution :

This property states that convolution is commutative operation. The convolution is given as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Here put $t-\tau = m$, then $d\tau = -dm$

When $\tau = \infty,\ m = -\infty$

and $\tau = -\infty,\ m = \infty$

$$\therefore \quad y(t) = -\int_{\infty}^{-\infty} x(t-m)\, h(m)\, dm$$

$$= \int_{-\infty}^{\infty} x(t-m)\, h(m)\, dm$$

$$= \int_{-\infty}^{\infty} h(m)\, x(t-m)\, dm$$

Since 'm' is just a variable we can write τ in the above equation,

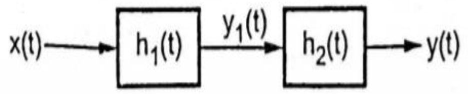
$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= h(t) * x(t)$$

Thus $\quad y(t) = x(t) * h(t) = h(t) * x(t)$

2. Associative property of convolution :

Consider the cascade connection of the two systems as shown in Fig.



Cascade connection of two LTI systems

The output $y(t)$ of the second system can be given as,

$$y(t) = y_1(t) * h_2(t)$$

$$= \int_{-\infty}^{\infty} y_1(\tau)\, h_2(t-\tau)\, d\tau$$

The output of first system is $y_1(t)$. It can be given as,

$$y_1(\tau) = x(\tau) * h_1(\tau)$$

$$= \int_{-\infty}^{\infty} x(m)\, h_1(\tau - m)\, dm$$

Here separate variables τ and m are used. Putting above equation for $y_1(\tau)$ in equation 2.9.20.

$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(m)\, h_1(\tau - m) h_2(t - \tau)\, dm\, d\tau$$

Here put $\tau - m = n$, then we get

$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(m)\, h_1(n) \cdot h_2(t - m - n)\, dm\, dn$$

$$= \int_{-\infty}^{\infty} x(m) \left[\int_{-\infty}^{\infty} h_1(n) \cdot h_2((t-m) - n)\, dn \right] dm$$

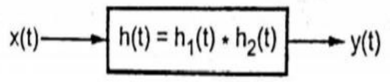
The integration in square brackets indicate convolution of $h_1(t)$ and $h_2(t)$ evaluated at t – m. i.e.,

$$\int_{-\infty}^{\infty} h_1(n)\, h_2((t-m)-n)\, dn = h(t-m)$$

Putting this value in equation

$$y(t) = \int_{-\infty}^{\infty} x(m)\, h(t-m)\, dm$$

$$= x(t) * h(t)$$



Equivalent of cascade connection of Fig. 2.9.21

Thus if the two systems are connected in cascade, the overall impulse response is equal to convolution of two impulse responses. This is shown in Fig.

We know that,

$$y_1(t) = x(t) * h_1(t)$$

and

$$y(t) = y_1(t) * h_2(t)$$

Putting for $y_1(t)$ in above equation,

$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

And from Fig. we can write,

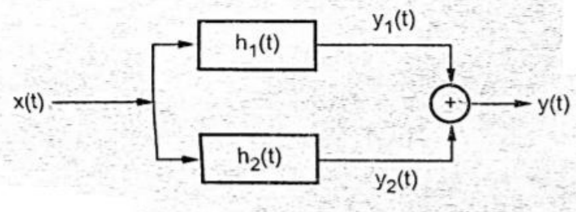
$$y(t) = x(t) * h(t)$$

$$= x(t) * [h_1(t) * h_2(t)]$$

Thus equation and above equation prove associative property. i.e.,

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

3. Distributive property of convolution :

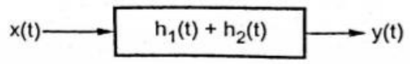


Parallel connection of the systems

Consider the two systems connected in parallel as shown in Fig.

The overall output is,

$$
\begin{aligned}
y(t) &= y_1(t) + y_2(t) \\
&= x(t) * h_1(t) + x(t) * h_2(t) \\
&= \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x(\tau)\, h_2(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\, \{h_1(t-\tau) + h_2(t-\tau)\}\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \\
&= x(t) * h(t)
\end{aligned}
$$

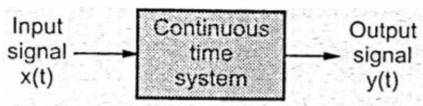


Here $h(t) = h_1(t) + h_2(t)$. Thus impulse responses of the parallel connected systems are added. i.e.

This proves the distributive property which can be stated as,

$$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}$$

Derivation of Convolution Integral



Continuous time system

Let us consider the continuous time system as shown in Fig. Let the signal x(t) be applied to the system. Consider that the system produces output y(t). This input and output is related as,

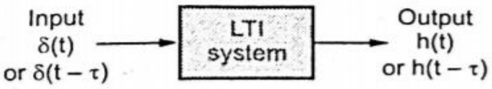
$$y(t) = T\{x(t)\}$$

Putting for x(t) from equation 2.9.1

$$y(t) = T\left\{\int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau\right\}$$

Here $x(\tau)$ is the amplitude of x(t) at $t = \tau$. It is constant. Since the system is linear we can write above equation as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, T[\delta(t-\tau)]\, d\tau$$



LTI system produces unit impulse response if δ(*t*) is input

If unit impulse is applied as input to the system, it produces unit *impulse response*. This is shown in Fig. The unit impulse response is denoted by h(t). Since the system is time invariant, the response to $\delta(t-\tau)$ will be $h(t-\tau)$. i.e.,

$$T[\delta(t-\tau)] = h(t-\tau)$$

Thus if unit impulse is delayed by τ, the impulse response is also delayed by the same amount. With the above result equation becoems,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

This equation relates the impulse response to the output. The output y(t) is equal to convolution of x(t) and h(t). Hence above equation is also called convolution integral. i.e.

$$\textit{Convolution integral}: y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The convolution of above equation can also be represented symbolically as,

$$y(t) = x(t) * h(t)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The impulse response of the LTI system is $h(t) = u(t)$. Determine the output of the system if input $x(t) = e^{-at} u(t)$, $a > 0$.

Solution : The given functions are,

Impulse response, $h(t) = u(t)$

Input, $x(t) = e^{-at} u(t), \quad a > 0$

Then output of the system will be equal to convolution of $x(t)$ and $h(t)$. i.e.,

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Fig shows the plots of $x(\tau)$ and $h(\tau)$.

Now let us determine y(t) at t = 0. Hence putting t = 0 in equation we get,

$$y(0) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau)$$

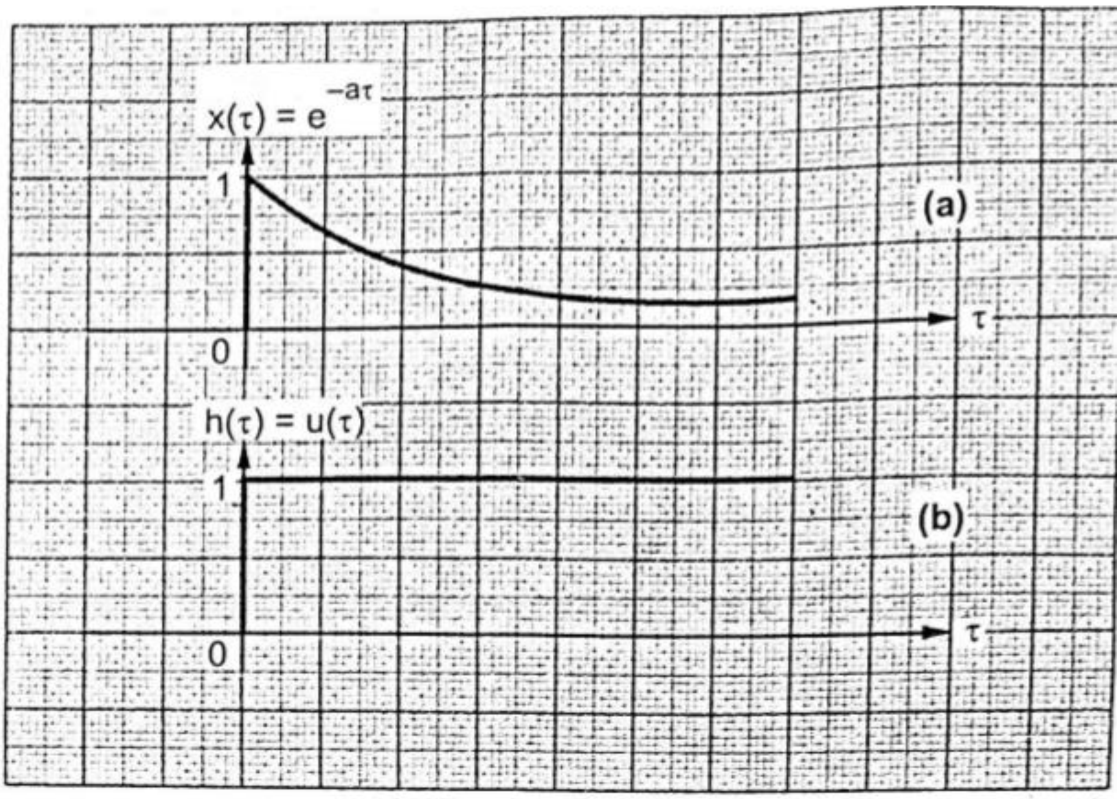


Fig. 2: Exponential signal

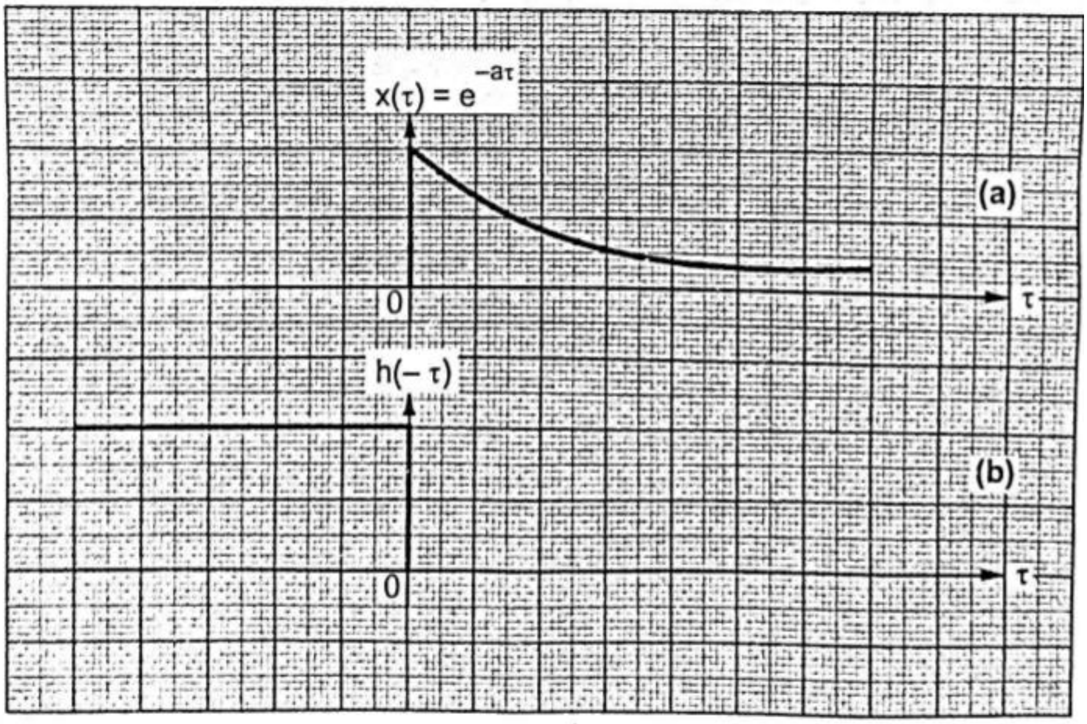


Fig. 3: Exponential signal

Now let us consider that $t > 0$. Then as per equation $y(t)$ will be,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad t > 0$$

$$y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau \qquad \text{for } t > 0$$

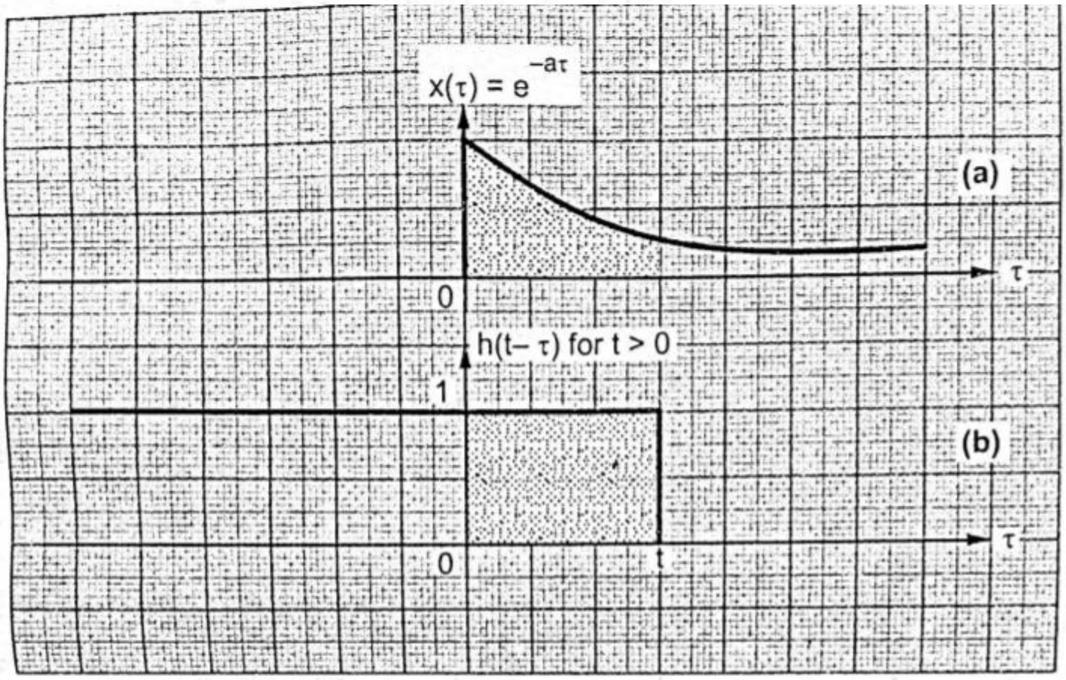


Fig. 4: Exponential signal

From Fig. it is clear that $h(t-\tau) = u(t-\tau) = 1$ from 0 to t. Hence above equation becomes,

$$y(t) = \int_{0}^{t} e^{-a\tau} \cdot 1\, d\tau = \int_{0}^{t} e^{-a\tau}\, d\tau$$

$$= \left[-\frac{1}{a} \cdot e^{-a\tau} \right]_{0}^{t}$$

$$= \frac{1}{a}\left(1 - e^{-at}\right) \qquad \textit{for } t > 0$$

Note that above result holds for t > 0.

Now let us consider that t < 0. The convolution equation will be,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau, \qquad t < 0$$

The plot of $h(t-\tau)$ for t < 0 is shown in Fig. below. In this figure observe that $h(t-\tau)$ is obtained gy shifting $h(-\tau)$ by t towards left.

