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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – I – Introduction and Linear Programming – SBAA5205

UNIT – 1 – INTRODUCTION AND LINEAR PROGRAMMING

Concept of Operations Research - Meaning and Models in OR - Utilization of OR Models for Managerial Decision Making. Linear Programming Problems(LPP) - Formulation, Graphical Method and Simplex Method of Solving LPP.

1. INTRODUCTION TO OPERATIONS

1.1 RESEARCH ORIGIN OF OPERATIONS RESEARCH (OR)

The term Operations Research (OR) was first coined by MC Closky and Trefthen in 1940 in a small town, Bowdsey of UK. The main origin of OR was during the second world war – The military commands of UK and USA engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations. Their mission was to formulate specific proposals and to arrive at the decision on optimal utilization of scarce military resources and also to implement the decisions effectively. In simple words, it was to uncover the methods that can yield greatest results with little efforts. Thus it had gained popularity and was called " An art of winning the war without actually fighting it."

The name Operations Research (OR) was invented because the team was dealing with research on military operations. The encouraging results obtained by British OR teams motivated US military management to start with similar activities. The work of OR team was given various names in US: Operational Analysis, Operations Evaluation, Operations Research, System Analysis, System Research, Systems Evaluation and so on. The first method in this direction was simplex method of linear programming developed in 1947 by G.B Dantzig, USA. Since then, new techniques and applications have been developed to yield high profit from least costs. Now OR activities has become universally applicable to any area such as transportation, hospital management, agriculture, libraries, city planning, financial institutions, construction management and so forth. In India many of the industries like Delhi cloth mills, Indian Airlines, Indian Railway, etc are making use of OR techniques.

1.2 HISTORY OF OR

The term OR coined by Mc.Clostcy and Tref in the year 1940 in U.K. OR was first used in military operations for optimum utilization of resources.

YEAR	EVENTS
1940	Term OR was coined by Mc.Closky and Trefthen in U.K
1949	- OR unit was set up in India in Hyderabad. (The Regional Research Lab) - OR unit was set up at defence science lab.
1951	- The National Research Council (NRC) in US formed a committee on OR. - The first book was published called "Methods on OR" by Morse and Kimball.
1952	- OR Society of America was formed.
1953	- OR unit was set up in Calcutta in the "Indian Statistical Institute".
1995	- OR society of India was established.

OR gained its significance first in the defence during the World War II (1939-1945) in order to make the best use of limited military resources and win the war. The effectiveness of OR in defence spread interest in Government departments and industry.

1.3 CONCEPT AND DEFINITION OF OR

Operations research signifies research on operations. It is the organized application of modern science, mathematics and computer techniques to complex military, government, business or industrial problems arising in the direction and management of large systems of men, material, money and machines.

The purpose is to provide the management with explicit quantitative understanding and assessment of complex situations to have sound basics for arriving at best decisions. Operations research seeks the optimum state in all conditions and thus provides optimum solution to organizational problems.

DEFINITION

"OR is defined as the application of Scientific methods, tools and techniques to problems involving the operations of a system so as to provide to those in control of the system, with optimum solutions to the problem".

"OR is defined as the application of Scientific methods by interdisciplinary team to problems involving control of organized system, so as to provide solutions which serve best to the organization as a whole."

OR is, otherwise, called as the "Science of use".

OR is the combination of management principles and mathematical concepts (Quantitative techniques) for managerial decision-making purpose.

1.4 CHARACTERISTICS OF OR

- Aims to find solutions for problems of organized systems.
- Aims to provide optimum solution. Optimization means the best minimum or maximum for the criteria under consideration.
- It is the application of scientific methods, tools and techniques.
- Interdisciplinary team approach is used to solve the problems.
- The solutions that serve best to the organization as a whole is taken into consideration.

1.5 APPLICATION OF OR

1. **Production:**
   - Production scheduling
   - Project scheduling
   - Allocation of resources
   - Equipment replacement
   - Inventory policy
   - Factory size and location
2. **Marketing**
   - Product introduction with timing
   - Product mix selection
   - Competitive strategies
   - Advertising strategies
   - Pricing strategies.
3. **Accounts**
   - Cash flow analysis (optimum cash balance)
   - Credit policies (optimum receivables)
4. **Finance**
   - Optimum dividend policy
   - Portfolio analysis
5. **Personnel Management**
   - Recruitment and selection
   - Assignment of jobs
   - Scheduling of training programs

6. **Purchasing**

- Rules for purchasing
- EOQ-Economic Order Quantity (how much to order)
- Timing of purchase (when to purchase)

7. **Distribution**

- Deciding number of warehouses.
- Location of warehouses
- Size of warehouses
- Transportation strategies

8. **Defence**

- Budget allocation
- Allocation of resources

9. **Government Departments**

- Transportation
- Budget fixation
- Fiscal policies.

10. **R & D (Research and Development)**

- Project introduction
- Project control
- Budget allocation for projects

1.6 THE MAIN PHASES OF OR

- Formulation of the problem
- Construction of a model (Mathematical model)
- Solve the model
- Control and update the model
- Test the model and validate it
- Implement the model

1.6.1. FORMULATION OF THE PROBLEM

- The first task is to study the relevant system and develop a well-defined statement of the problem. This includes determining appropriate objectives, constraints, interrelationships and alternative course of action.
- The OR team normally works in an advisory capacity . The team performs a detailed technical analysis of the problem and then presents recommendations to the management.
- Ascertaining the appropriate objectives is very important aspect of problem definition. Some of the objectives include maintaining stable price, profits, increasing the share in market, improving work morale etc.
- OR team typically spends huge amount of time in gathering relevant data.
  - To gain accurate understanding of problem
  - To provide input for next phase.
- OR teams uses Data mining methods to search large databases for interesting patternsthat may lead to useful decisions.

1.6.2. CONSTRUCTION OF A MATHEMATICAL MODEL

This phase is to reformulate the problem in terms of mathematical symbols and expressions. The mathematical model of a business problem is described as the system of equations and related mathematical expressions. Thus

1. Decision variables(x_1, x_2… x_n) – ‘n’ related quantifiable decisions to be made.
2. Objective function– measure of performance (profit) expressed as mathematical function of decision variables. For example $P=3x_1+5x_2+ \dots + 4x_n$
3. Constraints– any restriction on values that can be assigned to decision variables in terms of inequalities or equations. For example $x_1+2x_2 \geq 20$
4. Parameters– the constant in the constraints (right hand side values)The advantages of using mathematical models are

- Describe the problem more concisely
- Makes overall structure of problem comprehensible
- Helps to reveal important cause-and-effect relationships
- Indicates clearly what additional data are relevant for analysis
- Forms a bridge to use mathematical technique in computers to analyze

1.6.3. SOLVE THE MODEL

This phase is to develop a procedure for deriving solutions to the problem. A common theme isto search for an optimal or best solution. The main goal of OR team is to obtain an optimalsolution which minimizes the cost and time and maximizes the profit

1.6.4. TESTING THE MODEL

After deriving the solution, it is tested as a whole for errors if any. The process of testing and improving a model to increase its validity is commonly referred as Model validation. The OR group doing this review should preferably include at least one individual who did not participate in the formulation of model to reveal mistakes. A systematic approach to test the model is to use Retrospective test. This test uses historical data to reconstruct the past and then determine the model and the resulting solution. Comparing the effectiveness of this hypothetical performance with what actually happened, indicates whether the model tends to yield a significant improvement over current practice.

1.6.5. IMPLEMENTATION

The last phase of an OR study is to implement the system as prescribed by the management. The success of this phase depends on the support of both top management and operating management. The implementation phase involves several steps

1. OR team provides a detailed explanation to the operating management
2. If the solution is satisfied, then operating management will provide the explanation to the personnel, the new course of action.
3. The OR team monitors the functioning of the new system
4. Feedback is obtained
5. Documentation

1.7 SCOPE OF OR

- Linear programming model
- Transportation
- Sequencing and scheduling
- Assignment of jobs to minimize cost or maximize profit
- Game theory
- Inventory model
- Maintenance and Replacement
- Waiting line models
- Network analysis
- Shortest route problems like traveling sales person problem
- Resource allocation problems

1.8 LIMITATIONS OF OR

1. **Magnitude of computation:** In order to arrive at an optimum solutions OR takes into account all the variables that affect the system. Hence the magnitude of computation is very large.

2. **<u>Non-Quantifiable variables</u>:** OR can give an optimum solution to a problem only if all the variables are quantified. Practically all variables in a system cannot be quantified.
3. **<u>Time and Cost</u>:** To implement OR in an organization, it consumes more time and cost. If the basic decision variables change, OR becomes too costly for an organization to handle it.
4. **<u>Implementation of OR</u>:** Implementation of OR may lead to HR problems. The psychology of employees should be considered and the success of OR depends on co-operation of the employees.
5. **<u>Distance between Manager and OR Specialist</u>:** Managers may not be having a complete overview of OR techniques and has to depend upon an OR Specialist. Only if good link is established OR can be a success.

2. MODELS IN OR

Model is a reasonably simplified representation of reality. It is an abstraction of reality. It helps to arrive at a well-structured view of reality.

2.1 TYPES OF MODELS

I -BASED ON NATURE

Iconic Model	**Analogue Model (or) Schematic model**	**Mathematical Model (or) Symbolic model**	**Quantitative Model**

- **ICONIC MODELS:**
  - It is a pictorial representation or a physical representation of a system. A look alike correspondence is present.

Eg: miniature of a building, toys, globe etc.

- Iconic Models are either scaled up or scaled down. Scaled up - eg: Atom. Scaled down – eg: globe.
- Iconic models are either two-dimensional or three-dimensional.

- **ANALOGUE MODEL OR SCHEMATIC MODEL**

This model uses one set of properties to describe another set of properties. There is no look alike correspondence. It is more abstract.
Eg: a set of water pipes that are used to describe current flow. Eg: Maps, (different colors are used to depict water, land etc Eg: Organizational chart.

- **MATHEMATICAL MODEL**

This uses a set of mathematical symbols (letters and numbers) to represent a system.

$$\underset{\text{(Voltage)}}{V} = \underset{\text{(Current)}}{I} * R \longrightarrow \text{(Resistance)}$$

- **QUANTITATIVE MODELS**

Quantitative models are those, which can measure the observation. Eg: Models that measure temperature.

II -BASED ON VARIABILITY

STATIC DYNAMIC

- **STATIC MODEL**

  This model assumes the values of the variables to be constant (do not change with time) eg: Assignment Model.

- **DYNAMIC MODEL**

  This model assumes that the values of the variable change with time. Eg: Replacement model.

III -BASED ON RISK COSIDERATION

DETERMINISTIC **STOCHASTIC (PROBABILISTIC)**

- **DETERMINISTIC MODEL**

  This model does not take uncertainty into consideration.

  Eg: Linear programming, Assignment etc

- **STOCHASTIC (PROBABILISTIC) MODEL**

  This model considers uncertainty as an important factor. Eg: Stochastic Inventory models.

IV -BASED ON PRESENTATION

DESCRIPTIVE **PREDICTIVE** **PRESCRIPTIVE**

- **DESCRIPTIVE MODEL**

  This model just describes the situation under consideration. Eg: collecting an opinion regarding tendency to vote.

- **PREDICTIVE MODEL**

  This is a model, which predicts the future based on the data collected. Eg: predicting the election results before actual counting.

- **PRESCRIPTIVE MODEL**

  This is a model, which prescribes the course of action to be followed. Eg: Linear programming.

V -BASED ON ACCURACY OR EXACTNESS

ANALYTICAL **SIMULATION**

- **ANALYTICAL MODEL**

  This is a model that gives an exact solution to the problem.

- **SIMULATION MODEL**

  Simulation model is a representation of reality through the use of some devices, which will react in the same manner under the given set of conditions.

  Eg: Simulation of a drive of an Airplane through computer.

3. LINEAR PROGRAMMING

3.1 DEFINITION

Samuelson, Dorfman, and Solow define LP as "the analysis of problems in which linear function of a number of variables is to be maximized (or minimized) when those variables are
subject to a number of constraints in the form of linear inequalities".

3.2 BASIC ASSUMPTIONS OF LINEAR PROGRAMMING:

The following four basic assumptions are necessary for all linear programming models:

1. LINEARITY:

The basic requirements of a LP problem are that both the objectives and constraints must be expressed in terms of linear equations or inequalities. It is well known that if the number of machines in a plant is increased, the production in the plant also proportionately increases. Such a relationship, giving corresponding increment in one variable for every increment in other, is called linear and can be graphically represented in the form of a straight line.

2. DETERMINISTIC (OR CERTAINTY):

In all LP models, it is assumed that all model parameters such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit decisions variable must be known and fixed. In other words, this assumptions means that all the **coefficients** in the objectives function and constraints are completely known with certainty and do not change during the period being studied.

3. ADDITIVITY:

The value of the objective function for the given values of decision variables and the total sum of resources used, must be equal to the sum of the contributions (profit or cost) earned from each decision variable and the sum of the resources used by each decision respectively. For example, the total profit earned by the sale of three products A, B and C must be equal to the profits earned separately from A, B and C and similarly, the amount of resources consumed by A, B, and C individually.

4. DIVISIBILITY:

This implies that solution values of decision variables and resources can take any non-negative values, i.e., **fractional values** of the decision variables are **permitted**. This, however, is not always desirable. For example, it is impossible to produce one-fourth of a bus. When it is necessary to have integer variables, a technique known as integer programming could be used.

3.4 APPLICATIONS OF LINEAR PROGRAMMING:

(i) **MANUFACTURING PROBLEMS:** to find the number of items of each type that should be manufactured so as to maximize the profit subject to production restrictions imposed by limitations on the use of machinery and labour.

(ii) **ASSEMBLING PROBLEMS:** To have the best combinations of basic components to produce goods according to certain specifications.

(iii) **TRANSPORTATION PROBLEMS**: to find the least costly way of transporting shipments from the warehouses to customers.

(iv) **BLENDING PROBLEM:** To determine the optimal amount of several constitutes to use in producing a set of products which determining the optimal quantity of each product to produce.

(v) **PRODUCTION PROBLEMS:** To decide the production schedule to satisfy demand and minimize cost in face of fluctuating rates and storage expenses.

(vii) **DIET PROBLEMS**: To determine the minimum requirement of nutrients subject to availability of foods and their prices.

(viii) **JOB ASSIGNING PROBLEMS:** To assign job to workers for maximum effectiveness and optimal results subject to restrictions of wages and other costs.

(ix) **TRIM-LOSS PROBLEMS:** To determine the best way to obtain a variety of smaller rolls of paper from a standard width of roll that it kept its stock and at the same time minimize wastage.

(x) **STAFFING PROBLEM**: To find optimal staff in hotels, police stations and hospitals to maximize the efficiency.

(xi) **TELEPHONE EXCHANGE PROBLEMS:** To determine optimal facilities in telephone exchange to have minimum breakdowns.

3.5 APPLICATIONS OF LINEAR PROGRAMMING

a) Personnel Assignment Problem
b) Transportation Problem
c) Efficiency on Operation of system of Dams
d) Optimum Estimation of Executive Compensation
e) Agriculture Applications
f) Military Applications
g) Production Management
h) Marketing Management
i) Manpower Management

3.6 KEY TERMS

Artificial variables: A variable that has no meaning in a physical sense, but acts as tool to help generate an initial LP solution.

Basic variables: The set of variables that are in the solution (i.e., have positive, on-zero values) are listed in the product mix column. The variables that normally take non-zero values to obtain a solution.

Basic solution: A solution to m simultaneous linear equations in n unknowns, m<n, with the property that n-m of the variables have the value zero and the values of the remaining m variables are unique determined; obtained when a set of non-basic variables are assigned the

value zero.

Basic feasible solution: A basic solution, for which the values of all variables are non-negative, corresponds to a corner of the LP feasible region.

Degeneracy : A condition that arises when there is a tie in the values used to determine which variables indicated will enter the solution next. It can lead to cycling back and forth between two non-optimal solutions.

Degenerate solution: The number of variables in the standard equality form (counting decision variables, surpluses, and slacks) with positive optimal value is less than the number of constraints.

Optimal solution: A solution that is optimal for the given solution.

Pivot column: The column with the largest positive number in the Ci-Zj row of a maximization problem, or the largest negative Cj-Zj value in a minimization problem. It indicates which variable will enter the solution next.

Pivot row: The corresponding to the variable that will leave the basis in order to make room for the variable entering (as indicated by the new pivot column). This is the smallest positive ratio found by dividing the quantity column values by the pivot column values for each row.

Slack variable: A variable added to less than or equal to constraints in order to create an equality for a simplex method. It represents a quantity of unused resources.

Surplus variable: A variable inserted in a greater than or equal to constraint to create equality. If represents the amount of resources usage above the minimum required usage.

Unboundedness: A condition describing LP maximization problems having solutions that can become infinitely large without violating any stated constraints.

3.7 ADVATNTAGES OF LPP:

- It provides an insight and perspective into the problem environment. This generally results in clear picture of the true problem.
- It makes a scientific and mathematical analysis of the problem situations.
- It gives an opportunity to the decision-maker to formulate his strategies consistent with the constraints and the objectives.
- It deals with changing situations. Once a plan is arrived through the LP it can also be

revaluated for changing conditions.

- By using LP, the decision maker makes sure that he is considering the best solution.

3.8 LIMITATIONS OF LPP:

- The major limitation of LP is that it treats all relationships as linear but it is not true in many real life situations.
- The decision variables in some LPP would be meaningful only if they have integer values. But sometimes we get fractional values to the optimal solution, where only integer values are meaningful.
- All the parameters in the LP model are assumed to be known constants. But in real life they may not be known completely or they may be probabilistic and they may be liable for changes from time to time.
- The problems are complex if the number of variables and constraints are quite large.
- It deals with only single objective problems, whereas in real life situations, there may be more than one objective.

3.9 FORMULATION OF LPP:

- Identify the objective function
- Identify the decision variables
- Express the objective function in terms of decision variables
- Identify the constraints and express them
- Value of decision variables is ≥ 0 (always non-negativity)

EXAMPLE PROBLEM:

An organization wants to produce Tables and Chairs. Profit of one table is ₹100 and profit of one Chair is ₹50

Particulars	Tables	Chairs	Maximum hours available
Cutting (hours)	4	1	300
Painting (hours)	1	.5	100

.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Solution:

1) Objective: Maximization of profit
2) Decision variables

   No. of Tables to be produced ‘x’

   No. of Chairs to be produced ‘y’
3) Objective function Maxi $Z = 100x + 50y$
4) Constraints $4x +$

   $1y \leq 300$

   $1x + 0.5y \leq 100$ $x \geq$

   $0, y \geq 0$
5) Formulate

   Maxi $Z = 100x + 50y$

   Subject to

   $4x + 1y \leq 300$

   $1x + 0.5y \leq 100$ $x, y \geq 0$

3.9 STEPS IN GRAPHICAL SOLUTION METHOD:

- Formulate the objective and constraint functions.
- Draw a graph with one variable on the horizontal axis and one on the vertical axis.
- Plot each of the constraints as if they are inequalities.
- Outline the solution area.
- Circle the potential solutions points. These are the intersections of the constraints on the perimeter of the solution area. (vertices of the solution space)
- Substitute each of the potential extreme point values of the two decision variables into the objective function and solve for Z.
- Select the solution that optimizes Z.

3.10 PROCEEDURE FOR SOLVING LPP PROBLEM USING SIMPLEX METHOD

STEP:1

Convert all the inequality functions into equality:

For converting all the inequalities into equalities, we should use slack and surplus variables. **In case of ≤ inequalities, we should add Slack variable** so as to convert that inequality into equality. For example, $3x + 2y \leq 6$ will become $3x + 2y + S1 = 6$, where S1 is the slack variable.

In case of ≥ inequalities, we should deduct Surplus variable

so as to convert that inequality into equation.

For example, $5x + 6y \geq 10$ will become $5x + 6y - S2 = 10$, where S2 is the surplus variable. **In case if the given constraint is an equation category**, we should not use either slack variable or surplus variable.

STEP 2:

Find out the basic and non basic variables: Non Basic variable is the variable whose value is zero. Basic variable is the variable which will have either positive or negative value.

After converting all the inequality into equality, we should assume some variables as Non basic variables and find out the values of the other (Basic) variables. This solution is called as initial solution. If all the basic variable values are positive, then that initial solution is called as BASIC FEASIBLE SOLUTION.

STEP:3

Preparation of simplex table: The format of the simplex table is as follows:
Coefficients of Variables in the Objective function

EVALUATION ROW				
Coefficients of Basic variables	**Basic Variables**	**Variables**	**Solution**	**Ratio**

STEP 4:

Calculation of values in Evaluation row: To calculate the values in the evaluation row, we should use the following formula for each variable column:

Evaluation row values = (Variable coefficients x coefficients of basic variables) – Coefficients of the variables in the objective function.

All the values in the Evaluation row should be either positive or zero. Then it indicates that we have reached the optimum stage and thereby we can derive the optimum solution.

If any negative persists, we should proceed further by doing the following steps.

STEP 5:

IDENTIFICATION OF KEY COLUMN: The column that represents least value in the evaluation row is known as KEY COLYMN. The variable in that column is known as ENTERING VARIABLE.

STEP 6:

IDENTIFICATION OF KEY ROW: To find out the Key row, we should calculate the ratio.

Ratio = solution column values / Key column values. The least ratio row is treated as KEY ROW and the value in that row is known as LEAVING VARIABLE. **THE VARIABLE THAT PREVAILS IN BOTH KEY ROW AND KEY COLUMN IS KNOWN AS KEY ELEMENT.** After finding the key element, we should prepare next simplex table. In that table, should bring the entering variable and should write the new values of the entering

Problems

Problem 1. A person requires 10, 12 and 12 units of chemicals A, B and C respectively for herbal garden. A liquid product contains 5, 2 and 1 units of A,B and C respectively per Jar. A dry product contains 1, 2 and 4 units of A, B and C per cartoon. If the liquid product sells for Rs. 3 per Jar and dry product sells for Rs. 2 per cartoon, how many of each should be purchased to minimise the cost and meet the requirements.

	A	B	C	
Solution : Requirement	10	12	12 units	
Liquid product	5	2	1	units Rs. 3/-per jar
Dry product	1	2	4	Rs. 2/-per Cartons

1. Select decision variable

   x_1 – no. of jars of liquid product

   x_2 – no. of cartoons of dry product

2. Objective function

   Minimize cost $(z) = 3x_1 + 2x_2$

3. Constraints :

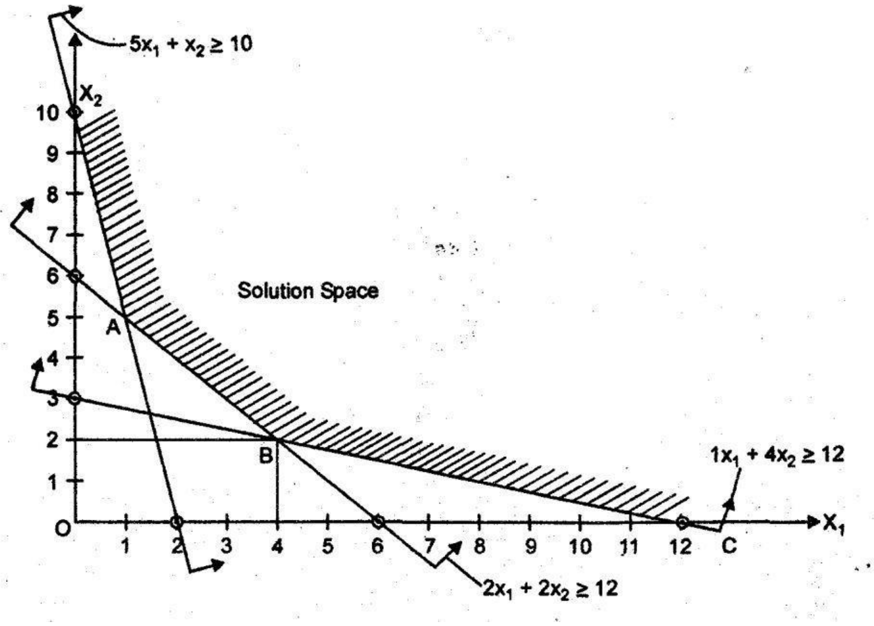
$$5x_1 + x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$1x_1 + 4x_2 \geq 12$$

4. Add non negativity constraints :

$$x_1 \geq 0 \quad ; \quad x_2 \geq 0$$

Graphical Method :

$$5x_1 + x_2 = 10 \Rightarrow x_1 = 0; x_2 = 10 \text{ and } x_2 = 0; x_1 = 2$$
$$2x_1 + 2x_2 = 12 \Rightarrow x_1 = 0; x_2 = 6 \text{ and } x_2 = 0; x_1 = 6$$
$$1x_1 + 4x_2 = 12 \Rightarrow x_1 = 0; x_2 = 3 \text{ and } x_2 = 0; x_1 = 12$$



Point A (1,5) $Z(A) = 3 \times 1 + 2 \times 5 = 13$

Point B (4, 2) $Z(B) = 3 \times 4 + 2 \times 2 = 16$

Point C (12,0) $Z(C) = 3 \times 12 + 2 \times 0 = 36$

Minimum cost at point A *i.e.* Rs. 13

x_1 (no. of Jar of Liquid product) = 1

x_2 (no. of carton of dry product) = 5

Minimum cost (Z) = Rs. 13.

Problem 2. A firm manufactures pain relieving pills in two sizes A and B, size A contains 4 grains of element a, 7 grains of element b and 2 grains of element c, size B contains 2 grains of element a, 10 grains of element b and 8 grains of c. It is found by users that it requires at least 12 grains of element a, 74 grains of element b and 24 grains of element c to provide immediate relief It is required to determine that least no. of pills a patient should take to get immediate relief. Formulate the problem as standard LPP.

Solution : Pain relieving pills

	a	b	c
Size A	4	7	2
Size B	2	10	8
Min. requirement	12	74	24

Step 1. Select decision variable

x_1 - no. of pills of size A

x_2 - no. of pills of size B

Step 2. Objective function

Minimum (no. of pills) $z = x_1 + x_2$

Step 3. Constraints

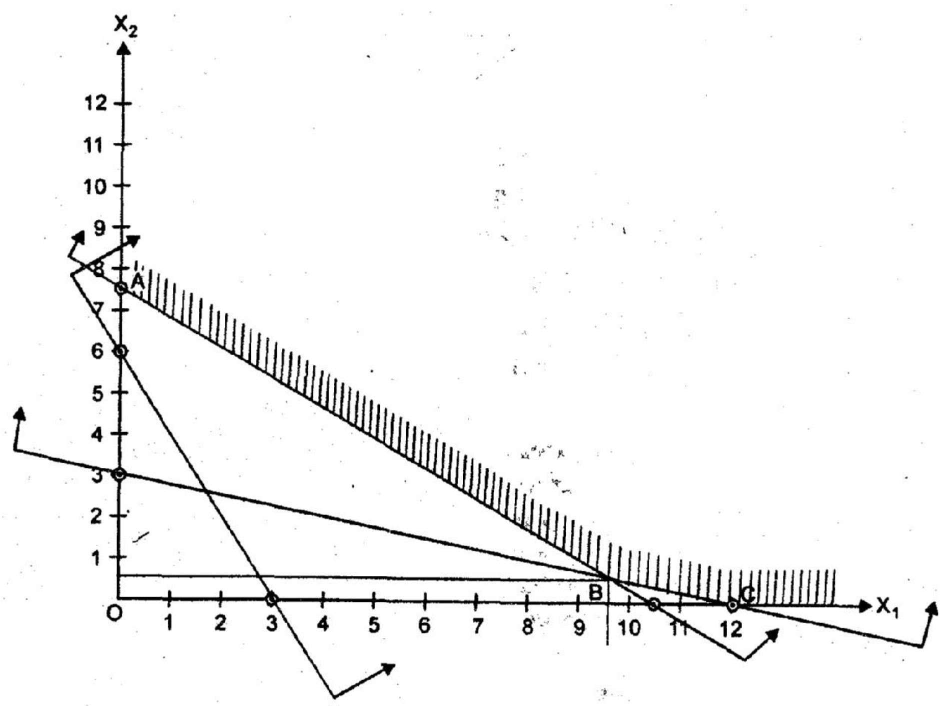
$$4x_1 + 2x_2 \geq 12$$
$$7x_1 + 10x_2 \geq 74$$
$$2x_1 + 8x_2 \geq 24$$

Step. 4. Add non negativity constraints

$$x_1 \geq 0 \quad ; \quad x_2 \geq 0$$

Determining the value of x_1 and x_2 by graphical method

$4x_1 + 2x_2 = 12$	$x_1 = 0; x_2 = 6$ and $x_2 = 0; x_1 = 3$
$7x_1 + 10x_2 = 74$	$x_1 = 0; x_2 = 7.4$ and $x_2 = 0; x_1 = 10.57$
$2x_1 + 8x_2 = 24$	$x_1 = 0; x_2 = 3$ and $x_2 = 0; x_1 = 12$



Point A (0, 7.4) Z(A) = 0 + 7.4 = 7.4 (Minimum)

Point C (12, 0) Z (C) = 12 + 0 = 12

Point B (9.6, 0.6) Z(B) = 9.6 + 0.6 = 10.2

No. of pills of size A = 0

No. of pills of size B = 7.4 ≈ 8 pills

Minimum no. of pills = 8 pills.

Problem 3. An automobile manufactuEer makes automobiles and trucks in a factory that is divided into two shops. Shop A which perform the basic assy operation must work 5 man days on each truck but only 2 man days on each automobile. Shop B which perform finishing operations must work 3 man days for each automobile or truck that it produces. Because of men and machine limitations shop A has 180 man days per week available while shop B has 135.man days per week. If the manufacturer makes a profit of Rs. 300 on each truck and Rs. 200 on each automobile; how many of each should be produced to maximize his profit?

