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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AUTOMOBILE ENGINEERING

SAUA1304 _ SOLID AND FLUID MECHANICS

UNIT I STRESS STRAIN AND DEFORMATION OF SOLIDS, STATES OF STRESS

UNIT 1 STRESS STRAIN AND DEFORMATION OF SOLIDS, STATES OF STRESS

Rigid bodies and deformable solids - stability, strength, stiffness - tension, compression and shear stresses - strain, elasticity, Hooke's law, limit of proportionately, modules of elasticity, stress-strain curve, lateral strain - temperature stresses deformation of simple and compound bars - shear modulus, bulk modulus, relationship between elastic constants - bi axial state of stress - stress at a point - stress on inclined plane - principal stresses and principal planes – Mohr's circle of stresses

When an external force acts on a body, the body tends to undergo some deformation. Due to cohesion between the molecules, the body resists deformation. This resistance by which material of the body opposes the deformation is known as strength of material, within a certain limit (i.e., in the elastic stage). Whenever a load is attached to a thin hanging wire, it elongates and the load moves downwards (sometimes through a negligible distance). The amount, by which the wire elongates, depends upon the amount of load and the nature as well as cross-sectional area of the wire material.

Elasticity

Whenever a force acts on a body, it undergoes some deformation and the molecules offer some resistance to the deformation. It will be interesting to know that when the external force is removed, the force of resistance also vanishes; and the body springs back to its original position. But it is only possible, if the deformation, caused by the external force, is within a certain limit. Such a limit is called elastic limit.

The property of certain materials of returning back to their original position, after removing the external force, is known as elasticity.

Stress

The force of resistance per unit area, offered by a body against deformation is known as stress. The external force acting on the body is called the load or force. The load is applied on the body while the stress is induced in the material of the body. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are equal.

$$Stress = \sigma = \frac{Force}{Area} = \frac{F}{A}$$

where F = Load or force acting on the body, and
A = Cross-sectional area of the body.

The unit of stress depends upon the unit of load (or force) and unit of area. In M.K.S. units, the force is expressed in kgf and area in metre square (i.e., m^2). Hence unit of stress becomes as kgf/m^2. In the S.L units, the force is expressed in newtons (written as N) and area is expressed as m^2. Hence unit of stress becomes as N/m^2.

Strain

Whenever a single force (or a system of forces) acts on a body, it undergoes some deformation. This deformation per unit length is known as strain. Mathematically strain may be defined as the deformation per unit length. *i.e.*, strain

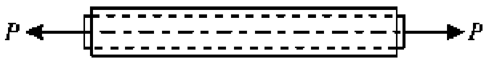
$$Strain = \varepsilon = \frac{x}{L}$$

Types of Stresses

Though there are many types of stresses, yet the following two types of stresses are important from the subject point of view: **1.** Tensile stress, **2.** Compressive stress.

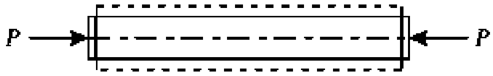
1. Tensile Stress

When a section is subjected to two equal and opposite pulls and the body tends to increase its Length. The stress induced is called tensile stress. The corresponding strain is called tensile strain. As a result of the tensile stress, the *cross-sectional area of the body gets reduced.



2. Compressive Stress

When a section is subjected to two equal and opposite pushes and the body tends to shorten its Length. The stress induced is called compressive stress. The corresponding strain is called compressive strain. As a result of the compressive stress, the cross-sectional area of the body gets increased.



Hooke's Law

It states, "*When a material is loaded, within its elastic limit, the stress is proportional to the strain.*"

$$\frac{\text{Stress}}{\text{Strain}} = E = \text{Constant}$$

Modulus of Elasticity or Young's Modulus (E)

Whenever a material is loaded, within its elastic limit, the stress is proportional to strain

$$\sigma \propto \varepsilon$$
$$= E \times \varepsilon$$
$$E = \frac{\sigma}{\varepsilon}$$

Where, σ = Stress,
ε = Strain, and
E = A constant of proportionality known as modulus of elasticity or Young's modulus.

Numerically, it is that value of tensile stress, which when applied to a uniform bar will increase its length to double the original length if the material of the bar could remain perfectly elastic throughout such an excessive strain.

S. No.	*Material*	*Modulus of elasticity (E) in GPa i.e. GN/m^2 or kN/mm^2*
1.	Steel	200 to 220
2.	Wrought iron	190 to 200
3.	Cast iron	100 to 160
4.	Copper	90 to 110
5.	Brass	80 to 90
6.	Aluminium	60 to 80
7.	Timber	10

Deformation of a Body Due to Force Acting on it

Consider a body subjected to a tensile stress.

Let P = Load or force acting on the body,

l = Length of the body,
A = Cross-sectional area of the body,
σ = Stress induced in the body,
E = Modulus of elasticity for the material of the body,
ε = Strain, and
δl = Deformation of the body.

$$\sigma = \frac{P}{A} \qquad \text{Strain,} \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

$$\delta l = \varepsilon . l = \frac{\sigma . l}{E} = \frac{Pl}{AE}$$

Example: *A steel rod 1 m long and 20 mm × 20 mm in cross-section is subjected to a tensile force of 40 kN. Determine the elongation of the rod, if modulus of elasticity for the rod material is 200 GPa.*

Given:

Length (l) = 1 m = 1×10^3 mm
Cross-sectional area (A) = 20 × 20 = 400 mm^2
Tensile force (P) = 40 kN = 40×10^3 N
Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2

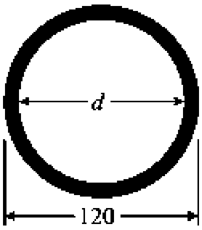
elongation of the road,

$$\delta l = \frac{P.l}{A.E} = \frac{(40 \times 10^3) \times (1 \times 10^3)}{400 \times (20 \times 10^3)} = 0.5 \text{ mm}$$

Example *A hollow steel tube 3.5 m long has external diameter of 120 mm. In order to determine the internal diameter, the tube was subjected to a tensile load of 400 kN and extension was measured to be 2 mm. If the modulus of elasticity for the tube material is 200 GPa, determine the internal diameter of the tube.*

Given:

Length (l) = 3.5 m = 3.5×10^3 mm
External diameter (D) = 120 mm
Load (P) = 400 kN = 400×10^3 N
Extension (δl) = 2 mm
Modulus of elasticity E = 200 GPa = 200×10^3 N/mm^2



area of the tube,

$$A = \frac{\pi}{4} [(120)^2 - d^2] = 0.7854 [(120)^2 - d^2]$$

extension of the tube (δl),

$$2 = \frac{P.l}{A.E} = \frac{(400 \times 10^3) \times (3.5 \times 10^3)}{0.7854 [(120)^3 - d^2 (200 \times 10^3)} = \frac{8913}{14400 - d^2}$$

$$\therefore \qquad 28800 - 2d^2 = 8913 \qquad \text{or} \qquad 2d^2 = 28800 - 8913 = 19887$$

$$\text{or} \qquad d^2 = \frac{19887}{2} = 9943.5 \qquad \text{or} \qquad d = 99.71 \text{ mm} \qquad \textbf{Ans.}$$

Example: Two wires, *one of steel and the other of copper, are of the same length and are subjected to the same tension. If the diameter of the copper wire is 2 mm, find the diameter of the steel wire, if they are elongated by the same amount. Take E for steel as 200 GPa and that for copper as 100 GPa.*

Given:

Diameter of copper wire (d_C) = 2 mm

Modulus of elasticity for steel (E_S) = 200 GPa = 200 × 10^3 N/mm^2

Modulus of elasticity for Copper (E_C) = 100 GPa = 100 × 10^3 N/mm^2

Let d_S = Diameter of the steel wire,

l = Lengths of both the wires and

P = Tension applied on both the wires.

$$A_C = \frac{\pi}{4} \times (d_C)^2 = \frac{\pi}{4} \times (2)^2 = 3.142 \text{ mm}^2$$

and area of steel wire, $A_S = \frac{\pi}{4} \times (d_S)^2 = 0.7854\, d_S^2 \text{ mm}^2$

We also know that increase in the length of the copper wire

$$\delta l_C = \frac{Pl}{A_C E_C} = \frac{Pl}{3.142 \times (100 \times 10^3)} = \frac{Pl}{314.2 \times 10^3} \quad ...(i)$$

and increase in the length of the steel wire,

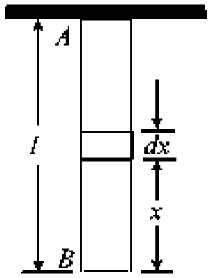
$$\delta l_S = \frac{Pl}{A_S E_S} = \frac{Pl}{0.7854\, d_S^2 \times (200 \times 10^3)} = \frac{Pl}{157.1 \times 10^3 \times d_S^2} \quad ...(ii)$$

Since both the wires are elongated by the same amount, therefore equating equations (*i*) and (*ii*).

$$\frac{Pl}{314.2 \times 10^3} = \frac{Pl}{157.1 \times 10^3 \times d_S^2} \quad \text{or} \quad d_S^2 = \frac{314.2}{157.1} = 2$$

$$\therefore \quad d_S = \sqrt{2} = 1.41 \text{ mm} \quad \textbf{Ans.}$$

Deformation of a Body Due to Self Weight



Consider a bar AB hanging freely under its own weight as shown.

Let l = Length of the bar.

A = Cross-sectional area of the bar.

E = Young's modulus for the bar material,

and w = Specific weight of the bar material.

Now consider a small section dx of the bar at a distance x from B. We know that weight of the bar for a length of x,

$$P = wAx$$

Elongation of the small section of the bar, due to weight of the bar for a small section of length x,

$$= \frac{Pl}{AE} = \frac{(wAx).dx}{AE} = \frac{wx\,.\,dx}{E}$$

Total elongation of the bar may be found out by integrating the above equation between zero and l. Therefore total elongation,

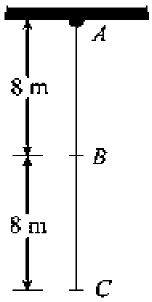
$$\delta l = \int_0^l \frac{wx.dx}{E}$$

$$= \frac{w}{E}\int_0^l x.dx$$

$$= \frac{w}{E}\left[\frac{x^2}{2}\right]_0^l$$

$$\delta l = \frac{wl^2}{2E} = \frac{Wl}{2AE} \qquad ...(\because W = wAl = \text{Total weight})$$

Example *A steel wire ABC 16 m long having cross-sectional area of 4 mm² weighs 20N as shown in Fig. If the modulus of elasticity for the wire material is 200 GPa, find the deflections at C and B.*



Given:

Length (l) = 16 m = 16 × 10^3 mm

Cross-sectional area (A) = 4 mm²

Weight of the wire *ABC* (W) = 20 N

Modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm²

Deflection of wire at C due to self-weight of the wire AC,

$$dl_C = \frac{Wl}{2AE} = \frac{20\times(16\times10^3)}{2\times4\times(200\times10^3)} = 0.2 \text{ mm } \textbf{Ans.}$$

Deflection at B consists of deflection of wire AB due to self-weight plus deflection due to weight of the wire BC. We also know that deflection of the wire at B due to self-weight of wire AB

$$\delta l_1 = \frac{(W/2)\times(l/2)}{2AE} = \frac{10\times(8\times10^3)}{2\times4\times(200\times10^3)} = 0.05 \text{ mm} \qquad ...(i)$$

and deflection of the wire at B due to weight of the wire BC.

$$\delta l_2 = \frac{(W/2)\times(l/2)}{AE} = \frac{10\times(8\times10^3)}{4\times(200\times10^3)} = 0.1 \text{ mm} \qquad ...(ii)$$

∴ Total deflection of the wire at *B*.

$$\delta l_B = \delta l_1 + \delta l_2 = 0.05 + 0.1 = 0.15 \text{ mm} \qquad \textbf{Ans.}$$

Principle of Superposition

A body is subjected to a number of forces acting on its outer edges as well as at some other sections, along the length of the body. In such a case, the forces are split up and their effects are considered on individual sections. The resulting deformation, of the body, is equal to the algebraic sum of the deformations of the individual sections. Such a principle, of finding out the resultant deformation, is called the principle of superposition. The relation for the resulting deformation may be modified as:

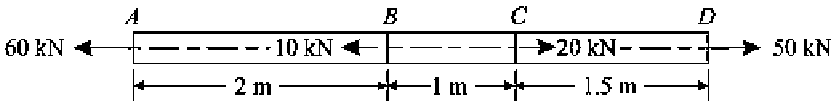
$$\delta l = \frac{Pl}{AE} = \frac{1}{AE}(P_1 l_1 + P_2 l_2 + P_3 l_3 + ...)$$

P_1 = Force acting on section 1,

l_1 = Length of section 1,

P_2, l_2 = Corresponding values of section 2, and so on.

***Example** A steel rod ABCD 4.5 m long and 25 mm in diameter is subjected to the forces as shown in Fig. If the value of Young's modulus for the steel is 200 GPa, determine its deformation.*



Given:

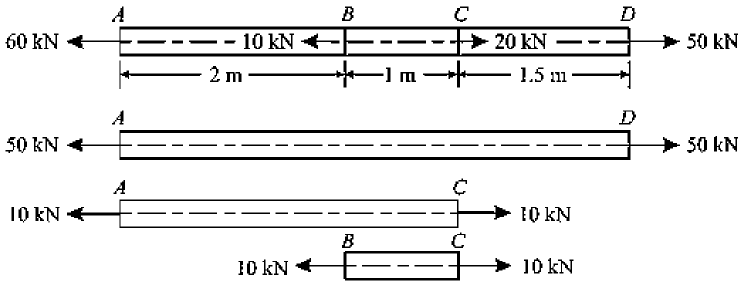
Diameter (D) = 25 mm and

Young's modulus (E) = 200 GPa = 200 kN/mm^2

We know that cross-sectional area of the steel rod.

$$A = \frac{\pi}{4}(D)^2 = \frac{\pi}{4} \times (25)^2 = 491 \text{ mm}^2$$

For the sake of simplification, the force of 60 kN acting at A may be split up into two forces of 50 kN and 10 kN respectively. Similarly the force of 20 kN acting at C may also be split up into two forces of 10 kN and 10 kN respectively.



Now it will be seen that the bar AD is subjected a tensile force of 50 kN, part AC is subjected to a tensile force of 10 kN and the part BC is subjected to a tensile force of 10 kN as shown in Fig. We know that deformation of the bar.

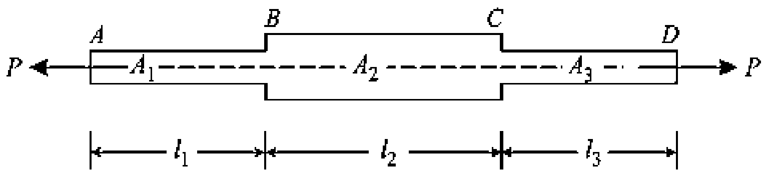
$$\delta l = \frac{1}{AE}[P_1 l_1 + P_2 l_2 + P_3 l_3]$$

$$= \frac{1}{491 \times 200}\left[[50 \times (4.5 \times 10^3)] + [10 \times (3 \times 10^3)] + [10 \times (1 \times 10^3)\right] \text{mm}$$

$$= \frac{1}{491 \times 200} \times (265 \times 10^3) = 2.70 \text{ mm} \qquad \textbf{Ans.}$$

Stresses in the Bars of Different Sections

A bar is made up of different lengths having different cross-sectional areas



In such cases, the stresses, strains and hence changes in lengths for each section is worked out separately as usual. The total change in length is equal to the sum of the changes of all the individual lengths. It may be noted that each section is subjected to the same external axial pull or push.

Let

P = Force acting on the body,

E = Modulus of elasticity for the body,

l_1 = Length of section 1,

A_1 = Cross-sectional area of section 1,

l_2, A_2 = Corresponding values for section 2 and so on.

We know that the change in length of section 1.

$$\delta l_1 = \frac{Pl_1}{A_1 E} \qquad \text{Similarly} \qquad \delta l_2 = \frac{Pl_2}{A_2 E} \quad \text{and so on}$$

∴ Total deformation of the bar,

$$\delta l = \delta l_1 + \delta l_2 + \delta l_3 + \ldots\ldots\ldots$$

$$= \frac{Pl_1}{A_1 E} + \frac{Pl_2}{A_2 E} + \frac{Pl_3}{A_3 E} + \ldots\ldots\ldots$$

$$= \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} + \ldots\ldots\ldots\right)$$

Note. Sometimes, the modulus of elasticity is different for different sections. In such cases, the total deformation,

$$\delta l = P\left(\frac{l_1}{A_1 E_1} + \frac{l_2}{A_2 E_2} + \frac{l_3}{A_3 E_3} + \ldots\ldots\ldots\right)$$

Example *A compound bar ABC 1.5 m long is made up of two parts of aluminium and steel and that cross-sectional area of aluminium bar is twice that of the steel bar. The rod is subjected to an axial tensile load of 200 kN. If the elongations of aluminium and steel parts are equal, find the lengths of the two parts of the compound bar. Take E for steel as 200 GPa and E for aluminium as one-third of E for steel.*

Given:

Total length (L) = 1.5 m = 1.5 × 103 mm

Cross-sectional area of aluminium bar (A_A) = 2 A_S

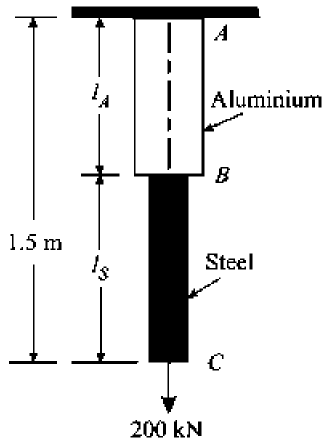
Axial tensile load (P) = 200 kN = 200 × 103 N

Modulus of elasticity of steel (ES) = 200 GPa = 200 × 103 N/mm2

Modulus of elasticity of aluminium (EA) = $\frac{E_S}{3} = \frac{200\ x\ 10^3}{3} N/mm^2$

Let, l_A = Length of the aluminium part,

and l_S = Length of the steel part.



We know that elongation of the aluminium part AB,

$$\delta l_A = \frac{P.l_A}{A_A.E_A} = \frac{(200\times10^3)\times l_A}{2A_S \times \left(\frac{200\times10^3}{3}\right)}$$

$$= \frac{1.5\, l_A}{A_S} \qquad \ldots(i)$$

and elongation of the steel part BC,

$$\delta l_S = \frac{P.l_S}{A_S.E_S} = \frac{(200\times10^3)\times l_S}{A_S\times(200\times10^3)} = \frac{l_S}{A_S}$$

Since elongations of aluminium and steel parts are equal, therefore equating equations (i) and (ii),

$$\frac{1.5 l_A}{A_S} = \frac{l_S}{A_S} \quad \text{or} \quad l_S = 1.5\, l_A$$

We also know that total length of the bar ABC (L)

$$1.5\times10^3 = l_A + l_S = l_A + 1.5\, l_A = 2.5\, l_A$$

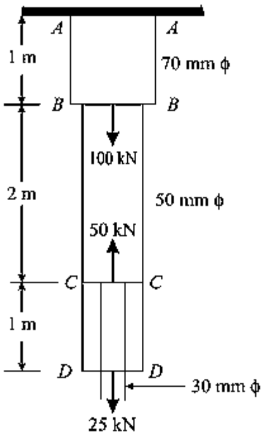
∴ $$l_A = \frac{1.5\times10^3}{2.5} = 600 \text{ mm} \qquad \textbf{Ans.}$$

and $$l_S = (1.5\times10^3) - 600 = 900 \text{ mm} \qquad \textbf{Ans.}$$

Example *A circular steel rod ABCD of different cross-sections is loaded as shown in Fig. Find the maximum stress induced in the rod and its deformation. Take E = 200 GPa.*

Given:

Length of first part AB (l_1) $= 1\text{ m} = 1 \times 10^3$ mm
Diameter of first part AB (D_1) $= 70$ mm
Length of second part BC (l_2) $= 2\text{ m} = 2 \times 10^3$ mm
Diameter of second part BC (D_2) $= 50$ mm
Length of third part CD (l_3) $= 1\text{ m} = 1 \times 10^3$ mm
Diameter of third part CD (D_3) $= 50$ mm
Internal diameter of hole (d_3) $= 30$ mm.



Maximum stress induced in the rod

We know that area of the first part (AB) of the rod,

$$A_1 = \frac{\pi}{4}(D_1)^2 = \frac{\pi}{4}(70)^2 \text{ mm}^2$$
$$= 3848.5 \text{ mm}^2$$

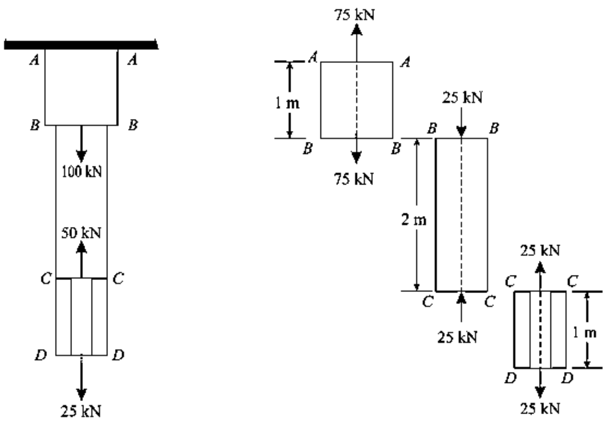
Similarly area of the second part (BC) of the rod,

$$A_2 = \frac{\pi}{4}(D_2)^2 = \frac{\pi}{4}(50)^2 = 1963.5 \text{ mm}^2$$

and area of the third part CD of the rod,

$$A_3 = \frac{\pi}{4}[(D_3)^2 - d_3^{\,2}]$$

For simplification, the force of 100 kN acting at B-B may be split up into two forces of 75 kN and 25 kN. Similarly the force of 50 kN acting at C-C may be split up into two forces of 25 kN and 25 kN respectively as shown in Fig.



(a) (b)

Now it will be seen that the bar AB is subjected to a tensile load of 75 kN, part BC is subjected to a compressive load of 25 kN and the part CD is subjected to a tensile load of 25 kN as shown in Fig. We know that tensile stress in part 1,

$$\sigma_1 = \frac{P_{AB}}{A_1} = \frac{75 \times 10^3}{3848.5} = 19.49 \text{ N/mm}^2 = 19.49 \text{ MPa}$$

Similarly, $$\sigma_2 = \frac{P_{BC}}{A_2} = \frac{25 \times 10^3}{1963.5} = 12.73 \text{ N/mm}^2 = 12.73 \text{ MPa}$$

and $$\sigma_3 = \frac{P_{CD}}{A_3} = \frac{25 \times 10^3}{1256.6} = 19.89 \text{ N/mm}^2 = 19.89 \text{ MPa}$$

From the above three values of the stresses, we find that maximum stress induced in the rod is in CD and is equal to 19.89 MPa. **Ans.**

We also know that elongation of the part AB, due to tensile load of 75 kN,

$$\delta l_1 = \frac{P_1 l_1}{A_1 E} = \frac{(75\times10^3)\times(1\times10^3)}{3848.5\times(200\times10^3)} = 0.097 \text{ mm}$$

Similarly shortening of the part BC due to compressive load of 25 kN.

$$\delta l_2 = \frac{P_2 l_2}{A_2 E} = \frac{(25\times10^3)\times(2\times10^3)}{1963.5\times(200\times10^3)} = 0.127 \text{ mm}$$

and elongation of the part CD due to tensile load of 25 kN.

$$\delta l_3 = \frac{P_3 l_3}{A_3 E} = \frac{(25\times10^3)\times(1\times10^3)}{1256.6\times(200\times10^3)} = 0.099 \text{ mm}$$

∴ Deformation of the rod,

$$\delta l = \delta l_1 - \delta l_2 + \delta l_3 = 0.097 - 0.127 + 0.099 = 0.069 \text{ mm}$$

Stresses in the Bars of Uniformly Tapering Circular Sections

Consider a circular bar AB of uniformly tapering circular section as shown in Fig.

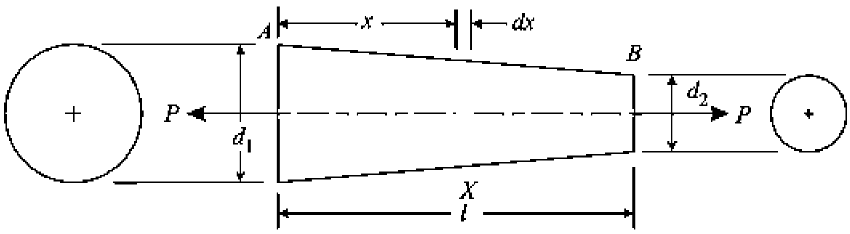
Let P = Pull on the bar.

l = Length of the bar,

$d1$ = Diameter of the bigger end of the bar, and

$d2$ = Diameter of the smaller end of the bar.

Now consider a small element of length dx of the bar, at a distance x from the bigger end as shown in Fig. We know that diameter of the bar at a distance x, from the left end A,



$$d_x = d_1 - (d_1 - d_2)\frac{x}{l} = d_1 - kx, \qquad \ldots\left(\text{where } k = \frac{d_1 - d_2}{l}\right)$$

and cross-sectional area of the bar at this section,

$$A_X = \frac{\pi}{4}(d_1 - kx)^2$$

∴ Stress,

$$\sigma_X = \frac{P}{\frac{\pi}{4}(d_1 - kx)^2} = \frac{4P}{\pi(d_1 - kx)^2}$$

and strain,

$$\varepsilon_X = \frac{\text{Stress}}{E} = \frac{\frac{4P}{\pi(d_1 - kx)^2}}{E} = \frac{4P}{\pi(d_1 - kx)^2 E}$$

∴ Elongation of the elementary length

$$= \varepsilon_X \,.\, dx = \frac{4P \,.\, dx}{\pi(d_1 - kx)^2 E}$$

Total extension of the bar may be found out by integrating the above equation between the limit 0 and l. Therefore total elongation,

$$\delta l = \int_0^l \frac{4P \,.\, dx}{\pi(d_1 - kx)^2 E}$$

$$= \frac{4P}{\pi E}\int_0^l \frac{dx}{(d_1 - kx)^2}$$

$$= \frac{4P}{\pi E}\left[\frac{(d_1 - kx)^{-1}}{-1 \times -k}\right]_0^l$$

$$= \frac{4P}{\pi E k}\left[\frac{1}{d_1 - kx}\right]_0^l$$

$$= \frac{4P}{\pi E k}\left[\frac{1}{d_1 - kl} - \frac{1}{d_1}\right]$$

Substituting the value of $k = \frac{d_1 - d_2}{l}$ in the above equation,

$$\delta l = \frac{4P}{\pi E \frac{(d_1 - d_2)}{l}}\left[\frac{1}{d_1 - \frac{(d_1 - d_2)\, l}{l}} - \frac{1}{d_1}\right]$$

$$= \frac{4Pl}{\pi E (d_1 - d_2)}\left[\frac{1}{d_2} - \frac{1}{d_1}\right] = \frac{4Pl}{\pi E (d_1 - d_2)}\left[\frac{d_1 - d_2}{d_2 d_1}\right]$$

$$\delta l = \frac{4Pl}{\pi E d_2 d_1}$$

Example *If the tension test bar is found to taper from (D + a) diameter to (D – a) diameter, prove that the error involved in using the mean diameter to calculate Young's modulus is* $(\frac{10a}{D})^2$ *per cent.*

Given:

Larger diameter ($d1$) = ($D + a$)

Smaller diameter ($d2$) = ($D – a$).

Let P = Pull on the bar,

l = Length of the bar,

$E1$ = Young's modulus by the tapering formula,

$E2$ = Young's modulus by the mean diameter formula and

δl = Extension of the bar.

First of all, let us find out the values of Young's modulus for the test bar by the tapering formula and then by the mean diameter formula. We know that extension of the bar by uniformly varying formula

$$\delta l = \frac{4\,Pl}{\pi E_1 d_1 d_2} = \frac{4\,Pl}{\pi E_1\,(D + a)\,(D - a)} = \frac{4\,Pl}{\pi E_1\,(D^2 - a^2)}$$

or
$$E_1 = \frac{4\,Pl}{\pi (D^2 - a^2)\,.\,\delta l} \qquad ...(i)$$

and extension of the bar by mean diameter (D) formula,

$$\delta l = \frac{Pl}{AE_2} = \frac{Pl}{\frac{\pi}{4}(D)^2 \times E_2} = \frac{4\,Pl}{\pi\, D^2\, E_2}$$

or
$$E_2 = \frac{4\,Pl}{\pi D^2\,.\,\delta l} \qquad ...(ii)$$

$\therefore$ Percentage error involved (in using the mean diameter to calculate the Young's modulus)

$$= \left(\frac{E_1 - E_2}{E_1}\right) \times 100 = \frac{\left(\dfrac{4\,Pl}{\pi(D^2 - a^2)\,\delta l}\right) - \left(\dfrac{4\,Pl}{\pi\, D^2 \,.\, \delta l}\right)}{\dfrac{4\,Pl}{\pi\,(D^2 - a^2)\,\delta l}} \times 100$$

$$= \frac{\dfrac{1}{(D^2 - a^2)} - \dfrac{1}{D^2}}{\dfrac{1}{(D^2 - a^2)}} \times 100 = \frac{\dfrac{D^2 - (D^2 - a^2)}{(D^2 - a^2)\,(D^2)}}{\dfrac{1}{(D^2 - a^2)}} \times 100$$

$$= \frac{a^2}{D^2} \times 100 = \left(\frac{10\,a}{D}\right)^2 \qquad \textbf{Ans.}$$

Example *A steel plate of 20 mm thickness tapers uniformly from 100 mm to 50 mm in a length of 400 mm. What is the elongation of the plate, if an axial force of 80 kN acts on it? Take E = 200 Gpa.*

Given :

Plate thickness	= 20 mm ;
Width at A	= 100 mm ; Width at B = 50 mm;
Length (l)	= 400 mm ;
Axial force (P)	= 80 kN = 80 × 10^3 N
Modulus of elasticity (E)	= 200 GPa = 200 × 10^3 N/mm^2

Now consider a small element of length *dx*, of the bar, at a distance *x* from *A* as shown in Fig. From the geometry of the figure, we find that the width of the plate at a distance *x* from *A*.

$$= 100 - (100 - 50) \times \frac{x}{400} = 100 - 0.125\,x \text{ mm}$$

$\therefore$ Cross-sectional area of the plate at this section.

$$A_x = 20 \times (100 - 0.125\,x)$$

and stress,
$$\sigma_x = \frac{P}{A_x} = \frac{80 \times 10^3}{20 \times (100 - 0.125\,x)} = \frac{4 \times 10^3}{100 - 0.125\,x}$$

$\therefore$ Strain,
$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{\dfrac{4 \times 10^3}{100 - 0.125\,x}}{200 \times 10^3} = \frac{1}{50\,(100 - 0.125\,x)}$$

and increase in the length of the small element

$$= \varepsilon_x \,.\, dx = \frac{dx}{50(100 - 0.125\,x)}$$

Now total elongation of the plate may be found out by integrating the above equation between 0 and 400.

$\therefore$
$$\delta l = \int_0^{400} \frac{dx}{50(100 - 0.125\,x)}$$

$$= \frac{1}{50}\int_0^{400} \frac{dx}{(100 - 0.125\,x)}$$

$$= \frac{1}{50(-0.125)}\left[\log_e (100 - 0.125\,x)\right]_0^{400}$$

$$= -\frac{1}{6.25}\left[\log_e (50 - \log_e 100)\right]$$

$$= 0.16\,[\log_e 100 - \log_e 50] \qquad \text{...(Taking minus sign outside)}$$

$$= 0.16 \times \log_e\left(\frac{100}{50}\right) = 0.16 \times \log_e 2 = 0.16 \times 2.3 \log 2$$

$$\text{...}(\because \log_e = 2.3 \log_{10})$$

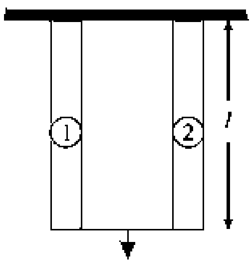
$$= 0.16 \times 2.3 \times 0.3010 = 0.11 \text{ mm} \qquad \textbf{Ans.}$$

Stresses in the Bars of Composite Structures

A bar made up of two or more different materials, joined together is called a composite bar. The bars are joined in such a manner, that the system extends or contracts as one unit, equally, when subjected to tension or compression. Following two points should always be kept in view, while solving example on composite bars:

1. Extension or contraction of the bar is equal.
2. The total external load, on the bar, is equal to the sum of the loads carried by the different materials.

Consider a composite bar made up of two different materials as shown in Fig.



Let P = Total load on the bar,

l_1 = Length of the bar 1

l_2 = Length of the bar 2

A_1 = Area of bar 1,

E_1 = Modulus of elasticity of bar 1.

P_1 = Load shared by bar 1, and

A_2, E_2, P_2= Corresponding values for bar 2,

Total load on the bar,

$$P = P_1 + P_2 \qquad \text{...}(i)$$

$\therefore$ Stress in bar 1, $$\sigma_1 = \frac{P_1}{A_1}$$

and strain in bar 1, $$\varepsilon_1 = \frac{\sigma_1}{E_1} = \frac{P_1}{A_1 E_1}$$

$\therefore$ Elongation, $$\delta l_1 = \varepsilon_1 . l_1 = \frac{\sigma_1 l_1}{E_1} = \frac{P_1 l_1}{A_1 E_1} \qquad \text{...}(ii)$$

Similarly, elongation of bar 2,

$$\delta l_2 = \varepsilon_2 l_2 = \frac{\sigma_2 l_2}{E_1} = \frac{P_2 l_2}{A_2 E_2} \qquad \text{...}(iii)$$

Since both the elongations are equal, therefore equating (*ii*) and (*iii*), we get $\delta l_1 = \delta l_2$

$$\frac{P_1 l}{A_1 E_1} = \frac{P_2 l}{A_2 E_2} \quad \text{or} \quad \frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \qquad \text{...}(iv)$$

or $$P_2 = P_1 \times \frac{A_2 E_2}{A_1 E_1}$$

But $$P = P_1 + P_2 = P_1 + P_1 \times \frac{A_2 E_2}{A_1 E_1}$$

$$= P_1\left(1 + \frac{A_2 E_2}{A_1 E_1}\right) = P_1\left(\frac{A_1 E_1 + A_2 E_2}{A_1 E_1}\right)$$

or $$P_1 = P \times \frac{A_1 E_1}{A_1 E_1 + A_2 E_2} \qquad \text{...}(v)$$

Similarly, $$P_2 = P \times \frac{A_2 E_2}{A_1 E_1 + A_2 E_2} \qquad \text{...}(vi)$$

From these equations we can find out the loads shared by the different materials. We have also seen in equation (*iv*) that

$$\frac{Pl_1}{A_1 E_1} = \frac{Pl_2}{A_2 E_2}$$

or $$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \quad \cdots \left(\because \frac{P}{A} = \sigma = \text{Stress}\right)$$

$\therefore$ $$\sigma_1 = \frac{E_1}{E_2} \times \sigma_2 \quad \ldots(vii)$$

Similarly, $$\sigma_2 = \frac{E_2}{E_1} \times \sigma_1 \quad \ldots(viii)$$

From the above equations, we can find out the stresses in the different materials. We also know that the total load,

$$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$$

Example *A reinforced concrete circular column of 400 mm diameter has 4 steel bars of 20 mm diameter embedded in it. Find the maximum load which the column can carry, if the stresses in steel and concrete are not to exceed 120 MPa and 5 MPa respectively. Take modulus of elasticity of steel as 18 times that of concrete.*

Given:

Diameter of column (D) = 400 mm

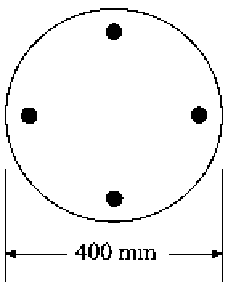
No. of reinforcing bars = 4

Diameter of bars (d) = 20 mm

Maximum stress in steel ($\sigma_{S(max)}$) = 120 MPa = 120 N/mm^2

Maximum stress in concrete ($\sigma_{C(max)}$) = 5 MPa = 5 N/mm^2

Modulus of elasticity of steel (E_S) = 18 E_C



Total area of the circular column.

$$= \frac{\pi}{4} \times (D)^2 = \frac{\pi}{4} \times (400)^2 = 125\,660 \text{ mm}^2$$

and area of reinforcement (*i.e.*, steel),

$$A_S = 4 \times \frac{\pi}{4} \times (d)^2 = 4 \times \frac{\pi}{4} \times (20)^2 \text{ mm}^2$$
$$= 1257 \text{ mm}^2$$

$\therefore$ Area of concrete,

$$A_C = 125\,660 - 1257 = 124\,403 \text{ mm}^2$$

First of all let us find out the maximum stresses developed in the steel and concrete. We know that if the stress in steel is 120 N/mm^2, then stress in the concrete.

$$\sigma_C = \frac{E_C}{E_S} \times \sigma_S = \frac{1}{18} \times 120 = 6.67 \text{ N/mm}^2 \quad \ldots(i)$$

It is more than the stress in the concrete (*i.e.*, 5 N/mm^2). Thus these stresses are not accepted. Now if the stress in concrete is 5 N/mm^2, then stress in steel,

$$\sigma_S = \frac{E_S}{E_C} \times \sigma_C = 18 \times 5 = 90 \text{ N/mm}^2 \quad \ldots(ii)$$

It is less than the stress is steel (*i.e.*, 120 N/mm^2). It is thus obvious that stresses in concrete and steel will be taken as 5 N/mm^2 and 90 N/mm^2 respectively. Therefore maximum load, which the column can carry.

$$P = (\sigma_C \cdot A_C) + (\sigma_S \cdot A_S) = (5 \times 124\,403) + (90 \times 1257) \text{ N}$$
$$= 735\,150 \text{ N} = 735.15 \text{ kN} \quad \textbf{Ans.}$$

Stresses and Strains in Statically Indeterminate Structures

Simple equations of statics were sufficient to solve the examples. But, sometimes, the simple equations are not sufficient to solve such problems. Such problems are called statically indeterminate problems and the structures are called statically indeterminate structures. For solving statically indeterminate problems, the deformation characteristics of the structure are also taken into account along with the statical equilibrium equations. Such equations, which contain the deformation characteristics, are called compatibility equations.

Types of Statically Indeterminate Structures

1. Simple statically indeterminate structures.
2. Indeterminate structures supporting a load.
3. Composite structures of equal lengths.
4. Composite structures of unequal lengths.

Stresses in Simple Statically Indeterminate Structures

Example *A square bar of 20 mm side is held between two rigid plates and loaded by an axial force P equal to 450 kN as shown. Find the reactions at the ends A and C and the extension of the portion AB. Take E = 200 Gpa*

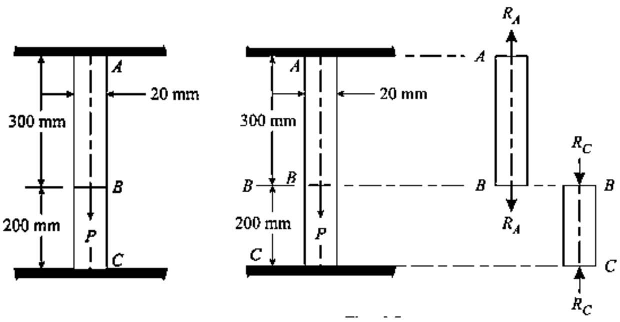
Given:

Area of bar (A) = $20 \times 20 = 400\ mm^2$

Axial force (P) = $450\ kN = 450 \times 10^3$ N

Modulus of elasticity (E) = $200\ GPa$

$= 200 \times 10^3$ N/mm^2



Length of AB (l_{AB}) = 300 mm and

length of BC (l_{BC}) = 200 mm.

R_A = Reaction at A, *and*

R_C = Reaction at C.

Since the bar is held between the two rigid plates *A and C, therefore, the upper portion will be C* subjected to tension, while the lower portion will be subjected to compression as shown

Moreover, the increase of portion AB will be equal to the decrease of the portion BC.

We know that sum of both the reaction is equal to the axial force, *i.e.*,

$$R_A + R_C = 450 \times 10^3 \qquad ...(i)$$

Increase in the portion AB,

$$\delta l_{AB} = \frac{R_A\, l_{AB}}{A\,E} = \frac{R_A \times 300}{A\,E}$$

and decrease in the portion BC,

$$\delta l_{BC} = \frac{R_C\, l_{BC}}{A\,E} = \frac{R_C \times 200}{A\,E} \qquad ...(ii)$$

Since the value δl_{AB} is equal to that of δl_{BC}, therefore equating the equations (ii) and (iii),

$$\frac{R_A \times 300}{A\,E} = \frac{R_C \times 200}{A\,E}$$

$$R_C = \frac{R_A \times 300}{200} = 1.5\ R_A$$

Now substituting the value of R_C in equation (ii),

$$R_A + 1.5\ R_A = 450 \qquad \text{or} \qquad 2.5\ R_A = 450$$

$$\therefore \qquad R_A = \frac{450}{2.5} = 180\ \text{kN} \qquad \textbf{Ans.}$$

and $R_C = 1.5\, R_A = 1.5 \times 180 = 270$ kN **Ans.**

Extension of the portion AB

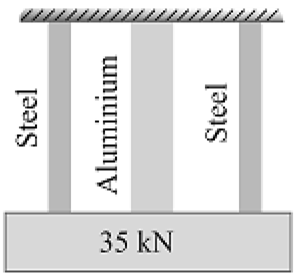
Substituting the value of R_A in equation (*ii*)

$$\delta_{AB} = \frac{R_A \times 300}{A\,E} = \frac{(180 \times 10^3) \times 300}{400 \times (200 \times 10^3)} = 0.675 \text{ mm} \qquad \textbf{Ans.}$$

Stresses in Indeterminate Structures Supporting a Load

Example *A block weighing 35 kN is supported by three wires. The outer two wires are of steel and have an area of 100 mm^2 each, whereas the middle wire of aluminium and has an area of 200 mm^2. If the elastic modulii of steel and aluminium are 200 GPa and 80 GPa respectively, then calculate the stresses in the aluminium and steel wires.*

Given:



Total load (P) = 35 kN = 35×10^3 N

Total area of steel rods (A) = 2×100 = 200 mm^2

Area of aluminium rod (A_A) = 200 mm^2

Modulus of elasticity of steel (E) = 200 Gpa = 200×10^3 N/mm^2

Modulus of elasticity of aluminium (E_A) = 80 GPa = 80×10^3 N/mm^2

Load supported by wires (P) = 35 kN = 35×10^3 N

Let σ_S = Stress in steel wires,

σ_A = Stress in aluminium wire and

l = Length of the wires.

We know that increase in the length of steel wires,

$$\delta l_S = \frac{\sigma_S \times l_S}{E_S} = \frac{\sigma_S \times l}{200 \times 10^3}$$

Similarly,

$$\delta l_A = \frac{\sigma_A \times l_A}{E_A} = \frac{\sigma_A \times l}{80 \times 10^3}$$

Since increase in the lengths of steel and aluminium wires is equal, therefore equating equations (*i*) and (*ii*), we get

$$\frac{\sigma_S \times l}{200 \times 10^3} = \frac{\sigma_A \times l}{80 \times 10^3} \quad \text{or} \quad \sigma_S = \frac{200}{80} \times \sigma_A = 2.5\, \sigma_A$$

We also know that load supported by the three wires (P),

$$35 \times 10^3 = (\sigma_S \,.\, A_S) + (\sigma_A \,.\, A_A) = (2.5\, \sigma_A \times 200) + (\sigma_A \times 200) = 700\, \sigma_A$$

$$\therefore \quad \sigma_A = \frac{35 \times 10^3}{700} = 50 \text{ N/mm}^2 = 50 \text{ MPa} \qquad \textbf{Ans.}$$

and $\sigma_S = 2.5\, \sigma_A = 2.5 \times 50 = 125$ MPa **Ans.**

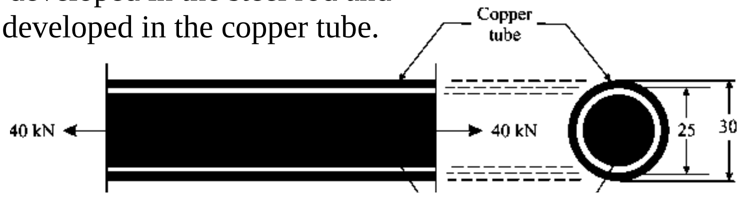
Stresses in Composite Structures of Equal Lengths

Example *A mild steel rod of 20 mm diameter and 300 mm long is enclosed centrally inside a hollow copper tube of external diameter 30 mm and internal diameter 25 mm. The ends of the rod and tube are brazed together, and the composite bar is subjected to an axial pull of 40 kN as shown. If E for steel and copper is 200 GPa and 100 GPa respectively, find the stresses developed in the rod and the tube.*

Given :

Diameter of steel rod = 20 mm;
External diameter of copper tube = 30 mm;
Internal diameter of copper tube = 25 mm;
Total load (P) = *40 kN = 40 × 103* N;
Modulus of elasticity of steel (E_S) = 200 GPa and
Modulus of elasticity of copper (E_C) = 100 GPa

Let σ_s = Stress developed in the steel rod and
σ_c= Stress developed in the copper tube.



We know that area of steel rod,

$$A_S = \frac{\pi}{4} \times (20)^2 = 314.2 \text{ mm}^2$$

and area of copper tube,

$$A_C = \frac{\pi}{4} [(30)^2 - (25)^2] = 216 \text{ mm}^2$$

We also know that stress in steel,

$$\sigma_S = \frac{E_S}{E_C} \times \sigma_C = \frac{200}{100} \times \sigma_C = 2\,\sigma_C$$

and total load (P), $$40 \times 10^3 = (\sigma_S . A_S) + (\sigma_C . A_C)$$
$$= (2\sigma_C \times 314.2) + (\sigma_C \times 216) = 844.4\,\sigma_C$$

$\therefore$ $$\sigma_C = \frac{40 \times 10^3}{844.4} = 47.4 \text{ N/mm}^2 = 47.4 \text{ MPa} \quad \textbf{Ans.}$$

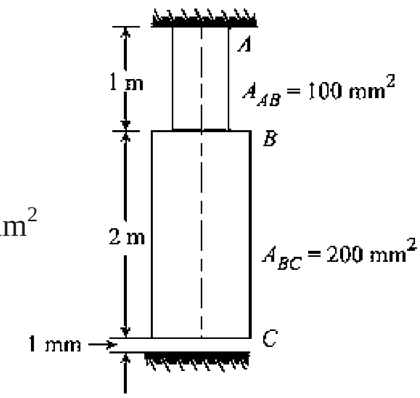
and $$\sigma_S = 2\,\sigma_C = 2 \times 47.4 = 94.8 \text{ MPa} \quad \textbf{Ans.}$$

Stresses in Composite Structures of Unequal Lengths

Example *A composite bar ABC, rigidly fixed at A and 1 mm above the lower support, is subjected to an axial load of 50 kN at B as shown. If the cross-sectional area of the section AB is 100 mm² and that of section BC is 200 mm², find the reactions at both the ends of the bar. Also find the stresses in both the section. Take E = 200 GPa.*

Given:

Length of AB (l_{AB}) = 1 m = 1 × 10^3 mm
Area of AB (A_{AB}) = 100mm²
Length of BC (l_{BC}) = 2 m = 2 × 10^3 mm
Area of BC (A_{BC}) = 200 mm²
Axial load (P) = 50 kN = 50 × 10^3 N
Modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm²



Reactions at both the ends of the bar

The bar is rigidly fixed at A and loaded at B, therefore, upper portion AB is subjected to tensions. We also know that increase in length of the portion AB due to the load at B

$$\delta l = \frac{P . l_{AB}}{A_{AB} . E} = \frac{(50 \times 10^3) \times (1 \times 10^3)}{100 \times (200 \times 10^3)} = 2.5 \text{ mm}$$

We find that of increase in the length of the portion *AB* would have been less than 1 mm (*i.e.*, gap between *C* and lower support), then the lower portion of the bar *BC* should not have been subjected to any stress. Now it will be interesting to know that as the increase in length *AB* is 2.5 mm, therefore, first action of the 50 kN load will be to increase the length AB by 1 mm, till the end C touches the lower support. And a part of the load will be required for this increase. Then the remaining load will be shared by both the portions of the bar *AB* and *BC* of the bar.

Let P = Load required to increase 1 mm length of the bar *AB*,

We know that increase in length

$$1 = \frac{P_1 \times l_{AB}}{A_{AB} \cdot E} = \frac{P_1 \times (1 \times 10^3)}{100 \times (200 \times 10^3)} = 0.05 \times 10^{-3}\, P_1$$

$$\therefore \quad P_1 = \frac{1}{0.05 \times 10^{-3}} = 20 \times 10^3 \text{ N} = 20 \text{ kN}$$

and the remaining loas, which will be shared by the portion *AB* and *CD*

$$= 50 - 20 = 30 \text{ kN}$$

Let R_A = Reaction at *A* due to 30 kN load, and

R_C = Reaction at *C* due to 30 kN load.

Thus, $$R_A + R_C = 30 \text{ kN} = 30 \times 10^3 \text{ N} \qquad ...(i)$$

We know that increase in length *AB* due to reaction R_A (beyond 1 mm),

$$\delta l_1 = \frac{R_A \cdot l_{AB}}{A_{AB} \cdot E} = \frac{R_A \times (1 \times 10^3)}{100 \times (200 \times 10^3)} = 0.05 \times 10^{-3}\, R_A \qquad ...(ii)$$

and decrease in length *BC* due to reaction R_C,

$$\delta l_2 = \frac{R_C \cdot l_{BC}}{A_{BC} \cdot E} = \frac{R_C \times (2 \times 10^3)}{200 \times (200 \times 10^3)} = 0.05 \times 10^{-3}\, R_C \qquad ...(iii)$$

Since δl_1 is equal to δl_2, therefore equating equations (*i*) and (*ii*),

$$0.05 \times 10^{-3}\, R_A = 0.05 \times 10^{-3}\, R_C \quad \text{or} \quad R_A = R_C$$

Now substituting the value of R_C in equation (*i*)

$$R_A + R_A = 30 \quad \text{or} \quad R_A = R_C = \frac{30}{2} = 15 \text{ kN}$$

$\therefore$ Total reaction at $A = (20 + 15) = 35$ kN **Ans.**

and total reaction at $C = 15$ kN **Ans.**

Stresses in both the sections

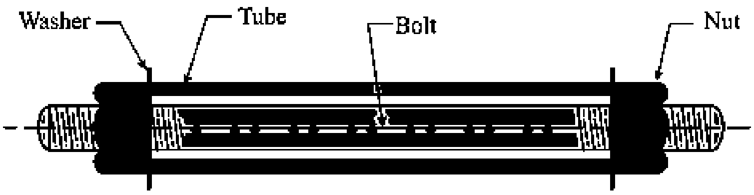
We know that stress in the bar *AB*,

$$\sigma_{AB} = \frac{35 \times 10^3}{100} = 350 \text{ N/mm}^2 = 350 \text{ MPa} \qquad \textbf{Ans.}$$

and $$\sigma_{BC} = \frac{15 \times 10^3}{200} = 75 \text{ N/mm}^2 = 75 \text{ MPa} \qquad \textbf{Ans.}$$

Stresses in Nuts and Bolts

Nuts and bolts to tighten the components of a machine or structure. It is generally done by placing washers below the nuts as shown. A nut can be easily tightened, till the space between the two washers becomes exactly equal to the body placed between them. It will be interesting to know that if we further tighten the nut, it will induce some load in the assembly. As a result of this, bolt will be subjected to some tension, whereas the washers and body between them will be subjected to some compression. And the induced load will be equally shared between the bolt and the body. Now consider an assembly consisting of two nuts and a bolt along with a tube as shown



Let P = Tensile load induced in the bolt as a result of tightening the nut,
l = Length of the bolt,
A_1= Area of the bolt,
σ_1 = Stress in the bolt due to induced load,
E_1 = Modulus of elasticity for the bolt material.
A_2, σ_2, E_2 = Corresponding values for the tube

The tensile load on the bolt is equal to the compressive load on the tube, therefore

$$\sigma_1 . A_1 = \sigma_2 . A_2$$

$$\therefore \quad \sigma_1 = \frac{A_2}{A_1} \times \sigma_2 \qquad \text{Similarly,} \quad \sigma_2 \frac{A_1}{A_2} \times \sigma_1$$

and the total load $(P) = \sigma_1 A_1 + \sigma_2 A_2$

We also know that increase in the length of the bolt due to tensile stress in it,

$$\delta l_1 = \frac{\sigma_1 . l}{E_1} \qquad ...(i)$$

and decrease in the length of the tube due to compressive stress in it,

$$\delta l_2 = \frac{\sigma_2 . l}{E_2} \qquad ...(ii)$$

$\therefore$ Axial advancement (*i.e.*, movement) of the nut

$$= \delta l_1 + \delta l_2$$

Example *A solid copper rod 300 mm long and 40 mm diameter passes axially inside a steel tube of 50 mm internal diameter and 60 mm external diameter. The composite bar is tightened by using rigid washers of negligible thickness. Determine the stresses in copper rod and steel tube, when the nut is tightened so as to produce a tensile load of 100 kN in the copper rod.*

Given:

Length of copper rod (l) = 300 mm
Diameter of copper rod (DC) = 40 mm
Internal diameter of steel tube (dS) = 50 mm
External diameter of steel tube (DS) = 60 mm
Tensile load in copper rod (P) = 100 kN = 100×10^3 N

Let σ_C = Stress in the copper rod and
σ_S = Stress in the steel rod.