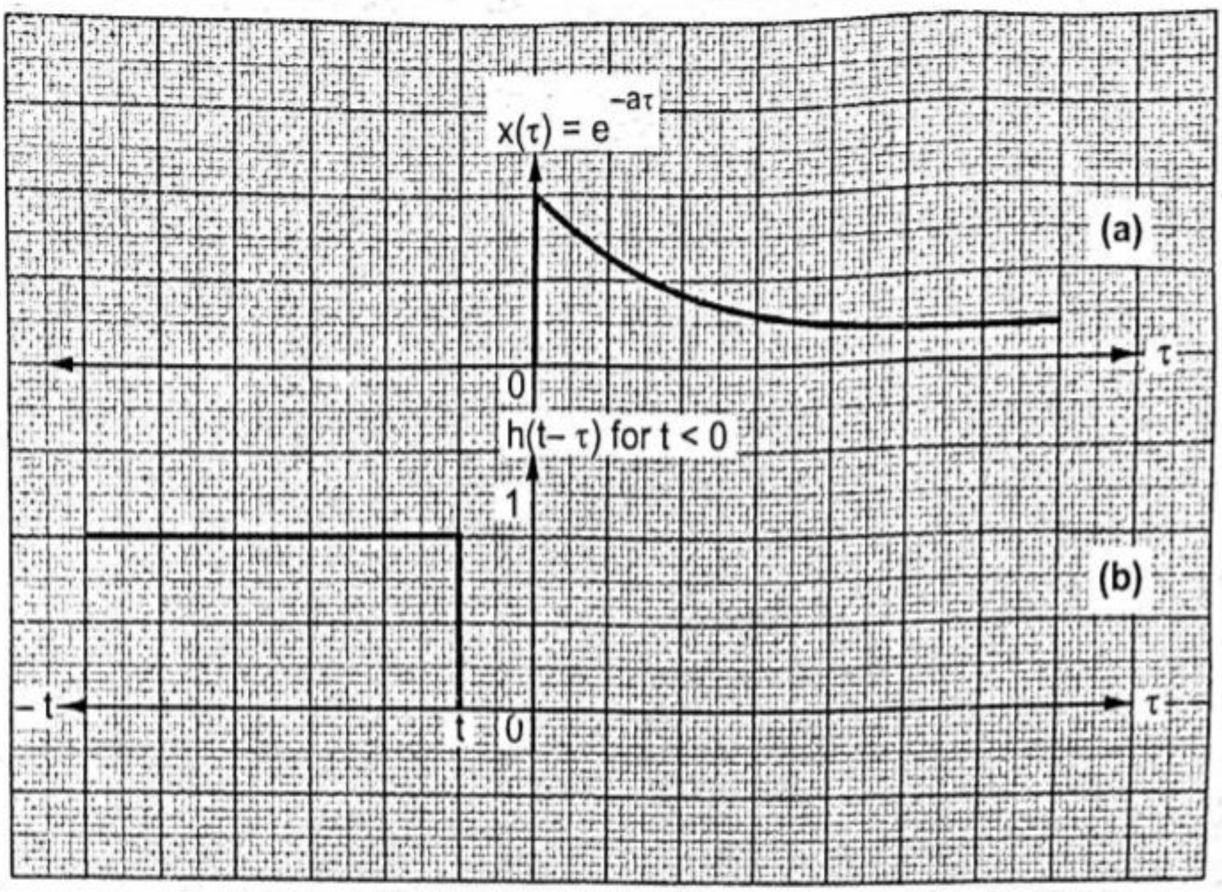


Fig. 5: Exponential signal

For t > 0, $h(-\tau)$ is shifted right and

For t < 0, $h(-\tau)$ is shifted left.

The product of $x(\tau)$ and $h(t-\tau)$ will be zero since there is no overlap between them. Hence equation will be zero. i.e.,

$$y(t) = 0 \quad \text{for } t < 0$$

Thus we obtained the output $y(t)$ as,

$$y(t) = \begin{cases} \frac{1}{a}\left(1-e^{-at}\right) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

The impulse response and the input to the system is given as,

$x(t) = u(t+1)$

$h(t) = u(t-2)$ *Determine the output of the system.*

Solution : The output $y(t)$ is given as the convolution of $x(t)$ and $h(t)$. i.e.,

$$y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The two functions $x(\tau)$ and $h(\tau)$ will be,

$$x(\tau) = u(\tau+1) = 1 \quad \text{for } \tau \geq -1$$

and $$h(\tau) = u(\tau-2) = 1 \quad \text{for } \tau \geq 2$$

These two functions are plotted in Fig. below. In equation we need the function $h(t-\tau)$. We know that

$$h(\tau) = u(\tau-2)$$

$$\therefore \quad h(\tau) = 1 \qquad \textit{for } \tau - 2 \geq 0 \;\; \textit{or } \tau \geq 2$$

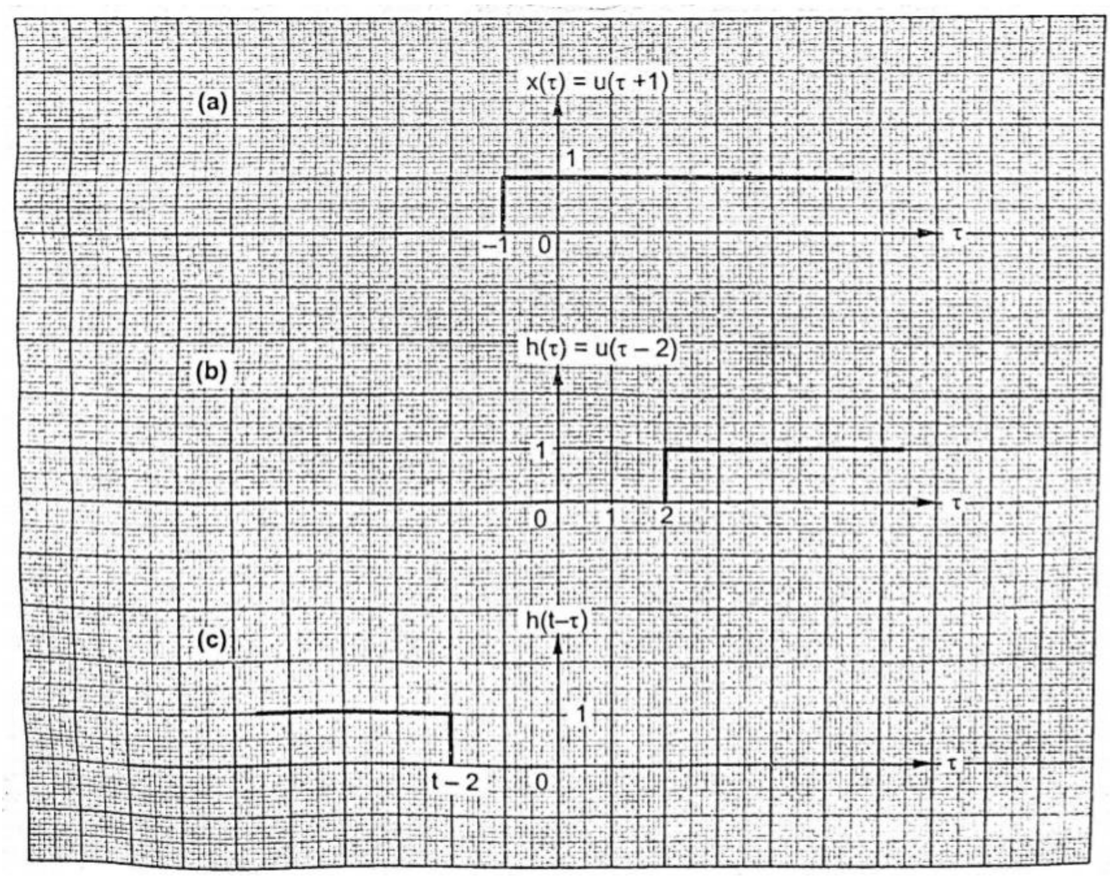


Fig. 6: Convolution integral

This waveform is shown in Fig.   (b) above. From the above equation we can write,

$$h(t-\tau) = 1 \qquad \text{for } t-\tau \geq 2$$

Here $t-\tau \geq 2$ can also be written as $t-2 \geq \tau$ or $\tau \leq t-2$. Thus

$$h(t-\tau) = 1 \qquad \text{for } \tau \leq t-2$$

This waveform is shown in Fig.   (c) above.

As per equation   the product $x(\tau)\,h(t-\tau)$ is integrated for every value of t. The variation of t shifts $h(t-\tau)$. Fig.   shows the plot of $h(t-\tau)$ for arbitrary value of t. In this figure observe that $h(t-\tau) = 1$ for $\tau \leq t-2$ as per equation

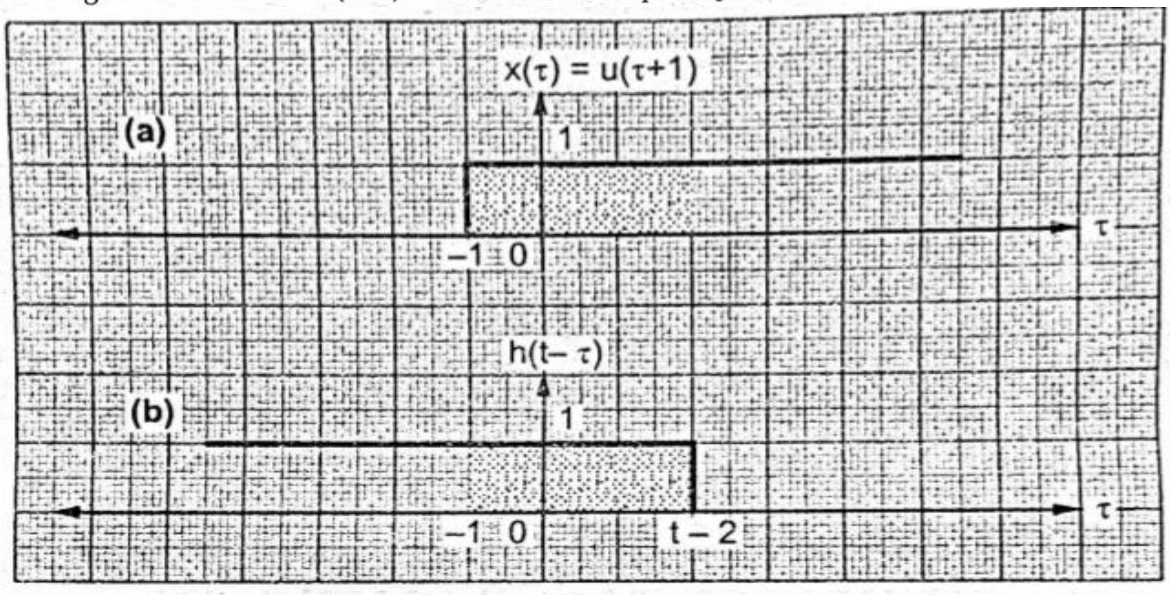


Fig. 7: Convolution integral

From the above figure we have three separate regions as follows,

From $-\infty < \tau < -1$, $x(\tau)\, h(t-\tau) = 0$. Since there is no overlap.

From $-1 \le \tau \le t-2$, $x(\tau)\, h(t-\tau) \ne 0$. Since there is overlap.

From $t-2 < \tau < \infty$, $x(\tau)\, h(t-\tau) = 0$. Since there is no overlap.

Based on the above we can write convolution integration

$$y(t) = \int_{-\infty}^{-1} (0 \times 1)\, d\tau + \int_{-1}^{t-2} (1 \times 1)\, d\tau + \int_{t-2}^{\infty} (1 \times 0)\, d\tau$$

$$= \int_{-1}^{t-2} d\tau = [\tau]_{-1}^{t-2}$$

$$= t-1$$

Thus the result of convolution of the two functions is,

$$y(t) = t-1 \qquad \text{for} \quad t-2 \ge -1 \quad \text{i.e. } t \ge 1$$

This function can also be written as,

$$y(t) = t-1\ u(t-1)$$

Solution : Here the input and impulse response are

$$x(t) = e^{-3t}\{u(t)-u(t-2)\}$$

Here $u(t)-u(t-2)$ has the value of 1 from $t=0$ to 2. Hence above equation can be written as,

$$x(t) = e^{-3t} \quad for \;\; 0 \le t \le 2$$

Similarly impulse response is,

$$h(t) = e^{-t} \quad for \;\; t \ge 0$$

The output of the circuit can be obtained by convolution of $x(t)$ and $h(t)$. From equation we can write convolution as,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Let us write $x(t)$ and $h(t)$ in terms of τ. i.e.,

$$x(\tau) = e^{-3\tau} \quad for \; 0 \le \tau \le 2$$

and

$$h(\tau) = e^{-\tau} \quad for \; \tau \ge 0$$

The plots of $x(\tau)$ and $h(\tau)$ are shown in Fig.

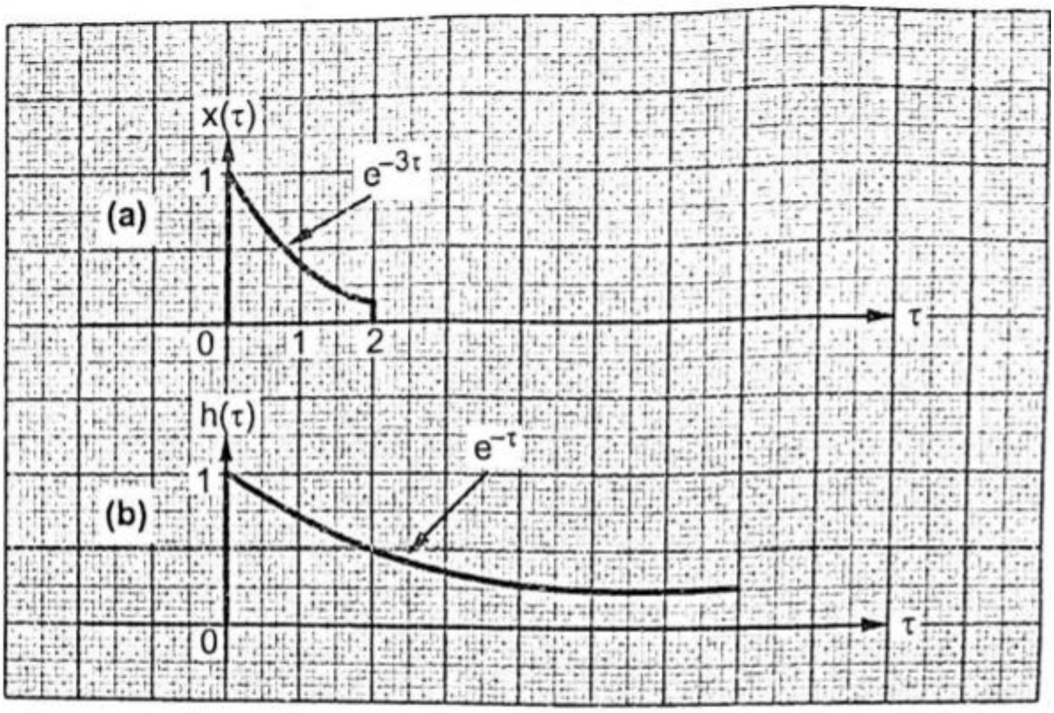


Fig. 8: Convolution integral

$$h(t-\tau) = e^{-(t-\tau)} \qquad for\ t-\tau \geq 0$$

$$h(t-\tau) = e^{-(t-\tau)} \qquad for\ \tau \leq t$$

$$y(t) = \int_{-\infty}^{0} 0 \times h(t-\tau)\, d\tau + \int_{0}^{t} x(\tau) \cdot h(t-\tau)\, d\tau + \int_{t}^{\infty} x(\tau) \times 0\, d\tau$$

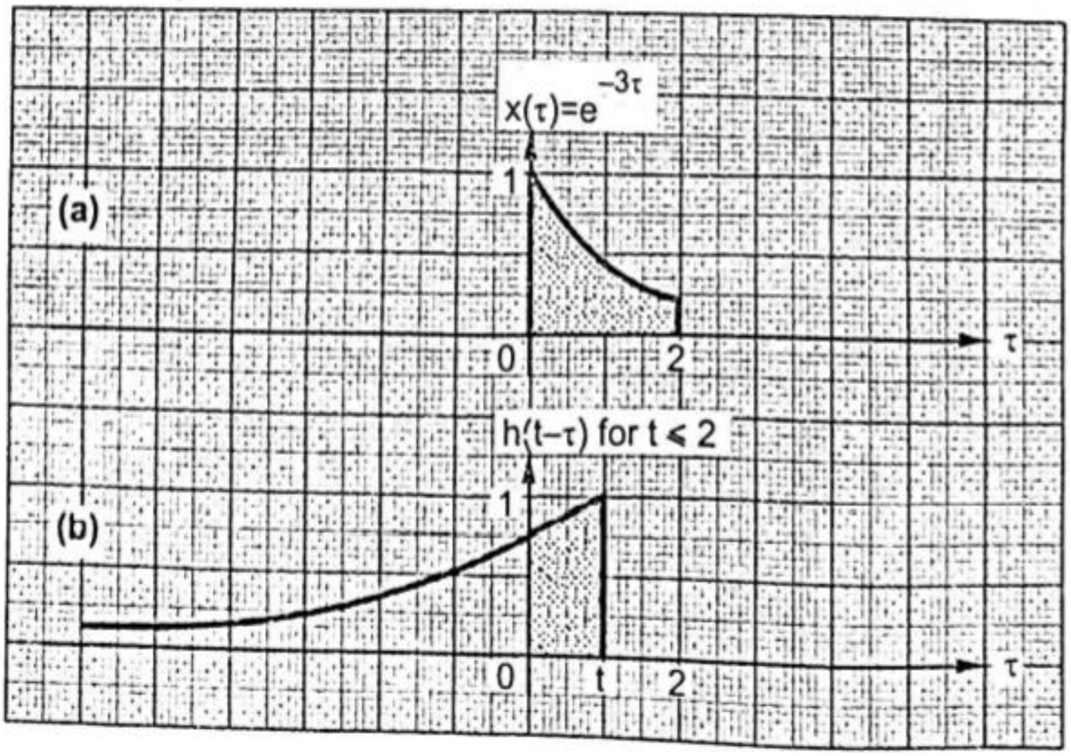


Fig. 9: Convolution integral

$$= \int_0^t x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_0^t e^{-3\tau}\cdot e^{-(t-\tau)}\,d\tau$$

$$= \int_0^t e^{-3\tau}\cdot e^{-t}\cdot e^{\tau}\,d\tau$$

$$= e^{-t}\int_0^t e^{-2\tau}\,d\tau$$

$$= e^{-t}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t$$

$$= \frac{1}{2}\left(1-e^{-2t}\right)e^{-t} \qquad \textit{for } 0\le t\le 2$$

Now consider the case when $t > 2$.