Solution :

	Shop A	Shop B	Profit
Automobile	2 man days	3 man days	Rs. 200
Trucks	5 man days	3 man days	Rs. 300
Availability	180 man days/week	135 man days/week	

1. Select decision variable
   x_1 - no. of automobile to be produced/week
   x_2 - no. of trucks to be produced/week
2. Objective function
   Maximize $Z = 200x_1 + 300x_2$
3. Constraints

$$2x_1 + 5x_2 \le 180$$
$$3x_1 + 3x_2 \le 135$$

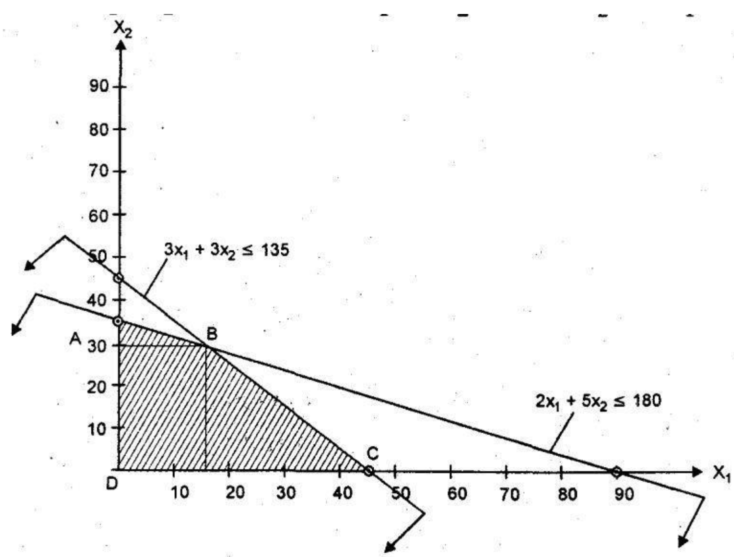
4. Add non negativity constraints

$$x_1 \ge 0; x_2 \ge 0$$

Determine the value of x_1 and x_2 by graphical method

$2x_1+5x_2 = 180$ $\quad x_1 = 0; x_2 = 36$ and $x_2 = 0; x_1 = 90$

$3x_1+3x_2 = 135$ $\quad x_1 = 0; x_2 = 45$ and $x_2 = 0; x_1 = 45$



Point D (0, 0) Z (D) $= 200 \times 0 + 300 \times 0 = 0$
Point A (0, 36) Z (A) $= 200 \times 0 + 300 \times 36 = 10800$
Point C (45, 0) Z (C) $= 200 \times 45 + 300 \times 0 = 9000$
Point B (15 , 30) Z (B) $= 200 \times 15 + 300 \times 30 = 3000 + 9000 = 12000$
Maximum Profit at Point B (15 , 30) *i.e.* Rs.12000/-

x_1 = no. of automobile/week = 15

x_2 = no. of trucks/week = 30

Maximum profit = 12000/-

Problem 4. On completing the construction of house a person discovers that J square feet of plywood scrap and 80 square feet of white pine scrap are in use!able m for the construction of tables and book cases. It takes 16 square feet of plywood 8 square feet of white pine to make a table, 12 square feet of plywood and 16 Llare feet of white pine are required to contruct a book case. By selling the finishing duct to a local furniture store the person can realize a profit of Rs. 25 on each table d Rs. 290 on each book case. How may the man most profitably use the left over ood ? Use graphical method to solve problem.

Solution :

	Plywood	White pine	Profit	
Table	16	8	Rs. 25	each table
Book case	12	16	Rs. 290	each book case
Availability	100	80		

1. Select decision variable
   x_1 – no. of table
   x_2 – no. of book case
2. Objective function
   Maximize profit (Z) = $25x_1 + 290x_2$
3. Constraints

$$16x_1 + 12x_2 \leq 100$$
$$8x_1 + 16x_2 \leq 80$$

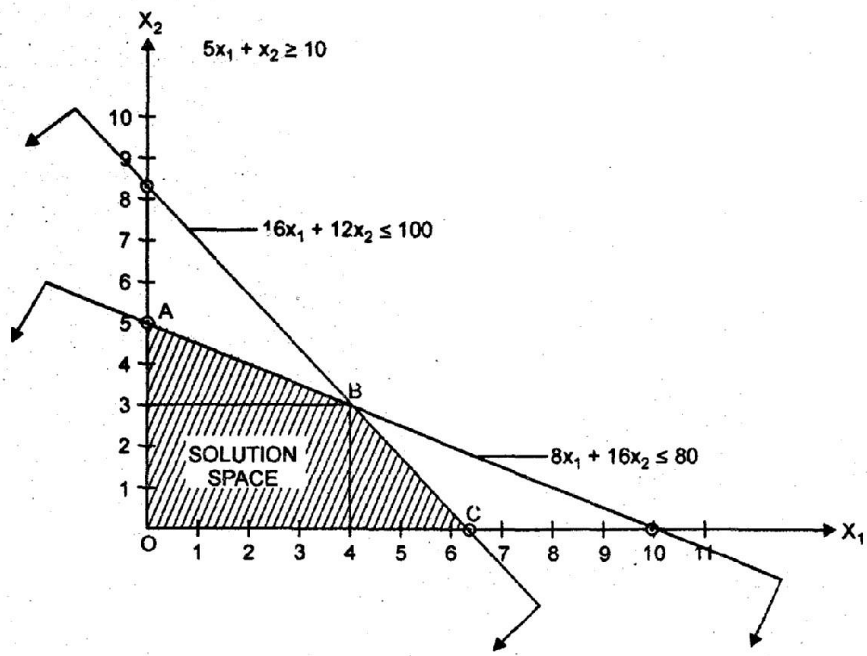
4. Add non negativity constraints

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Determine the value of x_1 and x_2 using graphical method

$16x_1 + 12x_2 = 100$ $\quad$ $x_1 = 0; x_2 = 8.3$ and $x_2 = 0; x_1 = 6.25$

$8x_1 + 16x_2 = 80$ $\quad$ $x_1 = 0; x_2 = 5$ and $x_2 = 0; x_1 = 10$



Point O (0, 0) Z(O) $= 25 \times 0 + 290 \times 0 = 0$

Point A (0, 5) Z (A) $= 25 \times 0 + 290 \times 5 = 1450$

Point C (6.25, 0) Z (C) $= 25 \times 6.25 + 290 \times 0 = 156.25$

Point B (4, 3) Z (B) $= 25 \times 4 + 290 \times 3 = 100 + 870 = 970$

Maximum profit (Z) at point A *i.e.* Rs. 1450.

x_1 – no. of table = 0 Max. Profit (Z) = Rs. 1450/–

x_2 – no.of bookcase = 5.

Problem 5. A truck can carry a total of 10 tons of product. Three types of produts are available for shipment. Their weight and values are tabulated. Assuming that at. least one of each type must be shipped. Determine the loading which will maximize the total value. Formulate the problem.

Type	Value (Rs)	Weight (tons)
A	20	1
B	50	2
C	60	2

Solution : 1. Select decision variable

x_1 – no. of type A products

x_2 – no. of type B products

x_3 – no. of type C products

2. Objective function

Maximize (total value) $Z = 20x_1 + 50x_2 + 60x_3$

3. Constraints

$$x_1 + 2x_2 + 2x_3 \leq 10$$
$$x_1 \geq 1 \Rightarrow x_1 = 1+x$$
$$x_2 \geq 1 \Rightarrow x_2 = 1+y$$
$$x_3 \geq 1 \Rightarrow x_3 = 1+z$$

Put these values in constraint 1.

$$(1 + x) + 2(1 + y) + 2(1 + z) \leq 10$$
$$1 + x + 2 + 2y + 2 + 2z \leq 10$$
$$x + 2y + 2z \leq 5$$

4. Add non negativity constraints

$$x \geq 0; y \geq 0; Z \geq 0$$

Objective function in terms of x, y, z

Maximize $\quad (V) = 20(1 + x) + 50(1 + y) + 60(1+z)$

$\quad = 20 + 20x + 50 + 50y + 60 + 60z$

Max. $\quad (U) = 20x + 50y + 60z + 130$

Subject to

$$x + 2y + 2z \leq 5$$
$$x \geq 0; y \geq 0; z \geq 0;$$

Simplex method

2. Find initial basic feasible solution

Substituting $x_1 = x_2 = x_3 = 0$ in equation (*i*), (*ii*) and (*iii*)

$$S_1 = 8$$
$$S_2 = 10$$
$$S_3 = 15$$

3. Perform optimality test

		C_J	3	5	4	0	0	0		
FR	C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b	θ
				key element						
Key row	0	S_1	2	3	0	1	0	0	8	8/3 ← Key row
2/3	0	S_2	0	2	5	0	1	0	10	5
2/3	0	S_3	3	2	4	0	0	1	15	15/2
		Z_j	0	0	0	0	0	0	$\boxed{0}$	
		C_j-Z_j	3	5	4	0	0	0		
				key column ↑						

4. Iterate towards optimal solution

		C_J	3	5	4	0	0	0		
FR	C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b	θ
	5	x_2	2/3	1	0	1/3	0	0	8/3	∝
	0	S_2	-4/3	0	5	-2/3	1	0	14/3	14/15 ← Key row
4/5	0	S_3	5/3	0	4	-2/3	0	1	29/3	29/12
		Z_J	10/3	5	0	5/3	0	0	$\boxed{40/3}$	
		C_J-Z_J	-1/3	0	4	-5/3	0	0		
					Key column ↑					

FR	C_B	C_J Basis	3 x_1	5 x_2	4 x_3	0 S_1	0 S_2	0 S_3	b	θ
$\frac{10}{41}$	5	x_2	2/3	1	0	1/3	0	0	8/3	4
$\frac{-4}{41}$	4	x_3	-4/15	0	1	-2/15	1/5	0	14/15	-7/2
	0	S_3	41/15	0	0	-2/15	-4/5	1	89/15	89/41
		Z_J	34/15	5	4	17/15	4/5	0	$\boxed{256/15}$	
		$C_J - Z_J$	11/15	0	0	-17/15	-4/5	0		

Key row

C_B	C_J Basis	3 x_1	5 x_2	4 x_3	0 S_1	0 S_2	0 S_3	b	θ
5	x_2	0	1	0	45/123	8/41	-10/41	50/41	
4	x_3	0	0	1	-6/41	33/205	4/41	62/41	
3	x_1	1	0	0	-2/41	-12/41	15/41	89/41	
	Z_J	3	5	4	135/41	152/41	11/41	$\boxed{765/41}$	
	$C_J - Z_J$	0	0	0	-135/41	-152/41	-11/41		

Since all element of $C_J - Z_J$ row are negative or zero so optimality test is passed

$$x_1 = \frac{89}{41}$$

$$x_2 = \frac{50}{41}$$

$$x_3 = \frac{62}{41}$$

$$\text{Max } z = 3\times\frac{89}{41} + 5\times\frac{50}{41} + 4\times\frac{62}{41}$$

$$\text{Ans. } z = \frac{765}{41}.$$

Problem 6. **Show that there is an unbounded solution to the following L.P. problem.**

$$\text{Maximize } Z = 4x_1+x_2+3x_3+5x_4$$

$$\text{Subject to } 4x_1 - 6x_2 - 5x_3 - 4x_4 \geq -20$$

$$-3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$$

$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution : Standard form

Multiplying the first constraint by –1

$$-4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 20$$

$$-4x_1 + 6x_2 + 5x_3 + 4x_4 + S_1 = 20$$

$$-3x_1 - 2x_2 + 4x_3 + x_4 + S_2 = 10$$

$$-8x_1 - 3x_2 + 3x_3 + 2x_4 + S_3 = 20$$

$$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$$

2. ibfs

$$x_1 = x_2 = x_3 = x_4 = 0$$

$$S_1 = 20; S_2 = 10; S_3 = 20; Z = 0$$

3. Iterations for optimal solution

	C_J	4	1	3	5	0	0	0			
C_B	Basis	x_1	x_2	x_3	x_4	S_1	S_2	S_3	b	θ	
0	S_1	-4	6	5	4	1	0	0	20	5	← Key row
0	S_2	-3	-2	4	1	0	1	0	10	10	
0	S_3	-8	-3	3	2	0	0	1	20	10	
	Z_J	0	0	0	0	0	0	0	0		
	C_J-Z_J	4	1	3	5	0	0	0			

	C_J	4	1	3	5	0	0	0		
C_B	Basis	x_1	x_2	x_3	x_4	S_1	S_2	S_3	b	θ
5	x_4	-1	3/2	5/4	1	1/4	0	0	5	-5
0	S_2	-2	-7/2	11/4	0	-1/4	1	0	5	-5/2
0	S_3	-6	-6	1/2	0	-1/2	0	1	10	-5/3
	Z_J	-5	15/2	25/4	5	5/4	0	0	25	
	C_J-Z_J	9	-13/2	-13/4	0	-5/4	0	0		

Since all replacement ratios (θ) are negative the problem has no bounded solution and further computation stop. (Unbounded Solution)

Problem 7. 6. Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Solution : Step 1. Standard form

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3$

Subject to $x_1 + 2x_2 + 3x_3 + 0x_4 + A_1 + 0A_2 + 0A_3 = 15$

$2x_1 + x_2 + 5x_3 + 0x_4 + 0A_1 + A_2 + 0A_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 + 0A_1 + 0A_2 + A_3 = 10$

$x_1, x_2, x_3, x_4, A_1, A_2, A_3 \geq 0$

Step 2. ibfs

$x_1 = x_2 = x_3 = x_4 = 0$

$A_1 = 15$

$A_2 = 20$

$A_3 = 10$

$Z = -45M$

Step 3. Check for optimality test

	C_J	1	2	3	-1	-M	-M	-M			
C_B	Basis	x_1	x_2	x_3	x_4	A_1	A_2	A_3	b	θ	
-M	A_1	1	2	3	0	1	0	0	15	5	
-M	A_2	2	1	(5)	0	0	1	0	20	4	← Key row
-M	A_3	1	2	1	1	0	0	1	10	10	
	Z_J	-4M	-5M	-9M	-M	-M	-M	-M	-45M		
	$C_J - Z_J$	1 + 4M	2 + 5M	3 + 9M	-1 + M	0	0	0			

	C_J	1	2	3	-1	-M	-M		
C_B	Basis	x_1	x_2	x_3	x_4	A_1	A_3	b	θ
- M	A_1	- 1/5	(7/5)	0	0	1	0	3	15/7 ← Key row
3	x_3	2/5	1/5	1	0	0	0	4	20
- M	A_3	3/5	9/5	0	1	0	1	6	10/3
	Z_J	$\frac{6-2M}{5}$	$\frac{3-16M}{5}$	3	- M	- M	- M	12 - 9M	
	$C_J - Z_J$	$\frac{-1+2M}{5}$	$\frac{7+16M}{5}$ ↑	0	- 1 + M	0	0		

	C_J	1	2	3	- 1	- M		
C_B	Basis	x_1	x_2	x_3	x_4	A_3	b	θ
2	x_2	- 1/7	1	0	0	0	15/7	∝
3	x_3	3/7	0	1	0	0	25/7	∝
- M	A_3	6/7	0	0	(1)	1	15/7	15/7 ← Key row
	Z_J	$\frac{7-6M}{7}$	2	3	- M	- M	$\frac{105-15M}{7}$	
	$C_J - Z_J$	$\frac{6M}{7}$	0	0	- 1 + M	0		

	C_J	1	2	3	-1		
C_B	Basis	x_1	x_2	x_3	x_4	b	θ
2	x_2	-1/7	1	0	0	15/7	-15
3	x_3	3/7	0	1	0	25/7	25/3
-1	x_4	6/7	0	0	1	15/7	5/2 ← Key row
	Z_J	1/7	2	3	-1	90/7	
	C_J-Z_J	6/7 ↑	0	0	0		

	C_J	1	2	3	-1	
C_B	Basis	x_1	x_2	x_3	x_4	b
2	x_2	0	1	0	1/6	5/2
3	x_3	0	0	1	-1/2	5/2
1	x_1	1	0	0	7/6	5/2
	Z_J	1	2	3	0	15
	C_J-Z_J	0	0	0	-1	

C_J-Z_J is either zero or negative under all columns, The optimal feasible solution has been obtained

$$x_1 = \frac{5}{2},\ x_2 = \frac{5}{2},\ x_3 = \frac{5}{2},\ x_4 = 0$$

$$A_1 = A_2 = A_3 = 0$$

$$\boxed{Z_{max} = 15}$$

Problem 8. **Use penalty Method to**

minimize $Z = x_1 + 2x_2 + x_3$

Subject to $x_1 + \frac{x_2}{2} + \frac{x_3}{2} \leq 1$

$$\frac{3}{2}x_1 + 2x_2 + x_3 \geq 8$$

$$x_1, x_2, x_3 \geq 0$$

Solution : 1. Standard form

minimize $z = x_1 + 2x_2 + x_3 + 0S_1 + 0S_2$

Subject to $x_1 + \frac{x_2}{2} + \frac{x_3}{2} + S_1 = 1$

$$\frac{3}{2}x_1 + 2x_2 + x_3 - S_2 + A_1 = 8$$

$$x_1, x_2, x_3, S_1, S_2 \geq 0$$

2. ibfs

Setting $x_1 = x_2 = x_3 = S_2 = 0$

$S_1 = 1, A_1 = 8$

3. Iterations for optimal Solution

	C_J	1	2	1	0	0	M		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	b	θ
0	S_1	1	1/2	1/2	1	0	0	1	2 ←
M	A_1	3/2	2	1	0	-1	1	8	4
	Z_J	3/2M	2M	M	0	-M	M	$\boxed{8M}$	
	$C_J - Z_J$	$1 - \frac{3}{2}M$	2 - 2M ↑	1 - M	0	M	0		

C_B	Basis (C_J)	x_1 (1)	x_2 (2)	x_3 (1)	S_1 (0)	S_2 (0)	A_1 (M)	b
2	x_2	2	1	1	2	0	0	2
M	A_1	-5/2	0	- 1	- 4	- 1	1	4
	Z_J	$4-\frac{5}{2}M$	2	2 - M	4 - 4M	- M	M	4 + 4M
	$C_J - Z_J$	$-3+\frac{5}{2}M$	0	- 1 + M	- 4 + 4M	M	0	

Since C_Z is non negative under all columns so optimality test passed since A1appears in he basis at a positive value, the given problem has no feasible solution

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Problem 9. **Use the two phase simplex method to**

Maximize $Z = 5x_1 - 4x_2 + 3x_3$

Subject to $2x_1 + x_2 - 6x_3 = 20$

$6x_1 + 5x_2 + 10x_3 \leq 76$

$8x_1 - 3x_2 + 6x_3 \leq 50$

$x_1, x_2, x_3 \geq 0$

Solution : Step 1. Standard form

$2x_1 + x_2 - 6x_3 + A_1 = 20$

$6x_1 + 5x_2 + 10x_3 + S_1 = 76$

$8x_1 - 3x_2 + 6x_3 + S_2 = 50$

$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$

The new objective function

minimize $w = A_1$

Subject to

$2x_1 + x_2 - 6x_3 + 0S_1 + 0S_2 + A_1 = 20$

$6x_1 + 5x_2 + 10x_3 + S_1 + 0S_2 + 0A_1 = 76$

$8x_1 - 3x_2 + 6x_3 + 0S_1 + S_2 + 0A_1 = 50$

$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$

Step 2. ibfs

$x_1 = x_2 = x_3 = 0$

$A_1 = 20$

$S_2 = 76$

$S_3 = 50$

Step 3. Iterations for optimal Solution :

	C_J	0	0	0	0	0	1		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	b	θ
1	A_1	2	1	–6	0	0	1	20	10
0	S_1	6	5	10	1	0	0	76	$\frac{38}{3}$
0	S_2	(8)	–3	6	0	1	0	50	$\frac{25}{4}$ Key row
	Z_J	2	1	–6	0	0	1	20	
	C_J-Z_J	–2	–1	6	0	0	0		

	C_J	0	0	0	0	0	1.		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	b	θ
1	A_1	0	(7/4)	–15/2	0	–1/4	1	15/2	30/7 Key row
0	S_2	0	29/4	11/2	1	–3/4	0	77/2	154/29
0	x_1	1	–3/8	3/4	0	1/8	0	25/4	–50/3
	Z_J	0	7/4	–15/2	0	–1/4	1	15/2	
	C_J–Z_J	0	–7/4	15/2	0	1/4	0		

	C_J	0	0	0	0	0	1	
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	b
0	x_2	0	1	– 30/7	0	– 1/7	4/7	30/7
0	S_2	0	0	256/7	1	2/7	– 29/7	52/7
0	x_1	1	0	– 6/7	0	1/14	3/14	55/7
	Z_J	0	0	0	0	0	0	0
	C_J - Z_J	0	0	0	0	0	1	

Since C-Z is non negative under all columns and no artificial variable appears in the basis.

2nd phase (deleting artificial variable column)

	C_J	5	-4	3	0	0	
C_B	**Basis**	x_1	x_2	x_3	S_1	S_2	b
-4	x_2	0	1	-30/7	0	-1/7	30/7
0	S_2	0	0	256/7	1	2/7	52/7
5	x_1	1	0	-6/7	0	1/14	55/7
	Z_J	5	-4	90/7	0	13/14	155/7
	$C_J - Z_J$	0	0	-69/7	0	-13/14	

Since $C_J - Z_J$ is either negative or zero under all columns, optimality test has passed.

$$x_1 = \frac{55}{7}, x_2 = \frac{30}{7}, x_3 = 0$$

$$\boxed{Z_{max} = \frac{155}{7}}$$

variable.

New values of the Entering Variable = Old values of the leaving variable / Key element.

Thereafter, we should write the new values of the other left out rows. The formula is New values of the Left out row= Old values of the left out row – (New values of the entering variable X value in the key column of the Old left out row)

UNIT – 1 – INTRODUCTION AND LINEAR PROGRAMMING

QUESTION BANK

PART – A

1. Define Operations Research.
2. Write the stages in operations research?
3. What are the areas in which operations research is being applied?
4. Name the models being classified based on nature.
5. What do you mean by simulation model?
6. Enumerate the limitations of operations research.
7. What is meant by Linear programming problem?
8. Write the steps involved in formulation of linear programming problem.
9. What are the decision variables in LPP?

10. A person wants to decide the constituents of a diet which will fulfill his daily requirement of protein, fat, and carbohydrates at minimum cost. The choice is to be made from 4

11.

different types of food.

Food type	(Yield/unit)			Cost/unit
	protein	fat	carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Min	800	200	700	

requirement

12. Consider food stuff A&B. These contain three vitamins V1, V2, V3. Minimum daily requirement of V1 is 1mg, V2 is 50mg and V3 is 10mg. Suppose food A contain 1mg of V1, 100mg of V2 and 10mg of V3. and food B contain 1mg of V1,10mg of V2. Cost of 1 unit of food A is ₹1 and food B ₹1.5.

13. Formulate the LPP. An organization wants to produce Tables & Chairs Profit of 1 table is ₹100 and profit of 1 Chair is ₹50.

	Tables	Chairs	Maximum hours available
Cutting (hours)	4	1	300
Painting (hours)	1	½	100

PART - B

11. Briefly discuss about Models in operations research.

OR

12. Solve the LPP using Graphical method for the given formulation.

MAX $Z=28x+30y$

Subject to $x+3y\leq 18$, $3x+y\leq 8$, $4x+5y\leq 30$ $(x, y \geq 0)$.

13. Define operations research. Give the scope, characteristics of operations research.

OR

14. Solve the given LPP using Simplex method.
MAX $Z=6X_1+12X_2$
Subject to $3X_1+4X_2\leq 12$, $10X_1+5X_2\leq 20$, $(X_1, X_2 \geq 0)$.

15. Solve the LPP using Graphical method for the given formulation. MAX $Z=5x-2y$

Subject to $2x+y\leq 9$, $2x-4y48$, $3x+2y\leq 3$ $(x, y \geq 0)$.

OR

16. Solve the given LPP using Simplex method. MAX $Z=6X_1+12X_2$

Subject to $3X_1+4X_2\leq12$, $10X_1+5X_2\leq20$, $(X_1, X_2 \geq 0)$.

17. Solve the LPP using Graphical method for the given formulation. MAX $Z=4X_1+2X_2$
Subject to $2X_1+3X_2\leq18$, $X_1+X_2\geq10$ $(X_1,X_2 \geq 0)$.

OR

18. Solve the given LPP using Simplex method. MINZ= $-40X1 - 100X2$
Subject to $10X_1+5X_2\leq250$, $2X_1+5X_2\leq100$, $2X_1+3X_2\leq90$ $(X_1, X_2 \geq 0)$.

19. Solve the LPP using Graphical method for the given formulation. MAX $Z=2X_1+4X_2$
Subject to $X_1+X_2\leq14$, $3X_1+2X_2\geq30$, $2X_1+X_2\leq18$ $(X_1, X_2 \geq 0)$.

OR

20. Solve the LPP using Graphical method for the given formulation
MAX $Z=2X1 + 4X2$
Subject to $X1+X2\leq14$, $3X1+2X2\geq30$,
$2X1+X2\leq18$, $(X1, X2\geq0)$

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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – II – Transportation and Assignment Problems – SBAA5205

UNIT II TRANSPORTATION AND ASSIGNMENT PROBLEMS 12 Hrs.

Transportation Problems - Initial Solution by North West Corner Method, Least Cost Method and VAM Method; MODI Method of Deriving Optimum Solution. Assignment Problems-Hungarian Method of Solving Minimization and Maximization Problems – Restricted Assignment problems – Travelling Salesman Problem.

1. TRANSPORTATION PROBLEMS

1.1. INTRODUCTION

quantities to be shifted from each source to destination, so that the total transportation cost is minimum.

Suppose a factory owns ware houses in 3 different locations in a city and has to dispatch the monthly requirement of the product manufactured by them to 5 different wholesale markets located in the same city. The cost of transporting one unit of the product from the i-th warehouseto the j-th market is known and is cij. It is assumed that the total cost is a linear function so that the total transportation cost of transporting xij, units of the product from the i-th warehouse to the j-th market is given by Σcijxij.

It is clear that the factory management will be interested in obtaining a solution that minimizes the total cost of transportation. During the process of transportation they will also face the constraints that from a warehouse they cannot transport more than what is stored or available in the warehouse (supply) and that they need to transport to a market the total monthly requirement of the market (demand).

1.2. ASSUMPTIONS

- Quantity of supply at each source is known.
- Quantity demanded at each destination is known.

The cost of transportation of a commodity from each source to destination is known.

1.3. PROCEDURE TO SOLVE TRANSPORTATION PROBLEM

Step I : Deriving the initial basic feasible solution.

Step II : Deriving the final optimal solution.

1.4 DERIVING THE INITIAL BASIC FEASIBLE SOLUTION

North West corner method.

Matrix minimum method.

Vogel's approximation method (VAM Method) penalty method.

1.5 DERIVING THE FINAL SOLUTION

Modified distribution method / Modi method / UV method.

If total demand = total supply, then it is a balanced transportation problem.

If the total supply not equal to total demand, then the transportationproblem is unbalanced transportation problem.

2. NORTH WEST CORNER METHOD

1. Check if Demand=Supply. If not add dummy row or column.
2. Select the North West (upper left hand) corner cell.
3. Allocate as large as possible in the North West corner cell.
4. If demand is satisfied, strike off the respective column and deduct supply accordingly If supply is exhausted, strike off the respective row and deduct demand accordingly
5. From the resultant array, locate the North West corner cell and repeat the procedure

Note : The assignment done is not taking cost into consideration.

6. Continue allocation until all demand is satisfied and all supply is exhausted.
7. Multiply the allocated quantity *cost of transportation for each occupied cell and add it to find the total cost.

3. LEAST COST METHOD

1. Check if Demand=Supply. If not add a dummy row /column.
2. The lowest cost cell in the matrix is allocated as much as possible based on demand and supply requirement.
    - If there are more than one least cost cell, select the one where maximum units can be allocated.

- If the tie exist, follow the serial order.

3. If demand is satisfied, strike off the respective column and deduct supply accordingly. If supply is exhausted, strike off the respective row and deduct demand accordingly.
4. From the resultant array, locate the least cost cell and repeat the procedure.
5. Continue allocation until all demand is satisfied and all supply is exhausted.
6. Find total cost.

4. VOGEL'S APPROXIMATION METHOD (VAM)

This method gives better initial solution in terms of less transportation cost through the concept of 'penalty numbers' which indicate the possible cost penalty associated with not assigning an allocation to given cell.

4.1. STEPS IN VOGEL'S APPROXIMATION METHOD (VAM)

1. Check if demand = supply, if not add a dummy row or column.
2. Calculate penalty of each row & column by taking the difference between the lowest unit transportation cost. This difference indicates the penalty or extra cost which has to be paid for not assigning an allocation to the cell with the minimum transportation cost.
3. Select the row or column which has got the largest penalty number.(If there is a tie it can be broken by selecting the cell where the maximum allocation can be made.)
4. In that row or column choose the minimum cost cell and allocate accordingly.
   - If there are more than one minimum cost cell, select the one where maximumunits can be allocated.
   - If the tie exists, follow the serial order.
5. If demand is satisfied, strike off the respective column and deduct supply accordingly. If supply is exhausted, strike off the respective row and deduct demand accordingly.
6. From the resultant array, calculate penalty and repeat the procedure.
7. Continue allocation until all demand is satisfied and all supply is exhausted.
8. Find the total cost.

4.2. OPTIMAL SOLUTION

Work out the basic feasible solution using by any one method

a) Northwest corner method

b) Least cost method

c) VAM/Penalty method. (preferably VAM)

STEP 1:

Check if the number of occupied cells is m+n-1 (i.e., number of rows +number of columns-1) Note : Rows & columns include dummy rows & columns.

- If number of occupied cells = m+n-1, then the solution to the transportation problem is **basic feasible solution**.
- If number of occupied cells < m+n-1, then the solution is **degenerate solution**.

Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

- In case of degeneracy, we allocate an extremely small amount, close to zero [(ξ) epsilon] to one or more empty cells of the transportation table (unoccupied least cost cell). So that total no of occupied cells equals to m+n-1.

STEP 2:

If the basic feasible solution is achieved then MODI method is used to obtain final optimal solution

□Defining the occupied cells.

cij= ui +vj where, cij □co st.

ui □row.

Vj □ column.

□Assume any one ui or vj is to be zero such that max. no of allocaitons are done in that row(i) or column (j) & find value of all other ui's & vj's

STEP 3:

□Evaluate the unoccupied cells.

dij= ui +vj - cij

□If all evaluation values are either negative or zero, then the initial solution is optimal solution.

□If any positive value exist, initial solution is not an optimal solution.

STEP 4:

□Identify the **entering variable.**

The highest **positive** evaluated value (dij) cell is treated as entering variable cell.

STEP 5:

□Identify leaving variable.

- To identify the leaving variable, construct a closed loop.
- Loop starts at the entering variable cell.
- Loop can go clockwise or anticlockwise.
- The turning point should be occupied cell.
- Loop can cross each other.