We know that area of the copper rod,

$$A_C = \frac{\pi}{4} \times (D_C)^2 = \frac{\pi}{4} \times (40)^2 = 400\,\pi \text{ mm}^2$$

and area of the steel tube,

$$A_S = \frac{\pi}{4} \times [D_S^2 - d_C^2] = \frac{\pi}{4} \times [(60)^2 - (50)^2 = 275\,\pi \text{ mm}^2$$

We also know that tensile load on the copper rod is equal to the compressive load on the steel tube. Therefore stress in steel rod,

$$\sigma_S = \frac{A_C}{A_S} \times \sigma_C = \frac{400\pi}{275\pi} \times \sigma_C = \frac{16\sigma_C}{11} = 1.455\ \sigma_C$$

and load (P) $\quad 100 \times 10^3 = (\sigma_C \cdot A_C) + (\sigma_S \cdot A_S) = (\sigma_C \times 400\,\pi) + (1.455\,\sigma_C \times 275\,\pi)$

$$= 800\,\pi\,\sigma_C$$

$$\therefore \quad \sigma_C = \frac{100 \times 10^3}{800\pi} = 39.8 \text{ N/mm}^2 = 39.8 \text{ MPa (tension)} \quad \textbf{Ans.}$$

and $\quad \sigma_S = 1.455\,\sigma_C = 1.455 \times 39.8 \text{ N/mm}^2 = 57.9 \text{ N/mm}^2$ **Ans.**

$$= 57.9 \text{ MPa (compression)} \quad \textbf{Ans.}$$

Thermal Stresses and Strains

Whenever there is some increase or decrease in the temperature of a body, it causes the body to expand or contract. A little consideration will show that if the body is allowed to expand or contract freely, with the rise or fall of the temperature, no stresses are induced in the body. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called thermal stresses or temperature stresses. The corresponding strains are called thermal strains or temperature strains.

Thermal Stresses in Simple Bars

The thermal stresses or strains, in a simple bar, may be found out as discussed below:

1. Calculate the amount of deformation due to change of temperature with the assumption that bar is free to expand or contract.

2. Calculate the load (or force) required to bring the deformed bar to the original length.

3. Calculate the stress and strain in the bar caused by this load.

The thermal stresses or strains may also be found out first by finding out amount of deformation due to change in temperature, and then by finding out the thermal strain due to the deformation. The thermal stress may now be found out from the thermal strain as usual. Now consider a body subjected to an increase in temperature.

Let $\quad l$ = Original length of the body,

t = Increase of temperature and

α = Coefficient of linear expansion.

We know that the increase in length due to increase of temperature

$$\delta l = l.\alpha.t$$

If the ends of the bar are fixed to rigid supports, so that its expansion is prevented, then compressive strain induced in the bar.

$$\varepsilon = \frac{\delta l}{l} = \frac{l.\alpha.t}{l} = \alpha.t$$

$\therefore$ Stress $\quad \sigma = \varepsilon.E = \alpha.t.E.$

Cor. If the supports yield by an amount equal to Δ, then the actual expansion that has taken place,

$$\delta l = l\alpha t - \Delta$$

and strain, $\quad \varepsilon = \dfrac{\delta l}{l} = \dfrac{l\alpha t - \Delta}{l} = \left(\alpha t - \dfrac{\Delta}{l}\right)$

$\therefore$ Stress, $\quad \sigma = \varepsilon.E = \left(\alpha t - \dfrac{\Delta}{l}\right) E$

Example *Two parallel walls 6 m apart are stayed together by a steel rod 25 mm diameter passing through metal plates and nuts at each end. The nuts are tightened home, when the rod is at a temperature of 100°C. Determine the stress in the rod, when the temperature falls down to 60°C, if (a) the ends do not yield, and (b) the ends yield by 1 mm. Take E = 200 GPa and $\alpha = 12 \times 10^{-6}$ /°C*

Given:

Length (l) = 6 m = 6×10^3 mm

Diameter (d) = 25 mm
Decrease in temperature (t) = 100° – 60° = 40°C
Amount of yield in ends (Δ) = 1 mm
Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2
Coefficient of linear expansion (α) = 12×10^{-6}/°C.

(*a*) ***Stress in the rod when the ends do not yield***

We know that stress in the rod when the ends do not yield,

$$\sigma_1 = \alpha.t.E = (12 \times 10^{-6}) \times 40 \times (200 \times 10^3) \text{ N/mm}^2$$
$$= 96 \text{ N/mm}^2 = 96 \text{ MPa} \qquad \textbf{Ans.}$$

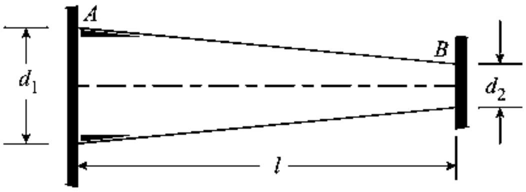
(*b*) ***Stress in the rod when the ends yield by 1 mm***

We also know that stress in the rod when the ends yield,

$$\sigma_2 = \left[\alpha t - \frac{\Delta}{l}\right] E = \left[(12 \times 10^{-6})\, 40 - \frac{1}{6 \times 10^3}\right] 200 \times 10^3 \text{ N/mm}^2$$
$$= 62.6 \text{ N/mm}^2 = 62.6 \text{ MPa} \qquad \textbf{Ans.}$$

Thermal Stresses in Bars of Circular Tapering Section

Consider a circular bar of uniformly tapering section fixed at its ends A and B and subjected to an increase of temperature as shown



Let l = Length of the bar.
d_1= Diameter at the bigger end of the bar,
d_2= Diameter at the smaller end of the bar,
t = Increase in temperature and
a = Coefficient of linear expansion.

The increase in temperature, the bar AB will tend to expand. But since it is fixed at both of its ends, therefore it will cause some compressive stress. We also know that the increase in length due to increase in temperature,

$$\delta l = l.\alpha.t \qquad ...(i)$$

Now let P = Load (or force) required to bring the deformed bar to the original length.

We know that decrease in the length of the circular bar due to load P

$$\delta l = l.\alpha.t \qquad ...(i)$$

Now let P = Load (or force) required to bring the deformed bar to the original length.

We know that decrease in the length of the circular bar due to load P

$$\delta l = \frac{4Pl}{\pi E d_1 d_2} \qquad ...(ii)$$

Equating equations (i) and (ii),

$$l.\alpha.t = \frac{4Pl}{\pi E d_1 d_2} \quad \text{or} \quad P = \frac{\pi E d_1 d_2 .\alpha t}{4}$$

$\therefore$ *Max. stress,
$$\sigma_{max} = \frac{P}{\frac{\pi}{4} \times d_2^2} = \frac{\pi E d_1 d_2 .\alpha t}{4 \times \frac{\pi}{4} \times d_2^2} = \frac{\alpha t E d_1}{d_2}$$

NOTE. If we substitute $d_1 = d_2$, the above relation is reduced to

$$\sigma = \alpha.t.E \qquad \text{...(Same as for simple bars)}$$

Example *A circular bar rigidly fixed at its both ends uniformly tapers from 75 mm at one end to 50 mm at the other end. If its temperature is raised through 26 K, what will be the maximum stress developed in the bar. Take E as 200 GPa and α as 12 × 10–6 /K for the bar material.*

Given:

Diameter at end 1 (d_1) = 75 mm
Diameter at end 2 (d_2) = 50 mm
Rise in temperature (t) = 26 K
E = 200 GPa = 200 × 10^3 N/mm^2
α = 12 × 10^{-6} /K
maximum stress developed in the bar,

$$\alpha_{max} = \frac{\alpha t.E.d_1}{d_2} = \frac{(12\times10^{-6})\times26\times(200\times10^3)\times75}{50} \text{ N/mm}^2$$

$$= 93.6 \text{ N/mm}^2 = 93.6 \text{ MPa} \qquad \textbf{Ans.}$$

Thermal Stresses in Bars of Varying Section

Consider a bar ABC fixed at its ends A and C and subjected to an increase of temperature as shown

Let

l_1 = Length of portion *AB*,
σ_1 = Stress in portion *AB*,
A_1 = Cross-sectional area of portion *AB*,
l_2, σ_2, A_2 = Corresponding values for the portion *BC*,
α= Coefficient of linear expansion and
t = Increase in temperature

We know that as a result of the increase in temperature, the bar *ABC* will tend to expand. But since it is fixed at its ends *A* and *C*, therefore it will cause some compressive stress in the body. Moreover, as the thermal stress is shared equally by both the portions, therefore

$$\sigma_1 A_1 = \sigma_2 A_2$$

Moreover, the total deformation of the bar (assuming it to be free to expand),

$$\delta l = \delta l_1 + \delta l_2 = \frac{\sigma_1 l_1}{E} + \frac{\sigma_2 l_2}{E} = \frac{l}{E}(\sigma_1 l_1 + \sigma_2 l_2)$$

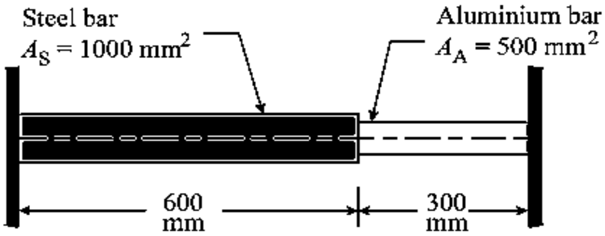
NOTE. Sometimes, the modulus of elasticity is different for different sections. In such cases, the total deformation.

$$\delta l = \left(\frac{\sigma_1 l_1}{E_1} + \frac{\sigma_2 l_2}{E_2}\right)$$

Example *A composite bar made up of aluminium and steel, is held between two supports as shown. The bars are stress-free at a temperature of 38°C. What will be the stresses in the two bars, when the temperature is 21°C, if (a) the supports are unyielding, (b) the supports come nearer to each other by 0.1 mm? It can be assumed that the change of temperature is uniform all along the length of the bar. Take E for steel as 200 GPa; E for aluminium as 75 GPa and coefficient of expansion for steel as 11.7 × 10–6 per °C and coefficient of expansion for aluminium as 23.4 × 10-6 per °C.*

Given:

Length of steel bar (l_S) = 600 mm
Area of steel bar (A_S) = *1000 mm*^{2}
Length of aluminium bar (l_A) = 300 mm
Area of aluminium bar (A_A) = *500 mm*^{2}
Decrease in temperature (t) = 38 – 21 = 17°C
Modulus of elasticity of steel (E_S) = 200 GPa = 200 × 10^3 N/mm^2
Modulus of elasticity of aluminium (E_A) = 75 GPa = 75 x 10^3 N/mm^2
Coefficient of expansion for steel (α_S) = 11.7 × 10^{-6}/°C
Coefficient of expansion for aluminium (α_A) = 23.4 × 10^{-6}/°C.



Let σ_S = Stress in the steel bar, and
σ_A = Stress in the aluminium bar.

(*a*) ***Stresses when the supports are unyielding***

$$\sigma_S \,.\, A_S = \sigma_A \,.\, A_A \quad \text{or} \quad \sigma_S \times 1000 = \sigma_A \times 500$$

$$\therefore \quad \sigma_S = \sigma_A \times 500/1000 = 0.5\,\sigma_A$$

We know that free expansion of steel bar due to increase in temperature,

$$\delta l_S = l_S.\alpha_S.t = 600 \times (11.7 \times 10^{-6}) \times 17 = 0.119 \text{ mm}$$

and

$$\delta l_A = l_A.\alpha_A.t = 300 \times (23.4 \times 10^{-6}) \times 17 = 0.119 \text{ mm}$$

$\therefore$ Total contraction of the bar,

$$\delta l = \delta l_S + \delta l_A = 0.119 + 0.119 = 0.238 \text{ mm}$$

Now let us assume a tensile force to be applied at *A* and *C*, which will cause an expansion of 0.238 mm of the rod (*i.e.*, equal to the total contraction). Therefore

$$0.238 = \frac{\sigma_S \,.\, l_S}{E_S} + \frac{\sigma_A \,.\, l_A}{E_A} = \frac{(0.5\,\sigma_A) \times 600}{200 \times 10^3} + \frac{\sigma_A \times 300}{75 \times 10^3} = 5.5 \times 10^{-3}\,\sigma_A$$

$$\therefore \quad \sigma_A = \frac{0.238}{5.5 \times 10^{-3}} = 43.3 \text{ N/mm}^2 = 43.3 \text{ MPa} \qquad \textbf{Ans.}$$

and

$$\sigma_S = 0.5\,\sigma_A = 0.5 \times 43.3 = 21.65 \text{ MPa} \qquad \textbf{Ans.}$$

(*b*) ***Stresses when the supports come nearer to each other by 0.1 mm***

In this case, there is an expansion of composite bar equal to 0.238 – 0.1 = 0.138 mm. Now let us assume a tensile force, which will cause an expansion of 0.138 mm. Therefore

$$0.138 = \frac{\sigma_S \,.\, l_S}{E_S} + \frac{\sigma_A \,.\, l_A}{E_A} = \frac{(0.5\,\sigma_A) \times 600}{200 \times 10^3} + \frac{\sigma_A \times 300}{75 \times 10^3} = 5.5 \times 10^{-3}\,\sigma_A$$

$$\therefore \quad \sigma_A = \frac{0.138}{5.5 \times 10^{-3}} = 25.1 \text{ N/mm}^2 = 25.1 \text{ MPa} \qquad \textbf{Ans.}$$

and

$$\sigma_S = 0.5\,\sigma_A = 0.5 \times 25.1 = 12.55 \text{ MPa} \qquad \textbf{Ans.}$$

Superposition of Thermal Stresses

Example *A rigid slab weighing 600 kN is placed upon two bronze rods and one steel rod each of 6000 mm^2 area at a temperature of 15°C as shown in Fig. Find the temperature, at which the stress in steel rod will be zero. Take: Coefficient of expansion for steel = 12 × 10^{-6} /°C, Coefficient of expansion for bronze = 18 × 10^{-6}/°C Young's modulus for steel = 200 Gpa, Young's modulus for bronze = 80 GPa.*

Given:

Weight = 600 kN = 600 × 10^3 N

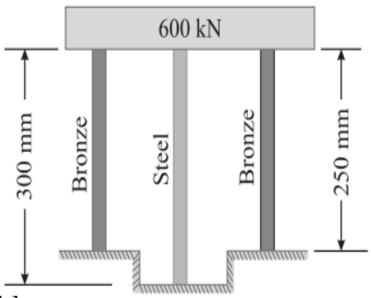
Area of bronze rod (A_B) = A_S = *6000mm^2*

Coefficient of expansion for steel (α_s) = 12 × 10^{-6} /°C

Coefficient of expansion for bronze (α_B) = 18 × 10^{-6} /°C

Modulus of elasticity of steel (E_S) = 200 GPa = 200 × 10^3 N/mm^2

Modulus of elasticity of bronze (E_B) = 80 GPa = 80 × 10^3 N/mm^2



Let t = Rise in temperature, when the stress in the steel rod will be zero.

Due to increase in temperature all the three rods will expand. The expansion of bronze rods will be more than the steel rod (because α_B is greater than α_S). If the stress in the steel rod is to be zero, then the entire load should be shared by the two bronze rods. Or in other words, the decrease in the length of two bronze rods should be equal to the difference of the expansion of the bronze rods and steel rod. We know that free expansion of the steel rod

$$= l_S.\alpha_S.t = 300 \times 12 \times 10^{-6} \times t = 3.6 \times 10^{-3}\, t$$

Similarly, free expansion of the bronze rods,

$$= l_B.\alpha_B.t = 250 \times 18 \times 10^{-6} \times t = 4.5 \times 10^{-3}\, t$$

∴ Difference in the expansion of the two rods

$$= (4.5 \times 10^{-3})\, t - (3.6 \times 10^{-3})\, t = 0.9 \times 10^{-3}\, t \qquad ...(i)$$

We also know that the contraction of the bronze rods due to load of 600 kN

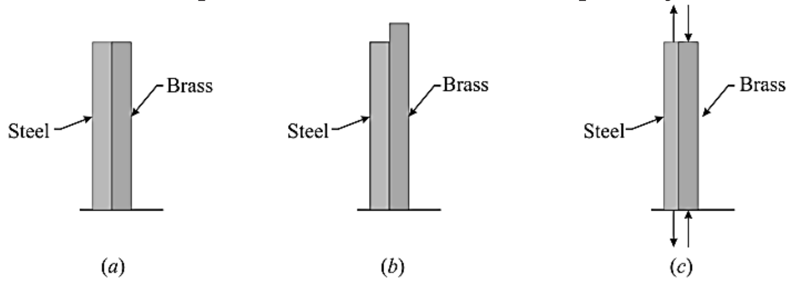
$$= \frac{Pl}{AE} = \frac{(600 \times 10^3) \times 250}{(2 \times 6000) \times (80 \times 10^3)} = 0.156 \text{ mm} \qquad ...(ii)$$

Now equating equations (*i*) and (*ii*),

$$0.9 \times 10^{-3} \times t = 0.156 \qquad \text{or} \qquad t = \frac{0.156}{9 \times 10^{-4}} = 173.3°\text{C} \qquad \textbf{Ans.}$$

Thermal Stresses in Composite Bars

Whenever there is some increase or decrease in the temperature of a bar, consisting of two or more different materials, it causes the bar to expand or contract. The different coefficients of linear expansions the two materials do not expand or contract by the same amount, but expand or contract by different amounts. The steel and brass could have been free to expand, and then no internal stresses would have induced. The two members are rigidly fixed, therefore the composite bar, as a whole, will expand by the same amount.



We know that the brass expands more than the steel (because the coefficient of linear expansion of the brass is greater than that of the steel). Therefore the free expansion of the brass will be more than that of the steel. But since both the members are not free to expand, therefore the expansion of the composite bar, as a whole, will be less than that of the brass; but more than that of the steel as shown. *It is thus obvious that the brass will be subjected to compressive force, whereas* the steel will be subjected to tensile force as shown.

σ_1 = Stress in brass

ε_1 = Strain in brass,

α_1 = Coefficient of linear expansion for brass,

A_1 = Cross-sectional area of brass bar,

$\sigma_2, \varepsilon_2, \alpha_2\, A_2$ = Corresponding values for steel, and

ε = Actual strain of the composite bar per unit length.

As the compressive load on the brass is equal to the tensile load on the steel, therefore

$$\sigma_1.A_1 = \sigma_2.A_2$$

Now strain in brass,

$$\varepsilon_1 = \alpha_1.t - \varepsilon \qquad ...(i)$$

and strain in steel,

$$\varepsilon_2 = \alpha_2.t - \varepsilon \qquad ...(ii)$$

Adding equation (*i*) and (*ii*), we get

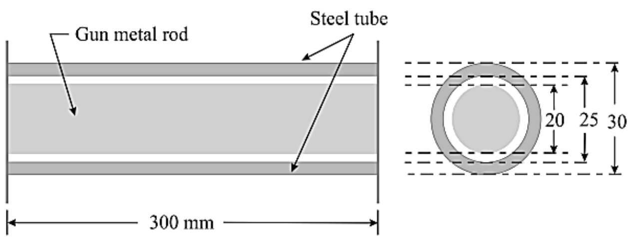
$$\varepsilon_1 + \varepsilon_2 = -t\,(\alpha_1 + \alpha_2)$$

Notes : 1. In the above equation the value of α_1 is taken as greater of the two values of α_1 and α_2.

Example *A gun metal rod 20 mm diameter, screwed at the ends, passes through a steel tube 25 mm and 30 mm internal and external diameters respectively. The nuts on the rod are screwed tightly home on the ends of the tube. Find the intensity of stress in each metal, when the common temperature rises by 200°F. Take. Coefficient of expansion for steel = 6 × 10–6/°F Coefficient of expansion for gun metal = 10 × 10 –6 /°F Modulus of elasticity for steel = 200 Gpa, Modulus of elasticity for gun metal = 100 GPa.*

Given:

Diameter of gun metal rod = 20 mm
Internal diameter of steel tube = 25 mm
External diameter of steel tube = 30 mm
Rise in temperature (t) = 200°F
Coeff of expansion for steel (α_S) = 6×10^{-6} /°F
Coeff of expansion for gun metals (α_G) = 10×10^{-6} /°F
(E_S) = 200 GPa = 200×10^3 N/mm^2
(E_G) = 100 GPa = 100×10^3 N/mm^2



The temperature of the gun metal rod and steel tube will increase; the free expansion of gun metal rod will be more than that of steel tube. Thus the gun metal rod will be subjected to compressive stress and the steel tube will be subjected to tensile stress.

$$A_G = \frac{\pi}{4} \times (20)^2 = 100\,\pi \text{ mm}^2$$

$$A_S = \frac{\pi}{4}\,[(30)^2 - (25)^2] = 68.75\,\pi \text{ mm}^2$$

$$\sigma_S = \frac{A_G}{A_S} \times \sigma_S = \frac{100\pi}{68.75\pi} \times \sigma_G = 1.45\ \sigma_G$$

We know that strain in steel tube,

$$\varepsilon_S = \frac{\sigma_S}{E_S} = \frac{\sigma_S}{200 \times 10^3}$$

and $$\varepsilon_G = \frac{\sigma_G}{E_G} = \frac{\sigma_G}{100 \times 10^3}$$

We also know that total strain,

$$\varepsilon_S + \varepsilon_G = t\,(\alpha_G - \alpha_S)$$

$$\frac{\sigma_S}{200 \times 10^3} + \frac{\sigma_G}{100 \times 10^3} = 200\ [(10 \times 10^{-6}) - (6 \times 10^{-6})]$$

$$\frac{1.45\sigma_G}{200 \times 10^3} + \frac{\sigma_G}{100 \times 10^3} = 200 \times (4 \times 10^{-6})$$

$$\frac{3.45\sigma_G}{200 \times 10^3} = 800 \times 10^{-6}$$

$$3.45\ \sigma G = (800 \times 10^{-6}) \times (200 \times 10^3) = 160$$

∴ $$\sigma_G = \frac{160}{3.45} = 46.4\ \text{N/mm}^2 = 46.4\ \text{MPa}$$ **Ans.**

and $$\sigma_S = 1.45\ \sigma_G = 1.45 \times 46.4 = 67.3\ \text{MPa}$$ **Ans.**

Elastic constant

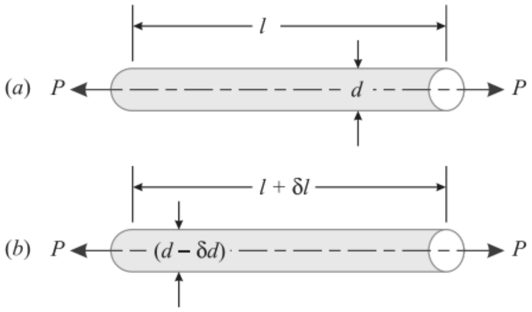
The axial deformation of a body, when it is subjected to a direct tensile or compressive stress. But we have not discussed the lateral or side effects of the pulls or pushes. It has been experimentally found, that the axial strain of a body is always followed by an opposite kind of strain in all directions at right angle to it. Thus, in general, there is always a set of the following two types of strains in a body, when it is subjected to a direct stress.

- Primary or linear strain, and
- Secondary or lateral strain

Whenever some external force acts on a body, it undergoes some deformation. Now consider a circular bar subjected to a tensile force as shown. Let

l = Length of the bar,
d = Diameter of the bar,
P = Tensile force acting on the bar, and
dl = Increase in the length of the bar

The deformation of the bar per unit length in the direction of the force is known as **linear strain**. The linear deformation of a circular bar of length l and diameter d subjected to a tensile force P. The deformation of the bar, we will find that bar has extended through a length dl, which will be followed by the decrease of diameter from d to (d – δd) as shown. Similarly, if the bar is subjected to a compressive force, the length of the bar will decrease by dl which will be followed by the increase of Diameter from d to (d + δd). It is thus obvious that every direct stress is always accompanied by a strain in its own direction and an opposite kind of strain in every direction at right angles to it. Such a strain is known as **secondary or lateral strain.**



Poisson's ratio

If a body is stressed within its elastic limit, the lateral strain bears a constant ratio to the linear strain.

$$\frac{\text{Lateral strain}}{\text{Linear strain}} = (\text{constant})$$

This constant is known as Poisson's ratio and is denoted by $\frac{1}{m}$ or μ. Mathematically,

$$\text{Lateral strain} = \frac{1}{m} \times \varepsilon = \mu\,\varepsilon$$

***Example** A steel bar 2 m long, 40 mm wide and 20 mm thick is subjected to an axial pull of 160 kN in the direction of its length. Find the changes in length, width and thickness of the bar. Take E = 200 GPa and Poisson's ratio = 0.3.*

Given: Length (l) = 2 m = 2 × 103 mm

Width (b) = 40 mm ;

Thickness (t) = 20 mm;

Axial pull (P) = 160 kN = 160 × 103 N ;

Modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2

poisson's ratio (1/m) = 0.3

Change in length

We know that change in length,

$$\delta l = \frac{Pl}{AE} = \frac{(160\times10^3)\times(2\times10^3)}{(40\times20)\times(200\times10^3)} = 2\text{ mm} \qquad \textbf{Ans.}$$

Change in width

We know that linear strain,

$$\varepsilon = \frac{\delta l}{l} = \frac{2}{2\times10^3} = 0.001$$

and lateral strain $= \frac{1}{m} \times \varepsilon = 0.3 \times 0.01 = 0.0003$

∴ Change in width,

$$\delta b = b \times \text{Lateral strain} = 40 \times 0.0003 = 0.012\text{ mm} \qquad \textbf{Ans.}$$

Change in thickness

We also know that change in thickness,

$$\delta t = t \times \text{Lateral strain} = 20 \times 0.0003 = 0.006\text{ mm} \qquad \textbf{Ans.}$$

Volumetric strain

Whenever a body is subjected to a single force (or a system of forces), it undergoes some changes in its dimensions. The change in dimensions of a body will cause some changes in its volume. The ratio of change in volume, to the original volume, is known as **volumetric strain**

The following are important from the subject point of view:

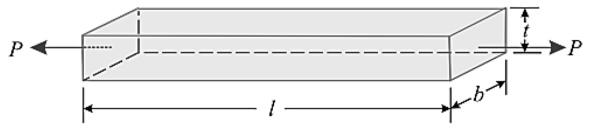
1. A rectangular body subjected to an axial force.
2. A rectangular body subjected to three mutually perpendicular force

$$\varepsilon_V = \frac{\delta V}{V}$$

δV = Change in volume, and

V = Original volume.

Volumetric Strain of a Rectangular Body Subjected to an Axial Force



Consider a bar, rectangular in section, subjected to an axial tensile force as shown in Fig. 6.2.

Let l = Length of the bar,

b = Breadth of the bar,

t = Thickness of the bar,

P = Tensile force acting on the bar,

E = Modulus of elasticity and

$\frac{1}{m}$ = Poisson's ratio.

We know that change in length,

$$\delta l = \frac{Pl}{AE} = \frac{Pl}{btE} \qquad ...(i)$$

and linear stress, $\sigma = \frac{\text{Force}}{\text{Area}} = \frac{P}{bt}$

$\therefore$ Linear strain $= \frac{\text{Stress}}{E} = \frac{P}{btE}$

and lateral strain $= \frac{1}{m} \times \text{Linear strain} = \frac{1}{m} \times \frac{P}{btE}$

$\therefore$ Change in thickness,

$$\delta t = t \times \frac{1}{m} \times \frac{P}{btE} = \frac{P}{mbE} \qquad ...(ii)$$

and change in breadth,

$$\delta b = b \times \frac{1}{m} \times \frac{P}{btE} = \frac{P}{mtE} \qquad ...(iii)$$

As a result of this tensile force, let the final length

$= l + \delta l$

Final breadth $= b - \delta b$ ...(Minus sign due to compression)

and final thickness $= t - \delta t$ ...(Minus sign due to compression)

We know that original volume of the body,

$V = l.b.t.$

and final volume $= (l + \delta l)\,(b - \delta b)\,(t - \delta t)$

$$= lbt\left(1 + \frac{\delta l}{l}\right)\left(1 - \frac{\delta b}{b}\right)\left(1 - \frac{\delta t}{t}\right)$$

$$= lbt\left[1+\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right] \qquad \text{...(Ignoring other negligible values)}$$

$\therefore$ Change in volume,

$$\delta V = \text{Final volume} - \text{Original volume}$$

$$= lbt\left(1+\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)-lbt = lbt\left(\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)$$

$$= V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)$$

and volumetric strain,

$$\frac{\delta V}{V} = \frac{V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)}{V} = \frac{P}{btE}\left(1-\frac{2}{m}\right)$$

$$= \varepsilon\left(1-\frac{2}{m}\right) \qquad \ldots\left(\because \frac{P}{btE}=\varepsilon=\text{Strain}\right)$$

Example *A steel bar 2 m long, 20 mm wide and 15 mm thick is subjected to a tensile load of 30 kN. Find the increase in volume, if Poisson's ratio is 0.25 and Young's modulus is 200 GPa.*

Given: Length (l) = *2 m = 2 × 10* 3 mm ; Width (b) = *20 mm* ; *Thickness (t) = 15 mm*

Tensile load (P) = 30 kN = 30×10^3 N ; Poisson's ratio $\left(\frac{1}{m}\right)$ = 0.25 or $m = 4$ and Young's modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

Let δV = Increase in volume of the bar.

We know that original volume of the bar,

$$V = l.b.t = (2\times10^3)\times20\times15 = 600\times10^3 \text{ mm}^3$$

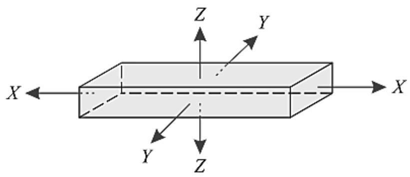
and

$$\frac{\delta V}{V} = \frac{P}{btE}\left(1-\frac{2}{m}\right) = \frac{30\times10^3}{20\times15\times(200\times10^3)}\left(1-\frac{2}{4}\right) = 0.000\,25$$

$$\therefore \quad \delta V = 0.000\,25\times V = 0.000\,25\times(600\times10^3) = 150 \text{ mm}^3 \qquad \textbf{Ans.}$$

Volumetric Strain of a Rectangular Body Subjected to Three Mutually Perpendicular Forces

Consider a rectangular body subjected to direct tensile stresses along three mutually perpendicular axes as shown



Let σ_x = Stress in x-x direction,

σ_y = Stress in y-y direction,

σ_z = Stress in z-z direction and

E = Young's modulus of elasticity.

$\therefore$ Strain in x-x direction due to stress σ_x,

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Similarly, $$\varepsilon_y = \frac{\sigma_y}{E} \quad \text{and} \quad \varepsilon_z = \frac{\sigma_z}{E}$$

The resulting strains in the three directions may be found out by the principle of superposition, *i.e.*, *by adding algebraically the strains in each direction due to each individual stress*. For the three tensile stresses shown. (taking tensile strains as +ve and compressive strains as –ve) the resultant strain in *x-x direction*,

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_x - \frac{\sigma_y}{m} - \frac{\sigma_z}{m}\right]$$

Similarly,

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_y - \frac{\sigma_x}{m} - \frac{\sigma_z}{m}\right]$$

and

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_y}{mE} = \frac{1}{E}\left[\sigma_z - \frac{\sigma_x}{m} - \frac{\sigma_y}{m}\right]$$

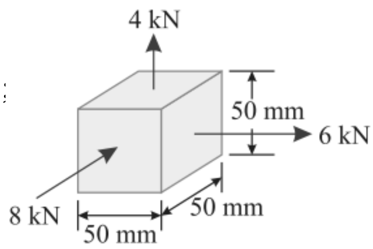
The volumetric strain may then be found by the relation;

$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Example *A steel cube block of 50 mm side is subjected to a force of 6 kN (Tension), 8kN (Compression) and 4 kN (Tension) along x, y and z direction respectively. Determine the change in volume of the block. Take E as 200 GPa and m as 10/3.*

Given:

Side of the cube = 50 mm ;
Force in *x*- direction (*Px*) = 6 kN = 6 × 10^3 N (Tension) ;
Force in *y-direction*(*Py*) = 8 kN = 8 × 10^3 N (Compression) ;
Force in *z-direction* (*Pz*) = 4 kN = 4 × 10^3 N (Tension) and
modulus of elasticity (*E*) = *200*GPa = 200 × 10^3 N/mm^2
and *m = 10 / 3*



δV = Change in volume of the block.

original volume of the steel cube,

$$V = 50 \times 50 \times 50 = 125 \times 10^3 \text{ mm}^3$$

and stress in *x-x* direction,

$$\sigma_x = \frac{P_x}{A} = \frac{6\times10^3}{2500} = 2.4 \text{ N/mm}^2 \text{ (Tension)}$$

Similarly

$$\sigma_y = \frac{P_y}{A} = \frac{8\times10^3}{2500} = 3.2 \text{ N/mm}^2 \text{ (Compression)}$$

and

$$\sigma_z = \frac{P_z}{A} = \frac{4\times10^3}{2500} = 1.6 \text{ N/mm}^2 \text{ (Tension)}$$

We also know that resultant strain in *x-x* direction considering tension as positive and compression as negative,

$$\varepsilon_x = \frac{\sigma_x}{E} + \frac{\sigma_y}{mE} - \frac{\sigma_z}{mE} = \frac{2.4}{E} + \frac{3.2\times3}{10\,E} - \frac{1.6\times3}{10\,E} = \frac{2.88}{E}$$

Similarly,

$$\varepsilon_y = -\frac{\sigma_y}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_z}{mE} = -\frac{3.2}{E} - \frac{2.4\times3}{10\,E} - \frac{1.6\times3}{10\,E} = -\frac{4.4}{E}$$

and

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\sigma_x}{mE} + \frac{\sigma_y}{mE} = \frac{1.6}{E} - \frac{2.4\times3}{10\,E} + \frac{3.2\times3}{10\,E} = \frac{1.84}{E}$$

volumetric strain,

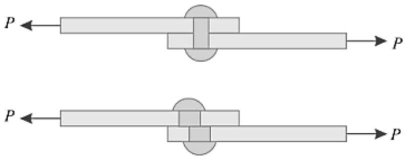
$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\frac{\delta V}{125\times10^3} = \frac{2.88}{E} - \frac{4.4}{E} + \frac{1.84}{E} = \frac{0.32}{E} = \frac{0.32}{200\times10^3}$$

$$\delta V = 125 \times 10^3 \times \frac{0.32}{200\times10^3} = 0.2 \text{ mm}^3 \qquad \textbf{Ans.}$$

Shear Stress

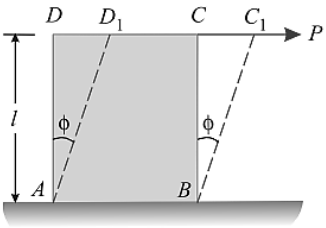
When a section is subjected to two equal and opposite forces, acting tangentially across the resisting section, as a result of which the body tends to shear off across the section as shown. The stress induced is called shear stress. The corresponding strain is called shear strain.



$$\text{Shear strain} = \frac{\text{Deformation}}{\text{Original length}}$$

$$= \frac{CC_1}{l} = \phi$$

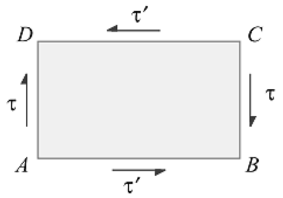
$$\tau = \frac{P}{AB}$$



Principle of Shear Stress

It states, "*A shear stress across a plane, is always accompanied by a balancing shear stress across the plane and normal to it.*

$$P = \tau \times .AD = \tau \times CB$$



Consider a rectangular block *ABCD, subjected to a shear stress of* intensity t on the faces *AD and CB as shown. Now consider* a unit thickness of the block. Therefore force acting on the faces *AD* and *CB,*

These forces will form a couple, whose moment is equal to τ × *AD* × *AB i.e., force* × *distance. If the* block is in equilibrium, there must be a restoring couple, whose moment must be equal to this couple. Let the shear stress of intensity t be set up on the faces *AB* and *CD* as shown. Therefore forces acting on the faces *AB* and *CD*,.

$$\tau \times AD \times AB = \tau' \times AD \times AB$$

$$\tau = \tau'$$

Relation between Modulus of Elasticity and Modulus of Rigidity

Consider a cube of length *l subjected to a shear stress of τ as shown.* due to these stresses the cube is subjected to some distortion, such that the diagonal *BD will be elongated and the diagonal AC will be shortened. Let this shear stress t cause* shear strain φ as shown. *We see that the diagonal BD is now distorted to BD*1.

$$\text{Strain of } BD = \frac{BD_1 - BD}{BD} \quad \ldots\left(\because \text{ Strain} = \frac{\delta l}{l}\right)$$

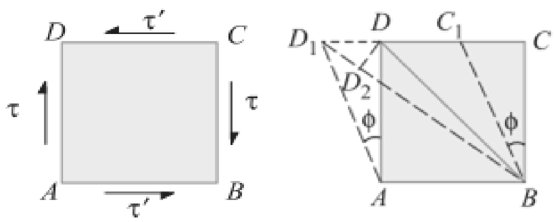
$$= \frac{D_1 D_2}{BD} = \frac{DD_1 \cos 45°}{AD\sqrt{2}} = \frac{DD_1}{2\,AD} = \frac{\phi}{2}$$

Linear strain of the diagonal *BD*

$$= \frac{\phi}{2} = \frac{\tau}{2C}$$

τ = Shear stress and

C = Modulus of rigidity.



(*a*) Before distortion (*b*) After distortion

Let us now consider this shear stress t acting on the sides AB, CD, CB and AD. We know that the effect of this stress is to cause tensile stress on the diagonal BD and compressive stress on the diagonal AC. Therefore tensile strain on the diagonal BD due to tensile stress on the diagonal BD

$$= \frac{\tau}{E} \quad \text{...}(ii)$$

and the tensile strain on the diagonal BD due to compressive stress on the diagonal AC

$$= \frac{1}{m} \times \frac{\tau}{E} \quad \text{...}(iii)$$

The combined effect of the above two stresses on the diagonal BD

$$= \frac{\tau}{E} + \frac{1}{m} \times \frac{\tau}{E} = \frac{\tau}{E}\left(1 + \frac{1}{m}\right) = \frac{\tau}{E}\left(\frac{m+1}{m}\right) \quad \text{...}(iv)$$

Equating equations (i) and (iv),

$$\frac{\tau}{2C} = \frac{\tau}{E}\left(\frac{m+1}{m}\right) \quad \text{or} \quad C = \frac{mE}{2(m+1)}$$

Example *An alloy specimen has a modulus of elasticity of 120 GPa and modulus of rigidity of 45 GPa. Determine the Poisson's ratio of the material.*

Given:

Modulus of elasticity *(E) = 120 GPa*

Modulus of rigidity (C) = 45 GPa.

Let $\frac{1}{m}$ = Poisson's ratio of the material.

We know that modulus of rigidity (C),

$$45 = \frac{mE}{2(m+1)} = \frac{m \times 120}{2\,(m+1)} = \frac{120m}{2m+2}$$

$$90\,m + 90 = 120\,m \quad \text{or} \quad 30\,m = 90$$

$$\therefore \quad m = \frac{90}{30} = 3 \quad \text{or} \quad \frac{1}{m} = \frac{1}{3} \quad \textbf{Ans.}$$

Strain Energy and Impact Loading

When the load moves downwards, it loses its *potential energy. This energy is absorbed (or stored) in the stretched wire, which may be released by removing the load. On removing the load, the wire will spring back to its original position.

Resilience

It is a common term used for the total strain energy stored in a body. Sometimes the resilience is also defined as the capacity of a strained body for doing work (when it springs back) on the removal of the straining force.

Proof Resilience

It is also a common term, used for the maximum strain energy, which can be stored in a body. (This happens when the body is stressed up to the elastic limit). The corresponding stress is known as proof stress.

Modulus of Resilience

The proof resilience per unit volume of a material, is known as modulus of resilience and is a important property of the material.

A load may act in either of the following three ways:

1. Gradually 2. suddenly 3. with impact

Strain Energy Stored in a Body, when the Load is Gradually Applied

When loading a body, in which the loading starts from zero and increases gradually till the body is fully loaded. *e.g., when we lower a body with the help of a* crane, the body first touches the platform on which it is to be placed. On further releasing the chain, the platform goes on loading till it is fully loaded by the body. This is the case of a gradually applied load. Now consider a metallic bar subjected to a gradual load.

Let P = Load gradually applied,

A = Cross-sectional area of the bar,

l = Length of the bar,

E = Modulus of elasticity of the bar material and

d = Deformation of the bar due to load.

Since the load applied is gradual, and varies from zero to P, therefore the average load is equal to P/2

$$\therefore \text{ Work done } = \text{Force} \times \text{Distance}$$

$$= \text{Average load} \times \text{Deformation}$$

$$= \frac{P}{2} \times \delta l = \frac{P}{2}(\varepsilon . l) \qquad ...(\because \delta l = \varepsilon . l)$$

$$= \frac{1}{2}\, \sigma . \varepsilon A . l \qquad ...(\because P = \sigma A)$$

$$= \frac{1}{2} \times \text{Stress} \times \text{Strain} \times \text{Volume}$$

$$= \frac{1}{2} \times \sigma \times \frac{\sigma}{E} \times A l \qquad ...(\because \varepsilon = \frac{\sigma}{E})$$

$$= \frac{1}{2} \times \frac{\sigma^2}{E} \times A l$$

Since the strain energy stored is also equal to the work done, therefore strain energy stored,

$$U = \frac{\sigma^2}{2E} \times Al = \frac{\sigma^2}{2E} \times V \qquad ...(\because Al = \text{Volume} = V)$$

We also know that modulus of resilience

$$= \text{Strain energy per unit volume}$$

$$= \frac{\sigma^2}{2E}$$

Example *Calculate the strain energy stored in a bar 2 m long, 50 mm wide and 40 mm thick when it is subjected to a tensile load of 60kN. Take E as 200 GPa.*

Given:

Length of bar (*l*) = *2 m* = *2* × *10*^{3} mm

Width of bar (*b*) = *50 mm*

Thickness of bar (*t*) = *40 mm*

Tensile load on bar (P) = *60 kN* = *60* × *10*^{3} N and

Modulus of elasticity (*E*) = *200*GPa = 200 × 10^3 N/mm^2

We know that stress in the bar

$$\sigma = \frac{P}{A} = \frac{60 \times 10^3}{50 \times 40} = 30 \text{ N/mm}^2$$

$\therefore$ Strain energy stored in the bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(30)^2}{2 \times (200 \times 10^3)} \times 4 \times 10^6 \text{ N-mm}$$

$$= 9 \times 10^3 \text{ N-mm} = 9 \text{ kN-mm} \qquad \textbf{Ans.}$$

Strain Energy Stored in a Body when the Load is Suddenly Applied

The load is suddenly applied on a body. *e.g.*, *when we* lower a body with the help of a crane, the body is, first of all, just above the platform on which it is to be placed. If the chain breaks at once at this moment the whole load of the body begins to act on the platform. This is the case of a suddenly applied load. Now consider a bar subjected to a sudden load.

P = Load applied suddenly,
A = Cross-sectional area of the bar,
l = Length of the bar,
E = Modulus of elasticity of the material,
δ = Deformation of the bar, and
σ = Stress induced by the application of the sudden load

Since the load is applied suddenly, therefore the load (P) is constant throughout the process of deformation of the bar.

$\therefore$ Work done = Force × Distance = Load × Deformation ...(*i*)

$= P \times \delta l$

We know that strain energy stored,

$$U = \frac{\sigma^2}{2E} \times Al \quad \text{...}(ii)$$

Since the strain energy stored is equal to the work done, therefore

$$\frac{\sigma^2}{2E} \times Al = P \times \delta l = P \times \frac{\sigma}{E} l \quad \text{...}\left(\delta l = \frac{\sigma}{E} l\right)$$

or
$$\sigma = 2 \times \frac{P}{A}$$

***Example** An axial pull of 20 kN is suddenly applied on a steel rod 2.5 m long and 1000 mm2 in cross-section. Calculate the strain energy, which can be absorbed in the rod. Take E = 200 GPa.*

Given:

Axial pull on the rod (P) = 20 kN = 20×10^3 N;
Length of rod (l) = 2.5 m = 2.5×10^3 mm
Cross-sectional area of rod (A) =1000 mm^2
and modulus of elasticity (E) = 200GPa = 200×10^3 N/mm^2

We know that stress in the rod, when the load is suddenly applied

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{20 \times 10^2}{1000} = 440 \text{ N/mm}^2$$

and volume of the rod,

$$V = l \,.\, A = (2.5 \times 10^3) \times 1000 = 2.5 \times 10^6 \text{ mm}^3$$

$\therefore$ Strain energy which can be absorbed in the rod,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(40)^2}{2 \times (200 \times 10^3)} \times (2.5 \times 10^6) \text{ N-mm}$$

$$= 10 \times 10^3 \text{ N-mm} = 10 \text{ kN-mm} \quad \textbf{Ans.}$$

Strain Energy Stored in a Body, when the Load is applied with Impact

The impact load is applied on a body *e.g., when we lower a body with the help of a crane, and the* chain breaks while the load is being lowered the load falls through a distance, before it touches the platform. This is the case of a load applied with impact. Now consider a bar subject to a load applied with impact as shown.

Let P = Load applied with impact,
A = Cross-sectional area of the bar,
E = Modulus of elasticity of the bar material,
l = Length of the bar,
δl = Deformation of the bar, as a result of this load,
σ = Stress induced by the application of this load with impact, and

$$h = \text{Height through which the load will fall, before impacting on the collar of the bar.}$$

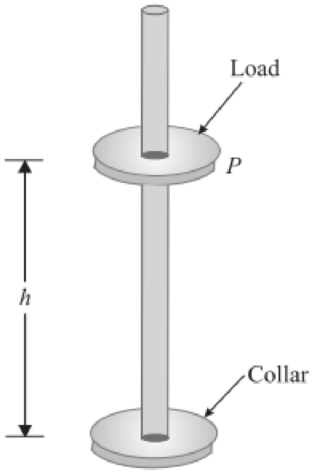
$\therefore$ Work done $= \text{Load} \times \text{Distance moved}$

$$= P\,(h + \delta l)$$

and energy stored, $U = \dfrac{\sigma^2}{2E} \times Al$

Since energy stored is equal to the work done, therefore

$$\frac{\sigma^2}{2E} \times Al = P\,(h + \delta l) = P\left(h + \frac{\sigma}{E}.l\right) \qquad \left(\because \delta l = \frac{\sigma}{E}.l\right)$$



$$\frac{\sigma^2}{2E} \times Al = Ph + \frac{P\sigma l}{E}$$

$$\therefore \quad \sigma^2\left(\frac{Al}{2E}\right) - \sigma\left(\frac{Pl}{E}\right) - Ph = 0$$

Multiplying both sides by $\left(\dfrac{E}{Al}\right)$,

$$\frac{\sigma^2}{2} - \sigma\left(\frac{P}{A}\right) - \frac{PEh}{Al} = 0$$

This is a quadratic equation. We know that

$$\sigma = \frac{P}{A} \pm \sqrt{\left(\frac{P}{A}\right)^2 + \left(4 \times \frac{1}{2}\right)\left(\frac{PEh}{Al}\right)}$$

$$= \frac{P}{A}\left[1 \pm \sqrt{1 + \frac{2AEh}{Pl}}\right]$$

Once the stress (σ)is obtained, the corresponding instantaneous deformation (δl) or the strain energy stored may be found out as usual.

Cor. When δ is very small as compared to h, then

$$\text{Work done} = Ph$$

$$\therefore \quad \frac{\sigma^2}{2E} Al = Ph$$

or

$$\sigma^2 = \frac{2EPh}{Al}$$

$$\therefore \quad \sigma = \sqrt{\frac{2EPh}{Al}}$$

Example *A copper bar of 12 mm diameter gets stretched by 1 mm under a steady load of 4 kN. What stress would be produced in the bar by a weight 500 N, the weight falls through 80 mm before striking the collar rigidly fixed to the lower end of the bar? Take Young's modulus for the bar material as 100 GPa.*

Given :

Diameter of bar (d) = *12 mm*

Change in length of bar (dl) = 1 mm

Load on bar ($P1$) = 4 kN = 4×10^3 N

Weight falling on collar ($P2$) = 500 N

Height from which weight falls (h) = *80 mm*

Modulus of elasticity (E) = 100 GPa = 100 $\times$ *10^3* N/mm^2

Let $\qquad l$ = Length of the copper bar.

We know that cross-sectional area of the bar,

$$A = \frac{\pi}{4} \times (d)^2 = \frac{\pi}{4} \times (12)^2 = 113.1 \text{ mm}^2$$

and stretching of the bar (δl),

$$l = \frac{P.l}{A.E} = \frac{(4 \times 10^3)}{113.1 \times (100 \times 10^3)} = \frac{l}{2.83 \times 10^3}$$

$$\therefore \quad l = 1 \times (2.83 \times 10^3) = 2.83 \times 10^3 \text{ mm}$$

We also know that stress produced in the bar by the falling weight.

$$\sigma = \frac{P_2}{A}\left(1 + \sqrt{1 + \frac{2AEh}{P_2 l}}\right)$$

$$= \frac{1500}{113.1}\left(1 + \sqrt{1 + \frac{2 \times 113.1 \times (100 \times 10^3) \times 80}{500 \times (2.83 \times 10^3)}}\right) \text{ N/mm}^2$$

$$= 4.2\ (1 + 35.77) = 162.52 \text{ N/mm}^2 = 162.52 \text{ MPa} \qquad \textbf{Ans.}$$

Strain Energy Stored in a Body of Varying Section

Sometimes, we come across bodies of varying section. The strain energy in such a body is obtained by adding the strain energies stored in different parts of the body. Mathematically total strain energy stored in a body.

$U = U_1 + U_2 + U_3 + ...$

Where U_1 = Strain energy stored in part 1,

U_2 = Strain energy stored in part 2,

U_3 = Strain energy stored in part 3

Example *A non-uniform tension bar 5 m long is made up of two parts as shown. Find the total strain energy stored in the bar, when it is subjected to a gradual load of 70 kN. Also find the total strain energy stored in the bar, when the bar is made of uniform cross-section of the same volume under the same load. Take E = 200 GPa.*

Given:

Total length of bar (L) = $5\ m = 5 \times 10^3$ mm

Length of part 1 ($L1$) = 3 m = 3×10^3 mm

Length of part 2 ($L2$) = 2 m = 2×10^3 mm

Area of part 1 ($A1$) = 1000 mm^2

Area of part 2 ($A2$) = 2000 mm^2

Pull (P) = $70\ kN = 70 \times 10^3$ N

Modulus of elasticity (E) = $200\ Gpa$ = 200×10^3 N/mm^2

Total strain energy stored in the non-uniform bar

We know that stress in the first part,

$$\sigma_1 = \frac{P}{A_1} = \frac{70 \times 10^3}{1000} = 70 \text{ N/mm}^2$$

and volume of the first part,

$$V_1 = (3 \times 10^3) \times 1000 = 3 \times 10^6 \text{ mm}^3$$

$\therefore$ Strain energy stored in the first part,

$$U_1 = \frac{\sigma_1^2}{2E} \times V_1 = \frac{(70)^2}{2 \times (200 \times 10^3)} \times (3 \times 10^6) = 36.75 \times 10^3 \text{ N-mm} \qquad ...(i)$$

Similarly, stress in the second part,

$$\sigma_2 = \frac{P}{A_2} = \frac{70 \times 10^3}{2000} = 35 \text{ N/mm}^2$$

and volume of the second part,

$$V_2 = (2 \times 10^3) \times 2000 = 4 \times 10^6 \text{ mm}^3$$

∴ Strain energy stored in the second part,

$$U_2 = \frac{\sigma_2^2}{2E} \times V_2 = \frac{(35)^2}{2 \times (200 \times 10^3)} \times (4 \times 10^6) = 12.25 \times 10^3 \text{ N-mm} \quad ...(ii)$$

and total strain energy stored in the non-uniform bar,

$$U = U_1 + U_2 = (36.75 \times 10^3) + (12.25 \times 10^3) = 49 \times 10^3 \text{ N=mm} = 49 \text{ N-m} \quad \textbf{Ans.}$$

Total strain energy in the uniform bar

We know that total volume of the bar,

$$V = V_1 + V_2 = (3 \times 10^6) + (4 \times 10^6) = 7 \times 10^6 \text{ mm}^3$$

and cross-sectional area of the circular bar,

$$A = \frac{\text{Volume of the bar}}{\text{Length of the bar}} = \frac{7 \times 10^6}{5 \times 10^3} = 1400 \text{ mm}^2$$

∴ Stress in the bar

$$\sigma = \frac{70 \times 10^3}{1400} = 50 \text{ N/mm}^2$$

and strain energy storad in the uniform bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(50)^2}{2 \times (200 \times 10^3)} \times (7 \times 10^6) = 43.75 \times 10^3 \text{ N-mm}$$

$$= 43.75 \text{ N-m} \quad \textbf{Ans.}$$

Strain Energy Stored in a Body due to Shear Stress

Consider a cube *ABCD* of length *l* fixed at the bottom face *AB* as shown in Fig 8.5.

Let P = Force applied tangentially on the face *DC*,

τ = Shear stress

ϕ = Shear strain, and

N = Modulus of rigidity or shear modulus.

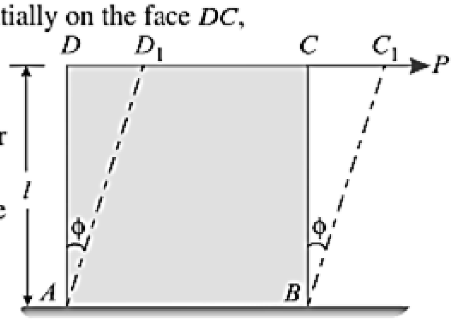


Fig. 8.5. Strain energy due to shear stress

If the force P is applied gradually then the average force is equal to $P/2$.

∴ Work done = Average force × Distance

$$= \frac{P}{2} \times DD_1$$

$$= \frac{1}{2} \times P \times AD \times \phi \quad ...(\because DD_1 = AD \times \phi)$$

$$= \frac{1}{2} \times \tau \times DC \times l \times AD \times \phi \quad ...(\because P = \tau \times DC \times l)$$

$$= \frac{1}{2} \times \tau \times \phi \times DC \times AD \times l$$

$$= \frac{1}{2} \text{ (stress} \times \text{strain} \times \text{volume)}$$

$$= \frac{1}{2} \times \tau \times \frac{\tau}{N} \times V \quad ...\left(\because \phi = \frac{\tau}{N}\right)$$

$$= \frac{\tau^2}{2N} \times V \qquad \text{...(where } V \text{ is the volume)}$$

Since energy stored is also equal to the work done, therefore energy stored,

$$U = \frac{\tau^2}{2N} \times V$$

We also know that modulus of resilience

$$= \text{Strain energy per unit volume}$$

$$= \frac{\tau^2}{2N}$$

Example *A rectangular body 500 mm long, 100 mm wide and 50 mm thick is subjected to a shear stress of 80 MPa. Determine the strain energy stored in the body. Take N = 85 GPa.*

Given:

Length of rectangular body (l) = *500 mm*
Width of rectangular body (b) = 100 mm
Thickness of rectangular body (t) = *50 mm*
Shear stress (t) = 80 MPa = 80 N/mm² and
modulus of rigidity (N) = *85 N/mm²*

We know that volume of the bar,

$$V = l.b.t = 500 \times 100 \times 50 = 2.5 \times 10^6 \text{ mm}^3$$

and strain energy stored in the body,

$$U = \frac{\tau^2}{2N} \times V = \frac{(80)^2}{2 \times (85 \times 10^3)} \times 2.5 \times 10^6 \text{ N-mm}$$

$$= 94.1 \times 10^3 \text{ N-mm} = 94.1 \text{ N-m} \qquad \textbf{Ans.}$$

Principal Stresses and Strains

At a time one type of stress, acting in one direction only. But the majority of engineering, component and structures are subjected to such loading conditions (or sometimes are of such shapes) that there exists a complex state of stresses; involving direct tensile and compressive stress as well as shear stress in various directions.