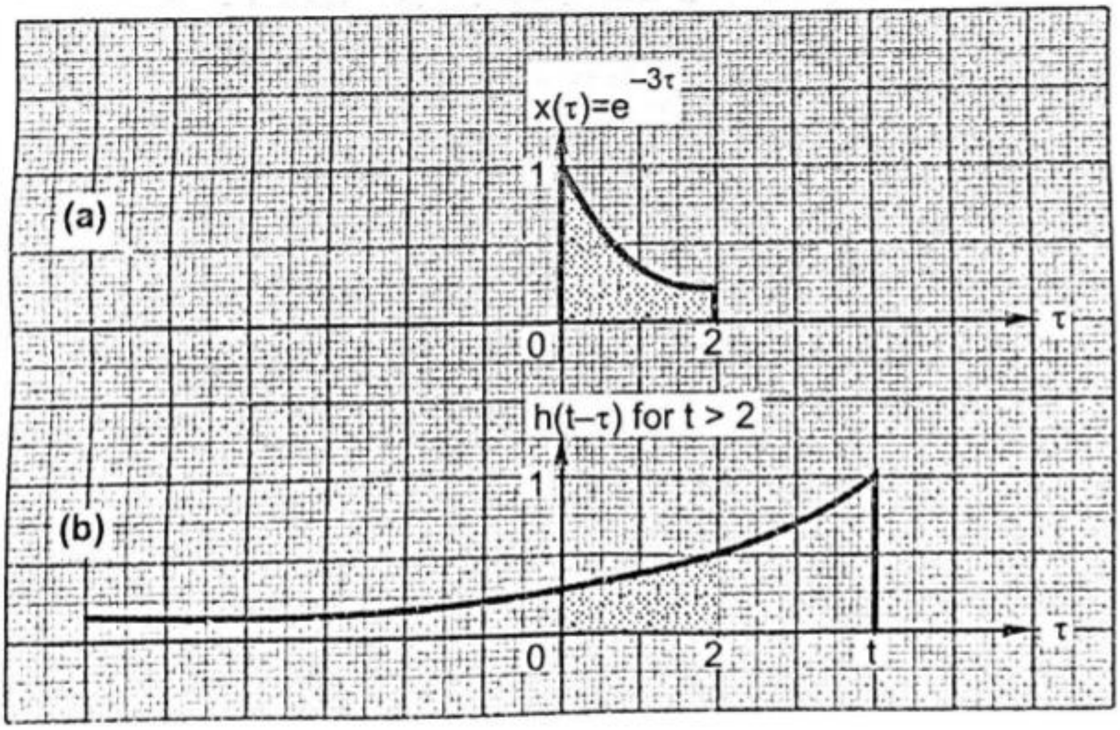


Fig. 10: Convolution integral

Hence we can write the convolution equation as,

$$y(t) = \int_{-\infty}^{0} 0 \times h(t-\tau)\, d\tau + \int_{0}^{2} x(\tau)\cdot h(t-\tau)\, d\tau + \int_{2}^{t} 0 \times h(t-\tau)\, d\tau$$

$$= \int_{0}^{2} x(\tau)\cdot h(t-\tau)\, d\tau$$

$$= \int_{0}^{2} e^{-3\tau}\cdot e^{-(t-\tau)}\, d\tau$$

$$= \int_{0}^{2} e^{-3\tau}\cdot e^{-t}\cdot e^{\tau}\, d\tau$$

$$= e^{-t}\int_{0}^{2} e^{-2\tau}\, d\tau$$

$$= e^{-t}\left[-\frac{1}{2}\cdot e^{-2\tau}\right]_{0}^{2}$$

$$= \frac{1}{2}\left(1-e^{-4}\right)e^{-t} \qquad \textit{for } t > 2$$

Thus we obtained convoluton as,

$$y(t) = \begin{cases} 0 & \text{for } t<2 \\ \frac{1}{2}\left(1-e^{-2t}\right)e^{-t} & \text{for } 0 \le t \le 2 \\ \frac{1}{2}\left(1-e^{-4}\right)e^{-t} & \text{for } t>2 \end{cases}$$

Obtain the convolution of two functions given below.

$$x(t) = \begin{cases} 2 & \text{for } -2 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(t) = \begin{cases} 4 & \text{for } 0 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution : The two functions are plotted in Fig. Observe that $x(t)$ is a pulse of amplitude 2 from − 2 to 2. And $h(t)$ is the pulse of amplitude 4 from 0 to 2.

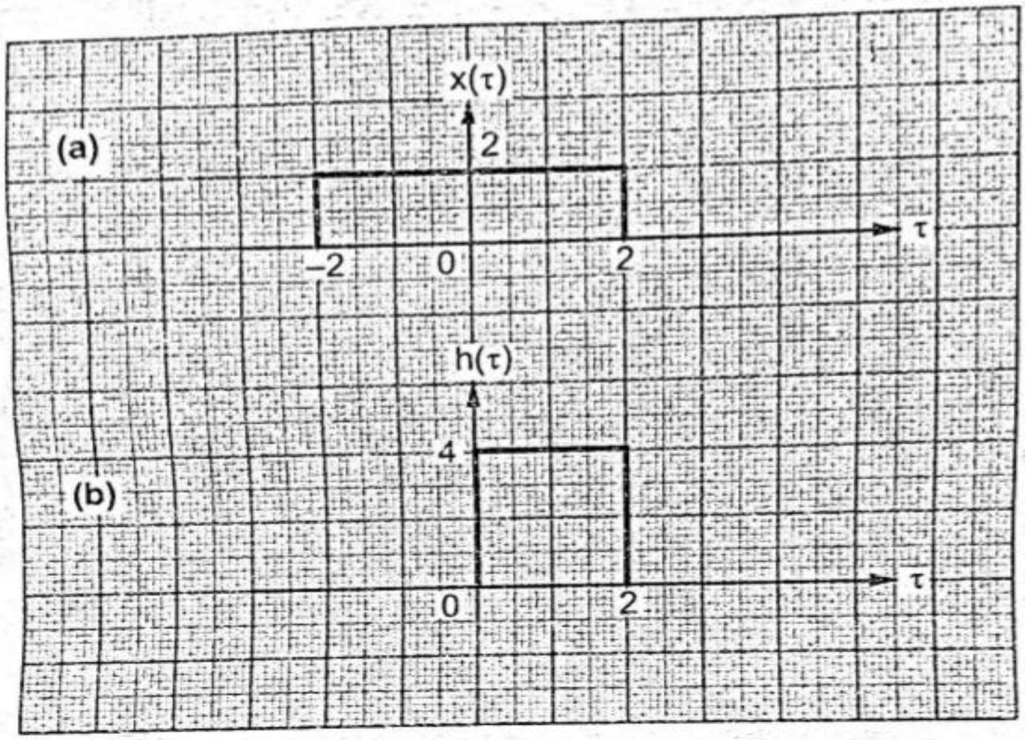


Fig. 11: Convolution integral

The convolution is given as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Here we require $h(t-\tau)$. Hence we can write from equation

$$h(t-\tau) = \begin{cases} 4 & \text{for } 0 \le t-\tau \le 2 \\ & \text{i.e. } -t \le -\tau \le -t+2 \\ & \text{i.e. } t-2 \le \tau \le t \\ 0 & \text{elsewhere} \end{cases}$$

Fig. shows the plots of $x(\tau)$ and $h(t-\tau)$ as per above equation.

Now the next step is to actually evaluate convolution integral.

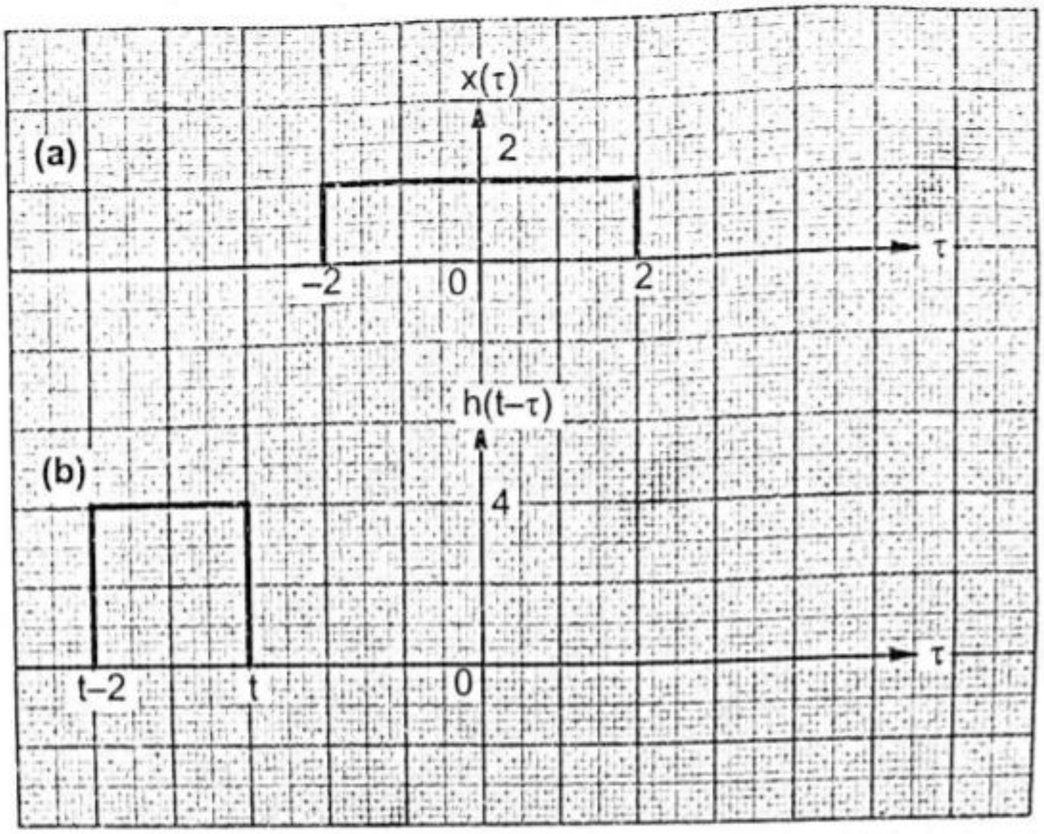


Fig. 12: Convolution integral

(i) To evaluate $y(t)$ for t < − 2

In the above figure observe that t < − 2. Hence there is no overlap between $x(\tau)$ and $h(t-\tau)$. Therefore the product $x(\tau)\ h(t-\tau)$ will be always zero.

$$\therefore \qquad y(t) = 0 \qquad for\ t < -2$$

(ii) To evaluate $y(t)$ for $-2 < t \le 0$

Fig.   shows the waveforms of $x(\tau)$ and $h(t-\tau)$ for $-2 \le t < 0$. The overlap of $x(\tau)$ and $h(t-\tau)$ starts from t = – 2. For $-2 \le t < 0$, both the pulses partially overlap. The shaded area shows the overlap of $x(\tau)$ and $h(t-\tau)$.

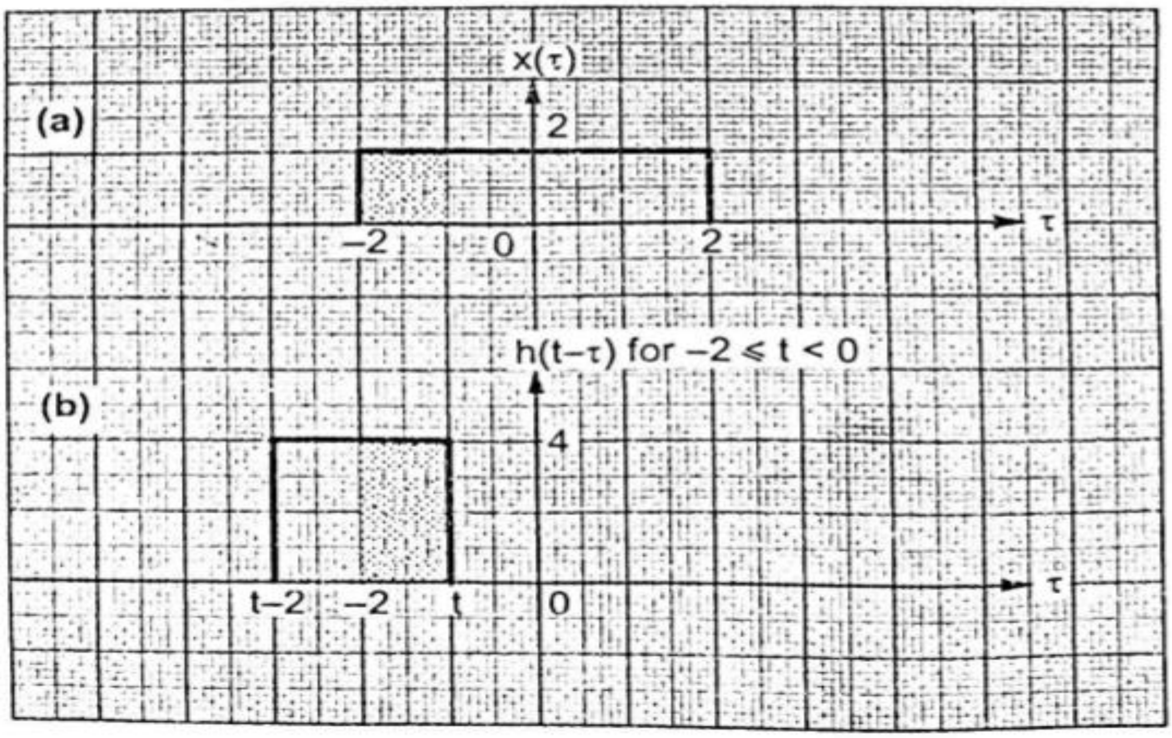


Fig. 13: Convolution integral

The convolution can be written as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{-2} 0 \times h(t-\tau)\, d\tau + \int_{-2}^{t} x(\tau) \cdot h(t-\tau)\, d\tau + \int_{t}^{\infty} x(\tau) \times 0\, d\tau$$

$$= \int_{-2}^{t} x(\tau) \cdot h(t-\tau) d\tau$$

$$= \int_{-2}^{t} 2 \times 4\, d\tau$$

$$= 8 \int_{-2}^{t} d\tau$$

$$= 8\, [\tau]_{-2}^{t}$$

$$= 8\,(t+2) \qquad \textit{for } -2 \le t < 0$$

(iii) To evaluate $y(t)$ for $0 \le t \le 2$

Fig. : shows the waveforms of $x(\tau)$ and $h(t-\tau)$ for $0 \le t \le 2$. For this range of t, both the pulses fully overlap each other. The shaded area shows the overlap of the pulses.