STEP 6:

□Start assigning positive & negative.

□Assign positive (+) for the entering variable & negative (-) alternatively. □Where is the **minimum allocation quantity** among the negative □cells.

□Add □to the allocated va lue in the positive □cells and deduct □to the allocated value in the negative □ cells.

□Cell/cells which have zero allocation (after deducting □) is the leaving rvi**a**ble.

STEP 7:

□Prepare a new transportation table.

□The values in the loop will get c hanged as per the step 6 and all other allocations not in the loop remains the same.

STEP 8:

□Check for optimality using step 1 to step 3

- If the solution is optimal calculate the minimum transportation cost from the allocations and the unit costs given.
- Repeat the procedure from step 4 to step 8, if the solution is not optimal.

5. ASSIGNMENT

In a printing press there is one machine and one operator is there to operate. How would you employ the worker? Your immediate answer will be, the available operator will operate the machine. Again suppose there are two machines in the press and two operators are engaged at different rates to operate them. Which operator should operate which machine for maximizing profit?

Similarly, if there are n machines available and n persons are engaged at different rates to operate them. Which operator should be assigned to which machine to ensure maximum efficiency? While answering the above questions we have to think about the interest of the press, so we have to find such an assignment by which the press gets maximum profit on minimum investment. Such problems are known as "assignment problems".

Assignment problem is a particular case of the transportation problem in which objective is to assign number of task to equal number of facilities at minimum cost and maximum profit. Suppose there are ‘m' facilities and ‘n' jobs and the effectiveness of each facility for each job are given, the objective is to assign one facility to one job so that the given measure of effectiveness is optimized.

If the matrix contains the cost involved in assignment the aim is to **minimize the cost**.

If the matrix contains revenue or profit the aim is to **maximize the revenue or profit**.

5.1. HUNGARIAN METHOD OR ASSIGNMENT ALGORITHM

STEP 1: **Balancing the problem**

- Check if the No. of Rows is equal to the No. of Columns, if not add a dummy row or a dummy column.

STEP 2: Row wise calculation (row reduced matrix)

- Select the min cost element from each row and subtract it from all the elements in the same row.

STEP 3: Column wise calculation (column reduced matrix)

- From the resultant matrix, select the minimum cost element from each column and subtract it from all the other elements in the same column.

STEP 4: Assigning the zeroes

- Starting with first row of the resultant matrix received in first step, examine the rows one by one until a row containing exactly one zero is found. Then an experimental assignment indicated by ' ' is marked to that zero. Now cross all the zeros in the column in which the assignment is made. This procedure should be adopted for each row assignment.
- When the set of rows has been completely examined, an identical procedure is applied successively to columns. Starting with column 1, examine all columns until a column containing exactly one zero is found. Then make an experimental assignment in that position andcross other zeros in the row in which the assignment was made. Continue these successive operations on rows and columns until all zero's have either been assigned or crossed-out.
- If all the zeros are assigned or crossed out, i.e., we get the maximal assignment.

Note: In case, if two zeros are remained by assignment or by crossing out in each row or column. In this situation we try to exclude some of the zeros by trial and error method.

STEP 5: Check for optimality

.

- ✓ If each job is assigned to each facility, then assignment is optimal. If any job or facility is left without assignment move to step 6

STEP 6: Draw of minimum lines to cover zeros

Draw the minimum possible straight lines covering all the zeros in the matrix by the following procedure

- ✓ Mark (√) rows in which the assignment has not been done.
- ✓ Locate zero in marked (√) row and then mark (√) the corresponding column.

- ✓ In the marked (√) column, locate assigned zeros & then mark (√) the corresponding rows.
- ✓ Repeat the procedure, till the completion of marking.
- ✓ Draw the lines through **unmarked rows and marked columns**.

Note: If the above method does not work then make an **arbitrary assignment**. If the number of these lines is equal to the order of the matrix then it will be an optimal solution and then go to step9 Otherwise proceed to step 7.

STEP 7: Modified Matrix

- ➢ Identify covered elements, uncovered elements and junction point
  - **Covered Elements –** where the lines passes through
  - **Uncovered Elements –** where the line does not pass through.
  - **Junction Point-** where the lines intersects
- ➢ Select the smallest element from the uncovered elements.
- ➢ Subtract this smallest element from the uncovered elements.
- ➢ Add this smallest element to the junction point
- ➢ Covered elements remain untouched

Thus we have increased the number of zero's

STEP: 8

- ➢ Repeat the procedure of assigning the zeroes as step 4.
- ➢ Repeat the procedure of checking for optimality as step 5.
- ➢ If optimality is arrived move to step 9 otherwise repeat steps 6 to 8.

STEP : 9

- ➢ Write separately the assignment (ONE TO ONE) and calculate the total cost taking corresponding values from the problem data.

NOTE:

Multiple optimal solutions

- ➢ If the final matrix (for zero assignment) is having more than one zero on rows and columns at independent positions (not possible to assign or cancel row-wise or column-wise) choose arbitrarily one zero for assignment and cancel all zeros in the corresponding rows and columns.

- Repeat the procedure by choosing another zero for assignment till all such zeroes are considered.
- Each assignment by this procedure will provide different set of assignments keeping the total minimum cost as constant. This implies multiple optimal solutions with the same optimal assignment cost.

5.2. SOLVING MAXIMISATION PROBLEMS IN ASSIGNMENT USING HUNGARIAN METHOD

- The maximization problem can be converted in to a minimization problem by subtracting all the elements of the matrix from the highest value.
- Follow the steps 1 to 9 of Hungarian Algorithm.

Note: While calculating the total profits take corresponding values from initial assignment problem (data before conversion of the problem)

5.3. RESTRICTED ASSIGNMENT PROBLEMS

The assignment technique assumes that the problem is free from practical restrictions and any task could be assigned to any facility. But in some cases, it may not be possible to assign a particular task to a particular facility due to space, size of the task, process capability of the facility, technical difficulties or other restrictions. This can be overcome by assigning a very high processing time of cost element ($\propto$ infinity) to the corresponding cell.

- Use Hungarian method for assignment steps 1 to 9.

NOTE:

- For maximization problems in restricted assignments, convert the problem in to a minimization problems given in the procedure above.
- Substitute $\propto$ (infinity) in the matrix for the restricted assignments.

- Use Hungarian method for assignment steps 1 to 9.

5.4. TRAVELLING SALESMAN PROBLEM

A salesman normally visits numbers of cities starting from high head quarters. The distance (or time or cost) between every pair of cities are assumed to be known. If a salesman has to visit ‘n' cities, then he will have a total of (n-1)! Possible round trips. The problem of finding the shortest distance (or minimum time or minimum cost) if the salesman starts from his headquarters and passes through each city under his jurisdiction exactly once and returns to the headquarters is called the Travelling salesman problem or A Travelling Salesperson problem. A travellingsalesman problem is very similar to the assignment problem with the additional constraints.

a) **Route Conditions**:

- The salesman should go through every city exactly once except the starting city (headquarters).
- The salesman starts from one city (headquarters) and comes back to that city (headquarters).

b) Obviously going from any city to the same city directly is not allowed (i.e., no assignments should be made along the diagonal line).

5.2.1. Steps to solve travelling salesman problem:

i. Assigning an infinitely large element ($\propto$) in each of the squares along the diagonal line in the cost matrix.

ii. Solving the problem as a routine assignment problem.

iii. Scrutinizing the solution obtained under (ii) to see if the ‘route' conditions are satisfied.

iv. If not, making adjustments in assignments to satisfy the condition with minimum increase in total cost (i.e. to satisfy route condition, ‘next best solution' may require to be considered).

i.

Problems

Problem 1. Find the optimum solution to the following problem.

I	3	4	6	8	8	20
II	2	10	1	5	30	30
From III	7	11	20	40	15	15
IV	2	1	9	14	18	13
	40	6	8	18	6	

Solution:

1. Make a transportation model

I	3	4	6	8	8	20
II	2	10	1	5	30	30
III	7	11	20	40	15	15
IV	2	1	9	14	18	13
	40	6	8	18	6	78

Find basic feasible solution (VAM method)

1.

$\boxed{14}$ 3	4	6	8	$\boxed{6}$ 8	20/14/0	[1] [1] [5]
$\boxed{4}$ 2	10	$\boxed{8}$ 1	$\boxed{18}$ 5	30	30/22/18	[1] [3] [3]
$\boxed{15}$ 7	11	20	40	15	15/0	[4] [4] ←
$\boxed{7}$ 2	$\boxed{6}$ 1	9	14	18	13/7/0	[1] [1] [12] ←
40/25 18/4/0	6/0	8/0	18/0	6/0	78	
[0]	[3]	[5]	[3]	[7]		
[1]	[3]	↑	[3]	↑		

Transportation cost = (14 × 3) + (6 × 8) + (4 × 2) + (8 × 1) +(18 × 5) + (15 × 7) + (7 × 2) + (6 × 1)

= 42 + 48 + 8 + 8 + 90 + 105 + 14 + 6

= Rs. 321.

3. Check for optimality (MODI Test) m (a) Cost matrix of allocated cell.

+ n - 1 = 8 (no. of allocation)

u_j \ V_i	1	2	3	4	5
1	3				8
2	2		1	5	
3	7				
4	2	1			

	$V_1 = 0$
$u_1 + v_1 = 3$	$u_1 = 3$
$u_1 + v_5 = 8$	$v_5 = 5$
$u_2 + v_1 = 2$	$u_2 = 2$
$u_2 + v_3 = 1$	$v_3 = -1$
$u_2 + v_4 = 5$	$v_4 = 3$
$u_3 + v_1 = 7$	$v_3 = 7$
$u_4 + v_1 = 2$	$u_4 = 2$
$u_4 + v_2 = 1$	$v_2 = -1$

(*b*) Opp. cost matrix

	0	–1	–1	3	5
3	·	2	2	6	·
2	·	1	·	·	7
7	·	6	6	10	12
2	·	·	1	5	7

(c) Cell evaluation matrix

·	2	4	2	·
·	9	·	·	23
·	5	14	30	3
·	·	8	9	11

Since all the elements of cell evaluation matrix are positive so optimality test is passed.

Minimum Transportation Cost = Rs. 321.

Problem 2. Solve the following cost-minimizing transportation problem.

	D_1	D_2	D_3	D_4	D_5	D_6	Available
O_1	2	1	3	3	2	5	50
O_2	3	2	2	4	3	4	40
O_3	3	5	4	2	4	1	60
O_4	4	2	2	1	2	2	30
Required	30	50	20	40	30	10	180

Ans. 1. Make a transportation model.

1. Find basic feasible solution

2	$\boxed{50}$ 1	3	3	2	5	50/0	[1]
3	2	$\boxed{20}$ 2	4	$\boxed{20}$ 3	4	40/20/0	[0][1][1][0]
$\boxed{30}$ 3	5	4	$\boxed{10}$ 2	$\boxed{10}$ 4	$\boxed{10}$ 1	60/50/40/10/0	[1][1][1][1]
4	2	2	$\boxed{30}$ 1	2	2	30/0	[1][1]
30/0	50/0	20/0	40/10/0	30/20/0	10/0		
[1]	[1]	[0]	[1]	[0]	[1]		
[0]		[0]	[1]	[1]	[1]		
[0]		[2]	[2]	[1]	[3]		

Check for optimality test (m + n -1) > no. of allocation (8)

2	$\boxed{50}$ 1	3	3	$\boxed{\varepsilon}$ 2	5
3	2	$\boxed{20}$ 2	4	$\boxed{20}$ 3	4
$\boxed{30}$ 3	5	4	$\boxed{10}$ 2	$\boxed{10}$ 4	$\boxed{10}$ 1
			$\boxed{30}$		

m + n- 1 = no. of allocation 9

9. Cost matrix of allocated cell

	1			2	
		2		3	
3			2	4	1
·			1		

$u_1 + v_2 = 1$	$v_1 = 0$
$u_1 + v_5 = 2$	$v_2 = 0$
$u_2 + v_3 = 2$	$u_1 = 1$
$u_2 + v_5 = 3$	$v_3 = 0$
$u_3 + v_1 = 3$	$u_2 = 2$
$u_3 + v_4 = 2$	$u_3 = 3$
$u_3 + v_5 = 2$	$v_4 = -1$
$u_3 + v_6 = 1$	$v_5 = 1$
$u_3 + v_4 = 1$	$v_6 = -2$
	$u_4 = 2$

(*b*) Opportunity cost matrix

	0	0	0	–1	1	–2
1	1	·	1	0	·	–1
2	2	2	·	1	·	0
3	·	3	3	·	·	·
2	2	2	2	·	3	0

(c) Cell evaluation matrix

1	·	2	3	·	6
1	0	·	3	·	4
·	2	1	·	·	·
2	0	0	·	–1√	2

identified cell

Iteration for optimal solution.

1.

	50			ε	
		20		20	
30			$^{+}10$	10^{-}	10
			$^{-}30$	√ +	

Check for optimality test

	50			ε	
		20		20	
30			20		10
			20	10	

2nd feasible solution

(a). Cost matrix of allocated cell.

	1			2	
		2		3	
3			2		1
			1	2	

$$\begin{array}{ll}
u_1 + v_2 = 1 & v_1 = 0 \\
u_1 + v_5 = 2 & v_2 = -1 \\
u_2 + v_3 = 2 & u_1 = 2 \\
u_2 + v_5 = 3 & v_3 = -1 \\
u_3 + v_1 = 3 & u_2 = 3 \\
u_3 + v_6 = 1 & u_3 = 3 \\
u_4 + v_4 = 1 & v_4 = -1 \\
u_4 + v_5 = 2 & v_6 = -2 \\
 & u_4 = 2 \\
 & v_5 = 0
\end{array}$$

(*b*) Opp. cost matrix

	0	−1	-1	−1	0	−2
2	2	·	1	1	·	0
3	3	2	·	2	·	1
3	·	2	2	·	3	·
2	2	1	1	·	·	0

(*c*) Cell evaluation matrix

0	·	2	2	·	5
0	0	·	2	·	3
·	3	2	·	1	·
2	1	1	·	·	2

Since all elements of cell evaluation matrix are non negative so 2hldI feasible solution is the optimum solution.

Transportation cost

$= 50 \times 1 + 20 \times 2 + 20 \times 3 + 30 \times 3 + 20 \times 2 + 10 \times 1 + 20 \times 1 + 10 \times 2$
$= 50 + 40 + 60 + 90 + 40 + 10 + 20 + 20$
$= 330/-$

Problem 3. **Goods have to be transported from factories F_1, F_2, F_3 to ware house W_1, W_2, W_3 and W_4. The transportation cost per unit capacities and requirement of the ware house are given in the following table**

	W_1	W_2	W_3	W_4	Capacity
F_1	95	105	80	15	12
F_2	115	180	40	30	7
F_3	195	180	95	70	5
Requrement	5	4	4	11	

Solution. 1. Make a transportation model

	W_1	W_2	W_3	W_4	
F_1	95	105	80	15	12
F_2	115	180	40	30	7
F_3	195	180	95	70	5
	5	4	4	11	24 **Balanced Model**

2. Find a basic feasible solution
VAM method

F_1	95	$\boxed{4}$ 105	80	$\boxed{8}$ 15	12/8/0	[65] ←
F_2	$\boxed{5}$ 115	180	$\boxed{2}$ 40	30	7/2/0	[10]
F_3	195	180	$\boxed{2}$ 95	$\boxed{3}$ 70	5/3/0	[25]
	5/0	4/0	4/2/0	11/3/0		
	[20]	[75]	[40]	[15]		
	[80]	↑	[55]	[50]		
	↑		↑			

3. Optimality test

$$m + n - 1 = \text{number of allocations}$$
$$6 = 6$$

(a) Cost matrix of allocated cell

$u_i \backslash v_j$	1	2	3	4
1	·	105		15
2	115		40	
3		·	95	70

$V_1 = 0$

$u_1 + V_2 = 105$	$V_2 = -10$
$u_1 + V_4 = 15$	$u_1 = 115$
$u_2 + V_1 = 115$	$u_2 = 115$
$u_2 + V_3 = 40$	$V_3 = -75$
$u_3 + V_3 = 95$	$u_3 = 170$
$u_3 + V_4 = 70$	$V_4 = -100$

(*b*) Opp. cost matrix

	0	–10	–75	–100
115	115	·	40	·
115	·	105	·	15
170	170	160	·	·

(*c*) Cell evaluation matrix

–20	·	40	·
·	75	·	15
25	20	·	·

4. Iteration for optimal solution

+	4		8 –
5 –		2 +	
		2 –	3 +

2	4		6
3		4	
			5

2nd feasible solution

5. Check for optimality

(*a*) Cost matrix of allocated cell

$u_i \backslash v_j$	1	2	3	4
1	95	105		15
2	115		40	
3				70

$V_1 = 0$

$u_1 + V_1 = 95$	$u_1 = 95$
$u_1 + V_2 = 105$	$V_2 = 10$
$u_1 + V_4 = 15$	$V_4 = -80$
$u_2 + V_1 = 115$	$u_2 = 115$
$u_2 + V_3 = 40$	$V_3 = -75$
$u_3 + V_4 = 70$	$u_3 = 150$

1. Opp. cost matrix

	0	10	–75	–80
95	·	·	20	·
115	·	125	·	35
150	150	160	75	·

Cell evaluation matrix

·	·	60	·
·	55	·	√ –5
45	20	20	·

Iteration for optimal solution.

+ 2	4		6 –
– 3		4	√ +
			5

5	4		3
		4	3
			5

3rd feasible solution

Check for optimality test (a)

Cost matrix of allocated cell

95	105		15
		40	30
			70

$u_1 + V_1 = 95$ $v_1 = 0$

$u_1 + V_2 = 105$ $u_1 = 95$

$u_1 + V_4 = 15$ $V_2 = 10$

$u_2 + V_3 = 40$ $V_4 = -80$

$u_2 + V_4 = 30$ $V_3 = -70$

$u_3 + V_4 = 70$ $u_2 = 110$

$u_3 = 150$

(*b*) Opp. cost matrix

	0	10	–70	–80
95	·	·	15	·
110	110	120	·	·
150	150	160	80	

(*c*) Cell evaluation matrix

·	·	65	·
5	60	·	·
45	20	15	·

Since all elements of cell evaluation matrix are non negative. Hence 3rd feasible solution is the optimum solution.

Transportation cost = (5 × 95) + (4 × 105) + (3 × 15) + (4 × 40) + (3 × 30) + (5 × 70) = 475 + 420 + 45 + 160 + 90 + 350 = Rs 1540/-

Problem 4. **Solve the following assignment problem.**

		1	2	3	4
	A	12	30	21	15
work	B	18	33	9	31
	C	44	25	24	21
	D	23	30	28	14

Solution:

1. Prepare a square matrix.

2. Reduce the matrix

0	5	12	1
6	8	0	17
32	0	15	7
11	15	19	0

→

3. Check if optimal assignment can be made in the current solution or not

	1	2	3	4
A	$\boxed{0}$	5	12	1
B	6	8	$\boxed{0}$	17
C	32	$\boxed{0}$	15	7
D	11	5	19	$\boxed{0}$

Since there is one assignment in each row and each column, the optimal assignment can be made in the current solution.

Minimum total cost = 12 ×1 + 9 ×1+ 25 ×1 + 14 ×1
= 12 +9 + 25+14
= 60

Ans.
A ⟶ 1
B ⟶ 3
C ⟶ 2
D ⟶ 4

Minimum cost = Rs. 60.

Problem 5. **. Find the optimal assignment for the assignment problem with the following cost matrix.**

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

Solution: 1. Prepare a square matrix

2. Prepare a reduced matrix.

0	0	0	2
2	6	1	0
1	1	4	1
0	4	6	0

0	0	0	2
2	6	1	0
0	0	3	0
0	4	6	0

3. Check if optimal assignment can be made in the current solution.

	I	II	III	IV
A	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
B	2	6	1	$\boxed{0}$
C	$\cancel{0}$	$\boxed{0}$	3	$\cancel{0}$
D	$\boxed{0}$	4	6	$\cancel{0}$

since each row and each column have assignment so optimal assignment can be made.

A – III
B – IV
C – II
D – I
Cost = 1+ 6 + 4 + 5 = 16.

Problem 6. **Four different jobs are to be done on four different machines. Table below indicate the cost of producing job i on machine j in rupees.**

		m/c 1	2	3	4
Job	1	5	7	11	6
	2	8	5	9	6
	3	4	7	10	7
	4	10	4	8	3

Solution : 1. Reduced matrix

0	2	6	1
3	0	4	1
0	3	6	3
7	1	5	0

0	2	2	1
3	0	0	1
0	3	2	3
7	1	1	0

3. Check if optimal assignment can be made in the current solution or not

$\boxed{0}$	2	2	1	✓
3	$\boxed{0}$	$\cancel{0}$	1	
$\cancel{0}$	3	2	3	✓
7	1	1	$\boxed{0}$	

Cross marked column and unmaked row.
Since no. of lines
$\leq$ Rank of matrix
$3 \leq 4$

4. Iterate towards optimality

$\cancel{0}$	1	1	$\cancel{0}$	✓
4	$\boxed{0}$	$\boxed{0}$	1	
$\boxed{0}$	2	1	2	✓
8	1	1	$\boxed{0}$	✓
✓			✓	

number of lines (3) ≤ Rank of matrix (4)

$$
\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{m/c} \\
 & & 1 & 2 & 3 & 4 \\
\hline
 & 1 & \cancel{0} & \cancel{0} & \boxed{0} & \cancel{0} \\
\hline
\text{Job} & 2 & 5 & \boxed{0} & \cancel{0} & 2 \\
\hline
 & 3 & \boxed{0} & 1 & \cancel{0} & 2 \\
\hline
 & 4 & 8 & \cancel{0} & \cancel{0} & \boxed{0} \\
\hline
\end{array}
$$

Since each row and column have assignment so optimality condition is satisfied.

Job 1 – M/c 3
Job 2 – M/c 2
Job 3 – M/c 1
Job 4 – M/c 4

Cost = 11 +5 + 4 +3 = 23

QUESTIONS BANK

TRANSPORTATION
INITIAL SOLUTION
NORTH WEST CORNER METHOD

1.

	Destination				Supply
Sources	D1	D2	D3	D4	
S1	2	4	6	2	100
S2	8	6	5	2	60
S3	9	10	7	5	40
Demand	40	60	80	20	

2.

	Destination				Supply
Sources	D1	D2	D3	D4	
S1	2	4	6	2	70
S2	8	6	5	2	30
S3	9	10	7	5	50
Demand	80	10	20	30	

LEAST COST METHOD

3.

	Destination				Supply
Sources	D1	D2	D3	D4	
S1	2	4	6	2	100
S2	8	6	5	2	60
S3	9	10	7	5	40
Demand	40	60	80	20	

4. .

	Destination				Supply
Sources	D1	D2	D3	D4	
S1	6	1	9	3	70
S2	11	5	2	8	55
S3	10	12	14	7	70
Demand	85	35	50	45	

VOGEL'S APPROXIMATION METHOD

5.

	Destination				Supply
Sources	D1	D2	D3	D4	
S1	7	3	6	8	60
S2	4	2	5	0	100
S3	2	6	5	1	40

Demand	20	50	50	80

6.

	Destination			Supply	
Sources	D1	D2	D3	D4	
S1	6	1	9	3	70
S2	11	5	2	8	55
S3	10	12	14	7	70
Demand	85	35	50	45	

7.

	Destination			Supply
Sources	D1	D2	D3	
S1	2	7	4	5
S2	3	3	1	8
S3	5	4	7	7
S4	1	6	2	14
Demand	7	9	18	

FINAL OPTIMAL SOLUTION (UV METHOD)

8.

	Destination				Supply
Sources	D1	D2	D3	D4	
S1	7	3	8	6	60
S2	4	2	5	10	100
S3	2	6	5	1	40
Demand	20	50	50	80	

9.

		Destination		Supply
Sources	D1	D2	D3	
S1	5	1	7	10
S2	6	4	6	80
S3	3	2	5	15
S4	5	3	2	40
Demand	75	20	50	

10.

	Destination			Supply	
Sources	D1	D2	D3	D4	
S1	6	1	9	3	70
S2	11	5	2	8	55
S3	10	12	4	7	70
Demand	85	35	50	45	

11.

Sources	D1	D2	D3	D4	D5	Supply
S1	3	5	8	9	11	20
S2	5	4	10	7	10	40
S3	2	5	8	7	5	30
Demand	10	15	25	30	40	

DEGENERACY

12.

Sources	D1	D2	D3	Supply
S1	16	20	12	50
S2	14	8	18	50
S3	26	24	16	50
Demand	50	50	50	

13.

Sources	D1	D2	D3	D4	Supply
S1	13	25	12	21	18
S2	18	23	14	9	27
S3	23	15	12	16	21
Demand	14	12	23	27	

14.

Sources	D1	D2	D3	D4	Supply
S1	42	48	38	37	160
S2	40	49	52	51	150
S3	39	38	40	43	190
Demand	80	90	110	160	

ASSIGNMENT

1.

JOBS	Persons 1	2	3	4
A	10	5	13	15
B	3	9	18	3
C	10	7	3	2
D	5	11	9	7

2.

JOBS	Persons 1	2	3	4
A	5	8	4	2
B	1	4	6	3
C	0	4	2	6
D	4	7	5	4

3.

JOBS	Persons 1	2	3	4
A	8	8	4	3
B	4	2	1	6
C	6	8	10	12
D	14	18	20	22

UNBALANCED ASSIGNMENT MODELS

4.

JOBS	Persons 1	2	3	4
A	24	27	18	20
B	26	23	20	31
C	24	22	34	26
D	19	21	21	22
E	30	25	28	27

MAXIMIZATION CASE IN ASSIGNMENT PROBLEMS

5.

	Territories			
	T1	T2	T3	T4
P1	60	50	40	30
P2	40	30	20	15
P3	40	20	35	10
P4	30	30	25	20

6.

	Territories			
	T1	T2	T3	T4
P1	10	22	12	4
P2	16	18	22	10
P3	24	20	12	18
P4	16	14	24	20

RESTRICTED ASSIGNMENT MODEL

7.

	Territories			
	T1	T2	T3	T4
R1	4	-	-	8
R2	9	-	4	3
R3	8	1	2	-

TRAVELLING SALESMAN PROBLEM

8.

		TO		
	A	B	C	D
A	-	46	16	40
B	41	-	50	40
FROM C	82	32	-	60
D	40	40	36	-

9.

		TO			
	A	B	C	D	E
A	-	3	6	2	3
B	3	-	5	2	3
FROM C	6	5	-	6	4
D	2	2	6	-	6
E	3	3	4	6	-

PROBLEMS FOR PRACTICE

10.

		Persons			
JOBS	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

11.

		Persons		
	1	2	3	4
JOBS				
A	8	3	2	1
B	4	5	6	3
C	2	2	9	4
D	1	3	6	5
E	9	3	6	5

MAXIMIZATION CASE IN ASSIGNMENT PROBLEMS

12.

	Territories		
	T1	T2	T3
S1	80	40	30
S2	20	10	10
S3	40	40	60
S4	90	30	40

13.

	Territories		
	T1	T2	T3
P1	20	26	42
P2	24	32	50
P3	32	34	44

RESTRICTED ASSIGNMENT MODEL

14.

	Territories			
	R1	R2	R3	R4
C1	4000	5000	-	-
C2	-	4000	-	4000
C3	3000	-	2000	-
C4	-	-	4000	5000

TRAVELLING SALESMAN PROBLEM

15.

			TO			
		A	B	C	D	E
	A	-	4	7	3	4
	B	4	-	6	3	4
FROM	C	7	6	-	7	5
	D	3	3	7	-	7
	E	4	4	5	7	-

UNIT – II – TRANSPORTATION AND ASSIGNMET MODEL
MODEL QUESTION PAPER

PART - A

1. Define and specify the objective of transportation model.
2. What is meant by unbalanced problem in transportation? How will you convert unbalanced problem into balanced problem in transportation?

3. List the methods used to find initial solution in transportation?
4. What is degeneracy in transportation?
5. Define and list out the objectives of assignment?
6. Specify the route conditions in travelling salesman problem.
7. How to convert maximization problem into minimization problem in assignment?
8. Differentiate between assignment problem and transportation problem.
9. Find the initial solution for the given transportation problem using North-West corner method.

Destination

		D1	D2	D3	D4	Supply
	O1	6	1	9	3	100
Origins	O2	11	5	2	8	60
	O3	10	12	4	7	40
	Demand	40	60	80	20	

10. A Computer centre has got 4 programmers. The centre needs 4 application programmes to be developed. The centre head, after studying carefully the programmes to be developed, estimates the computer time (in minutes) required by the respective experts todevelop the application programmes as follows:

Programmes

		A	B	C	D
	1	120	100	80	90
Programmers	2	80	90	110	70
	3	110	140	120	100
	4	90	90	80	90

Assign the programmers to the programmes in such a way that the total computer time gets minimized.

PART - B

1. What is meant by transportation? Specify the objectives of transportation tool. Write the procedure for making unbalanced problem into balanced problem with an example.

OR

2. Solve the transportation problem using MODI method.

	Destination				
Sources	D1	D2	D3	D4	Supply
S1	7	3	8	6	60
S2	4	2	5	10	100
S3	2	6	5	1	40
Demand	20	50	50	80	

3. Write the procedure to solve transportation problem using MODI method.

OR

4. For the given transportation problem, find the initial solution using North-west corner method and final optimal solution using MODI method.

		Destination				
Sources		D1	D2	D3	D4	Supply
S1	6	1	9	3	70	
S2	11	5	2	8	55	
S3	10	12	4	7	70	
Demand	85	35	50	45		

5. Write the procedure for a) North-West corner method b) Least-Cost method c) Vogel's approximation method.

OR

6. Using U-V method, solve the given transportation problem.

	Destination			
Sources	D1	D2	D3	Supply
S1	5	1	7	10
S2	6	4	6	80
S3	3	2	5	15
S4	5	3	2	40
Demand	75	20	50	

l

7. Write Hungarian algorithm.

OR

8. Given are the costs for assigning jobs to the persons working in an organization, find the minimum cost using the given information.

		Persons			
		P1	P2	P3	P4
Jobs	A	8	8	4	3
	B	4	2	1	6
	C	6	8	10	12
	D	14	18	20	22

9. Write the procedure for a) Maximization problem in assignment with an example b) Restricted assignment problem with an example c) Conditions to solve Travelling salesman assignment problem.