At any point in a strained material, there are three planes, mutually perpendicular to each other, which carry direct stresses only, and no shear stress. These three direct stresses one will be maximum, the other minimum, and the third and intermediate between the two. These particular planes, which have no shear stress, are known as ***principal planes.***

The magnitude of direct stress, across a principal plane, is known as **principal stress**. The determination of principal planes, and then principal stress is an important factor in the design of various structures and machine components.

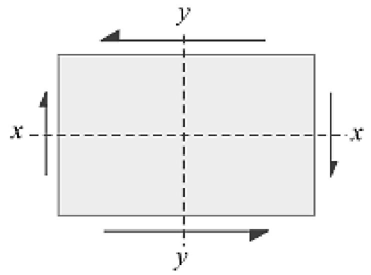
The following two methods for the determination of stresses on an oblique section of a strained body are important from the subject point of view: **1. Analytical method and 2. Graphical method.**

Analytical Method for the Stresses on an Oblique Section of a Body

The analytical method for the determination of stresses on an oblique section in the following cases, which are important from the subject point of view:

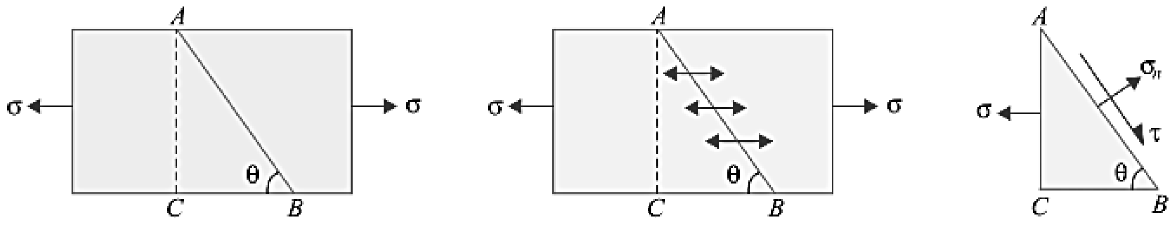
1. A body subjected to a direct stress in one plane.
2. A body subjected to direct stresses in two mutually perpendicular directions

In the element shown, the shear stress on the vertical faces (or *x-x axis) is taken as positive, whereas the shear stress on the* horizontal faces (or *y-y axis) is taken as negative*



Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a direct tensile stress along *x-x axis as shown. Now let us consider an oblique section AB* inclined with the *x-x axis.*



(*a*) (*b*) (*c*)

Let σ = Tensile stress across the face AC and

θ = Angle, which the oblique section AB makes with BC *i.e.* with the x-x axis in the clockwise direction.

First of all, consider the equilibrium of an element or wedge ABC whose free body diagram is shown in fig 7.2 (*b*) and (*c*). We know that the horizontal force acting on the face AC,

$$P = \sigma . AC \; (\leftarrow)$$

Resolving the force perpendicular or normal to the section AB

$$P_n = P \sin\theta = \sigma . AC \sin\theta \qquad ....(i)$$

and now resolving the force tangential to the section AB,

$$P_t = P \cos\theta = \sigma . AC \cos\theta \qquad ....(ii)$$

We know that normal stress across the section AB*,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma AC \sin\theta}{AB} = \frac{\sigma . AC \sin\theta}{\frac{AC}{\sin\theta}} = \sigma \sin^2\theta$$

$$= \frac{\sigma}{2}(1 - \cos 2\theta) = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta \qquad ...(iii)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma . AC \cos\theta}{AB} = \frac{\sigma . AC \cos\theta}{\frac{AC}{\sin\theta}} = \sigma \sin\theta \cos\theta$$

$$= \frac{\sigma}{2}\sin 2\theta \qquad ...(iv)$$

The face *AC will carry* the maximum direct stress. Similarly, the shear stress across the section *AB will be maximum when* sin 2θ = 1 or 2θ = 90° or 270°. Or in other words, the shear stress will be maximum on the planes inclined at 45° and 135° with the line of action of the tensile stress. Therefore maximum shear stress when θ is equal to 45°,

$$\tau_{max} = \frac{\sigma}{2}\sin 90° = \frac{\sigma}{2}\times 1 = \frac{\sigma}{2}$$

and maximum shear stress, when θ is equal to 135°,

$$\tau_{max} = -\frac{\sigma}{2}\sin 270° = -\frac{\sigma}{2}(-1) = \frac{\sigma}{2}$$

It is thus obvious that the magnitudes of maximum shear stress is half of the tensile stress. Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

NOTE : The planes of maximum and minimum normal stresses (*i.e. principal planes) may also be found out by* equating the shear stress to zero. This happens as the normal stress is either maximum or minimum on a plane having zero shear stress. Now equating the shear stress to zero, $\sigma \sin\theta \cos\theta = 0$

Example *Two wooden pieces 100 mm × 100 mm in cross-section are joined together along a line AB as shown. Find the maximum force (P), which can be applied if the shear stress along the joint AB is 1.3 MPa.*

Given:

Section = 100 mm × 100 mm ;
Angle made by section with the
Direction of tensile stress (θ) = 60° and
Permissible shear stress (t) = 1.3 MPa = 1.3 N/mm^2

Let σ = Safe tensile stress in the member

We know that cross- sectional area of the wooden member,

$$A = 100 \times 100 = 10\,000 \text{ mm}^2$$

and shear stress (τ),

$$1.3 = \frac{\sigma}{2}\sin 2\theta = \frac{\sigma}{2}\sin(2\times 60°) = \frac{\sigma}{2}\sin 120° = \frac{\sigma}{2}\times 0.866$$

$$= 0.433\,\sigma$$

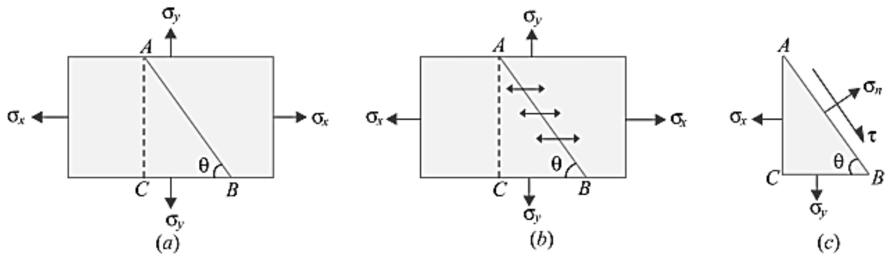
or $$\sigma = \frac{1.3}{0.433} = 3.0 \text{ N/mm}^2$$

∴ Maximum axial force, which can be applied,

$$P = \sigma.A = 3.0 \times 10\,000 = 30\,000 \text{ N} = 30 \text{ kN} \quad \textbf{Ans.}$$

Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along *x-x and y-y axes as shown*. Now let us consider an oblique section *AB inclined with x-x axis*



Let σ_x = Tensile stress along x-x axis (also termed as major tensile stress),

σ_y = Tensile stress along y-y axis (also termed as a minor tensile stress), and

θ = Angle which the oblique section AB makes with x-x axis in the clockwise direction.

First of all, consider the equilibrium of the wedge ABC. We know that horizontal force acting on the face AC (or x-x axis).

$$P_x = \sigma_x \,.\, AC \;(\leftarrow)$$

and vertical force acting on the face BC (or y-y axis),

$$P_y = \sigma_y \,.\, BC \;(\downarrow)$$

Resolving the forces perpendicular or normal to the section AB,

$$P_n = P_x \sin\theta + P_y \cos\theta = \sigma_x \,.\, AC \sin\theta + \sigma_y \,.\, BC \cos\theta \qquad ...(i)$$

and now resolving the forces tangential to the section AB,

$$P_t = P_x \cos\theta - P_y \sin\theta = \sigma_x \,.\, AC \cos\theta - \sigma_y \,.\, BC \sin\theta \qquad ....(ii)$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta + \sigma_y\, BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} = \frac{\sigma_x . AC \sin\theta}{\dfrac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\dfrac{BC}{\cos\theta}}$$

$$= \sigma_x \sin^2\theta + \sigma_y \,.\, \cos^2\theta = \frac{\sigma_x}{2}\,(1 - \cos 2\theta) + \frac{\sigma_y}{2}\,(1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2}\cos 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad ...(iii)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta - \sigma_y . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{AB} - \frac{\sigma_y . BC \sin\theta}{AB} = \frac{\sigma_x . AC \cos\theta}{\dfrac{AC}{\sin\theta}} - \frac{\sigma_y . BC \sin\theta}{\dfrac{BC}{\cos\theta}}$$

$$= \sigma_x \,.\, \sin\theta\cos\theta - \sigma_y \sin\theta\cos\theta$$

$$= (\sigma_x - \sigma_y)\sin\theta\cos\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta \qquad ...(iv)$$

It will be interesting to know from equation (*iii*) the shear stress across the section AB will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $\theta = 45°$. Therefore maximum shear stress,

$$\tau_{max} = \frac{\sigma_x - \sigma_y}{2}$$

Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

Example: *The stresses at point of a machine component are 150 MPa and 50 Mpa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress. Also find the magnitude of the maximum shear stress in the component.*

Given: Tensile stress along *x-x axis* (s_x) = 150 MPa;

Tensile stress along *y-y axis* (s_y) = 50 MPa and

Angle made by the plane with the major tensile stress (θ) = 55°.

Normal stress on the inclined plane

We know that the normal stress on the inclined plane

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} \quad \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$= \frac{150+50}{2} - \frac{150-50}{2} \cos (2 \times 55°) \text{ MPa}$$

$$= 100 - 50 \cos 110° = 100 - 50 (-0.342) \text{ MPa}$$

$$= 10 + 17.1 = 117.1 \text{ MPa} \qquad \textbf{Ans.}$$

Shear stress on the inclined plane

We know that the shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \frac{150-50}{2} \times \sin (2 \times 55°) \text{ MPa}$$

$$= 50 \sin 110° = 50 \times 0.9397 = 47 \text{ MPa} \qquad \textbf{Ans.}$$

Resultant stress on the inclined plane

We know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(117.1)^2 + (47.0)^2} = 126.2 \text{ MPa} \qquad \textbf{Ans.}$$

Maximum shear stress in the component

We also know that the magnitude of the maximum shear stress in the component,

$$\tau_{max} = \pm \frac{\sigma_x - \sigma_y}{2} = \pm \frac{150-50}{2} = \pm 50 \text{ MPa} \qquad \textbf{Ans.}$$

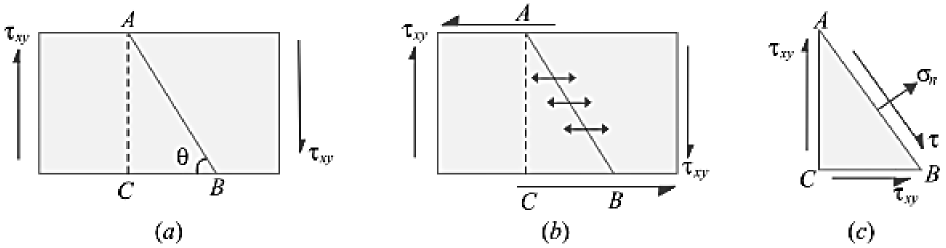
Stresses on an Oblique Section of a Body Subjected to a Simple Shear stress

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a positive (*i.e., clockwise) shear stress along x-x axis as shown. Now let us consider an* oblique section *AB inclined with x-x axis on which we are required to find out the stresses as shown.*

Let τ_{xy} = Positive (*i.e., clockwise) shear stress along x-x axis, and*

θ = Angle , which the oblique section *AB makes with x-x axis in* the anticlockwise direction.

First of all, consider the equilibrium of the wedge *ABC. We know that as per the principle of* simple shear, the face *BC, of the wedge will be subjected to an anticlockwise shear stress equal to* τ_{xy} as shown. *We know that vertical force acting on the face AC,*



$$P_1 = \tau_{xy} \cdot AC\ (\uparrow)$$

and horizontal force acting on the face BC,

$$P_2 = \tau_{xy} . BC\ (\rightarrow)$$

Resolving the forces perpendicular or normal to the AB,

$$P_n = P_1 \cos\theta + P_2 \sin\theta = \tau_{xy} . AC \cos\theta + \tau_{xy} . BC \sin\theta$$

and now resolving the forces tangential to the section AB,

$$P_t = P_2 \sin\theta - P_1 \cos\theta = \tau_{xy} . BC \sin\theta - \tau_{xy} . AC \cos\theta$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\tau_{xy} . AC \cos\theta + \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\tau_{xy} . AC \cos\theta}{AB} + \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \tau_{xy} . \sin\theta \cos\theta + \tau_{xy} . \sin\theta \cos\theta$$

$$= 2\,\tau_{xy} . \sin\theta \cos\theta = \tau_{xy} . \sin 2\theta$$

and shear stress (*i.e.* tangential stress) across the section AB

$$\tau = \frac{P_t}{AB} = \frac{\tau_{xy} . BC \sin\theta - \tau_{xy} . AC \cos\theta}{AB}$$

$$= \frac{\tau_{xy} . BC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} = \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\cos\theta}}$$

$$= \tau_{xy} \sin^2\theta - \tau_{xy} \cos^2\theta$$

$$= \frac{\tau_{xy}}{2}(1 - \cos 2\theta) - \frac{\tau_{xy}}{2}(1 + \cos 2\theta)$$

$$= \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta$$

$$= -\tau_{xy} \cos 2\theta$$ ...(Minus sign means that normal stress is opposite to that across AC)

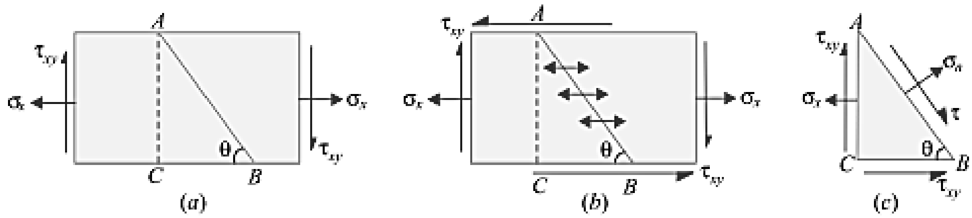
Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*

$$-\tau_{xy} \cos 2\theta = 0$$

The above equation is possible only if $2\theta = 90°$ or $270°$ (because $\cos 90°$ or $\cos 270° = 0$) or in other words, $\theta = 45°$ or $135°$.

Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane and Accompanied by a Simple Shear Stress

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a tensile stress along x-x axis accompanied by a positive (i.e. clockwise) shear stress along x-x axis as shown. Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses as shown in the figure.



Let σ_x = Tensile stress along x-x axis,

τ_{xy} = Positive (*i.e.* clockwise) shear stress along x-x axis, and

θ = Angle which the oblique section AB makes with x-x axis in clockwise direction.

First of all, consider the equilibrium of the wedge ABC. We know that as per the principle of simple shear, the face BC of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in Fig. 7.7 (*b*). We know that horizontal force acting on the face AC,

$$P_x = \sigma_x \, . \, AC \; (\leftarrow) \qquad ...(i)$$

Similarly, vertical force acting on the face AC,

$$P_y = \tau_{xy} \, . \, AC \; (\uparrow) \qquad ...(ii)$$

and horizontal force acting on the face BC,

$$P = \tau_{xy} \, . \, BC \; (\rightarrow) \qquad ...(iii)$$

Resolving the forces perpendicular to the section AB,

$$P_n = P_x \sin\theta - P_y \cos\theta - P \sin\theta$$
$$= \sigma_x \, . \, AC \sin\theta - \tau_{xy} \, . \, AC \cos\theta - \tau_{xy} \, . \, BC \sin\theta$$

and now resolving the forces tangential to the section AB,

$$P_t = P_x \cos\theta + P_y \sin\theta - P \cos\theta$$
$$= \sigma_x \, . \, AC \cos\theta + \tau_{xy} \, . \, AC \sin\theta - \tau_{xy} \, . \, BC \cos\theta$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta - \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} - \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{\dfrac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\dfrac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC \sin\theta}{\dfrac{BC}{\cos\theta}}$$

$$= \sigma_x \, . \, \sin^2\theta - \tau_{xy} \sin\theta \cos\theta - \tau_{xy} \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} (1 - \cos 2\theta) - 2\,\tau_{xy} \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \qquad ...(iv)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta + \tau_{xy} . AC \sin\theta - \tau_{xy} . BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{AB} + \frac{\tau_{xy} \, AC \sin\theta}{AB} - \frac{\tau_{xy} . BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{\dfrac{AC}{\sin\theta}} + \frac{\tau_{xy} \, AC \sin\theta}{\dfrac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC \cos\theta}{\dfrac{BC}{\cos\theta}}$$

$$= \sigma_x \sin\theta \cos\theta + \tau_{xy} \sin^2\theta - \tau_{xy} \cos^2\theta$$

$$= \frac{\sigma_x}{2} \sin 2\theta + \frac{\tau_{xy}}{2} (1 - \cos 2\theta) - \frac{\tau_{xy}}{2} (1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} \sin 2\theta + \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2} \cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2} \cos 2\theta$$

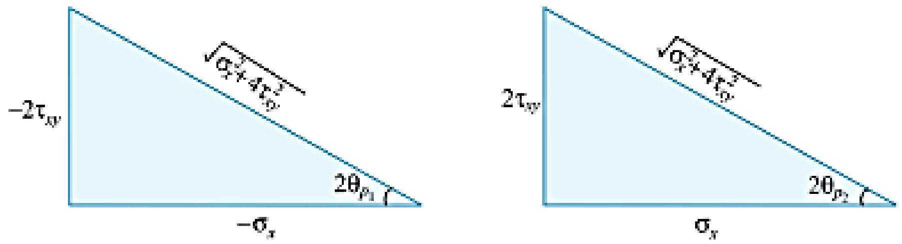
$$= \frac{\sigma_x}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \qquad ...(v)$$

Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*, from the above equation, we find that the shear stress on any plane is a function of σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_x and τ_{xy} are constant and thus the shear stress varies with the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

$$\therefore \quad \frac{\sigma_x}{2}\sin 2\theta_p - \tau_{xy}\cos 2\theta_p = 0 \qquad \text{or} \qquad \frac{\sigma_x}{2}\sin 2\theta_p = \tau_{xy}\cos 2\theta_p$$

$$\therefore \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x}$$

From the above equation we find that the following two cases satisfy this condition as shown in Fig 7.8 (*a*) and (*b*)



(a) case 1 (b) case 2

Fig. 7.8

Thus we find that these are two principal planes at right angles to each other, their inclination with *x-x* axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{\sigma_x^2+4\tau_{xy}^2}} \qquad \text{and} \qquad \cos 2\theta_{p_1} = \frac{-\sigma_x}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{\sigma_x^2+4\tau_{xy}^2}} \qquad \text{and} \qquad \cos 2\theta_{p_2} = \frac{\sigma_x}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (*iv*).

Maximum principal stress,

$$\sigma_{p_1} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\times\frac{-\sigma_x}{\sqrt{\sigma_x^2+4\tau_{xy}^2}} - \tau_{xy}\times\frac{-2\tau_{xy}}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2}{2\sqrt{\sigma_x^2+4\tau_{xy}^2}} + \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2+4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2+4\tau_{xy}^2}{2\sqrt{\sigma_x^2+4\tau_{xy}^2}} = \frac{\sigma_x}{2} + \frac{\sqrt{\sigma_x^2+4\tau_{xy}^2}}{2}$$

$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x^2}{2}\right) + \tau_{xy}^2}$$

Minimum principal stress, $\sigma_{p_2} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \times \frac{\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \tau_{xy} \times \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}{2} = \frac{\sigma_x}{2} - \frac{\sigma_x^2 + 4\tau_{xy}^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Example *An element in a strained body is subjected to a tensile stress of 150 MPa and a shear stress of 50 MPa tending to rotate the element in an anticlockwise direction. Find (i) the magnitude of the normal and shear stresses on a section inclined at 40° with the tensile stress; and (ii) the magnitude and direction of maximum shear stress that can exist on the element.*

Given:

Tensile stress along horizontal *x-x* axis (σx) = 150 MPa

Shear stress (τxy) – 50 MPa (Minus sign due to anticlockwise) and angle made by section with the tensile stress (θ) = 40°.

Normal and Shear stress on the inclined section

We know that magnitude of the normal stress on the section

$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{150}{2} - \frac{150}{2}\cos(2 \times 40°) - (-50)\sin(2 \times 40°) \text{ MPa}$$

$$= 75 - (75 \times 0.1736) + (50 \times 0.9848) \text{ MPa}$$

$$= 75 - 13.02 + 49.24 = 111.22 \text{ MPa} \quad \textbf{Ans.}$$

and shear stress on the section

$$\tau = \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{150}{2}\sin(2 \times 40°) - (-50)\cos(2 \times 40°) \text{ MPa}$$

$$= (75 \times 0.9848) + (50 \times 0.1736) \text{ MPa}$$

$$= 73.86 + 8.68 = 82.54 \text{ MPa} \quad \textbf{Ans.}$$

(*ii*) *Maximum shear stress and its direction that can exist on the element*

We know that magnitude of the maximum shear stress.

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{150}{2}\right)^2 + (-50)^2} = \pm 90.14 \text{ MPa} \quad \textbf{Ans.}$$

Let θ_s = Angle which plane of maximum shear stress makes with *x-x* axis.

We know that, $\tan 2\theta_s = \frac{\sigma_x}{2\tau_{xy}} = \frac{150}{2 \times 50} = 1.5$ or $2\theta_s = 56.3°$

$\therefore$ $\theta_s = 28.15°$ or $118.15°$ **Ans.**

Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress

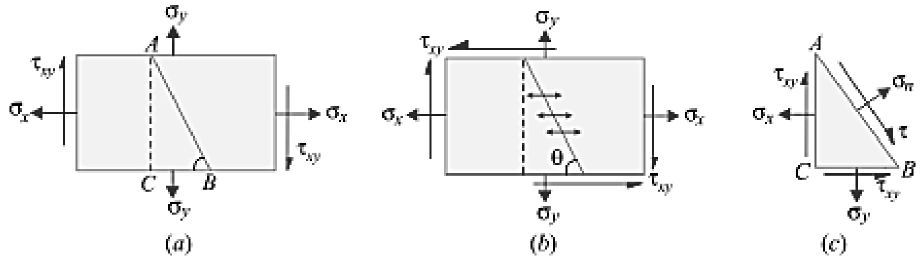


Fig. 7.9

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to tensile stresses along *x-x* and *y-y* axes and accompanied by a positive (*i.e.*, clockwise) shear stress along *x-x* axis as shown in Fig.7.9 (*b*). Now let us consider an oblique section *AB* inclined with *x-x* axis on which we are required to find out the stresses as shown in the figure.

Let
σ_x = Tensile stress along *x-x* axis,
σ_y = Tensile stress along *y-y* axis,
τ_{xy} = Positive (*i.e.* clockwise) shear stress along *x-x* axis, and
θ = Angle, which the oblique section *AB* makes with *x-x* axis in an anticlockwise direction.

First of all, consider the equilibrium of the wedge *ABC*. We know that as per the principle of simple shear, the face *BC* of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in Fig. 7.9 (*b*). We know that horizontal force acting on the face *AC*,

$$P_1 = \sigma_x \, . \, AC \, (\leftarrow) \qquad \text{... } (i)$$

and vertical force acting on the face *AC*,

$$P_2 = \tau_{xy} \, . \, AC \, (\uparrow) \qquad \text{...}(ii)$$

Similarly, vertical force acting on the face *BC*,

$$P_3 = \sigma_y \, . \, BC \, (\downarrow) \qquad \text{...}(iii)$$

and horizontal force on the face *BC*,

$$P_4 = \tau_{xy} \, . \, BC \, (\rightarrow) \qquad \text{...}(iv)$$

Now resolving the forces perpendicular to the section *AB*,

$$P_n = P_1 \sin\theta - P_2 \cos\theta + P_3 \cos\theta - P_4 \sin\theta$$
$$= \sigma_x \, . \, AC \sin\theta - \tau_{xy} \, . \, AC \cos\theta + \sigma_y \, . \, BC \cos\theta - \tau_{xy} \, . \, BC \sin\theta$$

and now resolving the forces tangential to *AB*,

$$P_t = P_1 \cos\theta + P_2 \sin\theta - P_3 \sin\theta - P_4 \cos\theta$$
$$= \sigma_x \, . \, AC \cos\theta + \tau_{xy} \, . \, AC \sin\theta - \sigma_y \, . \, BC \sin\theta - \tau_{xy} \, . \, BC \cos\theta$$

Normal Stress (across the inclined section *AB*)

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta + \sigma_y . BC \cos\theta - \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} - \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\frac{BC}{\cos\theta}} - \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \, . \, \sin^2\theta - \tau_{xy} \sin\theta \cos\theta + \sigma_y \, . \, \cos^2\theta - \tau_{xy} \, . \, \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} (1 - \cos 2\theta) + \frac{\sigma_y}{2} (1 + \cos 2\theta) - 2\,\tau_{xy} \, . \, \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

or

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \qquad \text{...}(v)$$

Shear Stress or Tangential Stress (across inclined the section AB)

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x.AC\cos\theta + \tau_{xy}.AC\sin\theta - \sigma_y.BC\sin\theta - \tau_{xy}\ BC\cos\theta}{AB}$$

$$= \frac{\sigma_x.AC\cos\theta}{AB} + \frac{\tau_{xy}.AC\sin\theta}{AB} - \frac{\sigma_y.BC\sin\theta}{AB} - \frac{\tau_{xy}\ BC\cos\theta}{AB}$$

$$= \frac{\sigma_x.AC\cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy}.AC\sin\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y.BC\sin\theta}{\frac{BC}{\cos\theta}} - \frac{\tau_{xy}.BC\cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin\theta\cos\theta + \tau_{xy}\sin^2\theta - \sigma_y\sin\theta\cos\theta - \tau_{xy}\cos^2\theta$$

$$= (\sigma_x - \sigma_y)\sin\theta\cos\theta + \frac{\tau_{xy}}{2}(1 - \cos 2\theta) - \frac{\tau_{xy}}{2}(1 + \cos 2\theta)$$

or

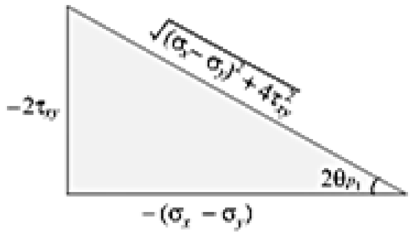
$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad ...(vi)$$

Now the planes of maximum and minimum normal stresses (*i.e.* principal planes) may be found out by equating the shear stress to zero. From the above equations, we find that the shear stress to any plane is a function of σ_y, σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_y, σ_x and τ_{xy} are constant and thus the shear stress varies in the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

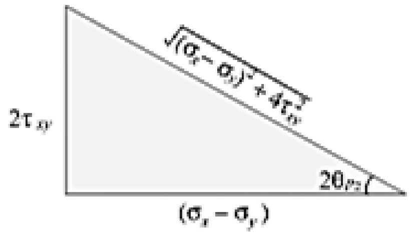
$$\therefore \quad \frac{\sigma_x - \sigma_y}{2}\sin 2\theta_p - \tau_{xy}\cos 2\theta_p = 0$$

or
$$\frac{\sigma_x - \sigma_y}{2}\sin 2\theta_p = \tau_{xy}\cos 2\theta_p \qquad \text{or} \qquad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

From the above equation, we find that the following two cases satisfy this condition as shown in Fig 7.10 (a) and (b).



(a) Case 1



(b) Case 2

Thus we find that there are two principal planes, at right angles to each other, their inclinations with x-x axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_1} = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_2} = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (v).

Maximum Principal Stress,

$$\sigma_{p_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2} \times \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y^2) + 4\tau_{xy}^2}}\right) - \left(\tau_{xy} \times \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\right)$$

or $$\sigma_{p_1} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum Principal Stress

$$\sigma_{p2} = \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x+\sigma_y}{2} - \left(\frac{\sigma_x-\sigma_y}{2} \times \frac{(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right) - \left(\tau_{xy} \times \frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right)$$

$$= \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}{2\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} = \frac{\sigma_x-\sigma_y}{2} - \frac{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}{2}$$

or $$\sigma_{p_2} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Example *A point is subjected to a tensile stress of 250 MPa in the horizontal direction and another tensile stress of 100 MPa in the vertical direction. The point is also subjected to a simple shear stress of 25 MPa, such that when it is associated with the major tensile stress, it tends to rotate the element in the clockwise direction. What is the magnitude of the normal and shear stresses on a section inclined at an angle of 20° with the major tensile stress?*

Given:

Tensile stress in horizontal *x-x* direction (σx) = 250 MPa

Tensile stress in vertical *y-y* direction (σy) = 100 MPa

Shear stress (τxy) = 25 MPa and angle made by section with the major tensile stress (θ) = 20°.

Magnitude of normal stress

We know that magnitude of normal stress,

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$= \frac{250+100}{2} - \frac{250-100}{2}\cos(2\times 20^\circ) - 25\sin(2\times 20^\circ)$$
$$= 175 - 75\cos 40^\circ - 25\sin 40^\circ \text{ MPa}$$
$$= 175 - (75\times 0.766) - (25\times 0.6428) \text{ MPa}$$
$$= 175 - 57.45 - 16.07 = 101.48 \text{ MPa} \quad \textbf{Ans.}$$

Magnitude of shear stress

We also know that magnitude of shear stress,

$$\tau = \frac{\sigma_x-\sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$
$$= \frac{250-100}{2}\sin(2\times 20^\circ) - 25\cos(2\times 20^\circ)$$
$$= 75\sin 40^\circ - 25\cos 40^\circ \text{ MPa}$$
$$= (75\times 0.6428) - (25\times 0.766) \text{ MPa}$$
$$= 48.21 - 19.15 = 29.06 \text{ MPa} \quad \textbf{Ans.}$$

Graphical Method for the Stresses on an Oblique Section of a Body

The Mohr's Circle of Stresses for the following cases:

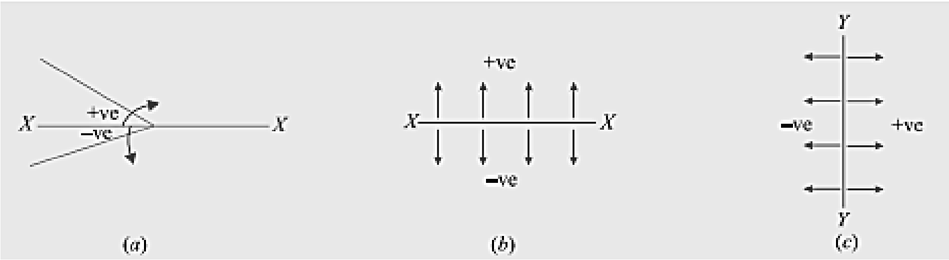
1. A body subjected to a direct stress in one plane.

2. A body subjected to direct stresses in two mutually perpendicular directions.

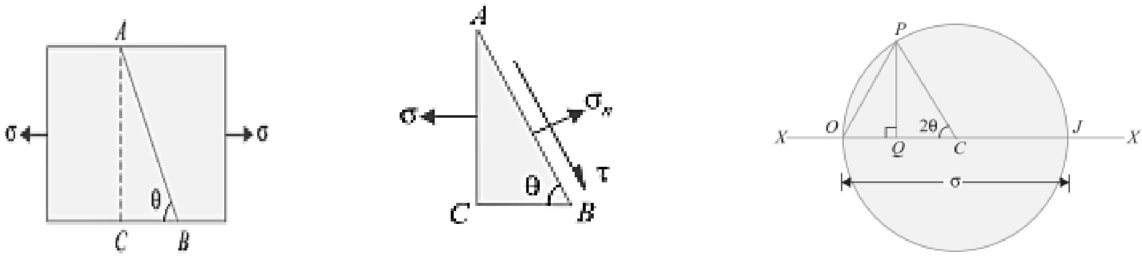
3. A body subjected to a simple shear stress.

4. A body subjected to a direct stress in one plane accompanied by a simple shear stress.

5. A body subjected to direct stresses in two mutually perpendicular directions accompanied by a simple shear stress.



Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane



Proof

From the geometry of the Mohr's Circle of Stresses, we find that,

$$OC = CJ = CP = \sigma/2 \qquad \text{... (Radius of the circle)}$$

∴ Normal Stress.

$$\sigma_n = OQ = OC - QC = \left(\frac{\sigma}{2}\right) - \left(\frac{\sigma}{2}\right)\cos 2\theta \qquad \text{...(Same as in Art. 7.7)}$$

and shear stress

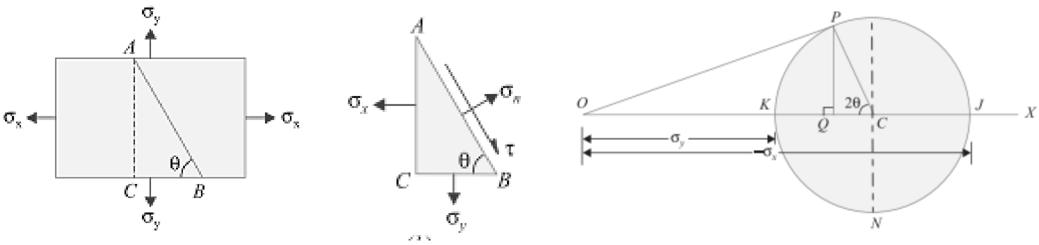
$$\tau = QP = CP \sin 2\theta = \frac{\sigma}{2}\sin 2\theta \qquad \text{...(Same as in Art. 7.7)}$$

We also find that maximum shear stress will be equal to the radius of the Mohr's Circle of Stresses *i.e.*, $\frac{\sigma}{2}$. It will happen when 2θ is equal to 90° or 270° *i.e.*, θ is equal to 45° or 135°.

However when θ = 45° then the shear stress is equal to $\frac{\sigma}{2}$.

And when θ = 135° then the shear stress is equal to $-\frac{\sigma}{2}$.

Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Direction



Proof

From the geometry of the Mohr's Circle of Stresses, we find that

$$KC = CJ = CP = \frac{\sigma_x - \sigma_y}{2}$$

or $$OC = OK + KC = \sigma_y + \frac{\sigma_x - \sigma_y}{2} = \frac{2\sigma_y + \sigma_x - \sigma_y}{2} = \frac{\sigma_x + \sigma_y}{2}$$

$\therefore$ Normal stress, $$\sigma_n = OQ = OC - CQ = \frac{\sigma_x - \sigma_y}{2} - CP\cos 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad \text{...(Same as Art. 7.8)}$$

and shear stress, $$\tau = QP = CP\sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2}\sin 2\theta \qquad \text{...(Same as Art. 7.8)}$$

We also find that the maximum shear stress will be equal to the radius of the Mohr's Circle of Stresses. *i.e.*, $\frac{\sigma_x - \sigma_y}{2}$. It will happen when 2θ is equal to 90° or 270° *i.e.*, when θ is equal to 45° or 135°.

However when $\theta = 45°$ then the shear stress is equal to $\frac{\sigma_x - \sigma_y}{2}$

And when $\theta = 135°$ then the shear stress will be equal to $\frac{-(\sigma_x - \sigma_y)}{2}$ or $\frac{\sigma_y - \sigma_x}{2}$.

Example The *stresses at a point of a machine component are 150 MPa and 50 MPa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress. Also find the magnitude of the maximum shear stresses in the component.*

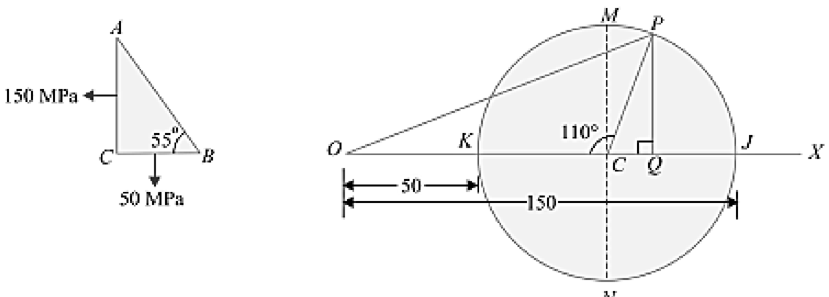
Given:

Tensile stress along horizontal *x-x* axis (s*x*) = 150 MPa

Tensile stress along vertical *y-y* axis (s*y*) = 50 MPa and

Angle made by the plane with the axis of major tensile stress (θ) = 55°.

The given stresses on the planes *AC* and *BC* in the machine component are shown.

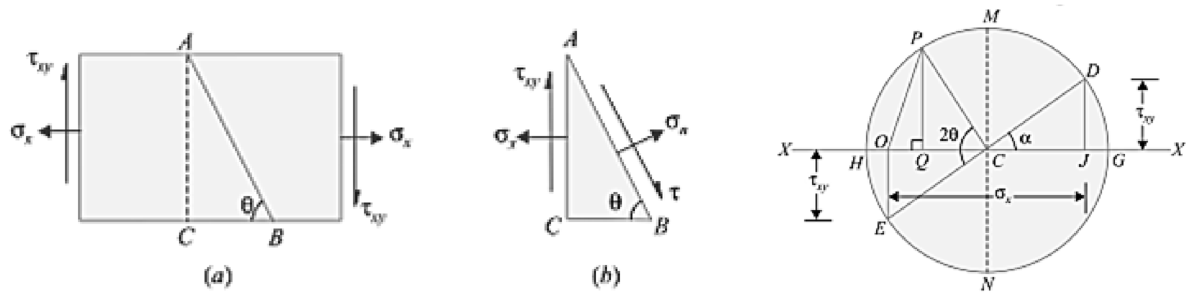


(*a*) (*b*)

1. First of all, take some suitable point *O* and draw a horizontal line *OX*.
2. Cut off *OJ* and *OK* equal to the tensile stresses σ_x and σ_y respectively (*i.e.* 150 MPa and 50 MPa) to some suitable scale towards right. The point *J* represents the stress system on the plane *AC* and the point *K* represents the stress system on the plane *BC*. Bisect *KJ* at *C*.
3. Now with *C* as centre and radius equal to *CJ* or *CK* draw the Mohr's Circle of Stresses.
4. Now through *C* draw two lines *CM* and *CN* at right angles to the line *OX* meeting the circle at *M* and *N*. Also through *C* draw a line *CP* making an angle of 2 × 55° = 110° with *CK* in clockwise direction meeting the circle at *P*. The point *P* represents the stress system on the plane *AB*.
5. Through *P*, draw *PQ* perpendicular to the line *OX*. Join *OP*.

   By measurement, we find that the normal stress (σ_n) = *OQ* = 117.1 MPa ; Shear stress (τ) = *QP* = 47.0 MPa ; Resultant stress (σ_R) = *OP* = 126.2 MPa and maximum shear stress (τ_{max}) = *CM* = ± 50 MPa **Ans.**

Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to a Direct Stresses in One Plane Accompanied by a Simple Shear Stress



Proof

From the geometry of the Mohr's Circle of Stresses, we find that

$$OC = \frac{\sigma_x}{2}$$

and radius of the circle,

$$R = EC = CD = CP = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Now in the right angled triangle DCJ,

$$\sin\alpha = \frac{DJ}{CD} = \frac{\tau_{xy}}{R} \qquad \text{and} \qquad \cos\alpha = \frac{JC}{CD} = \frac{\sigma_x}{2}\times\frac{1}{R} = \frac{\sigma_x}{2R}$$

and similarly in right angled triangle CPQ,

$$\angle PCQ = (2\theta - \alpha)$$

$$\begin{aligned}
\therefore \quad CQ &= CP\cos(2\theta-\alpha) = R\,[\cos(2\theta-\alpha)] \\
&= R\,[\cos\alpha\cos 2\theta + \sin\alpha\sin 2\theta] \\
&= R\cos\alpha\cos 2\theta + R\sin\alpha\sin 2\theta \\
&= R\times\frac{\sigma_x}{2R}\cos 2\theta + R\times\frac{\tau_{xy}}{R}\sin 2\theta \\
&= \frac{\sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta
\end{aligned}$$

We know that normal stress across the section AB,

$$\begin{aligned}
\sigma_n &= OQ = OC - CQ = \frac{\sigma_x}{2} - \left(\frac{\sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right) \\
&= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad \text{...(Same as in Art. 7.10)}
\end{aligned}$$

and shear stress,

$$\begin{aligned}
\tau &= QP = CP\sin(2\theta-\alpha) = R\sin(2\theta-\alpha) \\
&= R\,(\cos\alpha\sin 2\theta - \sin\alpha\cos 2\theta) \\
&= R\cos\alpha\sin 2\theta - R\sin\alpha\cos 2\theta \\
&= R\times\frac{\sigma_x}{2R}\sin 2\theta - R\times\frac{\tau_{xy}}{2}\cos 2\theta \\
&= \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad \text{...(Same as in Art. 7.10)}
\end{aligned}$$

We also know that maximum stress,

$$\sigma_{max} = OG = OC + CG = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

and minimum stress

$$\sigma_{min} = OH = OC - CH = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

We also find that the maximum shear stress will be equal to the radius of the Mohr's circle of stresses *i.e.*, $\sqrt{\left(\frac{\sigma_x}{2}\right)^2+\tau_{xy}^2}$. It will happen when $(2\theta-\alpha)$ is equal to 90° or 270°.

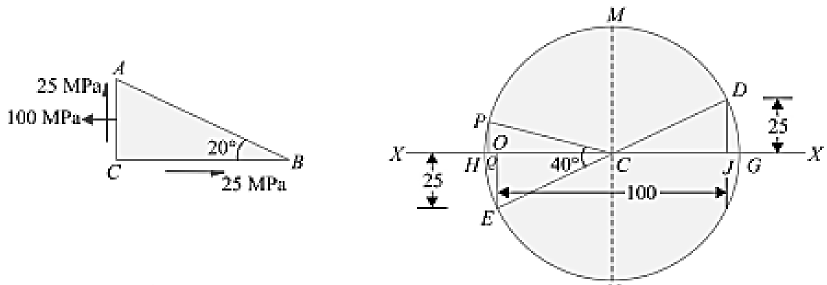
However when $(2\theta-\alpha)$ is equal to 90° then the shear stress is equal to $+\sqrt{\left(\frac{\sigma_x}{2}\right)^2+\tau_{xy}^2}$.

And when $(2\theta-\alpha)=270°$ then the shear stress is equal to $-\sqrt{\left(\frac{\sigma_x}{2}\right)^2+\tau_{xy}^2}$.

Example *A plane element in a body is subjected to a tensile stress of 100 MPa accompanied by a clockwise shear stress of 25 MPa. Find (i) the normal and shear stress on a plane inclined at an angle of 20° with the tensile stress; and (ii) the maximum shear stress on the plane.*

Given:

Tensile stress along horizontal *x-x* axis (σx) = 100 MPa
Shear stress (τxy) = 25 MPa and
angle made by plane with tensile stress (θ) = 20°

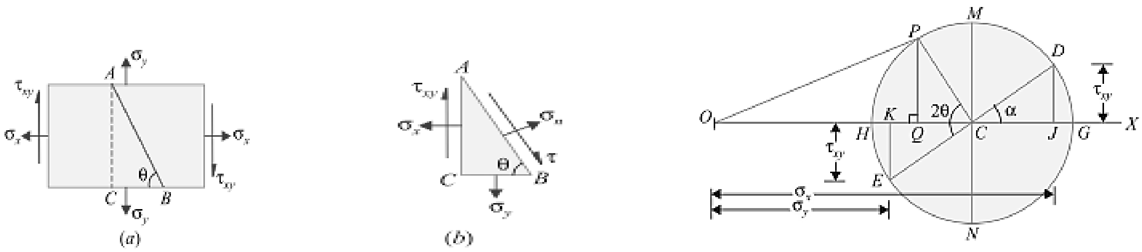


(*a*) (*b*)

1. First of all, take some suitable point *O*, and through it draw a horizontal line *XOX*.
2. Cut off *OJ* equal to the tensile stress on the plane *AC* (*i.e.*, 100 MPa) to some suitable scale towards right.
3. Now erect a perpendicular at *J* above the line *X-X* and cut off *JD* equal to the positive shear stress on the plane *BC* (*i.e.*, 25 MPa) to the scale. The point *D* represents the stress system on the plane *AC*. Similarly erect a perpendicular at *O* below the line *X-X* and cut off *OE* equal to the negative shear stress on the plane *BC* (*i.e.*, 25 MPa) to the scale. The point *E* represents the stress system on the plane *BC*. Join *DE* and bisect it at *C*.
4. Now with *C* as centre and radius equal to *CD* or *CE* draw the Mohr's Circle of Stresses.
5. Now through *C*, draw two lines *CM* and *CN* at right angle to the line *OX* meeting the circle at *M* and *N*. Also through *C*, draw a line *CP* making an angle of 2 × 20° = 40° with *CE* in clockwise direction meeting the circle at *P*. The point *P* represents the stress system on the section *AB*.
6. Through *P*, draw *PQ* perpendicular to the line *OX*.

   By measurement, we find that the normal stress (σ_n) = *OQ* = 4.4 MPa (compression) ; Shear stress (τ) = *QP* = 13.0 MPa and maximum shear stress (τ_{max}) = *CM* = 55.9 MPa **Ans.**

Mohr's Circle for Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress



Proof

From the geometry of the Mohr's Circle of Stresses, we find that

$$OC = \frac{\sigma_x + \sigma_y}{2}$$

and radius of the circle

$$R = EC = CD = CP = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Now in the right angled triangle *DCJ*

$$\sin\alpha = \frac{JD}{DC} = \frac{\tau_{xy}}{R} \quad \text{and} \quad \cos\alpha = \frac{JD}{DC} = \frac{\sigma_x - \sigma_y}{2} \times \frac{1}{R} = \frac{\sigma_x - \sigma_y}{2R}$$

Similarly in right angled triangle *CPQ*

$$\therefore \quad \angle PCQ = (2\theta - \alpha)$$

$$\begin{aligned}
CQ &= CP \cos 2\theta - \alpha \\
&= R\,[\cos(2\theta - \alpha)] \\
&= R\,[\cos\alpha \cos 2\theta + \sin\alpha \sin 2\theta] \\
&= R\cos\alpha \cos 2\theta + R\sin\alpha \sin 2\theta \\
&= R \times \frac{\sigma_x - \sigma_y}{2R} \cos 2\theta + R \times \frac{\tau_{xy}}{R} \sin 2\theta \\
&= \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\end{aligned}$$

Normal Stress (across the inclined section *AB*)

$$\sigma_n = OQ = OC - CQ$$

or

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

...(Same as in Art. 7.11)

Shear Stress or Tangential Stress (across the inclined section *AB*)

$$\begin{aligned}
\tau &= QP = CP \sin[(2\theta - \alpha)] = R \sin(2\theta - \alpha) \\
&= R(\cos\alpha \sin 2\theta - \sin\alpha \cos 2\theta) \\
&= R\cos\alpha \sin 2\theta - R\sin\alpha \cos 2\theta \\
&= R \times \frac{\sigma_x - \sigma_y}{2R} \sin 2\theta - R \times \frac{\tau_{xy}}{R} \cos 2\theta
\end{aligned}$$

or

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau_{xy} \cos 2\theta$$

...(Same as in Art. 7.11)

Maximum Principal Stress

$$\sigma_{max} = OG = OC + CG = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum Principal Stress

$$\sigma_{min} = OH = OC - CH = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

We also find the maximum shear stress will be equal to the radius of the Mohr's circle of Stresses.

i.e., $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$. It will happen when $(2\theta - \alpha)$ is equal to 90° or 270°.

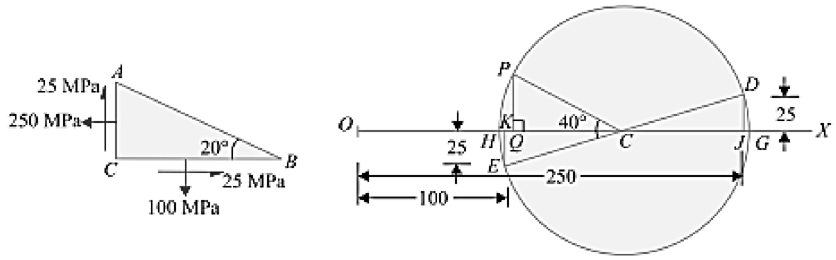
However when $(2\theta - \alpha) = 90°$ then the shear stress is equal to $+\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

And when $(2\theta - \alpha) = 270°$ then the shear stress is equal to $-\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

Example *A point is subjected to a tensile stress of 250 MPa in the horizontal direction and another tensile stress of 100 MPa in the vertical direction. The point is also subjected to a simple shear stress of 25 MPa, such that when it is associated with the major tensile stress, it tends to rotate the element in the clockwise direction. What is the magnitude of the normal and shear stresses inclined on a section at an angle of 20° with the major tensile stress?*

Given:

Tensile stress in horizontal direction (σx) = 250 MPa
Tensile stress in vertical direction (σy) = 100 MPa
Shear stress (τ) = 25 MPa and
angle made by section with major tensile stress (θ) = 20°



(*a*) (*b*)

The given stresses on the face *AC* of the point alongwith a tensile stress on the plane *BC* and a complimentary shear stress on the plane *BC* are shown in Fig 7.27 (*a*). Now draw the Mohr's Circle of Stresses as shown in Fig. 7.27 (*b*) and as discussed below :

1. First of all, take some suitable point *O*, and through it draw a horizontal line *OX*.
2. Cut off *OJ* and *OK* equal to the tensile stresses σ_x and σ_y respectively (*i.e.*, 250 MPa and 100 MPa) to some suitable scale towards right.
3. Now erect a perpendicular at *J* above the line *OX* and cut off *JD* equal to the positive shear stress on the plane *AC* (*i.e.*, 25 MPa) to the scale. The point *D* represents the stress system on the plane *AC*. Similarly, erect a perpendicular at *K* below the *OX* and cut off *KE* equal to the negative shear stress on the plane *BC* (*i.e.*, 25 MPa) to the scale. The point *E* represents the stress system on the plane *BC*. Join *DE* and bisect it at *C*.
4. Now with *C* as centre and radius equal to *CD* or *CE* draw the Mohr's Circle of Stresses.
5. Now through *C* draw a line *CP* making an angle of 2 × 20° = 440° with *CE* in clockwise direction meeting the circle at *P*. The point *P* represents the stress system on the section to *AB*.
6. Through *P*, draw *PQ* perpendicular to the line *OX*.

   By measurement, we find that the normal stress, $(\sigma_x) = OQ = 101.5$ MPa and shear stress $\tau = QP$ = 29.0 MPa **Ans.**

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DEPARTMENT OF AUTOMOBILE ENGINEERING

SAUA1304 _ SOLID AND FLUID MECHANICS

UNIT II BENDING AND SHEAR STRESS DISTRIBUTION, TORSION AND BEAMS

UNIT 2 BENDING AND SHEAR STRESS DISTRIBUTION, TORSION AND BEAMS

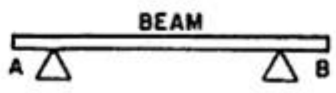
Stresses in Beams - Simple bending theory: bending stresses in Symmetrical and Unsymmetrical sections - Composite Beams - Combined bending and Direct stress – Shear Stress Distribution for Different Sections - Simple Torsion theory - Stresses and deformations in Solid and Hollow circular shafts- Double integration method – Shear force and bending moment diagram: Simply supported, Cantilever and overhanging beam – various loading condition

A beam may be defined as a structural element which has one dimension considerably larger than the other two dimensions, namely breadth and depth, and is supported a few points. The distance between two adjacent supports is called span. It is usually loaded normal to its axis. The applied loads make every cross-section to face bending and shearing.

The load finally gets transferred to supports. The system of forces consisting of applied loads and reactions keep the beam in equilibrium. The reactions depend upon the type of supports and type of loading. The types of beams are:

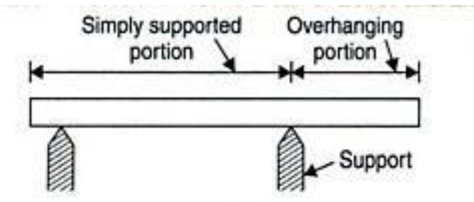
TYPES OF BEAMS

Simple supported beam: A beam supported or resting freely on the supports at its both ends, is known as simply supported beam.

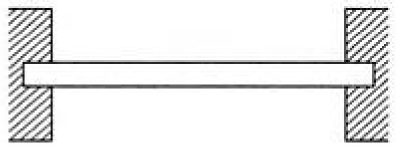


Cantilever beam: A beam which is fixed at one end and free at the other end is known as cantilever beam.

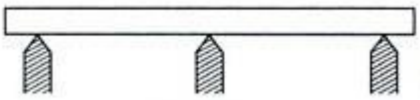
Over hanging beam: If the end portion of a beam is extended beyond the support such beam is known as Overhanging beam



Fixed beam: A beam whose both ends are fixed or built in walls is known as fixed beam.



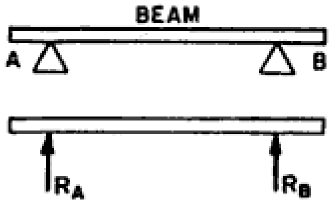
Continuous beam: A beam which is provided more than two supports is known as continuous beam.



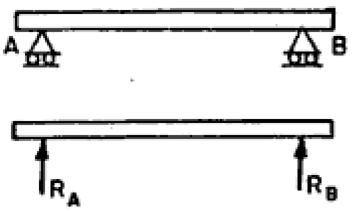
TYPES OF SUPPORTS

The Various types of supports and reactions developed are listed below:

Simple supports or knife edged support: in this case support will be normal to the surface of the beam. If AB is a beam with knife edges A and B, then R_A and R_B will be the reaction.