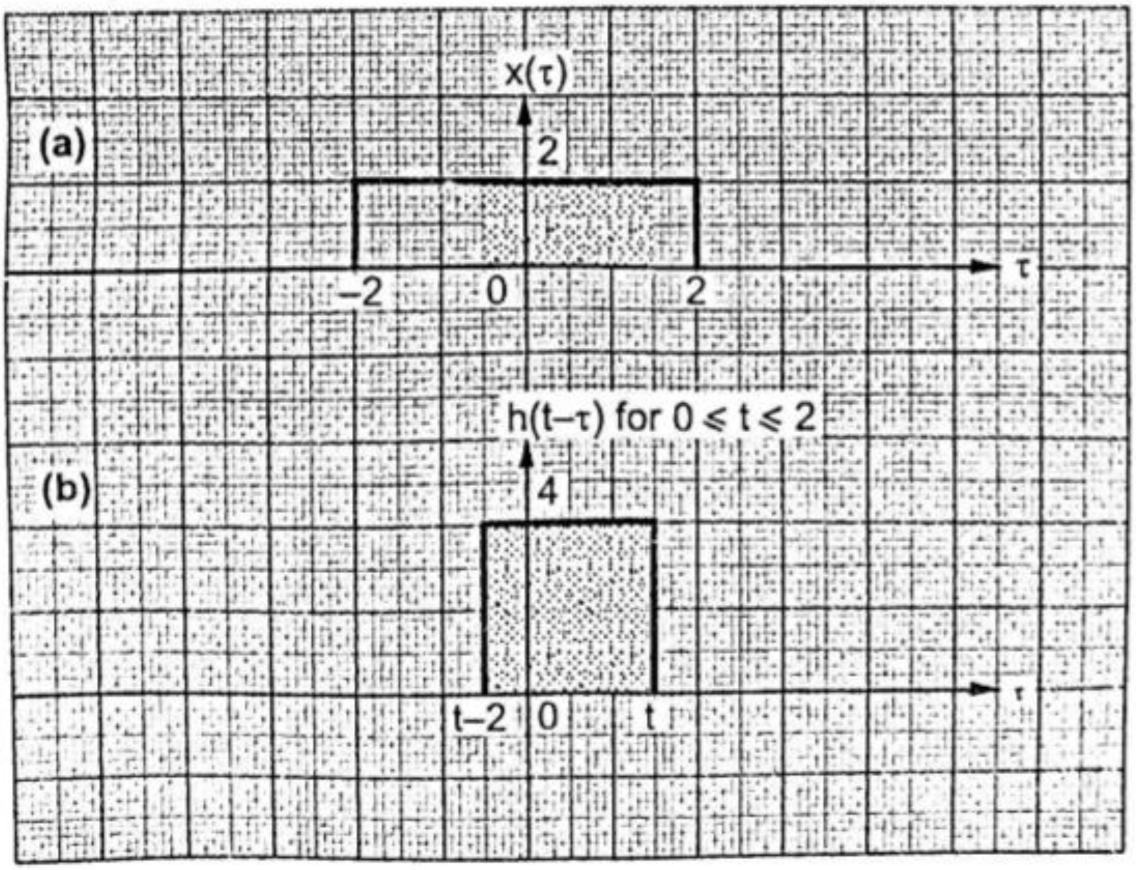


Fig. 14: Convolution integral

We can write convolution for this range as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t-2} x(\tau) \times 0\, d\tau + \int_{t-2}^{t} x(\tau) \cdot h(t-\tau)\, d\tau + \int_{t}^{\infty} x(\tau) \times 0\, d\tau$$

$$= \int_{t-2}^{t} x(\tau) \cdot h(t-\tau) d\tau$$

$$= \int_{t-2}^{t} 2 \times 4\, d\tau = \int_{t-2}^{t} 8\, d\tau$$

$$= 8\, [\tau]_{t-2}^{t} = 8[t-t+2] = 16$$

$$y(t) = 16 \qquad for\ 0 \le t \le 2$$

(iv) To evaluate $y(t)$ for $2 \leq t \leq 4$

The plots of $x(\tau)$ and $h(t-\tau)$ are shown in Fig. for $2 \leq t \leq 4$. For this range $x(\tau)$ and $h(t-\tau)$ partially overlap each other. The shaded region shows the overlap in figure.

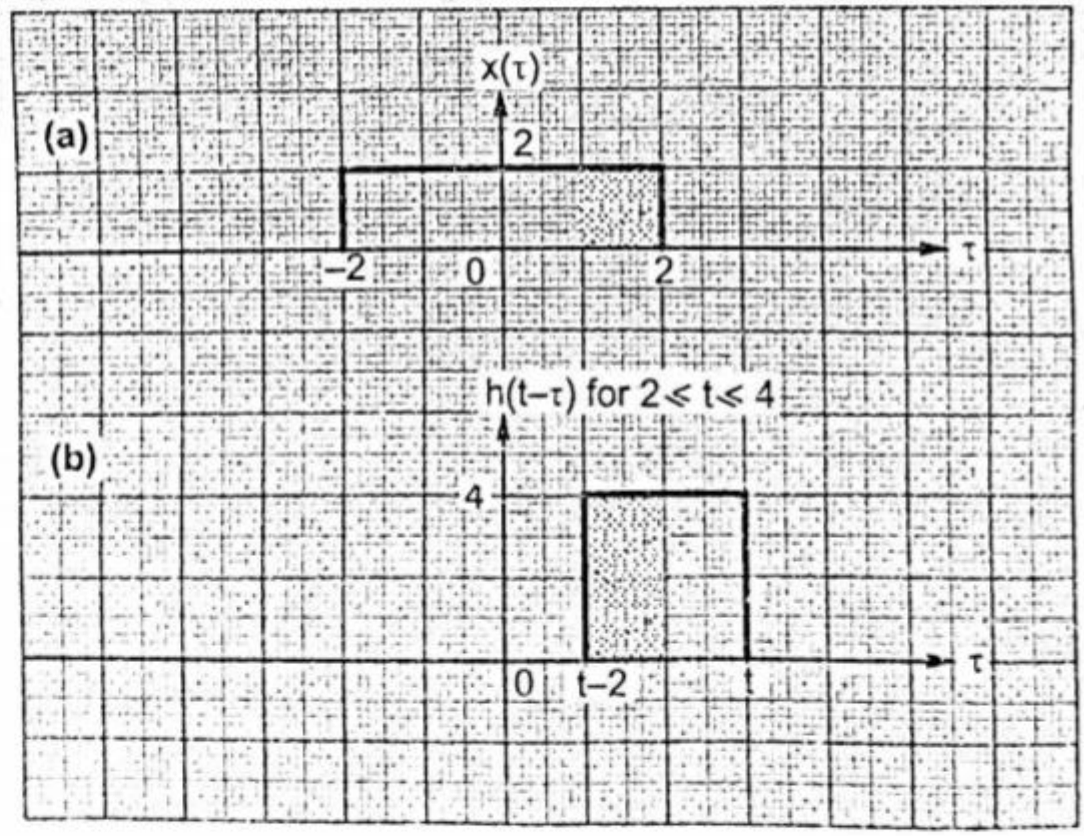


Fig. 15: Convolution integral

For this range we can write the convolution as,

$$y(t) = \int_{-\infty}^{t-2} x(\tau) \times 0 \, d\tau + \int_{t-2}^{2} x(\tau) \cdot h(t-\tau) \, d\tau + \int_{2}^{\infty} 0 \times h(t-\tau) \, d\tau$$

$$= \int_{t-2}^{2} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-2}^{t} 2 \times 4\, d\tau = \int_{t-2}^{2} 8\, d\tau$$

$$= 8 \int_{t-2}^{2} d\tau = 8\, [\tau]_{t-2}^{t}$$

$$= 8\,(4-t) \qquad \text{for } 2 \le t \le 4$$

For t > 4, there will be no overlap between $x(\tau)$ and $h(\tau)$ and result of convolution will be zero. i.e.,

$$y(t) = 0 \qquad \text{for } t > 4$$

Let us write all the values of $y(t)$ combinely,

$$y(t) = \begin{cases} 0 & \text{for } t < -2 \\ 8(t+2) & \text{for } -2 \le t < 0 \\ 16 & \text{for } 0 \le t < 2 \\ 8(4-t) & \text{for } 2 \le t \le 4 \\ 0 & \text{for } t > 4 \end{cases}$$

This is the result of convolution.

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SCHOOL OF BIO AND CHEMICAL ENGINEERING

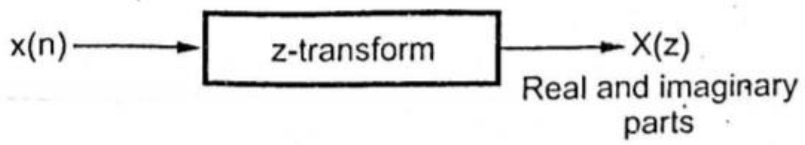
DEPARTMENT OF BIOMEDICAL ENGINEERING

UNIT – IV – Analysis of Discrete Time Signals – SBMA1304

ANALYSIS OF DISCRETE TIME SIGNALS

4.1. Z TRANSFORM

For any input sequence, the z-transform is complex. It has real and imaginary parts.



It can be used for

i) Analysis of DT signals and systems
ii) Digital filter design
iii) Digital filter/systems synthesis

4.2. Definition

The z-transform of x(n) is denoted by X(z). It is defined as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Here z-is complex variable. x(n) and X(z) is called z-transform pair. It is represented as,

z-transform pair : $x(n) \overset{z}{\longleftrightarrow} X(z)$

Unilateral or one sided z-transform : It is defined as,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

It is also known as Bilateral Z transform

4.3 Region of Convergence

Definition : ROC is the region where z-transform converges. From definition, it is clear that z-transform is an infinite power series. This series is not convergent all values of z. Hence ROC is useful in mentioning z-transform.

i) ROC gives an idea about values of z for which z-transform can be calculated.

ii) ROC can be used to determine causality of the system.

iii) ROC can be used to determine stability of the system.

4.4. PROBLEMS

Determine z-transform of following sequences

i) $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

ii) $x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

$$x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$$

i.e. $x_1(0) = 1$, $x_1(1) = 2$, $x_1(2) = 3$, $x_1(3) = 4$, $x_1(4) = 5$, $x_1(5) = 0$, $x_1(6) = 7$

By definition, $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$\therefore \quad X_1(z) = \sum_{n=0}^{6} x_1(n) z^{-n}$$

Putting for $x_1(n)$, $= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 0z^{-5} + 7z^{-6}$

$$X_1(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + \frac{7}{z^6}$$

Result : i) $X_1(z)$ is as calculated above.

ii) $X_1(z) = \infty$ for $z = 0$, i.e. $X_1(z)$ is convergent for all values of z, except $z = 0$.

iii) Hence ROC : Entire z-plane except $z = 0$.

ii) $x_2(n) = \{1, 2, 3, 4, 5, 0, 7\}$
$\qquad\qquad\qquad\quad\uparrow$

i.e. $x_2(0) = 4,\ x_2(1) = 5,\ x_2(2) = 0,\ x_2(3) = 7$ *and*

$$x_2(-1) = 3,\ x_2(-2) = 2,\ x_2(-3) = 1$$

$$\therefore \qquad X_2(z) = \sum_{n=-3}^{3} x_2(n) z^{-n}$$

Putting for $x_2(n)$,

$$= 1 \cdot z^3 + 2 \cdot z^2 + 3z^1 + 4z^0 + 5z^{-1} + 0z^{-2} + 7z^{-3}$$

$$= z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{7}{z^3}$$

Result : i) Above equation gives $X_2(z)$.

ii) $X_2(z) = \infty$ for $z = 0$ and $z = \infty$.

iii) Hence ROC : Entire z-plane except $z = 0$ and ∞.

z-transform of $\delta(n)$.

We know that $\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$

$$\therefore \qquad X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$$= 1 \cdot z^0 = 1$$

This is fixed value for any z. Hence ROC will be entire z-plane.

$\delta(n) \overset{z}{\longleftrightarrow} 1,$ *ROC : Entire z-plane*

z-transform of unit step sequence, u(n).

Unit step sequence, $u(n) = \begin{cases} 1 & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$

$$\therefore \quad X(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

Putting u(n),

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= 1 + (z^{-1}) + (z^{-1})^2 + (z^{-1})^3 + (z^{-1})^4 + \cdots$$

Here use, $1 + A + A^2 + A^3 + A^4 + \cdots = \frac{1}{1-A}, |A| < 1$. Then above equation will be,

$$X(z) = \frac{1}{1-z^{-1}}, |z^{-1}| < 1$$

$$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}}, \qquad ROC : |z| > 1$$

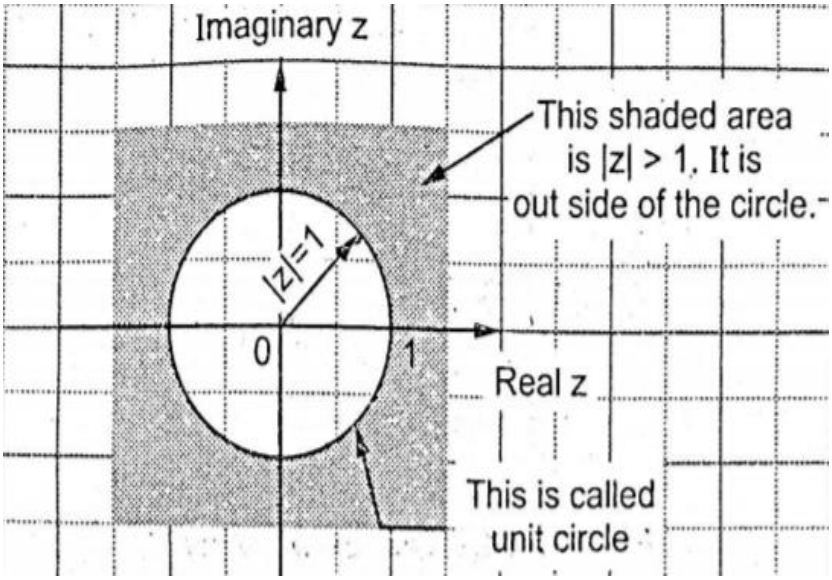


Fig. 1: Region of convergence

z-transform of right hand sided sequence $x(n) = a^n u(n)$.

By definition of z-transform, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \quad \textit{since } u(n) = 1 \textit{ for } n = 0 \textit{ to } \infty$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + (az^{-1}) + (az^{-1})^2 + (az^{-1})^3 + (az^{-1})^4 + \cdots$$

Here use $1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A}, |A| < 1$. Then above equation will be,

ROC of the right hand sided sequence (i.e. causal sequence) is outside the circle.

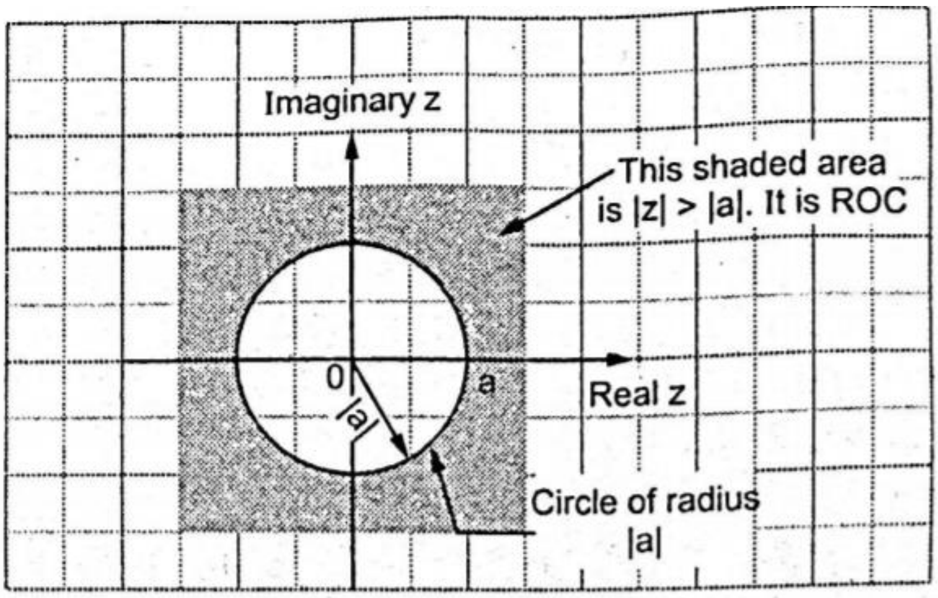


Fig. 2: Region of convergence

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}, \qquad ROC : |z| > |a|$$

z-transform of left handsided sequence.