OR

10. Solve the given assignment problem in which profits are given for various territories.

		Territories			
		T1	T2	T3	T4
Profits	P1	10	22	12	4
	P2	16	18	22	10
	P3	24	20	12	18
	P4	16	14	24	20

SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – III – NETWORK ANALYSIS and SEQUENCING – SBAA5205

UNIT III NETWORK ANALYSIS and SEQUENCING 12 Hrs.

Network Analysis- CPM-Network Diagram Construction, Identification of Critical Path, Calculation of Floats. PERT- Calculation of Estimated Time, Standard Deviation and Probability. Sequencing- Sequencing of 'N' number of jobs on Two, Three, Four and N Machines.

NETWORK ANALYSIS

3.1 PROJECT

A **project** defines a combination of interrelated **activities** that must be executed in a certain order before the entire task can be completed. The activities are interrelated in a logical sequencein the sense that some activities cannot start until others are completed. An activity in a project isusually viewed as a job requiring time and possibly resources (like manpower, money, material, machinery etc.) for its completion.

3.1.1 PHASES OF PROJECT

Any type of project scheduling consists of three basic phases namely:

- **PLANNING**

The planning phase is initiated by breaking down the project into distinct activities with their associated logical sequence. The time estimates for each of the activities are then determined.

- **SCHEDULING**

The scheduling phase constructs a time table giving the start and finish times of each activity as well as its relationship to the other activities in the project.

- **CONTROLLING**

The final phase is project control where periodic progress is reviewed and, depending upon the situation revised time-table for the remaining part of the project is worked out.

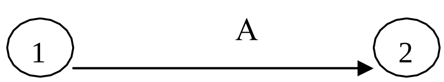
With growing sophistication of technology, the projects at organizational level have tended to become more and more complex, demanding efficient method of planning. Considering the inherent adequacies for planning big and complex projects, some efforts were made in USA and other western countries during 1950s to develop certain more efficient techniques. The outcome was the development of **CPM** (Critical Path Method) and **PERT** (Project Evaluation and Report Technique), which are two important techniques for planning and scheduling of large projects. These techniques are most widely used in industry and services around the globe.

CPM was first developed by E. I. du Pontde Bemours & Company as an application to construction projects and was later extended to a more advanced status by Mauchly Associates. However, PERT was developed for the U.S. Navy by a consulting firm for scheduling the research and Project Scheduling development activities for the Polaris missile program.

Although these two methods were developed independently, they are similar. The most important difference is that the time estimates for the activities are assumed to be deterministic in CPM and probabilistic in PERT. But, the underlying basis of both the techniques is the Network diagram.

3.1.2 Network Diagram

A network (or arrow) diagram is a graphic representation of the project, describing the logical sequence and the interdependence of the activities. Moreover, construction of network diagram helps studying all the activities more critically. The basic elements of a network diagram are Arrowand Node



Tail Event Head Event

So, in a network diagram, an arrow is used to represent an activity, with its head indicating the direction of progress of the concerned project.

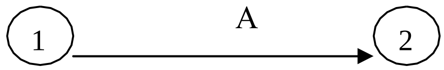
3.1.3 Terminologies related to network diagram

- **Activity**

An activity represents a job or an individual operation for a project. It consumes time, money, or resources in doing the work.

Every activity has a head event and tail event. Event 1(tail event) indicates start and event 2 (head event) indicates completion of activity A. Activity B can start only after completion of activity A.

Activity A is the predecessor activity and Activity B is the successor activity.



Tail Event Head Event

- **Dummy activity**

An activity which does not consume time, money and resources but merely depicts the technological dependence. It is an imaginary activity represented by a dotted line. Purpose for having a dummy activity is to create logic and avoid ambiguity.

Ex. Sending invitation cards for a function:

A

Printing of cards

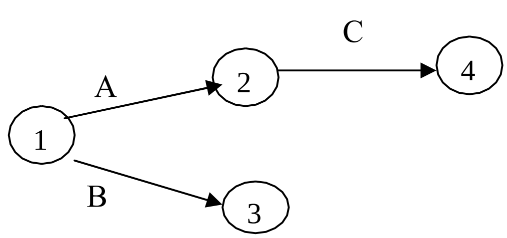
B

Collection of address

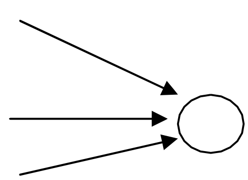
C

Despatching

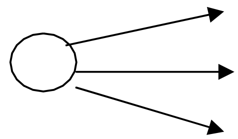
A & B can be done simultaneously but C can be done only after A & B hence to get the network logic we draw dummy activity.



Two or more activities ending in a single node is **merged.**



Two or more activities starting in a single node is **Burst**



3.1.4 HINTS FOR DRAWING A NETWORK DIAGRAM

(a) In network diagram, an arrow represents an activity a d each node signifies either of the two events - start time or completion time. The length of an arrow is arbitrary. It has no relationship with the duration of the activity. The orientation of an arrow indicates the direction of its completion.

(b) The tail-event and head-event of an arrow represent the start and completion of the concerned activity respectively.

(c) Only one activity can span across any given pair of events.

(d) Event numbers should not get repeated.

(e) A dummy activity follows the same logic of precedence relationship as any other normal activity, but consumes no resource (including time).

(f) No unnecessary dummies.

(g) Crossing of arrows should be avoided.

(h) The start-time as well as the completion time of the project must be represented by unique events.

(i) No dangling of arrows. There should not be more than one terminal node.

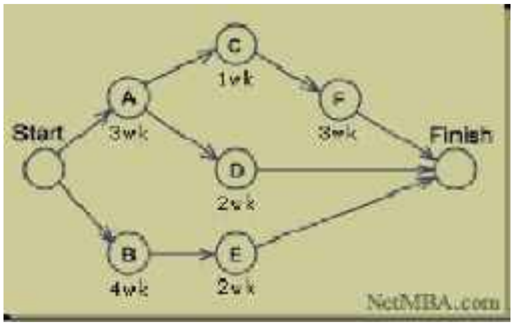
(j) The logic of inter-dependencies between the activities is governed by the following rules.

(i) An event can occur only when all activities leading into it, have been completed.

(ii) No activity can commence until its tail event has occurred.

3.2 Critical Path Method (CPM)

A path in a network diagram is a continuous chain of activities that connects the initial event to the terminal event. The length of a path is the sum of the durations of all the activities those lie on it. Critical Path defines the longest path consisting of critical activities that connects the start and end nodes of the network. To shorten the time for completion of the project, we must reduce the duration of the activities lying on the critical path. In order to complete the project in specified time, no delay is allowed in execution of the critical activities. It may be achieved by diverting allocated resources of non-critical activities to critical activities. However, this calls for information on the slack of each non-critical activity and Critical Path Method finds the same.They are extremely useful to a project-manager.



3.2.1. Steps in CPM Project Planning

1. Specify the individual activities.
2. Determine the sequence of those activities.
3. Draw a network diagram.
4. Estimate the completion time for each activity.
5. Identify the critical path (longest path through the network)
6. Update the CPM diagram as the project progresses.

1. Specify the individual activities

All the activities in the project are listed. This list can be used as the basis for adding sequence and duration information in later steps.

2. Determine the sequence of the activities

Some activities are dependent on the completion of other activities. A list of the immediate predecessors of each activity is useful for constructing the CPM network diagram.

3. Draw the Network Diagram

Once the activities and their sequences have been defined, the CPM diagram can be drawn. CPM originally was developed as an *activity on node* network.

4. Estimate activity completion time

The time required to complete each activity can be estimated using past experience. CPM does not take into account variation in the completion time.

5. Identify the Critical Path

The critical path is the longest-duration path through the network. The significance of the critical path is that the activities that lie on it cannot be delayed without delaying the project. Because of its impact on the entire project, critical path analysis is an important aspect of project planning.

The critical path can be identified by determining the following four parameters for each activity:

- ES - earliest start time: the earliest time at which the activity can start given that its precedent activities must be completed first.

- EF - earliest finish time, equal to the earliest start time for the activity plus the time required completing the activity.
- LF - latest finish time: the latest time at which the activity can be completed without delaying the project.
- LS - Latest start time, equal to the latest finish time minus the time required to complete the activity.

The *slack time* for an activity is the time between its earliest and latest start time, or between its earliest and latest finish time. Slack is the amount of time that an activity can be delayed past its earliest start or earliest finish without delaying the project.

The critical path is the path through the project network in which none of the activities have slack, that is, the path for which ES=LS and EF=LF for all activities in the path. A delay in the critical path delays the project. Similarly, to accelerate the project it is necessary to reduce the total time required for the activities in the critical path.

6. Update CPM diagram

As the project progresses, the actual task completion times will be known and the network diagram can be updated to include this information. A new critical path may emerge, and structural changes may be made in the network if project requirements change.

3.2.2 FLOATS IN CPM CALCULATION

Total Float

1. Total float is the time by which a particular activity can be delayed for non-critical activity.
2. It is a difference between latest finish & earliest finish or latest start & earliest start.
3. If the total float is positive, it indicates resources are more than adequate.
4. If the total float is negative, it indicates resources are inadequate.
5. If the total float is zero, it indicates resources are thus adequate.

Free Float:

It is the portion of Total Float. It is that amount of time where the activity can be rescheduled without affecting succeeding activity.

Free Float = Total Float – Slack of Head event

Where Slack = Latest Occurrence time – Earliest occurrence time

Individual Float

It is that amount of time where activity can be rescheduled without affecting both preceding & succeeding activity. It is a portion of Free Float

3.2.3 CPM Benefits

- Provides a graphical view of the project.
- Predicts the time required to complete the project.
- Shows which activities are critical to maintaining the schedule and which are not.

3.2.4 CPM Limitations

While CPM is easy to understand and use, it does not consider the time variations that can have a great impact on the completion time of a complex project. CPM was developed for complex but fairly routine projects with minimum uncertainty in the project completion times. For less routine projects there is more uncertainty in the completion times, and this uncertainty limits its usefulness.

3.3 Project Evaluation &Review Technique(PERT)

The *Program Evaluation and Review Technique* (PERT) is a network model that allows for randomness in activity completion times.

A distinguishing feature of PERT is its ability to deal with uncertainty in activity completion times. For each activity, the model usually includes three time estimates:

- **Optimistic estimate** (t_o) is **a** minimum time duration of any activity when everything goes on well about the project. It can be also written as '**a**'.
- **Pessimistic estimate** (t_p) is **maximum** time duration of any activity when everything goes against our will and lot of difficulties is faced in the project. It can be also written as '**b**'.
- **Most likely estimate** ($t_{m,}$) means the time required in normal course when something goes on very well and something goes on bad during the project. It can be also written as '**m**'.

Then, given any activity, we compute its expected duration and variance induration are given by the following relations.

(a) **Expected duration (t_e)** = $\dfrac{t_o + 4t_m + t_p}{6}$

(b) **Standard deviation** = $\dfrac{t_p - t_o}{6}$

(c) **Variance** = $\left[\dfrac{t_p - t_o}{6}\right]^2$

3.3.1 Benefits of PERT

PERT is useful because it provides the following information:

- Expected project completion time.
- Probability of completion before a specified date.
- The critical path activities that directly impact the completiontime.
- The activities that have slack time and that can lend resources to critical path activities.
- Activities start and end dates.

3.3.2 Limitations of PERT

The following are some of PERT's limitations:

- The activity time estimates are somewhat subjective and depend on judgment. In cases where there is little experience in performing an activity, the numbers may be only a guess. In other cases, if the person or group performing the activity estimates the time there may be bias in the estimate.

- The underestimation of the project completion time due to alternate paths becoming critical is perhaps the most serious.

3.3.3 Basic difference between PERT and CPM

Difference Point	PERT	CPM
Stands for	PERT stands for "***Program Evaluation and Review Technique***".	CPM stands for "**Critical Path Method".**
Model	It is a **probabilistic** model under which the result estimated in a manner of probability.	It is a **deterministic** model under which the result is ascertained in a manner of certainty
Time	It deals with the activities of **uncertain** time.	It deals with the activities of precise well known time.
Jobs	It is used for onetime projects that involve activities of **non-repetitive** nature .	It is used for completing of projects that involve activities of **repetitive** nature.
Orientation	It is **activity oriented** in as much as its result is calculated on the basis of the activities.	It is **even oriented**, in as much as its results are calculated on the basis of the events.
Dummy Activities	It does **not** make use of **dummy activities**.	It makes use of **dummy activities** to represent the proper sequencing of the activities.
Cost	It has nothing to do with cost of a project.	It deals with the cost of a project schedules and their minimization.
Estimation	t finds out expected time of each activity on the basis of three types of estimates.	Its calculation is based on one type of time estimation that is precisely known.

Time	PERT is restricted to time variable.	CPM includes **time-cost trade off.**

3.4 DEFINITION OF SEQUENCING

The selection of an appropriate order for a series of jobs to be done on a finite number of service facilities is called sequencing. The objective is to determine the optimal order of performing the jobs in such a way that the total elapsed time will be minimum. The total cost involved may be minimum if the total elapsed time is made minimum in the business situation.

Consider there are jobs 1,2,3,…..................n to be processed through m machines. (The machines may be A, B, C ....)There are actually (n!)^m combinations. The objective is to find the technologically feasible solution, such that the total elapsed time is minimum.

.: Consider 5 jobs and 2 machines.

Possible sequences = $(5!)^2$ = 14400. From these (14400) sequences the best sequence(having minimum total elapsed time) has to be selected.

Consider a printing press. Each job is processed through two machines MI and M2. Documents arrive there for printing books, articles, magazines etc. Printing is done with desired number of copies on machine MI. Binding of the materials is done on machine M2.The press has at present, five jobs on hand. The time estimates for printing and binding for each job are worked out as follows:

	Time (hours) for	
Job No.	Printing	Binding
1	22	50
2	18	25
3	55	45
4	42	50
5	35	20

How do you sequence the jobs in order to minimize the finish time (the total time devoted by the press) of all the jobs?

3.4.1 IMPORTANT TERMS

- **No of machines** means the number of service facilities through which the jobs must be passed for processing.
- **Processing order** – is the order in which the machines are required for processing the job.
- **Processing time** – is the time taken by each job at each machine.
- **Total elapsed time** – is the time interval between starting the first job and completing the last job.
- **Idle time** – is the time during which the machine remains idle during the total elapsed time.
- **No passing rule** –refers to the rule of maintaining the same order of processing for all the jobs. Each job should be processed in the particular order.

3.4.2 ASSUMPTIONS OF SEQUENCING:

- Only one operation is carried out in a machine at a time.
- Processing times are known and they do not change.
- Each operation as well as the job once started must be completed.
- Only one machine of each type is available.
- The transportation time in moving jobs from one machine to another is negligible.

- No inventory aspect of the problem is considered.
- Only on completion of an operation, the next operation can start.
- Processing times are independent of the order in which the jobs are performed.
- Jobs are completely known and are ready for processing when the period under consideration starts.

3.4.3 SEQUENCING FOR PROCESSING OF 'n' JOBS THROUGH TWO MACHINES [JOHNSON'S ALGORITHM]

- Let the jobs be 1,2,3,............................... n
- Let the two machines be A & B.
- Let the processing order be A-B.
- Let the processing time in A be A_1, A_2, A_3 .... A_n
- Let the processing time in B be B_1, B_2, B_3, ... B_n

STEP 1:

Examine the available processing time on Machine A & Machine B and find the smallest Value.

STEP 2:

a) If the minimum value falls on A schedule it first. If it occurs in B schedule it last.

b) If there is a tie of equal minimum values, one in A and one in B for different jobs then schedule the job in A first and schedule the job in B last.

c) If there is a tie equal minimum values both in A, choose the job with the minimum value in B and schedule it first and the next job consequently.

d) If there is a tie of equal minimum values both in B, choose the job with the minimum value in A and schedule it last and the next job previously.

e) If there is a tie of equal min values both in A and B for the same job, choose the job and schedule it either first or last. (Preferably first)

STEP 3:

Cancel the scheduled job along with the processing times Repeat the same procedure from step 1 till all the jobs are scheduled, to get the optimum sequence.

3.4.4 SEQUENCING FOR PROCESSING OF 'n' JOBS THROUGH THREE MACHINES

- Let the 3 machines be A, B and C.
- Let the processing order be ABC
- Let the jobs be 1, 2, 3,… .... n.
- Let the processing time in A be A_1, A_2, A_3 .... A_n
- Let the processing time in B be B_1, B_2, B_3,… .... B_n
- Let the processing time in C be C_1, C_2, C_3 .... C_n

The three-machine problem can be converted in to a two-machine problem and Johnson's method can be applied for finding the optimum sequence if either of the following condition is satisfied:

[Min Processing time in A >= Max processing time in B]

OR

[Min Processing time in C >= Max processing time in B]

Convert the 3-machines in to 2 fictitious (imaginary) machines to apply Johnson's method for finding the optimum sequence. Let the two fictitious machines be X and Y.

$\mathbf{X_i = A_i + B_i}$

$\mathbf{Y_i = B_i + C_i}$

Follow the same procedure of Johnson's method as for 2 machines to find out the sequence.

Note : Consider all the three actual machines(A, B & C) to find out the total elapsed time & find idle time.

3.4.5 SEQUENCING FOR PROCESSING OF 'n' JOBS THROUGH 'm' MACHINES

- Let the machines be A,B ,C............m
- Let the processing order be ABC..m
- Let the jobs be 1,2,3,.............................. n.
- Let the processing time in A be A_1, A_2, A_3...A_n
- Let the processing time in B be B_1, B_2, B_3,… ....B_n
- Let the processing time in C be C_1, C_2, C_3...C_n
- Let the processing time in m be m_1, m_2, m_3...m_n

The m machine problem can be converted in to a two-machine problem and Johnson's method can be applied for finding the optimum sequence if either of the following condition is satisfied:

[Min Processing time in A >= Max processing time in B,C,D….m-1]

OR

[Min Processing time in m >= Max processing time in B,C,D….m-1]

Convert the m machines in to 2 fictitious (imaginary) machines to apply Johnson's method for finding the optimum sequence. Let the two fictitious machines be X and Y.

$\mathbf{X_i = A_i + B_i ......(m-1)_i}$

$\mathbf{Y_i = B_i + C_i ....m_i}$

Follow the same procedure of Johnson's method as for 2 machines to find out the sequence.

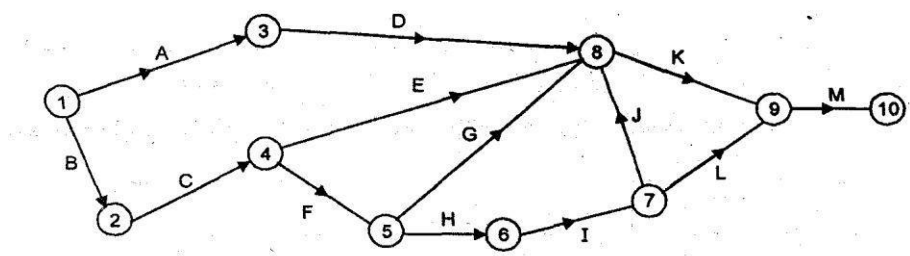
Note : Consider all the actual machines(A, B,C,D,E, …) to find out the total elapsed time & find idle time.

Problems

Problem 11.15. Construct the network for the following activity data:

Activity	Preceded by	Activity	Preceded by
A	–	–	–
B	–	H	F
C	B	I	H
D	A	J	I
E	C	K	D,E,G,J
F	C	L	I
G	F	M	K,L

Solution. Network:

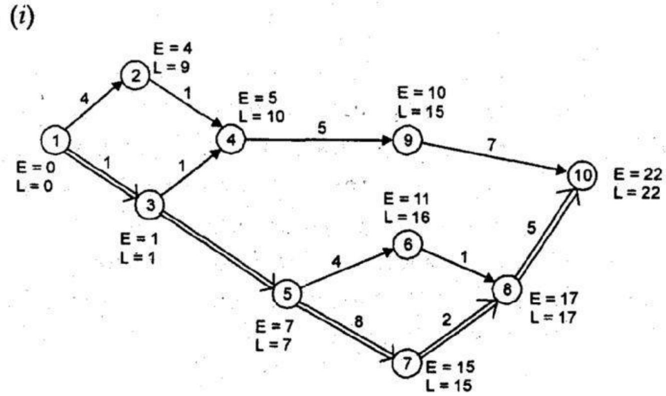


Problem 11.16. A project has the following time schedule :

Activity	1 – 2	1 – 3	2 – 4	3 – 4	3 – 5	4 – 9	5 – 6	5 – 7	6 – 8	7 – 8	8 – 9	8 – 10	9 – 10
Time weeks	4	1	1	1	6	5	4	8	1	2	1	5	7

1. Draw Network diagram and find the critical paths.
2. Calculate float on each activity.

Solution. (*i*)



2.

Activity	Duration (weeks)	Start Time		Finish Time		Total Float
		E	T_{LS}	T_{EF}	L	
1 – 2	4	0	5	4	9	5
1 – 3	1	0	0	1	1	0
2 – 4	1	4	9	5	10	5
3 – 4	1	1	9	2	10	8
3 – 5	6	1	1	7	7	0
4 – 9	5	5	10	10	15	5
5 – 6	4	7	12	11	16	5
5 – 7	8	7	7	15	15	0
6 – 8	1	11	16	12	17	5
7 – 8	2	15	15	17	17	0
8 – 10	5	17	17	22	22	0
9 – 10	7	10	15	17	22	5

Critical path 1—3—5—7—8—10 with project duration of 22 weeks.

Problem. 11.17. The time estimate for the activities of a PERT network are given

below :

Activity	t_0	t_m	t_p
1 – 2	1	1	7
1 – 3	1	4	7
1 – 4	2	2	8
2 – 5	1	1	1
3 – 5	2	5	14
4 – 6	2	5	8
5 – 6	3	6	15

(a) Draw the project network and identify all the path through it.

(b) Determine the expected project length.

(c) Calculate the standard deviation and variance of the project length.

(d) What is the probability that the project will be completed

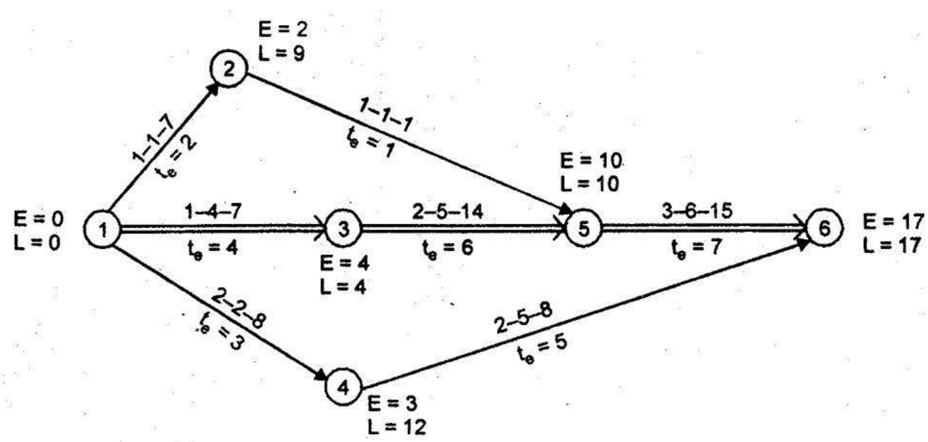
1. At least 4 weeks earlier than expected time.

2. No more than 4 weeks later than expected time.

(e) The probability that the project will be completed on schedule if the schedule completion time is 20 weeks.

(f) What should be the scheduled completion time for the probability of completion to be 90%.

Solution. (a) Network



Activity	t_0	t_m	t_p	$t_e = \frac{t_0 + 4t_m + t_p}{6}$	$\sigma^2 = \frac{(t_p - t_0)^2}{6}$
1 – 2	1	1	7	2	1
1 – 3	1	4	7	4	1
1 – 4	2	2	8	3	1
2 – 5	1	1	1	1	0
3 – 5	2	5	14	6	4
4 – 6	2	5	8	5	1
5 – 6	3	6	15	7	4

Critical path—1 —3—5—6

Project duration = 17 weeks.

(c) Variance of the project length is the sum of the variance of the activities on the critical.

$$V_{cp} = V_{1-3} + V_{3-5} + V_{5-6} = 1 + 4 + 4 = 9$$
$$\sigma^2 = V \Rightarrow \sigma^2 = 9 \Rightarrow \sigma = 3 \text{ weeks.}$$

(d) (i) Probability that the project will be completed at least 4 week earlier than expected time

Expected time (E_p) = 17 weeks

Scheduled time = 17 – 4 = 13 weeks

$$Z = \frac{13-17}{3} = -1.33$$

$$P(-1.33) = 1 - 0.9082 = 0.0918$$

2. Probability that the project will be completed at least 4 weeks later than expected

Time

Expected time = 17 weeks Scheduled time =17+ 4 =21 weeks

$$Z = \frac{21-17}{3} = 1.33$$

$$P(1.33) = 0.9082 = 90.8\%.$$

(e) Scheduled time = 20 weeks

$$Z = \frac{20-17}{3} = 1$$

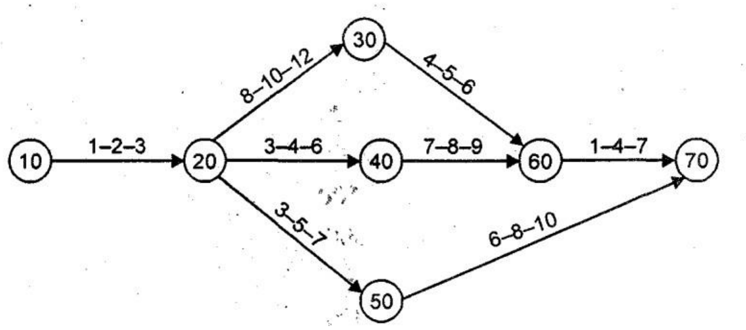
$$P(1) = 84.13\%$$

(f) Value of Z for P = 0.9 is 1.28 (from probability table)

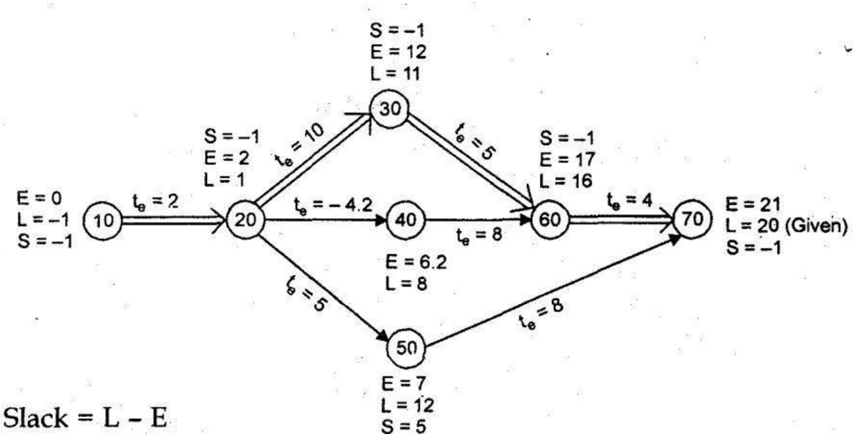
$$1.28 = \frac{T-17}{3}$$

$$T = 17 + 3.84 = 20.84 \text{ weeks.}$$

Problem 11.18. Consider the PERT network given in fig. Determine the float of each activity and identify the critical path if the scheduled completion time for the project is 20 weeks.



Solution.



(s) Slack = L – E

Activity	$t_e = \frac{t_0 + 4tm + t_p}{6}$	Start Time		Finish Time		Total
		E	T_{ES}	T_{EF}	L	Float
10 – 20	2	0	–1	2	1	–1
20 – 30	10	2	1	12	11	–1
20 – 40	4.2	2	3.8	6.2	8	1.8
20 – 50	5	2	7	7	12	5
30 – 60	5	12	11	17	16	–1
40 – 60	8	6.2	8	14.2	16	1.8
50 – 70	8	7	12	15	20	5
60 – 70	4	17	16	21	20	–1

Critical path 10 - 20- 30 - 60 — 70.

Problem. 3.25. There are five jobs each of which just go through two machines A and B in the order of AB.

Processing times are given below. Determine a sequence for five jobs that will

minimize the elapse time and also calculate the total time.

Job	1	2	3	4	5
Time for A	5	1	9	3	10
Time for B	2	6	7	8	4

Determine the sequence for the jobs so as to minimize the process time. Find

total elapsed time.

Solution : Examine the columns of processing time on rn/c A and B and find the

smallest value. If this value falls in column A, schedule the job first on M/c, A, if this

value falls in column B, schedule the jobs last on M/c A. In this way sequence of jobs so as to minimize the process time is

2	4	3	5	1

Job	Machine A		Machine B	
	Time in	Time out	Time in	Time out
2	0	1	1	7
4	1	4	7	15
3	4	13	15	22
5	13	23	23	27
1	23	28	28	30

Ans. Sequence

2	4	3	5	1

Total elapsed time = 30 hours. V

Problem 3.26. Find the sequence that minimize the total elapsed time to complete the following Jobs. Each Job is processed in the order of AB.

Job (Processing time in minutes)

		1	2	3	4	5	6	7
m/c	**A**	**12**	**6**	**5**	**11**	**5**	**7**	**6**
	B	**7**	**8**	**9**	**4**	**7**	**8**	**3**

Determine the sequence for the jobs so as to minimize the process time. Find the total elapsed time and idle time of M/c A and M/c B.

Solution : The sequence of jobs so as to minimize the process time is

5	3	2	6	1	4	7

Job	**Machine A**		**Machine B**	
	Time in	**Time out**	**Time in**	**Time out**
5	0	5	5	12
3	5	10	12	21
2	10	16	21	29
6	16	23	29	37
1	23	35	37	44
4	35	46	46	50
7	46	52	52	55

Min elapsed time = 55 mins
Idle time of M/c A = 3 mins
Idle time of M/c B = 9 mins.

QUESTIONS

SEQUENCING

1. Find out the optimum sequence for the jobs which are to be processed through two machines. Machines A and B.

	Jobs			
	1	2	3	4
Machine A	1	6	6	5
Machine B	2	8	10	3

2.

	Jobs			
	1	2	3	4
Machine A	1	6	8	5
Machine B	2	10	6	3

3.

	Jobs					
	1	2	3	4	5	6
Machine A	2	4	6	3	3	10
Machine B	4	4	8	4	9	12

4. Find out the appropriate sequence total elapse time and total idle time for jobs to be processed through 2 machines.

	Jobs					
	A	B	C	D	E	F
Machine X	11	7	12	4	6	7
Machine Y	11	11	11	11	11	15

5.