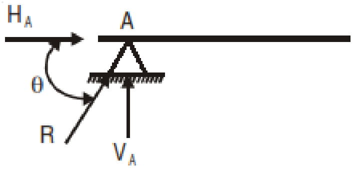


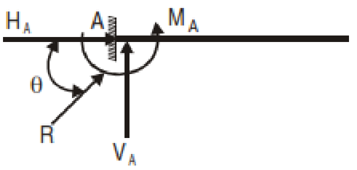
Roller support: here beam AB is supported on the rollers. The reaction will be normal to the surface on which rollers are placed.



Pin joint (or hinged) support: here the beam AB is hinged at point A. the reaction at the hinged end may be either vertical or inclined depending upon the type of loading. If load is vertical, then the reaction will also be vertical. But if the load is inclined, then the reaction at the hinged end will also be inclined.

Fixed or built-in support: in this type of support the beam should be fixed. The reaction will be inclined. Also the fixed support will provide a couple.



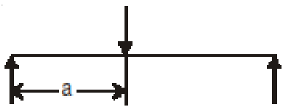


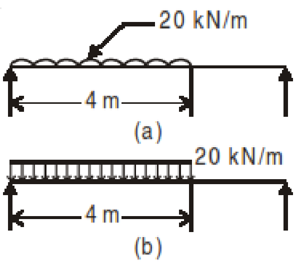
Types of Loading

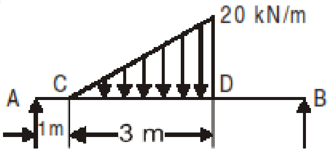
Concentrated Loads: If a load is acting on a beam over a very small length, it is approximated as acting at the midpoint of that length and is represented by an arrow as shown

Uniformly Distributed Load (UDL): Over considerably long distance such load has got uniform intensity. For finding reaction, this load may be assumed as total load acting at the centre of gravity of the loading (middle of the loaded length). For example, in the beam the load may be replaced by a $20 \times 4 = 80$ kN concentrated load acting at a distance 2 m from the left support.

Uniformly Varying Load: The load varies uniformly from *C* to *D*. Its intensity is zero at *C* and is 20kN/m at *D*. In the load diagram, the ordinate represents the load intensity and the abscissa represents the position of load on the beam.

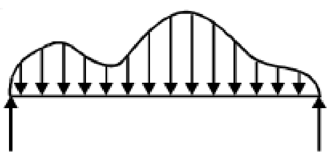


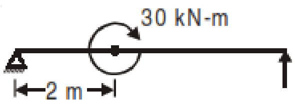




General Loadings: *T*he ordinate represents the intensity of loading and abscissa represents position of the load on the beam. For simplicity in analysis such loadings are replaced by a set of equivalent concentrated loads.

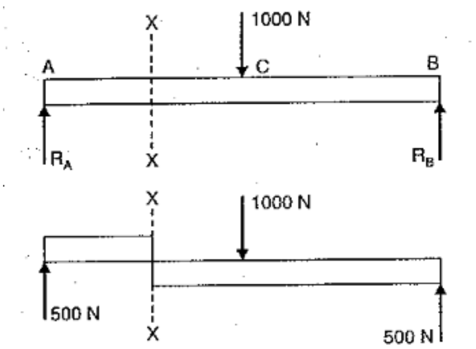
External Moment: A beam may be subjected to external moment at certain points. The beam is subjected to clockwise moment of 30 kN-m at a distance of 2 m from the left support.

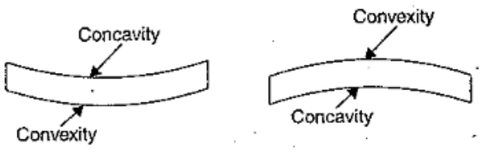




Concept and significance of shear force and bending moment Sign conventions for shear force and bending moment

Shear force: A simply supported beam AB. carrying a load of 1000 N at its middle point. The reactions at the supports will be equal to 500 N. Hence $R_A = R_B = 500$ N. Now imagine the beam to be divided into two portions by the section X-X. The resultant of the load and reaction to the left of X-X is 500 N vertically upwards. And the resultant of the load and reaction to the right of X-X is (1000↓ -500 ↑= 500↓N) 500 N downwards. The resultant force acting on any one of the parts normal to the axis of the beam is called the shear force at the section X-X is 500N.

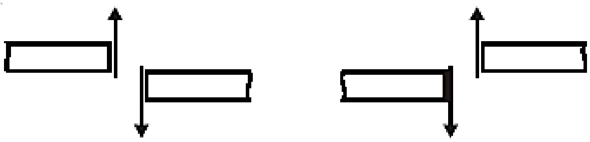




(a) Positive B.M. (b) Negative B.M.

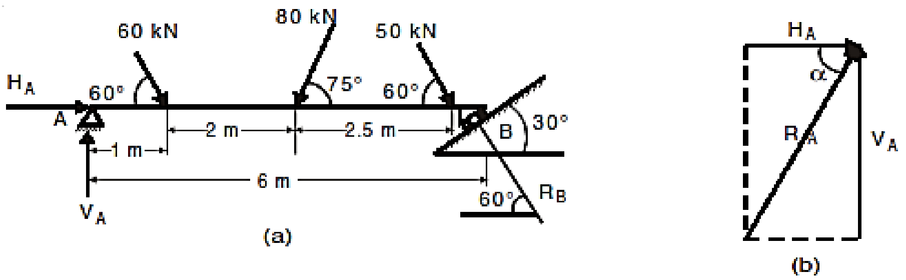
The shear force at a section will be considered positive when the resultant of the forces to the left to the section is upwards, or to the right of the section is downwards. Similarly the shear force at a section will be considered negative if the resultant of the forces to the left of the section is downward, or to the right of the section is upwards. Here the resultant force to the left of the section is upwards and hence the shear force will be positive.

Bending moment: The bending moment at a section is considered positive if the bending moment at that section is such that it tends to bend the beam to a curvature having concavity at the top. Similarly the bending moment at a section is considered negative if the bending moment at that section is such that it tends to bend the beam to a curvature haling convexity at the top. The positive B.M. is often called sagging moment and negative B.M. as hogging Moment.



+ve S.F. – ve S.F.

Example: Find the reactions at supports A and B in the beam AB shown.



Solution: The reaction at *B* will be at right angles to the support, *i.e.,* at 60° to horizontal as shown in the figure. Let the components of the reactions at *A* be H_A and V_A. Then

$$\Sigma M_A = 0 \text{ gives}$$

$$R_B \sin 60° \times 6 - 60 \sin 60° \times 1 - 80 \times \sin 75° \times 3 - 50 \times \sin 60° \times 5.5 = 0$$

$$\therefore \quad \mathbf{R_B = 100.4475\ kN.}$$

$$\Sigma H = 0, \text{ gives}$$

$$H_A + 60 \cos 60° - 80 \cos 75° + 50 \cos 60° - R_B \cos 60° = 0$$

$$H_A = -60 \cos 60° + 80 \cos 75° - 50 \cos 60° + 100.4475 \cos 60°$$
$$= 15.9293 \text{ kN}$$

$$\Sigma V = 0, \text{ gives}$$

$$V_A + R_B \sin 60° - 60 \sin 60° - 80 \sin 75° - 50 \sin 60° = 0$$

$$V_A = -100.4475 \sin 60° + 60 \sin 60° + 80 \sin 75° + 50 \sin 60°$$
$$= 85.5468 \text{ kN}$$

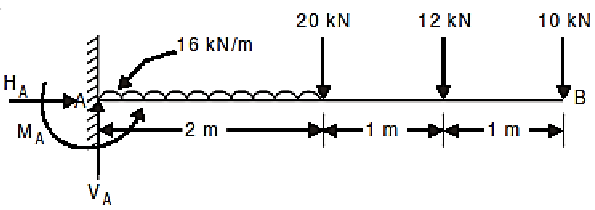
$$\therefore \quad R_A = \sqrt{15.9293^2 + 85.5468^2}$$

i.e., $$\mathbf{R_A = 87.0172\ kN.}$$

$$\alpha = \tan^{-1} \frac{85.5468}{15.9293}$$

i.e., $\boldsymbol{\alpha} = \mathbf{79.45°}$, as shown in Fig. 9.18(*b*).

Example: The cantilever is fixed at A and is free at B. Determine the reactions, when it is loaded as shown



Solution: Let the reactions at A be H_A, V_A and M_A as shown in the figure

Now $\Sigma H = 0$, gives

$$\boldsymbol{H_A = 0.}$$

$$\Sigma V = 0, \text{ gives}$$

$$V_A - 16 \times 2 - 20 - 12 - 10 = 0$$

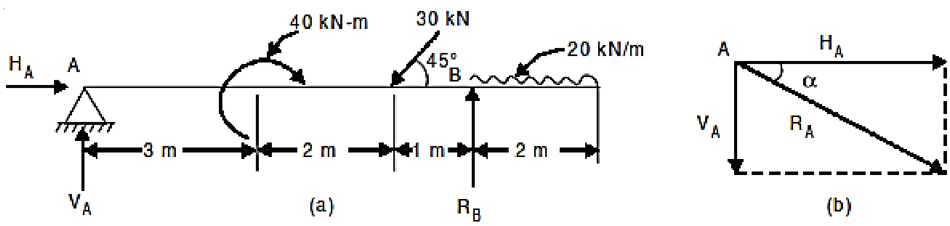
$$\therefore \quad \boldsymbol{V_A = 74 \text{ kN.}}$$

$$\Sigma M = 0, \text{ gives}$$

$$M_A - 16 \times 2 \times 1 - 20 \times 2 - 12 \times 3 - 10 \times 4 = 0$$

$$\therefore \quad \boldsymbol{M_A = 148 \text{ kN-m.}}$$

Example: Determine the reactions at A and B of the overhanging beam shown



Solution:

$$\sum M_A = 0$$

$$R_B \times 6 - 40 - 30 \sin 45° \times 5 - 20 \times 2 \times 7 = 0$$

$$\boldsymbol{R_B = 71.0110 \text{ kN.}}$$

$$\sum H = 0$$

$$H_A = 30 \cos 45° = 21.2132 \text{ kN}$$

$$\sum V = 0$$

$$V_A - 30 \sin 45° + R_B - 20 \times 2 = 0$$

$$V_A = 30 \sin 45° - R_B + 40$$

$$V_A = -9.7978$$

(Negative sign show that the assumed direction of V_A is wrong. In other words, V_A is acting vertically downwards).

$$R_A = \sqrt{V_A^2 + H_A^2}$$

$$\boldsymbol{R_A = 23.3666 \text{ kN.}}$$

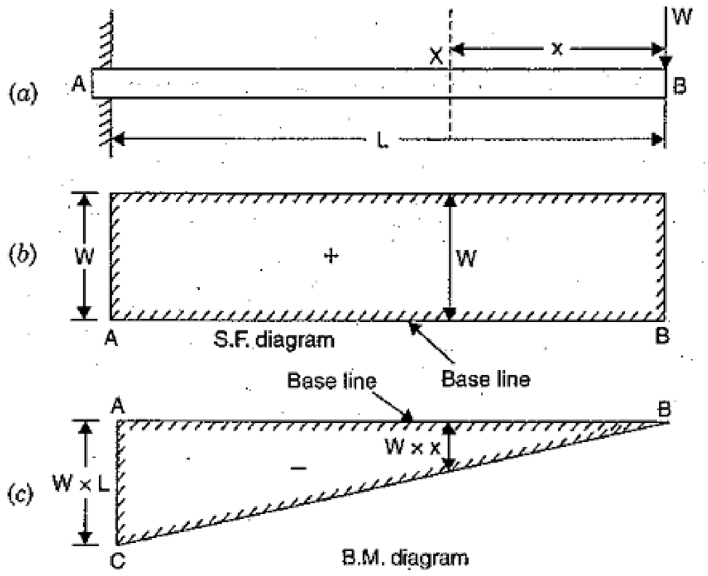
$$\alpha = \tan^{-1} \frac{V_A}{H_A}$$

$\boldsymbol{\alpha = 24.79°}$, as shown in Fig. 9.24(**b**).

Shear force and Bending moment diagram

The following are the important points for drawing shear force and bending moment diagrams:

1. Consider the left or the right portion of the section.
2. Add the forces (including reaction) normal to the beam on one of the portion. If right portion of the section is chosen, a force on the right portion acting downwards is positive while force acting upwards is negative.
3. If the left portion of the section is chosen, a force on the left portion acting upwards is positive while force acting downwards is negative.
4. The positive values of shear force and bending moments are plotted above the base line, and negative values below the base line.
5. The shear force diagram will increase or decrease suddenly i.e., by a vertical straight line at a section where there is a vertical point load.
6. The shear force between any two vertical loads will be constant and hence the shear force diagram between two vertical loads will be horizontal.
7. The bending moment at the two supports of a simply supported beam and at the free end of a cantilever will be zero.


W
X
x
(a)
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+
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S.F. diagram
B
Base line
Base line
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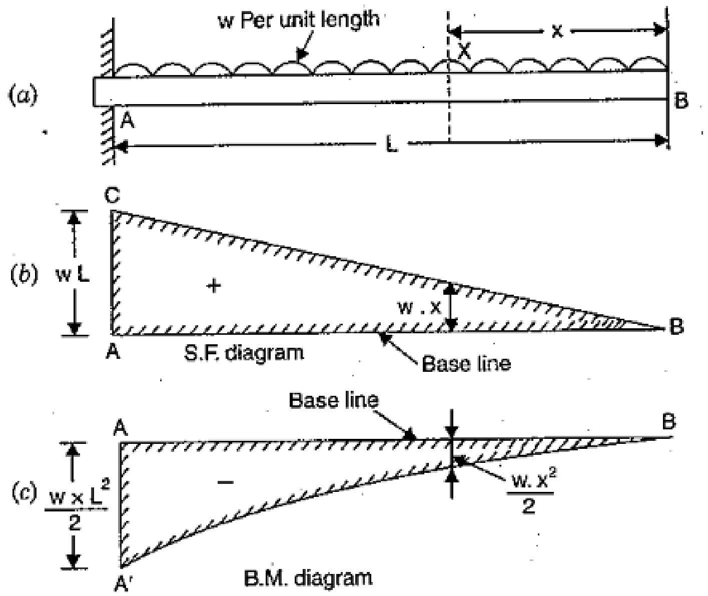
W × L
C
B.M. diagram


w Per unit length
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S.F. diagram
Base line
Base line
A
B
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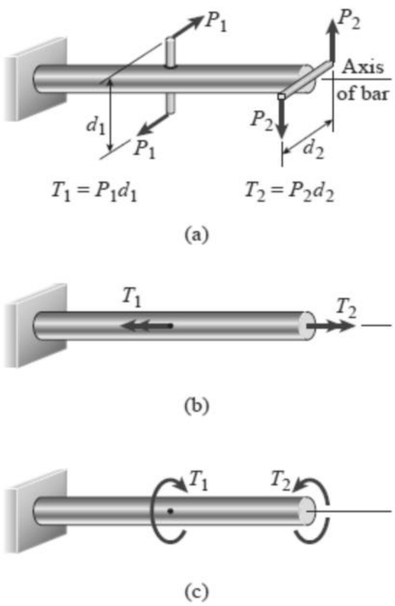
(c)
w × L²/2
w. x²/2
A′
B.M. diagram

INTRODUCTION: TORSION

In machinery, the general term “shaft” refers to a member, usually of circular cross section, which supports gears, sprockets, wheels, rotors, etc., and which is subjected to torsion and to transverse or axial loads acting singly or in combination. An “axle” is a rotating/non-rotating member that supports wheels, pulley and carries no torque. A “spindle” is a short shaft. Terms such as line shaft, head shaft, stub shaft, transmission shaft, countershaft, and flexible shaft are names associated with special usage.

Analysis of torsion

In a slender member under the action of a torsional moment (also called twisting moment or torque) shearing stresses appear, whose moment about the bar axis is equal to the applied torque. In the same way as the shearing stresses caused by the shear force, these stresses must be tangent to the contour in the points lying close the boundary of the cross-section. These two conditions are not sufficient to determine the distribution of shearing stresses in the cross-section. Furthermore, the twisting moment is not a symmetrical loading with respect to the middle cross-section of a piece of bar.



An idealized case of torsional loading is a straight bar supported at one end and loaded by two pairs of equal and opposite forces. The first pair consists of the forces P1 acting near the midpoint of the bar and the second pair consists of the forces P2 acting at the end. Each pair of forces forms a couple that tends to twist the bar about its longitudinal axis. As we know from statics, the moment of a couple is equal to the product of one of the forces and the perpendicular distance between the lines of action of the forces; thus, the first couple has a moment T1 = P1d1 and the second has a moment T2 = P2d2.

Torsion refers to the twisting of a straight bar when it is loaded by moments (or torques) that tends to produce rotation about the longitudinal axis of the bar. For instance, when you turn a screwdriver, your hand applies a torque *T* to the handle and twists the shank of the screwdriver. Other examples of bars in torsion are drive shafts in automobiles, axles, propeller shafts, steering rods, and drill bits.

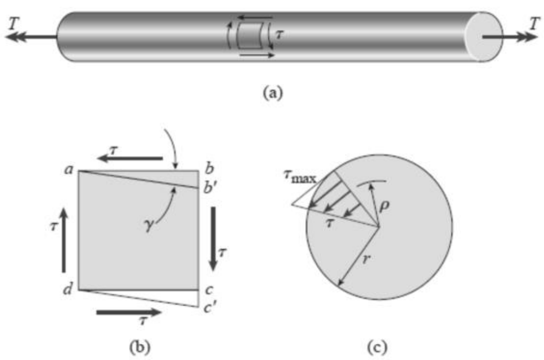
The moment of a couple may be represented by a **vector** in the form of a double-headed arrow. The arrow is perpendicular to the plane containing the couple, and therefore in this case both arrows are parallel to the axis of the bar. The direction (or *sense*) of the moment is indicated by the *right-hand rule* for moment vectors—namely, using your right hand, let your fingers curl in the direction of the moment, and then your thumb will point in the direction of the vector. An alternative representation of a moment is curved arrow acting in the direction of rotation. The choice depends upon convenience and personal preference. Moments that produce twisting of a bar, such as the moments *T*1 and *T*2, are called **torques** or **twisting moments**. Cylindrical members that are subjected to torques and transmit power through rotation are called **shafts**; for instance, the drive shaft of an automobile or the propeller shaft of a ship. Most shafts have circular cross sections, either solid or tubular. In this chapter we begin by developing formulas for the deformations and stresses in circular bars subjected to torsion. We then analyze the state of stress known as *pure shear* and obtain the relationship between the moduli of elasticity *E* and *G* in tension and shear, respectively. Next, we analyze rotating shafts and determine the power they transmit. Finally, we cover several additional topics related to torsion, namely, statically indeterminate members, strain energy, thin-walled tubes of noncircular cross section, and stress concentrations.

Torsional deformations of a circular bar

A prismatic bar with a circular cross-section has a symmetrical geometry with respect to any plane passing through the bar axis. If, in addition, the material also has symmetrical rheological properties with respect to these planes, which happens if the material is isotropic or monotropic with the monotropy direction parallel to the bar axis, the bar is totally symmetric with respect to the bar axis, i.e., it is axisymmetric. As a consequence of this type of symmetry, all the points of a cross-section lying on a circumference with the centre in the bar axis, are in the same conditions with respect to the centre of the cross-section. If we consider a vector applied at the centre of the cross-section, representing the torque acting on the bar, all the points of that circumference are also in the same conditions with respect to that vector. As a consequence, all the points will undergo the same displacement in relation to the bar axis, i.e., the radial, circumferential and longitudinal components of the displacement will be the same in all points of the circumference. This means that the circumference will remain on a plane perpendicular to the bar axis and that its centre will remain on that axis.

The shear strains in a circular bar in torsion, we are ready to determine the directions and magnitudes of the corresponding shear stresses. The directions of the stresses can be determined by inspection. We observe that the torque T tends to rotate the right-hand end of the bar counterclockwise when viewed from the right. The magnitudes of the shear stresses can be determined from the strains by using the stress-strain relation for the material of the bar. If the material is linearly elastic, we can use Hooke’s law in shear, in which G is the shear modulus of elasticity and γ is the shear strain in radians. Combining this equation with the equations for the shear strains, in which τ max is the shear stress at the outer surface of the bar (radius r), τ is the shear stress at an interior point (radius r), and θ is the rate of twist. (In these equations, θ has units of radians per unit of length.)

$$\tau = G\gamma$$



$$\tau_{max} = Gr\theta \qquad \tau = G\rho\theta = \frac{\rho}{r}\tau_{max}$$

Equations show that the shear stresses vary linearly with the distance from the center of the bar, illustrated by the triangular stress diagram. This linear variation of stress is a consequence of Hooke's law. If the stress-strain relation is nonlinear, the stresses will vary nonlinearly and other methods of analysis will be needed.

The shear stresses acting on a cross-sectional plane are accompanied by shear stresses of the same magnitude acting on longitudinal planes. This conclusion follows from the fact that equal shear stresses always exist on mutually perpendicular planes. If the material of the bar is weaker in shear on longitudinal planes than on cross-sectional planes, as is typical of wood when the grain runs parallel to the axis of the bar, the first cracks due to torsion will appear on the surface in the longitudinal direction. The state of pure shear at the surface of a bar is equivalent to equal tensile and compressive stresses acting on an element oriented at an angle of 45. Therefore, a rectangular element with sides at 45° to the axis of the shaft will be subjected to tensile and compressive stresses. If a torsion bar is made of a material that is weaker in tension than in shear, failure will occur in tension along a helix inclined at 45° to the axis.

Torsion of circular shafts

Equation for shafts subjected to torsion "T"

$$\frac{\tau}{R} = \frac{T}{J} = \frac{G\theta}{L}$$

Torsion Equation

Where J = Polar moment of inertia,τ = Shear stress induced due to torsion T.

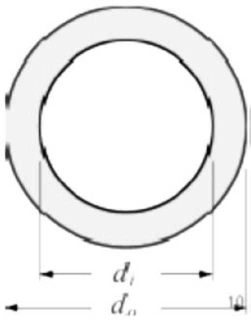
G = Modulus of rigidity,θ = Angular deflection of shaft, R, L = Shaft radius & length respectively.

Assumptions

- The bar is acted upon by a pure torque.
- The section under consideration is remote from the point of application of the load and from a change in diameter.
- Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.
- The material obeys Hooke's law
- Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle

Polar moment of Inertia

As stated above, the polar second moment of area, J is defined as



$$J = \int_0^R 2\pi r^3 \, dr$$

For a solid shaft
$$J = 2\pi \left[\frac{r^4}{4}\right]_0^R = \frac{2\pi R^4}{4} = \frac{\pi D^4}{32}$$

For a hollow shaft of internal radius r:

$$J = \int_0^R 2\pi r^3 \, dr = 2\pi \left[\frac{r^4}{4}\right]_r^R = \frac{\pi}{2}(R^4 - r^4) = \frac{\pi}{32}(D^4 - d^4)$$

Where D is the external and d is the internal diameter.

- Solid shaft "J" = $\frac{\pi d^4}{32}$
- Hollow shaft, "J" = $\frac{\pi}{32}(d_o^4 - d_i^4)$

Polar section Modulus

$$Z_p = J / c, \text{ where } c = r = D/2$$

For a solid circular cross-section, $Z_p = \pi D^3 / 16$

For a hollow circular cross-section, $Z_p = \pi (D_o^4 - D_i^4) / (16D_o)$

Then, $\tau_{max} = T / Z_p$

If design shears stress, τ_d is known, required polar section modulus can be calculated from:

$Z_p = T / \tau_d$

Polar Moment of Inertia and Section Modulus.

The polar moment of inertia, J, of a cross-section with respect to a polar axis, that is, an axis at right angles to the plane of the cross-section, is defined as the moment of inertia of the cross-section with respect to the point of intersection of the axis and the plane. The polar moment of inertia may be found by taking the sum of the moments of inertia about two perpendicular axes lying in the plane of the cross-section and passing through this point. Thus, for example, the polar moment of inertia of a circular or a square area with respect to a polar axis through the center of gravity is equal to two times the moment of inertia with respect to an axis lying in the plane of the cross-section and passing through the center of gravity. The polar moment of inertia with respect to a polar axis through the center of gravity is required for problems involving the torsional strength of shafts since this axis is usually the axis about which twisting of the shaft takes place.

The polar section modulus

(also called section modulus of torsion), Zp, for circular sections may be found by dividing the polar moment of inertia, J, by the distance c from the center of gravity to the most remote fiber. This method may be used to find the approximate value of the polar section modulus of sections that are nearly round. For other than circular cross-sections, however, the polar section modulus does not equal the polar moment of inertia divided by the distance c.

Power Transmission

$$P \text{ (in Watt)} = \frac{2\pi NT}{60}$$

P (in hp) $= \dfrac{2\pi NT}{4500}$ (1 hp = 75 Kgm/sec).

[Where N = rpm; T = Torque in N-m.]

Safe diameter of a shaft (d)

- Stiffness consideration

$$\frac{T}{J} = \frac{G\theta}{L}$$

- Shear Stress consideration

$$\frac{T}{J} = \frac{\tau}{R}$$

We take higher value of diameter of both cases above for overall safety if other parameters are given.

In Twisting

- Solid shaft, $\tau_{max} = \dfrac{16T}{\pi d^3}$
- Hollow shaft, $\tau_{max} = \dfrac{16Td_o}{\pi(d_o^4 - d_i^4)}$
- Diameter of a shaft to have a maximum deflection "α" d = $4.9 \times \sqrt[4]{\dfrac{TL}{G\alpha}}$

  [Where T in N-mm, L in mm, G in N/mm²]

Problems on Solid and hollow circular section

1. What torque, applied to a hollow circular shaft of 25 cm outside diameter and 17.5 cm inside
   diameter will produce a maximum shearing stress of 75 MN/m2 in the material.

We have

$$r_1 = 12.5 \text{ cm}, \qquad r_2 = 8.75 \text{ cm}$$

Then

$$J = \frac{\pi}{2}\left[(0.125)^4 - (0.0875)^4\right] = 0.292 \times 10^{-3} \text{ m}^4$$

If the shearing stress is limited to 75 MN/m^2, the torque is

$$T = \frac{J\tau}{r_1} = \frac{(0.292 \times 10^{-3})(75 \times 10^6)}{(0.125)} = 175.5 \text{ kNm}$$

2. A ship's propeller shaft has external and internal diameters of 25 cm and 15 cm. What power can be

transmitted at 1 10 rev/minute with a maximum shearing stress of 75 MN/m2, and what will then

be the twist in degrees of a 10 m length of the shaft? G = 80 GN/m2

$$r_1 = 0.125 \text{ m}, \qquad r_2 = 0.075 \text{ m}, \qquad l = 10 \text{ m}$$

$$J = \frac{\pi}{2}\left[(0.125)^4 - (0.075)^4\right] = 0.335 \times 10^{-3} \text{ m}^4$$

and

$$\tau = 75 \text{ MN/m}^2$$

Then

$$T = \frac{J\tau}{r_1} = \frac{(0.335 \times 10^{-3})(75 \times 10^6)}{0.125} = 201 \text{ kNm}$$

At 110 rev/min the power generated is

$$(201 \times 10^3)\left(2\pi \times \frac{110}{60}\right) = 2.32 \times 10^6 \text{ Nm/s}$$

The angle of twist is

$$\theta = \frac{TL}{GJ} = \frac{(201 \times 10^3)(10)}{(80 \times 10^9)(0.335 \times 10^{-3})} = 0.075 \text{ radians} = 4.3°$$

3. A solid circular shaft of 25 cm diameter is to be replaced by a hollow shaft, the ratio of the external to internal diameters being 2 to 1. Find the size of the hollow shaft if the maximum

shearing stress is to be the same as for the solid shaft. What percentage economy in mass will this change effect?

Let r be the inside radius of the new shaft; then $= 2r$ the outside radius of the new shaft

$$J \text{ for the new shaft} = \frac{\pi}{2}\left(16r^4 - r^4\right) = 7.5\pi r^4$$

$$J \text{ for the old shaft} = \frac{\pi}{2} \times (0.125)^4 = 0.384 \times 10^{-3} \text{ m}^4$$

If T is the applied torque, the maximum shearing stress for the old shaft is

$$\frac{T(0.125)}{0.384 \times 10^{-3}}$$

and that for the new one is

$$\frac{T(2r)}{7.5\pi r^4}$$

If these are equal,

$$\frac{T(0.125)}{0.384 \times 10^{-3}} = \frac{T(2r)}{7.5\pi r^4}$$

Then

$$r^3 = 0.261 \times 10^{-3} \text{ m}^3$$

or $\quad r = 0.640 \text{ m}$

Hence the internal diameter will be 0.128 m and the external diameter 0.256 m.

$$\frac{\text{area of new cross-section}}{\text{area of old cross-section}} = \frac{(0.128)^2 - (0.064)^2}{(0.125)^2} = 0.785$$

Thus, the saving in mass is about 21%.

4. A ship's propeller shaft transmits 7.5 x 106 W at 240 rev/min. The shaft has an internal diameter of 15 cm. Calculate the minimum permissible external diameter if the shearing stress in the shaft is to be limited to 150 MN/m2.

If T is the torque on the shaft, then

$$T\left(\frac{2\pi \times 240}{60}\right) = 7.5 \times 10^6$$

Thus

$$T = 298 \text{ kNm}$$

If d_1 is the outside diameter of the shaft, then

$$J = \frac{\pi}{32}\left(d_1^4 - 0.150^4\right) \text{ m}^4$$

If the shearing stress is limited to 150 MN/m², then

$$\frac{Td_1}{2J} = 150 \times 10^6$$

Thus,

$$Td_1 = \left(300 \times 10^6\right)J$$

On substituting for J and T

$$\left(298 \times 10^3\right)d_1 = \left(300 \times 10^6\right)\left(\frac{\pi}{32}\right)\left(d_1^4 - 0.150^4\right)$$

This gives

$$\left(\frac{d_1}{0.150}\right)^4 - 3\left(\frac{d_1}{0.150}\right) - 1 = 0$$

On solving this by trial-and-error, we get

$$d_1 = 1.54(0.150) = 0.231 \text{ m}$$

or $\quad d_1 = 23.1$ cm

Problems for practice

1. A solid steel bar of circular cross section has diameter d =1.5 in., length L =54 in., and shear modulus of elasticity G = 11.5 x10^6 psi. The bar is subjected to torques T acting at the ends.

(a) If the torques has magnitude T =250 lb-ft, what is the maximum shear stress in the bar? What is the angle of twist between the ends?

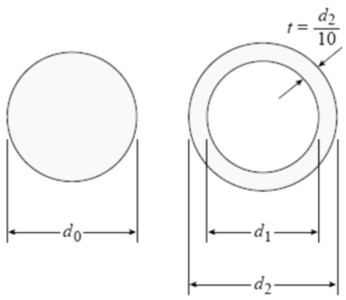
(b) If the allowable shear stress is 6000 psi and the allowable angle of twist is 2.5°, what is the maximum permissible torque?

2. A steel shaft is to be manufactured either as a solid circular bar or as a circular tube. The shaft is required to transmit a torque of 1200 N_m without exceeding an allowable shear stress of 40 MPa nor an allowable rate of twist of 0.75°/m. (The shear modulus of elasticity of the steel is 78 GPa.)

(a) Determine the required diameter d0 of the solid shaft.

(b) Determine the required outer diameter d2 of the hollow shaft if the thickness t of the shaft is specified as one-tenth of the outer diameter.

(c) Determine the ratio of diameters (that is, the ratio d2/d0) and the ratio of weights of the hollow and solid shafts.



3. A hollow shaft and a solid shaft constructed of the same material have the same length and the same outer radius R . The inner radius of the hollow shaft is 0.6R. (a) Assuming that both shafts are subjected to the same torque, compare their shear stresses, angles of twist, and weights. (b) Determine the strength-to-weight ratios for both shafts.

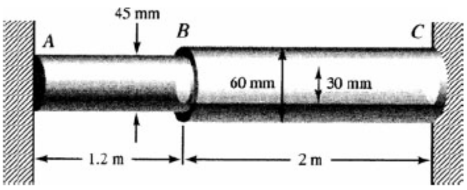
Stepped shafts

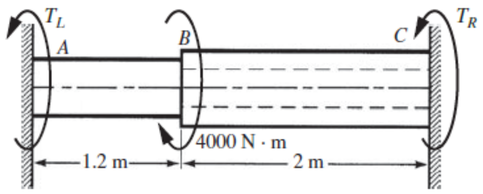
When a shaft is made of different lengths and of different diameters, it is termed as shaft as varying cross section. For such a shaft, the torque induced in its individual sections should be calculated first. The strength of the shaft is the minimum of all these torques.

Problems

A stepped shaft has the appearance as shown in figure. The region AB is aluminum, having G = 28 GPa, and the region BC is steel, having G = 84 GPa. The aluminum portion is of solid circular cross section 45 mm in diameter, and the steel region is circular with 60-mm outside diameter and 30-mm inside diameter. Determine the maximum shearing stress in each material as well as the angle of twist at B where a torsional load of 4000 N - m is applied. Ends A and C are rigidly clamped.

SOLUTION: The free-body diagram of the system is shown. The applied load of 4000 N-m as well as the unknown end reactive torques are as indicated. The only equation of static equilibrium is





$$\Sigma M_x = T_L + T_R - 4000 = 0$$

Since there are two unknowns TL and TR, another equation (based upon deformations) is required. This is set up by realizing that the angular rotation at B is the same if we determine it at the right end of AB or the left end of BC. We thus have

$$\frac{T_L \times 1.2}{(28 \times 10^9)\pi \times 0.045^4/32} = \frac{T_R \times 2.0}{(84 \times 10^9)\pi(0.06^4 - 0.03^4)/32} \qquad \text{or} \qquad T_L = 0.1875\, T_R$$

Solving for T_L and T_R, we find

$$T_L = 632 \text{ N} \cdot \text{m} \qquad \text{and} \qquad T_R = 3368 \text{ N} \cdot \text{m}$$

The maximum shearing stress in AB is given by

$$\tau_{AB} = \frac{T\rho}{J} = \frac{(632)(0.0225)}{\pi(0.045)^4/32} = 35.6 \text{ MPa}$$

and in BC by

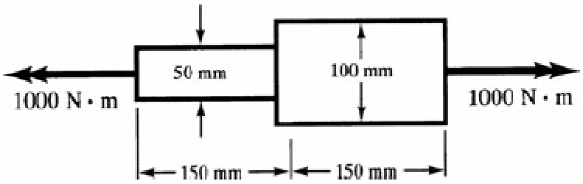
$$\tau_{BC} = \frac{T\rho}{J} = \frac{(3370)(0.030)}{\pi(0.06^4 - 0.03^4)/32} = 85.0 \text{ MPa}$$

The angle of twist at B, using parameters of the region AB, is

$$\theta_B = \frac{TL}{GJ} = \frac{(632)(1.2)}{(28 \times 10^9)(\pi \times 0.045^4/32)} = 0.0673 \text{ rad} \qquad \text{or} \qquad 3.86°$$

Problems for practice

A circular cross-section steel shaft is of diameter 50 mm over the left 150 mm of length and of diameter 100 mm over the right 150 mm, as shown in Fig. 5-21. Each end of the shaft is loaded by a twisting moment of 1000 N · m (as indicated by the double-headed arrows). If $G = 80$ GPa, determine the angle of twist between the ends of the shaft as well as the peak shearing stress. *Ans.* 1.09°, 40.7 MPa

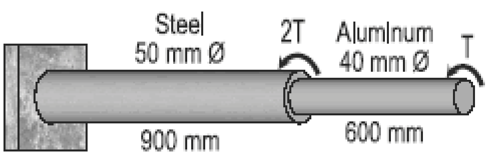


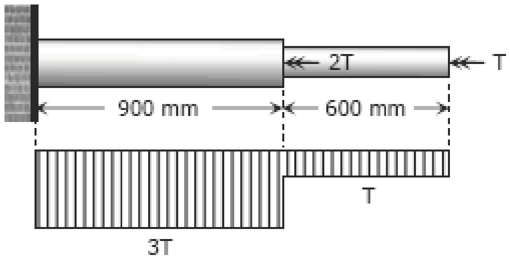
A compound shaft is composed of a 70-cm length of solid copper 10 cm in diameter, joined to 90-cm length of solid steel 12 cm in diameter. A torque of 14 kN · m is applied to each end of the shaft. Find the maximum shear stress in each material and the total angle of twist of the entire shaft. For copper $G = 40$ GPa, for steel $G = 80$ GPa. *Ans.* In the copper, 71.3 MPa; in the steel, 41.3 MPa; $\theta = 0.0328$

Compound shafts – fixed and simply supported shafts

A compound shaft is made of two or more different materials joined together in such a way that the shaft is elongated or compressed as a single shaft. The total torque transmitted by a compound shaft is the sum of the torques transmitted by each individual shaft and the angle of twist in each shaft will be equal.

1. A compound shaft consisting of a steel segment and an aluminum segment is acted upon by two torques as shown. Determine the maximum permissible value of T subject to the following conditions: $\tau_{st} = 83$ MPa, $\tau_{al} = 55$ MPa, and the angle of rotation of the free end is limited to 6°. For steel, G = 83 GPa and for aluminum, G = 28 GPa.





Based on maximum shearing stress $\tau_{max} = 16T / \pi d^3$:

$$\tau_{st} = \frac{16(3T)}{\pi(50^3)} = 83$$

$T = 679\,042.16$ N·mm

$T = 679.04$ N·m

$$\tau_{al} = \frac{16T}{\pi(40^3)} = 55$$

$T = 691\,150.38$ N·mm

$T = 691.15$ N·m

Based on maximum angle of twist:

$$\theta = \left(\frac{TL}{JG}\right)_{st} + \left(\frac{TL}{JG}\right)_{al}$$

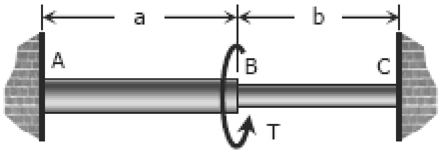
$$6°\left(\frac{\pi}{180°}\right) = \frac{3T(900)}{\frac{1}{32}\pi(50^4)(83\,000)} + \frac{T(600)}{\frac{1}{32}\pi(40^4)(28\,000)}$$

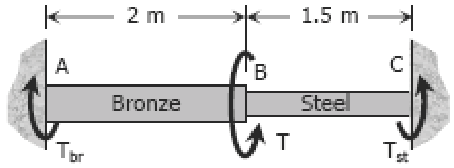
$T = 757\,316.32$ N·mm

$T = 757.32$ N·m

Use T = **679.04 N·m**

2. The compound shaft shown is attached to rigid supports. For the bronze segment AB, the diameter is 75 mm, $\tau \leq 60$ MPa, and G = 35 GPa. For the steel segment BC, the diameter is 50 mm, $\tau \leq 80$ MPa, and G = 83 GPa. If a = 2 m and b = 1.5 m, compute the maximum torque T that can be applied.





$\Sigma M = 0$

$T = T_{br} + T_{st}$ → Equation (1)

$\theta_{br} = \theta_{st}$

$$\left(\frac{TL}{JG}\right)_{br} = \left(\frac{TL}{JG}\right)_{st}$$

$$\frac{T_{br}(2)(1000)}{\frac{1}{32}\pi(75^4)(35000)} = \frac{T_{st}(1.5)(1000)}{\frac{1}{32}\pi(50^4)(83000)}$$

$T_{br} = 1.6011T_{st}$
$T_{st} = 0.6246T_{br}$ } Equations (2)

$$\tau_{\max} = \frac{16T}{\pi D^3}$$

Based on $\tau_{br} \leq 60$ MPa

$$60 = \frac{16T_{br}}{\pi(75^3)}$$

$T_{br} = 4\,970\,097.75$ N·mm

$T_{br} = 4.970$ kN·m → Maximum allowable torque for bronze

$T_{st} = 0.6246(4.970)$ → From one of Equations (2)

$T_{st} = 3.104$ kN·m

Based on $\tau_{st} \leq 80$ MPa

$$80 = \frac{16T_{st}}{\pi(50^3)}$$

$T_{st} = 1\,963\,495.41$ N·mm

$T_{st} = 1.963$ kN·m → maximum allowable torque for steel

$T_{br} = 1.6011(1.963)$ → From Equations (2)

$T_{br} = 3.142$ kN·m

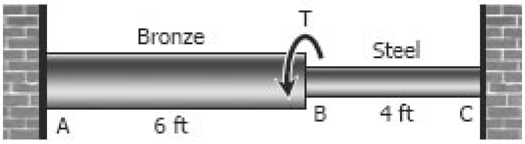
Use $T_{br} = 3.142$ kN·m and $T_{st} = 1.963$ kN·m

$T = 3.142 + 1.963$ → From Equation (1)

$T =$ **5.105 kN·m**

3. The compound shaft shown is attached to rigid supports. For the bronze segment AB, the maximum shearing stress is limited to 8000 psi and for the steel segment BC, it is limited to

12 ksi. Determine the diameters of each segment so that each material will be simultaneously stressed to its permissible limit when a torque T = 12 kip·ft is applied. For bronze, G = 6 × 106 psi and for steel, G = 12 × 106 psi.



$$\tau_{max} = \frac{16T}{\pi D^3}$$

For bronze:

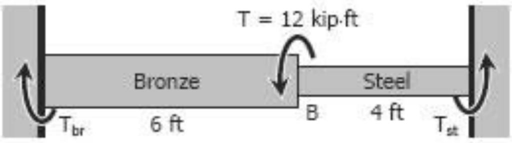
$$8000 = \frac{16T_{br}}{\pi D_{br}^{\ 3}}$$

$$T_{br} = 500\pi\, D_{br}^{\ 3}\ \text{lb·in}$$

For steel:

$$12\,000 = \frac{16T_{st}}{\pi D_{st}^{\ 3}}$$

$T_{st} = 750\pi D_{st}^3$ lb·in



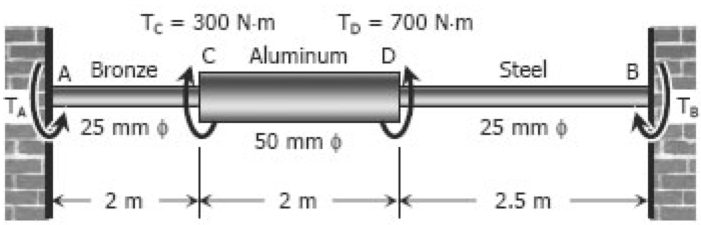
$$\Sigma M = 0$$
$$T_{br} + T_{st} = T$$
$$T_{br} + T_{st} = 12(1000)(12)$$
$$T_{br} + T_{st} = 144\ 000 \text{ lb·in}$$
$$500\pi D_{br}^3 + 750\pi D_{st}^3 = 144\ 000$$
$$D_{br}^3 = 288/\pi - 1.5 D_{st}^3 \quad \rightarrow \text{equation (1)}$$

$$\theta_{br} = \theta_{st}$$
$$\left(\frac{TL}{JG}\right)_{br} = \left(\frac{TL}{JG}\right)_{st}$$
$$\frac{T_{br}(6)}{\frac{1}{32}\pi D_{br}^4 (6\times 10^6)} = \frac{T_{st}(4)}{\frac{1}{32}\pi D_{st}^4 (12\times 10^6)}$$
$$\frac{T_{br}}{D_{br}^4} = \frac{T_{st}}{3D_{st}^4}$$
$$\frac{500\pi D_{br}^3}{D_{br}^4} = \frac{750\pi D_{st}^3}{3D_{st}^4}$$
$$D_{st} = 0.5 D_{br}$$

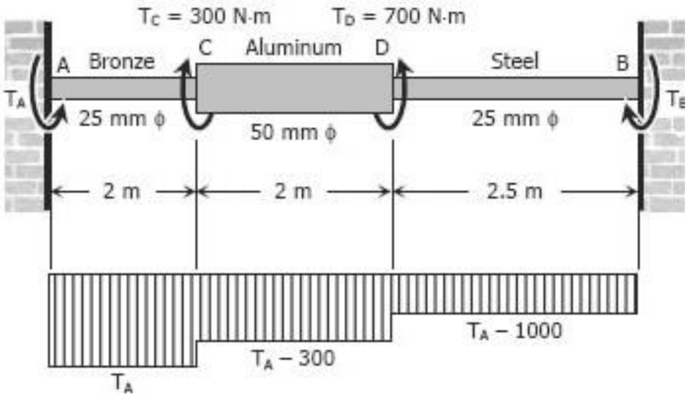
From Equation (1)

$$D_{br}^3 = 288/\pi - 1.5(0.5D_{br})^3$$
$$1.1875\, D_{br}^3 = 288/\pi$$
$$D_{br} = \mathbf{4.26\ in.}$$
$$D_{st} = 0.5(4.26) = \mathbf{2.13\ in.}$$

4. A shaft composed of segments AC, CD, and DB is fastened to rigid supports and loaded as shown. For bronze, G = 35 GPa; aluminum, G = 28 GPa, and for steel, G = 83 GPa. Determine the maximum shearing stress developed in each segment.



Stress developed in each segment with respect to T_A:



The rotation of B relative to A is zero.

$\theta_{A/B} = 0$

$$\left(\sum \frac{TL}{JG}\right)_{A/B} = 0$$

$$\frac{T_A(2)(1000^2)}{\frac{1}{32}\pi(25^4)(35000)} + \frac{(T_A - 300)(2)(1000^2)}{\frac{1}{32}\pi(50^4)(28000)} + \frac{(T_A - 1000)(2.5)(1000^2)}{\frac{1}{32}\pi(25^4)(83000)} = 0$$

$$\frac{2T_A}{(25^4)(35)} + \frac{2(T_A - 300)}{(50^4)(28)} + \frac{2.5(T_A - 1000)}{(25^4)(83)} = 0$$

$$\frac{16T_A}{35} + \frac{T_A - 300}{28} + \frac{20(T_A - 1000)}{83} = 0$$

$\frac{16}{35} T_A + \frac{1}{28} T_A - \frac{75}{7} + \frac{20}{83} T_A - \frac{20000}{83} = 0$

$\frac{8527}{11620} T_A = 251.678$

$T_A = 342.97$ N·m

$\Sigma M = 0$

$T_A + T_B = 300 + 700$

$342.97 + T_B = 1000$

$T_B = 657.03$ N·m

$T_{br} = 342.97$ N·m

$T_{al} = 342.97 - 300 = 42.97$ N·m

$T_{st} = 342.97 - 1000 = -657.03$ N·m $= -T_B$ (*ok!*)

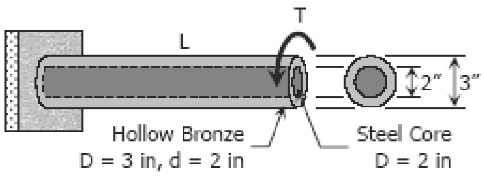
$$\tau_{max} = \frac{16T}{\pi D^3}$$

$$\tau_{br} = \frac{16(342.97)(1000)}{\pi(25^3)} = \mathbf{111.79\ MPa}$$

$$\tau_{al} = \frac{16(42.97)(1000)}{\pi(50^3)} = \mathbf{1.75\ MPa}$$

$$\tau_{st} = \frac{16(657.03)(1000)}{\pi(25^3)} = \mathbf{214.16\ MPa}$$

5. A hollow bronze shaft of 3 in. outer diameter and 2 in. inner diameter is slipped over a solid steel shaft 2 in. in diameter and of the same length as the hollow shaft. The two shafts are then fastened rigidly together at their ends. For bronze, G = 6 × 106 psi, and for steel, G = 12 × 106 psi. What torque can be applied to the composite shaft without exceeding a shearing stress of 8000 psi in the bronze or 12 ksi in the steel?



$$\theta_{st} = \theta_{br}$$

$$\left(\frac{TL}{JG}\right)_{st} = \left(\frac{TL}{JG}\right)_{br}$$

$$\frac{T_{st}L}{\frac{1}{32}\pi(2^4)(12\times10^6)} = \frac{T_{br}L}{\frac{1}{32}\pi(3^4-2^4)(6\times10^6)}$$

$$\frac{T_{st}}{192\times10^6} = \frac{T_{br}}{390\times10^6} \qquad \rightarrow \text{Equation (1)}$$

Applied Torque = Resisting Torque

$$T = T_{st} + T_{br} \qquad \rightarrow \text{Equation (2)}$$

Equation (1) with T_{st} in terms of T_{br} and Equation (2)

$$T = \frac{192\times10^6}{390\times10^6}T_{br} + T_{br}$$

$$T_{br} = 0.6701T$$

Equation (1) with T_{br} in terms of T_{st} and Equation (2)

$$T = T_{st} + \frac{390\times10^6}{192\times10^6}T_{st}$$

$$T_{st} = 0.3299T$$

Based on hollow bronze ($T_{br} = 0.6701T$)

$$\tau_{max} = \left[\frac{16TD}{\pi(D^4 - d^4)}\right]_{br}$$

$$8000 = \frac{16(0.6701T)(3)}{\pi(3^4 - 2^4)}$$

$$T = 50\ 789.32\ \text{lb}\cdot\text{in}$$

$$T = 4232.44\ \text{lb}\cdot\text{ft}$$

Based on steel core ($T_{st} = 0.3299T$):

$$\tau_{max} = \left[\frac{16T}{\pi D^3}\right]_{st}$$

$$12\ 000 = \frac{16(0.3299T)}{\pi(2^3)}$$

$$T = 57\ 137.18\ \text{lb}\cdot\text{in}$$

$$T = 4761.43\ \text{lb}\cdot\text{ft}$$

Use T = **4232.44 lb·ft**

6. The two steel shaft shown in Fig. P-325, each with one end built into a rigid support have flanges rigidly attached to their free ends. The shafts are to be bolted together at their flanges. However, initially there is a 6° mismatch in the location of the bolt holes as shown in the figure. Determine the maximum shearing stress in each shaft after the shafts are bolted together. Use G = 12 × 106 psi and neglect deformations of the bolts and flanges.

$$\theta_{\text{of 6.5' shaft}} + \theta_{\text{of 3.25' shaft}} = 6°$$

$$\left(\frac{TL}{JG}\right)_{\text{of 6.5' shaft}} + \left(\frac{TL}{JG}\right)_{\text{of 3.25' shaft}} = 6°\left(\frac{\pi}{180°}\right)$$

$$\frac{T(6.5)(12^2)}{\frac{1}{32}\pi(2^4)(12\times10^6)} + \frac{T(3.25)(12^2)}{\frac{1}{32}\pi(1.5^4)(12\times10^6)} = \frac{\pi}{30}$$

$$T = 817.32\ \text{lb}\cdot\text{ft}$$

$$\tau_{max} = \frac{16T}{\pi D^3}$$

$$\tau_{\text{of 6.5' shaft}} = \frac{16(817.32)(12)}{\pi(2^3)} = \mathbf{6243.86\ psi}$$

$$\tau_{\text{of 3.25' shaft}} = \frac{16(817.32)(12)}{\pi(1.5^3)} = \mathbf{14\ 800.27\ psi}$$

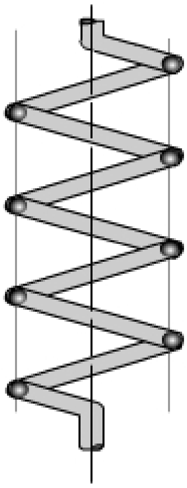
Closed Coiled helical springs subjected to axial loads:

Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released. Also Springs are energy absorbing units whose function is to store energy and to restore it slowly or rapidly depending on the particular application.

Important types of springs are:

There are various types of springs such as

(i) helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is Torsional shear stress due to twisting. They are both used in tension and compression.



(ii) Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.
In this the major stresses are tensile and compression due to bending.

(iii) Leaf springs: They are composed of flat bars of varying lengths clamped together so as to obtain greater efficiency. Leaf springs may be full elliptic, semi elliptic or cantilever types, In these type of springs the major stresses which come into picture are tensile & compressive.

Uses of springs:

(a) To apply forces and to control motions as in brakes and clutches.

(b) To measure forces as in spring balance.

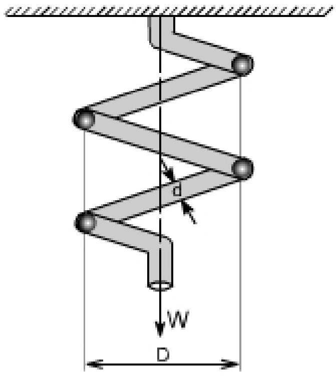
(c) To store energy as in clock springs.

(d) To reduce the effect of shock or impact loading as in carriage springs.

(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

Derivation of the Formula :

In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load W.