$x(n) = -a^n u(-n-1)$

Here $x(n) = \begin{Bmatrix} -a^n & for\ n \leq -1 \\ 0 & for\ n \geq 0 \end{Bmatrix}$ $u(-n-1) = 1$ for n = − 1 to − ∞

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$n = -l, \ X(z) = -\sum_{l=\infty}^{1} a^{-l} z^{l}$$

$$= -\sum_{l=1}^{\infty} (a^{-1} z)^l$$

$$= -\{(a^{-1} z) + (a^{-1}z)^2 + (a^{-1} z)^3 + (a^{-1} z)^4 + \cdots\}$$

$$= -(a^{-1} z)\{1 + a^{-1} z + (a^{-1}z)^2 + (a^{-1} z)^3 + (a^{-1} z)^4 + \cdots\}$$

For the term in bracket use,

$$1+A+A^2+A^3+\cdots = \frac{1}{1-A}, |A|<1. \text{ i.e.,}$$

$$X(z) = -(a^{-1}z)\cdot\frac{1}{1-a^{-1}z}, |a^{-1}z|<1$$

$$= \frac{1}{1-az^{-1}}, |a^{-1}z|<1$$

Here $|a^{-1}z|<1$ is equal to $|z|<|a|$. This ROC is the area that lies inside the circle of radius $|a|$. It is shown in Fig.

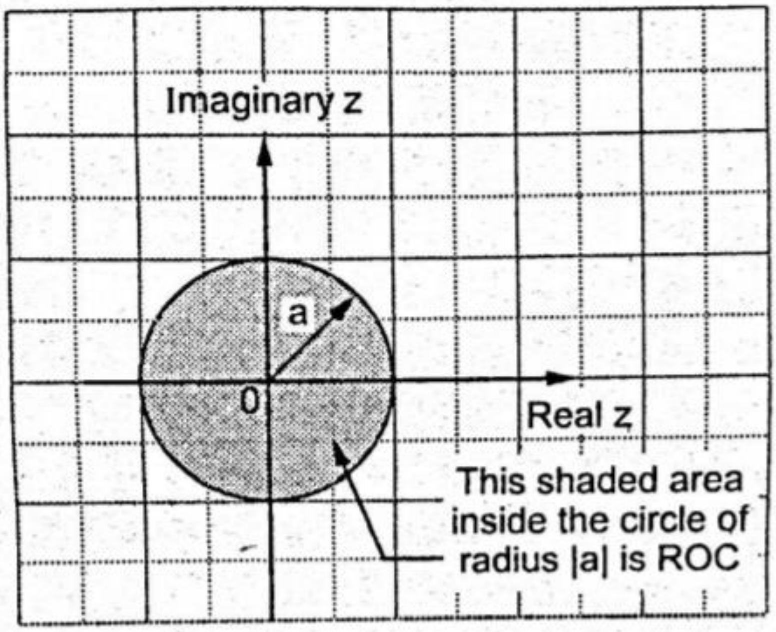


Fig.3: Region of convergence

z-transform of both sided sequence.

$x(n) = a^n u(n) + b^n u(-n-1)$

Here let $x_1(n) = a^n u(n)$ and $x_2(n) = b^n u(-n-1)$

$$x(n) = x_1(n) + x_2(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} [x_1(n) + x_2(n)] z^{-n}$$

From the previous problem's result

$$= \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}, \; ROC: |z|>|a| \text{ and } |z|<|b|$$

$$\text{i.e. } |a| < |z| < |b|$$

When $|a|>|b|$

As shown Fig. there is no overlap between the shaded areas for $|z|>|a|$ and $|z|<|a|$. Hence both the terms of X(z) do not converge simultaneously. Therefore ROC is not possible.

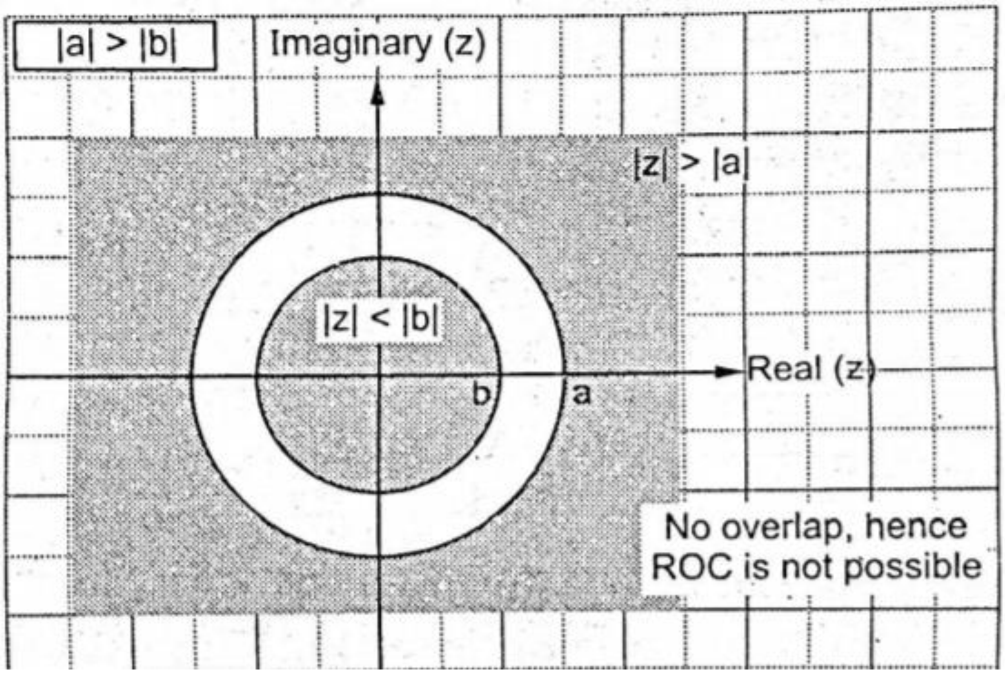


Fig. 4: Region of convergence

When $|a|<|b|$

For this case, as shown in Fig. the shaded area shows the overlap of $|z|>|a|$ and $|z|<|b|$. This area is $|a|<|z|<|b|$. In this area both the terms of X(z) converge simultaneously. Hence the ring shown by $|a|<|z|<|b|$ is ROC of X(z).

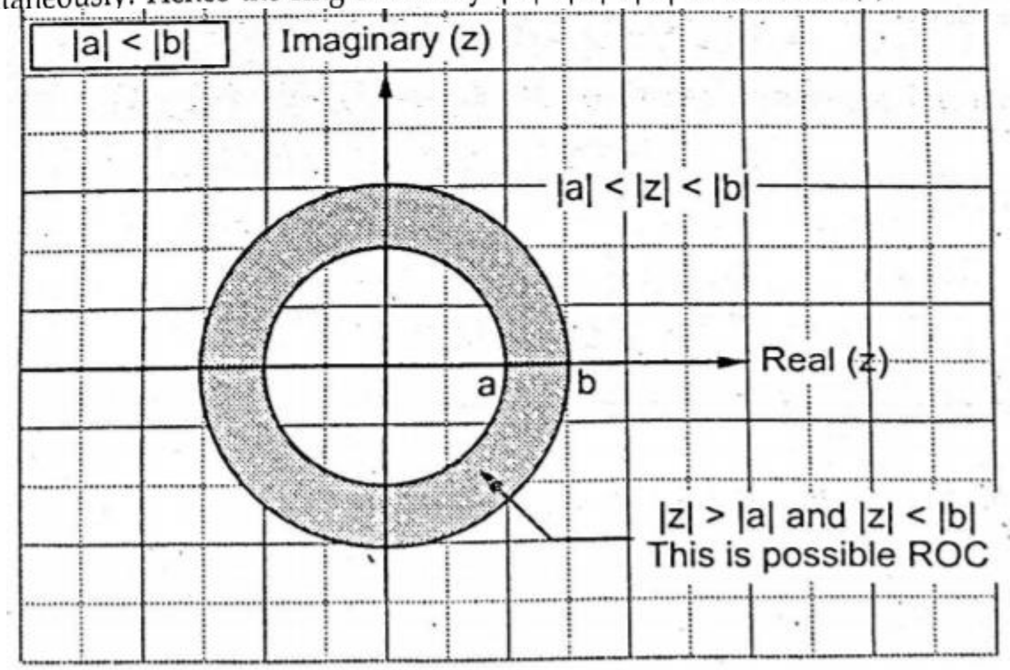


Fig. 5: Region of convergence

4.5. PROPERTIES OF ROC

Property 1 : The ROC for a finite duration sequence includes entire z-plane, except $z=0$, and/or $|z| = \infty$.

Proof : Consider the finite duration sequence $x(n) = \{1\ 2\ 1\ 2\}$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\uparrow$$

$$\therefore \qquad X(z) = 1\cdot z^2 + 2\cdot z + 1z^0 + 2z^{-1} = z^2 + 2z + 1 + \frac{2}{z}$$

Here $X(z) = \infty$ for $z = 0$ and ∞. This proves first property.

Property 2 : ROC does not contain any poles.

Proof : The z-transform of $a^n\,u(n)$ is calculated as,

$$X(z) = \frac{1}{1-az^{-1}}$$

$$= \frac{z}{z-a} \quad \because ROC : |z|>|a|$$

This function has pole at $z = a$. Note that ROC is $|z|>|a|$. This means poles do not lie in ROC. Actually $X(z) = \infty$ at poles by definition of pole.

Property 3 : ROC is the ring in the z-plane centered about origin.

Proof : Consider $a^n\,u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$, ROC : $|z|>|a|$

or $\qquad -a^n\,u(-n-1) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$, ROC : $|z|<|a|$

Here observe that $|z|$ is always a circular region (ring) centered around origin.

Property 4 : ROC of causal sequence (right hand sided sequence) is of the form $|z|>r$.

Proof : Consider right hand sided sequence $a^n u(n)$. It's ROC is $|z| > |a|$. Thus the ROC of right hand sided sequence is of the form of $|z|>r$ where 'r' is the radius of the circle.

Property 5 : ROC of left sided sequence is of the form $|z|<r$.

Proof : Consider left sided sequence $-a^n\,u(-n-1)$. Its ROC is $|z|<|a|$. Thus the ROC of left sided sequence is inside the circle of radius 'r'.

Property 6 : ROC of two sided sequence is the concentric ring in z-plane.

Proof : We know that ROC of $x(n) = a^n\,u(n) + b^n\,u(-n-1)$ is $|a|<|z|<|b|$, which is the concentric ring

4.6. PROPERTIES OF Z TRANSFORM

Linearity

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} a_1 x_1(z) + a_2 x_2(z)$$

Proof :

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \quad \text{Since } a_1 \text{ and } a_2 \text{ are constants}$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

Time Shifting

$$x(n-k) \xleftrightarrow{z} z^{-k} X(z)$$

Proof : $Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n}$

Let $n-k = m$. Hence $n = k+m$ and $m = -\infty$ to $+\infty$. i.e.,

$$Z\{x(n-k)\} = \sum_{m=-\infty}^{\infty} x(m) z^{-(k+m)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-k} \cdot z^{-m} = z^{-k} \sum_{m=-\infty}^{\infty} x(m) z^{-m}$$

$$= z^{-k} X(z)$$

Scaling in z-Domain

Let $x(n) \xleftrightarrow{z} X(z), \quad ROC : r_1 < |z| < r_2$

then $a^n\, x(n) \xleftrightarrow{z} X\left(\frac{z}{a}\right), \quad \text{ROC} : |a| r_1 < |z| < |a| r_2$

Proof : $Z\{a^n\, x(n)\} = \sum_{n=-\infty}^{\infty} a^n\, x(n)\, z^{-n}$

$= \sum_{n=-\infty}^{\infty} x(n)\,(a^{-1} z)$

$= X(a^{-1} z)$

$= X\left(\frac{z}{a}\right), \quad ROC : r_1 < \left|\frac{z}{a}\right| < r_2 \text{ i.e. } |a| r_1 < |z| < |a| r_2$

Time Reversal

Let $x(n) \xleftrightarrow{z} X(z), \; ROC : r_1 < |z| < r_2$

then $x(-n) \xleftrightarrow{z} X\left(z^{-1}\right), \quad \text{ROC} : \frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof : $Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$

with $n = -m$, $= \sum_{m=\infty}^{-\infty} x(m)\, z^{m} = \sum_{m=-\infty}^{\infty} x(m)\,(z^{-1})^{-m}$

$= X(z^{-1}) \quad ROC : r_1 < |z^{-1}| < r_2 \text{ i.e. } \frac{1}{r_2} < |z| < \frac{1}{r_1}$

Differentiation in z-Domain

$$n\,x(n) \xleftrightarrow{z} -z\frac{d}{dz}X(z)$$

Proof : $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

$$\therefore \quad \frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} \frac{d}{dz}[x(n)\,z^{-n}] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cdot(-n)\cdot z^{-n-1} = -\sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n}\cdot z^{-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty}[n\,x(n)]z^{-n} = -z^{-1}\cdot Z\{n\;x(n)\}$$

or $$Z\{n\;x(n)\} = -z\frac{d}{dz}X(z), \quad ROC : \text{Same as that of } x(n)$$

Convolution in Time Domain

$$x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z)\cdot X_2(z)$$

Proof : $$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)$$

$$\therefore \quad Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)\right]z^{-n}$$

Interchanging orders of summation,

$$= \sum_{k=-\infty}^{\infty} x_1(k)\left\{\sum_{n=-\infty}^{\infty} x_2(n-k)\,z^{-n}\right\}$$

Since $x_2(n-k) \xleftrightarrow{z} z^{-k}X_2(z)$,

$$= \sum_{k=-\infty}^{\infty} x_1(k)\left\{z^{-k}X_2(z)\right\}$$

$$= \left\{\sum_{k=-\infty}^{\infty} x_1(k)\,z^{-k}\right\}\cdot X_2(z) = X_1(z)\cdot X_2(z)$$

Correlation of Two Sequences

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l) \xleftrightarrow{z} X_1(z)\, X_2(z^{-1})$$

Proof : Correlation of two sequences is given as,

$$r_{x_1 x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\, x_2[-(l-n)] = x_1(l) * x_2(-l)$$

$$\therefore \quad Z\left\{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)\right\} = Z\{x_1(l) * x_2(-l)\}$$

$$= Z\{x_1(l)\} \cdot Z\{x_2(-l)\}$$

$$= X_1(z) \cdot X_2(z^{-1})$$

Multiplication of Two Sequences

$$x_1(n)\,x_2(n) \xleftrightarrow{z} \frac{1}{2\pi j}\oint_c X_1(v)\,X_2\left(\frac{z}{v}\right)v^{-1}\,dv$$

Here 'c' is the closed contour. It encloses the origin and lies in the ROC which is common to both $X_1(v)$ and $X_2\left(\frac{1}{v}\right)$.