	Jobs				
	1	2	3	4	5
Machine A	4	8	6	8	1
Machine B	3	4	7	8	5

6. Find out the appropriate sequence, idle time, and total elapsed time for processing through 3 machines.

	Jobs				
	1	2	3	4	5
Machine A	4	8	6	4	6
Machine B	2	3	1	1	4
Machine C	6	8	2	4	3

7. Find out the appropriate sequence, idle time, and total elapsed time for processing through 3 machines.

	Jobs					
	A	B	C	D	E	F
Machine 1	8	7	3	2	5	1
Machine 2	3	4	5	2	1	6
Machine 3	8	7	6	9	10	9

8. Find out the optimum sequence, idle time and total elapsed time for the jobs to be processed through 4 machines.

	Jobs			
	A	B	C	D
Machine 1	8	8	4	3
Machine 2	4	2	1	6
Machine 3	6	8	10	12
Machine 4	14	18	20	22

9.

	Machines			
	M1	M2	M3	M4
JOB1	11	8	7	14
JOB 2	10	6	8	19
JOB 3	9	7	5	18
JOB 4	8	5	5	18

NETWORK ANALYSIS

NETWORK CONSTRUCTION AND SCHEDULING

1. Draw the network for the project given :

Activities	**Predecessor**
A	-
B	-
C	-
D	A
E	B
F	B
G	C
H	D
I	E
J	H, I
K	F, G

2. Draw the network for the project given :

Activities	**Predecessor**
P	-
Q	-
R	-
S	P, Q
T	R, Q

3. Draw the network for the project given :

Activities	Predecessor
A	-
B	A
C	A
D	-
E	D
F	B, C, E
G	F
H	E
I	G, H

4. Draw the network for the project given :

Activities	Predecessor
A	-
B	-
C	A, B
D	B
E	B
F	A, B
G	F, D
H	F, D
I	C, G

5. Draw the network for the project given :

Activities: A, D, and E can start simultaneously. Activities B, C is greater than A; G,F greater than D, C; H> E, F.

Hint:

Activities	Predecessor
A	-
B	A
C	A
D	-
E	-
F	D, C
G	D, C
H	E, F

6. A< C, D; B< E; C, E < F, G; D < H; G < I; H, I, < J.

Hint:

Activities	Predecessor
A	-
B	-
C	A
D	A
E	B

F	C, E
G	C, E
H	D
I	G
J	H, I

CRITICAL PATH METHOD

7. Draw the network and also find the critical path. Duration of each activity is given below

A < C, D, I; B < G, F; D < G, F; F < H, K; G, H, < J; I, J, K < E.

Activities	**Predecessor**	**Duration**
A	-	5 days
B	-	3
C	A	10
D	A	2
E	I, J, K	8
F	B, D	4
G	B, D	5
H	F	6
I	A	12
J	G & H	8
K	F	9

8. Draw the network and find the critical path. Also find earliest start, earliest finish, latest start and latest finish of each activity.

Activity	Duration
1-2	8 days
1-3	4
2-4	10
2-5	2
3-4	5
4-5	3

9. Draw the network and find the critical path, and also calculate floats

Activity	Duration
1-2	8 days
1-3	7
1-5	12
2-3	4
2-4	10
3-4	3
3-5	5
3-6	10
4-6	7
5-6	4

PROGRAM EVALUATION AND REVIEW TECHNIQUE

10. Draw the network; find the expected duration and the variance of the project. Also find the standard deviation of the project.

Activity	Optimistic time	Moderate time	Pessimistic time
1-2	3	5	8
1-3	3	4	9
1-4	8	10	12
2-4	14	15	16
3-4	3	4	6
2-5	1	3	5
3-5	2	4	6
4-5	3	4	6

11. Draw the network; find the expected duration and the variance of the project. Also find the standard deviation of the project

Activity	a	m	b
1-2	3	6	15
1-3	2	5	14
1-4	6	12	30
2-5	2	5	8
2-6	5	11	17
3-6	3	6	15
4-7	3	9	27
5-7	1	4	7
6-7	2	5	8

What is the probability that project will be completed within 27 days.

- What is the probability that project will be completed within 33 days.
- What is the probability that project will take above 33 days.
- What is the probability that project will be completed within 25 days or probability that the project is just completed on the expected duration.
- What is the probability that project will be completed between 20-25 days.

12. Draw the network; find the expected duration and the variance of the project. Also find the standard deviation of the project

Activity	to	tp	tm
1-2	0.8	1.2	1
2-3	3.7	9.9	5.6
2-4	6.2	15.4	6.6
3-4	2.1	6.1	2.7
4-5	0.8	3.6	3.4
5-6	0.9	1.1	1

UNIT – III RESOURCE SCHEDULING AND NETWORK ANALYSIS

MODEL QUESTION PAPER

PART - A

1. What do you mean by sequencing? Explain the objectives of sequencing.
2. Explain the procedure of sequencing.
3. What is meant by a) Total elapsed time b) Idle time.
4. Write the conditions to convert (i) 3 machines problem into 2 machines problem (ii) 4 machines problem into 2 machines problem in sequencing.
5. Write short notes on i) Total float ii) Free float iii) Independent float
6. What do you mean by critical path?
7. Explain the procedure for constructing network diagram
8. Explain the steps in CPM project planning.
9. Differentiate between PERT and CPM.
10. What is meant by a) Project b) Earliest Start and Earliest Finish b) Latest start and Latest finish?
11. Find the optimum Sequence for the following tasks:

Tasks

		A	B	C	D	E	F	G	H	I
	M1	2	5	4	9	6	8	7	5	4
Machines	M2	6	8	7	4	3	9	3	8	11

12. Construct the network for the project whose activities and their precedence relationships are as given below.

    A, B, C can start simultaneously

    A<F, E: B<D; C, E, D<G.

PART - B

1. The time in hours to process six known batches J1 – J6 through the washer and cooker is given below:

Batches

	J1	J2	J3	J4	J5	J6
Washer (M1)	4	7	3	12	11	9
Cooker (M2)	11	7	10	8	10	13

Find out the optimum sequence and also find out total elapsed time and idle time.

2. There are six jobs which must go through two machines A and B. Processing time is given below. Find out the optimum sequence, idle time, and total elapsed time.

Job	1	2	3	4	5	6
Machine A	8	9	11	12	16	20
Machine B	7	15	10	14	13	9

3. Find out the optimum sequence, idle time, and total elapsed time for the given 3 machines problem.

		Jobs				
Machines		1	2	3	4	5
Machine A	4	8	6	4	6	
Machine B	2	3	1	1	4	
Machine C	6	8	2	4	3	

4. Find out the optimum sequence, idle time and total elapsed time for the jobs to be processed through 4 machines.

	Jobs			
	A	B	C	D
Machine 1	8	8	4	3
Machine 2	4	2	1	6
Machine 3	6	8	10	12
Machine 4	14	18	20	22

5. Draw the network and find the critical path. Find earliest start, earliest finish, latest start, and latest finish of each activity.

Activity	Duration	Preceding Activity

A	6	-
B	8	A
C	4	A
D	9	B
E	2	C
F	7	D

6. Draw the network and find the critical path, and also calculatefloats

Activity	Duration
1-2	8 days
1-3	7
1-5	12
2-3	4
2-4	10
3-4	3
3-5	5
3-6	10
4-6	7
5-6	4

7. Draw the network; find the expected duration and the variance of the project. Also find the standard deviation of the project

Activity	a	m	b
1-2	3	6	15
1-3	2	5	14
1-4	6	12	30
2-5	2	5	8
2-6	5	11	17
3-6	3	6	15
4-7	3	9	27
5-7	1	4	7
6-7	2	5	8

What is the probability that project will be completed within 27 days.

- What is the probability that project will be completed within 33 days.
- What is the probability that project will take above 33 days.
- What is the probability that project will be completed within 25 days or probability that the project is just completed on the expected duration.
- What is the probability that project will be completed between 20-25 days.

8. Draw the network and find the critical path. Find earliest start, earliest finish, latest start, latest finish, total float, free float and independent float for each activity.

Activity	Preceding Activity	Duration
A	-	2
B	A	6
C	A	6
D	B	5
E	C,D	3
F	-	3
G	E,F	1

9. Draw the network and find the critical path. Find earliest start, earliest finish, latest start, latest finish A< C, D; B< E; C, E < F, G; D < H; G < I; H, I, < J.

Hint:

Activities	**Predecessor**
A	-
B	-
C	A
D	A
E	B
F	C, E
G	C, E
H	D
I	G
J	H, I

10. Draw the network; find the expected duration and the variance of the project. Also find the standard deviation of the project

Activity	To	tp	tm
1-2	0.8	1.2	1
2-3	3.7	9.9	5.6
2-4	6.2	15.4	6.6
3-4	2.1	6.1	2.7
4-5	0.8	3.6	3.4
5-6	0.9	1.1	1

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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – IV – REPLACEMENT AND GAME THEORY – SBAA5205

UNIT IV: REPLACEMENT AND GAME THEORY 12 Hrs.

Replacement Problems - Replacement of Assets that Deteriorates with Time and without Time Value of Money Consideration; Replacement of Assets that Fail Suddenly. Theory of Games - Pure and Mixed Strategies - Saddle Point, Dominance Property, Modified Dominance property and Graphical Method of Solving Games.

4.1 REPLACEMENT MODEL

If any equipment or machine is used for a long period of time, due to wear and tear, the item tends to worsen. A remedial action to bring the item or equipment to the original level is desired. Then the need for replacement becomes necessary. This may be due physical impairment, due to normal wear and tear, obsolescence etc. The resale value of the item goes on diminishing with the passage of time.

The depreciation of the original equipment is a factor, which is responsible not to favor replacement because the capital is being spread over a long time leading to a lower average cost. Thus there exists an economic trade-off between increasing and decreasing cost functions. We strike a balance between the two opposing costs with the aim of obtaining a minimum cost.

Replacement model aims at identifying the **time** at which the assets must be replaced in order to minimize the cost.

4.2 REASONS FOR REPLACEMENT OF EQUIPMENT:

1. Physical impairment or malfunctioning of various parts refers to

- The physical condition of the equipment itself
- Leads to a decline in the value of service rendered by the equipment
- Increasing operating cost of the equipment
- Increased maintenance cost of the equipment
- Or a combination of the above.

2. Obsolescence of the equipment, caused due to improvement in the existing tools and machinery mainly when the technology becomes advanced.

3. When there is sudden failure or breakdown.

4.3 REPLACEMENT MODELS:

- **Assets that fails Gradually:**

  Certain assets wear and tear as they are used. The efficiency of the assets decline with time. The maintenance cost keeps increasing as the years pass by eg. Machinery, automobiles, etc.

  1. Gradual failure without taking time value of money into consideration
  2. Gradual failure taking time value of money into consideration

- **Assets which fail suddenly**

  Certain assets fail suddenly and have to be replaced from time to time eg. bulbs.

  1. Individual Replacement policy (IRP)
  2. Group Replacement policy (GRP)

4.3.1 Assests that fails Gradually

4.3.1.1 Gradual failure without taking time value of money into consideration

As mentioned earlier the equipments, machineries and vehicles undergo wear and tear with the passage of time. The cost of operation and the maintenance are bound to increase year by year. A stage may be reached that the maintenance cost amounts prohibitively large that it is better and economical to replace the equipment with a new one. We also take into account the salvage value of the items in assessing the appropriate or opportune time to replace the item. We assume that the details regarding the costs of operation, maintenance and the salvage value of the item are already known

- **Procedure for replacement of an asset that fails gradually (without considering Time value of money):**

a) Note down the years

b) Note down the running cost 'R' (Running cost or operating cost or Maintenance cost or other expenses)

c) Calculate Cumulative the running cost '∑R'

d) Note down the capital cost 'C'

e) Note down the scrap or resale value 'S'

f) Calculate Depreciation = Capital Cost – Resale value

g) Find the Total Cost

Total Cost = Cumulative Running cost + Depreciation

h) Find the average cost

Average cost = Total cost/No. of corresponding year

i) Replacement decision: Average cost is minimum (Average cost will decrease and reach minimum, later it will increase)

Year	Running Cost	Cumulative Running Cost	Capital cost	Salvage value Or Resale value	Depn. = Capital cost – salvage value	Total cost= Cumulative running cost + Depreciation	Average annual cost P_n = Total cost / no. of corresponding year
N	R_n	$\sum R_n$	C	S_n	$C - S_n$	$\sum R_n + C - S_n$	$(\sum R_n + C - S_n)/n$
1	2	3	4	5	6 (4-5)	7 (3+6)	8 (7/1)

4.3.1.2 Gradual failure taking time value of money into consideration

In the previous section we did not take the interest for the money invested, the running costs and resale value. If the effect of time value of money is to be taken into account, the analysis must be based on an equivalent cost. This is done with the present value or present worth analysis.

For example, suppose the interest rate is given as 10% and Rs. 100 today would amount to Rs. 110 after a year's time. In other words the expenditure of Rs. 110 in year's time is equivalent to Rs. 100 today. Likewise one rupee a year from now is equivalent to (1.1)-1 rupees today and one-rupee in '*n*' years from now is equivalent to (1.1)-n rupees today. This quantity (1.1)-n is called the present value or present worth of one rupee spent 'n'years from now.

- **Procedure for replacement of an asset that fails gradually (with considering Time value of money):**

Assumption:

i. Maintenance cost will be calculated at the beginning of the year
ii. Resale value at the end of the year

Procedure:

a) Note down the years
b) Note down the running cost 'R' (Running cost or operating cost or Maintenance cost or other expenses)
c) Write the present value factor at the beginning for running cost
d) Calculate present value for Running cost
e) Calculate Cumulative the running cost '$\sum$ R'
f) Note down the capital cost 'C'
g) Note down the scrap or resale value 'S'
h) Write the present value factor at the end of the year and also calculate present value for salvage or scrap or resale value.
i) Calculate Depreciation = Capital Cost – Resale value

j) Find the Total Cost = Cumulative Running cost + Depreciation
k) Calculate annuity factor (Cumulative present value factor at the beginning)

l) Find the Average cost = Total cost / Annuity

m) Replacement decision: Average cost is minimum (Average cost will decrease and reach minimum, later it will increase)

Year n	R_n	Pv^{n-1}	R_nPv^{n-1}	$\sum R_nPv^{n-1}$	C	S_n	Pv^n	S_nPv^n	C - S_nPv^n	$\sum R_nv^{n-1}$ + C - S_nPv^n	$\sum PV^{n-1}$	W_n
1	2	3	4(2*3)	5	6	7	8	9(7*8)	10	11(5+10)	12	13

4.3.2 ITEMS THAT FAIL COMPLETELY AND SUDDENLY

There is another type of problem where we consider the items that fail completely. The item fails such that the loss is sudden and complete. Common examples are the electric bulbs, transistors and replacement of items, which follow sudden failure mechanism.

4.3.2.1 INDIVIDUAL REPLACEMENT POLICY (IRP):

Under this strategy equipments or facilities break down at various times. Each breakdown can be remedied as it occurs by replacement or repair of the faulty unit.

Examples: Vacuum tubes, transistors

Calculation of Individual Replacement Policy (IRP): n

Average life of an item = $\sum_{i-1} i * P_i$

Pi denotes Probability of failure during that week **i** denotes no. of weeks

$$\text{No. of failures} = \frac{\text{Total no. of items}}{\text{Average life of an item}}$$

Total IRP Cost = No. of failures * IRP cost

4.3.2.2 GROUP REPLACEMENT

As per this strategy, an optimal group replacement period '*P*' is determined and common preventive replacement is carried out as follows.

(a) Replacement an item if it fails before the optimum period '*P*'.

(b) Replace all the items every optimum period of '*P*' irrespective of the life of individual item. Examples: Bulbs, Tubes, and Switches.

Among the three strategies that may be adopted, the third one namely the group replacement policy turns out to be economical if items are supplied cheap when purchased in bulk quantities. With this policy, all items are replaced at certain fixed intervals.

4.3.4.1 Procedure for Group Replacement Policy (GRP):

1. Write down the weeks
2. Write down the individual probability of failure during that week
3. Calculate No. of failures:

   N_0 - No. of items at the beginning

   N_1 - No. of failure during 1st week (N_0P_1)

   N_2 - No. of failure during 2nd week ($N_0P_2 + N_1P_1$)

   N_3 - No. of failure during 3rd week ($N_0P_3 + N_1P_2 + N_2P_1$)
4. Calculate cumulative failures
5. Calculate IRP Cost = Cumulative no. of failures * IRP cost
6. Calculate and write down GRP Cost = Total items * GRP Cost
7. Calculate Total Cost = IRP Cost + GRP Cost
8. Calculate Average cost = Total cost / no. of corresponding year

4.4 GAME THEORY

A competitive situation in business can be treated similar to a **game**. There are two or more players and each player uses a strategy to out play the opponent.

A strategy is an action plan adopted by a player in-order to counter the other player.In our of game theory we have two players namely Player A and Player B.
The basic objective would be that

Player A – plays to **Maximize profit** (offensive) - Maxi (min) criteria

Player B – plays to **Minimize losses** (defensive) - Mini (max) criteria

The Maxi (Min) criteria is that – Maximum profit out of minimum possibilities

The Mini (max) criteria is that – Minimze losses out of maximum possibilities.

Game theory helps in finding out the best course of action for a firm in view of the anticipated counter-moves from the competing organizations.

4.4.1 Characteristics of a game

A competitive situation is a competitive game if the following properties hold good

1. The number of competitors is finite, say N.
2. A finite set of possible courses of action is available to each of the N competitors.

3. A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that all the players select their courses of action simultaneously. As a result no competitor will be in a position to know the choices of his competitors.

4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

4.4.2 TERMINOLOGIES

Zero Sum game because the Gain of A – Loss of B = 0. In other words, the gain of Player A is the Loss of Player B.

Pure strategy If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to minimize the gain Therefore the pure strategy is a decision rule always to select a particular course of action.

Mixed strategy If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain. Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Optimal strategy The strategy that puts the player in the most preferred position irrespective of the strategy of his opponents is called an optimal strategy Any deviation from this strategy would reduce his payoff.

Saddle Point : If the Maxi (min) of A = Mini (max) of B then it is known as the Saddle Point Saddle point is the number, which is lowest in its row and highest in its column. When minimax value is equal to maximin value , the game is said to have saddle point. It is the cell in the payoff matrix which satisfies minimax to maximin value

Value of the Game : It is the average wining per play over a long no. of plays. It is the expected pay off when all the players adopt their optimum strategies .If the value of game is zero it is said to be a fair game , If the value of game is not zero it is said to be a unfair game . In all problems relating to game theory, first look for saddle point, then check out for rule of dominance and see if you can reduce the matrix.

Rule of Dominance:

The dominance and modified dominance principles and their applications for reducing the size of a game with or without a saddle point. If every value of one strategy of A is lesser than that of the other strategy of A, Then A will play the strategy with greater values and remove the strategywith the lesser payoff values.

If every value of one strategy of B is greater than that of other strategy of B, B will play the lesser value strategy and remove the strategy with higher payoff values.

Dominance rule for the row

If all the elements in a particular row is lower than or equal to all the elements in another row, then the row with the lower items are said to be dominated by row with higher ones, Then the row with lower elements will be eliminated.

Dominance rule for the column

If all the elements in a particular column is higher than or equal to all the elements in another column, then the column with the higher items are said to be dominated by column with lower ones, Then the column with higher elements will be eliminated.

Modified Dominance Rule

In few cases, if the given strategy is inferior to the average of two or more pure strategies, then the inferior strategy is deleted from the pay-off matrix and the size of the matrix is reduced considerably. In other words, if a given row has lower elements than the elements of average of two rows then particular row can be eliminated. Similarly if a given column has higher elements than the elements of average of two columns then particular column can be eliminated. Average row/column cannot be eliminated under any circumstances.. This type of dominance property is known as the modified dominance property

4.4.3 Graphical Method

If one of the players, play only two strategies or if the game can be reduced such that one of the players play only two strategies. Then the game can be solved by the graphical method.

In case the pay-off matrix is of higher order (say m x n), then we try to reduce as much as possible using dominance and modified dominance ,f we get a pay-off matrix of order 2 x n or n x 2 we try to reduce the size of the pay-off matrix to that of order 2 x 2 with the graphical method so that the value of game could be obtained

.

4.4.4 Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

1. Analysis of the market strategies of a business organization in the long run.
2. Evaluation of the responses of the consumers to a new product.
3. Resolving the conflict between two groups in a business organization.
4. Decision making on the techniques to increase market share.
5. Material procurement process.
6. Decision making for transportation problem.
7. Evaluation of the distribution system.
8. Evaluation of the location of the facilities.
9. Examination of new business ventures and
10. Competitive economic environment

QUESTION BANK

Problems

Problem 1. The cost of a machine is Rs. 6100/- and its scrap The maintenance costs found from experience are as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost	100	250	400	600	900	1200	1600	2000

When should the machine be replaced ?

Ans. Let it is profitable to replace the machine after n years. The n is determined by the minimum value of T_{avg}.

Years service	Purchase price-scrap value	Annual maintenance cost	Summation of maintenance cost	Total cost	Avg. annual cost (T_{avg})
1.	6000	100	100	6100	6100
2.	6000	250	350	6350	3175
3.	6000	400	750	6750	2250
4.	6000	600	1350	7350	1837.50
5.	6000	900	2250	8250	1650
6.	6000	1200	3450	9450	$\boxed{1575}$ Min
7.	6000	1600	5050	11050	1578
8.	6000	2000	7050	13050	1631

The avg. annual cost is minimum Rs. should be replaced after 6 years of use.

(1575/-) during the sixth year. Hence the m/c

Problem 2. A machine owner finds from his past records that the costs per year of maintaining a machine whose purchase price is Ks. 6000 are as given below

Year	1	2	3	4	5	6	7	8
Maintenance cost	1000	1200	1400	1800	2300	2800	3400	4000
Cost Resale price	3000	1500	750	375	200	200	200	200

Determine at what age is a replacement due?

Ans. Capital cost C = 6000/-. Let it be profitable to replace the. machine after n

years. Then n should be determined by the minimum value of Tav•

Year of service	Resale value	Purchase Price Resale value	Annual Maintenance cost	Summation of maintenance cost	Total Cost	Average annual cost
1.	3000	3000	1000	1000	4000	4000
2.	1500	4500	1200	2200	6700	3350
3.	750	5250	1400	3600	8850	2950
4.	375	5625	1800	5400	11025	2756.25
5.	200	5800	2300	7700	13500	$\boxed{2700}$
6.	200	5800	2800	10500	16300	2716.66
7.	200	5800	3400	13900	19700	2814.28
8.	200	5800	3400	17300	23100	2887.5

We observe from the table that avg. annual cost is minimum (Rs. 2700/-). Hence the m/c should replace at the end of 5th year.

Type B. Replacement of items whose maintenance costs increase with time and value of money also changes with time.

The machine should be replaced if the next period's cost is greater than weighted average of previous cost.

Discount rate [Present worth factor (PWF)

$$V = \frac{1}{1+i}$$

$$V_n = (V)^{n-1}$$

n – no. of year

i – annual interest rate

V_n – PWF of n^{th} year.

Problem 3. A machine costs Rs. 500/— Operation and Maintenance cost are zero for the first year and increase by Rs. 100/— every year. If money. is worth 5% every year, determine the best age at which the machine should be replaced. The resale value of the machine is negligible small. What is the weighted average cost of owning and operating the machine?

Ans. Discount rate $V = \frac{1}{1+i} = \frac{1}{1+0.05} = 0.9524$

Discount rate for Ist year $V_n = \left(\frac{1}{1+i}\right)^{n-1}$

$$V_1 = (0.9524)^0 = 1$$

$$2^{nd} \text{ year } V_2 = (0.9524)^1 = 0.9524$$

$$3^{rd} \text{ year } V_3 = (0.9524)^2 = 0.9070$$

$$4^{th} \text{ year } V_4 = (0.9524)^3 = 0.8638$$

$$5^{th} \text{ year } V_5 = (0.9524)^4 = 0.8227$$

Years of service (n)	Maintenance cost (Rs)	Discount factor $(V)^{n-1}$	Discounted cost	Summation of cost of m/c and maint. Cost	Summation of discount factor	Weighted average cost
1	0	1.0000	0.00	500.00	1.0000	500
2	100	0.9524	95.24	595.24	1.9524	304.88
3	200	0.9070	181.40	776.64	2.8594	$\boxed{217.61}$ min
4	300	0.8638	259.14	1035.78	3.7232	278.20
5	400	0.8227	329.08	1364.86	4.5459	300.25

M/c su1d be replaced at the end of 3C1 year.

Problem 3. Purchase price of a machine is Rs. 3000/— and its running cost is given in the table below. If should be replaced. the discount rate is 0.90. Find at what age the machine

Year	1	2	3	4	5	6	7
Running cost (Rs.)	500	600	800	1000	1300	1600	2000

Ans. V (Discount rate) = 0.90

Year of service (n)	Running cost (Rs.)	Discount factor $(V)^{n-1}$	Discounted cost	Summation of cost of m/c and maint. cost	Summation of discount factor	Weighted average cost
1	500	1	500	3500	1	3500
2	600	0.90	540	4040	1.9	2126.31
3	800	0.81	648	4688	2.71	1729.88
4	1000	0.729	729	5417	3.439	1575.16
5	1300	0.6561	852.93	6269.93	4.0951	$\boxed{1531.08}$ min.
6	1600	0.59049	944.78	7214.71	4.6855	1539.79
7	2000	0.5314	1062.8	8277.51	5.2169	1586.6

M/c should be replaced at the end of 5th year.

Problem 4. The following mortality ratio have been observed for a certain type

of light bulbs in an installation with 1000 bulbs

End of week	1	2	3	4	5	6
Probability of failure to date	0.09	0.25	0.49	0.85	0.97	1.00

There are a large no. of such bulbs which are to be kept in working order. If a bulb fails in service, it cost Rs. 3 to replace but if all the bulbs all replaced in the same operation it can be done for only Rs. 0.70/— a bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out and continue replacing burnt out bulb as they fail.

(a) What is the best interval between group replacement?

(b) Also establish if the policy, as determined by you is superior to the policy of replacing bulbs as and when they, fail, there being nothing like group replacement.

(c) At what group replacement price per bulb, would a policy of strictly individual replacement become preferable to the adopted policy?

Solution : Let p. be the probability that a new light bulbs fail during the 1th wek of the life.

$P_1 = 0.09$

$P_2 = 0.25 - 0.09 = 0.16$

$P_3 = 0.49 - 0.25 = 0.24$

$P_4 = 0.85 - 0.49 = 0.36$

$P_5 = 0.97 - 0.85 = 0.12$

$P_6 = 1.00 - 0.97 = 0.03$

Week	Expected no. of failure (N)	
0	$N_0 = N_0$	
1	$N_1 = 1000 \times 0.09$	= 90
2	$N_2 = 1000 \times 0.16 + 90 \times 0.09$	= 168
3	$N_3 = 1000 \times 0.24 + 90 \times 0.16 + 168 \times 0.09$	= 269
4	$N_4 = 1000 \times 0.36 + 90 \times 0.24 + 168 \times 0.16 + 269 \times 0.09$	= 432
5	$N_5 = 1000 \times 0.12 + 90 \times 0.36 + 168 \times 0.24 + 269 \times 0.16 + 432 \times 0.09$	= 274
6.	$N_6 = 1000 \times 0.03 + 90 \times 0.12 + 168 \times 0.36 + 269 \times 0.24 + 432$	
	$\times 0.16 + 274 \times 0.09$	= 260
and so on		

(*a*) Determination of optimum group replacement interval

Week	Total cost of group replacement	Avg cost/week
1.	$1000 \times 0.70 + 90 \times 3 = 970$	970.00
2.	$1000 \times 0.70 + 3\,(90+168) = 1474$	$\boxed{737.00}$
3.	$1000 \times 0.70 + 3\,(90 + 168 + 269) = 2281$	760.33

Problem 5. Find the value of games shown below also indicator whether they are fair or strictly determinable

Solution.

(*a*) A vs B:

1	9	6	0
2	3	8	−1
−5	−2	10	−3
7	4	−2	−5

(*b*) A vs B:

6	−2	−3	8
−1	−2	−7	0
8	9	−6	−7
9	5	−7	7

Solution.

(a)

Player B

Player A	I	II	III	IV	Minimum of row
I	1	$\boxed{9}$	6	$\boxed{0}$	(0)maximum
II	2	3	8	− 1	− 1
III	−5	− 2	$\boxed{10}$	− 3	− 5
IV	$\boxed{7}$	4	− 2	−5	− 5
Max.of column	7	9	10	(0) Minimax	

Saddle point = (I, IV)
Game value 0 Strategy of A = Al
Strategy of B = B IV
Since maximum = Minimax = 0
So game = Fair.

(*b*)

B

A					Minimum of row
	6	− 2	$\boxed{-3}$	$\boxed{8}$	(−3) maximin ←
	− 1	− 2	−7	0	− 7
	8	$\boxed{9}$	− 6	−7	− 7
	$\boxed{9}$	5	−7	7	− 7
Max. of column	9	9	(−3) Minimax ↑	8	

Saddle point = A1, B3
Game value (V) = −3
Strategy of A = A1
Strategy of B = B 3
Maximum = minimax = −3 ≠ 0
Hence Game = Strictly determinable but not fair.

Problem 6. In a game of matching coins, player A wins Rs. 2. If there are two heads, win nothing if there are two tails and loses Rs. 1. When there are one head and one tail. Determine the pay off matrix, best strategies for each player and the value of game to A.