Let

W = axial load

D = mean coil diameter

d = diameter of spring wire

n = number of active coils

C = spring index = D / d For circular wires

l = length of spring wire

G = modulus of rigidity

x = deflection of spring

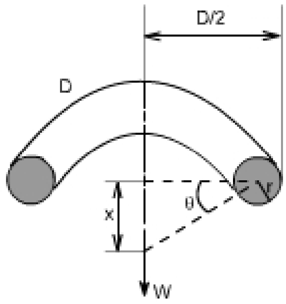
q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that

x = D / 2 . q

again l = p D n [consider ,one half turn of a close coiled helical spring]



Assumptions: (1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a spring will be assumed to lie in a plane which is nearly perpendicular to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical. Hence to maintain equilibrium of a segment of the spring, only a shearing force V = F and Torque T = F. r are required at any X – section. In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is
uniformly distributed and is negligible
so applying the torsion formula. Using the torsion formula i.e

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

$$\text{and substitituting } J = \frac{\pi d^4}{32}; T = w.\frac{d}{2}$$

$$\theta = \frac{2.x}{D}; l = \pi D.x$$

SPRING DEFLECTION

$$\frac{\frac{w.d/2}{\frac{\pi d^4}{32}}}{} = \frac{G.2x/D}{\pi D.n}$$

Thus,

$$x = \frac{8w.D^3.n}{G.d^4}$$

Spring striffness: The stiffness is defined as the load per unit deflection therefore

$$k = \frac{w}{x} = \frac{w}{\frac{8w.D^3.n}{G.d^4}}$$

Therefore

$$k = \frac{G.d^4}{8.D^3.n}$$

Shear stress

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{\tau_{max^m}}{d/2}$$

$$\text{or } \tau_{max^m} = \frac{8wD}{\pi d^3}$$

WAHL'S FACTOR :

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

K = Wahl' s factor and is defined as

$$K = \frac{4c-1}{4c-4} + \frac{0.615}{c}$$

Where C = spring index

= D/d

if we take into account the Wahl's factor than the formula for the shear stress becomes

$$\tau_{max^m} = \frac{16.T.k}{\pi d^3}$$

Strain Energy : The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion

$$U = \frac{T^2 L}{2EI}$$

$$L = \pi D n$$

$$I = \frac{\pi d^4}{64}$$

so after substitution we get

$$U = \frac{32T^2 Dn}{E.d^4}$$

Worked examples:

1. A close coiled helical spring is to carry a load of 5000N with a deflection of 50 mm and a maximum shearing stress of 400 N/mm2 if the number of active turns or active coils is 8.Estimate the following:

(i) wire diameter
(ii) mean coil diameter
(iii) weight of the spring.
Assume G = 83,000 N/mm2 ; r = 7700 kg/m3

solution :

(i) for wire diametre if W is the axial load, then

$$\frac{\frac{w.d/2}{\frac{\pi d^4}{32}}}{} = \frac{\tau_{max} m}{d/2}$$

$$D = \frac{400}{d/2} \cdot \frac{\pi d^4}{32} \cdot \frac{2}{W}$$

$$D = \frac{400.\pi d^3.2}{5000.16}$$

$$D = 0.0314\, d^3$$

Further, deflection is given as

$$x = \frac{8wD^3.n}{G.d^4}$$

on substituting the relevant parameters we get

$$50 = \frac{8.5000.(0.0314\, d^3)^3.8}{83{,}000.d^4}$$

$$d = 13.32\text{mm}$$

Therefore,

D = .0314 x (13.317)3mm

=74.15mm

D = 74.15 mm

2. Determine the maximum shearing stress and elongation in a helical steel spring composed of 20 turns of 20-mm-diameter wire on a mean radius of 90 mm when the spring is supporting a load of 1.5 kN. G = 83 GPa.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(\frac{4m-1}{4m-4} + \frac{0.615}{m}\right) \rightarrow$$

Where P = 1.5 kN = 1500 N; R = 90 mm
d = 20 mm; n = 20 turns
m = 2R/d = 2(90)/20 = 9

$$\tau_{max} = \frac{16(1500)(90)}{\pi(20^3)}\left[\frac{4(9)-1}{4(9)-4} + \frac{0.615}{9}\right]$$

$$\tau_{max} = \mathbf{99.87\ MPa}$$

$$\delta = \frac{64PR^3n}{Gd^4} = \frac{64(1500)(90^3)(20)}{83\,000(20^4)}$$

$$\delta = \mathbf{105.4\ mm}$$

3. Determine the maximum shearing stress and elongation in a bronze helical spring composed of 20 turns of 1.0-in.-diameter wire on a mean radius of 4 in. when the spring is supporting a load of 500 lb. G = 6 × 106 psi.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(\frac{4m-1}{4m-4}+\frac{0.615}{m}\right)$$

Where P = 500 lb; R = 4 in
d = 1 in; n = 20 turns
m = 2R/d = 2(4)/1 = 8

$$\tau_{max} = \frac{16(500)(4)}{\pi(1^3)}\left[\frac{4(8)-1}{4(8)-4}+\frac{0.615}{8}\right] \qquad \delta = \frac{64PR^3n}{Gd^4} = \frac{64(500)(4^3)(20)}{(6\times10^6)(1^4)}$$

τ_{max} = 12 060.3 psi = **12.1 ksi** $\qquad \delta$ = **6.83 in**

4. A helical spring is fabricated by wrapping wire ¾ in. in diameter around a forming cylinder 8 in. in diameter. Compute the number of turns required to permit an elongation of 4 in. without exceeding a shearing stress of 18 ksi. G = 12 × 106 psi.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(1+\frac{d}{4R}\right)$$

$$18000 = \frac{16P(4)}{\pi(3/4)^3}\left[1+\frac{3/4}{4(4)}\right]$$

P = 356.07 lb

$$\delta = \frac{64PR^3n}{Gd^4}$$

$$4 = \frac{64(356.07)(4^3)n}{(12\times10^6)(3/4)^3}$$

n = 13.88 say **14 turns**

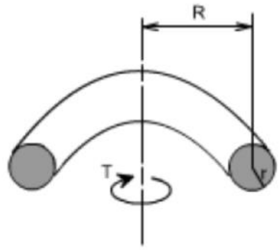
Weight

mass or weight = volume. density
= area. length of the spring. density of spring material

$$= \frac{\pi d^2}{4}.\pi D n.\rho$$

On substituting the relevant parameters we get
Weight = 1.996 kg
= 2.0kg

Close – coiled helical spring subjected to axial torque T or axial couple.



In this case the material of the spring is subjected to pure bending which tends to reduce Radius R of the coils. In this case the bending moment is constant through out the spring and is equal to the applied axial Torque T. The stresses i.e. maximum bending stress may

$$\sigma_{max} = \frac{M.y}{I}$$
$$= \frac{T.d/2}{\frac{\pi d^4}{64}}$$
$$\sigma_{max} = \frac{32T}{\pi d^3}$$

thus be determined from the bending theory.

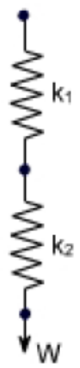
Springs in Series: If two springs of different stiffness are joined endon and carry a common load W, they are said to be connected in series and the combined stiffness and deflection are given by the following equation

.

$$\frac{W}{k} = x_1 + x_2 = \frac{W}{k_1} + \frac{W}{k_2}$$

or

$$\boxed{\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}}$$



Springs in parallel: If the two spring are joined in such a way that they have a common deflection ‘x' ; then they are said to be connected in parallel. In this care the load carried is shared between the two springs and total load W = W1 + W2

$$x = \frac{W}{k} = \frac{W_1}{k_1} = \frac{W_2}{k_2}$$

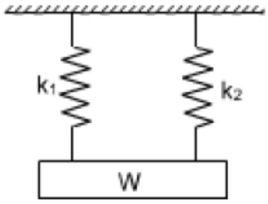
Thus $W_1 = \dfrac{Wk_1}{k}$

$W_2 = \dfrac{Wk_2}{k}$

Futher

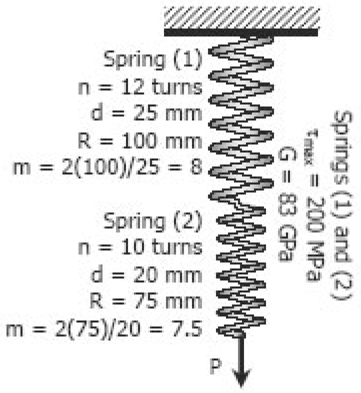
$W = W_1 + W_2$

thus $\boxed{k = k_1 + k_2}$



1. Two steel springs arranged in series as shown supports a load P. The upper spring has 12 turns of 25-mm-diameter wire on a mean radius of 100 mm. The lower spring consists of 10 turns of 20-mmdiameter wire on a mean radius of 75 mm. If the maximum shearing stress in either spring must not exceed 200 MPa, compute the maximum value of P and the total elongation of the assembly. G = 83 GPa. Compute the equivalent spring constant by dividing the load by the total elongation.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(\frac{4m-1}{4m-4} + \frac{0.615}{m}\right)$$



For Spring (1)

$$200 = \frac{16P(100)}{\pi(25^3)}\left[\frac{4(8)-1}{4(8)-4} + \frac{0.615}{8}\right]$$

$P = 5182.29$ N

For Spring (2)

$$200 = \frac{16P(75)}{\pi(20^3)}\left[\frac{4(7.5)-1}{4(7.5)-4} + \frac{0.615}{7.5}\right]$$

$P = 3498.28$ N

Use $P = \mathbf{3498.28\ N}$

Total elongation:

$$\delta = \delta_1 + \delta_2$$

$$\delta = \left(\frac{64PR^3n}{Gd^4}\right)_1 + \left(\frac{64PR^3n}{Gd^4}\right)_2$$

$$\delta = \frac{64(3498.28)(100^3)12}{83\,000(25^4)} + \frac{64(3498.28)(75^3)(10)}{83\,000(20^4)}$$

$\delta = \mathbf{153.99\ mm}$

Equivalent spring constant, $k_{equivalent}$:

$$k_{equivalent} = \frac{P}{\delta} = \frac{3498.28}{153.99}$$

$k_{equivalent} = \mathbf{22.72\ N/mm}$

Design of helical coil springs – stresses in helical coil springs under torsion loads

Worked problems

Design a close-coiled helical compression spring with a following data :

- Service load range. = 2250N to 2750N
- Axial deflection of spring for load range = 6mm
- Spring index = 5
- Permissible shear stress for spring = 420N/mm^2
- Modulus of rigidity for spring material = 84 KN/mm^2
- Neglect the effect of stress concentration. Draw a dimensioned sketch of the spring

Given : $F_{min} = 2250N$; $F_{max} = 2750N$;

$\delta = 6$ mm ; $C = 5$;

$\tau = 420$ N/mm^2 ; $G = 84 \times 10^3$ N/mm^2.

- **Wire diameter :**
- Neglecting effect of stress concentration,

$$K_s = \left[1+\frac{0.5}{C}\right] = \left[1+\frac{0.5}{5}\right] = 1.1$$

$$\text{Now,} \quad \tau = K_s\left[\frac{8F_{max}\,C}{\pi d^2}\right]$$

$$\therefore \quad 420 = 1.1 \times \left[\frac{8\times 2750 \times 5}{\pi\, d^2}\right]$$

$$\therefore \quad d = 9.58 \text{ mm or } 9.6 \text{ mm}$$

$$\mathbf{d = 9.6\ mm}$$

- **Mean coil diameter :**

$$D = C \cdot d = 5 \times 9.6$$

$$\text{or} \quad \mathbf{D = 48\ mm}$$

- **Number of coils :**

Spring stiffness, $$K = \frac{F_{max} - F_{min}}{\delta}$$

$$= \frac{2750 - 2250}{6}$$

or, $$K = 83.33 \text{ N/mm}$$

Now, $$K = \frac{Gd}{8C^3 n}$$

$$\therefore \quad 83.33 = \frac{84 \times 10^3 \times 9.6}{8 \times 5^3 \times n}$$

$\therefore \qquad n = 9.68$ or 9.7 turns

$$\mathbf{n = 9.7}$$

Assuming square and ground ends,

$$n' = n + 2 = 9.7 + 2 = 11.7 \text{ turns}$$

$$\mathbf{n' = 11.7}$$

- **Solid length :**

$$L_s = (n + 2)\, d = (9.7 + 2) \times 9.6$$

or $\qquad \mathbf{L_s = 112.32\ mm}$

- **Free length :**

maximum deflection, $\quad \delta_{max} = \dfrac{F_{max}}{K} = \dfrac{2750}{83.33}$

$$\therefore\ \delta_{max} = 33 \text{ mm}$$

Free length, L_F = solid length + maximum deflection + total clearance

$$= L_s + \delta_{max} + 0.15\, \delta_{max}$$

- (Assume total clearance as 15 % of maximum deflection)

$$= 112.32 + 33 + 0.15 \times 33$$

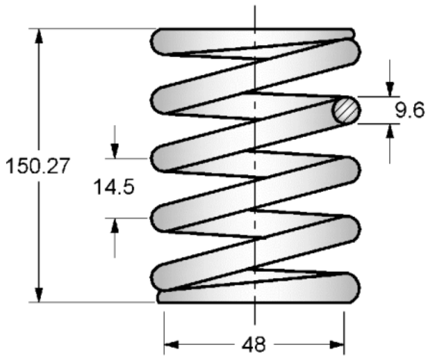
or $\qquad \mathbf{L_F = 150.27\ mm}$ $\qquad$ **...Ans.**

- **Pitch of coil :**

Now, $\qquad L_F = p\,n + d$

$$150.27 = p \times 9.7 + 9.6$$

$\therefore\ \mathbf{p = 14.5\ mm}$ $\qquad$ **...Ans.**



The following data refers to a helical compression spring :

- Mean coil diameter = 125 mm
- Maximum axial load = 8000 N
- Spring rate = 72 kN/m
- Allowable shear stress for string = 275 N/mm^2
- Modulus of rigidity for spring material = 80×10^3 N/mm^2

Determine :

(i) Wire diameter; and

(ii) Number of active turns.

Given : $\quad D = 125$ mm ; $\quad F_{max} = 8000$ N;

$K = 72$ kN/m = 72 N/mm ; $\quad \tau = 275$ N/mm^2 ;

$G = 80 \times 10^3$ N/mm^2.

(i) Wire diameter :

- **Trial 1 :**

As spring index is not known, initially assuming $K_w = 1$,

$$\tau = K_w \left[\frac{8 F_{max} D}{\pi d^3} \right]$$

$$275 = \frac{1 \times 8 \times 8000 \times 125}{\pi d^3}$$

$$\therefore \quad d = 21 \text{ mm}$$

- **Trial 2 :**

The initial value of wire diameter d = 21 mm is used to estimate C and K_w . Taking the new value of K_w, the wire diameter is determined as follows :

$$C = D/d = \frac{125}{21} = 5.95$$

$$K_w = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} = \frac{4 \times 5.95 - 1}{4 \times 5.95 - 4} + \frac{0.615}{5.95}$$

$$\text{or} \quad K_w = 1.255$$

$$\tau = K_w \left[\frac{8 F_{max} D}{\pi d^3} \right]$$

$$275 = \frac{1.255 \times 8 \times 8000 \times 125}{\pi d^3}$$

$$\therefore \quad d = 22.65 \text{ mm or } 23 \text{ mm}$$

- **Check for shear stress induced in spring wire :**

$$C = D/d = \frac{125}{23} = 5.43$$

$$K_w = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} = \frac{4 \times 5.43 - 1}{4 \times 5.43 - 4} + \frac{0.615}{5.43}$$

$$\text{or} \quad K_w = 1.2826$$

$$\tau = K_{w\,G} \left[\frac{8 F_{max} D}{\pi d^3} \right] = \frac{1.2826 \times 8 \times 8000 \times 125}{\pi \times (23)^3}$$

$$\text{or} \quad \tau = 268.44 \text{ N/mm}^2 < 275 \text{ N/mm}^2$$

design is safe $\quad$ **d = 23 mm**

Hence, $\quad$ **C = 5.43**

(ii) Number of active coils :

Now,

$$K = \frac{Gd}{8 C^3 n}$$

$$72 = \frac{80 \times 10^3 \times 23}{8 \times (5.43)^3 \times n}$$

$$\therefore \quad n = 19.95 \text{ or } 20 \text{ turns}$$

$$\mathbf{n = 20}$$

Design a helical compression for a spring operated pressure relief valve with following data :

- Operating pressure = 1.25 N/mm^2
- Valve lift = 3.5 mm at 10% pressure rise over operating pressure
- Diameter of valve = 25 mm
- Limiting mean coil diameter = 40 mm

- Permissible shear stress for spring = 500 N/mm^2
- Modulus of rigidity for spring material = 834 Pa
- The available standard spring wire diameters are : 2, 3, 4, 5, 6, 7, 8 and 10 mm.

Given : p_o = 1.25 N/mm^2 ; δ_o = 3.5 mm;

p_{max} = 1.1 p_o = 1.25 × 1.1 = 1.375 N/mm^2 ; dv = 25 mm;

D = 40 mm ; τ = 500 N /mm^2;

G = 83 × 10^3 N/mm^2.

- **Maximum spring force :**

Cross-sectional area of valve, $A_V = \pi dv^2/4 = \pi \times (25)^2/4$

or $A_V = 490.87 \text{ mm}^2$

The spring force at operating pressure,

$$F_o = p_o A_V = 1.25 \times 490.87 = 613.59 \text{ N}$$

The maximum spring force,

$$F_{max} = p_{max} A_V = 1.375 \times 490.87 = 674.95 \text{ N}$$

- **Wire diameter :**

$$\tau = \frac{K_w 8 F_{max} C}{\pi d^2}$$

$$C = D/d$$

$$\therefore \quad d = D/C$$

Substituting value of **'d'** from Equation (b) in Equation (a),

$$\tau = \frac{K_w 8F_{max} C}{\pi (D/C)^2} = \frac{K_w 8 F_{max} C^3}{\pi D^2}$$

$$500 = \frac{K_w \times 8 \times 674.95 \times C^3}{\pi \times (40)^2}$$

$$\therefore \quad K_w C^3 = 465.45$$

$$\left[\frac{4C-1}{4C-4} + \frac{0.615}{C}\right] C^3 = 465.45$$

Solving Equation (c) by trial and error, we get,

$$C = 7.3$$

$$\therefore \quad d = D/C = 40/7.3 = 5.48 \text{ mm or } 5.5 \text{ mm}$$

The next standard wire diameter selected is,

$$\mathbf{d = 6 \text{ mm}}$$

$$\therefore \quad C = D/d = 40/6 = 6.667$$

$$\mathbf{C = 6.667}$$

- **Number of coils :**

The spring stiffness is, $K = \dfrac{F_{max} - F_o}{\delta_o} = \dfrac{674.95 - 613.59}{3.5}$

or $K = 17.53 \text{ N/mm}$

Now, $K = \dfrac{Gd}{8C^3 n}$

$$17.53 = \frac{83 \times 10^3 \times 6}{8 \times (6.677)^3 \times n}$$

$$\therefore \quad n = 11.98 \text{ or } 12 \text{ turns}$$

$$\mathbf{n = 12}$$

Assuming square and ground ends,

$$n' = n + 2 = 12 + 2 = 14 \text{ turns}$$

$$\mathbf{n' = 14}$$

- **Solid length :**

$$\text{Solid length, } L_s = (n+2)\,d = (12+2) \times 6 = 84 \text{ mm}$$

$$\text{or} \quad \mathbf{L_s = 84 \text{ mm}}$$

- **Free length :**

$$\text{Maximum deflection, } \delta_{max} = \frac{F_{max}}{K} = \frac{675.95}{17.53} = 38.5 \text{ mm}$$

Free length, L_F = solid length + maximum deflection + total clearance

(Assuming total clearance as 15% of δ_{max})

$$L_F = L_s + \delta_{max} + 0.15\,\delta_{max}$$

$$= 84 + 38.5 + 0.15 \times 38.5 = 128.275 \text{ mm}$$

$$\text{or} \quad \mathbf{L_F = 128.275 \text{ mm}}$$

- **Pitch of coil :**

$$\text{Now,} \quad L_F = pn + 2d$$

$$128.275 = p \times 12 + 2 \times 6$$

$$\mathbf{p = 9.69 \text{ mm}}$$

Two helical springs are arranged in a concentric manner, with one inside the other. Both the springs have same free length and carry a total load of 5500 N. The outer spring has 8 coils with mean coil diameter of 128 mm and wire diameter of 16 mm. The inner spring has 12 coils with mean coil diameter of 84 mm and wire diameter of 12 mm. Determine :

(i) the maximum load carried by each spring;

(ii) the total deflection of each spring; and

(iii) the maximum stress in each spring.

Assume G = 81 GPa.

Given : $L_{F1} = L_{F2}$; $F = 5500$ N ; $n_1 = 8$; $D_1 = 128$ mm ; $d_1 = 16$ mm ; $n_2 = 12$; $D_2 = 84$ mm ; $d_2 = 12$ mm ; $G = 81 \times 10^3$ N/mm^2.

1. Stiffness of outer spring :

$$C_1 = \frac{D_1}{d_1} = \frac{128}{16} = 8$$

$$\therefore \quad K_1 = \frac{Gd_1}{8\,C_1^3\,n_1} = \frac{81 \times 10^3 \times 16}{8 \times (8)^3 \times 8} = 39.55 \text{ N/mm}$$

2. Stiffness of inner spring :

$$C_2 = \frac{D_2}{d_2} = \frac{84}{12} = 7$$

$$\therefore \quad K_2 = \frac{Gd_2}{8\,C_2^3\,n_2} = \frac{81 \times 10^3 \times 12}{8 \times (7)^3 \times 12} = 29.52 \text{ N/mm}$$

3. Load shared by each spring :

$$F_1 + F_2 = F$$
$$F_1 + F_2 = 5500 \text{ N}$$

$$\delta_1 = \delta_2 \qquad \frac{F_1}{K_1} = \frac{F_2}{K_2}$$

$$\frac{F_1}{39.55} = \frac{F_1}{29.52} \qquad \therefore \; F_1 = 1.34 \; F_2$$

Substituting Equation (b) in Equation (a),

$$1.34 \, F_2 + F_2 = 5500 \qquad 2.34 \, F_2 = 5500$$
$$\therefore \quad F_2 = 2350.61 \text{ N} \qquad \therefore \quad F_1 = 5500 - 2350.61$$
$$\text{or} \quad F_1 = 3149.39 \text{ N}$$

The load shared by outer spring, $\mathbf{F_1 = 3149.39 \; N}$

The load shared by inner spring, $\mathbf{F_2 = 2350.61 \; N}$

4. Deflection of each spring :

$$\delta_1 = \frac{F_1}{K_1} = \frac{3149.39}{39.55} = 79.63 \text{ mm}$$
$$\delta_2 = \frac{F_2}{K_2} = \frac{2350.61}{29.52} = 79.63 \text{ mm}$$
$$\therefore \; \boldsymbol{\delta_1 = \delta_2 = 79.63 \text{ mm}}$$

5. Maximum stress in each spring :

$$K_{w1} = \frac{4\,C_1 - 1}{4\,C_1 - 4} + \frac{0.615}{C_1} = \frac{4 \times 8 - 1}{4 \times 8 - 4} + \frac{0.615}{8} = 1.184$$
$$\tau_1 = K_{w1} \left[\frac{8\,F_1\,C_1}{\pi\,d_1^2} \right] = \frac{1.184 \times 8 \times 3149.39 \times 8}{\pi \times (16)^3}$$

The Maximum stress in outer spring, $\boldsymbol{\tau_1 = 296.73 \text{ N/mm}^2}$

$$K_{w2} = \frac{4\,C_2 - 1}{4\,C_2 - 4} + \frac{0.615}{C_2} = \frac{4 \times 7 - 1}{4 \times 7 - 4} + \frac{0.615}{7} = 1.2128$$
$$\tau_2 = K_{w2} \left[\frac{8\,F_2\,C_2}{\pi\,d_2^2} \right] = \frac{1.2128 \times 8 \times 2350.61 \times 7}{\pi \times (12)^2}$$

The Maximum stress inner spring, $\boldsymbol{\tau_2 = 352.91 \text{ N/mm}^2}$

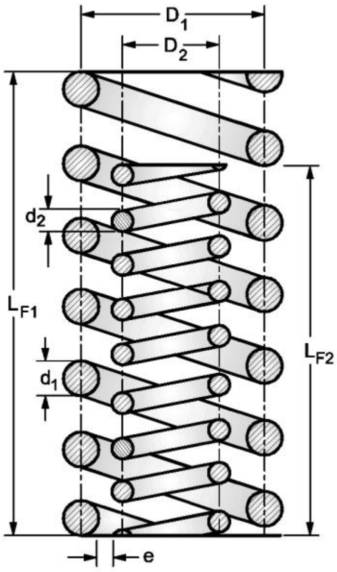
A composite compression spring has two closed coil helical springs and is subjected to an axial load of 400 N. The outer spring is 15 mm longer than the inner spring. The outer spring has 10 coils of 40 mm mean diameter and 5 mm wire diameter. The inner spring has 8 coils of 30 mm mean diameter and 4 mm wire diameter. If the modulus of rigidity for spring material is 84 GPa, determine :

(i) the compression of each spring;
(ii) the load carried by each spring; and
(iii) the shear stress induced in each spring.

Given : $F = 400$ N ; $G = 84 \times 10^3$ N/mm^2 ;

For outer spring : For inner spring :

$n_1 = 10$; $n_2 = 8$;



$D_1 = 240$ mm ; $D_2 = 30$ mm ;

$d_1 = 5$ mm ; $d_2 = 4$ mm ;

$L_{F1} = h + 15$ mm ; $L_{F2} = h$ mm

Referring Fig. 12.22.1;

- **Deflection of outer spring :**

$$\therefore \quad C_1 = \frac{D_1}{d_1} = \frac{40}{5} = 8$$

$$\frac{F_1}{\delta_1} = \frac{Gd_1}{8\,C_1^3\,n_1}$$

$$\frac{F_1}{\delta_1} = \frac{84 \times 10^3 \times 5}{8 \times 8^3 \times 10}$$

$$\therefore \quad \delta_1 = 9.75 \times 10^{-2}\,F_1, \text{ mm}$$

- **Deflection of inner spring :**

$$\therefore \quad C_2 = \frac{D_2}{d_2} = \frac{30}{4} = 7.5$$

$$\frac{F_2}{\delta_2} = \frac{Gd_2}{8\,C_2^3\,n_2} = \frac{84 \times 10^3 \times 4}{8 \times 7.5^3 \times 8}$$

$$\therefore \quad \delta_2 = 8.04 \times 10^{-2}\,F_2, \text{ mm}$$

$$\text{Now, } \delta_1 = \delta_2 + 15$$

- **Load carried by each spring :**

Substituting Equations (a) and (b) in Equation (c),

$$9.75 \times 10^{-2}\,F_1 = 8.04 \times 10^{-2}\,F_2 + 15$$

$$F_1 = 0.824\,F_2 + 153.846$$

$$\text{Now,} \quad F_1 + F_2 = 400$$

Substituting Equation (d) in Equation (e),

$$0.824\,F_2 + 153.846 + F_2 = 400$$

$$1.824\,F_2 = 246.15$$

$$\therefore \quad F_2 = 134.95 \text{ N}$$

$$\text{and} \quad F_1 = 400 - F_2 = 400 - 134.95 = 265.05 \text{ N} \quad \text{[from Equation (e)]}$$

$$F_1 = 265.05 \text{ N}$$

$$\mathbf{F_2 = 134.95\ N}$$

- **Compression of each spring :**

From Equation (a),

$$\delta_1 = 9.75 \times 10^{-2}\, F_1 = 9.75 \times 10^{-2} \times 265.05$$

or $$\boldsymbol{\delta_1 = 25.84 \text{ mm}}$$

From Equation (b),

$$\delta_2 = 8.04 \times 10^{-2}\, F_2 = 8.04 \times 10^{-2} \times 134.95$$

or $$\boldsymbol{\delta_2 = 10.85 \text{ mm}}$$

- **Shear stress in outer spring :**

$$K_{w1} = \frac{4C_1 - 1}{4C_1 - 4} + \frac{0.615}{C_1} = \frac{4 \times 8 - 1}{4 \times 8 - 4} + \frac{0.615}{8}$$

$$K_{w1} = 1.16$$

$$\tau_1 = \frac{K_{W1}\, 8\, F_1\, C_1}{\pi\, d_1^2} = \frac{1.16 \times 8 \times 265.05 \times 8}{\pi \times (5)^2}$$

or $$\boldsymbol{\tau_1 = 250.58 \text{ N/mm}^2}$$

- **Shear stress in inner spring :**

$$K_{w2} = \frac{4C_2 - 1}{4C_2 - 4} + \frac{0.615}{C_2} = \frac{4 \times 7.5 - 1}{4 \times 7.5 - 4} + \frac{0.615}{7.5}$$

$$K_{w2} = 1.2$$

$$\tau_2 = \frac{K_{w2}\, 8\, F_2\, C_2}{\pi\, d_2^2} = \frac{1.2 \times 8 \times 134.95 \times 7.5}{\pi \times (4)^2}$$

or $$\boldsymbol{\tau_2 = 193.3 \text{ N/mm}^2}$$

A composite compression spring has two closed coil helical springs. The outer spring is 15 mm longer than the inner spring. The outer spring has 10 coils of mean diameter 40 mm and wire diameter 5 mm. The inner spring has 8 coils of mean diameter 30 mm and wire diameter 4 mm. When the spring is subjected to an axial load of 400 N, find :

(i) Compression of each spring;

(ii) Load shared by each spring;

(iii) Shear stress induced in each spring

Modulus of rigidity may be taken as 84 kN/mm^2.

Given :

$L_{F1} = L_{F2} + 15$	;	$n_1 = 10$;
$D_1 = 40$ mm	;	$d_1 = 5$ mm;
$n_2 = 8$	;	$D_2 = 30$ mm;
$d_2 = 4$ mm	;	$F = 400$ N;
$G = 84 \times 10^3$ N/mm^2.		

- **Stiffness of outer spring :**

$$C_1 = D_1/d_1 = \frac{40}{5} = 8$$

$$\therefore \quad K_1 = \frac{Gd_1}{8\, C_1^3\, n_1} = \frac{84 \times 10^3 \times 5}{8 \times (8)^3 \times 10} = 10.2539 \text{ N/mm}^2$$

$$\boldsymbol{K_1 = 10.2539 \text{ N/mm}^2}$$

- **Stiffness of inner spring :**

$$C_2 = \frac{D_2}{d_2} = \frac{30}{4} = 7.5$$

$$\therefore \quad K_2 = \frac{Gd_2}{8\,C_2^3\,n_2} = \frac{84 \times 10^3 \times 4}{8 \times (7.5)^3 \times 8} = 12.444 \text{ N/mm}^2$$

$$\mathbf{K_2 = 12.444 \text{ N/mm}^2}$$

- **Load shared by each spring :**

$$F_1 + F_2 = F$$

$$\therefore \quad F_1 + F_2 = 400$$

$$\therefore \quad F_2 = 400 - F_1$$

Again,

$$\delta_1 = \delta_2 + 15$$

$$\therefore \quad \frac{F_1}{K_1} = \frac{F_2}{K_2} + 15$$

$$\frac{F_1}{10.2539} = \frac{F_2}{12.444} + 15$$

$$1.2136\,F_1 = F_2 + 186.66$$

Substituting Equation (a) in Equation (b),

$$1.2136\,F_1 = 400 - F_1 + 186.66$$

$$2.2136\,F_1 = 586.66$$

$$\therefore \quad F_1 = 265 \text{ N}$$

$$\text{and} \quad F_1 + F_2 = 400$$

$$\therefore \quad F_2 = 400 - 265 = 135 \text{ N}$$

$$F_1 = 265 \text{ N}$$

$$\mathbf{F_2 = 135 \text{ N}}$$

- **Compression of each spring :**

$$\delta_1 = \frac{F_1}{K_1} = \frac{265}{10.2539} = 25.84 \text{ mm}$$

$$\text{and} \quad \delta_2 = \frac{F_2}{K_2} = \frac{135}{12.4444} = 10.84 \text{ mm}$$

$$\delta_1 = 25.84 \text{ mm}$$

$$\mathbf{\delta_2 = 10.84 \text{ mm}}$$

- **Shear stress induced in each spring :**

$$K_{W1} = \frac{4C_1 - 1}{4C_1 - 4} + \frac{0.615}{C_1} = \frac{4 \times 8 - 1}{4 \times 8 - 4} + \frac{0.615}{8} = 1.184$$

$$\tau_1 = K_{W1}\left[\frac{8\,F_1\,C_1}{\pi d_1^2}\right] = \frac{1.184 \times 8 \times 265 \times 8}{\pi\,(5)^2}$$

$$\therefore \quad \tau_1 = 255.67 \text{ N/mm}^2$$

$$K_{W2} = \frac{4C_2 - 1}{4C_2 - 4} + \frac{0.615}{C_2} = \frac{4 \times 7.5 - 1}{4 \times 7.5 - 4} + \frac{0.615}{7.5} = 1.1974$$

$$\tau_2 = K_{W2}\left[\frac{8\,F_2\,C_2}{\pi d_2^2}\right] = \frac{1.1974 \times 8 \times 135 \times 7.5}{\pi\,(4)^2}$$

$$\therefore \quad \tau_2 = 192.65\ \text{N/mm}^2$$

$$\boldsymbol{\tau_1 = 255.67\ \text{N/mm}^2}$$

$$\boldsymbol{\tau_2 = 192.65\ \text{N/mm}^2}$$

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DEPARTMENT OF AUTOMOBILE ENGINEERING

SAUA1304 _ SOLID AND FLUID MECHANICS

UNIT III SLOPE AND DEFLECTION OF BEAMS

UNIT 3 SLOPE AND DEFLECTION OF BEAMS
Deflection and Slope of a Beam – Radius of curvature - Deflection of a Simply Supported Beam (various load condition) - Macaulay's method - Moment area method – Mohr's Theorem – Conjugate beam method for simply supported and cantilever beams, (only point loads & Uniformly distributed loads.)

Introduction: Elastic Stability of Columns

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions. The analysis and design of compression members can differ significantly from that of members loaded in tension or in torsion. If you were to take a long rod or pole, such as a meter stick, and apply gradually increasing compressive forces at each end, nothing would happen at first, but then the stick would bend (buckle), and finally bend so much as to fracture. Try it. The other extreme would occur if you were to saw off, say, a 5-mm length of the meter stick and perform the same experiment on the short piece. You would then observe that the failure exhibits itself as a mashing of the specimen, that is, a simple compressive failure. For these reasons it is convenient to classify compression members according to their length and according to whether the loading is central or eccentric. The term column is applied to all such members except those in which failure would be by simple or pure compression.

General comments

The critical load of a column is proportional to the flexural rigidity EI and inversely proportional to the square of the length. Of particular interest is the fact that the strength of the material itself, as represented by a quantity such as the proportional limit or the The flexural rigidity can be increased by using a "stiffer" material (that is, a material with larger modulus of elasticity E) or by distributing the material in such a way as to increase the moment of inertia I of the cross section, just as a beam can be made stiffer by increasing the moment of inertia. The moment of inertia is increased by distributing the material farther from the centroid of the cross section. Hence, a hollow tubular member is generally more economical for use as a column than a solid member having the same cross-sectional area. Reducing the wall thickness of a tubular member and increasing its lateral dimensions (while keeping the cross-sectional area constant) also increases the critical load because the moment of inertia is increased. This process has a practical limit, however, because eventually the wall itself will become unstable. When that happens, localized buckling occurs in the form of small corrugations or wrinkles in the walls of the column. Thus, we must distinguish between overall buckling of a column, which is discussed in this chapter, and local buckling of its parts. yield stress, does not appear in the equation for the critical load. Therefore, increasing a strength property does not raise the critical load of a slender column. It can only be raised by increasing the flexural rigidity, reducing the length, or providing additional lateral support.

we assumed that the xy plane was a plane of symmetry of the column and that buckling took place in that plane. The latter assumption will be met if the column has lateral supports perpendicular to the plane of the figure, so that the column is constrained to buckle in the xy plane. If the column is supported only at its ends and is free to buckle in any direction, then bending will occur about the principal centroidal axis having the smaller moment of inertia. If the cross section is square or circular, all centroidal axes have the same moment of inertia and buckling may occur in any longitudinal plane.

Limitations

In addition to the requirement of small deflections, the Euler buckling theory used in this section is valid only if the column is perfectly straight before the load is applied, the column and its supports have no imperfections, and the column is made of a linearly elastic material that follows Hooke's law.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded. Columns can be categorized then as:

- Long columns with central loading
- Intermediate-length columns with central loading
- Columns with eccentric loading
- Struts or short columns with eccentric loading

Struts:

Long, slender columns are generally termed as struts; they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons. A short bar loaded in pure compression by a force P acting along the centroidal axis will shorten in accordance with Hooke's law, until the stress reaches the elastic limit of the material. At this point, permanent set is introduced and usefulness as a machine member may be at an end. If the force P is increased still more, the material either becomes "barrel-like" or fractures. When there is eccentricity in the loading, the elastic limit is encountered at smaller loads.

(a) The strut may not be perfectly straight initially.

(b) The load may not be applied exactly along the axis of the Strut.

(c) One part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties throughout the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as $I = Ak^2$,

Where I = area of moment of inertia

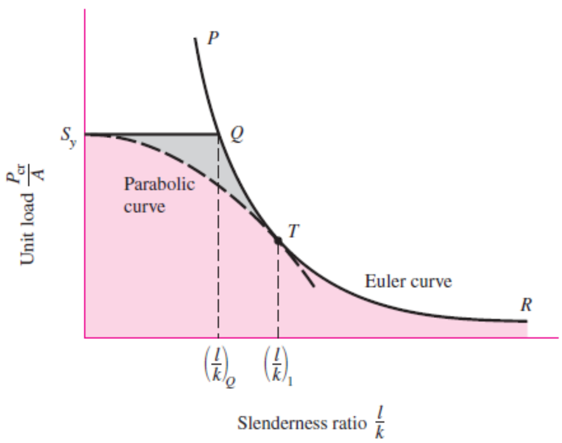
A = area of the cross-section

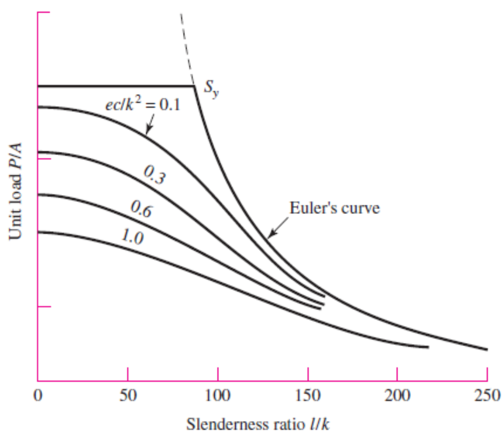
k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k. There will be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

is called the slenderness ratio. Its numerical value indicates whether the member falls into the class of columns or struts.



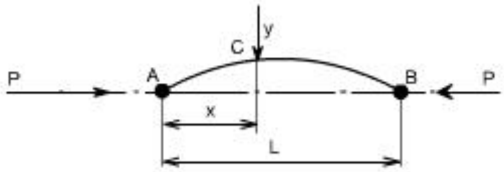


Euler's Theory: The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

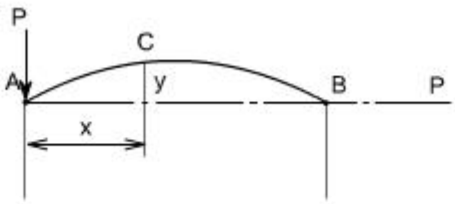
Consider an axially loaded strut, shown below, and is subjected to an axial load 'P' this load 'P' produces a deflection 'y' at a distance 'x' from one end.

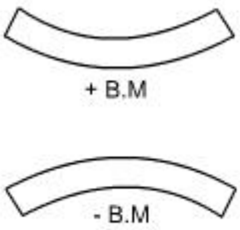
Assume that the ends are either pin jointed or rounded so that there is no moment at either end.



Assumption:

The strut is assumed to be initially straight, the end load being applied axially through centroid.





$$B.M|_C = -Py$$

Futher,we know that

$$EI\,\frac{d^2y}{dx^2} = M$$

$$EI\,\frac{d^2y}{dx^2} = -P.y = M$$

In this equation 'M' is not a function 'x'. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI\,\frac{d^2y}{dx^2} + Py = 0$$

Though this equation is in 'y' but we can't say at this stage where the deflection would be maximum or minimum.

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

So the above differential equation can be arranged in the following form

Let us define a operator

D = d/dx

$(D^2 + n^2)\ y = 0$ where $n^2 = P/EI$

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus y = A cos (nx) + B sin (nx)

Where A and B are some constants.

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at x = 0; y = 0

(ii) at x = L ; y = 0

Applying the first boundary condition yields A = 0.

Applying the second boundary condition gives

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Thus either $B = 0$, or $\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$

if B=0,that y0 for all values of x hence the strut has not buckled yet.Therefore,the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \text{ or } \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \text{ or } P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the “ **Euler Crippling Load** ” P_e from which w obtain.

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0 \qquad \text{i.e. } \sin\ nL = 0$$

The interpretation of the above analysis is that for all the values of the load P, other than those which make sin nL = 0; the strut will remain perfectly straight since

$y = B \sin nL = 0$

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

$$\sin nL = 0 \quad \text{or } nL = \pi$$

$$\text{Therefore } \; n = \frac{\pi}{L}$$

$$\text{Hence } y = B \sin nx = B \sin \frac{\pi x}{L}$$

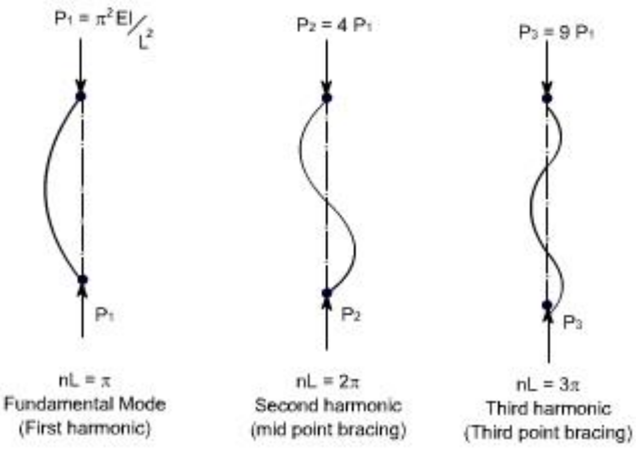
Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that ‘L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; like wise it will be found that the maximum stress is not proportional to load.

The solution chosen of nL = p is just one particular solution; the solutions nL= 2p, 3p, 5p etc are equally valid mathematically and they do, infact, produce values of ‘P_e' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling.

The value selected above is so called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.

The solution nL = 2p produces buckling in two half – waves, 3p in three half-waves etc.



$$L\sqrt{\frac{P}{EI}} = \pi \quad \text{or} \quad P_1 = \frac{\pi^2 EI}{L^2}$$

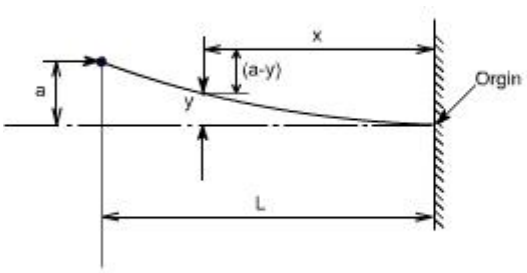
$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \text{ or } P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \text{ or } P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse.

<u>struts and columns with other end conditions:</u> Let us consider the struts and columns having different end conditions

Case b: One end fixed and the other free:



writing down the value of bending moment at the point C

$$B.M|_c = P(a - y)$$

Hence, the differential equation becomes,

$$EI\frac{d^2y}{dx^2} = P(a - y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

$$\text{Let } \frac{P}{EI} = n^2$$

Hence in operator form, the differential equation reduces to $(D^2 + n^2) y = n^2a$

The solution of the above equation would consist of complementary solution and particular solution, therefore

$y_{gen} = A \cos(nx) + \sin(nx) + P. I$

where

P.I = the P.I is a particular value of y which satisfies the differential equation

Hence $y_{P.I} = a$

Therefore the complete solution becomes

$Y = A \cos(nx) + B \sin(nx) + a$

Now imposing the boundary conditions to evaluate the constants A and B

(i) at $x = 0$; $y = 0$

This yields $A = -a$

(ii) at $x = 0$; $dy/dx = 0$

This yields $B = 0$

Hence

$y = -a \cos(nx) + a$

Futher, at $x = L$; $y = a$

Therefore $a = - a \cos(nx) + a$ or $0 = \cos(nL)$

Now the fundamental mode of buckling in this case would be

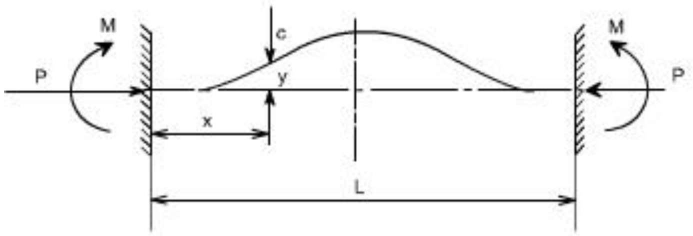
$$nL = \frac{\pi}{2}$$

$$\sqrt{\frac{P}{EI}}\, L = \frac{\pi}{2}, \text{Therefore, the Euler's crippling load is given as}$$

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3

Strut with fixed ends:



Due to the fixed end supports bending moment would also appears at the supports, since this is the property of the support.

Bending Moment at point C = M – P.y

$$EI\frac{d^2y}{dx^2} = M - Py$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$$

$n^2 = \frac{P}{EI}$,Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\ integral}$$

$$y|_{P.I} = \frac{M}{n^2 EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\ Cosnx + A\ Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at x=0:y=0

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

A=0

Therefore,

$$y = -\frac{M}{P}\ Cos\ nx + \frac{M}{P}$$

$$y = \frac{M}{P}\left(1 - Cos\ nx\right)$$

Futher,it maybe noted that at $x = L; y = 0$

$$\text{Then } 0 = \frac{M}{P}\left(1 - Cos\ nL\right)$$

Thus,either $\frac{M}{P} = 0$ or $\left(1 - Cos\ nL\right) = 0$

obviously,$\left(1 - Cos\ nL\right) = 0$

$\cos nL = 1$

Hence the least solution would be

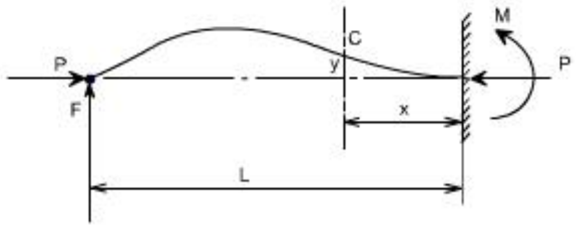
$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\ L = 2\pi$,Thus,the buckling load or crippling load is

Thus, $$P_e = \frac{4\pi^2.EI}{L^2}$$

Case 4

<u>One end fixed, the other pinned</u>



In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin. The moment of F about the built in end then balances the fixing moment.

With the origin at the built in end, the B,M at C is given as

$$EI\frac{d^2y}{dx^2} = -Py + F(L-x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L-x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L-x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L-x)$$

$$y_{particular} = \frac{F}{n^2EI}(L-x) \text{ or } y = \frac{F}{P}(L-x)$$

The full solution is therefore

$$y = A\,\mathrm{Cos}\,mx + B\,\mathrm{Sin}\,nx + \frac{F}{P}(L-x)$$

The boundry conditions relevants to the problem are at x=0;y=0

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x = 0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P}\mathrm{Cos}\,nx + \frac{F}{nP}\mathrm{Sin}\,nx + \frac{F}{P}(L-x)$$

$$y = \frac{F}{nP}\left[\mathrm{Sin}\,nx - nL\,\mathrm{Cos}\,nx + n(L-x)\right]$$

Also when x = L ; y = 0

Therefore

nL Cos nL = Sin nL or tan nL = nL

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is nL = 4.49radian

$$\text{or} \sqrt{\frac{P}{EI}}\, L = 4.49$$

$$\frac{P_e}{EI} L^2 = 20.2$$

$$P_e = \frac{2.05\pi^2\, EI}{L^2}$$

Equivalent Strut Length:

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.

$$\text{i.e. } P_e = \frac{\pi^2 EI}{L^2}$$

Where L is the equivalent length of the strut and can be related to the actual length of the strut depending on the end conditions.

The equivalent length is found to be the length of a simple bow(half sine wave) in each of the strut deflection curves shown. The buckling load for each end condition shown is then readily obtained. The use of equivalent length is not restricted to the Euler's theory and it will be used in other derivations later.

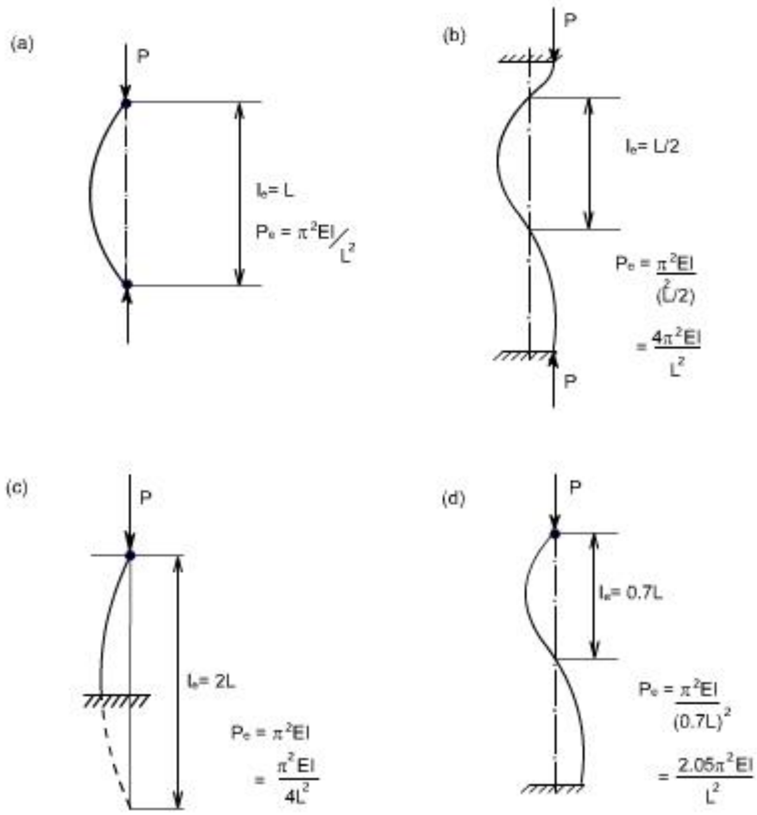
The critical load for columns with other end conditions can be expressed in terms of the critical load for a hinged column, which is taken as a fundamental case.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the freebody diagram would indicates that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we have considered so far are:

(a) Both ends pinned (c) One end fixed, other free

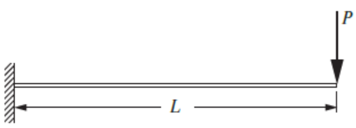
(b) Both ends fixed (d) One end fixed and other pinned

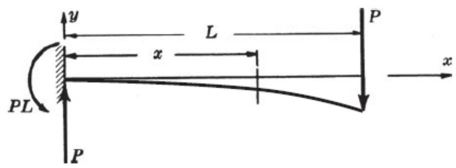


Solved Problems on deflection of beams

1. Determine the deflection at every point of the cantilever beam subject to the single concentrated force *P*, as shown in Figure shown below

SOLUTION: The *x*-*y* coordinate system shown is introduced, where the *x*-axis coincides with the original unbent position of the beam. The deformed beam has the appearance indicated by the heavy line in Fig It is first necessary to find the reactions exerted by the supporting wall upon the bar, and these are easily found from statics to be a vertical force reaction *P* and a moment *PL*, as shown.





According to the sign convention of Chap. 6, the bending moment M at the section x is

$$M = -PL + Px$$

The differential equation (8.4) of the bent beam is then

$$EI\frac{d^2y}{dx^2} = -PL + Px \tag{1}$$

This equation is readily integrated once to yield

$$EI\frac{dy}{dx} = -PLx + \frac{Px^2}{2} + C_1 \tag{2}$$

which represents the equation of the slope, where C_1 denotes a constant of integration. This constant may be evaluated by use of the condition that the slope dy/dx of the beam at the wall is zero since the beam is rigidly clamped there. Equation (2) is true for all values of x and y, and if the condition $x = 0$ is substituted we obtain $0 = 0 + 0 + C_1$ or $C_1 = 0$.

Next, integration of Eq. (2) yields

$$EIy = -PL\frac{x^2}{2} + \frac{Px^3}{6} + C_2 \tag{3}$$

where C_2 is a second constant of integration. Again, the condition at the supporting wall will determine this constant. At $x = 0$, the deflection y is zero since the bar is rigidly clamped. We find $0 = 0 + 0 + C_2$ or $C_2 = 0$.

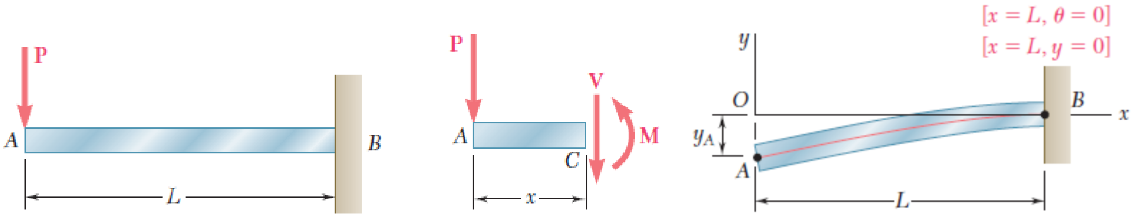
Thus Eqs. (2) and (3) with $C_1 = C_2 = 0$ give the slope dy/dx and deflection y at any point x in the beam. The deflection is maximum at the right end of the beam ($x = L$), under the load P, and from Eq. (3),

$$EIy_{\max} = \frac{-PL^3}{3} \tag{4}$$

where the negative value denotes that this point on the deflection curve lies below the x-axis. If only the magnitude of the maximum deflection at $x = L$ is desired, it is usually denoted by $\Delta_{\max}$ and we have

$$\Delta_{\max} = \frac{PL^3}{3EI} \tag{5}$$

2. The cantilever beam AB is of uniform cross section and carries a load P at its free end A). Determine the equation of the elastic curve and the deflection and slope at A.



Using the free-body diagram of the portion AC of the beam , where C is located at a distance x from end A, we find

$$M = -Px$$

Substituting for M and multiplying both members by the constant EI, we write

$$EI\,\frac{d^2y}{dx^2} = -Px$$

Integrating in x, we obtain

$$EI\,\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1$$

We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ Substituting these values and solving for C_1, we nave

$$C_1 = \tfrac{1}{2}PL^2$$

$$EI\,\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2$$

Integrating both members we write

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2$$

But, at B we have $x = L$, $y = 0$. Substituting we have

$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$
$$C_2 = -\tfrac{1}{3}PL^3$$

Carrying the value of C_2 , we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

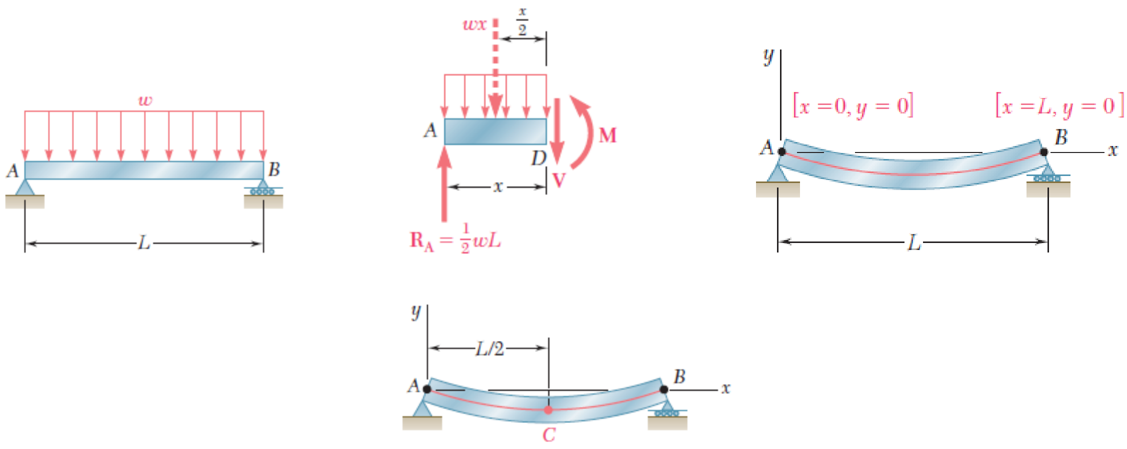
or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

The deflection and slope at A are obtained by letting $x = 0$ We find

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

3. The simply supported prismatic beam AB carries a uniformly distributed load w per unit length. Determine the equation of the elastic curve and the maximum deflection of the beam.