Proof : Inverse z-transform is given as, $x(n) = \frac{1}{2\pi j}\oint X(v)v^{n-1}\,dv$

Let $\quad x(n) = x_1(n)\,x_2(n)$

Putting inverse z-transform of $x_1(n)$ in above equation,

$$x(n) = \frac{1}{2\pi j}\oint_c X_1(v)v^{n-1}\,dv\cdot x_2(n)$$

$$\therefore \quad X(z) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2\pi j}\oint_c X_1(v)v^{n-1}\,dv\cdot x_2(n)\right\}z^{-n}$$

Interchanging the order of integration and summation,

$$X(z) = \frac{1}{2\pi j}\oint_c X_1(v)\sum_{n=-\infty}^{\infty} v^n\cdot v^{-1}\,x_2(n)z^{-n}\,dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v)\left\{\sum_{n=-\infty}^{\infty} x_2(n)\left(\frac{z}{v}\right)^{-n}\right\}v^{-1}\,dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v)\cdot X_2\left(\frac{z}{v}\right)\cdot v^{-1}\,dv$$

Conjugation of a Complex Sequence

$$x^*(n) \xleftrightarrow{z} X^*(z^*)$$

Proof : $$Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n)z^{-n} = \sum_{n=-\infty}^{\infty}[x(n)(z^*)^{-n}]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^*$$

$$= [X(z^*)]^* = X^*(z^*)$$

z-Transform of Real Part of a Sequence

$$\mathrm{Re}[x(n)] \xleftrightarrow{z} \frac{1}{2}\,[X(z) + X^*(z^*)]$$

Proof : $x(n) = Re[x(n)] + j\ Im\ [x(n)]$ and $x^*(n) = Re\ [x(n)] - j\ Im\ [x(n)]$

$$\therefore \quad Re[x(n)] = \frac{1}{2}\,[x(n) + x^*(n)]$$

$$\therefore \quad Z\,\{\mathrm{Re}[x(n)] = Z\left\{\frac{1}{2}[x(n) + x^*(n)]\right\}$$

$$= \frac{1}{2}\{Z\,[x(n)] + Z\,[x*(n)]\}$$

$$= \frac{1}{2}[X(z) + X^*(z*)]$$

z-Transform of Imaginary Part of Sequence

$$\mathrm{Im}[x(n)] \xleftrightarrow{z} \frac{1}{2j}[X(z) - X^*(z*)]$$

Proof : $\mathrm{Im}[x(n)] = \frac{1}{2j}[x(n) - x*(n)]$

$$Z\,\{\mathrm{Im}[x(n)]\} = Z\left\{\frac{1}{2j}[x(n) - x*(n)]\right\} = \frac{1}{2j}\left\{Z\,[x(n) - Z\,[x*(n)]\right\}$$

$$= \frac{1}{2j}\left\{X(z) - X^*(z^*)\right\}$$

Parseval's Relation

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_c X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

Proof : Inverse z-transform of $X_1(z)$ is, $x_1(n) = \frac{1}{2\pi j}\oint_c X_1(v)\, v^{n-1}\, dv$

$$\therefore \quad \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi j}\oint_c X_1(v)\, v^{n-1}\, dv\, x_2^*(n)$$

$$= \frac{1}{2\pi j}\oint_c X_1(v) \left\{ \sum_{n=-\infty}^{\infty} x_2^*(n)\, v^{n-1} \right\} dv$$

Here $v^{n-1} = v^n \cdot v^{-1} = (v^{-1})^{-n} \cdot v^{-1} = \left(\frac{1}{v}\right)^{-n} \cdot v^{-1}$ then above equation will be,

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_c X_1(v) \left\{ \sum_{n=-\infty}^{\infty} x_2^*(n) \cdot \left(\frac{1}{v}\right)^{-n} \cdot v^{-1} \right\} dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n) \cdot \left(\frac{1}{v^*}\right)^{-n} \right]^* v^{-1} dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v) \left[X_2\left(\frac{1}{v^*}\right) \right]^* v^{-1} dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

Initial Value Theorem

$$x(0) = \lim_{z \to \infty} X(z)$$

Proof : z-transform of a causal sequence is given as,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \text{ since } x(n) = 0 \text{ for } n < 0$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

$$\therefore \quad \lim_{z \to \infty} X(z) = \lim_{z \to \infty} x(0) + \lim_{z \to \infty} x(1)z^{-1} + \lim_{z \to \infty} x(2)z^{-2} + \cdots$$

$$= x(0) + 0 + 0 + 0 + 0 \cdots$$

$$\therefore \quad x(0) = \lim_{z \to \infty} X(z)$$

4.7. PROBLEMS

$x_1(n) = \delta(n-k)$

$\delta(n) \overset{z}{\longleftrightarrow} 1$, ROC : entire z-plane

$x(n-k) \overset{z}{\longleftrightarrow} z^{-k} X(z)$, By time delay property.

$$Z\{\delta(n-k)\} = z^{-k} Z\{\delta(n)\}$$

$$= z^{-k} \cdot 1 = z^{-k}, \quad \textit{ROC : entire z-plane except } z = 0.$$

$x_2(n) = \delta(n+k)$

$$Z\{\delta(n+k)\} = z^{k} Z\{\delta(n)\}$$

$$= z^{k} \cdot 1 = z^{k}, \quad \textit{ROC : entire z-plane except } z = \infty$$

$\boldsymbol{x_3(n) = u(-n)}$

$x(n) \xleftrightarrow{z} X(z)$, ROC : $r_1 < |z| < r_2$

$x(-n) \xleftrightarrow{z} X(z^{-1})$, ROC : $\frac{1}{r_2} < |z| < \frac{1}{r_1}$, By time reversal property

$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}}$, ROC : $|z| > 1$. Here $r_1 = 1$

$\therefore u(-n) \xleftrightarrow{z} \frac{1}{1-z}$, ROC : $|z| < 1$, By time reversal property

$\boldsymbol{x_4(n) = n\,a^n\,u(n)}$

$$Z\{a^n u(n)\} = \frac{1}{1-az^{-1}}, \quad ROC : |z| > |a|$$

And $Z\{n\,x(n)\} = -z\frac{d}{dz}X(z)$, differentiation in z-domain property

$$\therefore \quad Z\{n \cdot a^n u(n)\} = -z\frac{d}{dz}\frac{1}{1-az^{-1}}, \quad \textit{Here } x(n) = a^n u(n)$$

$$= -z \cdot \frac{(1-az^{-1})\frac{d}{dz}1 - 1 \cdot \frac{d}{dz}(1-az^{-1})}{(1-az^{-1})^2}$$

$$= -z \cdot \frac{0 + a(-1)z^{-2}}{(1-az^{-1})^2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}, \quad ROC : |z| > |a|$$

Thus ,

$$\boxed{n a^n u(n) \xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}, \quad ROC : |z| > |a|}$$

$$x(n) = \cos(\omega_0 n)\, u(n)$$

$$= \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} u(n)$$

$$X(z) = Z\left\{\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right\} u(n)$$

$$= \frac{1}{2} Z\left\{e^{j\omega_0 n} u(n)\right\} + \frac{1}{2} Z\left\{e^{-j\omega_0 n} u(n)\right\}$$

Here use $a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$, ROC $|z|>|a|$ i.e.,

$$X(z) = \frac{1}{2}\cdot\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}}, \quad ROC: |z|>|e^{j\omega_0}| \text{ and } |z|>|e^{-j\omega_0}|$$

$$X(z) = \frac{1}{2}\left\{\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right\} ROC: |z|>1$$

$$= \frac{1}{2}\left\{\frac{1-e^{-j\omega_0}z^{-1} + 1 - e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right\}$$

$$= \frac{1}{2}\left\{\frac{2-z^{-1}(e^{j\omega_0}+e^{-j\omega_0})}{1-z^{-1}(e^{j\omega_0}+e^{-j\omega_0})+z^{-2}}\right\} = \frac{1}{2}\left\{\frac{2-z^{-1}\cdot 2\cos\omega_0}{1-z^{-1}\cdot 2\cos\omega_0+z^{-2}}\right\}$$

$$= \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}, \quad ROC: |z|>1$$

$$x(n) = \sin\omega_0 n\, u(n)$$

$$= \frac{e^{j\omega_0 n} - e^{-j\omega_0 n})}{2j} u(n)$$

$$X(z) = Z\left\{\frac{e^{j\omega_0 n} - e^{-j\omega_0 n})}{2j}\right\} u(n)$$

$$= \frac{1}{2j}\left[Z\{e^{j\omega_0 n}u(n)\} - Z\{e^{-j\omega_0 n}u(n)\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right],$$

$ROC: |z| > |e^{j\omega_0}|$ *and* $|z| > |e^{-j\omega_0}|$

i.e. $|z| > 1$

$$= \frac{1}{2j}\left[\frac{1-e^{-j\omega_0}z^{-1} - 1 + e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right], \ ROC: |z|>1$$

$$= \frac{1}{2j}\left[\frac{(e^{j\omega_0} - e^{-j\omega_0})z^{-1}}{1-z^{-1}(e^{j\omega_0} + e^{-j\omega_0}) + z^{-2}}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\sin\omega_0 z^{-1}}{1-z^{-1}\cdot 2\cos\omega_0 + z^{-2}}\right]$$

$$= \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}, \ ROC: |z|>1$$

$x(n) = a^n \cos(\omega_0 n) u(n)$

Let $\quad x_1(n) = \cos(\omega_0 n) u(n)$, hence $X_1(z) = \dfrac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$

From above equations, $\quad x(n) = a^n x_1(n)$

$\therefore \quad X(z) = Z\{a^n x_1(n)\}$

$= X_1\left(\frac{z}{a}\right)$, ROC : $|a|r_1 < |z| < |a|r_2$, By scaling in z-domain

Replacing z by $\frac{z}{a}$ in $X_1(z)$, $= \dfrac{1 - \left(\frac{z}{a}\right)^{-1}\cos\omega_0}{1 - 2\left(\frac{z}{a}\right)^{-1}\cos\omega_0 + \left(\frac{z}{a}\right)^{-2}}$, ROC : $|z| > 1\,|a|$ i.e. $|z| > |a|$

$x(n) = a^n \sin(\omega_0 n) u(n)$

Let $\quad x_1(n) = \sin(\omega_0 n) u(n)$, hence $X_1(z) = \dfrac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$

From above equations, $x(n) = a^n x_1(n)$

$\therefore \quad X(z) = z\{a^n x_1(n)\}$

$= X_1\left(\frac{z}{a}\right)$, ROC : $|a|r_1 < |z| < |a|r_2$, By scaling in z-domain

Replacing z by $\frac{z}{a}$ in $X_1(z)$, $= \dfrac{\left(\frac{z}{a}\right)^{-1}\sin\omega_0}{1 - 2\left(\frac{z}{a}\right)^{-1}\cos\omega_0 + \left(\frac{z}{a}\right)^{-2}}$, ROC : $|z| > 1\,|a|$ i.e. $|z| > |a|$

4.8. INVERSE Z TRANSFORM

The inverse z-transform can be obtained by,

i) Power series expansion

ii) Partial fraction expansion

iii) Contour integration.

iv) Convolution method

Inverse z-Transform using Power Series Expansion

By definition z-transform of the sequence x(n) is given as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \cdots + x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

From above expansion of z-transform, the sequence x(n) can be obtained as,

$$x(n) = \{\cdots, x(-2), x(-1), x(0), x(1), x(2), \cdots\}$$

The power series expansion can be obtained directly or by long division method.

Determine inverse z-transform of the following :

$$X(z) = \frac{1}{1-az^{-1}}, \; ROC : |z| > |a|$$

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} \leftarrow \textit{Negative power of 'z'} \\
1 - az^{-1} & 1 \\
 & 1 - az^{-1} \\
\hline
 & az^{-1} \\
 & az^{-1} - a^2 z^{-2} \\
\hline
 & a^2 z^{-2} \\
 & a^2 z^{-2} - a^3 z^{-3} \\
\hline
 & a^3 z^{-3} \\
 & a^3 z^{-3} - a^4 z^{-4} \\
\hline
 & a^4 z^{-4} \ldots
\end{array}
$$

Thus we have, $X(z) = \dfrac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$

Taking inverse z-transform, $x(n) = \{1, a, a^2, a^3, \cdots\}$

$= a^n u(n)$

$$X(z) = \frac{1}{1-az^{-1}}, \; ROC: |z|<|a|$$

$$\begin{array}{r l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 \quad \leftarrow \text{Positive powers of 'z'} \\
-az^{-1}+1 \,\big) & 1 \\
 & 1 - a^{-1}z \\
\hline
 & a^{-1}z \\
 & a^{-1}z - a^{-2}z^2 \\
\hline
 & a^{-2}z^2 \\
 & a^{-2}z^2 - a^{-3}z^3 \\
\hline
 & a^{-3}z^3 \\
 & a^{-3}z^3 - a^{-4}z^4 \\
\hline
 & a^{-4}z^4 \ldots
\end{array}$$

Thus we have, $X(z) = \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 \ldots$

Rearranging above equation, $= \cdots\cdots - a^{-4}z^4 - a^{-3}z^3 - a^{-2}z^2 - a^{-1}z$

Taking inverse z-transform, $x(n) = \{\cdots -a^{-4}, -a^{-3}, -a^{-2}, -a^{-1}\}$

$\uparrow$

$= -a^n u(-n-1)$

Inverse z-Transform using Partial Fraction Expansion

Following steps are to be performed for partial fraction expansions :

Step 1 : Arrange the given X(z) as,

$$\frac{X(z)}{z} = \frac{\textit{Numerator polynomial}}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

Step 2 :

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \cdots \frac{A_N}{z-p_N}$$

Where, $A_k = (z-p_k) \cdot \left.\frac{X(z)}{z}\right|_{z=p_k}, \; k = 1, 2, \ldots N$

If $\frac{X(z)}{z}$ has the pole of multiplicity 'n' i.e.,

$$\frac{X(z)}{z} = \frac{\textit{Numerator polynomial}}{(z-p)^n}$$

$$= \frac{A_1}{z-p} + \frac{A_2}{(z-p)^2} + \cdots \frac{A_n}{(z-p)^n}$$

Where $A_1, A_2, \cdots A_n$ are given as,

$$A_k = \frac{1}{(n-k)!} \cdot \frac{d^{n-k}}{dz^{n-k}} \left\{ (z-p)^n \cdot \frac{X(z)}{z} \right\} \Bigg|_{z=p}$$

k = 1, 2, 3, ... n

Step 3 : Equation (3.7.1) can be written as,

$$X(z) = \frac{A_1 z}{z-p_1} + \frac{A_2 z}{z-p_2} + \cdots + \frac{A_N z}{z-p_N}$$

$$= \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \cdots + \frac{A_N}{1-p_N z^{-1}}$$

Step 4 : All the terms in above step are of the form $\frac{A_k}{1-p_k z^{-1}}$. Depending upon ROC, following standard z-transform pairs must be used.

$$p_k^n u(n) \xleftrightarrow{z} \frac{1}{1-p_k z^{-1}}, \; ROC : |z| > |a| \text{ i.e. causal sequence}$$

$$-(p_k)^n u(-n-1) \xleftrightarrow{z} \frac{1}{1-p_k z^{-1}}, \; ROC : |z| < |a| \text{ i.e. noncausal sequence}$$

Determine inverse z-transform of $X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$.

For (i) ROC : $|z| > 1$, *(ii) ROC :* $|z| < 0.5$ *and (iii) ROC :* $0.5 < |z| < 1$

Step 1 : First convert X(z) to positive powers of z. i.e.,

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

$$\frac{X(z)}{z} = \frac{z}{z^2-1.5z+0.5}$$

$$= \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$A_1 = (z-1) \cdot \left. \frac{z}{(z-1)(z-0.5)} \right|_{z=1} = \frac{1}{1-0.5} = 2$$

and $$A_2 = (z-0.5) \cdot \left. \frac{z}{(z-1)(z-0.5)} \right|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

Equation will be, $\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$

Step 3 : $$X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

$$= \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

Step 4 : i) x(n) for ROC of $|z| > 1$

- Here the poles are at $z = 1$ and $z = 0.5$ from equation
- Now ROC of $|z| > 1$ indicates that sequence corresponding to the term $\frac{2}{1-z^{-1}}$ in equation (3.7.7) must be causal.
- Fig. 3.7.1 shows the ROCs of $|z| > 1$ and $|z| > 0.5$. Observe that $|z| > 1$ includes $|z| > 0.5$.