Solution. The payoff matrix for A will be

$$\begin{array}{cc} & \begin{array}{cc} H & T \end{array} \\ \text{Player A} \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \end{array}$$

There is no saddle point
By Arithmetic method

$$\begin{array}{ccc} & \begin{array}{cc} H & T \end{array} & \\ A\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} & \begin{array}{l} 1(0.25) \\ 3(0.75) \end{array} \\ & \begin{array}{cc} 1 & 3 \\ (0.25) & (0.75) \end{array} & \end{array}$$

Player A best strategy (0.25, 0.75)
Player B best strategy (0.25, 0.75)
Game value
Let B plays H; Value of the game

$$(V) = \text{Rs.}\left(\frac{1\times 2-3\times 1}{3+1}\right) = \text{Rs.}\left(-\frac{1}{4}\right).$$

Problem 7. (By Dominance) Two players P and Q play a game. Each of them has to choose one of three colours, white (W) Black (B) and Red (R) independently of the other. There after the colours are compared. If both P and Q have choosen white (W,W) neither win anything. If player P selects white and player Q black (W, B), player P loses Rs. 2 or player Q wins the same amount and so on. The complete payoff table is shown. Find the optimum strategies for P and Q and the value of the game.

$$\begin{array}{cc} & \begin{array}{ccc} W & B & R \end{array} \\ (P)\ \begin{array}{c} W \\ B \\ R \end{array} & \begin{bmatrix} 0 & -2 & 7 \\ 2 & 5 & 6 \\ 3 & -3 & 8 \end{bmatrix} \end{array}$$

Solution.

Colour choosen by Q

	W	B	R
W	0	−2	7
B	2	5	6
R	3	−3	8

Colour choosen by P

There is no saddle point

By dominance rule for column, 3' column may be removed. The resulting matrix is

(Q)

	W	B
W	0	−2
(P) B	2	5
R	3	−3

By dominance rule for row, row may be removed The resultmg matrix (2 x 2) is

	W	B
B	2	5
R	3	−3

Applying Arithmatic method

Q

$$
\begin{array}{cc|cc|cc}
 & & W & B & & \\
\hline
P & B & 2 & 5 & 6 & \frac{6}{9} \\
 & R & 3 & -3 & 3 & \frac{3}{9} \\
\hline
 & & 8 & 1 & & \\
 & & \frac{8}{9} & \frac{1}{9} & &
\end{array}
$$

Optimum strategies for P

$$= \left(0, \frac{6}{9}, \frac{3}{9}\right)$$

Optimum strategies for Q $= \left(\frac{8}{9}, \frac{1}{9}, 0\right)$

$$\text{Game value} = \frac{2\times6+3\times3}{9} = \frac{12+9}{9} = \frac{21^{7}}{9_{3}}$$

(Let Q plays W)

$$\text{Game value (V)} = \frac{7}{3}.$$

Problem 8. Solve the following games by reducing them to 2 x 2 games by graphical method.

(a) Player A (rows) vs. Player B (columns):

3	0	6	−1	7
−1	5	−2	2	1

(b) Player A (rows) vs. Player B (columns):

−4	3
−7	1
−2	−4
−5	−2
−1	−6

Solution.

(a) Player A (rows) vs. Player B (columns):

3	0	6	−1	7
−1	5	−2	2	1

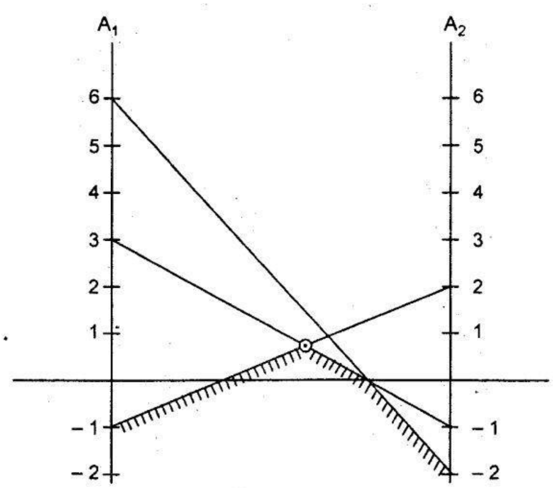
— reduce by dominance rule for column 2 and 5th column may be deleted as dominated by 4th and 3' column.

The resulting matrix is

1. No saddle point

A	B_1	B_3	B_4
	3	6	−1
	−1	−2	2

Since player A wishes to minimize his minimum expected payoff, the two lines which intersect at highest point of lower bound show the two cause of action B should choose in his best strategy.



Resulting matrix

$$
\begin{array}{cc|c|c|cc}
 & & B_1 & B_4 & & \\
\hline
A & A_1 & 3 & -1 & 3 & \frac{3}{7} \\
\hline
 & A_2 & -1 & 2 & 4 & \frac{4}{7} \\
\hline
 & & 3 & 4 & & \\
 & & \frac{3}{7} & \frac{4}{7} & & \\
\end{array}
$$

Ans. Optimum st. for A $= \left(\frac{3}{7}, \frac{4}{7}\right)$

Optimum st. for B $= \left(\frac{3}{7}, 0, 0, \frac{4}{7}\right)$

$$\text{Game value} = \frac{3\times3+(-1)\times4}{7} = \frac{5}{7}$$

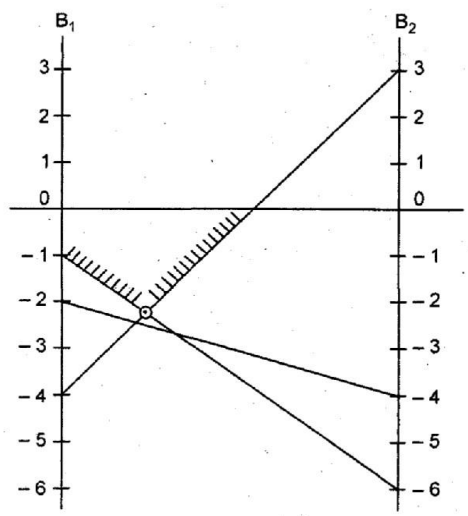
Let B plays B_1.

(*b*)

$$
\begin{array}{c|c|c|}
 & \multicolumn{2}{c}{B} \\
\hline
 & -4 & 3 \\
\hline
 & -7 & 1 \\
\hline
A & -2 & -4 \\
\hline
 & -5 & -2 \\
\hline
 & -1 & -6 \\
\hline
\end{array}
$$

– Reduced by dominance rule for row. 2nd and 4th row may be removed as it is dominated by 1st row. The resulting matrix is

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & B_1 & B_2 \\
\hline
 & A_1 & -4 & 3 \\
\hline
A & A_3 & -2 & -4 \\
\hline
 & A_5 & -1 & -6 \\
\hline
\end{array}
$$



Since B wishes to minimize his minimum expected payoff the two lines which intersect at lowest pomt of upper bound show the two course of action A should choose in his best strategy.

The resulting matrix is

$$\begin{array}{cccccc} & & B_1 & B_2 & & \\ A & A_1 & -4 & 3 & 5 & \frac{5}{12} \\ & A_5 & -1 & -6 & 7 & \frac{7}{12} \\ & & 9 & 3 & & \\ & & \frac{9}{12} & \frac{3}{12} & & \end{array}$$

Optimum strategy for A = $\left(\frac{5}{12}, 0, 0, 0, \frac{7}{12}\right)$

Optimum strategy for B = $\left(\frac{9}{12}, \frac{3}{12}\right)$

$$\text{Game value (V)} = \frac{-4 \times 5 + (-1) \times 7}{12} = \frac{-20 - 7}{12} = \frac{-27}{12} = \frac{-9}{4}$$

$$V = -\frac{9}{4}.$$

Problem 9. Solve the following game.

		(B)			
		B_1	**B_2**	**B_3**	**B_4**
(A)	**A_1**	2	1	0	−2
	A_2	1	0	3	3

Solution There is no saddle point in the game By rule of dommance for column 1st and 3' column may be deleted as dominated by 2nd and 4th column respectively.
Thus the resulting matrix is

	B_2	B_4
A_1	1	−2
A_2	0	3

Solving by Arithmatic method.

$$\begin{array}{ccccc} & B_2 & B_4 & & \\ A_1 & 1 & -2 & 3 & \frac{3}{6} \\ A_2 & 0 & 3 & 3 & \frac{3}{6} \\ & 5 & 1 & & \\ & \frac{5}{6} & \frac{1}{6} & & \end{array}$$

Optimum strategy for A $\left(\frac{3}{6},\frac{3}{6}\right)$

Optimum strategy for B $\left(0,\frac{5}{6},0,\frac{1}{6}\right)$

$$\text{Game value (V)} = \frac{1\times3+3\times0}{6} = \frac{3}{6} = \frac{1}{2}$$

(Let B plays B_2)

$$\text{Game value (V)} = \frac{1}{2}.$$

Problem 10. Obtain the optimal strategies for both persons and the value of the game for zero sum two person game whose payoff matrix is given as follows:

Player A

Player B	1	3	−1	4	2	−5
	−3	5	6	1	2	0

Solution. There is no saddle point m the game by rule of dominance for column 2nd, 4th and 5th column are dominated by 1st column and 3rd column dommated by 6th column hence 2nd, 4th, 5th and 3rd column may be removed. The resulting matrix is (2x2).

B_1	1	−5
B_2	−3	0

B_1	1	−5
B_2	−3	0
	5	4
	$\frac{5}{9}$	$\frac{4}{9}$

$$\text{Optimal strategy for A} = \left(\frac{5}{9},0,0,0,0,\frac{4}{9}\right)$$

$$\text{Optimal strategy for B} = \left(\frac{3}{9},\frac{6}{9}\right)$$

$$\text{Game value (V)} = \frac{3\times1-3\times6}{9} = \frac{3-18}{9} = \frac{-15}{9}$$

(Let A play A_1)

$$\text{Game value (V)} = \frac{-5}{3}.$$

QUESTION BANK

GAME THEORY

PURE STRATEGIES

1. Solve the game whose payoff matrix is given below

	B1	B2	B3
A1	-2	5	-3
A2	1	3	5
A3	-3	-7	11

2.

	B1	B2	B3
A1	0	-4	-2
A2	3	-5	1
A3	-2	-1	6
A4	1	0	4

MIXED STRATEGIES

3. Solve the game whose payoff matrix is given below

	B1	B2
A1	1	7
A2	5	1

4.

	B1	B2
A1	6	-3
A2	-3	7

MIXED STRATEGIES (DOMINANCE PRINCIPLE)/MODIFIED DOMINANCE

5.

	B1	B2	B3
A1	2	-2	4
A2	6	1	12
A3	-3	2	10

6.

	B1	B2	B3
A1	-3	7	4
A2	-2	-2	5
A3	3	-2	5

7.

	B1	B2	B3	B4	B5	B6
A1	4	2	0	2	1	1
A2	4	3	1	3	2	2
A3	4	3	7	-5	1	2
A4	4	3	4	-1	2	2
A5	4	3	3	-2	2	2

8.

	B1	B2	B3	B4
A1	3	2	4	0
A2	3	4	2	4
A3	4	2	4	0
A4	0	4	0	8

9.

	B1	B2	B3
A1	-1	2	8
A2	7	5	-1
A3	6	0	-12

10.

	B1	B2	B3
A1	19	20	23
A2	10	5	9
A3	21	14	10

GRAPHICAL METHOD

11.

	B1	B2	B3	B4
A1	2	4	-2	8
A2	3	6	5	-5

12.

	B1	B2
A1	1	-3
A2	3	5
A3	-1	6
A4	4	1
A5	2	2
A6	-5	0

REPLACEMENT

REPLACEMENT OF ASSET THAT FAIL GRADUALLY (WITHOUT TIME VALUE)

1. The purchase price of an asset is ₹8000, maintenance cost and resale value are given as follows.

Year	M.C	R.V
1	1000	4000
2	1200	2000
3	1700	1200
4	2200	600
5	2900	500
6	3800	400

   Find out optimum year and cost for replacement.

2. Cost of machine is ₹7000, maintenance cost is given by equation. 1000x (n-1), resale value is 4000, 2000, 1200, 600,500,400 thereafter. Find out when to replace the asset.

3. Purchase cost ₹4100, scrap value 100. Installation ₹2000

Year	1	2	3	4	5	6	7	8
M.C	50	125	200	300	450	600	800	1200
OP.C	50	125	200	300	450	600	800	800

4. There are 2 machines A and B. Machine A cost RS.45000, operating cost is Rs.1000 in first year and it increases by 10000 every year.

Machine B cost Rs.50000, operating cost is Rs.2000 and it increases by RS.4000 every year. Prove if Machine A must be replaced by Machine B. If yes, when? Assume both machines do not have resale value.

REPLACEMENT OF ASSET THAT FAILS GRADUALLY TAKING TIME VALUE INTO CONSIDERATION

5. Find out time of replacement if the maintenance cost is given by the equation 500(n-1). Discount rate is 15%. No resale value. The machinery cost RS. 5000.

6. A lorry cost RS. 80000, running cost and salvage value are given. Use 10% discount rate.

Year	1	2	3	4	5	6	7	8
R.C	6000	7500	9000	12000	15000	20000	20000	30000
SV	60000	40000	30000	25000	20000	2000	2000	2000

ASSETS THAT FAIL SUDDENLY-INDIVIDUAL & GROUP REPLACEMENT POLICY

7. Find out whether to use IRP Or GRP given the following details

Week	Cum. Prob
1	0.07
2	0.15
3	0.25
4	0.45
5	0.75
6	0.9
7	1

8.

Week	1	2	3	4	5
% of failure	10	25	50	80	100
IRP cost	Rs.2				
GRP cost	0.50ps.				

9.

Week		0	1	2	3	4	5	6
Survival rate		100	97	90	70	30	15	0
IRP cost	Rs.1							
GRP cost	0.35ps							
Assume 1000	bulbs.							

Hint: When cumulative probability is given, convert it in to individual probability
When probability of failure is given in percentage, convert it into decimals
In case survival rate is given, calculate failure rate which is equal to 1- survival rate

MODEL QUESTIONS

PART – A

1. What is meant by replacement? Explain the different types of replacements.
2. Explain the different methods of replacement of assets.
3. What do you mean by a) Individual replacement policy b) Group replacement policy?
4. Explain the procedure for replacement of an asset that fails gradually without considering Time value of money.
5. Explain the procedure for replacement of an asset that fails gradually with considering Time value of money
6. What is meant Game theory? Explain its importance.
7. Explain the characteristics of Game.
8. What do you mean by a) pay-off matrix b) saddle point c) pure strategy d) Mixed strategy e) Maximin principle f) Minimax principle.
9. What do you mean by Dominance principle? Explain its procedure.
10. Write the rules of dominance principle?

PART – B

11. Solve the Game using Dominance principle

Player B

Player A	B1	B2	B3	B4	B5
A1	2	4	3	8	4
A2	5	6	3	7	8
A3	3	7	9	8	7
A4	4	2	8	4	3

12. The cost of a new machine is Rs. 3000. Discounted factor is 10%. Find the Optimum period of replacement.

Year	1	2	3	4	5	6	7
Running Cost	500	600	800	1000	1300	1600	2000

13. Solve the Game using Dominance principle

Player A	B1	B2	B3	B4
A1	3	2	4	0
B2	3	4	2	4
C3	4	2	4	0
D4	0	4	0	4

14. There are 1000 bulbs. The following failure rates have been observed for a certain items.

End of week:	1	2	3	4	5
Prob. of failure:	0.10	0.30	0.55	0.85	1.00

The cost of replacing an individual item is Rs 1.25. The decision is made to replace all items simultaneously and also replace individual items as they fail. The cost of group replacement is 50 Paise. Which is better individual replacement or group replacement?

15. Solve the Game using Graphical method.

	B1	B2
A1	1	-3
A2	3	5
A3	-1	6
A4	4	1
A5	2	2
A6	-5	0

16. The following failure rates have been observed for a certain type of transistors in a digital computer.

End of week	1	2	3	4	5	6	7	8
Failure to date	.05	.13	.25	.43	.68	.88	.96	1

The cost of replacing an individual failed transistor is Rs1.25. The decision is made to replace all these transistors simultaneously at fixed intervals and to replace the individual transistor as they fail in service. If the cost of group replacement is 30 paise per transistor.

What is the interval between group replacements? It is preferable over individual replacement policy?

17. Purchase cost of a machie is ₹4100, scrap value ₹100 and installation charges ₹2000

Year	1	2	3	4	5	6	7	8
Maintenance Cost	50	125	200	300	450	600	800	1200
Operating Cost	50	125	200	300	450	600	800	800

Find the optimum period of replacement.

18. Solve the game whose payoff matrix is given below.

	B1	B2
A1	7	5
A2	3	2

19. Consider the following replacement schedule of a component in an electronic gadget

Hours in use	300	600	900	1200
Probability of failure	0.05	0.30	0.75	1

The cost of the replacement of the part is Rs. 500 whereas the failure would cost Rs. 3000. What should be the optimal replacement policy?

20. For the firm A and B, the payoff matrix is given below. Solve the game with dominance principles.

PLAYER	B1	B2	B3
A1	-4	-1	7
A2	2	0	3
A3	3	-2	1

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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – V – : INVENTORY AND QUEUING MODELS – SBAA5205

Unit – V: INVENTORY AND QUEUING MODELS 12 hrs

Inventory Models- ABC Analysis, Cost involved in inventory management- EOQ Calculation, Deterministic Demand Inventory Models, Quantity Discount Models. Waiting Line Models- Feature of Waiting Line Models- Kendall Notations- M/M/1; M/M/C; FIFO/N/N Models only.

5.1. INVENTORY

Inventory may be defined a stock of goods, commodities or other economic resources that are stored or reserved for smooth and efficient running of business. The inventory may be kept in any one of the following forms:

1. Raw material
2. Work-in progress
3. Finished goods

If an order for a product is receive, we should have sufficient stock of materials required for manufacturing the item in order to avoid delay in production and supply. Also there should not be over stock of materials and goods as it involves storage cost and wastage in storing. Therefore inventory control is essential to promote business. Maintaining inventory helps to run the business smoothly and efficiently and also to provide adequate service to the customer. Inventorycontrol is very useful to reduce the cost of transportation and storage.

A good inventory system, one has to address the following questions quantitatively and qualitatively.

- What to order?
- When to order?
- How much to order?
- How much to carry in an inventory?

5.1.1 Objectives of inventory management/Significance of inventory management

To maintain continuity in production.

To provide satisfactory service to customers.

To bring administrative simplicity.

To reduce risk.

To eliminate wastage.

To act as a cushion against high rate of usage.

To avoid accumulation of inventory.

To continue production even if there is a break down in few machinery.

To ensure proper execution of policies.

To take advantages of price fluctuations and buy economically.

5.1.2 Costs involved in inventory

1. **Holding Cost (Carrying or Storage Cost)**

   It is the cost associated with the carrying or holding the goods in stock. It includes storage cost, depreciation cost, rent for godown, interest on investment locked up, record keeping and administrative cost, taxes and insurance cost, deterioration cost, etc. It is denoted by ‘C'.

2. **Setup Cost/ Ordering Cost**

   Ordering cost is associated with cost of placing orders for procurement of material or finished goods from suppliers. It includes, cost of stationery, postage, telephones, travelling expenses, handling of materials, etc. (Purchase Model)Setup cost is associated with production. It includes, cost involved in setting up machines for production run. (Production Model). Both are denoted by ‘S'.

3. **Purchase Cost/Production Cost**

   When the organization purchases materials from other suppliers, the actual price paid for the material will be called the purchase cost.

   When the organization produces material in the factory, the cost incurred for production of material is called as production cost. Both are denoted by ‘P'.

4. **Shortage Cost**

   If the inventory on hand is not sufficient to meet the demand of materials or finished goods, then it results in shortage of supply. The cost may include loss of reputation, loss of customer, etc.

   Total incremental cost = Holding Cost + Setup Cost/ Ordering Cost

   Total Cost = Purchase Cost/ Production Cost + Shortage Cost + Total Incremental cost.

5.2 *Demand* is one of the most important aspects of an inventory system.

Demand can be classified broadly into two categories:

5.2.1 **Deterministic** i.e., a situation when the demand is known with certainty. And, deterministic demand can either be *static* (where demand remains constant over time) orit could be *dynamic* (where the demand, though known with certainty, may change with time).

5.2.2 **Probabilistic (Stochastic)** refers to situations when the demand is *random* and is governed by a *probability density function* or *probability mass function.* Probabilistic demand can also be of two types - *stationary*(in which the demand probability density function remains unchanged over time),and *non-stationary,* where the probability densities vary over time.

Deterministic Inventory Models

i. Model I: Purchasing model without shortages
ii. Model II: Production model without shortages
iii. Model III: Purchasing model with shortages

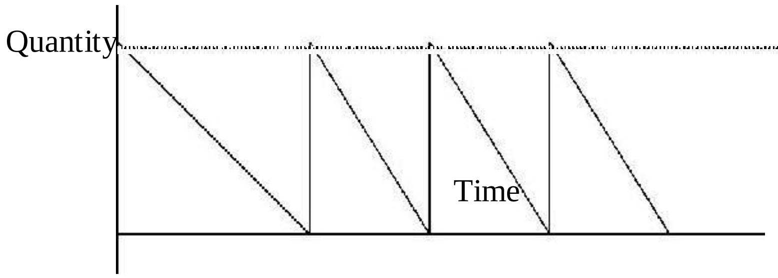
iv. Model IV: Production model with shortages

5.2.1.1 Model I: Purchasing model without shortages

<u>Assumptions</u>

- Demand(D) per year is known and is uniform
- Ordering cost(S) per order remains constant

- Carrying cost(C) per unit remains constant
- Purchase price(P) per unit remains constant
- No Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back. Lead time is Zero.

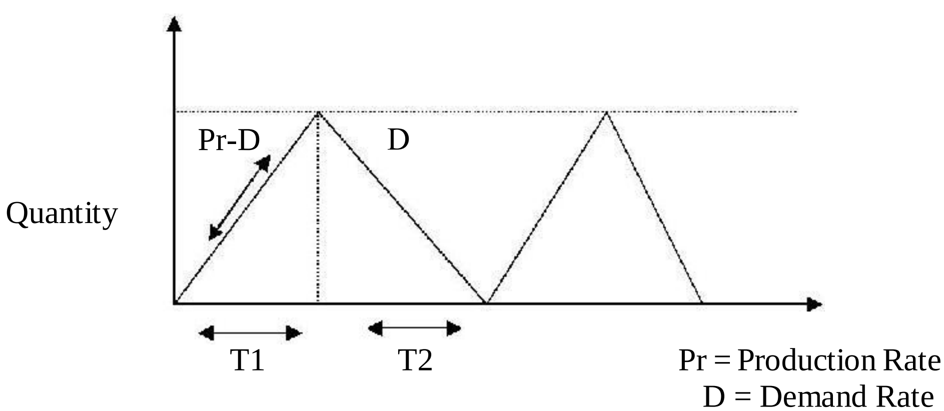


Inventory decreases at the rate of ‘D' As soon as the level of inventory reaches zero, the inventory is replenished back

5.2.1.2 Model II: Production model without shortages

Assumptions

- Demand(D) per year is known and is uniform
- Setup cost (S) per production run remains constant
- Carrying cost(C) per unit remains constant
- Production cost per unit(P) per unit remains constant
- No Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back.



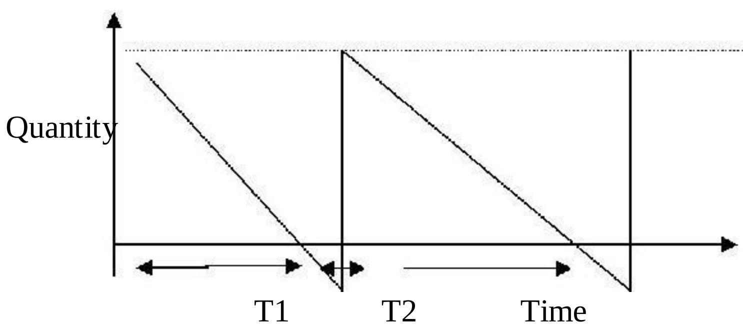
T1 is the time taken when manufacturing takes place at the rate of Pr and demand at the rate of

D. So the stock is built up at the rate of (Pr – D). During t2 there is no production only usage of stock. Hence, stock is decreased at the rate of 'D'. At the end of t2, stock will be nil.

5.2.1.3 Model III: Purchasing model with shortages

Assumptions

- Demand(D) per year is known and is uniform
- Ordering cost(S) per order remains constant
- Carrying cost(C) per unit remains constant
- Purchase price(P) per unit remains constant
- Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back with lead time.
- Shortage cost (sh) per unit remains constant



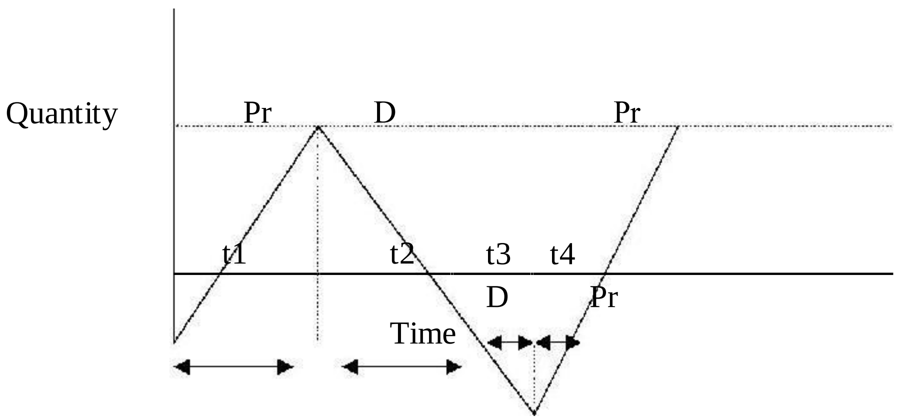
T1 is the time during which stock is nil.During T2 shortage occur and at the end of T2 stock is replenished back.

5.2.1.4 Model IV: Production model with shortages

Assumptions

Demand(D) per year is known and is uniform

- ❖ Setup cost (S) per production run remains constant
- ❖ Carrying cost(C) per unit remains constant
- ❖ Production cost per unit(P) per unit remains constant
- ❖ Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back with lead time.
- ❖ Shortage cost (Sh) per unit remains constant



T1 is the time taken when manufacturing takes place at the rate of Pr and demand at the rate of D. So the stock is built-up at the rate of (Pr – D). During t2 there is no production only usage of stock. Hence, stock is decreased at the rate of 'D'. At the end of t2, stock will be nil. During T3 shortage exists at the rate of 'D'. During T4 production begins stock builds and shortage decreases at the rate of 'Pr-D'

5.2.2.1 Inventory basic terminologies

- EOQ- Economic order quantity – The optimum order per order quantity for which total inventory cost is minimum.
- EBQ- Economic batch quantity – The optimum manufacturing quantity in one batch for which total inventory cost is minimum.
- Demand Rate – rate at which items are consumed
- Production rate- rate at which items are produced
- Stock replenishment rate
  - Finite rate – the inventory builds up slowly /step by step(production model)
  - Instantaneous rate – rate at which inventory builds up from minimum to maximum instantaneously (purchasing model)
- Lead time- Time taken by supplier to supply goods
- Lead time demand it is the demand for goods in the organization during lead time.
- Reorder level- the level between maximum and minimum inventory at which purchasing or manufacturing activities must start from replenishment.

  Reorder level = Buffer stock+ Lead time demand
- Buffer stock- to face the uncertainties in consumption rate and lead time , an extra stock is

maintained. This is termed as buffer stock:

Buffer stock = (Maximum Lead time – Average Lead time) x Demand per month

- Maximum Inventory Level: Maximum quantity that can be allowed in the stock: Maximum Inventory = EOQ + Buffer stock
- Minimum Inventory Level is the level that is expected to be available when thee supply is due: Minimum Inventory level = Buffer stock
- Average Inventory = (Minimum Inventory + Maximum Inventory)/2
- Order cycle is the period of time between two consecutive placements of orders.

5.3Inventory system followed in a organization:

- Q – System (fixed order quantity system)
- P - System (fixed period system)

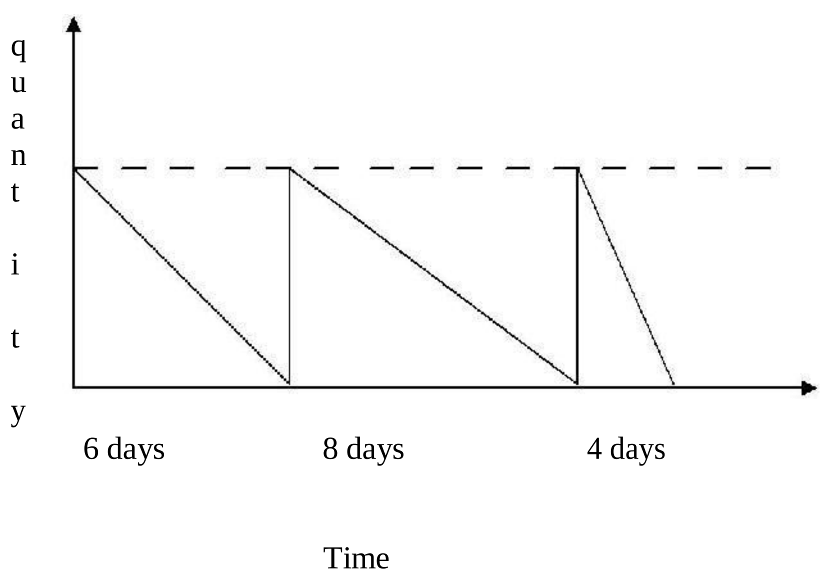
5.3.1 Q – System

In a fixed order quantity system means every time an order is placed the quantity order is EOQ.

In Q – System, the period between the orders is not constant:

Ex. 1st – 1 month – EOQ
2nd – 1 ½ month – EOQ
3rd - 2 month – EOQ
4th – 15 days - EOQ

Whenever the stock reaches reorder level, next order is placed.



- Reorder level- the level between maximum and minimum inventory at which purchasing or manufacturing activities must start from replenishment.

  Reorder level = Buffer stock+ Lead time demand
- Lead time is the time taken by supplier to supply goods
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- Buffer stock: To face the uncertainties in consumption rate and lead time, an extra stock is maintained. This is termed as buffer stock:

  Buffer stock = (Maximum Lead time – Average Lead time) x Demand per month
- Maximum Inventory Level: Maximum quantity that can be allowed in the stock: Maximum Inventory = EOQ + Buffer stock
- Minimum Inventory Level is the level that is expected to be available when the supply is due:

  Minimum Inventory level = Buffer stock

  Average Inventory = (Minimum Inventory + Maximum Inventory)/2

5.3.2 P – System

Time period between the orders is fixed; hence it is called as Fixed Period System. Period of order is fixed but the quantity will vary. Ex:

1^{st} – 1 month – 1000 units

2^{nd} – 1 month – 1200 units

3^{rd} - 1 month – 950 units

A predetermined level of inventory is fixed and thee order quantity is determined by deducting the level of stock at the time review from P determine level of inventory.