Drawing the free-body diagram of the portion AD of the beam and taking moments about D, we find that

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$$

Substituting for M and multiplying both members of this equation by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

Integrating twice in x, we have

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2$$

Observing that $y = 0$ at both ends of the beam we first let $x = 0$ and $y = 0$ and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_1L$$

$$C_1 = -\tfrac{1}{24}wL^3$$

Carrying the values of C_1 and C_2 we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 - \tfrac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x)$$

Substituting the value obtained for C_1, we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam . Letting $x = L/2$, we have

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{\max} = \frac{5wL^4}{384EI}$$

4. A steel rod 5 cm diameter protrudes 2 m horizontally from a wall. (i) Calculate the deflection due to a load of **1 kN** hung on the end of the rod. The weight of the rod may be neglected. (ii) If **a** vertical steel wire 3 m long, 0.25 cm diameter, supports the end of the cantilever, being taut but unstressed before the load is applied, calculate the end deflection on application of the load. TakeE = 200GN/m2.

The second moment of are of the cross-section is

$$I_x = \frac{\tau}{64}(0.050)^4 = 0.307 \times 10^{-6}\ \mathrm{m}^4$$

The deflection at the end is then

$$v = \frac{PL^3}{3EI} = \frac{(1000)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = 0.0434\ \mathrm{m}$$

Let T = tension in the wire; the area of cross-section of the wire is $4.90 \times 10^{-6}\ \mathrm{m}^2$. The elongation of the wire is then

$$e = \frac{Tl}{EA} = \frac{T(3)}{(200 \times 10^9)(4.90 \times 10^{-6})}$$

The load on the end of the cantilever is then $(1000 - T)$, and this produces a deflection of

$$v = \frac{(1000 - T)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})}$$

If this equals the stretching of the wire, then

$$\frac{(1000 - T)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = \frac{T(3)}{(200 \times 10^9)(4.90 \times 10^{-6})}$$

This gives $T = 934$ N, and the deflection of the cantilever becomes

$$v = \frac{(66)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = 0.00276 \text{ m}$$

5. **A** steel beam rests on two supports 6 m apart, and carries a uniformly distributed load of **10 kN** per metre run. The second moment of area of the cross-section is **1 x 10-3 m4** and E = 200 **GN/m2.** Estimate the maximum deflection.

The greatest deflection occurs at mid-length and has the value given by equation

$$v = \frac{5wL^4}{384EI} = \frac{5(100 \times 10^3)(6)^4}{384(200 \times 10^9)(1 \times 10^{-3})} = 0.00844 \text{ m}$$

Solved Problems on columns

1. A 2-m-long pin-ended column of square cross section is to be made of wood. Assuming E = 13 GPa, σ =12 MPa, and using a factor of safety of 2.5 in computing Euler's critical load for buckling, determine the size of the cross section if the column is to safely support (a) a 100-kN load, (b) a 200-kN load.

(a) For the 100-kN Load. Using the given factor of safety, we make

$$P_{cr} = 2.5(100 \text{ kN}) = 250 \text{ kN} \qquad L = 2 \text{ m} \qquad E = 13 \text{ GPa}$$

in Euler's formula (10.11) and solve for I. We have

$$I = \frac{P_{cr}L^2}{\pi^2 E} = \frac{(250 \times 10^3 \text{ N})(2 \text{ m})^2}{\pi^2(13 \times 10^9 \text{ Pa})} = 7.794 \times 10^{-6} \text{ m}^4$$

Recalling that, for a square of side a, we have $I = a^4/12$, we write

$$\frac{a^4}{12} = 7.794 \times 10^{-6} \text{ m}^4 \qquad a = 98.3 \text{ mm} \approx 100 \text{ mm}$$

We check the value of the normal stress in the column:

$$\sigma = \frac{P}{A} = \frac{100 \text{ kN}}{(0.100 \text{ m})^2} = 10 \text{ MPa}$$

Since σ is smaller than the allowable stress, a 100 × 100-mm cross section is acceptable.

(b) For the 200-kN Load. Solving again Eq. (10.11) for I, but making now $P_{cr} = 2.5(200) = 500$ kN, we have

$$I = 15.588 \times 10^{-6} \text{ m}^4$$

$$\frac{a^4}{12} = 15.588 \times 10^{-6} \qquad a = 116.95 \text{ mm}$$

The value of the normal stress is

$$\sigma = \frac{P}{A} = \frac{200 \text{ kN}}{(0.11695 \text{ m})^2} = 14.62 \text{ MPa}$$

Since this value is larger than the allowable stress, the dimension obtained is not acceptable, and we must select the cross section on the basis of its resistance to compression. We write

$$A = \frac{P}{\sigma_{all}} = \frac{200 \text{ kN}}{12 \text{ MPa}} = 16.67 \times 10^{-3} \text{ m}^2$$

$$a^2 = 16.67 \times 10^{-3} \text{ m}^2 \qquad a = 129.1 \text{ mm}$$

A 130 × 130-mm cross section is acceptable.

Deflection of Beams: Problems for practice

1. A cantilever steel beam has a free length of 3m. The moment of inertia of the section is $30\text{x}10^6$ mm4. A concentrated load of 50kN at the free end. Find the deflection at the free end using

   a. Double integration method

   b. Macauley's Method

   c. Moment Area Method

d. Conjugate Beam Method, Take E= $2x10^5$ N/mm^2

2. A cantilever Beam of 8m carries a UDL of 5kN/m run and a load of W at the free end. If the deflection at the free end is 30mm, calculate the magnitude of the load W, and the slope at the free end. Take E= $2x10^5$ N/mm^2, I = $5x10^7$ mm^4.

3. A cantilever beam of 6m long carries a UDL of 5kN/m throughout its length and a concentrated load of 80 kN. Determine the slope and deflection at the free eng by using moment area method. Take E= $2x10^5$ N/mm^2, I = $2x10^9$ mm^4.

4. A SSB of 6m span carries a concentrated load of 50 kN at 3m from left support. Find the slope at the supports and deflection under the load. EI = 2000 $kN\text{-}m^2$.

5. A SSB of 10 m span carries a concentrated load of 10 kN at its center. It carries a UDL of 2 kN/m over its length. Find the maximum Deflection of beam by

   a. Double integration method

   b. Macauley's Method

   c. Moment Area Method

   d. Conjugate Beam Method, Take E= $2x10^5$ N/mm^2, I = 200x106 mm^4.

6. A beam is simply supported at its ends over a span of 10 m and carries two concentrated loads of 100 kN and 60 kN at a distance of 2 m and 5 m respectively from the left support. Calculate (i) slope at the left support (ii) slope and deflection under the 100 kN load. Assume EI = 36 × 104 kN-m2.

7. (i) State Moment-Area Mohr's theorem.
   (ii) A simply supported beam AB uniform section, 4 m span is subjected to a clockwise moment of 10 kNm applied at the right hinge B. Derive the equation to the deflected shape of the beam. Locate the point of maximum deflection and find the maximum deflection.

Columns: Problems for practice

1. Find the Euler critical load for a hollow cylindrical cast iron column 150mm external diameter, 20 mm wall thickness if it is 6 m long with hinged at both ends. Assume Young's modulus of cast iron as 80 kN/mm2. Compare this load with that given by Rankine formula. Using Rankine constants α = 1/1600 and 567 N/mm2.

2. A column of solid circular section, 12 cm diameter, 3.6 m long is hinged at both ends. Rankine's constant is 1 / 1600, σc = 54 KN/cm2. Find the buckling load. ii) If another column of the same length, end conditions and rankine constant but of 12 cm X 12 cm square cross-section, and different material, has the same buckling load, find the value of σc of its material.

3. Determine the section of a hollow C.I. cylindrical column 5 m long with ends firmly built in. The column has to carry an axial compressive load of 588.6 KN. The internal diameter of the column is 0.75 times the external diameter. Use Rankine's constants.
   a = 1 / 1600, σc = 57.58 KN/cm2 and F.O.S = 6.

4. Find the euler critical load for a hollow cylindrical cast iron column 150mm external diameter, 20mm wall thick ness if it is 6m long with hinged at both ends. Assume young's modulus of cast iron as 80 KN/mm2.compare this load with that given by rankine constants. a=1/1600 and 567N/mm2.

5. A 1.2m long column has a cross section of 45mm diameter one of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using. I. Rankine's formula, take yield stress=560N/mm2 and a=1/1600 for pinned ends. II. Euler's formula Young's modulus for cast iron = 1.2X105 N/mm2.

6. The external and internal diameters of a hollow cast iron column are 50mm and 40mm respectively. If the length of this column is 3m and both of its ends are fixed, determine the crippling load using Euler formula taking E=100Gpa. Also determine the rankine load for the column assuming fc=550Mpa and α=1/1600.

7. An I section joists 400mmx200mmx20mm and 6m long is used as a strut with both ends fixed. What is Euler's crippling load for the column? Take E=200Gpa.

Deflection of Beams

In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value. In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

2. The curvature is always small.

3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.

Equation of the Elastic curve

We first recall from elementary calculus that the curvature of a plane curve at a point Q(x,y) of the curve can be expressed as

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

where dy/dx and d^2y/dx^2 are the first and second derivatives of the function y(x) represented by that curve. But, in the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity. We write, therefore,

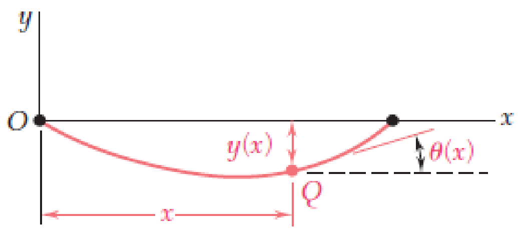
$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

It should be noted that, in this chapter, y represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve. The product EI is known as the flexural rigidity and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integrate. However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Equations by EI and integrate in x. We write

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1$$



where C_1 is a constant of integration. Denoting by u(x) the angle, measured in radians, that the tangent to the elastic curve at Q forms with the horizontal, and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x)$$

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1$$

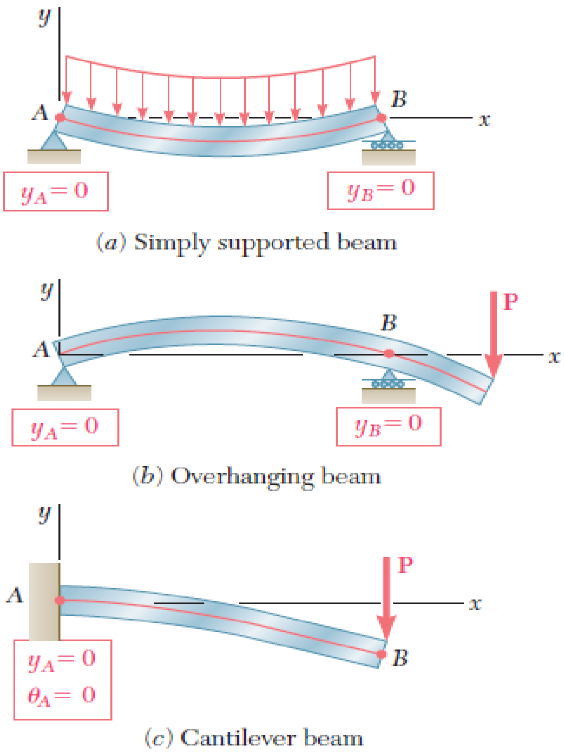
Integrating both members of Eq. (9.5) in x, we have

$$EI\,y = \int_0^x \left[\int_0^x M(x)\,dx + C_1\right]dx + C_2$$

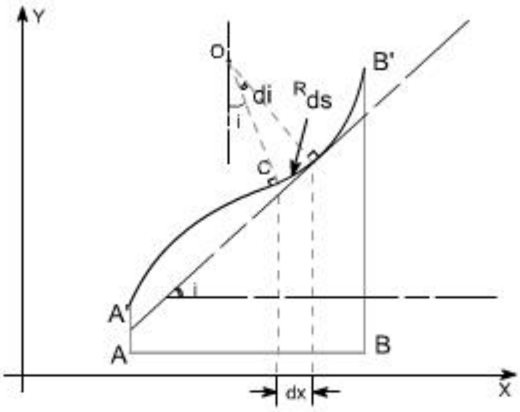
$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1x + C_2$$

where C_2 is a second constant, and where the first term in the right hand member represents the function of x obtained by integrating twice in x the bending moment M(x). If it were not for the fact that the constants C_1 and C_2 are as yet undetermined, would define the deflection of the beam at any given point Q, and define the slope of the beam at Q.

The constants C_1 and C_2 are determined from the boundary conditions or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to statically determinate beams, i.e., to beams supported in such a way that the reactions at the supports can be obtained by the methods of statics, we note that only three types of beams need to be considered here (a) the simply supported beam, (b) the overhanging beam, and (c) the cantilever beam.



In the first two cases, the supports consist of a pin and bracket at A and of a roller at B, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in the Equation, and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations that can be solved for C_1 and C_2. In the case of the cantilever beam, we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Equation and $x = x_A$, $u = u_A = 0$ in Equation, we obtain again two equations that can be solved for C_1 and C_2.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflects to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points. To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad \text{......(1)} \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

$$\text{or } \boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force, bending moment and deflection of the beam may be obtained as differentiating the equation as derived

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx} = F$$

Thus,

$$F = EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{\text{slope} = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods. The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

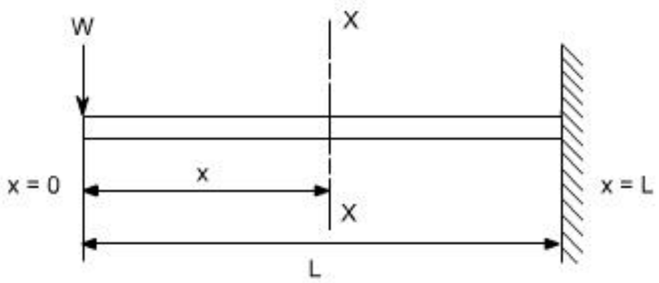
Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples: let us consider few illustrative examples to have a familiarity with the direct integration method

Case 1: Cantilever Beam with Concentrated Load at the end:- A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force and the bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

Therefore $M|_{x-x} = -W.x$

the governing equation $\dfrac{M}{EI} = \dfrac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}\,dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}\,dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

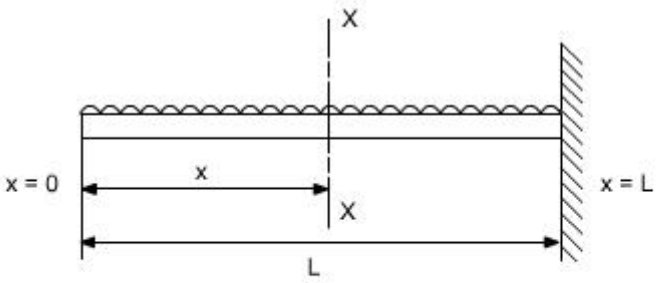
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max}m = +\frac{WL^2}{2EI}}$$

Case 2: A Cantilever with Uniformly distributed Loads:- In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length. The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}\,dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int \frac{dy}{dx} = \int -\frac{wx^3}{6EI}\,dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0

2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

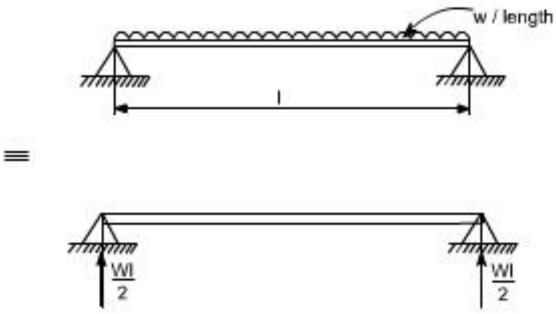
$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3 x}{6} - \frac{wL^4}{8}\right]$$

So $y_{max}m$ will be at $x = 0$

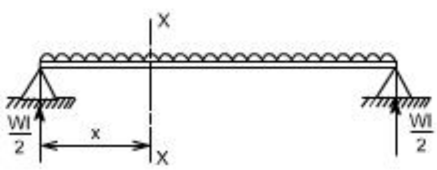
$$\boxed{y_{max}m = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{max}m = \frac{wL^3}{6EI}}$$

Case 3: Simply Supported beam with uniformly distributed Loads:- In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI} dx - \int \frac{wx^2}{2EI} dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad ----- (1)$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

$$\text{Then} \quad y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) The value of maximum deflection would be at the centre i.e. at x = L/2.

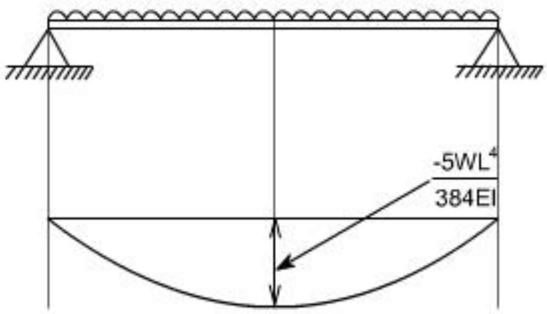
The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

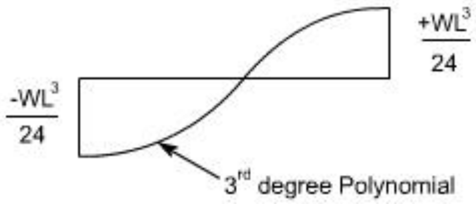
Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



Slope (dy/dx)

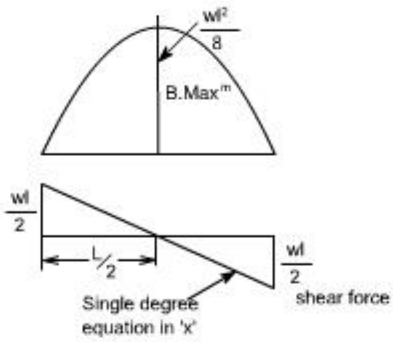
$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



So the bending moment diagram would be

Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$



Shear Force

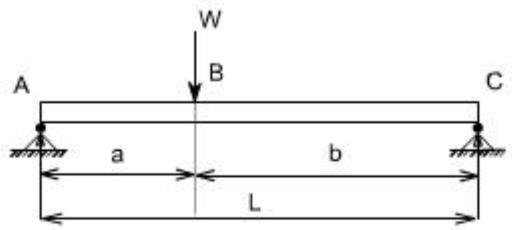
Shear force is obtained by taking third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

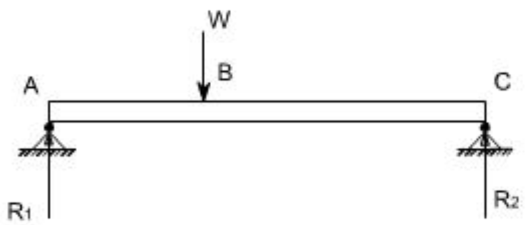
Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \leq x \leq a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x-a) \quad a \leq x \leq l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1 x$$

$$EI\frac{d^2y}{dx^2} = R_1 x - W(x-a)$$

These two equations can be integrated in the usual way to find ‘y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at x = 0; y = 0 in the portion AB i.e. $0 \leq x \leq a$

(b) at x = l; y = 0 in the portion BC i.e. $a \leq x \leq l$

(c) at x = a; dy/dx, the slope is same for both portion

(d) at x = a; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting $K_1 = K_2 = K$, Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{ ------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{ ------- (4)}$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{ ------(5)}$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{ ------(6)}$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y|_{\text{from equation 5}} = y|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{- - - -} \textbf{for } \mathbf{0 \le x \le a} \text{ - - - - - (7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{For } \textbf{for } \mathbf{a \le x \le l} \text{ - - - - - (8)}$$

so either of the equation (7) or (8) may be used to find the deflection at x = a

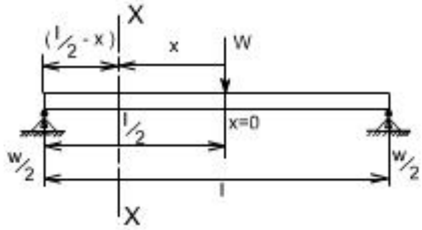
hence substituting x = a in either of the equation we get

$$Y|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD: There is also an alternative way to attempt this problem in a simpler way. Let us considering the origin at the point of application of the load,



$$S.F|_{xx} = \frac{W}{2}$$

$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$

$$B = -\frac{WL^3}{48}$$

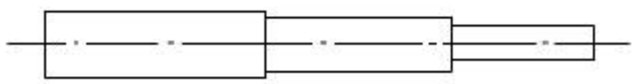
Hence he equation which governs the deflection would be

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$$\boxed{\begin{array}{lll} Y_{max^m}\Big|_{at\,x=0} = -\frac{WL^3}{48EI} & & \text{At the centre} \\ \left(\frac{dy}{dx}\right)_{max^m}\Big|_{at\,x=\pm\frac{L}{2}} = \pm\frac{WL^2}{16EI} & & \text{At the ends} \end{array}}$$

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,



i.e. it is having different cross-section then this method also fails. So there are other methods by which we find the deflection like

1. Macaulay's method in which we can write the different equation for bending moment for

different sections.

2. Area moment methods

3. Energy principle methods

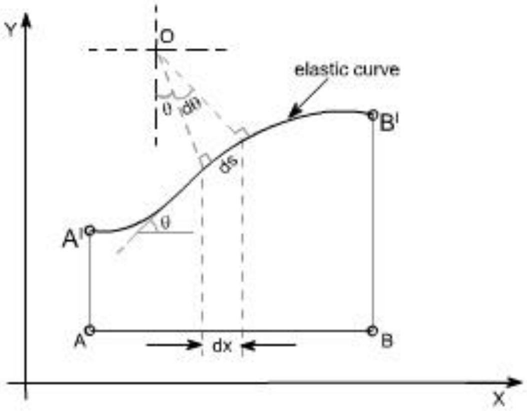
THE AREA-MOMENT / MOMENT-AREA METHODS

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

The moment-area method provides a semigraphical technique for finding the slope and displacement at specific points on the elastic curve of a beam or shaft. Application of the method requires calculating areas associated with the beam's moment diagram; and so if this diagram consists of simple shapes, the method is very convenient to use. Normally this is the case when the beam is loaded with concentrated forces and couple moments. To develop the moment-area method we will make the same assumptions we used for the method of integration: The beam is initially straight, it is

elastically deformed by the loads, such that the slope and deflection of the elastic curve are very small, and the deformations are only caused by bending. The moment-area method is based on two theorems, one used to determine the slope and the other to determine the displacement at a point on the elastic curve.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that dq is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected. We can assume, ds = dx [since the curvature is small] hence, R dq = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

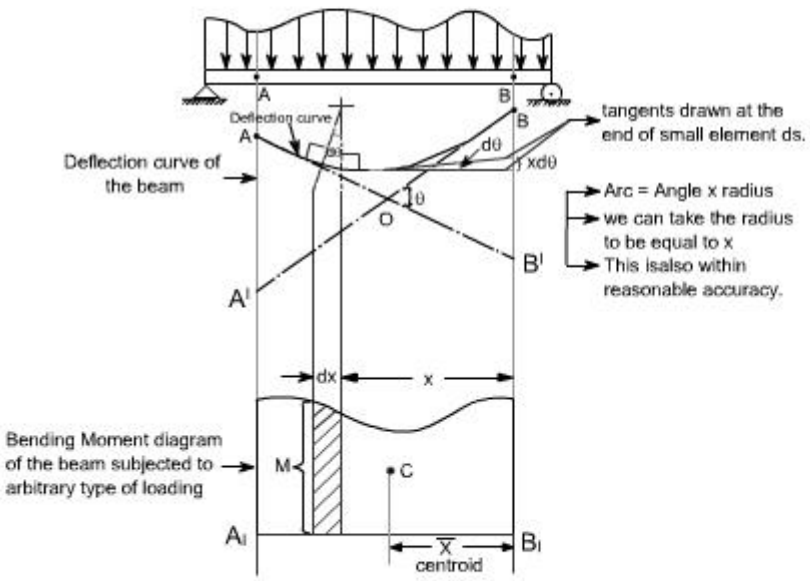
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle ,slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$,hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting ds ≈ dx]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \text{ - - - - - (1)}$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and A1B1is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are futher understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be dq. Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.

From the above relationship the total angle q between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or } \theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI} \times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x dq [each of this intercept may be considered as the arc of a circle of radius x subtended by the angle q

Hence the total distance B'B becomes $\delta = \int_A^B x\,d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of dq = M dx / EI as derived earlier

$$\delta = \int_A^B x\frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$$

[This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Therefore, we are in a position to state the above conclusion in the form of theorem as follows:

Theorem II:

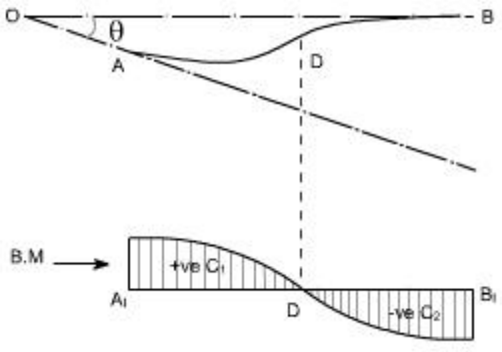
Deflection of point ‘B' relative to point A $= \frac{1}{EI} \times \left\{ \begin{array}{l} \text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram} \end{array} \right\}$

Futher, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

$$\boxed{\delta_A = \frac{1}{EI} A\bar{X}}$$

Therefore,the first moment of area may be obtained simply as a product of the total area of the B.M diagram betweenthe points A and B multiplied by the distance $\bar{x}$ to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids C1and C2. Then to find an angle q between the tangentsat the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

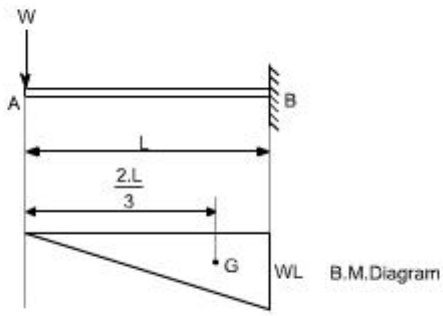
Illustrative Examples: Let us study few illustrative examples, pertaining to the use of these theorems

Example 1:

1. A cantilever is subjected to a concentrated load at the free end.It is required to find out the

deflection at the free end.

Fpr a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area - moment theorem

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B.M diagram between the points A and B}]$$
$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$
$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

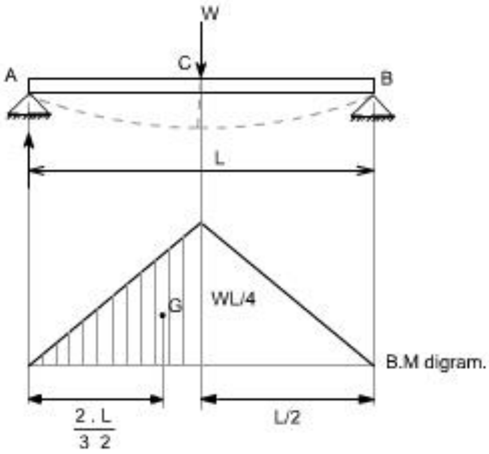
NOTE: In this case the point B is at zero slope.

Thus,

$$\delta = \frac{1}{EI}[\text{first moment of area of B.M diagram between A and B about A}]$$
$$= \frac{1}{EI}[A\bar{y}]$$
$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$
$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid span determine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.



Again working relative to the zero slope at the centre C.

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B.M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right] \quad \text{we are taking half area of the B.M because we have to work out this relative to a zero slope}$$

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

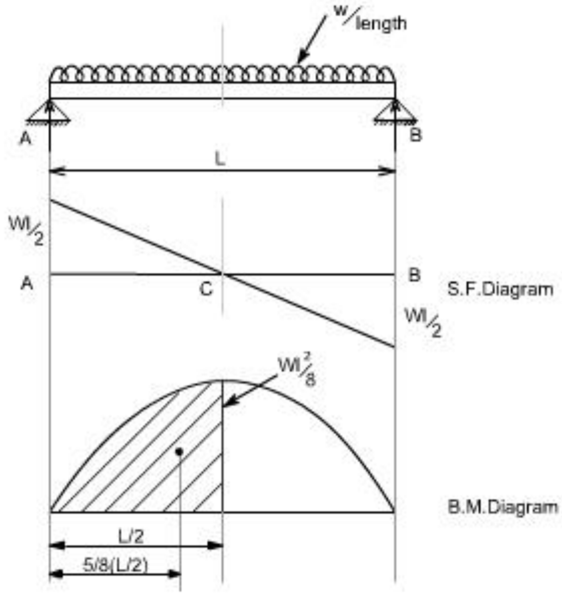
$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, with a intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam, the value of maximum B.M is equal to Wl2 / 8



So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}\left[\text{Area of B.M diagram between point A and C}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}\left[A\,\bar{y}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

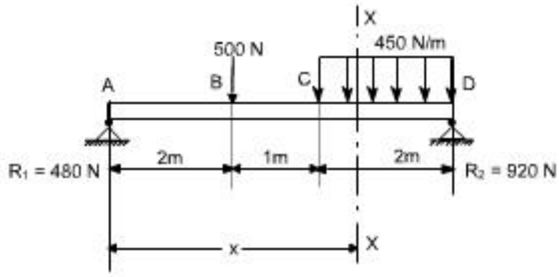
Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integrations be made for each such moment equation. Evaluation of the constants introduced each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

For example consider the beam shown in fig below:

Let us write the general moment equation using the definition M = (∑ M)L, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows

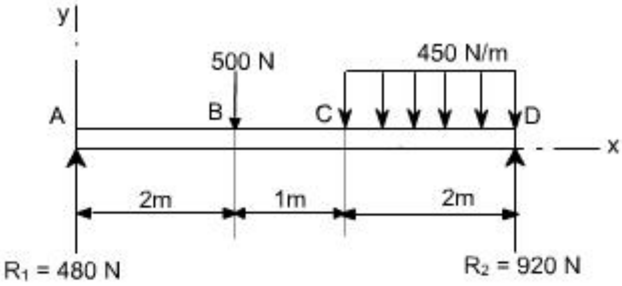


$$M_{AB} = 480\ x\ \text{N.m}$$

$$M_{BC} = \left[480\ x - 500(x-2)\right]\text{N.m}$$

$$M_{CD} = \left[480\ x - 500(x-2) - \frac{450}{2}(x-3)^2\right]\text{N.m}$$

It may be observed that the equation for MCD will also be valid for both MAB and MBC provided that the terms (x - 2) and (x - 3)2are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and (x - 3)2 are nonexistent for values of x for which the terms in parentheses are negative.

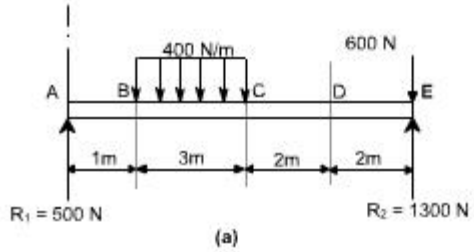


As an clear indication of these restrictions,one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

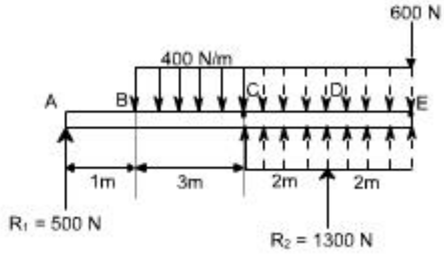
$$M = \left(480x - 500(x-2) - \frac{450}{2}(x-3)^2\right)\text{N.m}$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As an another example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward-distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:



(a)



$$M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) N.m$$

It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratary just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

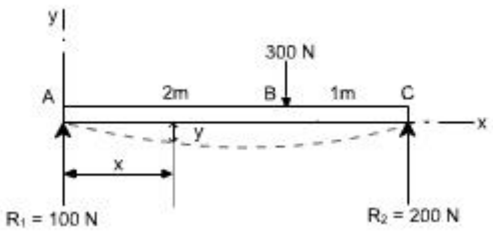
Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

llustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig. Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution : writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right) N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right) N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right) N.m^3 \qquad ......(3)$$

To evaluate the two constants of integration. Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find C2 = 0.keep in mind that < x -2 >3 is to be neglected for negative values.

2. At the other support where x = 3m, the value of deflection y is also zero.

substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50(3-2)^3 + 3.C_1\right) \text{or } C_1 = -133 N.m^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB $(0 \le x \le 2m)$

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right) N.m^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right) N.m^3 \qquad ....(5)$$

segment BC $(2m \le x \le 3m)$

$$EI\frac{dy}{dx} = \left(50x^2 - 150(x-2)^2 - 133x\right) N.m^2 \; ....(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50(x-2)^3 - 133x\right) N.m^3 \; ....(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

We obtain

50 x2– 133 = 0 or x = 1.63 m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x

= 1.63 m in Eq (5), which yields

$$EIy\big|_{max} m = -145 N.m^3 \qquad ....(8)$$

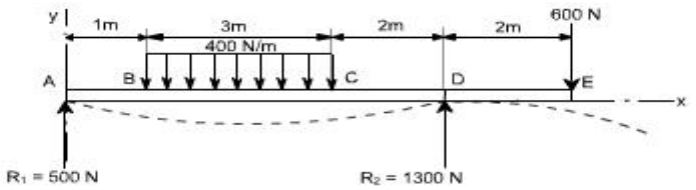
The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by d, the use of y may be reserved to indicate a directed value of deflection.

if E = 30 Gpa and I = 1.9 x 106 mm4 = 1.9 x 10 -6 m4 , Eq. (h) becomes

$$y\big|_{max} m = \left(30 \times 10^9\right)\left(1.9 \times 10^{-6}\right)$$
$$= -2.54 mm$$

Example 2:

It is required to determine the value of EIy at the position midway between the supports and at the overhanging end for the beam shown in figure below.



Solution:

Writing down the moment equation which is valid for the entire span of the beam and applying the differential equation of the elastic curve, and integrating it twice, we obtain

$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) N.m$$
$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}(x-1)^3 + \frac{200}{3}(x-4)^3 + 650(x-6)^2 + C_1\right) N.m$$
$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}(x-1)^4 + \frac{50}{3}(x-4)^4 + \frac{650}{3}(x-6)^3 + C_1x + C_2\right) N.m^3$$

To determine the value of C2, It may be noted that EIy = 0 at x = 0,which gives C2 = 0.Note that the negative terms in the pointed brackets are to be ignored Next,let us use the condition that EIy = 0 at the right support where x = 6m.This gives

$$0 = \frac{250}{3}(6)^3 - \frac{50}{3}(5)^4 + \frac{50}{3}(2)^4 + 6C_1 \quad \text{or} \quad C_1 = -1308 N.m^2$$

Finally, to obtain the midspan deflection, let us substitute the value of x = 3m in the deflection equation for the segment BC obtained by ignoring negative values of the bracketed terms á x - 4 ñ4 and á x - 6 ñ3. We obtain

$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941\ N.m^3$$

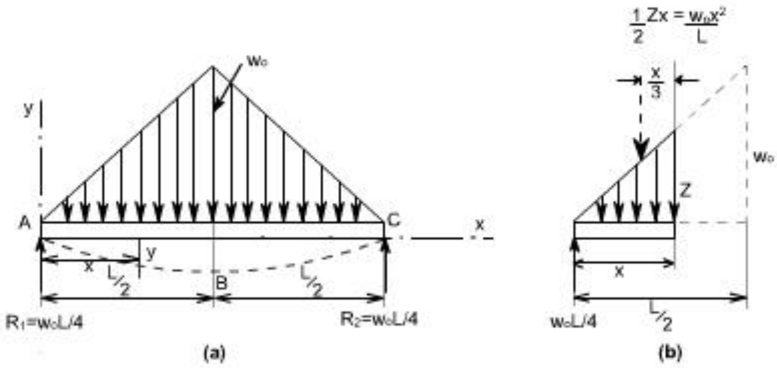
For the overhanging end where x=8 m,we have

$$EIy = \left(\frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)\right)$$

$$= -1814 N.m^3$$

Example 3:

A simply supported beam carries the triangularly distributed load as shown in figure. Determine the deflection equation and the value of the maximum deflection.



Solution:

Due to symmetry, the reactionsis one half the total load of 1/2w0L, or R1 = R2 = 1/4w0L.Due to the advantage of symmetry to the deflection curve from A to B is the mirror image of that from C to B. The condition of zero deflection at A and of zero slope at B do not require the use of a general moment equation. Only the moment equation for segment AB is needed, and this may be easily written with the aid of figure(b).

Taking into account the differential equation of the elastic curve for the segment AB and integrating twice, one can obtain

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{w_0L}{4}x - \frac{w_0x^2}{L}.\frac{x}{3} \qquad ......(1)$$

$$EI\frac{dy}{dx} = \frac{w_0Lx^2}{8} - \frac{w_0x^4}{12L} + C_1 \qquad ......(2)$$

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} + C_1x + C_2 ......(3)$$

In order to evaluate the constants of integration,let us apply the B.C'swe note that at the support A, y = 0 at x = 0.Hence from equation (3), we get C2 = 0. Also,because of symmetry, the slope dy/dx = 0 at midspan where x = L/2.Substituting these conditions in equation (2) we get

$$0 = \frac{w_0 L}{8}\left(\frac{L}{2}\right)^2 - \frac{w_0}{12L}\left(\frac{L}{2}\right)^4 + C_1 C_1 = -\frac{5 w_0 L^3}{192}$$

Hence the deflection equation from A to B (and also from C to B because of symmetry) becomes

$$EIy = \frac{w_0 L x^3}{24} - \frac{w_0 x^5}{60L} - \frac{5 w_0 L^3 x}{192}$$

Whichreducesto

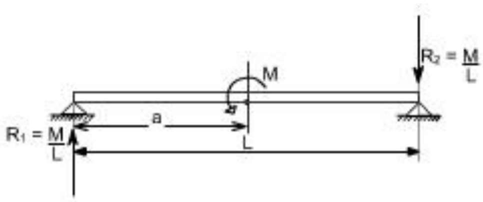
$$EIy = -\frac{w_0 x}{960L}\left(25L^4 - 40L^2 x^2 + 16x^4\right)$$

The maximum deflection at midspan,where $x = L/2$ is then found to be

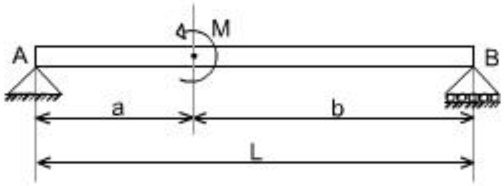
$$\boxed{EIy = -\frac{w_0 L^4}{120}}$$

Example 4: couple acting

Consider a simply supported beam which is subjected to a couple M at adistance 'a' from the left end. It is required to determine using the Macauley's method.



To deal with couples, only thing to remember is that within the pointed brackets we have to take some quantity and this should be raised to the power zero.i.e. M á x - a ñ0 . We have taken the power 0 (zero) ' because ultimately the term M á x - a ñ0 Should have the moment units.Thus with integration the quantity á x - a ñ becomes either á x - a ñ1or á x - a ñ2 Or



Therefore, writing the general moment equation we get

$$M = R_1 x - M\langle x - a\rangle \text{ or } EI\frac{d^2y}{dx^2} = M$$

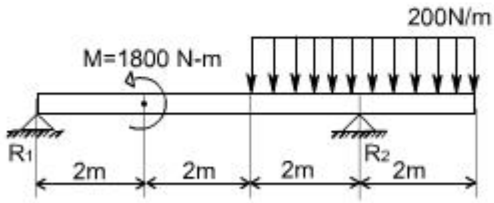
Integrating twice we get

$$EI\frac{dy}{dx} = R_1 . \frac{x^2}{2} - M\langle x - a\rangle^1 + C_1$$

$$EI.y = R_1 . \frac{x^3}{6} - \frac{M}{2}\langle x - a\rangle^2 + C_1 x + C_2$$

Example 5:

A simply supported beam is subjected to U.d.l in combination with couple M. It is required to determine the deflection.



This problem may be attemped in the some way. The general moment equation my be written as

$$M(x) \quad = R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle\langle x - 4\rangle}{2} + R_2\langle x - 6\rangle$$

$$= R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle$$

Thus,

$$EI\frac{d^2y}{dx^2} = R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle$$

Integrate twice to get the deflection of the loaded beam.

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SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF AUTOMOBILE ENGINEERING

SAUA1304 _ SOLID AND FLUID MECHANICS

UNIT IV FLUID PROPERTIES & EQUATION OF MOTION

UNIT 4 FLUID PROPERTIES & EQUATIONS OF MOTION
Fluid Properties: Density - Specific Weight - Specific Gravity - Viscosity - Surface tension - Capillarity - compressibility. Fluid Statics: Hydrostatic Law - Pressure Variation in static fluid - Hydrostatic force on submerged plane-surfaces - Location of hydrostatic force. Manometers - Simple U tube and differential manometers - Buoyancy - Meta-centric height – determination of stability of floating bodies and submerged bodies- Basic equations of motion: Types of fluid flow - Continuity, momentum and energy equations - Euler's and Bernoulli's Equation and its applications.-Flow Measurement: Orifice meter, Venturimeter, Piezometer, Pitot Tube.

Fluids

Substances capable of flowing are known as fluids. Flow is the continuous deformation of substances under the action of shear stresses. Fluids have no definite shape of their own, but confirm to the shape of the containing vessel. Fluids include liquids and gases.

Fluid Mechanics

Fluid mechanics is the branch of science that deals with the behavior of fluids at rest as well as in motion. The study of fluids at rest is called fluid statics. The study of fluids in motion, where pressure forces are not considered, is called fluid kinematics and if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics.

Fluid Properties

1. **DENSITY (or) MASS DENSITY**: Density or mass density of a fluid is defined as the ratio of the mass of the fluid to its volume.

$$Mass\ density,\ \rho = \frac{Mass\ of\ fluid}{Volume\ of\ fluid}$$

S.I unit of density is **kg/m^3**. The value of density for water is **1000 kg/m^3**.

2. **SPECIFIC WEIGHT (or) WEIGHT DENSITY (w):** Specific weight or weight density of a fluid is the ratio between the weight of a fluid to its volume.

$$Weight\ density = \frac{Weight\ of\ fluid}{Volume\ of\ fluid}$$

$$w = \frac{Mass\ of\ fluid\ X\ g}{Volume\ of\ fluid}$$

$$w = \rho g$$

S.I unit of specific weight is **N/m^3.** The value of specific weight or weight density of water is **9810N/m^3 or 9.81 kN/m^3**.

3. SPECIFIC VOLUME (υ):

Specific volume of a fluid is defined as the volume of a fluid occupied by unit mass.

$$Specific\ volume = \frac{Volume\ of\ a\ fluid}{Mass\ of\ fluid} = \frac{1}{\rho}$$

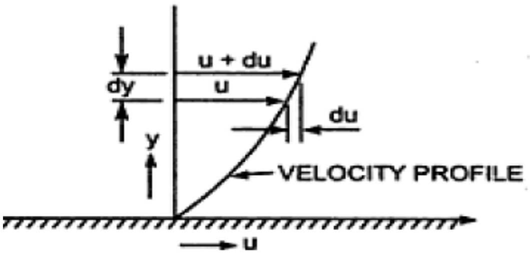
Thus specific volume is the reciprocal of mass density. **S.I unit: m^3 /kg**

4. **SPECIFIC GRAVITY or RELATIVE DENSITY (s):** Specific gravity is defined as the ratio of the specific weight of a fluid to the specific weight of a standard fluid.

$$\text{Specific gravity} = \frac{\text{Specific weight or density of liquid}}{\text{Specific weight or density of water}}$$

Example:
Specific gravity of water=1
Specific gravity of mercury=13.6

5. **VISCOSITY:** Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over adjacent layer of the fluid.



Velocity variation near a solid boundary.

NEWTONS LAW OF VISCOSITY: The shear stress between two layers is proportional to the rate of change of velocity with respect to y.

$$\tau \,\alpha\, \frac{du}{dy}$$

$$\tau = \mu \frac{du}{dy}$$

where, μ is co-efficient of dynamic viscosity or viscosity

du/dy rate of shear strain or rate of shear deformation or velocity gradient.

Thus the viscosity is also defined as the shear stress required to produce unit rate of shear strain.

$$\mu = \frac{\tau}{\left(\frac{du}{dy}\right)}$$

S.I unit: **Ns/m^2.** It is still expressed in **poise** (P) as well as **centipoises** (cP).

$$\text{One poise} = \frac{1}{10}\frac{Ns}{m^2}; \qquad 1 \text{ centipoise} = \frac{1}{100} \text{ poise}$$

Kinematic Viscosity (ν): It is defined as the ratio between the dynamic viscosity and density of the fluid.

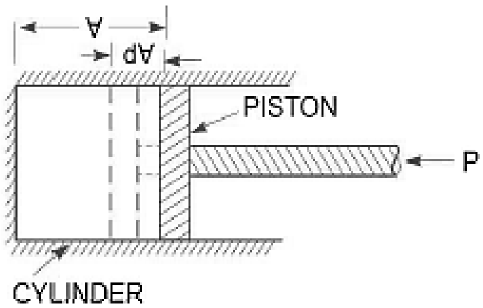
$$\nu = \frac{Dynamic\ viscosity}{Density} = \frac{\mu}{\rho}$$

SI unit: m^2 /s; **CGS unit** 'stoke'. 1 stoke = 1 cm^2 / sec = 10^{-4} m^2 /s

6. **COMPRESSIBILITY:** Compressibility is the reciprocal of the bulk modulus of elasticity, K, which is defined as the ratio of compressive stress to volumetric strain.

$$\text{Bulk modulus K} = \frac{Increase\ of\ pressure}{Volumetric\ Strain} = \frac{dp}{\frac{-dV}{V}}$$

$$\text{Compressibility} = \frac{1}{K}$$



Cohesion is due to the force of attraction between molecules of same liquid

Adhesion is defined as the force of attraction between the molecules of two different liquids or between the molecules of the liquid and molecules of the solid boundary surface.

7. **SURFACE TENSION:** Surface tension is defined as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension.

   Some important real life examples are

   (i) Formation of water bubbles.

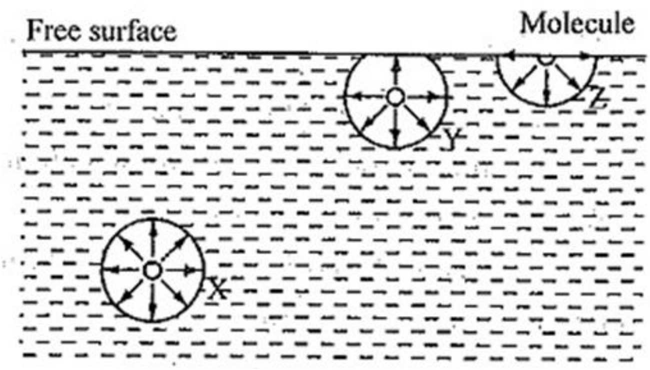
   (ii) Formation of rain droplets.

   (iii) Collection of dust particles on water surface.

   (iv) A small needle can gently place on the liquid surface without sinking.

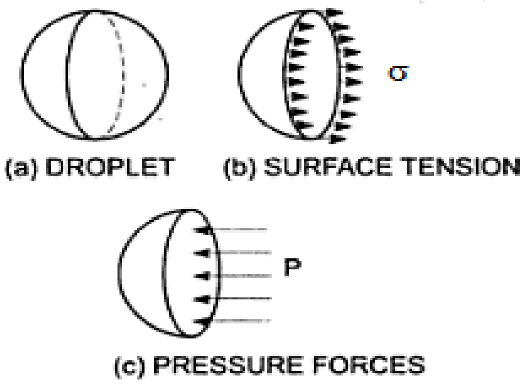
   (v) Breakup of liquid jets.

   (vi) Capillary rise and capillary siphoning.



Surface Tension on Liquid Droplet:

Consider a small spherical droplet of a liquid of diameter'd'. On the entire surface of the droplet, the tensile force due to surface tension will be acting.



Forces on droplet.

Where, σ = Surface tension of the liquid

p = Pressure intensity inside the droplet (in excess of the outside pressure intensity)

d = Dia. of droplet

Let the droplet is cut into two halves. The forces acting on one half will be

i) Tensile force (FT)due to surface tension acting around the circumference of the cut portion as shown in fig. and this is equal to = σ x Circumference = σ x π d

ii) Pressure force (Fp) on the area $\pi d^2/4$ is = p x $\pi d^2/4$ as shown in the figure. These two forces are equal under equilibrium conditions.

i.e., p x $\pi d^2/4 = \sigma \times \pi d$

Therefore, $p = 4\sigma/p$

Surface Tension on a Hollow Bubble:

A hollow bubble like a soap bubble in air has two surfaces in contact with air, one inside and other outside. Thus two surfaces arc subjected to surface tension. In that case,

$$p \times \frac{\pi d^2}{4} = 2 \times (\sigma \times \pi d)$$

$$\text{Therefore, } p = \frac{8\sigma}{d}$$

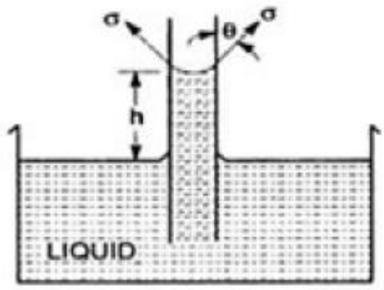
8. CAPILLARITY: Capillarity is defined as a phenomenon of rise or fall of a liquid surface in a small tube relative to the adjacent general level of liquid when the tube is held vertically in the liquid. The rise of liquid surface is known as **capillary rise** while the fall of the liquid surface is known as **capillary depression**. It is expressed in terms of cm or mm of liquid.

Its value depends upon the

- specific weight of the liquid
- diameter of the tube and
- surface tension of the liquid.

Expression for Capillary Rise

Consider a glass tube of small diameter 'd' opened at both ends and is inserted in a liquid. The liquid will rise in the tube above the level of the liquid.



Capillary rise

Let, h = height of the liquid in the tube.

σ = Surface tension of liquid

θ= Angle of contact between liquid and glass lube.

Under a state of equilibrium,

The weight of liquid of height h = Vertical component of surface tension force

(Area of tube x h) x ρ x g = σ x Circumference x cos θ

$$\frac{\pi d2}{4} \text{ x h x } \rho \text{ x g } = \sigma \text{ x } \pi d \text{ x } \cos\theta$$

$$h = \frac{4\sigma\cos\theta}{\rho \text{ x g} X\, d} = \frac{4\sigma\cos\theta}{wd}$$

Example: 5000 litres of an oil weighs 45 kN. Find its Specific weight, mass density and relative density.

Given: Volume, V = 5000 lit = 5000/1000 = 5 m^3 Weight, W= 45 kN = 45000 N

Specific Weight, w = W/V = 45000 / 5 = 9000 N/m^3 = 9 kN/m^3

Specific Weight, w = ρg

Mass density, ρ = w/g = 9000/ 9.81 = 917.43 kg/m^3

Relative density = Density of oil/density of water = 917.43/ 1000 = 0.917

2. The density of an oil is 850 kg/m^3. Find its relative density and Kinematic viscosity if the dynamic viscosity is 5 x 10^{-3} kg/ms

Density of oil, ρ_{oil} = 850 kg/m^3

Density of water, ρ_{water} = 1000 kg/m^3

Relative density of oil = 850/1000 = 0.85

Dynamic viscosity = $\mu = 5 \times 10^{-3}$ kg/ms = 5×10^{-3} N s/m^2

Kinematic viscosity = $\nu = \mu / \rho = 5 \times 10^{-3} / 850 = 5.882 \times 10^{-6}$ m^2/s

A flat plate of area 1.5×10^6 mm^2 is pulled with a speed of 0.4 m/s relative to another plate located at a distance of 0.15 mm from it. Find the force and power required to maintain this speed, if the fluid separating them is having viscosity as 1 poise.

Solution. Given :

Area of the plate, $A = 1.5 \times 10^6 \text{ mm}^2 = 1.5 \text{ m}^2$

Speed of plate relative to another plate, $du = 0.4$ m/s

Distance between the plates, $dy = 0.15 \text{ mm} = 0.15 \times 10^{-3}$ m

Viscosity $\mu = 1 \text{ poise} = \dfrac{1}{10} \dfrac{\text{Ns}}{\text{m}^2}$.

(*i*) $\therefore$ Shear force, $F = \tau \times \text{area} = 266.66 \times 1.5 = \mathbf{400\ N.\ Ans.}$

(*ii*) Power* required to move the plate at the speed 0.4 m/sec

$$= F \times u = 400 \times 0.4 = \mathbf{160\ W.\ Ans.}$$

Calculate the dynamic viscosity of an oil, which is used for lubrication between a square plate of size 0.8 m × 0.8 m and an inclined plane with angle of inclination 30° as shown in Fig. The weight of the square plate is 300 N and it slides down the inclined plane with a uniform velocity of 0.3 m/s. The thickness of oil film is 1.5 mm.

Solution. Given :

Area of plate, $A = 0.8 \times 0.8 = 0.64 \text{ m}^2$

Angle of plane, $\theta = 30°$

Weight of plate, $W = 300$ N

Velocity of plate, $u = 0.3$ m/s

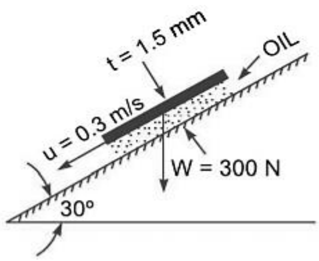
Thickness of oil film, $t = dy = 1.5 \text{ mm} = 1.5 \times 10^{-3}$ m

Let the viscosity of fluid between plate and inclined plane is μ.

Component of weight W, along the plane = $W \cos 60° = 300 \cos 60° = 150$ N

Thus the shear force, F, on the bottom surface of the plate = 150 N

$$\tau = \mu \frac{du}{dy}$$



$$\frac{150}{0.64} = \mu \frac{0.3}{1.5 \times 10^{-3}}$$

$$\mu = \frac{150 \times 1.5 \times 10^{-3}}{0.64 \times 0.3} = 1.17 \text{ N s/m}^2 = 1.17 \times 10 = \mathbf{11.7\ poise.\ Ans.}$$

du = change of velocity = $u - 0 = u = 0.3$ m/s

$$dy = t = 1.5 \times 10^{-3} \text{ m}$$

$$\tau = \frac{F}{\text{Area}} = \frac{150}{0.64} \text{ N/m}^2$$

Example Determine the viscosity of a liquid having kinematic viscosity 6 stokes and sp. Gravity of 1.9.