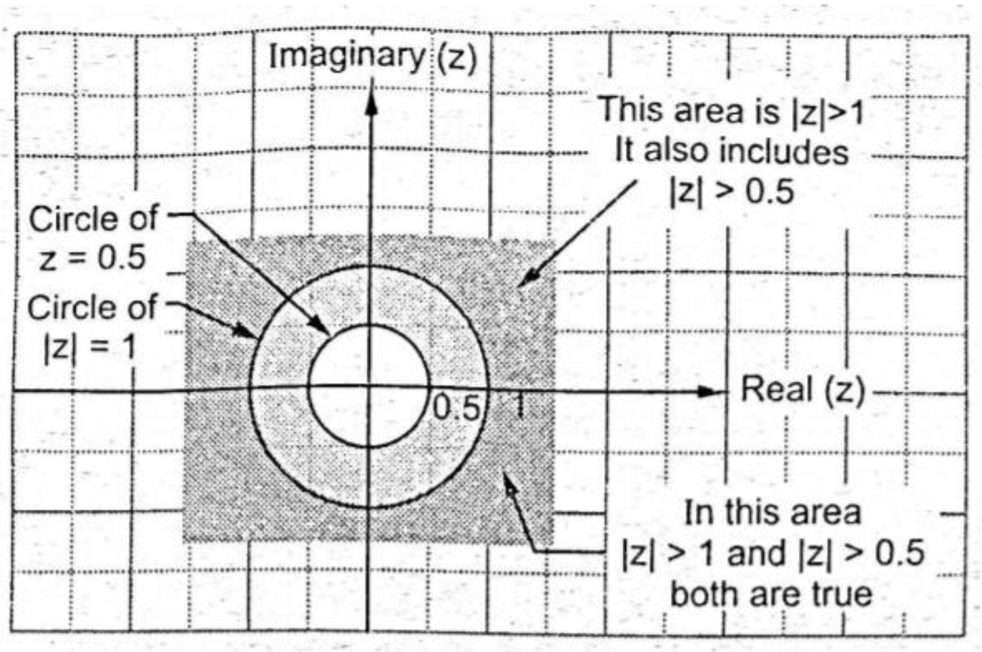


Fig. 6: Region of convergence

Hence the sequence corresponding to the term $\frac{1}{1-0.5z^{-1}}$ in equation

Therefore from equation (3.7.7) inverse z-transform becomes,

$$x(n) = 2(1)^n u(n) - 1.(0.5)^n u(n)$$
$$= [2-(0.5)^n]u(n)$$

x(n) for ROC : $|z| < 0.5$

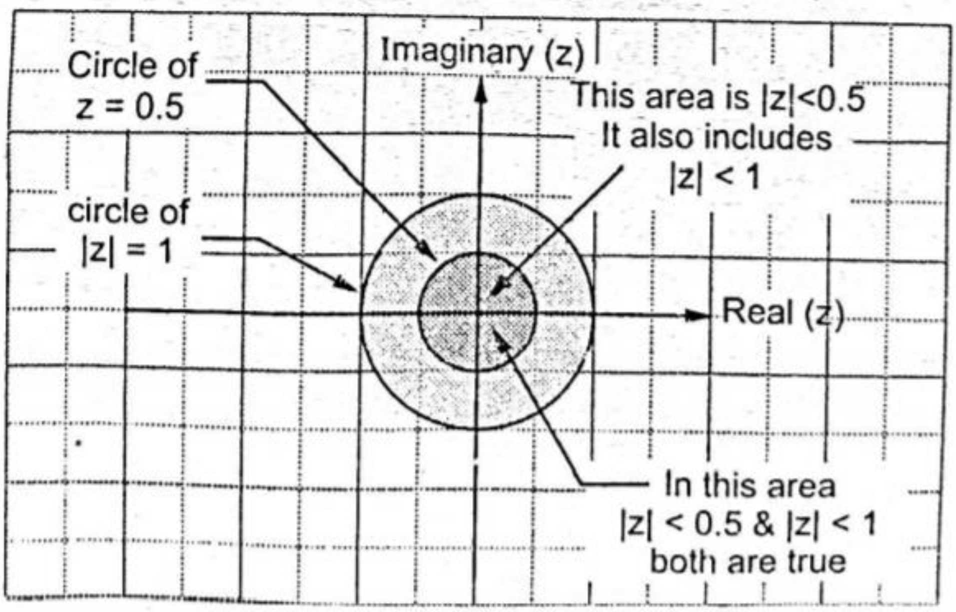


Fig. 7: Region of convergence

$$x(n) = 2[-1^n u(-n-1)] - [-0.5^n u(-n-1)]$$

$$= [-2 + 0.5^n] u(-n-1)$$

$x(n)$ for ROC : $0.5 < |z| < 1$

- This ROC can be written as $|z| > 0.5$ and $|z| < 1$.
- The sequence corresponding to $\frac{2}{1-z^{-1}}$ in equation

  ROC is $|z| < 1$.
- The sequence corresponding to $\frac{1}{1-0.5z^{-1}}$ in equation

  ROC is $|z| > 0.5$.

Taking inverse z-transform of equation

$$\begin{aligned} x(n) &= 2[-1^n u(-n-1)]-(0.5)^n u(n) \\ &= -2u(-n-1)-(0.5)^n u(n) \end{aligned}$$

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}, \; ROC : |z| > 1$$

Solution : **Step 1 :** Converting X(z) to positive powers of z,

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

Step 2 : Here there is multiple pole at z = 1. Therefore the partial fraction expansion will be,

$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

$$\therefore \quad A_1 = (z+1) \cdot \left. \frac{X(z)}{z} \right|_{z=-1} = \left. \frac{z^2}{(z-1)^2} \right|_{z=-1} = \frac{1}{4}$$

$$A_3 = (z-1)^2 \left. \frac{X(z)}{z} \right|_{z=1} = \left. \frac{z^2}{z+1} \right|_{z=1} = \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left\{(z-1)^2 \cdot \frac{X(z)}{z}\right\}\Bigg|_{z=1} \qquad \text{By equation (3.7.3)}$$

$$= \frac{d}{dz}\cdot\left(\frac{z^2}{z+1}\right)\Bigg|_{z=1} = \frac{(z+1)\,2z-z^2}{(z+1)^2}\Bigg|_{z=1} = \frac{3}{4}$$

Putting values in equation (3.7.8),

$$\frac{X(z)}{z} = \frac{1/4}{z+1}+\frac{3/4}{z-1}+\frac{1/2}{(z-1)^2}$$

Step 3 :

$$X(z) = \frac{1/4z}{z+1}+\frac{3/4z}{z-1}+\frac{1/2z}{(z-1)^2}$$

$$\therefore \quad X(z) = \frac{1/4}{1+z^{-1}}+\frac{3/4}{1-z^{-1}}+\frac{1/2z^{-1}}{(1-z^{-1})^2}$$

Step 4 : ROC is $|z| > 1$. Let us use following relations :

$$p_k^n\, u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - p_k z^{-1}}, \; ROC : |z| > |p_k|$$

Hence inverse z-transform of first two terms of X(z) will be,

$$\text{IZT}\left\{\frac{1/4}{1+z^{-1}}\right\} = \frac{1}{4}(-1)^n u(n) \text{ and IZT}\left\{\frac{3/4}{1-z^{-1}}\right\} = \frac{3}{4}(1)^n u(n)$$

For 3rd term of X(z) let us use,

$$n p_k^n\, u(n) \overset{z}{\longleftrightarrow} \frac{p_k z^{-1}}{(1 - p_k z^{-1})^2}, \; ROC : |z| > |p_k|$$

$$i.e. \;\; \text{IZT}\left\{\frac{1/2 z^{-1}}{(1-z^{-1})^2}\right\} = \frac{1}{2} IZT\left\{\frac{z^{-1}}{(1-z^{-1})^2}\right\} = \frac{1}{2} n(1)^n u(n)$$

Putting all the sequences together,

$$\begin{aligned} x(n) &= \frac{1}{4}(-1)^n u(n) + \frac{3}{4}(1)^n u(n) + \frac{1}{2} n(1)^n u(n) \\ &= \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{1}{2} n\right] u(n) \end{aligned}$$

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \; ROC : |z|>1$$

Solution : To arrange X(z) in proper form suitable for partial fraction expansion.

- The highest negative powers of numerator and denominator in X(z) are same. Hence such equation cannot be expanded in partial fractions.
- Let us rearrange X(z) as follows :

$$X(z) = \frac{z^{-2}+2z^{-1}+1}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1}$$

- Now perform the division,

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & z^{-2}-3z^{-1}+2 \\ \hline & \quad 5z^{-1}-1 \end{array}$$

$$\therefore \quad X(z) = 2+\frac{5z^{-1}-1}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1} = 2+\frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$= 2+X_1(z), \quad where\ X_1(z) = \frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

Now apply step 1 to step 3 for $X_1(z)$

Step 1 : $$X_1(z) = \frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$= \frac{z(-z+5)}{z^2-\frac{3}{2}z+\frac{1}{2}}$$ By converting to positive powers of z

$$\therefore \quad \frac{X_1(z)}{z} = \frac{-z+5}{z^2-\frac{3}{2}z+\frac{1}{2}}$$

$$= \frac{-z+5}{(z-1)\left(z-\frac{1}{2}\right)}$$

Step 2 : $$X_1(z) = \frac{A_1}{z-1}+\frac{A_2}{z-\frac{1}{2}}$$

where $$A_1 = (z-1)\cdot\frac{X_1(z)}{z}\bigg|_{z=1} = \frac{-z+5}{z-\frac{1}{2}}\bigg|_{z=1} = 8$$

and $$A_2 = \left(z-\frac{1}{2}\right)\cdot\frac{X_1(z)}{z}\bigg|_{z=\frac{1}{2}} = \frac{-z+5}{z-1}\bigg|_{z=\frac{1}{2}} = -9$$

$$\therefore \quad \frac{X_1(z)}{z} = \frac{8}{z-1}-\frac{9}{z-\frac{1}{2}}$$

Step 3 : $$X_1(z) = \frac{8}{1-z^{-1}}-\frac{9}{1-\frac{1}{2}z^{-1}}$$

Step 4 : Putting for $X_1(z)$ in equation

$$X(z) = 2+\frac{8}{1-z^{-1}}-\frac{9}{1-\frac{1}{2}z^{-1}}$$

- Here ROC is $|z|>1$. Hence sequence corresponding to $\frac{8}{1-z^{-1}}$ will be causal.
- The causal sequence corresponding to $\frac{9}{1-\frac{1}{2}z^{-1}}$ will be $9\cdot\left(\frac{1}{2}\right)^n u(n)$. It has ROC of $|z|>\frac{1}{2}$.
- Note that given ROC of $|z|>1$ includes $|z|>\frac{1}{2}$ also. Hence x(n) will be,

$$x(n) = 2\delta(n)+8(1)^n u(n)-9\left(\frac{1}{2}\right)^n u(n)$$

$$= 2\delta(n)+\left[8-9\left(\frac{1}{2}\right)^n\right]u(n)$$

Inverse z-Transform using Contour Integration

Cauchy integral theorem is used to calculate inverse z-transform. Following steps are to be followed :

Step 1 : Define the function $X_0(z)$, which is rational and its denominator is expanded into product of poles.

i.e.,

$$X_0(z) = X(z)z^{n-1} = \frac{N(z)}{\prod_{i=1}^{m}(z-p_i)^m}$$

Here 'm' is order of the pole.

Step 2 : i) For simple poles, i.e. m = 1, the residue of $X_0(z)$ at pole p_i is given as,

$$\operatorname*{Res}_{z=p_i} X_0(z) = \lim_{z\to p_i}[(z-p_i)X_0(z)] = (z-p_i)X_0(z)\Big|_{z=p_i}$$

ii) For multiple poles of order m_0, the residue of $X_0(z)$ can be calculated as,

$$\operatorname*{Res}_{z=p_i} X_0(z) = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}(z-p_i)^m X_0(z)\right\}_{z=p_i}$$

iii) If $X_0(z)$ has simple pole at the origin, i.e. n = 0, then x(0) can be calculated independently.

Step 3 : i) Using residue theorem, calculate x(n) for poles inside the unit circle. i.e.,

$$x(n) = \sum_{i=1}^{N}\operatorname*{Res}_{z=p_i} X_0(z)$$

ii) For poles outside the contour of integration,

$$x(n) = -\sum_{i=1}^{N}\operatorname*{Res}_{z=p_i} X_0(z) \text{ with } n < 0$$

Determine the inverse z-transform of $X(z) = \frac{z^2}{(z-a)^2}$,

ROC : $|z|>|a|$ *using contour integration (i.e. residue method).*

Step 1 : $$X_0(z) = X(z)\,z^{n-1} = \frac{z^2}{(z-a)^2} z^{n-1}$$

$$= \frac{z^{n+1}}{(z-a)^2}$$

Step 2 : Here the pole is at z = a and it has order m = 2. Hence using equation we can calculate residue of $X_0(z)$ at z = a as,

$$\underset{z=a}{\text{Res}}\; X_0(z) = \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}(z-a)^2 X_0(z)\right\}_{z=a}$$

$$= \frac{d}{dz} z^{n+1}\bigg|_{z=a} = (n+1)z^n\Big|_{z=a} = (n+1)\,a^n$$

Step 3 : By equation the sequence x(n) is given as,

$$x(n) = \sum_{i=1} \underset{z=a}{\text{Res}}\; X_0(z)$$

$$= (n+1)\,a^n\,u(n) \quad \text{since ROC : } |z|>|a|$$

Using residue method find the inverse z-transform of $X(z) = \frac{z+1}{(z+0.2)(z-1)}, \quad |z| > 1$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

$$= \sum \text{ residues of } X(z)z^{n-1} \text{ at poles of } X(z)z^{n-1} \text{ within C}$$

$$= \sum \text{ residues of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at poles of same within C}$$

The closed contour C, begin in the ROC $|z| > 1$ encloses the poles at $z = -0.2$, $z = 1$ and, for $n = 0$ the pole is at $z = 0$. Therefore for $n = 0$

$$x(0) = \sum \text{ residues of } \frac{z+1}{z(z+0.2)(z+1)} \text{ at poles } z = 0, z = 1 \text{ and } z = -0.2$$

$$= \cancel{(z)} \frac{z+1}{\cancel{(z)}(z+0.2)(z-1)}\Big|_{z=0} + \cancel{(z+0.2)} \frac{(z+1)}{\cancel{(z+0.2)}(z-1)}\Big|_{z=0.2}$$

$$+ \cancel{(z-1)} \frac{(z+1)}{z(z+0.2)\cancel{(z-1)}}\Big|_{z=1}$$

$$= -5 + \frac{10}{3} + \frac{5}{3} = 0$$

i.e., $x(0) = 0$.

For $n \geq 1$

$$x(n) = \sum \text{ residues of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at poles } z = -0.2 \text{ and } z = 1$$

$$= \text{ residue of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at } z = -0.2$$

$$+ \text{ residue of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at } z = 1$$

$$= (z+0.2)\frac{(z+1)z^{n-1}}{(z+0.2)(z-1)}\bigg|_{z=-0.2} + (z-1)\frac{(z+1)z^{n-1}}{(z+0.2)(z-1)}\bigg|_{z=1}$$

$$= -\frac{2}{3}(-0.2)^{n-1} + \frac{5}{3}$$

Therefore, $x(n) = -\frac{2}{3}(-0.2)^{n-1}u(n-1) + \frac{5}{3}u(n-1)$.