Order quantity = Predetermined level of inventory – level of stock at the time of review

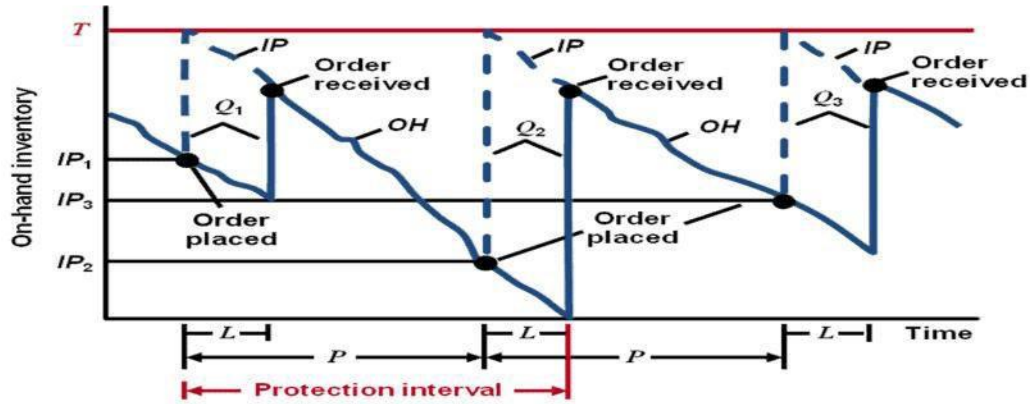


Figure 12.10 – P System When Demand Is Uncertain

5.4Inventory Selective Control Techniques

Every organization consumes several items of store. Since all the items are not of equal importance, a high degree control on inventories of each item is neither applicable nor useful. So it becomes necessary to classify items in group depending upon their utility importance. Such type of classification is name as the principle of selective control.

5.4.1 ABC Analysis (Always Better Control)

- A - High value items
- B - Moderate value items
- C - Low value items

ABC analysis is one of the methods for classification of materials. It is based on Parelo's law that a few high usage value items constitute a major part of the inventory while a large bulk of items constitute to very low usage value.

5.4.1.1 PROCEDURE FOR ABC ANALYSIS:

1. Note down the material code.
2. Note down the annual usage in terms of units.
3. Note down the price per unit.
4. Calculate the Annual usage value.

Annual usage value = Quantity used x Price per unit

5. Arrange the materials according to the value in descending order.
6. Find out the percentage contribution of each material to the total value.
7. Find out the percentage contribution of each material towards the total quantity.

8. Cumulate the % contribution towards value.
9. The classification is as follows.

A = 80% contribution B = 15% contribution C = 5% contribution.

<u>5.4.1.2</u> <u>SIGNIFICANCE OF ABC ANALYSIS</u>

ABC analysis is a very useful technique to classify the materials.

- The control procedure is based on which category the item belongs to. A = Tight control
  B = Moderate control
  C = Very little control.

- The inventory to be maintained is again based on the category
  A = Low Inventory
  B = Moderate Inventory C = High Inventory.

- The number of suppliers is also based on the category to which it belongs. A = Many suppliers
  B = Moderate No. of suppliers C = Few suppliers.

<u>5.4.2</u> VEDAnalysis

- V Vital items
- E Essential items
- D Desirable or Durable it ems

<u>5.4.3</u> HML Analysis

- High price items
- Moderate price items
- Low price items

<u>5.4.4</u> FNSD Analysis

- F Fast Moving items
- N Normal Moving items
- S Slow Moving items

- D Dead items

5.5Probabilistic Inventory Model

One such model is fixed order quantity model (FOQ).
In this model,

1. The demand (D) is uncertain, you can estimate the demand through any one of the forecasting techniques an d the probability of demand distribution is known.

2. Lead time (L) is uncertain, probability of lead time distribution is known.
3. Cost(C) all the costs are known.
   a. –Inventory holdin g costs C1

   b. –shortage cost C2
4. The optimum order level Z is determined by the following relationship

$$\sum_{d=0}^{z-1} p(d) < \frac{C2}{C1+C2} < \sum_{d=0}^{z} p(d)$$

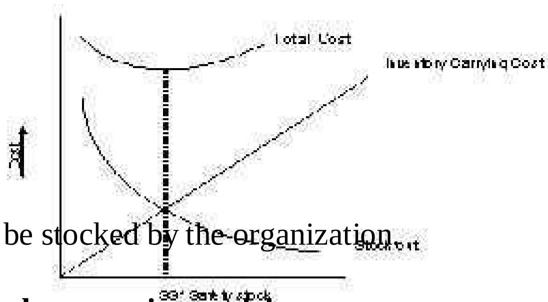
5.5.1 Stock out Cost/Shortage cost

It is difficult to calculate stock out cost because it consists of components difficult to quantify so indirect way of handling stock out cost is through service levels. Service levels means ability of organization to meet the requirements of the customer as on when he demands for the product. It is measured in terms of percentage.
For example: if an organization maintains 90% service level, this means that 10% is “stock out” level. This way the stock out level is addressed.

5.5.2 Safety stock

It is the extra stock or buffer stock or minimum stock. This is kept to take care of fluctuations in demand and lead time.

If you maintain more safety stock, this helps in reducing the chances of being “stock out”. But at the same time it increases the inventory carrying cost. Suppose the organization maintains less service level that results in more stock out cost but less inventory carrying cost. It requiresa tradeoff between inventory carrying cost and stock out cost. This is explained throughfollowing Fig.



Safety stock (S.S*) is to be stocked by the organization.

5.5.3 Working of fixed order quantity model

Fixed order quantity system is also known as continuous review system or perpetual inventory system or Q system.

In this system, the ordering quantity is constant. Time interval between the orders is thevariable. The system is said to be defined only when if the ordering quantity and time interval between the orders are specified. EOQ provides answer for ordering quantity. Reorder level provides answers for time between orders.

The working and the fixed order quantity model is shown in the below Fig

5.5.4 Application of Fixed Order Quantity System

1. It requires continuous monitoring of stock to know when the reorder point is reached.
2. This system could be recommended to" A" class because they are high consumption items. So we need to have fewer inventories. This system helps in keeping less inventory comparing to other inventory systems.

5.5.4.1 Advantages:

1. Since the ordering quantity is EOQ, comparatively it is meaningful. You need to have less safety stock. This model relatively insensitive to the forecast and the parameter changes.
2. Fast moving items get more attention because of more usage.

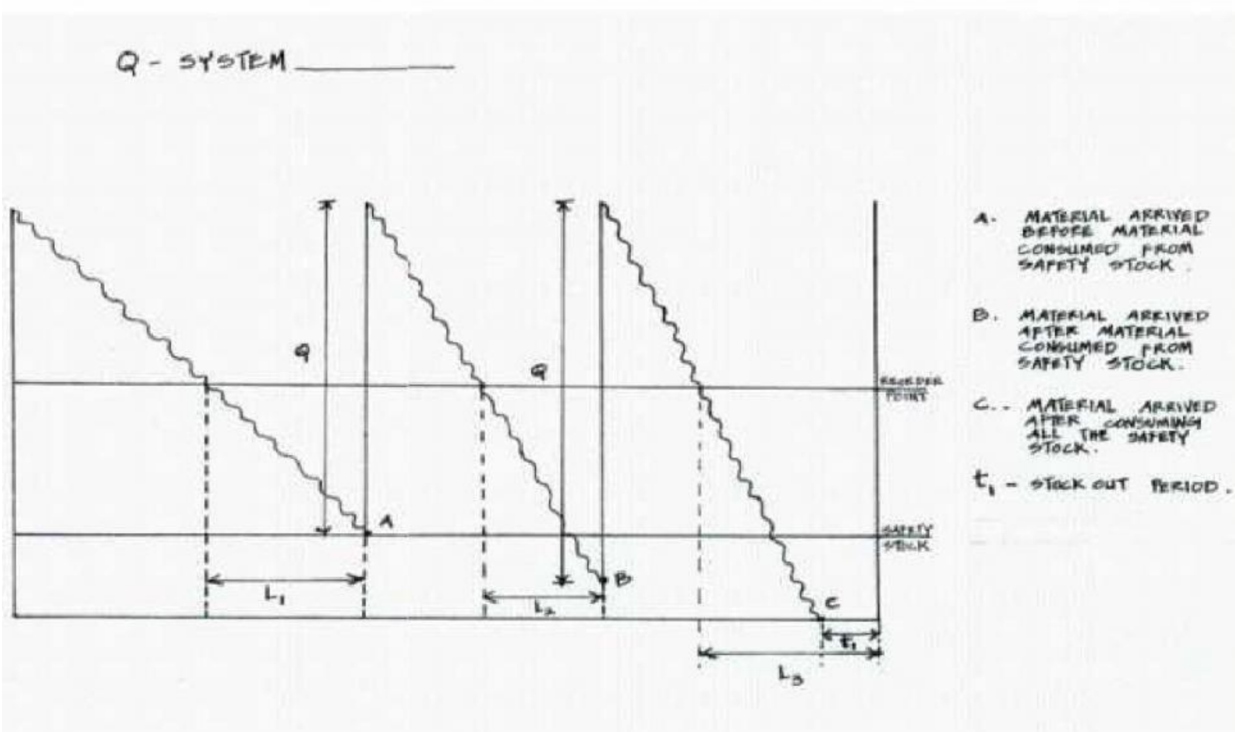
5.5.4.2 Weakness:

1. We can't club the order for items which are to be procured from one supplier to reduce the ordering cost.

2. There is more chance for high ordering cost and high transaction cost for the items, which follow different reorder level.

3. You can not avail supplier discount. While the reorder level fall in different time periods.

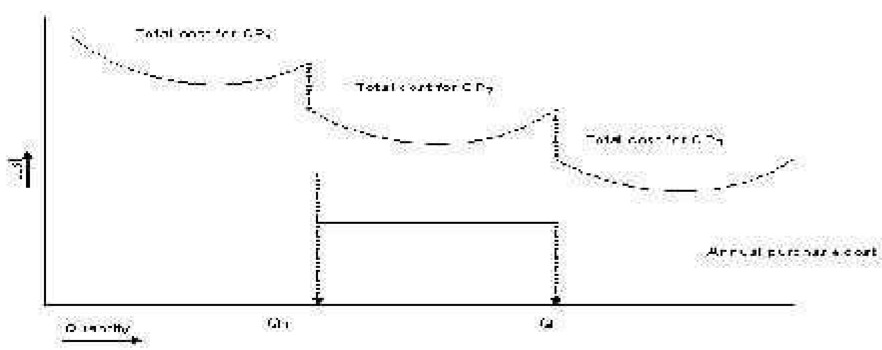
Figure –Fixed Order Quantity Model



<u>5.5.5</u> QUANTITY DISCOUNT MODEL

As it is mentioned already, the purchase cost becomes relevant with respect to the quantity of order only when the supplier offers discounts. Discounts means if the ordering quantity exceeds particular limit supplier offers the quantity at lesser price per unit. This is possible because the supplier produces more quantity. He could achieve the economy of scale the benefit achieved through economy of scale that he wants to pass it onto customer. This results in lesser price per unit if customer orders more quantity.

If you look at in terms of the customer's perspective customer has also to see that whether it is advisable to avail the discount offered, this is done through a trade off between his carrying inventory by the result of acquiring more quantity and the benefit achieved through purchase price. Suppose if the supplier offers discount schedule as follows,



If the ordering quantity is less than or equal to Q1 then purchase price is Cp1.

If the ordering quantity is more than Q1 and less than Q2 then purchase price is Cp2.

If the ordering quantity is greater than or equal to Q2 then purchase price is Cp3.

Then the curve you get cannot be a continuous total cost curve, because the annual purchase cost

breaks at two places namely at Q1 and Q2.

5.5.5.1STEPS TO FIND THE QUANTITY TO BE ORDERED

1. Find out EOQ for the all price break events. Start with lowest price
2. Find the feasible EOQ from the EOQ's we listed in step 1.
3. Find the total annual inventory cost using the formulae for feasible EOQ =$\sqrt{[2DSC]}$+D*P
4. Find the total annual inventory cost for the quantity at which price break took place using the following formula.

Total annual inventory cost = TC= (D/Q)*S + (Q/2)*C + D*P

5. Compare the calculated cost in steps 3 and 4. Choose the particular quantity as ordered Quantity at which the total annual inventory cost is minimum.

5.6QUEUING THEORY

Queuing theory concerns the mathematical study of queues or waiting lines (seen in banks, post offices, hospitals, airports etc.). The formation of waiting lines usually occurs whenever the current demand for a service exceeds the current capacity to provide that service.

The objective of the waiting line model is to minimize the cost of idle time & the cost of waiting time.

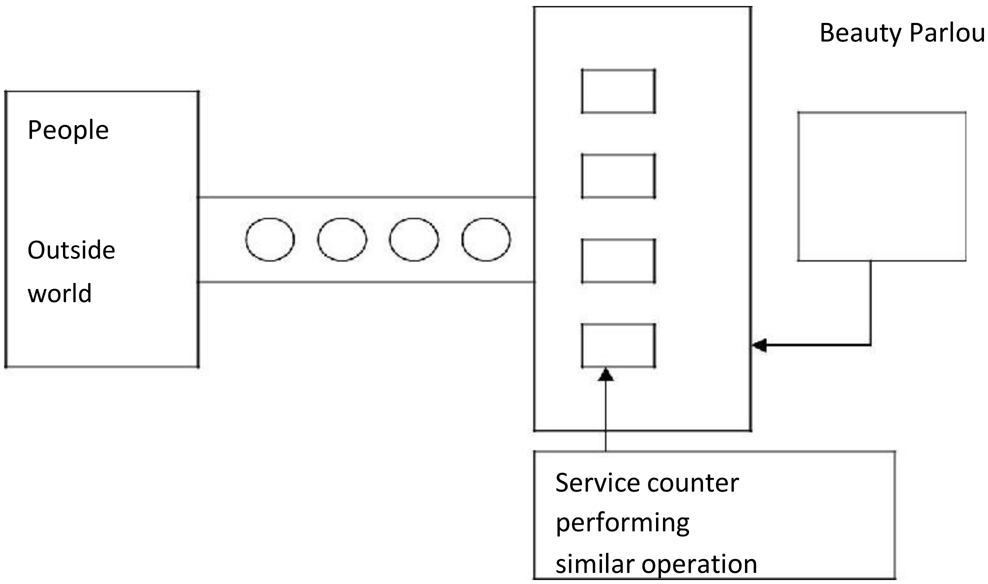
IDLE TIME COST: If an organization operates with many facilities and the demand from customers is very low, then the facilities are idle and the cost involved due to the idleness of the facilities is the *idle time cost*. The cost of idle service facilities is the payment to be made to the services for the period for which they remain idle.

*WAITING TIME COST:*If an organization operates with few facilities and the demand from customer is high and hence the customer will wait in queue. This may lead to dissatisfaction of customers, which leads to *waiting time cost*. The cost of waiting generally includes the indirect cost of lost business.

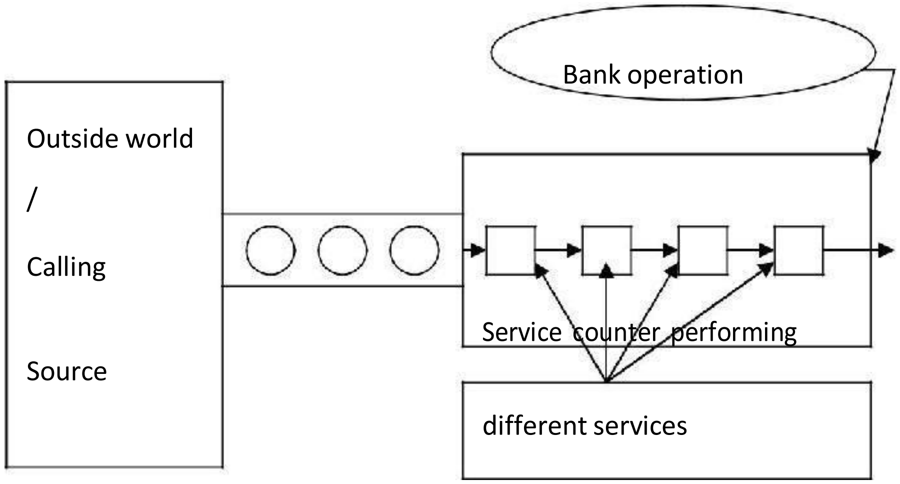
5.6.1 TYPE OF QUEUE

a) Parallel queues. b) Sequential queues.

5.6.1.1 PARALLEL QUEUES: If there is more than one server performing the same function, then queues are parallel.



5.6.1.2 SEQUENTIAL QUEUES : If there is one server performing one particular function or many servers performing sequential operations then the queue will be sequential.



a. Limited Queue:

In some facilities, only a limited number of customers are allowed in the system and new arriving customers are not allowed to join the system unless the number below less the limiting value. (Number of appointments in hospitals)

b. Unlimited Queue:

In some facilities, there is no limit to the number of customer allowed in the system. (Entertainment centers).

a. **Infinite queue:** If the customer who arrives and forms the queue from a very large population the queue is referred to as infinite queue.

b. **Finite Queue:** if the customer who arrives and forms the queue from a small population then the queue is referred to as finite queue.

DEFINITIONS:

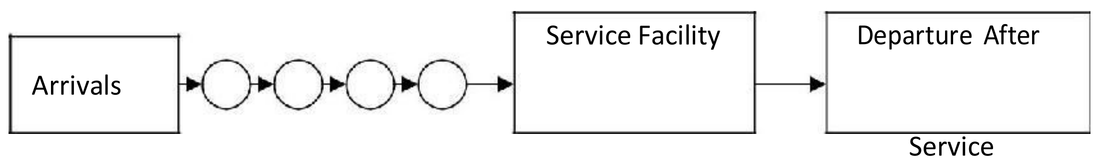
1. **The customer:** The arriving unit that requires some service to be provided.
2. **Server**: A server is one who provides the necessary service to the arrived customer.

3. **Queue (Waiting line):** The number of customers, waiting to be serviced. **The queue does not include the customer being serviced.**
4. **Service channel**: The process or system, which performs the service to the customer.

Based on the number of servers available.

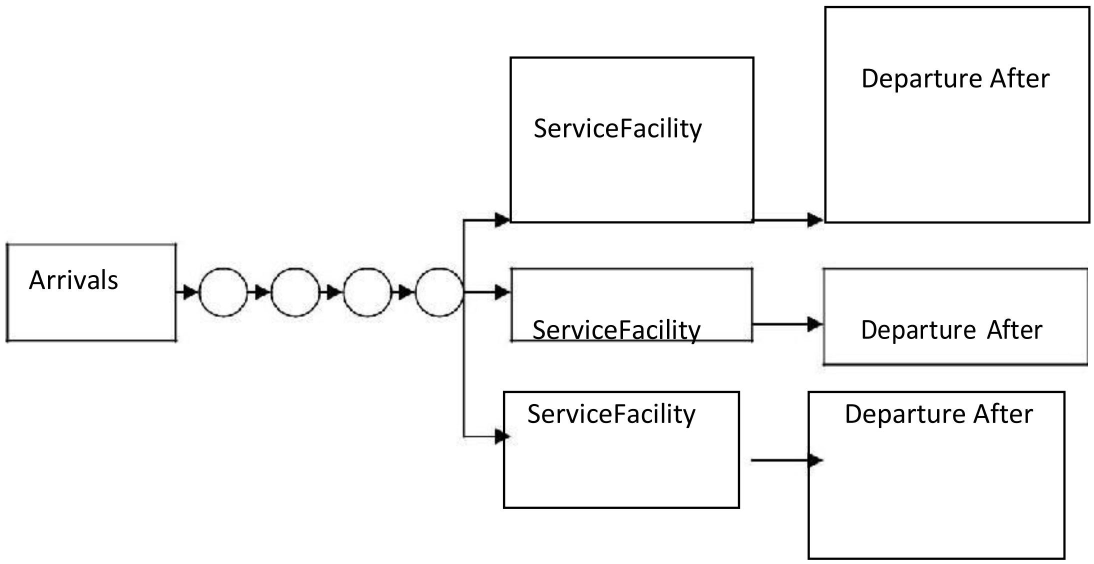
4A. **Single Channel:** If there is a single service station, customer arrivals from a single line to be serviced then the channel is said to Single Channel Model or Single Server Model.

Eg. Doctor's clinic



4B. **Multiple Channel Waiting Line Model:** If there are more than one service station to handle customer who arrive then it is called Multiple Channel Model. Symbol “c” is used.

E.g., Barber shop



5. **Arrival rate:** The rate at which the customers arrive to be serviced. It is denoted by λ. λ indicates take average number of customer arrivals per time period.

6. **Service rate:** The rate at which the customers are actually serviced. It in indicated by μ . μ indicates the mean value of customer serviced per time period.

7. **Infinite queue:** If the customers who arrive and form the queue from a very large population the queue is referred to as infinite queue.

8. **Priority:** This refers to method of deciding as to which customer will be serviced. Priority is said to occur when an arriving customer is chosen for service ahead of some other customer

already in the queue.

9. **Expected number in the queue**“**Lq**”: This is average or mean number of customer waiting to be serviced. This is indicated by “Lq”.

10. **Expected number in system Ls.:** This is average or mean number of customer either waiting to be serviced or being serviced. This is denoted by Ls.

11. **Expected time in queue Wq".:** This is the expected or mean time a customer spends waiting in the queue. This is denoted by "Wq".
12. **The Expected time in the system "Ws':** This is the expected time or mean timecustomers spends for waiting in the queue and for being serviced. This is denoted by "Ws'.
13. **Expected number in a non-empty queue:** Expected number of customer waiting in the line excluding those times when the line is empty.
14. **System utilization or traffic intensity**: This is ratio between arrival and service rate.
15. **Customer Behaviour:** The customer generally behaves in 4 ways:

a) **Balking:** A customer may leave the queue, if there is no waiting space or he has no time to wait.

b) **Reneging:** A customer may leave the queue due to impatience

c) **Priorities:** Customers are served before others regardless of their arrival

d) **Jockeying**: Customers may jump from one waiting line to another.

16. **Transient and Steady State**:

A system is said to be in Transient state when its operating characteristics are dependent on time.A system is said to be in Steady state when its operating characteristics are not dependent on time.

5.6.2 CHARACTERISTICS OF QUEUING MODELS:

a) Input or arrival (inter –arrival) distribution.

b) Output or Departure (Service) distribution.

c) Service channel

d) Service discipline.

e) Maximum number of customers allowed in the system.

f) Calling source or Population.

a) ARRIVAL DISTRIBUTION:

It represents the rate in which the customer arrives at the system. Arrival rate/interval rate:

► Arrival rate is the rate at which the customers arrive to be serviced per unit of time.

► Inter-arrival time is the time gap between two arrivals.

► Arrival may be separated

1) By **equal** interval of time
2) By **unequal** interval of time which is **definitely known.**
3) Arrival may be **unequal** interval of time whose **probability is known.**

► Arrival rate may be

1. Deterministic (D)
2. Probabilistic
    a. Normal (N)
    b. Binomial (B)
    c. Poisson (M/N)
    d. Beta (β)
    e. Gama (g)
    f. Erlongian (Eh)

The typical assumption is that arrival rate is randomly distributed according to Poisson distribution it is denoted by λ. λ indicates average number of customer arrival per time period.

b) SERVICE OR DEPARTURE DISTRIBUTON:

It represents the pattern in which the customer leaves the system. Service rate at which the customer are actually serviced. It indicated by μ. μ indicates the mean value of service per time period. Interdeparture is the rate time between two departures.

Service time may be

- **Constant.**
- **Variable** with **definitely known** probability.
- **Variable** with **known probability**.

Service Rate Or Departure Rate may be:

1. Deterministic
2. Probabilistic.
   a. Normal (N)

   b. Binomial (B)
   c. Poisson (M/N)
   d. Beat (β)
   e. Gama (g)
   f. Erlongian (Ek)
   g. Exponential (M/N)

The typical assumption used is that service rate is randomly distributed according to exponential distribution. Service rate at which the customer are actually serviced. It indicated by μ . μ indicates the mean value of serv ice per time period.

c) SERVICE CHANNELS:

The process or system, which is performing the service to the customer.

Based on the number of channels**:**

Single channel

If there is a single service station and customer arrive and from a single line to be serviced, the channel is said to single channel. **Single Channel – 1.**

Multiple channel

If there is more than one service station to handle customer who arrive, then it is called multiple channel model. **Multiple Channel - C.**

d) SERVICE DISCIPLINE: Service discipline or order of service is the rule by which customer are selected from the queue for service.

FIFO: First In First Out – Customer are served in the order of their arrival. Eg. Ticket counter, railway station, banks.

LIFO: Last In First Out – Items arriving last come out first.

Priority: is said to occur when a arriving customer is chosen ahead of some other customer for service in the queue.

SIRO: Service in random order

Here the common service discipline "First Come, First Served".

e) MAXIMUM NUMBER OF CUSTOMER ALLOWED IN THE SYSTEM:

Maximum number of customer in the system can be either finite or finite.

a. Limited Queue:

In some facilities, only a limited number of customers are allowed in the system and new arriving customers are not allowed to join the system unless the number below less the limiting value. (Number of appointments in hospitals)

b. Unlimited Queue:

In some facilities, there is no limit to the number of customer allowed in the system. (Entertainment centers).

f) POPULATION:

The arrival pattern of the customer depends upon the source, which generates them. a. **Finite population (<40):**

If there are a few numbers of potential customers the calling source is finite. b. **Infinite calling source or population:**

If there are large numbers of potential customer, it is usually said to be infinite.

5.6.3 KENDALL'S NOTATION: a/b/c;

d/e/f. Where, a – Arrival rate.

b – Service rate.

c – Number of service s 1 or c.

d – Service discipline (FIFO)

e - Number of persons allowed in the queue (N or∞)

f - Number of people in the calling source (∞ or N)

1. **M/M/1, FIFO/∞/∞:**

Means Poisson arrival rate, Exponential service rate/one server /FIFO service discipline/Unlimited queues & Unlimited queue in the calling source.

2. **M/M/C, FIFO/∞/∞:**

Poisson arrival rate, Exponential service rate, more than one server, FIFO service discipline Unlimited queues and unlimited persons in the calling source.

3. M/M/I, FIFO/N/∞:

Means Poisson arrival rate, Exponential service rate, One server, FIFO, Limited queue & Unlimited population.

5.6.4 SINGLE CHANNEL /MULTIPLE CHANNELPOPULATION MODEL:

1. Find an expression for probability of n customer in the system at time (Pn) in terms of λ and μ

2. Find an expression for probability of zero customers in the system at time t.(Po)
3. Having known Pn, find out the expected number of units in the Queue (Lq)
4. Find out the expected number of units in the system (Ls)
5. Expected waiting time in system (Ws)

6. Expected waiting time queue (Wq)

<u>5.6.5</u> SOLUTION PROCESS

1. Determine what quantities you need to know.
2. Identify the server
3. Identify the queued items
4. Identify the queuing model
5. Determine the service time
6. Determine the arrival rate
7. Calculate ρ
8. Calculate the desired values

Problems

Problem 1. A particular item has a demand of 9000 units/year. The cost of one procurement is Rs. 100 and the holding cost per unit is Rs. 2.40 per year. The replacement is instantaneous and no shortance are allowed.
Determine
1. Economic lot size.
2. The no. of order per year.
3. The time between orders.
4. Total cost per year if the cost of one unit is Rs. 1.

Solution.

$$R = 9000 \text{ unit/year}$$
$$C_3 = \text{Rs. } 100/\text{procurement}$$
$$C_1 = \text{Rs. } 2.40/\text{unit/year}$$

1. $$q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 100 \times 9000}{2.40}} = 866 \text{ units/Procurement.}$$

2. $$n_0 = \frac{1}{t_0} = \sqrt{\frac{C_1R}{2C_3}} = \sqrt{\frac{2.40 \times 9000}{2 \times 100}} = \sqrt{108} = 10.4 \text{ order/year.}$$

3. $$t_0 = \frac{1}{n_0} = \frac{1}{10.4} = 0.0962 \text{ years} = 1.15 \text{ months between procurement.}$$

4. $$C_0 = 9000 \times 1 + \sqrt{2C_1C_3R}$$
$$= 9000 + \sqrt{2 \times 2.40 \times 100 \times 9000}$$
$$= \text{Rs. } 11080 \text{ per year.}$$

Problem 2. A manufacturing company purchases 9000 parts of a machine for its annual requirements, ordering one month usuage at a time. Each part cost Rs. 20. The order cost per ordering is Rs. 15 and the carrying charges are 15% of the average inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer and how much would it same the company per year.

Solution. $R = 9000$ parts/year.

$$q = \frac{9000}{12} = 750 \text{ parts}$$

C = Rs. 20/ parts, C_3 = Rs. 15/order.

$$C_1 = \text{Rs.}20 \times \frac{15}{100} = \text{Rs. } 3/\text{part/year}$$

$$\text{Total annual variable cost} = \frac{q}{2}.C_1 + \frac{R}{q}.C_3$$

$$= \text{Rs.}\left[\frac{750}{2} \times 3 + \frac{9000}{750} \times 15\right]$$

$$= \text{Rs.}1305.$$

$$\text{E.O.Q. } (q) = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 9000 \times 15}{3}} = 300 \text{ units.}$$

$$\text{Total annual variable cost} = \sqrt{2RC_1C_3} = \sqrt{2 \times 9000 \times 3 \times 15}$$

(with E.O.Q.) = Rs. 900.

Hence if the company purchases 300 units each time and places 30 orders in the year, the net saving to the company will be Rs. (1305 — 900) = Rs. 405 a year.

Problem 3. You have to supply your customers 100 units of a certain product every monday- You obtain the product from a local supplier at Rs. 60 per unit. The cost of ordering and transportation from the supplier are Rs. 150 per order. The cost of carrying inventory is estimated at 15% per year of the cost of the product carried.

1. Find the lot size which will minimize the cost of the system.

2. Determine the optimal cost.

Solution. $R = 100$ units/week ; $C =$ Rs. 60.

$C_3 =$ Rs. 150 per order

$C_1 =$ 15% per year of the cost of the product.

$= (15 \times 60)/(100 \times 52)$ per unit per week.

$=$ Rs. 9/52 per unit per week.

1. $$q \text{ (EOQ)} = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 150 \times 100 \times 52}{9}} = 416 \text{ units.}$$

2. $$C_{min} = CR + \sqrt{(2C_1C_3R)}$$

$$= 60 \times 100 + \sqrt{2 \times \left(\frac{9}{52}\right) \times 150 \times 100}$$

$$= \text{Rs. } 6072.$$

Optimal cost $=$ Rs. 6072.