Solution. Given :

Kinematic viscosity $\nu = 6$ stokes $= 6$ cm^2/s $= 6 \times 10^{-4}$ m^2/s

Sp. gr. of liquid $= 1.9$

Now sp. gr. of a liquid $= \dfrac{\text{Density of the liquid}}{\text{Density of water}}$

or $$1.9 = \frac{\text{Density of liquid}}{1000}$$

$\therefore$ Density of liquid $= 1000 \times 1.9 = 1900 \dfrac{\text{kg}}{\text{m}^3}$

$\therefore$ Using the relation $\nu = \dfrac{\mu}{\rho}$, we get

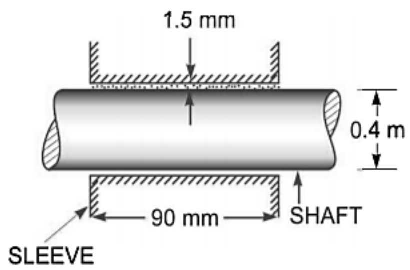
$$6 \times 10^{-4} = \frac{\mu}{1900}$$

$$\mu = 6 \times 10^{-4} \times 1900 = 1.14 \text{ Ns/m}^2$$

$$= 1.14 \times 10 = \textbf{11.40 poise. Ans.}$$

The dynamic viscosity of an oil, used for lubrication between a shaft and sleeve is 6 poise. The shaft is of diameter 0.4 m and rotates at 190 r.p.m. Calculate the power lost in the bearing for a sleeve length of 90 mm. The thickness of the oil film is 1.5 mm.

Solution. Given :



Viscosity $\mu = 6$ poise

$$= \frac{6}{10} \frac{\text{N s}}{\text{m}^2} = 0.6 \frac{\text{N s}}{\text{m}^2}$$

Dia. of shaft, $D = 0.4$ m

Speed of shaft, $N = 190$ r.p.m

Sleeve length, $L = 90$ mm $= 90 \times 10^{-3}$ m

Thickness of oil film, $t = 1.5$ mm $= 1.5 \times 10^{-3}$ m

Tangential velocity of shaft, $u = \dfrac{\pi D N}{60} = \dfrac{\pi \times 0.4 \times 190}{60} = 3.98$ m/s

Using the relation $\tau = \mu \dfrac{du}{dy}$

where du = Change of velocity $= u - 0 = u = 3.98$ m/s

dy = Change of distance $= t = 1.5 \times 10^{-3}$ m

$$\tau = 0.6 \times \frac{3.98}{1.5 \times 10^{-3}} = 1592 \text{ N/m}^2$$

This is shear stress on shaft

$\therefore$ Shear force on the shaft, F = Shear stress $\times$ Area

$$= 1592 \times \pi D \times L = 1592 \times \pi \times .4 \times 90 \times 10^{-3} = 180.05 \text{ N}$$

Torque on the shaft, $T = \text{Force} \times \frac{D}{2} = 180.05 \times \frac{0.4}{2} = 36.01 \text{ Nm}$

$\therefore$ *Power lost $= \frac{2\pi NT}{60} = \frac{2\pi \times 190 \times 36.01}{60} = $ **716.48 W. Ans.**

Determine the bulk modulus of elasticity of a liquid, if the pressure of the liquid is increased from 70 N/cm² to 130 N/cm². The volume of the liquid decreases by 0.15 per cent.

Solution. Given :

Initial pressure $= 70 \text{ N/cm}^2$

Final pressure $= 130 \text{ N/cm}^2$

$\therefore$ dp = Increase in pressure $= 130 - 70 = 60 \text{ N/cm}^2$

Decrease in volume $= 0.15\%$

$$\therefore \quad -\frac{d\forall}{\forall} = +\frac{0.15}{100}$$

BULK MODULUS $$K = \frac{dp}{-\frac{d\forall}{\forall}} = \frac{60 \text{ N/cm}^2}{\frac{.15}{100}} = \frac{60 \times 100}{.15} = \mathbf{4 \times 10^4 \text{ N/cm}^2}. \textbf{ Ans.}$$

The surface tension of water in contact with air at 20°C is 0.0725 N/m. The pressure inside a droplet of water is to be 0.02 N/cm² greater than the outside pressure. Calculate the diameter of the droplet of water.

Solution. Given :

Surface tension, $\sigma = 0.0725 \text{ N/m}$

Pressure intensity, p in excess of outside pressure is

$$p = 0.02 \text{ N/cm}^2 = 0.02 \times 10^4 \frac{\text{N}}{\text{m}^2}$$

we get $p = \frac{4\sigma}{d}$ or $0.02 \times 10^4 = \frac{4 \times 0.0725}{d}$

$$d = \frac{4 \times 0.0725}{0.02 \times (10)^4} = .00145 \text{ m} = .00145 \times 1000 = \mathbf{1.45 \text{ mm}}. \textbf{ Ans.}$$

Calculate the capillary effect in millimetres in a glass tube of 4 mm diameter, when immersed in (i) water, and (ii) mercury. The temperature of the liquid is 20°C and the values of the surface tension of water and mercury at 20°C in contact with air are 0.073575 N/m and 0.51 N/m respectively. The angle of contact for water is zero and that for mercury is 130°. Take density of water at 20°C as equal to 998 kg/m³.

Solution. Given :

Dia. of tube, $d = 4 \text{ mm} = 4 \times 10^{-3}$ m

The capillary effect (*i.e.*, capillary rise or depression) is given by equation (1.20) as

$$h = \frac{4\sigma \cos\theta}{\rho \times g \times d}$$

where σ = surface tension in N/m

θ = angle of contact, and ρ = density

(*i*) **Capillary effect for water**

$$\sigma = 0.073575 \text{ N/m}, \theta = 0°$$

$$\rho = 998 \text{ kg/m}^3 \text{ at } 20°\text{C}$$

$$\therefore \quad h = \frac{4 \times 0.073575 \times \cos 0°}{998 \times 9.81 \times 4 \times 10^{-3}} = 7.51 \times 10^{-3} \text{ m} = \mathbf{7.51 \text{ mm. Ans.}}$$

(*ii*) **Capillary effect for mercury**

$$\sigma = 0.51 \text{ N/m}, \theta = 130° \text{ and}$$

$$\rho = \text{sp. gr.} \times 1000 = 13.6 \times 1000 = 13600 \text{ kg/m}^2$$

$$\therefore \quad h = \frac{4 \times 0.51 \times \cos 130°}{13600 \times 9.81 \times 4 \times 10^{-3}} = -2.46 \times 10^{-3} \text{ m} = \mathbf{-2.46 \text{ mm. Ans.}}$$

9.VAPOUR PRESSURE

Vapour pressure is the pressure of the vapor over a liquid which is confined in a closed vessel at equilibrium. Vapour pressure increases with temperature. All liquids exhibit this phenomenon.

Types of fluid

i. **Ideal Fluid:**

A fluid, which is incompressible and is having no viscosity, is known as an ideal fluid.

ii. **Real Fluid:**

A fluid, which possesses viscosity, is known as real fluid. All the fluids, are real fluids in actual practice.

iii. **Newtonian Fluid:**

A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or) velocity gradient, is known as a Newtonian fluid

iv. **Non-Newtonian Fluid:**

A real fluid, in which the shear stress is not proportional to the rate of shear strain (or) velocity gradient, is known as a Non-Newtonian fluid.

v. **Ideal Plastic Fluid:**

A fluid, in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain (or) velocity gradient, is known as ideal plastic fluid

Fluid pressure

Fluid pressure is the force exerted by the fluid per unit area. Fluid pressure is transmitted with equal intensity in all directions and acts normal to any plane.

$$p = \frac{F}{A}$$

S.I unit of fluid pressure are N/m^2 or Pa,
where 1 N/m^2 = 1 Pa.
Many other pressure units are commonly used:
1 bar = 105 N/m^2
1 atmosphere = 101325 N/m^2 = 101.325kN/m^2 = 1.01325 bar= 760mm of mercury = 10.336m of water

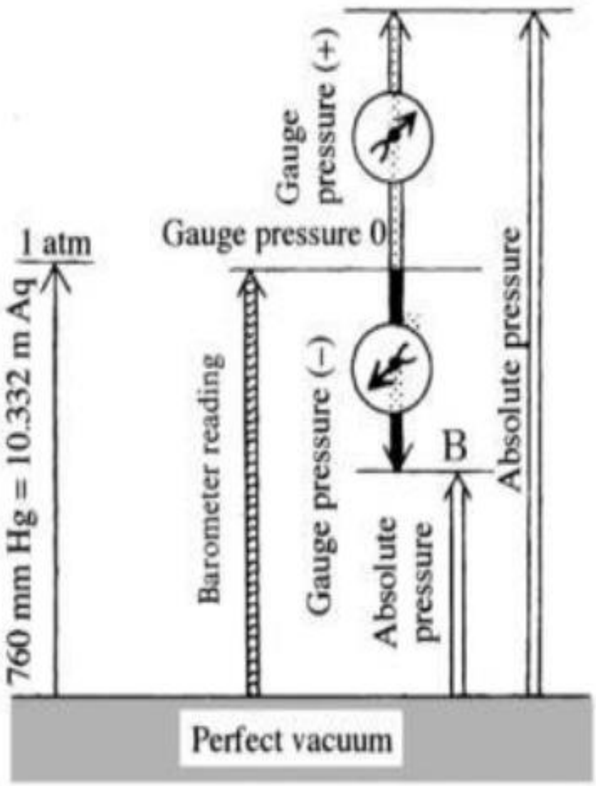
Pressure Head: The pressure intensity exerted at the base of a column of homogenous fluid of a given height in metres.

Atmospheric Pressure: The pressure at the surface of the earth exerted by the head of air above the surface

Gauge Pressure: The pressure measured by a pressure gauge above or below atmospheric pressure

Vacuum pressure: The gauge pressure less than atmospheric is called Vacuum pressure or negative pressure

Absolute Pressure: The pressure measured above absolute zero or vacuum.



Atmospheric, Gauge & Absolute pressure

Fig.5. Barometer, Atmospheric, Gauge and Absolute Pressure

Fluid Pressure

Fluid pressure is the force exerted by the fluid per unit area.

Fluid pressure or Intensity of pressure or pressure, = Fluids exert pressure on surfaces with which they are in contact.

Fluid pressure is transmitted with equal intensity in all directions and acts normal to any plane. In the same horizontal plane the pressure intensities in a liquid are equal.

Hydrostatic law

The **hydrostatic law** is a principle that identifies the amount of pressure exerted at a specific point in a given area of fluid.

It states that, "The rate of increase of pressure in the vertically downward direction, at a point in a static fluid, must be equal to the specific weight of the fluid."

Pressure Variation in static fluid

Consider a small vertical cylinder of static fluid in equilibrium.

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Consider a small vertical cylinder of static fluid in equilibrium.

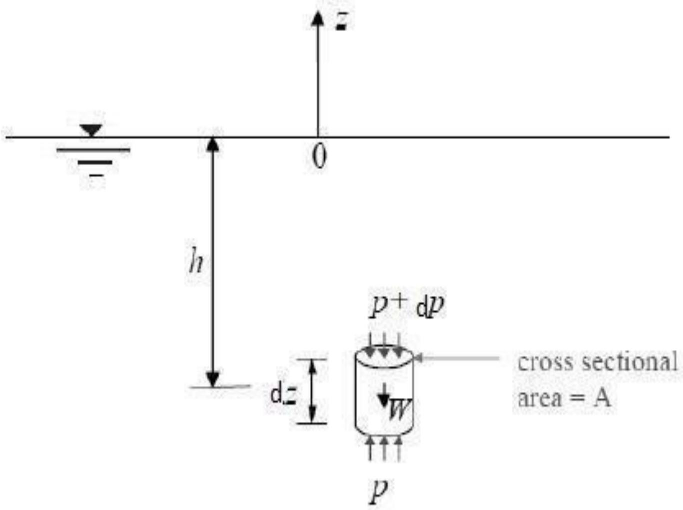


Fig.6. Pressure variation in static fluid

Assume that the sectional area is "**A**" and the pressure acting upward on the bottom surface is **p** and the pressure acting downward on the upper surface (dz above bottom surface) is (p + dp)dz.

Let the free surface of the fluid be the origin, i.e., Z = 0. Then the pressure variation at a depth Z = -h below the free surface is governed by

$(p + dp) A + W = pA$

⇨ $dpA + \rho gAdz = 0$ [W= *w* x volume = ρg Adz] $dp = -\rho gdz$

⇨ $= - \rho g = - w$

Therefore, the hydrostatic pressure increases linearly with depth at the rate of the specific weight, *w* = ρg of the fluid.

If fluid is homogeneous, ρ is constant. By simply integrating the above equation, $\int dp = - \int \rho g\, dz$ =>

$p = - \rho g Z + C$ Where C is constant of integration.

When z = 0 (on the free surface), p = C = po = the atmospheric pressure. Hence, $p = - \rho gZ + p_o$

Pressure given by this equation is called absolute pressure, i.e., measured above perfect vacuum.

However, it is more convenient to measure the pressure as gauge pressure by setting atmospheric pressure as datum pressure. By setting po = 0,

$p = -\rho gz+0 = -\rho gz = \rho gh$

$\mathbf{p = \mathit{w}h}$

The equation derived above shows that when the density is constant, the pressure in a liquid at rest increases linearly with depth from the free surface.

Here, **h** is known as ***pressure head*** or simply ***head*** of fluid.

In fluid mechanics, fluid pressure is usually expressed in height of fluids or head of fluids.

Hydrostatic force

Hydrostatic pressure is the force exerted by a static fluid on a plane surface, when the static fluid comes in contact with the surface. This force will act normal to the surface. It is also known as ***Total Pressure***.

The point of application of the hydrostatic or total pressure on the surface is known as ***Centre of pressure***.

The vertical distance between the free surface of fluid and the centre of pressure is called depth of centre of pressure or location of hydrostatic force.

Total Pressure on a Horizontally Immersed Surface

Consider a plane horizontal surface immersed in a liquid as shown in figure.

Let, w = Specific weight of the liquid, kN/m³ A = Area of the immersed surface in m²

= Depth of the horizontal surface from the liquid level in m We know that,

Total pressure on the surface, **P** = Weight of the liquid above the immersed surface

P = Specific weight of liquid x Volume of liquid

= Specific weight of liquid x Area of surface x Depth of liquid P = wA kN

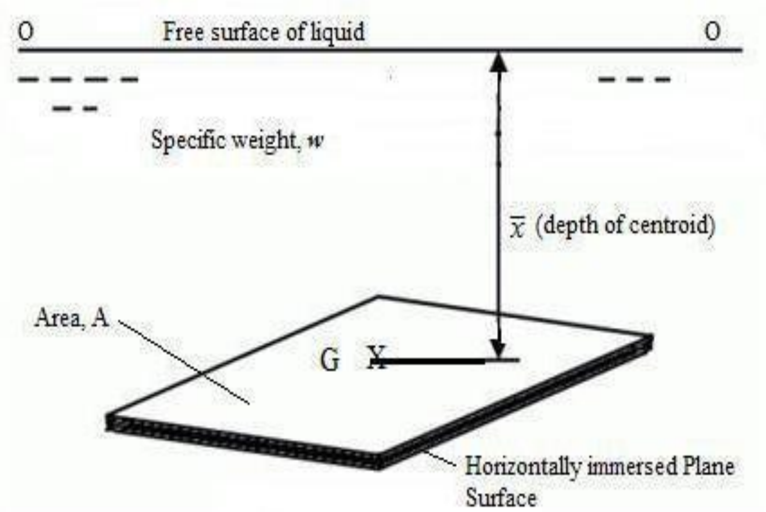


Fig:7. Horizontal Plane surface submerged in liquid

Total Pressure and depth of centre of pressure on a Vertically Immersed Surface

Consider an irregular plane vertical surface immersed in a liquid as shown in figure. Let,

w = Specific weight of liquid

A = Total area of the immersed surface

= Depth of the center of gravity of the immersed surface from the liquid surface

Now. consider a strip of width 'b', thickness 'dx' and at a depth x from the free surface of the liquid

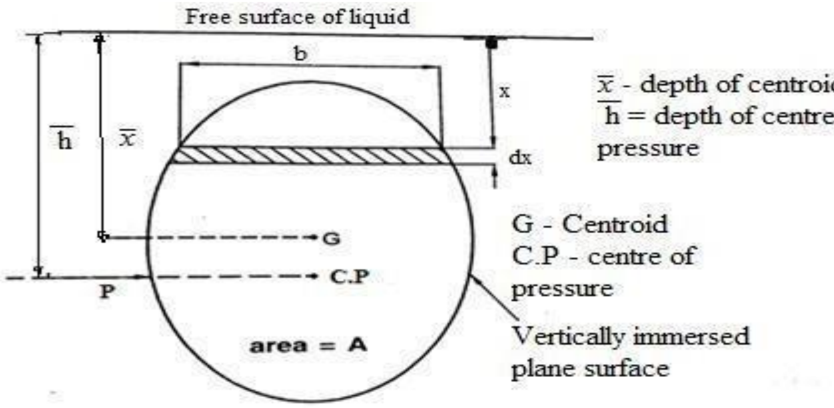


Fig: 9. Vertical Plan immersed in liquid

Moment of pressure on the strip about the free surface of liquid = w *x b dx* X *x* = w x^2 *b dx* Total moment on the entire plane immersed surface = $\int w$ x^2 *b dx*

M = $w \int$ ^{2}

But, $\int$ ^{2} = second moment of area about free liquid surface = Io

therefore, M = w Io

Io = IG + A x^2, according to parallel axis theorem.

Therefore, M = w (IG + A x^2) (1)

Also = x h = w A*x* xh (2)

Since equations 1 & 2 are equal,

A x h = (IG + A x^2)

Depth of centre of pressure, h= (IG + A x^2) / A

Total Pressure and depth of Centre of Pressure on an Inclined Immersed Surface

Consider a plane inclined surface, immersed in a liquid as shown in figure. Let,

w = Specific weight of the liquid

A = Total area of the immersed surface

x = Depth of the centroid of the immersed plane surface from the free surface of liquid. θ = Angle at which the immersed surface is inclined with the liquid

Surface h= depth of centre of pressure from the liquid surface

b = width of the considered thin strip dx = thickness of the strip

O = the reference point obtained by projecting the plane surface with the free surface of liquid

x = distance of the strip from O

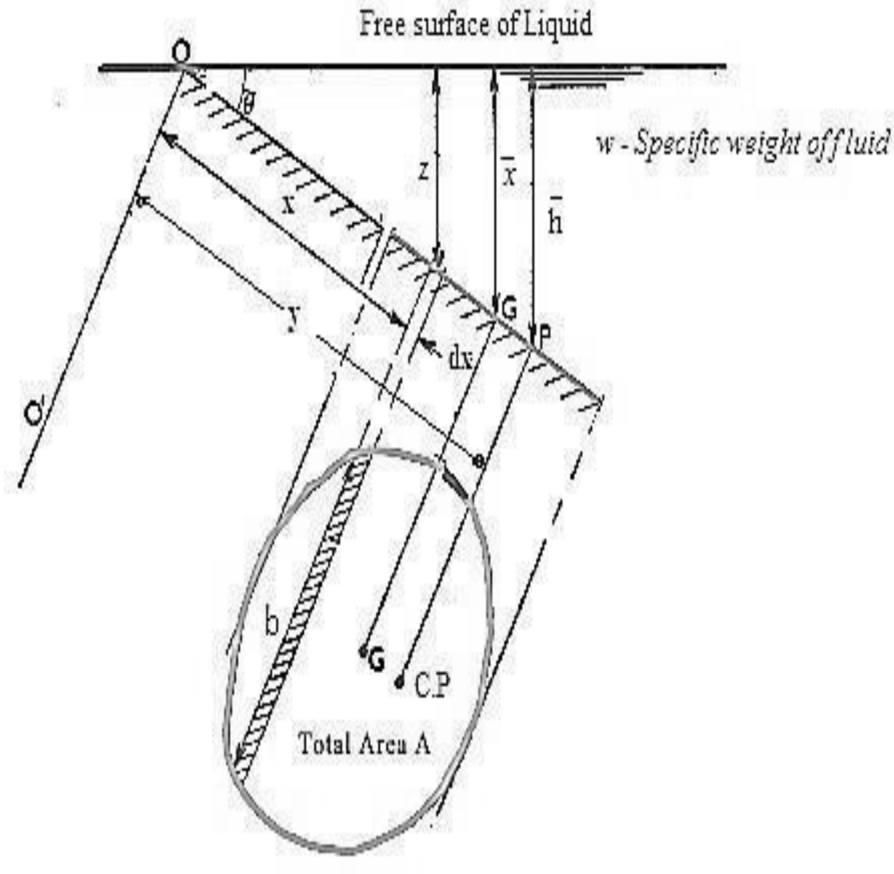


Fig: 10. Inclined Immersed Plain

Let the plane of the surface, if produced meet the free liquid surface at O. Then O-O is the axis perpendicular to the plane of the surface.

Let $\bar{y}$ = distance of the C.G. of the inclined surface from O-O

y^* = distance of the centre of pressure from O-O.

Consider a small strip of area dA at a depth 'h' from free surface and at a distance y from the axis O-O as shown in Fig. 3.18.

Pressure intensity on the strip, $p = \rho gh$

$\therefore$ Pressure force, dF, on the strip, $dF = p \times \text{Area of strip} = \rho gh \times dA$

Total pressure force on the whole area, $F = \int dF = \int \rho ghdA$

But from Fig. 3.18,
$$\frac{h}{y} = \frac{\bar{h}}{\bar{y}} = \frac{h^*}{y^*} = \sin\theta$$

$$\therefore \quad h = y\sin\theta$$

$$\therefore \quad F = \int \rho g \times y \times \sin\theta \times dA = \rho g \sin\theta \int ydA$$

But
$$\int ydA = A\bar{y}$$

where $\bar{y}$ = Distance of C.G. from axis O-O

$$\therefore \quad F = \rho g \sin\theta\ \bar{y} \times A$$

$$= \rho g A\bar{h} \qquad (\because \ \bar{h} = \bar{y}\sin\theta) \quad \text{...(3.6)}$$

Centre of Pressure (h*)

Pressure force on the strip, $dF = \rho ghdA$

$$= \rho gy\sin\theta\, dA \qquad [h = y\sin\theta]$$

Moment of the force, dF, about axis O-O

$$= dF \times y = \rho gy\sin\theta\, dA \times y = \rho g\sin\theta\, y^2 dA$$

Sum of moments of all such forces about O-O

$$= \int \rho g\sin\theta\, y^2\, dA = \rho g\sin\theta \int y^2\, dA$$

But
$$\int y^2\, dA = \text{M.O.I. of the surface about } O\text{-}O = I_0$$

$\therefore$ Sum of moments of all forces about O-O = $\rho g \sin\theta\, I_0$

Moment of the total force, F, about O-O is also given by

$$= F \times y^*$$

where y^* = Distance of centre of pressure from O-O.

Equating the two values given by equations (3.7) and (3.8)

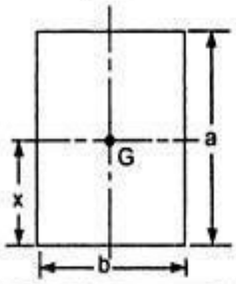
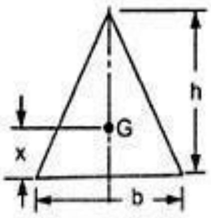
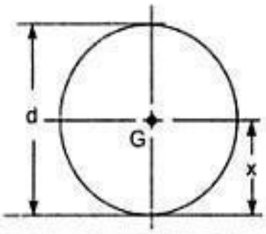
$$F \times y^* = \rho g\sin\theta\, I_0$$

or
$$y^* = \frac{\rho g \sin\theta\, I_0}{F}$$

Now
$$y^* = \frac{h^*}{\sin\theta},\ F = \rho g A\bar{h}$$

and I_0 by the theorem of parallel axis = $I_G + A\bar{y}^2$.

Table: M.I and Geometric Properties of some plane surfaces

Plane surface	C.G. from the base	Area	Moment of inertia about an axis passing through C.G. and parallel to base (I_G)
1. Rectangle 	$x = \frac{d}{2}$	bd	$\frac{bd^3}{12}$
2. Triangle 	$x = \frac{h}{3}$	$\frac{bh}{2}$	$\frac{bh^3}{36}$
3. Circle 	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$

Pascal's law

The basic property of a static fluid is pressure.

Pressure is the surface force exerted by a fluid against the walls of its container. Pressure also exists at every point within a volume of fluid.

For a static fluid, as shown by the following analysis, pressure turns to be independent direction.

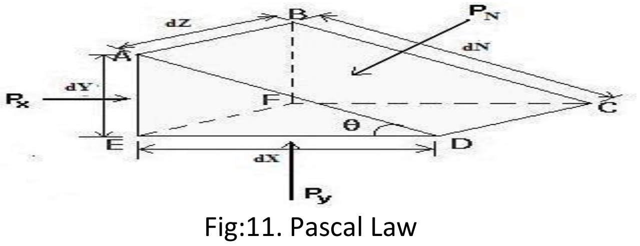


Fig:11. Pascal Law

Consider a triangular prism of small fluid element ABCDEF in equilibrium. Let P*x* is the intensity of pressure in the X direction acting at right angle on the face ABFE, Py is the

intensity of pressure in the Y direction acting at right angle on the face CDEF, and Ps is the intensity of pressure normal to inclined plane at an angle θ as shown in figure at right angle to ABC ..
For a fluid at rest there will be no shear stress, there will be no accelerating forces, and therefore the sum of the forces in any direction must be zero.
Thus the forces acting on the fluid element are the pressures on the surrounding and the gravity force. Force due to p*x* = px x Area ABFE = px dydz
Horizontal component of force due to pN = - (pN x Area ABC) sin(θ) = - pNdNdz dy/ds = - PNdydz As Py has no component in the x direction, the element will be in equilibrium, if
px dydz + (-pNdydz) = 0
i.e. p*x* = pN
Similarly in the y direction, force due to py = pydxdz
Component of force due to pN = - (pN x Area ABC) cos(θ) = - pNdsdz d*x*/ds = - pNd*x*dz
Force due to weight of element is negligible and the equation reduces to, py = pN
Therefore, p*x* = py = pN
Thus, Pressure at a point in a fluid at rest is same in all directions.

Manometers:

Manometer is an instrument for measuring the pressure of a fluid, consisting of a tube filled with a heavier gauging liquid, the level of the liquid being determined by the fluid pressure and the height of the liquid being indicated on a scale. A U-tube manometer consists of a glass tube bent in U-Shape, one end of which is connected to gauge point and the other end is exposed to atmosphere.

Manometric liquids:

1. Manometric liquids should neither mix nor have any chemical reaction with the liquid whose pressure intensity is to be measured.
2. It should not undergo any thermal variation.
3. Manometric liquid should have very low vapour pressure.
4. Manometric liquid should have pressure sensitivity depending upon the magnitude of pressure to be measured and accuracy requirement.

Simple U-Tube Manometer: It consist of glass tube in U shape one end of which is connected to a point at which pressure is to be measured and other end remains open to the atmosphere as shown in fig. The tube generally contains mercury or any other liquid whose specific gravity is greater than the specific gravity of the liquid whose pressure is to be measured.

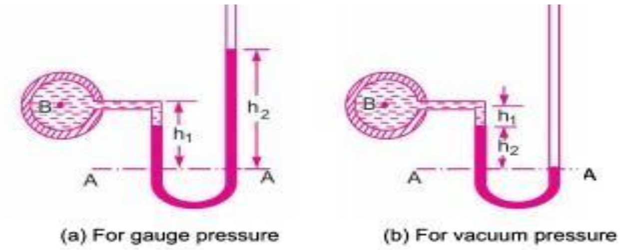


Fig: 12. Simple U tube Manometer

For Gauge Pressure. Let B is the point at which pressure is to be measured, whose value is p. The datum line is A-A Let,H_1 = Height of light liquid above the datum line

H_2 = Height of heavier liquid above the datum line S_1 = Specific gravity of light liquid
ρ_1 = Density of light liquid = 1000 x S_1 S_2 = Specific gravity of heavy liquid
ρ_2 = Density of heavy liquid = 1000 x S_2

As the pressure is the same for the horizontal surface. Hence pressure above the horizontal datum line *A-A* in the left column and in the right column of U-tube manometer should be same.

Pressure above *A-A* in the left column $= p + \rho_1 \times g \times h_1$

Pressure above *A-A* in the right column $= \rho_2 \times g \times h_2$

Hence equating the two pressures $p + \rho_1 g h_1 = \rho_2 g h_2$

$$\therefore \quad p = (\rho_2 g h_2 - \rho_1 \times g \times h_1).$$

(*b*) **For Vacuum Pressure.** For measuring vacuum pressure, the level of the heavy liquid in the manometer will be as shown in Fig. 2.9 (*b*). Then

Pressure above *A-A* in the left column $= \rho_2 g h_2 + \rho_1 g h_1 + p$

Pressure head in the right column above *A-A* $= 0$

$$\therefore \quad \rho_2 g h_2 + \rho_1 g h_1 + p = 0$$

$$\therefore \quad p = -(\rho_2 g h_2 + \rho_1 g h_1).$$

Differential U-Tube Manometer:

Let, A and B are the two pipes carrying liquids of specific gravity s1 and s3 & s2 = specific gravity of manometer liquid.

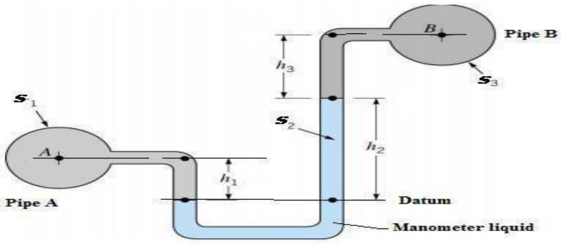


Fig:13. Differential U-tube Manometer

Let two point A & B are at different level and also contains liquids of different sp.gr. These points are connected to the U-tube differential manometer. Let the pressure at A and B are P_A and P_B

Let h = Difference of mercury level in the U-tube.
y = Distance of the centre of B, from the mercury level in the right limb.
x = Distance of the centre of A, from the mercury level in the right limb.
ρ_1 = Density of liquid at A.
ρ_2 = Density of liquid at B.
ρ_g = Density of heavy liquid or mercury.

Taking datum line at *X-X*.

Pressure above *X-X* in the left limb $= \rho_1 g(h + x) + p_A$

where p_A = pressure at A.

Pressure above *X-X* in the right limb $= \rho_g \times g \times h + \rho_2 \times g \times y + p_B$

where p_B = Pressure at B.

Equating the two pressure, we have

$$\rho_1 g(h + x) + p_A = \rho_g \times g \times h + \rho_2 g y + p_B$$

$$\therefore \quad p_A - p_B = \rho_g \times g \times h + \rho_2 g y - \rho_1 g(h + x)$$
$$= h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x$$

$\therefore$ Difference of pressure at A and $B = h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x$

In Fig. 2.18 (*b*), the two points A and B are at the same level and contains the same liquid of density ρ_1. Then

Pressure above *X-X* in right limb $= \rho_g \times g \times h + \rho_1 \times g \times x + p_B$

Pressure above *X-X* in left limb $= \rho_1 \times g \times (h + x) + p_A$

Equating the two pressure

$$\rho_g \times g \times h + \rho_1 g x + p_B = \rho_1 \times g \times (h + x) + p_A$$

$$\therefore \quad p_A - p_B = \rho_g \times g \times h + \rho_1 g x - \rho_1 g(h + x)$$
$$= g \times h(\rho_g - \rho_1).$$

Buoyant force: The upward force exerted by a liquid on a body when the body is immersed in the liquid is known as buoyancy or buoyant force.
The point through which force of buoyancy is supposed to act is called centre of buoyancy.
The buoyant force acting on a body is equal to the weight of the liquid displaced by the body.
For a fluid with constant density, the buoyant force is independent of the distance of the body from the free surface. It is also independent of the density of the solid body.
Archimedes principle: The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume. For floating bodies, the weight of the entire body must be equal to the buoyant force, which is the weight of the fluid whose volume is equal to the volume of the submerged portion of the floating body.

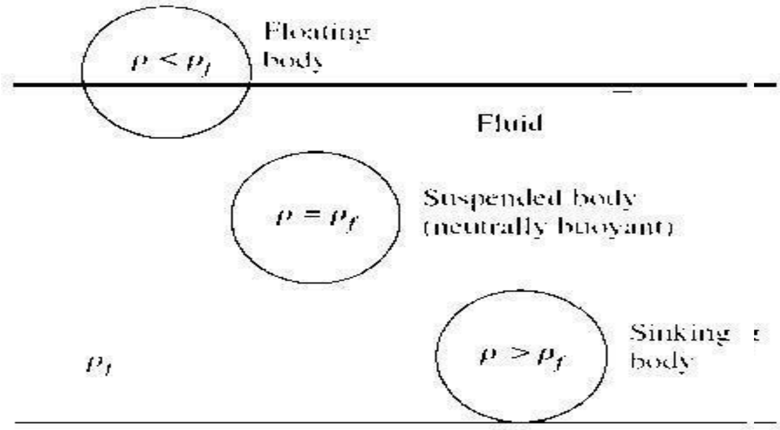


Fig:14. Floating Body

Stability of immersed and floating bodies

A floating body possesses vertical stability, while an immersed neutrally buoyant body is neutrally stable since it does not return to its original position after a disturbance.

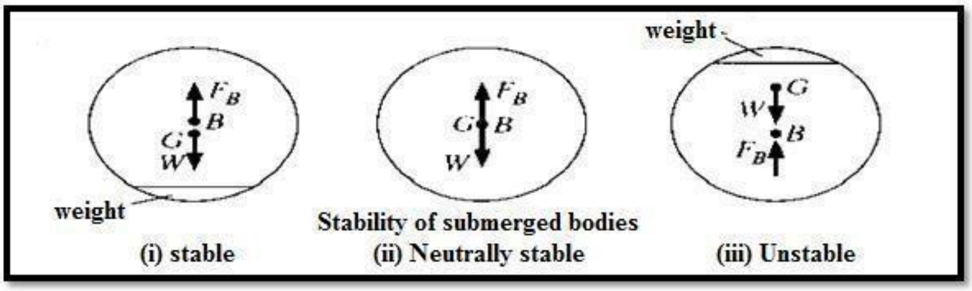
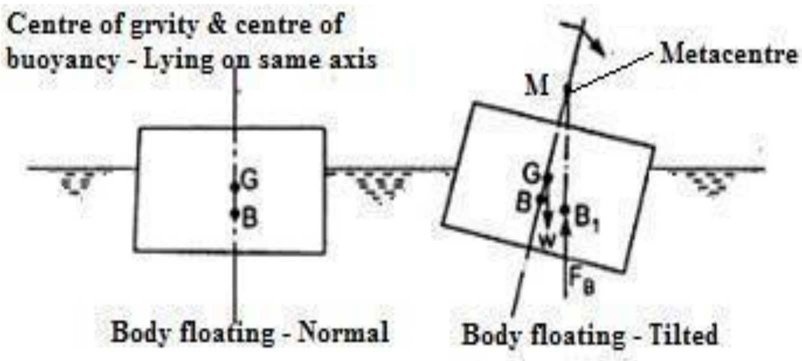


Fig:15. An immersed neutrally buoyant body is (a) stable if the center of gravity G is directly below the center of buoyancy B of the body, (b) neutrally stable if G and B are coincident, and (c) unstable if G is directly above B.

Metacentre: The point about which a body starts oscillating when the body is tilted is known meta- centre.

Metacentric height GM: The distance between the center of gravity G and the metacenter M is known as Meta centric height. It is the point of intersection of line of action of buoyant



force with the line passing through centre of gravity, when the body is slightly tilted.

Fig.16. Metacentric Height

The length of the metacentric height GM above G is a measure of the stability: If the metacentric height increases, then the floating body will be more.. The meta-centric height (GM) is.given by, GM = V - BGWhere, I = Moment of Inertia of the floating body (in plan) at water surface about the axis Y- Y V = Volume of ihe body sub merged in waterBG = Distance between centre of gravity and centre of buoyancy. Conditions of equilibrium of a floating and submerged body are :

Table.2. Condition of Equilibrium of a Floating bodies

Equilibrium	Floating Body	Sub-merged Body
(i) Stable Equilibrium (a) Unstable Equilibrium (Hi) Neutral Equilibrium	M is above G M is below G Af and G coincide	B is above G B is below G B and G coincide

Stability of floating bodies .A floating body is stable if the body is bottom-heavy and thus the center of gravity G is below the centroid B of the body, or if the metacentre M is above point G. However, the body is unstable if point M is below point G.

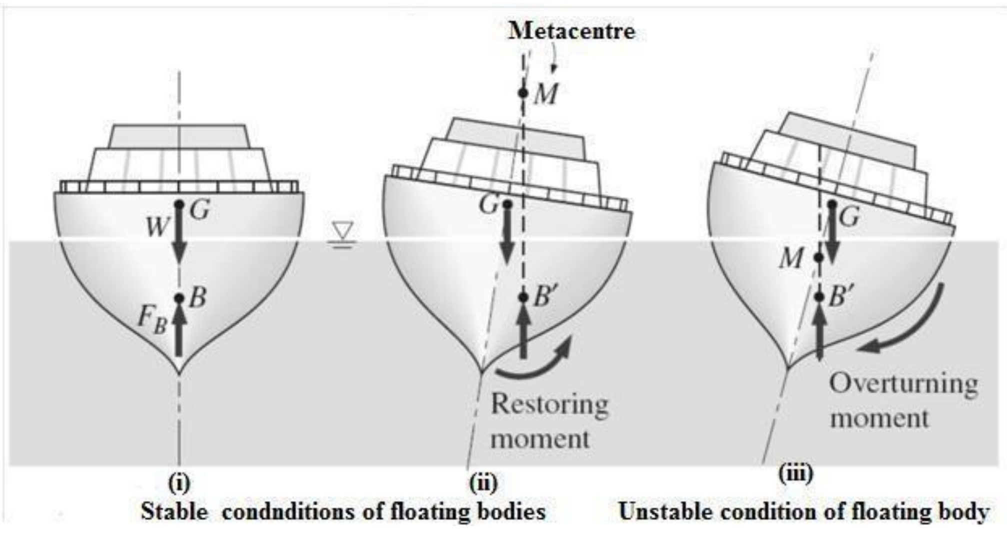


Fig.17.Stability of Floating Bodies

Problems:

1.Calculate the sp.weight, density and sp.gravity of one litre of liquid which weights 7N.

$$\text{Volume} = 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \quad \left(\because \; 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \text{ or } 1 \text{ litre} = 1000 \text{ cm}^3\right)$$

$$\text{Weight} = 7 \text{ N}$$

(*i*) Specific weight (w) $= \dfrac{\text{Weight}}{\text{Volume}} = \dfrac{7\text{N}}{\left(\dfrac{1}{1000}\right)\text{m}^3} =$ **7000 N/m³. Ans.**

(*ii*) Density (ρ) $= \dfrac{w}{g} = \dfrac{7000}{9.81}$ kg/m³ = **713.5 kg/m³. Ans.**

(*iii*) Specific gravity $= \dfrac{\text{Density of liquid}}{\text{Density of water}} = \dfrac{713.5}{1000}$ $\{\because$ Density of water = 1000 kg/m³$\}$

= **0.7135. Ans.**

2.Calculate the density, sp.weight and weight of one litre of petrol of specific gravity = 0.7

Solution. Given : Volume = 1 litre = $1 \times 1000 \text{ cm}^3 = \dfrac{1000}{10^6} \text{ m}^3 = 0.001 \text{ m}^3$

Sp. gravity $S = 0.7$

(*i*) *Density* (ρ)

Using equation (1.1*A*),

Density (ρ) $= S \times 1000 \text{ kg/m}^3 = 0.7 \times 1000 =$ **700 kg/m³. Ans.**

(*ii*) *Specific weight* (w)

Using equation (1.1), $w = \rho \times g = 700 \times 9.81 \text{ N/m}^3 =$ **6867 N/m³. Ans.**

(*iii*) *Weight (W)*

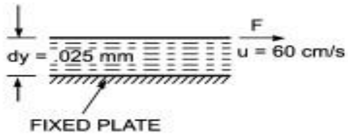
We know that specific weight $= \dfrac{\text{Weight}}{\text{Volume}}$

or $w = \dfrac{W}{0.001}$ or $6867 = \dfrac{W}{0.001}$

$\therefore$ $W = 6867 \times 0.001 =$ **6.867 N. Ans.**

3.A plate 0.023 mm distant from a fixed plate moves at 60 cm/s and requires a force of 2N per unit area i.e 2 N/m² to maintain this speed. Determine the fluid viscosity between the plates.

Solution. Given :



Distance between plates, $dy = .025$ mm $= .025 \times 10^{-3}$ m

Velocity of upper plate, $u = 60$ cm/s $= 0.6$ m/s

Force on upper plate, $F = 2.0 \dfrac{\text{N}}{\text{m}^2}$.

This is the value of shear stress *i.e.*, τ

Let the fluid viscosity between the plates is μ.

Using the equation (1.2), we have $\tau = \mu \dfrac{du}{dy}$.

where du = Change of velocity $= u - 0 = u = 0.60$ m/s

dy = Change of distance $= .025 \times 10^{-3}$ m

τ = Force per unit area $= 2.0 \dfrac{\text{N}}{\text{m}^2}$

$\therefore$ $2.0 = \mu \dfrac{0.60}{.025 \times 10^{-3}}$ $\therefore$ $\mu = \dfrac{2.0 \times .025 \times 10^{-3}}{0.60} = 8.33 \times 10^{-5} \dfrac{\text{Ns}}{\text{m}^2}$

$= 8.33 \times 10^{-5} \times 10$ poise = **8.33×10^{-4} poise. Ans.**

4.The dynamic viscosity of oil used for lubrication between a shaft and sleeve is 6 poise. The shaft is of diameter 0.4 m and rotates at 190 rpm. Calculate the power lost in the bearing for a sleeve length of 90mm. The thickness of the oil film is 1.5mm.

Solution. Given :

Viscosity $\mu = 6$ poise

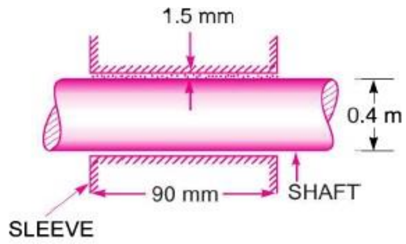
$$= \frac{6}{10}\frac{\text{N s}}{\text{m}^2} = 0.6\ \frac{\text{N s}}{\text{m}^2}$$

Dia. of shaft, $D = 0.4$ m

Speed of shaft, $N = 190$ r.p.m

Sleeve length, $L = 90 \text{ mm} = 90 \times 10^{-3}$ m

Thickness of oil film, $t = 1.5 \text{ mm} = 1.5 \times 10^{-3}$ m



Tangential velocity of shaft, $u = \dfrac{\pi D N}{60} = \dfrac{\pi \times 0.4 \times 190}{60} = 3.98$ m/s

Using the relation $\tau = \mu \dfrac{du}{dy}$

where du = Change of velocity $= u - 0 = u = 3.98$ m/s

dy = Change of distance $= t = 1.5 \times 10^{-3}$ m

$$\tau = 10 \times \frac{3.98}{1.5 \times 10^{-3}} = 1592 \text{ N/m}^2$$

This is shear stress on shaft

$\therefore$ Shear force on the shaft, F = Shear stress $\times$ Area

$$= 1592 \times \pi D \times L = 1592 \times \pi \times .4 \times 90 \times 10^{-3} = 180.05 \text{ N}$$

Torque on the shaft, $T = \text{Force} \times \dfrac{D}{2} = 180.05 \times \dfrac{0.4}{2} = 36.01$ Nm

$\therefore$ *Power lost $= \dfrac{2\pi NT}{60} = \dfrac{2\pi \times 190 \times 36.01}{60} = $ **716.48 W. Ans.**

5.The surface tension of water in contact with air at 20°C is 0.0725N/m. The pressure inside a droplet of water is to be 0.02 N/cm^2 greater then the outside pressure. Calculate the diameter of the droplet of water.

Solution. Given :

Surface tension, $\sigma = 0.0725$ N/m

Pressure intensity, p in excess of outside pressure is

$$p = 0.02 \text{ N/cm}^2 = 0.02 \times 10^4\ \frac{\text{N}}{\text{m}^2}$$

Let d = dia. of the droplet

$$\text{we get } p = \frac{4\sigma}{d} \text{ or } 0.02 \times 10^4 = \frac{4 \times 0.0725}{d}$$

$$d = \frac{4 \times 0.0725}{0.02 \times (10)^4} = .00145 \text{ m} = .00145 \times 1000 = \textbf{1.45 mm. Ans.}$$

6.Calculate the capillary rise in a glass tube of 2.5mm diameter when immersed vertically in a) water b) Mercury. Take surface tension of 2.5 mm diameter when immersed vertically in contact with air. The specific gravity for mercury is given as 13.6 and angle of contact = 130◦

$\therefore$ Density $= 13.6 \times 1000 \text{ kg/m}^3$.

(*a*) Capillary rise for water ($\theta = 0°$)

Using equation (1.20), we get $h = \dfrac{4\sigma}{\rho \times g \times d} = \dfrac{4 \times 0.0725}{1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= .0118 \text{ m} = \mathbf{1.18 \text{ cm. Ans.}}$

(*b*) For mercury

Angle of contact between mercury and glass tube, $\theta = 130°$

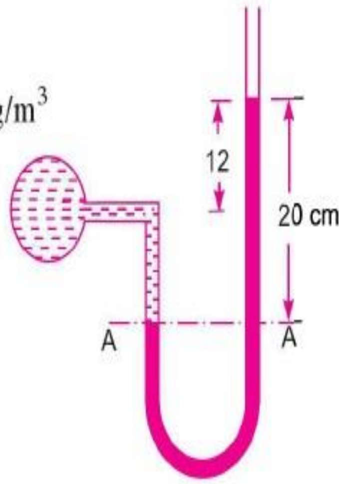
Using equation (1.21), we get $h = \dfrac{4\sigma \cos\theta}{\rho \times g \times d} = \dfrac{4 \times 0.52 \times \cos 130°}{13.6 \times 1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= -.004 \text{ m} = -\mathbf{0.4 \text{ cm. Ans.}}$

The negative sign indicates the capillary depression.

7.The right limb of a single U-tube manometer containing mercury is open to the atmosphere while the left limb is connected to a pipe in which a fluid of sp.gravity is 0.9 is flowing. The centre of the pipe is 12cm below the level of mercury in the right limb. Find the pressure of fluid in the pipe if the difference of mercury in the two limbs is 20cm.

Solution. Given :



Sp. gr. of fluid, $S_1 = 0.9$

$\therefore$ Density of fluid, $\rho_1 = S_1 \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3$

Sp. gr. of mercury, $S_2 = 13.6$

$\therefore$ Density of mercury, $\rho_2 = 13.6 \times 1000 \text{ kg/m}^3$

Difference of mercury level, $h_2 = 20 \text{ cm} = 0.2 \text{ m}$

Height of fluid from *A-A*, $h_1 = 20 - 12 = 8 \text{ cm} = 0.08 \text{ m}$

Let p = Pressure of fluid in pipe

Equating the pressure above *A-A*, we get

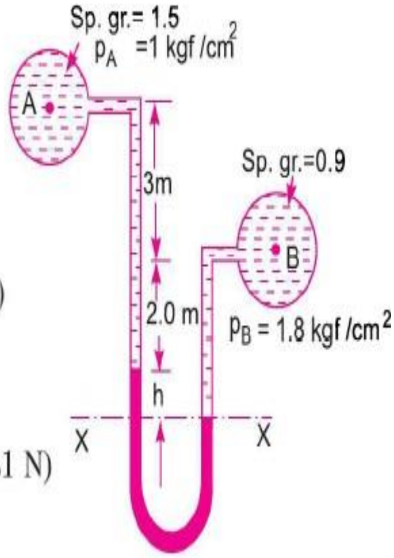
$$p + \rho_1 g h_1 = \rho_2 g h_2$$

$$p + 900 \times 9.81 \times 0.08 = 13.6 \times 1000 \times 9.81 \times .2$$

$$p = 13.6 \times 1000 \times 9.81 \times .2 - 900 \times 9.81 \times 0.08$$

$$= 26683 - 706 = 25977 \text{ N/m}^2 = \mathbf{2.597 \text{ N/cm}^2. \text{ Ans.}}$$

8.A differential manometer is connected at the two points A and B of two pipes as shown in fig. The pipe A contains a liquid of Sp.gravity = 1.5 while pipe B contains a liquid of sp.gravity = 0.9. The pressure at A and B are 1 Kgf/cm^2 and1.80 Kgf/cm^2 respectively. Find the difference in mercury level in the differential manometer.



Solution. Given :

Sp. gr. of liquid at A, $S_1 = 1.5$ $\therefore$ $\rho_1 = 1500$

Sp. gr. of liquid at B, $S_2 = 0.9$ $\therefore$ $\rho_2 = 900$

Pressure at A, $p_A = 1$ kgf/cm$^2 = 1 \times 10^4$ kgf/m^2

$= 10^4 \times 9.81$ N/m^2 ($\because$ 1 kgf = 9.81 N)

Pressure at B, $p_B = 1.8$ kgf/cm^2

$= 1.8 \times 10^4$ kgf/m^2

$= 1.8 \times 10^4 \times 9.81$ N/m^2 ($\because$ 1 kgf = 9.81 N)

Density of mercury $= 13.6 \times 1000$ kg/m^3

Taking X-X as datum line.

Pressure above X-X in the left limb

$$= 13.6 \times 1000 \times 9.81 \times h + 1500 \times 9.81 \times (2 + 3) + p_A$$

$$= 13.6 \times 1000 \times 9.81 \times h + 7500 \times 9.81 + 9.81 \times 10^4$$

Pressure above X-X in the right limb $= 900 \times 9.81 \times (h + 2) + p_B$

$$= 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Equating the two pressure, we get

$$13.6 \times 1000 \times 9.81h + 7500 \times 9.81 + 9.81 \times 10^4$$

$$= 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Dividing by 1000×9.81, we get

$$13.6h + 7.5 + 10 = (h + 2.0) \times .9 + 18$$

$$13.6h + 17.5 = 0.9h + 1.8 + 18 = 0.9h + 19.8$$

$$(13.6 - 0.9)h = 19.8 - 17.5 \text{ or } 12.7h = 2.3$$

$$\therefore \quad h = \frac{2.3}{12.7} = 0.181 \text{ m} = \mathbf{18.1\ cm.\ Ans.}$$

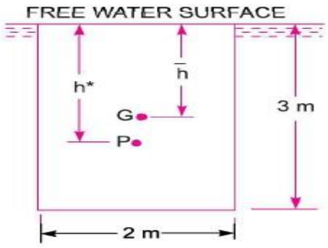
.A rectangular plane surface is 2m wide and 3m deep. It lies in vertical plane in water. Determine the total pressure and position of centre of pressure on the plane surface when its upper edge is horizontal and a) coincide with water surfaceb) 2.5 m below the free water surface.

Solution. Given :

Width of plane surface, $b = 2$ m

Depth of plane surface, $d = 3$ m

(*a*) **Upper edge coincides with water surface**



$$F = \rho g A \bar{h}$$

where $\rho = 1000$ kg/m^3, $g = 9.81$ m/s^2

$$A = 3 \times 2 = 6 \text{ m}^2,\ \bar{h} = \frac{1}{2}(3) = 1.5 \text{ m.}$$

$$\therefore \quad F = 1000 \times 9.81 \times 6 \times 1.5$$
$$= \mathbf{88290 \text{ N. Ans.}}$$

Depth of centre of pressure is given by equation (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where I_G = M.O.I. about C.G. of the area of surface

$$= \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

$$\therefore \quad h^* = \frac{4.5}{6 \times 1.5} + 1.5 = 0.5 + 1.5 = \mathbf{2.0 \text{ m. Ans.}}$$

(*b*) **Upper edge is 2.5 m below water surface**

Solution. Given :

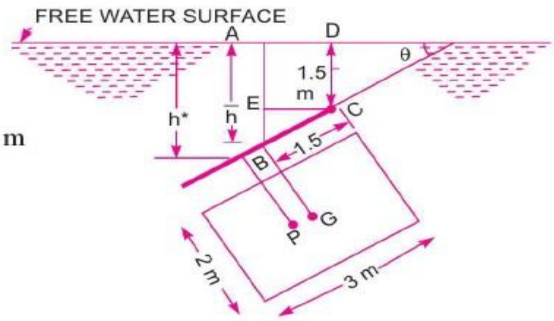
Width of plane surface, $b = 2$ m

Depth, $d = 3$ m

Angle, $\theta = 30°$

Distance of upper edge from free water surface = 1.5 m

(*i*) **Total pressure** force is given by equation



$$F = \rho g A \bar{h}$$

where $\rho = 1000$ kg/m^3

$A = b \times d = 3 \times 2 = 6$ m^2

$\therefore \quad \bar{h}$ = Depth of C.G. from free water surface

$= 1.5 + 1.5 \sin 30°$

$$\{\because \quad \bar{h} = AE + EB = 1.5 + BC \sin 30° = 1.5 + 1.5 \sin 30°\}$$

$= 1.5 + 1.5 \times \frac{1}{2} = 2.25$ m

$$\therefore \quad F = 1000 \times 9.81 \times 6 \times 2.25 = \mathbf{132435 \text{ N. Ans.}}$$

(*ii*) **Centre of pressure (h*)**

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}, \quad \text{where } I_G = \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

$$\therefore \quad h^* = \frac{4.5 \times \sin^2 30°}{6 \times 2.25} + 2.25 = \frac{4.5 \times \frac{1}{4}}{6 \times 2.25} + 2.25$$
$$= 0.0833 + 2.25 = \mathbf{2.3333 \text{ m. Ans.}}$$

10.A rectangular plane surface 2m wide and 3m deep lies in water in such a way that its plane makes an angle of 30◦ with the free surface of water. Determine the total surface and position of centre of pressure when the upper edge is 1.5m below the free water surface.