Use the residue method to find the inverse z-transform of $X(z) = \frac{z}{(z-2)(z-3)} \quad |z| < 2$

Solution In this case there are two poles $z = 3$ and $z = 2$ outside the ROC $|z| < 2$. So the sequence is non-causal. For $n < 0$

$$x(n) = -\sum \text{ residues of } X(z)z^{n-1} \text{ at poles } z = 2 \text{ and } z = 3$$

$$= -\left[(z-2)\frac{z \cdot z^{n-1}}{(z-2)(z-3)}\bigg|_{z=2} + (z-3)\frac{z \cdot z^{n-1}}{(z-2)(z-3)}\bigg|_{z=3}\right]$$

$$= -[-(2)^n + (3)^n]$$

$$= (2)^n - (3)^n$$

For $n < 0$ $x(n)$ can be written as

$$x(n) = [2^n - 3^n]u(-n-1)$$

Using Cauchy integral method find the inverse z-transform of $X(z) = \frac{z}{(z-1)(z-2)}$ $1 < |z| < 2$

Solution The contour of integration C lies in the annular region of ROC, and the inverse z-transform is

$$x(n) = -\sum \text{ residue of } X(z)z^{n-1} \text{ at pole } z = 2 \text{ for } n < 0$$
$$= \sum \text{ residue of } X(z)z^{n-1} \text{ at pole } z = 1 \text{ for } n \geq 0$$

For $n < 0$

$$x(n) = -(z-2)\frac{z^n}{(z-1)(z-2)}\Big|_{z=2} = -(2)^n$$

For $n \geq 0$

$$x(n) = (z-1)\frac{z^n}{(z-1)(z-2)}\Big|_{z=1} = -(1)^n$$

Therefore

$$x(n) = -u(n) - 2^n u(-n-1)$$

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DEPARTMENT OF BIOMEDICAL ENGINEERING

UNIT – V – Linear Time Invariant Discrete Time Systems – SBMA1304

LINEAR TIME INVARIANT DISCRETE TIME SYSTEMS

5.1. LINEAR CONVOLUTION

Linear convolution is a very powerful technique used for the analysis of LTI systems. In the last subsection we have seen that how the sequence $x(n)$ can be expressed as sum of weighted impulses. It is given by equation

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

If $x(n)$ is applied as an input to the discrete time system, then response $y(n)$ of the system is given as,

$$y(n) = T[x(n)]$$

Putting for $x(n)$ in above equation from equation (4.3.7),

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

The above equation we have written on the basis of scaling property. It states that if $y(n) = T[a\,x(n)]$, then $y(n) = aT[x(n)]$ for a = constant. The above equation can be written in compact form with the help of '$\sum$' sign. i.e.,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,T[\delta(n-k)]$$

The response of the system to unit sample sequence $\delta(n)$ is given as,

$$T[\delta(n)] = h(n)$$

Here $h(n)$ is called unit sample response or impulse response of the system. If the discrete time system is shift invariant, then above equation can be written as,

$$T[\delta(n-k)] = h(n-k)$$

Here 'k' is some shift in samples. The above equation indicates that; if the excitation of the shift invariant system is delayed, then its response is also delayed by the same amount. Putting for $T[\delta(n-k)] = h(n-k)$ in equation we get,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

This equation gives the response of linear shift invariant (LTI) system or LTI system to an input $x(n)$. The behaviour of the LTI system is completely characterized by the unit sample response $h(n)$. The above equation is basically linear convolution of $x(n)$ and $h(n)$. This linear convolution gives $y(n)$. Thus,

Convolution sum : $y(n) = x(n) * h(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Convolve the following two sequences $x(n)$ and $h(n)$ to get $y(n)$

$x(n) = \{1, 1, 1, 1\}$

$h(n) = \{2, 2\}$

Solution : Here upward arrow $(\uparrow)$ is not shown in $x(n)$ as well as $h(n)$ means, the first sample in the sequence is 0^{th} sample. Thus the sample values are :

$$x(k=0) = 1$$

$$x(k=1) = 1$$

$$x(k=2) = 1$$

$$x(k=3) = 1$$

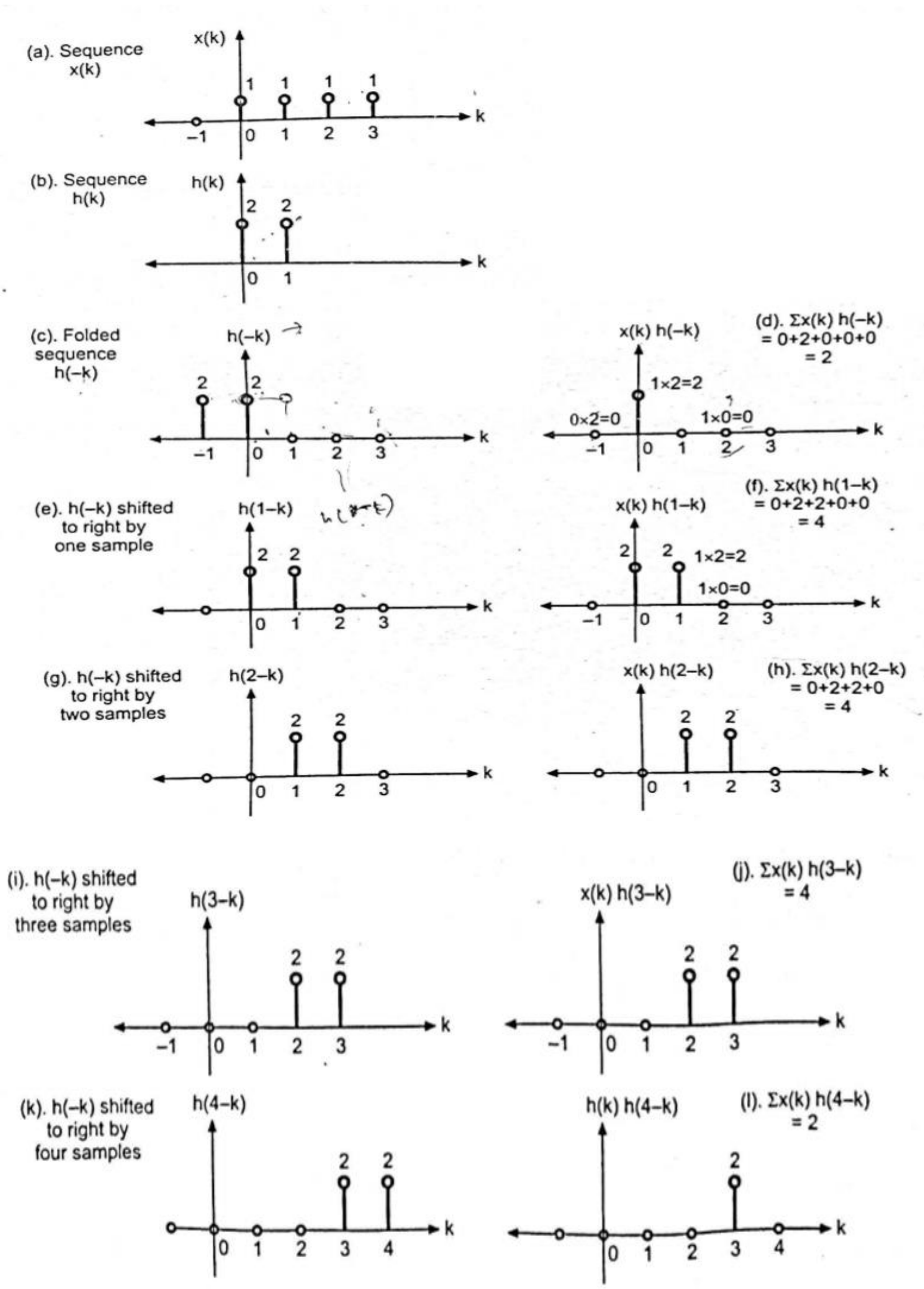
and

$$h(k=0) = 2$$

$$h(k=1) = 2$$

The convolution of $x(n)$ and $h(n)$ is given by equation

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$


(a). Sequence x(k)
x(k)
1 1 1 1
−1 0 1 2 3
k
(b). Sequence h(k)
h(k)
2 2
0 1
k
(c). Folded sequence h(−k)
h(−k)
2 2
−1 0 1 2 3
k
x(k) h(−k)
(d). Σx(k) h(−k)
= 0+2+0+0+0
= 2
1×2=2
0×2=0
1×0=0
−1 0 1 2 3
k
(e). h(−k) shifted to right by one sample
h(1−k)
2 2
0 1 2 3
k
x(k) h(1−k)
(f). Σx(k) h(1−k)
= 0+2+2+0+0
= 4
2 2 1×2=2
1×0=0
−1 0 1 2 3
k
(g). h(−k) shifted to right by two samples
h(2−k)
2 2
0 1 2 3
k
x(k) h(2−k)
(h). Σx(k) h(2−k)
= 0+2+2+0
= 4
2 2
0 1 2 3
k
(i). h(−k) shifted to right by three samples
h(3−k)
2 2
−1 0 1 2 3
k
x(k) h(3−k)
(j). Σx(k) h(3−k)
= 4
2 2
−1 0 1 2 3
k
(k). h(−k) shifted to right by four samples
h(4−k)
2 2
0 1 2 3 4
k
h(k) h(4−k)
(l). Σx(k) h(4−k)
= 2
2
0 1 2 3 4
k

$x_2(n)$ \ $x_1(n)$	2	1	2	1
3	6	3	6	3
3	6	3	6	3
4	8	4	8	4
4	8	4	8	4

sum of the elements in 1st strip = $y(0)$

2nd strip = $y(1)$

$$y(n) = \{\underset{\uparrow}{6}, 9, 17, 21, 15, 12, 4\}$$

2. Determine the o/p response of DTLTI sys. with the impulse response $h(n) = \{1, -2, 0, 4\}$ & the i/p $x(n) = \{2, \underset{\uparrow}{1}, 2\}$. Find the o/p response using conv sum

$$y(n) = x(n) * h(n)$$

Finite.

$$N = N_1 + N_2 - 1$$
$$= 4 + 3 - 1 = 6$$

$$n = n_1 + n_2$$
$$= 0 + (-1)$$

$$n = -1$$

$$y(n) = \{y(-1), \underset{\uparrow}{y(0)}, y(1), y(2), y(3), y(4)\}$$

x(n) \ h(n)	1	-2	0	4
2	2	-4	0	8
1	1	-2	0	4
2	2	-4	0	8

$$y(n) = \{2, \underset{\uparrow}{-3}, 0, 4, 4, 8\}$$

5.2. ANALYSIS OF LTI DT SYSTEMS

The general form of difference equation of a Nth order system is given by

$$1 + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad N > M$$

For input $x(n) = \delta(n)$, we obtain

$$1 + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k \delta(n-k)$$

For $n > M$, Eq. reduces to homogeneous equation

$$\sum_{k=1}^{N} a_k y(n-k) = 0 \; ; \quad a_0 = 1.$$

If $N = M$, we have to add an impulse function to the homogeneous solution.

PROBLEMS

Determine the impulse response $h(n)$ for the system described by the second-order difference equation

$y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$

Solution

Given ·

$$y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$$

We know the total response

$$y(n) = y_h(n) + y_p(n).$$

For impulse $x(n) = \delta(n)$, the particular solution

$$y_p(n) = 0$$
$$\Rightarrow \quad y(n) = y_h(n).$$

The homogeneous solution can be found by substituting $x(n) = 0$

$$\Rightarrow \quad y(n) - 0.6y(n-1) + 0.08y(n-2) = 0$$

Let the solution

$y_h(n) = \lambda^n$ Substituting

we obtain

$$\lambda^n - 0.6\lambda^{n-1} + 0.08\lambda^{n-2} = 0$$
$$\lambda^{n-2}[\lambda^2 - 0.6\lambda + 0.08] = 0$$
$$\Rightarrow \quad \lambda^2 - 0.6\lambda + 0.08 = 0$$

The roots of the characteristic equation are

$$\lambda_1 = 0.4; \lambda_2 = 0.2$$

The general form of the solution of the homogeneous equation is

$$y_h(n) = c_1\lambda_1^n + c_2\lambda_2^n$$
$$= c_1(0.4)^n + c_2(0.2)^n$$

$$y(0) = c_1 + c_2$$
$$y(1) = 0.4c_1 + 0.2c_2$$

From the difference equation we have

$$y(0) = 0.6y(-1) - 0.08y(-2) + x(0)$$
$$= 1$$

$y(-1)= y(-2) = 0$
$x(0)= \delta(0) = 1$

$$y(1) = 0.6y(0) - 0.08y(-1) + x(1)$$
$$= 0.6(1) - 0.08(0) + 0$$
$$= 0.6$$

$$\Rightarrow \quad y(0) = 1$$
$$y(1) = 0.6$$

Comparing

$$c_1 + c_2 = 1$$
$$0.4c_1 + 0.2c_2 = 0.6$$

and solving for c_1 and c_2 we get

$$c_1 = 2$$
$$c_2 = -1$$

Substituting the values in Eq. (1.235) yields

$$y(n) = 2(0.4)^n u(n) - (0.2)^n u(n)$$

Determine the impulse response $h(n)$ for the system described by difference equation

$$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

Solution

Given

$$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2).$$

Since $M = N = 2$, the homogeneous solution includes an impulse term.

The total response is given by

$$y(n) = y_h(n) + y_p(n)$$

For input $x(n) = \delta(n)$, the particular solution $y_p(n) = 0$

$$\Rightarrow y(n) = y_h(n)$$

The homogeneous solution can be found by equating the input terms to zero, that is

$$y(n) + y(n-1) - 2y(n-2) = 0$$

Let the homogeneous solution $y_h(n) = \lambda^n$. Substituting this solution we obtain the characteristic equation

$$\lambda^n + \lambda^{n-1} - 2\lambda^{n-2} = 0$$
$$\lambda^{n-2}[\lambda^2 + \lambda - 2] = 0$$
$$\Rightarrow \quad \lambda^2 + \lambda - 2 = 0$$

Therefore, the roots are $1, -2$ and the general form of the solution to the homogeneous equation is

$$y_h(n) = c_1(1)^n + c_2(-2)^n + A\delta(n)$$

From the difference equation

$$y(0) + y(-1) - 2y(-2) = x(-1) + 2x(-2)$$
$$y(0) = 0$$
$$y(1) + y(0) - 2y(-1) = x(0) + 2x(-1)$$
$$y(1) = 1$$
$$\Rightarrow \quad y(0) = 0$$
$$y(1) = 1$$
$$y(2) = 1$$

Substituting $n = 0, n = 1$ and $n = 2$ in Eq.

$$y(0) = c_1 + c_2 + A$$
$$y(1) = c_1 - 2c_2$$
$$y(2) = c_1 - 4c_2$$

from which $c_1 = 1; c_2 = 0; A = -1$.

Substituting these values in Eq.

$$\begin{aligned} y(n) &= u(n) - \delta(n) \\ &= u(n-1) \end{aligned}$$

Find the impulse response and step response of a discrete-time linear time invariant system whose difference equation is given by $y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1)$.

Solution

Given

$$y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1).$$

For impulse response the particular solution $y_p(n) = 0$.

Therefore

$$y(n) = y_h(n)$$

The homogeneous solution can be obtained by solving the homogeneous equation

$$\lambda^2 - \lambda - 0.5 = 0$$

from which

$$\begin{aligned} \lambda_1 &= 1.366 \\ \lambda_2 &= -0.366 \\ y_n(n) &= c_1(1.366)^n + c_2(-0.366)^n \end{aligned}$$

From the difference equation we can find $y(0) = 1$, $y(1) = 2$

$$y(0) = c_1 + c_2$$
$$y(1) = 1.366c_1 - 0.366c_2$$

Comparing Eq. we get

$$c_1 + c_2 = 1$$
$$1.366c_1 - 0.366c_2 = 2$$
$$\Rightarrow \quad c_1 = 1.366$$
$$c_2 = -0.366$$
$$y(n) = 1.366(1.366)^n - 0.366(-0.366)^n$$

Step response

For step input $x(n) = u(n)$, the particular solution $y_p(n) = ku(n)$. Substituting $x(n)$ and $y_p(n)$ in difference equation

$$ku(n) = ku(n-1) + 0.5ku(n-2) + u(n) + u(n-1)$$

For $n = 2$ where none of the terms vanish we get

$$k = k + 0.5k + 1 + 1$$
$$\Rightarrow \quad k = -4$$

Therefore

$$y_p(n) = -4u(n)$$

The total response

$$y(n) = y_h(n) + y_p(n)$$
$$= c_1(1.366)^n + c_2(-0.366)^n - 4u(n)$$

For step input from the difference equation

$$y(0) = 1$$
$$y(1) = 3$$

From Eq.

$$y(0) = c_1 + c_2 - 4$$
$$y(1) = 1.366c_1 - 0.366c_2 - 4$$

Comparing Eq.

$$c_1 + c_2 = 5$$
$$1.366c_1 - 0.366c_2 = 7$$
$$c_1 = 5.098$$
$$c_2 = -0.098$$

The step response

$$y(n) = 5.098(1.366)^n - 0.098(-0.366)^n - 4 \qquad n \geq 0$$
$$= 5.098(1.366)^n u(n) - 0.098(-0.366)^n u(n) - 4u(n).$$