E.O.Q. $= 416$ units.

Problem 4. Daily demand for a product is normally distributed with mean, 60 units and a standard deviation of 6 units. The lead time is constant at 9 days. The cost of placing an order is Rs. 200, and the annual holding costs are 20% of the unit price of Rs. 50. A 95% service level is desired for the customers, who place orders during the reorder period. Determine the order quantity and the reorder level for the item in question, assuming that there are 300 working days during a year.

Solution.

(R) Demand/day $= 60$ units

(C_3) order cost $=$ Rs. 200/order

(C_1) holding cost $= 0.20 \times 50 =$ Rs. 10/- per unit per year

working day/year $= 300$

Demand/year $= 60 \times 300 = 18000$ units

$$\text{EOQ } q = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 200 \times 18000}{10}} = 848.52 \text{ units.}$$

Lead time $= 9$ days

Standard deviation of daily demand = 6 units

Now variance of demand during the lead time is equal to the sum of variance of daily demand during the lead time period

$$\text{variance} = 6^2 + 6^2 + 6^2 + \ldots\ldots + 6^2 \text{ (9 times)}$$
$$= 9 \times 6^2 = 324$$

Standard deviation of demand during the lead time

$$\text{Period} = \sqrt{324} = 18.$$

With E.O.Q. of 848.52 the no. of order during the year

$$(n_0) = \frac{18000}{848.52} = 21.21$$

Service level = 0.95

the normal deviate Z as found from probability table is 1.65.

Safety stock = Z × standard deviation

= 1.65 × 18 = 29.7 units.

Reorder level = Expected demand during lead time + safety stock

= 60 × 9 + 29.7 = 569.7 units.

Ans. Economic order quantity EOQ = 848.52 units.

Reorder level = 569.7 units.

Problem 5. The demand per month for a product is distributed normally with a mean of 100 and standard deviation 25. The lead time distribution is given below. What service level will be offorded by a reorder level of 500 units?

Lead time (months)	:	**1**	**2**	**3**	**4**	**5**
Probability	:	**0.10**	**0.20**	**0.40**	**0.20**	**0.10**

Solution. It is given that the demand is distributed normally with

Mean $(\bar{D})$ = 100 units SD (σd) = 25 units

lead time (L) = 1, 2, 3, 4 and 5

Reorder level (M) = 500 units

We shall use iterative method of computing service level for the reorder level policy when the demand per unit time is distributed normally and distribution of lead time is known

$$Z = \frac{M - L\bar{D}}{\sigma d\sqrt{L}}$$

By iterative method

Lead time	Value of Z when M = 500	Probability of not running out of stock corresponding to the value of Z (from table)	Probability of this particular lead time occuring	Conditional Probability of not running out of stock
1.	$\frac{500-100\times 1}{25\sqrt{1}} = 16.0$	100	0.10	10
2.	$\frac{500-100\times 2}{25\sqrt{2}} = 8.49$	100	0.20	20
3.	$\frac{500-100\times 3}{25\sqrt{3}} = 4.49$	100	0.40	40
4.	$\frac{500-100\times 4}{25\sqrt{4}} = 2.00$	97.7	0.20	19.5
5.	$\frac{500-100\times 5}{25\sqrt{5}} = 0.00$	50.0	0.10	5.0

Total conditional probability of not running out of stock
=10 + 20+ 40+ 19.5 + 5= 94.5.
Hence a reorder level of 500 units will give 94.5% service level.

Problem 6. The annual demand for a product is 500 units. The cost of storage per unit per year is 10% of the unit cost, The ordering cost is Rs. 180 for each order. The unit cost depends upon the amount ordered. The range of amount ordered and the unit cost price are as follows

Range of amount ordered	$0 \leq Q_1 \leq 500$	$0 \leq Q_2 \leq 1500$	$1500 \leq Q_3 \leq 3000$	$3000 < Q_4$
Unit cost (Rs.)	25.00	24.80	24.60	24.4

Solution. Here $R = 500$ units

$I = 0.10$

$C_3 = \text{Rs. } 180$

$$\text{EOQ for unit price of Rs. 24.40} = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\times 180\times 500}{24.40\times 0.10}}$$

$$= 271.6 \text{ units.}$$

But this is not feasible because the unit price of Rs. 24.40 is not available for an order size of 271.6 units.

$$\text{EOQ for unit price of Rs. } 24.60 = \sqrt{\frac{2 \times 180 \times 500}{24.60 \times 0.10}} = 270.5 \text{ units (infeasible)}$$

$$\text{EOQ for unit price of Rs. } 24.80 = \sqrt{\frac{2 \times 180 \times 500}{24.80 \times 0.10}} = 269.4 \text{ units (infeasible)}$$

$$\text{EOQ for unit price of Rs. } 25.00 = \sqrt{\frac{2 \times 180 \times 500}{25 \times 0.10}} = 268.3. \text{ units (feasible)}$$

Total annual cost for order quantity of 268.3 units (optimal size)

$$= \sqrt{2C_1C_3R} + CR$$

$$= \sqrt{2 \times 25 \times 0.10 \times 180 \times 500} + 25 \times 500$$

$$= \text{Rs. } 13170.82.$$

Total annual cost for order quantity corresponding to cut off point of 500 units.

$$= \frac{q}{2}C_1 + C_3\frac{R}{q} + CR$$

$$= \frac{500}{2} \times 24.80 \times 0.10 + 180 \times \frac{500}{500} + 24.80 \times 500$$

$$= \text{Rs. } 13200.$$

Total annual cost for order quantity corresponding to cut of point of 3000 units.

$$= \text{Rs.}\left(\frac{1500}{2} \times 24.60 \times 0.10 + 180 \times \frac{500}{1500} + 24.60 \times 500\right)$$

$$= \text{Rs. } 14745.$$

Total annual cost for order quantity corresponding to cut off point of 3000 units.

$$= \text{Rs.}\left(\frac{3000}{2} \times 24.40 \times 0.10 + 180 \times \frac{500}{3000} + 24.40 \times 500\right)$$

$$= \text{Rs. } (3660 + 30 + 12200) = \text{Rs. } 15890.$$

Since the total cost is minimum at $q_0 = 271.6$ units. It represents the optimal order quantity.

List of Formulas

1. Expected number of units m the system (waiting + being served) (or)
Length of the system

$$\boxed{L_s = \frac{\lambda}{\mu - \lambda}}$$

λ . – mean arrival rate (no. of arrivals per unit of time)

μ – mean service rate per server (no. of customer served per unit time).

2. Expected number of units in the queue

Length of the queue $\boxed{L_q = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu-\lambda}\right).}$

3. Expected time per unit in the system
(Expected time a unit spends in the system)

$$\boxed{W_s = \frac{1}{\mu-\lambda}.}$$

4. Expected time per unit in the queue (Expected time a unit spends in the queue)

$$\boxed{W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right).}$$

5. Avg. length of non empty queue (length of the queue that is formed from time to time)

$$\boxed{L_n = \frac{\mu}{\mu-\lambda}}$$

6. Avg. waiting time is non empty queue Avg. waitmg time of an arrival who waits

$$\boxed{W_n = \frac{1}{\mu-\lambda}}$$

7. Traffic Intensity/Equipment utilization $= \frac{\lambda}{\mu}$.

8. Probability that a person will have to wait $= \frac{\lambda}{\mu}$.

9. Probability that equipment or service (person) remain idle $= 1 - \frac{\lambda}{\mu}$.

10. Probability that n customers (units) in the system

$$\boxed{p(n) = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n}$$

n = no. of customers in the system.

11. Probability that there are more than (n) customer in the system = 1- $(P_0 + P_1 + \text{..........} + P_n)$.

12. Probability of n customers arriving in time $t = \frac{(\lambda t)^n \cdot e^{-\lambda t}}{n!}$.

13. Proabability [Waiting time $\geq t$].

$$= \int_t^{\infty} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)t}\,dt$$

14. Probability [time in system $\geq t$]

$$= \int_t^{\infty} (\mu-\lambda)e^{-(\mu-\lambda)t}\,dt.$$

Problems

Problem 8.13. Auto vehicles arrive at a petrol pump, having one petrol unit, in poission fashion an avg. of 10 units per hour. The service is distributed exponentially with a mean of 3 minutes. Find the following:
(a) Avg. number of units in the system.
(b) Avg. waiting time for customer.
(c) Avg. length of queue
(d) Probability that a customer arriving at the pump will have to wait.
(e) The utilisation factor for the pump unit.
(t) Probability that the number of customers in the system is 2.
Ans.

Arrival rate (λ) = 10 units per hour

Service time = 3 minutes

$$\text{Service rate } (\mu) = \frac{1}{3} \times 60 = 20 \text{ units per hour.}$$

(*a*) Avg. number of units in the system

$$L_s = \frac{\lambda}{\mu-\lambda}$$

$$= \frac{10}{20-10} = 1 \text{ auto vehicle.}$$

(*b*) Avg. waiting time for customer

$$W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right)$$

$$= \frac{10}{20}\left(\frac{1}{20-10}\right) = \frac{1}{2} \times \frac{1}{10} = \frac{1}{20} \text{ hours}$$

$$= \frac{1}{20} \times 60 = 3 \text{ min.}$$

(*c*) Avg. length of queue

$$L_q = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu - \lambda}\right)$$

$$L_q = \frac{10}{20}\left(\frac{10}{20-10}\right) = \frac{1}{2} = 0.5 \text{ vehicles}$$

(*d*) Probability that a customer arriving at the pump will have to wait

$$= \frac{\lambda}{\mu} = \frac{10}{20} = 0.5$$

(*e*) The utilisation factor for the pump unit

$$= \frac{\lambda}{\mu} = \frac{10}{20} = 0.5$$

(*f*) Probability that the number of customer in the system is 2

$$P(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$P(2) = \left(1 - \frac{10}{20}\right)\left(\frac{10}{20}\right)^2$$

$$= \left(1 - \frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} = 0.125.$$

Problem. 8.14. Arrival at a telephone booth are considered are to be poission, with an average time of 10 minutes between one arrival and next. The length of phone call assumed to be distributed exponentially with mean 3 minutes then

(a) What is. the probability that a person arriving at the booth will have to wait?

(b) What is the average length of the queues that form from time to time.

Ans. Arrival rate $\lambda = \frac{1}{10}$ per minute

Service rate $\mu = \frac{1}{3}$ per minute

(*a*) Probability that a person arriving at the booth will have to wait

$$= \frac{\lambda}{\mu} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10} = 0.3.$$

(b) Average queue length that is formed from time to time

$$= \frac{\mu}{\mu-\lambda} = \frac{\frac{1}{3}}{\frac{1}{3}-\frac{1}{10}} = \frac{\frac{1}{3}}{\frac{7}{30}}$$

$$= \frac{30}{21} = 1.42 \text{ customer.}$$

Problem. 8.15. Customers arrive at one-window drive according to a poission distribution with mean of 10 mm and service time per customer is exponential with mean of 6 minutes. The space in front of the window can accommodate only three vehicles including the serviced one. Other vehicles have wait outside the space.
Calculate.
(a) Probability that an arriving customer can drive directly to the space in front of the window.
(b) Probability that an arriving customer will have to wait outside the directed space. -
(c) How long is an arriving customer expected to wait before starting service?

Ans. Arrival rate $\lambda = \frac{1}{10}$ customers/minute

$= 6$ customers/hour

Service rate $\mu = \frac{1}{6}$ customers/minute

$= 10$ customers/hour

(a) The probability that an arriving customer can drive to the space in front of the window can be obtained by summing up the probabilities of the events in which this can happen. A customer can drive directly to the space if
(1) three is no. customer car already.
(2) there is already 1 customer car.
(3) there are 2 cars in the space.

Thus the required probability $= P_0 + P_1 + P_2$

$$= \left(1-\frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}\right]$$

$$= \left(1 - \frac{6}{10}\right)\left[1 + \frac{6}{10} + \frac{36}{100}\right]$$

$$= \left(\frac{2}{5}\right)\left[\frac{196}{100}\right] = \frac{392}{500} = 0.78$$

(*b*) The probability that an arriving customer has to wait outside the directed space

$$= 1 - 0.78 = 0.22$$

(*c*) Avg. waiting time of a customer in the queue

$$= \frac{1}{\mu}\frac{\lambda}{\mu - \lambda} = \frac{1}{10}\left(\frac{6}{10-6}\right) = \frac{1}{10}\left(\frac{6}{4}\right) = \frac{6}{40} = \frac{3}{20}$$

$$= 0.15 \text{ hours} = 9 \text{ minutes.}$$

Problem 8.16. Arrival of machinists at a tool crib are considered to be poission distribution at an avg. rate of 6 per hour. The length of time the machinists must remain at the tool crib is exponentially distributed with an average time being 0.05 hours.
(a) What is the probability that the machinists arriving at tool crib will have to wait.
(b) What is the average number of machinists at the tool crib.
(c) The company will install a second tool crib when convinced that a machinist would expect to have spent at least 6 mins waiting and being serviced at the tool crib. By how much must the flow of machinists to toolcrib increase to justify the addition of second tool crib?

Ans. Arrival rate of machinist 2 = 6 per hour time spent by machinist at the tool crib = 0.05 hours.

$$\text{Service rate to machinist } \mu = \frac{1}{0.05} = 20 \text{ per hour}$$

Probability that the machinists arriving at tool crib will have to wait

$$= \frac{\lambda}{\mu} = \frac{6}{20} = \frac{3}{10} = 0.3$$

Avg. no. of machinists at the tool crib

$$(L_S) = \frac{\lambda}{\mu - \lambda} = \frac{6}{20-6} = \frac{6}{14} = \frac{3}{7} \text{ machinists}$$

(*c*) Waiting time + Service time:

$$\text{Time spent in the system } W_s = 6 \text{ minutes} = \frac{1}{10} \text{ hour}$$

λ_1 – new arrival rate of machinist

$$W_s = \frac{1}{\mu - \lambda_1} = \frac{1}{20 - \lambda_1}$$

$$\frac{1}{10} = \frac{1}{20 - \lambda_1} \Rightarrow 20 - \lambda_1 = 10 \qquad \Rightarrow \lambda_1 = 10 \text{ machinist/hour}$$

Increase in the flow of machinists to toolcrib increase to justify the addition of a second tool crib =10 — 6 = 4/hour.

Problem 8.17 On an average 96 patients per 24 hours day require the service of an emergency clinic. Also an average a patient requires 10 miii. of active attention. Assume that the facility can handle one emergency at a time. Suppose that it cost the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in his average time would cost Rs. 10/-per patient treated. How much would have to be budgeted by the clinic to decrease the average size of

the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patient.

Ans. $$\lambda = \frac{96}{24} = 4 \text{ patients/hour}$$

$$\mu = \frac{1}{10} \times 60 = 6 \text{ patients/hour}$$

Avg. no. of patients in the queue.

$$L_q = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu - \lambda}\right) = \frac{4}{6}\left(\frac{4}{6-4}\right)$$

$$L_q = \frac{4}{2}(2) = \frac{8}{6} = 1\frac{1}{3}$$

This number is to be reduced from $1\frac{1}{3}$ to $\frac{1}{2}$. This can be achieved by increasing the service rate to say μ'

$$L_q' = \frac{\lambda}{\mu'}\left(\frac{\lambda}{\mu' - \lambda}\right)$$

$$\frac{1}{2} = \frac{4}{\mu'}\left(\frac{4}{\mu' - 4}\right)$$

$$\mu'^2 - 4\mu' - 32 = 0 \text{ or } (\mu' - 8)(\mu' + 4) = 0$$

$$\mu' = 8 \text{ patients/hour} \quad (\mu' = -4 \text{ is illogical and hence neglected})$$

Avg. time required by each patient $= \frac{1}{8} hr$

$$= \frac{15}{2} \text{ minutes}$$

Therefore the budget required for each patient

$$= \text{Rs. } (100 + \frac{5}{2} \times 10) = \text{Rs. } 125/-$$

Thus to decrease the size of the queue, the budget per patient should be increased from Rs. 100 to Rs. 125/—

Problem 8.18. In a large maintenance department, fitters draw parts from the

parts stores which is at present staffed by one storeman. The maintenance foreman is concerned about the time spent by fitters getting parts and wants to know if the employment of a stores labourer to assist the storeman would be worth while. On investigation it is found that

(a) a simple queue situation exists.

(b) fitters cost Rs. 2.50 per hour.

(c) the storeman costs Rs. 2 per hour and can deal, on the avg. with 10 fitters per hour.

(d) a labourer could be employed at Rs. 1.75 per hour and would, increase the service capacity of the stores to 12 per hour.

(e) on the average 8 fitters visit the stores each hour.

Ans. We calculate the avg. number of customers in the system before and after the labouer is employed and compare the reduction in the resulting queuing cost with the increase in service cost.

Without labourer:

Number of customers in the system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{8}{10-8} = 4$$

$$\text{Cost/hr} = 4 \times \text{Rs. } 2.50 = \text{Rs. } 10/-$$

With labourer :

$$\lambda = 8/\text{hr}, \ \mu = 12/\text{hr}.$$

Number of customer in the system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{8}{12-8} = 2$$

Cost/hr = Cost of fitters per hour + cost of labourer per hour

= 2 * Rs. 2.50 + Rs. 1.75 = Rs. 6.75.

Since there is net saving of Rs. 3.25/- It is recommended to employs the labourer.

Problem 8.19. Customers arrive at the first class ticket counter of a theatre at the rate of 12 per hour. There is one clerk serving the customers at the rate of 30 per hour.

(a) What is the probability that there is no customer in the counter (i.e. that the system is idle) ?

(b) What is the probability that there are more than 2 customers in the counter?

(c) What is the probability that there is no customer waiting to be served?

(d) What is the probability that a customer is being served and no body is waiting.

Ans. Here $\lambda = 12/\text{hour}, \mu = 30/\text{hour}$

(*a*) Probability that there is no customer in the system $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{12}{30} = 0.6$

Probability that there are more than two customers in the counter

$$= P_3 + P_4 + P_5 + \dots\dots\dots$$
$$= 1 - (P_0 + P_1 + P_2)$$
$$= 1 - \left[\left(1-\frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)\right]$$
$$= 1 - \left[\left(1-\frac{\lambda}{\mu}\right) + \left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}\right]\right]$$
$$= 1 - \left[0.6\left(1+\frac{12}{30}+\frac{144}{900}\right)\right]$$
$$= 0.064$$

Probability that there is no customer waiting to be served = Probability that there is at most one customer in the counter.

$$= P_0 + P_1 = 0.6 + 0.6\left(\frac{12}{30}\right) = 0.84$$

Probability that a customer is being served and no body is waiting.

$$= P_1 = \left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}$$
$$= 0.6\left(\frac{12}{30}\right) = 0.24.$$

Problem 8.20. In a bank there,is only one window, a solitary employee performs all the service required and the window remains continuously open from 7 am to 1 pm. It has been discovered that average number of clients is 54 during the day and the average service-time is of 5 mins per person.
Calculate
(a)Average number of clients in the system (including the one bring served)
(b)The average number of clients in the waiting line. (including the one being served)
(c) Average waiting time.
(d) Average time spends in the system. Ans. Working hours per day = 6 hrs.
Ans.

Arrival rate λ = 54 clients/day

$$= \frac{54}{6} = 9 \text{ clients/hr}$$

Service rate μ = 5 min. per person

$$= \frac{1}{5} \text{ person/min}$$

$$= \frac{1}{5} \times 60 = 12 \text{ clients/hr.}$$

(*a*) Avg. no. of client in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{9}{12-9} = \frac{9}{3} = 3 \text{ clients}$$

(*b*) Avg. no. of clients in the Queue

$$L_q = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu - \lambda}\right) = \frac{9}{12}(3) = \frac{9}{4} = 2.25 \text{ clients}$$

(*c*) Avg. waiting time $w_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{12}(3) = \frac{1}{4}$ hr.

$$= 15 \text{ min per client.}$$

(*d*) Avg. time spends in the system

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12-9} = \frac{1}{3} \text{ hr.}$$

$$= \frac{1}{3} \times 60 = 20 \text{ min per client.}$$

QUESTION BANK

INVENTORY MANAGEMENT

INVENTORY MANAGEMENT

Deterministic cost Inventory Models

Model I: Purchasing model without shortages

(Demand rate Uniform, Production rate Infinite)

1. Find the economic order quantity and the number of orders if demand for the year is 2000 units. Ordering cost is Rs500 per order and the carrying cost for one unit per year is Rs2.50. calculate the Total Incremental Cost and Total cost if the purchase price of 1 unit isRs25/-.

2. A manufacturing company uses an item at a constant rate of 4000 per year. Each unit costs Rs2. The company estimates that it will cost Rs50 to place an order and the carrying cost is 20% of stock value per year. Find economic order quantity and the Total Cost.

Model II: Production model without shortages

(Demand rate Uniform, Production rate finite)

3. A company needs 12000 units per year. The set up cost is Rs 400 per production run. Holding cost per unit per month is Rs15. The production cost is Rs4. The company can produce 2000 units per month. Find out the economic batch quantity, total incrementalcost, total cost.

4. Demand = 2000 units/yr. The organization can produce @ 250 units per month. The set up cost is Rs1500/set up, running cost is 10% of average cost of the inventory pr year. If the organization incurs the cost of Rs100, determine how frequently the organization has to go for producing the required material.

Model III: Purchasing model with shortages

(Demand rate Uniform, Production rate Infinite, Shortages allowed)

5. The demand for an item is 20 units per month. The inventory carrying cost is Rs25 per item/month. The fixed cost (ordering cost) is Rs10 for each item a order is made. The purchase cost is Re.1 per item. The shortage cost is Rs15 per year. Determine how often a order should be made and what is the economic order quantity. Find the No. of orders, Total Incremental Cost and Total cost.

6. Demand = 9000 units. Cost of 1 procurement Rs100, holding cost – Rs2.40 per unit, shortage cost = Rs5 per unit. Find economic order quantity and how often should it be ordered. If price is Rs10 find Total Incremental Cost and Total Cost.

Model IV: Production model with shortages

(Demand rate Uniform, Production rate finite, Shortages allowed)

7. A company demands 12000 units per year. The set up cost is Rs 400 per production run. Holding cost per unit per month is Rs0.15. The shortage cost is Rs20 per year. The company can produce 2000 units per month. Find out the economic batch quantity, total incremental cost, total cost per year assuming cost of one unit is Rs 4.

8. The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3000 per month. The cost of one set up is Rs.500 and the holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs.20 per month. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and time between set-ups.

Buffer Stock - Deterministic Model

9. A Company uses annually 50,000 units, Each order costs Rs.45 and inventory carrying costs are .18 per unit. i) Find economic order quantity ii) If the company operates 250 days a year and the procurement lead time is 10 days and safety stock is 500 units, find reorder level, maximum, minimum and average inventory.

10. Annual Demand = 12000, Ordering cost = Rs 12, Carrying cost = 10% of inventory per unit cost per unit is Rs 10. The company operates for 250 days per year .The procurement lead time in the past is 10 days, 8 days, 12 days, 13 days and 7 days. find EOQ, Buffer stock reorder level, maximum, minimum and average inventory.

PROBABILISTIC INVENTORY MODEL

1. The probability distribution of the demand for certain items is as follows

Monthly sales	0	1	2	3	4	5	6
Probability	.01	.06	.25	.35	.20	.03	.10

The cost of carrying inventory is Rs 30 per unit per month and cost of unit short is Rs 70 per month. Determine the optimum stock level that would minimize the total expected cost.

2. A news paper boy buys paper for Rs 1.40 and sells them for Rs 2.45 .He cannot return unsold news papers .Daily Demand for the following distribution is as follows

Customers	25	26	27	28	29	30	31	32	33	34	35	36
Probability	.03	.05	.05	.10	.15	.15	.12	.10	.10	.07	.06	.02

If the days demand is independent of the previous day, how many papers he should order each day?

3. The probability distribution of the demand for certain items is as follows

Monthly sales	0	1	2	3	4	5	6
Probability	.02	.05	.30	.27	.20	.10	.06

The cost of carrying inventory is Rs 10 per unit per month .The current policy is to maintain a stock of 4 items at the beginning of each month. Determine the shortage cost per one unit for one time unit.

4. A company orders a new machine after certain fixed time. It is observed that one of the parts of the parts of the machine is very expensive if it is ordered without the machine. The cost of spare part when ordered with the machine is Rs 500 and the cost of down time of the machine and cost of arranging the new part is Rs10, 000. From the past records it is observed that spare parts required with probabilities mentioned below

Demand	0	1	2	3	4	5	6
Probability	.90	.05	.02	.01	.01	.01	0.00

Find the optimal no of spare parts which should be ordered along with the machine.

QUANTITY DISCOUNT MODEL

5. Find the optimal order quantity for a product for which price break up is as follows :

Quantity	**Unit Cost(Rs)**
$0 \leq Q1 < 50$	10
$50 \leq Q2 < 100$	9
$100 \leq Q3$	8

The monthly demand for the product is200 units, the cost of storage is 25%of the unit cost and ordering cost is Rs 20 per order.

6. Find the optimal order quantity for a product for which price break up is as follows :

Quantity	**Unit Cost(Rs)**
$0 \leq Q1 < 500$	10
$500 \leq Q2$	9.25

The monthly demand for the product is200 units, the cost of storage is 2%of the unit cost and ordering cost is Rs 350 per order.

M/M/1, FIFO/∞/∞:

SINGLE CHANNEL/INFINITE POPULATION

Arrival Rate: Poisson

Service Rate: Exponential

No of Channels: Single

Service Discipline: FIFO

Queue Discipline: Infinite

Population: Infinite

1. Consider a self-service store with one cashier. Assume Poisson arrival and exponential service times. Suppose 9 customers arrive on an average for every 5 minutes and the cashier can service 10 in 5 minutes. Find the average number of customer in the system and average time a customer spends in the store.

2. In a public telephone booth, the arrivals are on an average 15 per hour. A call on the average takes 3 minutes. If there are just one phone (Poisson arrivals and exponential service), find the expected number of customer in the booth and the idle time of the booth.

M/M/1, FIFO/N/∞:

SINGLE CHANNEL/FINITE POPULATION

Arrival Rate: Poisson

Service Rate: Exponential

No of Channels: Single

Service Discipline: FIFO

Queue Discipline: finite

Population: Infinite

4. At a one-man barbershop, the customer arrives according to Poisson process at an average rate of 5 per hour and they are served according to exponential distribution with an average service rate of 10 minutes. There are only 5 seats available for waiting of the customer and customer do not wait if they find no seat available. Find the averagenumber of customer in the system, average queue length and the average time a customer spends in the barbershop. Also find the idle time of the barber.

5. Consider a single server queuing system with poisson input and exponential servicetimes. Suppose mean arrival rate is 3 units per hour and expected service time is 0.25 hours and the maximum calling units in the system is two. Calculate expected number in the system .

5. In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter arrival time follows an exponential distribution and theservice time distribution is also exponential with an average 36 minutes. the line capacity is 9 trains Calculate the following:

   a) The probability that the yard is empty
   b) Average queue length

MODEL QUESTION PAPER

PART – A

1. Write short notes on i) Re-order level ii) Safety stock
2. Explain the different models of inventory.
3. What do you mean by Buffer stock and write the formula to find buffer stock?
4. Discuss the various types of deterministic inventory models.
5. Define a) EOQ b) EBQ c) Lead time d) Shortage cost.
6. List out the inventory selective control techniques.
7. What do you mean by a) Parallel queues b) Sequential queues.
8. Briefly explain the characteristics of queuing model.
9. Explain the objectives of waiting line model.
10. Describe the queuing models M/M/1 and M/M/C.

PART – B

11. From the following information calculate EOQ, frequency of orders, Number of orders, Total cost, and Total incremental cost:

    Annual Demand - 20000 units/yr
    Ordering cost – Rs.30 per order
    Carrying cost – 12.5% on inventory cost
    Purchase price – Rs.1.50 per unit per year

12. A company orders a new machine after certain fixed time. It is observed that one of the parts of the parts of the machine is very expensive if it is ordered without the machine. The cost of spare part when ordered with the machine is Rs 500 and the cost of down time of the machine and cost of arranging the new part is Rs10, 000. From the past records it is observed that spare parts required with probabilities mentioned below

Demand	0	1	2	3	4	5	6
Probability	.90	.05	.02	.01	.01	.01	0.00

Find the optimal no of spare parts which should be ordered along with the machine.

13. Find the optimal order quantity for a product for which price break up is as follows :

Quantity	Unit Cost(Rs)
$0 \le Q1 < 100$	20
$100 \le Q2$	19.25

The monthly demand for the product is100 units, the cost of storage is 2%of the unit cost and ordering cost is Rs 250 per order.

14. A news paper boy buys paper for 0.30p and sells them for 0.50p .He cannot return unsold news papers .Daily Demand for the following distribution is as follows

No. of copies sold	10	11	12	13	14
Probability	0.1	0.15	0.20	0.25	0.30

If the days demand is independent of the previous day, how many papers he should order each day?

15. A Company uses annually 15,000 units, Each order costs Rs.25 and inventory carrying costs are .9 per unit. i) Find economic order quantity ii) If the company operates 200 days a year and the procurement lead time is 15days and safety stock is 250 units, find reorder level, maximum, minimum and average inventory.

18. Find the optimal order quantity for a product for which price break up is as follows :

Quantity	Unit Cost(Rs)
$0 \le Q1 < 25$	5
$25 \le Q2 < 50$	4
$50 \le Q3$	3

The monthly demand for the product is200 units, the cost of storage is 25%of the unit cost and ordering cost is Rs 20 per order.

19. The demand for an item in a company is 20,000 units per year, and the company can produce the item at a rate of 5000 per month. The cost of one set up is Rs.500 and the holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs.15 per month. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and time between set-ups.

20. At a one-man barbershop, the customer arrives according to Poisson process at anaverage rate of 2 per hour and they are served according to exponential distribution with an average service rate of 5 minutes. There are only 4 seats available for waiting of the customerand customer do not wait if they find no seat available. Find the average number of customer in the system, average queue length and the average time a customer spends in the barbershop. Also find the idle time of the barber.

21. Consider a bank with one cashier. Assume Poisson arrival and exponential service times. Suppose 9 customers arrive on an average for every 5 minutes and the cashier can service 10 in 5 minutes. Find the average number of customer in the system and average time a customer spends in the bank.