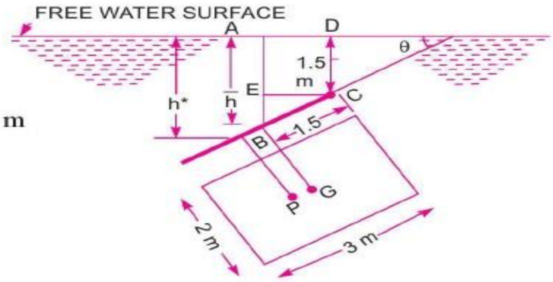
Solution. Given :

Width of plane surface, $b = 2$ m

Depth, $d = 3$ m

Angle, $\theta = 30°$

Distance of upper edge from free water surface = 1.5 m



(*i*) **Total pressure** force is given by equation

$$F = \rho g A \bar{h}$$

where $\rho = 1000$ kg/m^3

$A = b \times d = 3 \times 2 = 6$ m^2

$\therefore$ $\bar{h}$ = Depth of C.G. from free water surface
= 1.5 + 1.5 sin 30°

$\{\because \quad \bar{h} = AE + EB = 1.5 + BC \sin 30° = 1.5 + 1.5 \sin 30°\}$

$= 1.5 + 1.5 \times \frac{1}{2} = 2.25$ m

$\therefore \quad F = 1000 \times 9.81 \times 6 \times 2.25 =$ **132435 N. Ans.**

(*ii*) **Centre of pressure (h*)**

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}, \quad \text{where } I_G = \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

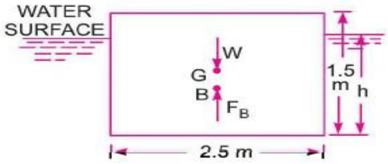
$$\therefore \quad h^* = \frac{4.5 \times \sin^2 30°}{6 \times 2.25} + 2.25 = \frac{4.5 \times \frac{1}{4}}{6 \times 2.25} + 2.25$$

$= 0.0833 + 2.25 =$ **2.3333 m. Ans.**

11. Find the volume of the water displaced and position of centre of buoyancy for a wooden block of width 2.5m and depth 1.5m. When it floats horizontally in water. The density of wooden block is 650 kg/m^3 and its length 6m.

Solution. Given :

Width	= 2.5 m
Depth	= 1.5 m
Length	= 6.0 m
Volume of the block	= 2.5 × 1.5 × 6.0 = 22.50 m^3
Density of wood,	ρ = 650 kg/m^3
$\therefore$ Weight of block	= $\rho \times g \times$ Volume
	= 650 × 9.81 × 22.50 N = 143471 N



For equilibrium the weight of water displaced = Weight of wooden block

= 143471 N

$\therefore$ Volume of water displaced

Solution. Given :

Dimension of pontoon = 5 m × 3 m × 1.20 m $= \dfrac{143471}{1000 \times 9.81} =$ **14.625 m^3. Ans.**

Depth of immersion = 0.8 m

(∵ ... density of water = 1000 × 9.81 N/m^3)

Position of Centre of Buoyancy. Volume of wooden block in water

= Volume of water displaced

$2.5 \times h \times 6.0 = 14.625$ m^3, where h is depth of wooden block in water

$$\therefore \quad h = \frac{14.625}{2.5 \times 6.0} = 0.975 \text{ m}$$

$\therefore$ Centre of Buoyancy $= \dfrac{0.975}{2} =$ **0.4875 m from base. Ans.**

12.A rectangular pontoon is 5m long, 3m wide and 1.20m high. The depth of immersion of the position is 0.80 m in sea water. If the centre of gravity is 0.6m above the bottom of the position, determine the meta centric height. The density for sea water is 1025 kg/m^3.

Types of fluid flow

The fluid flow is classified as :

(*i*) Steady and unsteady flows ;
(*ii*) Uniform and non-uniform flows ;
(*iii*) Laminar and turbulent flows ;
(*iv*) Compressible and incompressible flows
(*v*) Rotational and irrotational flows ; and
(*vi*) One, two and three-dimensional flows.

Steady and Unsteady Flows. Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\left(\frac{\partial V}{\partial t}\right)_{x_0,\, y_0,\, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0,\, y_0,\, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0,\, y_0,\, z_0} = 0$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0,\, y_0,\, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0,\, y_0,\, z_0} \neq 0 \text{ etc.}$$

Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where ∂V = Change of velocity
∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} \neq 0.$$

Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$ called the Reynold number,

where D = Diameter of pipe

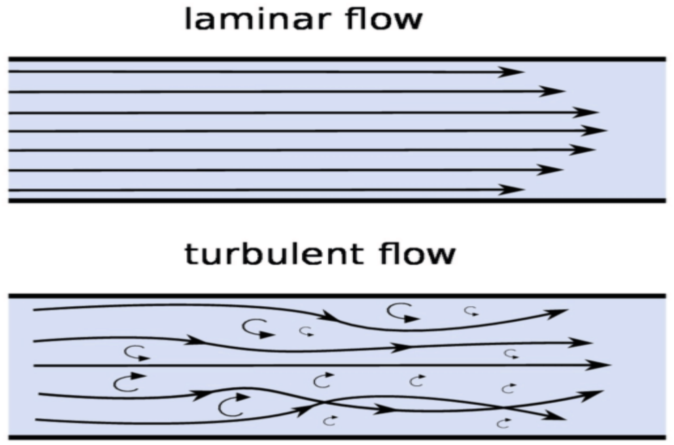
V = Mean velocity of flow in pipe

and ν = Kinematic viscosity of fluid.

If $Re < 2000$, the flow is Laminar

$Re > 4000$, the flow is turbulent

$2000 < Re < 4000$, the flow may be Laminar or turbulent



Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant}.$$

Rotational and Irrotational Flows. Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

One-, Two- and Three-Dimensional Flows. **One-dimensional flow** is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only,

$$u = f(x),\ v = 0 \text{ and } w = 0$$

Two-dimensional flow is that type of flow in which the velocity is a function of time and two rectangular space co-ordinates say x and y.

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$

Three-dimensional flow is that type of flow in which the velocity is a function of time and three mutually perpendicular directions.

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

Rate of flow or Discharge (Q)

It is defined as quantity of flow per second through the section of pipe or channel.

(*i*) For liquids the units of Q are m^3/s or litres/s

(*ii*) For gases the units of Q is kgf/s or Newton/s

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

V = Average velocity of fluid across the section

Then discharge $Q = A \times V$.

Continuity Equation

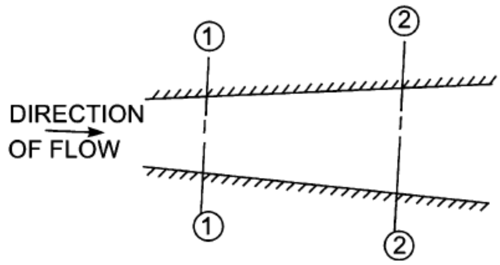
The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant.

Rate of flow at section 1 – 1 = rate of flow at section 2 – 2

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

If the fluid flow is incompressible, the $\rho_1 = \rho_2$

$$A_1 V_1 = A_2 V_2$$



Equation of motion

According to Newton's second law of motion,

$$F_x = m.a_x$$

In the fluid flow, the following forces are present :

(*i*) F_g, gravity force.

(*ii*) F_p, the pressure force.

(*iii*) F_v, force due to viscosity.

(*iv*) F_t, force due to turbulence.

(*v*) F_c, force due to compressibility.

the net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x.$$

(*i*) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion.**

(*ii*) For flow, where (F_t) is negligible, the resulting equations of motion are known as **Navier-Stokes Equation.**

(*iii*) If the flow is assumed to be ideal, viscous force (F_v) is zero and equation of motions are known as **Euler's equation of motion.**

Euler's equation of motion.

The forces acting on the cylindrical element are:

1. Pressure force pdA in the direction of flow.
2. Pressure force $\left(p+\frac{\partial p}{\partial s}ds\right)dA$ opposite to the direction of flow.
3. Weight of element $\rho gdAds$.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element × acceleration in the direction s.

$$pdA - \left(p+\frac{\partial p}{\partial s}ds\right)dA - \rho gdAds\cos\theta$$

$$= \rho dAds \times a_s$$

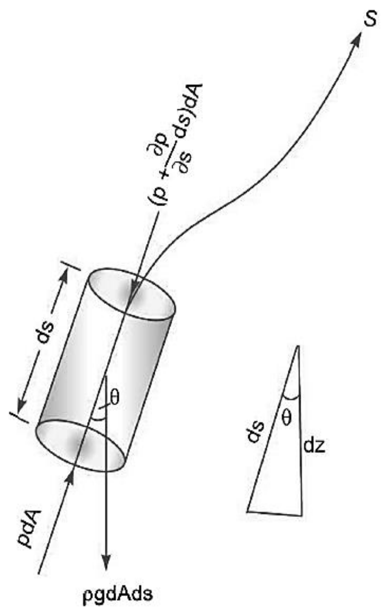
where a_s is the acceleration in the direction of s.

$$a_s = \frac{dv}{dt}, \text{ where } v \text{ is a function of } s \text{ and } t.$$

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{\because \frac{ds}{dt} = v\right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\therefore \quad a_s = \frac{v\partial v}{\partial s}$$



Substituting the value of a_s

$$-\frac{\partial p}{\partial s}dsdA - \rho g\,dAds\cos\theta = \rho dAds \times \frac{\partial v}{\partial s}$$

Dividing by $\rho dsdA$, $-\frac{\partial p}{\rho\partial s} - g\cos\theta = \frac{v\partial v}{\partial s}$

or $$\frac{\partial p}{\rho\partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0$$

we have $\cos\theta = \frac{dz}{ds}$

$$\therefore \quad \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + \frac{vdv}{ds} = 0$$

or $$\frac{dp}{\rho} + gdz + vdv = 0$$

is known as Euler's equation of motion.

BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli's equation is obtained by integrating the Euler's equation of motion

$$\int \frac{dp}{\rho} + \int g dz + \int v dv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\therefore \qquad \frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

$$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}$$

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

$v^2/2g$ = kinetic energy per unit weight or kinetic head.

z = potential energy per unit weight or potential head.

Statement of Bernoulli's Theorem. It states that in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant. The total energy consists of pressure energy, kinetic energy and potential energy or datum energy.

A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 4.0 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.

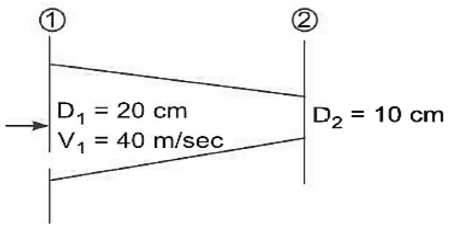
$$D_1 = 20 \text{ cm} = 0.2 \text{ m}$$

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$$

$$V_1 = 4.0 \text{ m/s}$$

$$D_2 = 0.1 \text{ m}$$

$$A_2 = \frac{\pi}{4} (.1)^2 = .00785 \text{ m}^2$$



(*i*) Velocity head at section 1

$$= \frac{V_1^2}{2g} = \frac{4.0 \times 4.0}{2 \times 9.81} = \mathbf{0.815 \text{ m.}}$$

(*ii*) Velocity head at section 2 = $V_2^2/2g$

To find V_2, apply continuity equation at 1 and 2

$$A_1 V_1 = A_2 V_2 \quad \text{or} \quad V_2 = \frac{A_1 V_1}{A_2} = \frac{.0314}{.00785} \times 4.0 = 16.0 \text{ m/s}$$

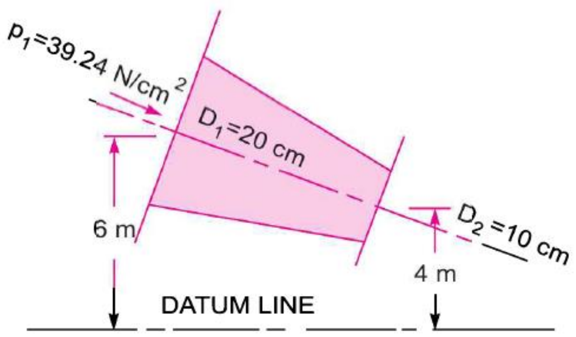
Velocity head at section 2 = $\frac{V_2^2}{2g} = \frac{16.0 \times 16.0}{2 \times 9.81}$ = **83.047 m**.

(*iii*) Rate of discharge

$= A_1V_1 \quad \text{or} \quad A_2V_2$
$= 0.0314 \times 4.0 = 0.1256 \text{ m}^3/\text{s}$
= **125.6 litres/s. Ans.** $\{\because 1 \text{ m}^3 = 1000 \text{ litres}\}$

Problem 2:

The water is flowing through a pipe having diameters 20 and 10 cm at sections 1 and 2 respectively. The rate of flow through the pipe is 35 litres/sec. the section 1 is 6 m above the datum and section 2 is 4 m above the datum. If the pressure at section 1 is 39.24 N/cm^2. Find the intensity of pressure at section 2



At section 1,

$D_1 = 20 \text{ cm} = 0.2 \text{ m}$

$A_1 = \frac{\pi}{4}(.2)^2 = .0314 \text{ m}^2$

$p_1 = 39.24 \text{ N/cm}^2$
$= 39.24 \times 10^4 \text{ N/m}^2$

$z_1 = 6.0 \text{ m}$

At section 2,

$D_2 = 0.10$ m

$A_2 = \frac{\pi}{4}(0.1)^2 = .00785 \text{ m}^2$

$z_2 = 4$ m

$p_2 = ?$

$$Q = 35 \text{ lit/s} = \frac{35}{1000} = .035 \text{ m}^3/\text{s}$$

$$Q = A_1V_1 = A_2V_2$$

$$V_1 = \frac{Q}{A_1} = \frac{.035}{.0314} = 1.114 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{.035}{.00785} = 4.456 \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

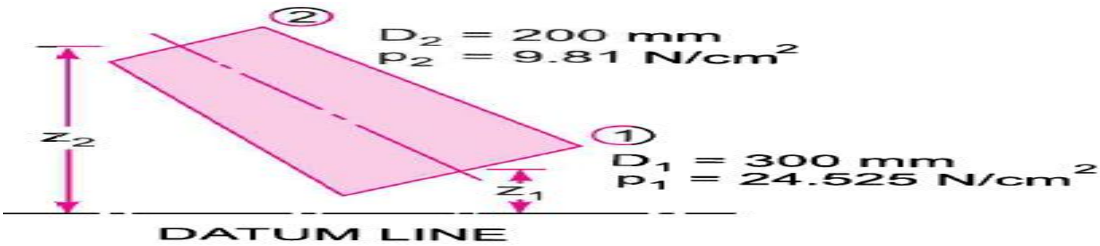
$$\frac{39.24 \times 10^4}{1000 \times 9.81} + \frac{(1.114)^2}{2 \times 9.81} + 6.0 = \frac{p_2}{1000 \times 9.81} + \frac{(4.456)^2}{2 \times 9.81} + 4.0$$

$$p_2 = 41.051 \times 9810 \text{ N/m}^2$$

$$= \frac{41.051 \times 9810}{10^4} \text{ N/cm}^2 = \mathbf{40.27\ N/cm^2}.$$

Problem 3:

Water is flowing through a pipe having diameter 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm² and the pressure at the upper end is 9.81 N/cm². Determine the difference in datum head if the rate of flow through pipe is 40 lit/s.



Section 1,

$D_1 = 300\text{ mm} = 0.3\text{ m}$
$p_1 = 24.525\text{ N/cm}^2 = 24.525 \times 10^4\text{ N/m}^2$

Section 2,

$D_2 = 200\text{ mm} = 0.2\text{ m}$
$p_2 = 9.81\text{ N/cm}^2 = 9.81 \times 10^4\text{ N/m}^2$

Rate of flow = 40 lit/s

$$Q = \frac{40}{1000} = 0.04\text{ m}^3/\text{s}$$

$A_1V_1 = A_2V_2 = \text{rate of flow} = 0.04$

$$V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4}D_1^2} = \frac{0.04}{\frac{\pi}{4}(0.3)^2} = 0.5658\text{ m/s}$$

$$\simeq 0.566\text{ m/s}$$

$$V_2 = \frac{.04}{A_2} = \frac{.04}{\frac{\pi}{4}(D_2)^2} = \frac{0.04}{\frac{\pi}{4}(0.2)^2} = 1.274\text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\frac{24.525\times10^4}{1000\times9.81}+\frac{.566\times.566}{2\times9.81}+z_1=\frac{9.81\times10^4}{1000\times9.81}+\frac{(1.274)^2}{2\times9.81}+z_2$$

$$25 + .32 + z_1 = 10 + 1.623 + z_2$$

$$25.32 + z_1 = 11.623 + z_2$$

$$z_2 - z_1 = 25.32 - 11.623 = 13.697 = 13.70 \text{ m}$$

Difference in datum head $= z_2 - z_1 =$ **13.70 m. Ans.**

Problem 4:

The water is flowing through a taper pipe of length 100 m having diameters 600 mm at the upper end and 300 mm at the lower end, at the rate of 50 litres/s. The pipe has a slope of 1 in 30. Find the pressure at the lower end if the pressure at the higher level is 19.62 N/cm^2.

$L = 100$ m

$D_1 = 600$ mm $= 0.6$ m

$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times (.6)^2$

$= 0.2827$ m^2

$p_1 =$ pressure at upper end

$= 19.62$ N/cm^2

$D_2 = 300$ mm $= 0.3$ m

$$A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}$$

$$Q = \text{rate of flow} = 50 \text{ litres/s} = \frac{50}{1000} = 0.05 \text{ m}^3\text{/s}$$

Let the datum line passes through the centre of the lower end.

Then $z_2 = 0$

As slope is 1 in 30 means $z_1 = \frac{1}{30} \times 100 = \frac{10}{3}$ m

Also we know $Q = A_1V_1 = A_2V_2$

$\therefore$ $V_1 = \frac{Q}{A} = \frac{0.05}{.2827} = 0.1768$ m/sec $= 0.177$ m/s

and $V_2 = \frac{Q}{A_2} = \frac{0.5}{.07068} = 0.7074$ m/sec $= 0.707$ m/s

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{.177^2}{2 \times 9.81} + \frac{10}{3} = \frac{p_2}{\rho g} + \frac{.707^2}{2 \times 9.81} + 0$$

$$20 + 0.001596 + 3.334 = \frac{p_2}{\rho g} + 0.0254$$

$$23.335 - 0.0254 = \frac{p_2}{1000 \times 9.81}$$

$$p_2 = 23.3 \times 9810 \text{ N/m}^2 = 228573 \text{ N/m}^2 = \mathbf{22.857\ N/cm^2}.$$

Practical applications of Bernoulli's equation:

Although Bernoulli's equation is applicable in all problems of incompressible flow where there is involvement of energy considerations. But we shall consider its application to the following measuring devices. 1) Venturimeter 2) Orifice meter 3) Pitot tube

Venturimeter: is a device used for measuring the rate of flow of a fluid flowing through a pipe. It consists of three parts:

• A short converging part

• Throat

• Diverging part

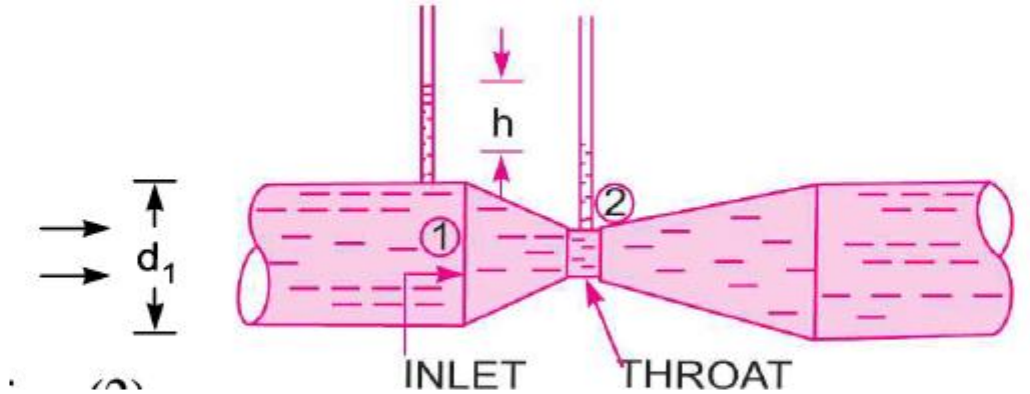


Figure 4.4. Venturimeter

Let d_1 = diameter at inlet or at section (1),
p_1 = pressure at section (1)
v_1 = velocity of fluid at section (1),

a = area at section (1) = $\frac{\pi}{4} d_1^{\,2}$

d_2, p_2, v_2, a_2 are corresponding values at section (2).

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

As pipe is horizontal, hence $z_1 = z_2$

$$\therefore \qquad \frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} \quad \text{or} \quad \frac{p_1 - p_2}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But $\frac{p_1 - p_2}{\rho g}$ is the difference of pressure heads at sections 1 and 2 and it is equal to h or $\frac{p_1 - p_2}{\rho g} = h$

Substituting this value of $\frac{p_1 - p_2}{\rho g}$ in the above equation, we get

$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

Now applying continuity equation at sections 1 and 2

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2 v_2}{a_1}$$

Substituting this value of v_1

$$h = \frac{v_2^2}{2g} - \frac{\left(\frac{a_2 v_2}{a_1}\right)^2}{2g} = \frac{v_2^2}{2g}\left[1 - \frac{a_2^2}{a_1^2}\right] = \frac{v_2^2}{2g}\left[\frac{a_1^2 - a_2^2}{a_1^2}\right]$$

$$v_2{}^2 = 2gh \frac{a_1^2}{a_1^2 - a_2^2}$$

$$v_2 = \sqrt{2gh \frac{a_1^2}{a_1^2 - a_2^2}} = \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

Discharge, $\qquad Q = a_2 v_2$

$$= a_2 \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

Equation gives the discharge under ideal conditions and is called, theoretical discharge. Actual discharge will be less than theoretical discharge.

$$\therefore \qquad Q_{\text{act}} = C_d \times \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

where C_d = Co-efficient of venturimeter and its value is less than 1.

Value of ‘h’ given by differential U-tube manometer

Case I. Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let

S_h = Sp. gravity of the heavier liquid
S_o = Sp. gravity of the liquid flowing through pipe
x = Difference of the heavier liquid column in U-tube

Then $$h = x\left[\frac{S_h}{S_o} - 1\right]$$

Case II. If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

$$h = x\left[1 - \frac{S_l}{S_o}\right]$$

where S_l = Sp. gr. of lighter liquid in U-tube
S_o = Sp. gr. of fluid flowing through pipe
x = Difference of the lighter liquid columns in U-tube.

Case III. Inclined Venturimeter with Differential U-tube manometer. The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_h}{S_o} - 1\right]$$

Case IV. Similarly, for inclined venturimeter in which differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[1 - \frac{S_l}{S_o}\right]$$

Problem 5:

A horizontal venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to the inlet and the throat is 20 cm of mercury. Determine the rate of flow. Take C_d = 0.98.

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ Area at inlet, $a_1 = \frac{\pi}{4} d_1{}^2 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \frac{\pi}{4} \times 15^2 = 176.7 \text{ cm}^2$

$C_d = 0.98$

Reading of differential manometer = x = 20 cm of mercury.

$\therefore$ Difference of pressure head is given by (6.9)

or $$h = x\left[\frac{S_h}{S_o} - 1\right]$$

where S_h = Sp. gravity of mercury = 13.6, S_o = Sp. gravity of water = 1

$$= 20\left[\frac{13.6}{1} - 1\right] = 20 \times 12.6 \text{ cm} = 252.0 \text{ cm of water.}$$

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 9.81 \times 252}$$

$$= \frac{86067593.36}{\sqrt{499636.9 - 31222.9}} = \frac{86067593.36}{684.4}$$

$$= 125756 \text{ cm}^3\text{/s} = \frac{125756}{1000} \text{ lit/s} = \mathbf{125.756 \text{ lit/s.}}$$

Problem 6:

A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of oil of sp. gr. 0.8. The discharge of oil through venturimeter is 60 litres/s. Find the reading of the oil-mercury differential manometer. Take $C_d = 0.98$.

$$d_1 = 20 \text{ cm}$$

$$a_1 = \frac{\pi}{4} 20^2 = 314.16 \text{ cm}^2$$

$$d_2 = 10 \text{ cm}$$

$$a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$$

$$C_d = 0.98$$

$$Q = 60 \text{ litres/s} = 60 \times 1000 \text{ cm}^3\text{/s}$$

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$60 \times 1000 = 9.81 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times h} = \frac{1071068.78\sqrt{h}}{304}$$

$$\sqrt{h} = \frac{304 \times 60000}{1071068.78} = 17.029$$

$$h = (17.029)^2 = 289.98 \text{ cm of oil}$$

$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

$$289.98 = x\left[\frac{13.6}{0.8} - 1\right] = 16x$$

$$x = \frac{289.98}{16} = 18.12 \text{ cm.}$$

Reading of oil-mercury differential manometer = **18.12 cm.**

Problem 7:

The inlet and throat diameters of a horizontal venturimeter are 30 cm and 10 cm respectively. The liquid flowing through the meter is water. The pressure intensity at inlet is 13.734 N/cm^2 while the vacuum pressure head at the throat is 37 cm of mercury. Find the rate of flow. Assume that 4% of the differential head is lost between the inlet and throat. Find also the value of C_d for the venturimeter.

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \dfrac{\pi}{4}(30)^2 = 706.85\text{ cm}^2$

Dia. at throat, $d_2 = 10$ cm

$\therefore$ $a_2 = \dfrac{\pi}{4}(10)^2 = 78.54\text{ cm}^2$

Pressure, $p_1 = 13.734\text{ N/cm}^2 = 13.734 \times 10^4\text{ N/m}^2$

$\therefore$ Pressure head, $\dfrac{p_1}{\rho g} = \dfrac{13.734 \times 10^4}{1000 \times 9.81} = 14$ m of water

$$\frac{p_2}{\rho g} = -37 \text{ cm of mercury}$$

$$= \frac{-37 \times 13.6}{100} \text{ m of water} = -5.032 \text{ m of water}$$

Differential head,
$$h = p_1/\rho g - p_2/\rho g$$
$$= 14.0 - (-5.032) = 14.0 + 5.032$$
$$= 19.032 \text{ m of water} = 1903.2 \text{ cm}$$

Head lost, $h_f = 4\%$ of $h = \dfrac{4}{100} \times 19.032 = 0.7613$ m

$\therefore$ $C_d = \sqrt{\dfrac{h - h_f}{h}} = \sqrt{\dfrac{19.032 - .7613}{19.032}} = 0.98$

$\therefore$ Discharge $= C_d \dfrac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$

$\therefore$ Discharge $= C_d \dfrac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$

$$= \frac{0.98 \times 706.85 \times 78.54 \times \sqrt{2 \times 981 \times 1903.2}}{\sqrt{(706.85)^2 - (78.54)^2}}$$

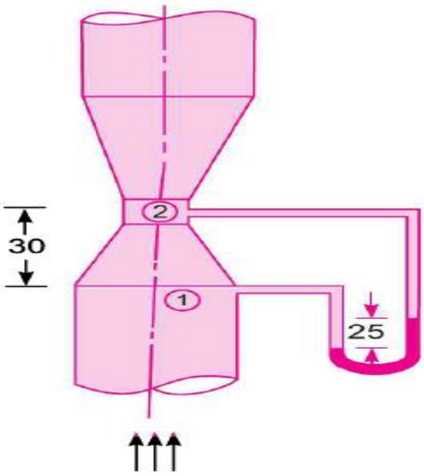
$$= \frac{105132247.8}{\sqrt{499636.9 - 6168}} = 149692.8 \text{ cm}^3\text{/s} = \mathbf{0.14969\ m^3/s.}$$

Problem 8:

A 30 cm × 15 cm venturimeter is provided in a vertical pipe line carrying oil of specific gravity 0.9, the flow being upwards. The difference in elevation of the throat section and entrance section of the venturimeter is 30 cm. The differential U-tube mercury manometer shows a gauge deflection of 25 cm. Calculate :

(i) the discharge of oil, and

(ii) the pressure difference between the entrance section and the throat section. Take the co-efficient of discharge as 0.98 and specific gravity of mercury as 13.6.



Dia. at inlet, $d_1 = 30$ cm

$\therefore$ Area, $a_1 = \dfrac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ Area, $a_2 = \dfrac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

Sp. gr. of oil, $S_o = 0.9$

Sp. gr. of mercury, $S_g = 13.6$

Reading of diff. manometer, $x = 25$ cm

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right)$$

$$= x\left[\frac{S_g}{S_o} - 1\right] = 25\left[\frac{13.6}{0.9} - 1\right] = 352.77 \text{ cm of oil}$$

(i) The discharge, Q of oil

$$= C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= \frac{0.98 \times 706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} = \sqrt{2 \times 981 \times 352.77}$$

$$= \frac{101832219.9}{684.4} = 148790.5 \text{ cm}^3\text{/s}$$

= 148.79 litres/s. Ans.

(ii) Pressure difference between entrance and throat section

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = 352.77$$

$$\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) + z_1 - z_2 = 352.77$$

$$z_2 - z_1 = 30 \text{ cm}$$

$$\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) - 30 = 352.77$$

$\dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g}$ = 352.77 + 30 = 382.77 cm of oil = **3.8277 m of oil.**

$$(p_1 - p_2) = 3.8277 \times \rho g$$

$$= \text{Sp. gr. of oil} \times 1000 \text{ kg/m}^3$$

$$= 0.9 \times 1000 = 900 \text{ kg/cm}^3$$

$$(p_1 - p_2) = 3.8277 \times 900 \times 9.81 \ \frac{\text{N}}{\text{m}^2}$$

$$= \frac{33795}{10^4} \text{ N/cm}^2 = \mathbf{3.3795 \ N/cm^2.}$$

Orifice Flow Measurement – History:

The first record of the use of orifices for the measurement of fluids was by Giovanni B.Venturi, an Italian Physicist, who in 1797 did some work that led to the development of the modern Venturi Meter by Clemons Herschel in 1886. It has been reported that an orifice meter, designed by Professor Robinson of Ohio State University was used to measure gas near Columbus, Ohio, about 1890. About 1903 Mr. T.B. Weymouth began a series of tests in Pennsylvania leading to the publication of coefficients for orifice meters with flange taps. At the same time Mr. E.O. Hickstein made a similar series of tests at Joplin, Missouri, from which he developed data for orifice meters with pipe taps. An orifice in a pipeline is shown in Figure 4.5 with a manometer for measuring the drop in pressure (differential) as the fluid passes thru the orifice. The minimum cross sectional area of the jet is known as the "vena contracta."

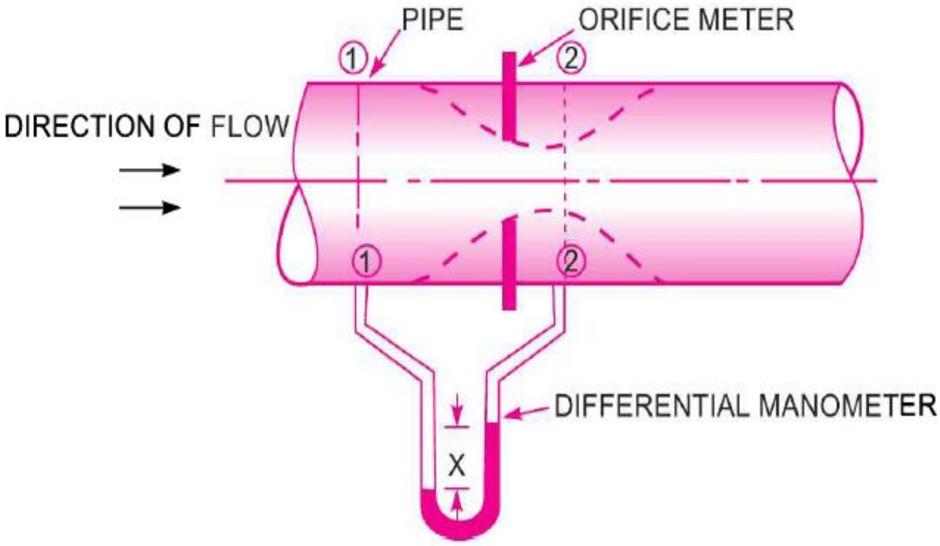


Figure 4.5. Orificemeter

The discharge, Q is given by equation

$$Q = C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

What is an Orifice Meter?

An orifice meter is a conduit and a restriction to create a pressure drop. An hour glass is a form of orifice. A nozzle, venturi or thin sharp edged orifice can be used as the flow restriction. In order to use any of these devices for measurement it is necessary to empirically calibrate them. That is, pass a known volume through the meter and note the reading in order to provide a standard for measuring other quantities. Due to the ease of duplicating and the simple construction, the thin sharp edged orifice has been adopted as a standard and extensive calibration work has been done so that it is widely accepted as a standard means of measuring fluids. Provided the standard mechanics of construction are followed no further calibration is required.

Major Advantages of Orifice Meter Measurement

Flow can be accurately determined without the need for actual fluid flow calibration. Well established procedures convert the differential pressure into flow rate, using empirically derived coefficients. These coefficients are based on accurately measurable dimensions of the orifice plate and pipe diameters as defined in standards, combined with easily measurable characteristics of the fluid, rather than on fluid flow calibrations. With the exception of the orifice meter, almost all flow meters require a fluid flow calibration at flow and temperature conditions closely approximating service operation in order to establish accuracy.

Problem 9:

An orifice meter with orifice diameter 10 cm is inserted in a pipe of 20 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter gives readings of 19.62 N/cm^2 and 9.81 N/cm^2 respectively. Co-efficient of discharge for the orifice meter is given as 0.6. Find the discharge of water through pipe.

Dia. of orifice, $d_0 = 10$ cm

$\therefore$ Area, $a_0 = \frac{\pi}{4}(10)^2 = 78.54\ \text{cm}^2$

Dia. of pipe, $d_1 = 20$ cm

$\therefore$ Area, $a_1 = \frac{\pi}{4}(20)^2 = 314.16 \text{ cm}^2$

$p_1 = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2$

$$\frac{p_1}{\rho g} = \frac{19.62 \times 10^4}{1000 \times 9.81} = 20 \text{ m of water}$$

$$\frac{p_2}{\rho g} = \frac{9.81 \times 10^4}{1000 \times 9.81} = 10 \text{ m of water}$$

$$h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 20.0 - 10.0 = 10 \text{ m of water} = 1000 \text{ cm of water}$$

$$Q = C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.6 \times \frac{78.54 \times 314.16}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 1000}$$

$$= \frac{20736838.09}{304} = 68213.28 \text{ cm}^3\text{/s} = \mathbf{68.21 \text{ litres/s.}}$$

Problem 10:

An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure difference measured by a mercury oil differential manometer on the two sides of the orifice meter gives a reading of 50 cm of mercury. Find the rate of flow of oil of sp. gr. 0.9 when the co-efficient of discharge of the orifice meter = 0.64.

Dia. of orifice, $d_0 = 15$ cm

$\therefore$ Area, $a_0 = \frac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

Dia. of pipe, $d_1 = 30$ cm

$\therefore$ Area, $a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Sp. gr. of oil, $S_o = 0.9$

Reading of diff. manometer, $x = 50$ cm of mercury

$\therefore$ Differential head, $h = x\left[\frac{S_g}{S_o} - 1\right] = 50\left[\frac{13.6}{0.9} - 1\right]$ cm of oil

$= 50 \times 14.11 = 705.5$ cm of oil

$$Q = C_d \cdot \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.64 \times \frac{176.7 \times 706.85}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 705.5}$$

$$= \frac{94046317.78}{684.4} = 137414.25 \text{ cm}^3\text{/s} = \mathbf{137.414 \text{ litres/s.}}$$

Pitot tube for Flow Measurement Construction:

The principle of flow measurement by Pitot tube was adopted first by a French Scientist Henri Pitot in 1732 for measuring velocities in the river. A right angled glass tube, large enough for capillary effects to be negligible, is used for the purpose. One end of the tube faces the flow while the other end is open to the atmosphere as shown in Fig.4.6.

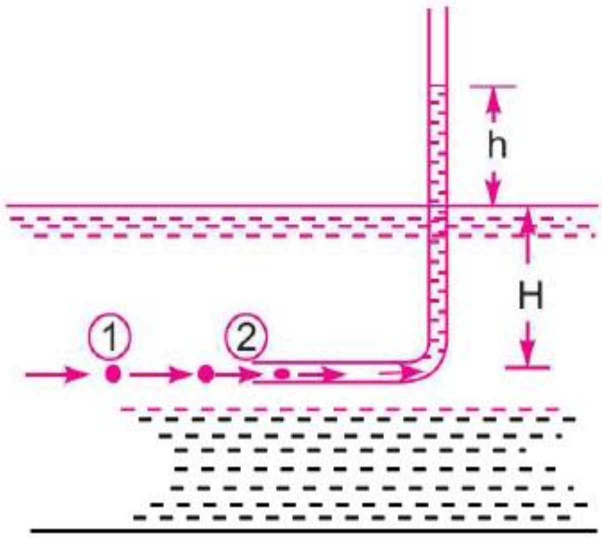


Figure 4.6. Pitot tube

Consider two points (1) and (2) at the same level in such a way that point (2) is just as the inlet of the pitot-tube and point (1) is far away from the tube.

Let p_1 = intensity of pressure at point (1)

v_1 = velocity of flow at (1)

p_2 = pressure at point (2)

v_2 = velocity at point (2), which is zero

H = depth of tube in the liquid

h = rise of liquid in the tube above the free surface.

Applying Bernoulli's equation at points (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $z_1 = z_2$ as points (1) and (2) are on the same line and $v_2 = 0$.

$$\frac{p_1}{\rho g} = \text{pressure head at (1)} = H$$

$$\frac{p_2}{\rho g} = \text{pressure head at (2)} = (h + H)$$

Substituting these values, we get

$$\therefore \qquad H + \frac{v_1^2}{2g} = (h + H) \quad \therefore \quad h = \frac{v_1^2}{2g} \quad \text{or} \quad v_1 = \sqrt{2gh}$$

This is theoretical velocity. Actual velocity is given by

$$(v_1)_{act} = C_v \sqrt{2gh}$$

where C_v = Co-efficient of pitot-tube

Velocity of flow in a pipe by pitot-tube. For finding the velocity at any point in a pipe by pitot-tube, the following arrangements are adopted :

1. Pitot-tube along with a vertical piezometer tube
2. Pitot-tube connected with piezometer tube
3. Pitot-tube and vertical piezometer tube connected with a differential U-tube manometer

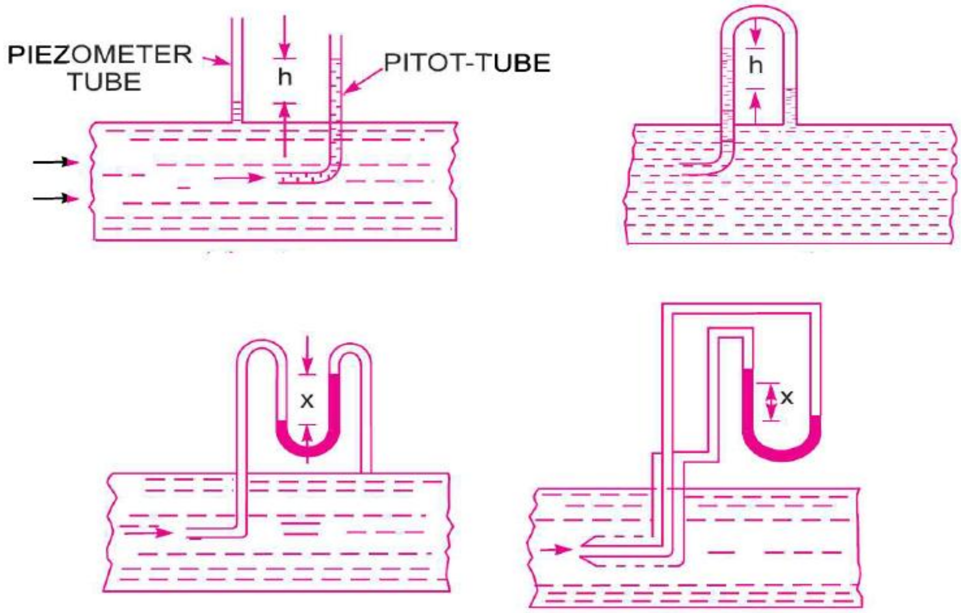


Figure 4.7. Velocity of flow in a pipe by Pitot tube

Problem 11:

Find the velocity of the flow of an oil through a pipe, when the difference of mercury level in a differential U-tube manometer connected to the two tappings of the pitot-tube is 100 mm. Take co-efficient of pitot-tube 0.98 and sp. gr. of oil = 0.8.

Diff. of mercury level, $x = 100 \text{ mm} = 0.1 \text{ m}$

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_g = 13.6$

$C_v = 0.98$

Diff. of pressure head, $h = x\left[\frac{S_g}{S_o} - 1\right] = .1\left[\frac{13.6}{0.8} - 1\right] = 1.6 \text{ m of oil}$

$\therefore$ Velocity of flow $= C_v\sqrt{2gh} = 0.98\sqrt{2 \times 9.81 \times 1.6} =$ **5.49 m/s. Ans.**

Problem 12:

A sub-marine moves horizontally in sea and has its axis 15 m below the surface of water. A pitot-tube properly placed just in front of the sub-marine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 170 mm. Find the speed of the sub-marine knowing that the sp. gr. of mercury is 13.6 and that of sea-water is 1.026 with respect of fresh water.

Diff. of mercury level, $x = 170 \text{ mm} = 0.17 \text{ m}$

Sp. gr. of mercury, $S_g = 13.6$

Sp. gr. of sea-water, $S_o = 1.026$

$$\therefore \quad h = x\left[\frac{S_g}{S_o} - 1\right] = 0.17\left[\frac{13.6}{1.026} - 1\right] = 2.0834 \text{ m}$$

$$\therefore \quad V = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2.0834} = 6.393 \text{ m/s}$$

$$= \frac{6.393 \times 60 \times 60}{1000} \text{ km/hr} = \mathbf{23.01 \text{ km/hr. Ans.}}$$

Problem 13:

A pitot-tube is inserted in a pipe of 300 mm diameter. The static pressure in pipe is 100 mm of mercury (vacuum). The stagnation pressure at the centre of the pipe, recorded by the pitot-tube is 0.981 N/cm^2. Calculate the rate of flow of water through pipe, if the mean velocity of flow is 0.85 times the central velocity. Take $C_v = 0.98$.

Dia. of pipe, $d = 300 \text{ mm} = 0.30 \text{ m}$

$\therefore$ Area, $a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$

Static pressure head $= 100$ mm of mercury (vacuum)

$$= -\frac{100}{1000} \times 13.6 = -1.36 \text{ m of water}$$

Stagnation pressure $= .981 \text{ N/cm}^2 = .981 \times 10^4 \text{ N/m}^2$

$\therefore$ Stagnation pressure head $= \frac{.981 \times 10^4}{\rho g} = \frac{.981 \times 10^4}{1000 \times 9.81} = 1 \text{ m}$

$\therefore$ h = Stagnation pressure head – Static pressure head

$= 1.0 - (-1.36) = 1.0 + 1.36 = 2.36$ m of water

$\therefore$ Velocity at centre $= C_v \sqrt{2gh}$

$$= 0.98 \times \sqrt{2 \times 9.81 \times 2.36} = 6.668 \text{ m/s}$$

Mean velocity, $\overline{V} = 0.85 \times 6.668 = 5.6678 \text{ m/s}$

$\therefore$ Rate of flow of water $= \overline{V} \times$ area of pipe

$= 5.6678 \times 0.07068 \text{ m}^3\text{/s} = \mathbf{0.4006 \text{ m}^3\text{/s}}$. **Ans.**

Force exerted by a flowing fluid on a pipe bend

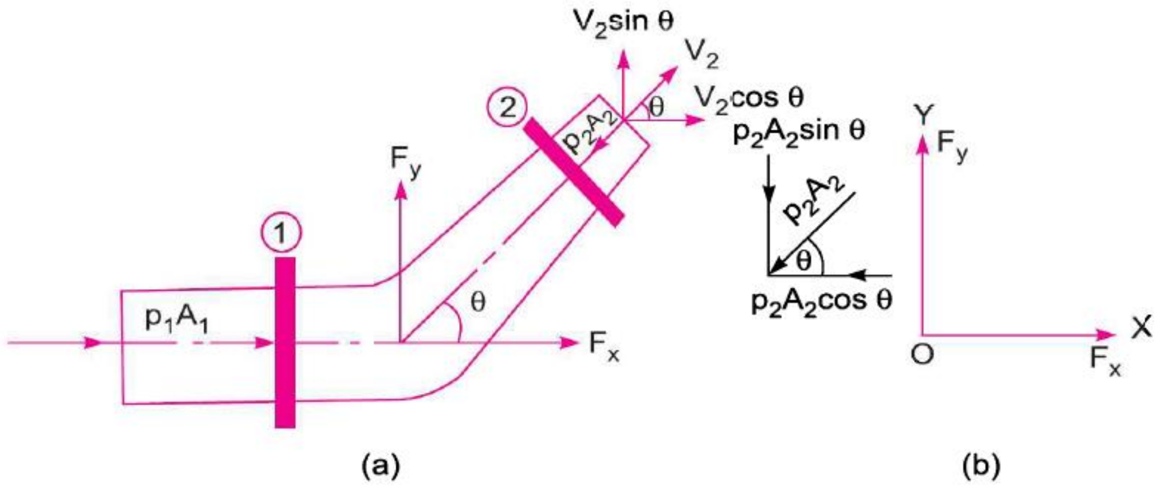


Figure4.8. Forces on bend

Let v_1 = velocity of flow at section (1),

p_1 = pressure intensity at section (1),

A_1 = area of cross-section of pipe at section (1) and

v_2, p_2, A_2 = corresponding values of velocity, pressure and area at section (2)

Net force acting on fluid in the direction of x = Rate of change of momentum in x-direction

$\therefore \quad p_1A_1 - p_2A_2 \cos\theta - F_x$ = (Mass per sec) (change of velocity)

= ρQ (Final velocity in the direction of x – Initial velocity in the direction of x)

$$= \rho Q\,(V_2 \cos\theta - V_1)$$

$$F_x = \rho Q\,(V_1 - V_2\cos\theta) + p_1A_1 - p_2A_2\cos\theta$$

Similarly the momentum equation in y-direction gives

$$0 - p_2A_2\sin\theta - F_y = \rho Q\,(V_2\sin\theta - 0)$$

$$\therefore \quad F_y = \rho Q\,(-V_2\sin\theta) - p_2A_2\sin\theta$$

Now the resultant force (F_R) acting on the bend

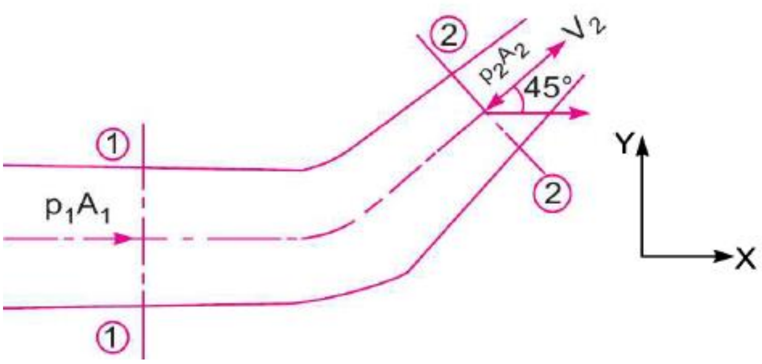
$$= \sqrt{F_x^2 + F_y^2}$$

And the angle made by the resultant force with horizontal direction is given by

$$\tan\theta = \frac{F_y}{F_x}$$

Problem 14:

A 45° reducing bend is connected in a pipe line, the diameters at the inlet and outlet of the bend being 600 mm and 300 mm respectively. Find the force exerted by water on the bend if the intensity of pressure at inlet to bend is 8.829 N/cm² and rate of flow of water is 600 litres/s.



Angle of bend, $\theta = 45°$

Dia. at inlet, $D_1 = 600 \text{ mm} = 0.6 \text{ m}$

$\therefore$ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.6)^2$

$= 0.2827 \text{ m}^2$

Dia. at outlet, $D_2 = 300 \text{ mm} = 0.30 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$

Pressure at inlet, $p_1 = 8.829 \text{ N/cm}^2 = 8.829 \times 10^4 \text{ N/m}^2$

$Q = 600 \text{ lit/s} = 0.6 \text{ m}^3/\text{s}$

$$V_1 = \frac{Q}{A_1} = \frac{0.6}{.2827} = 2.122 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.6}{.07068} = 8.488 \text{ m/s.}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$z_1 = z_2$

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \quad \text{or} \quad \frac{8.829 \times 10^4}{1000 \times 9.81} + \frac{2.122^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{8.488^2}{2 \times 9.81}$$

$9 + .2295 = p_2/\rho g + 3.672$

$$\frac{p_2}{\rho g} = 9.2295 - 3.672 = 5.5575 \text{ m of water}$$

$$p_2 = 5.5575 \times 1000 \times 9.81 \text{ N/m}^2 = 5.45 \times 10^4 \text{ N/m}^2$$

Forces on the bend in x- and y-directions are given by equations

$$F_x = \rho Q\ [V_1 - V_2 \cos \theta] + p_1 A_1 - p_2 A_2 \cos \theta$$
$$= 1000 \times 0.6\ [2.122 - 8.488 \cos 45°]$$
$$+ 8.829 \times 10^4 \times .2827 - 5.45 \times 10^4 \times .07068 \times \cos 45°$$
$$= -2327.9 + 24959.6 - 2720.3 = 24959.6 - 5048.2$$
$$= 19911.4 \text{ N}$$

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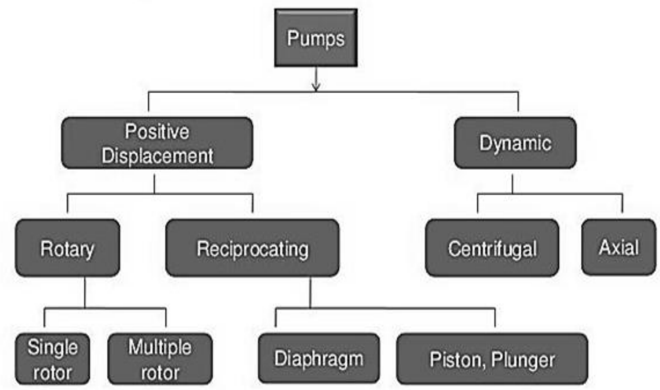
UNIT V PUMPS AND TURBINES

UNIT 5 PUMPS & TURBINES
Centrifugal Pumps: Definition - Operations - Velocity Triangles - Performance curves - Cavitations - Multistaging. Reciprocating Pumps: Operation - Slip - indicator Diagram - Separation - Air vessels. Hydraulic Turbines: Classification of hydraulic turbines - Working principle of Pelton wheel, Francis and Kaplan turbines - velocity triangles - draft tube – hydraulic turbine characteristics. Dimensional Analysis: Buckingham's Theorem, Non-Dimension Numbers, Similarities of Flow- Model studies

Hydraulic Pump

A hydraulic pump is a mechanical source of power that converts mechanical power into hydraulic energy. It generates flow with enough power to overcome pressure induced by the load at the pump outlet. When a hydraulic pump operates, it creates a vacuum at the pump inlet, which forces liquid from the reservoir into the inlet line to the pump and by mechanical action delivers this liquid to the pump outlet and forces it into the hydraulic system.

Classifications of Pump



Centrifugal Pump

The main components of a centrifugal pump are:

i) Impeller
ii) Casing
iii) Suction pipe
iv) Foot valve with strainer,
v) Delivery pipe
vi) Delivery valve.

Impeller is the rotating component of the pump. It is made up of a series of curved vanes. The impeller is mounted on the shaft connecting an electric motor.

Casing is an air tight chamber surrounding the impeller. The shape of the casing is designed in such a way that the kinetic energy of the impeller is gradually changed to potential energy. This is achieved by gradually increasing the area of cross section in the direction of flow.

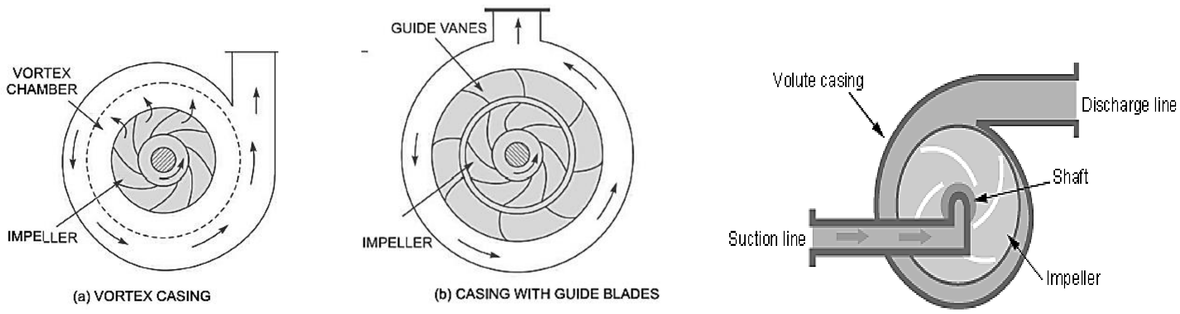


Fig. Types of Casing

Suction pipe: It is the pipe connecting the pump to the sump, from where the liquid has to be lifted up.

Foot valve with strainer: The foot valve is a non-return valve which permits the flow of the liquid from the other words the foot valve opens only in the upward direction. The strainer is a mesh surrounding the valve, it p debris and silt into the pump.

Delivery pipe is a pipe connected to the pump to the overhead tank. Delivery valve is a valve which can regulate the pump.

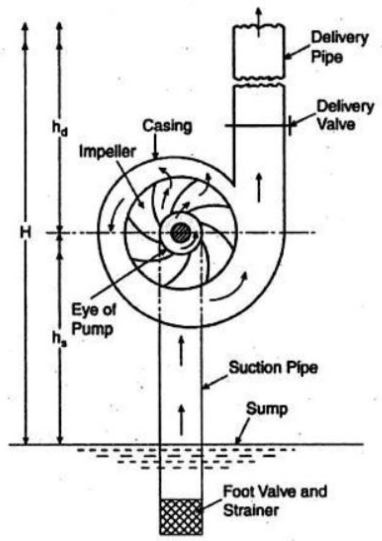


Fig. Main parts of a centrifugal pump

Working

A centrifugal pump works on the principle that when a certain mass of fluid is rotated by an external source, it is thrown away from the central axis of rotation and a centrifugal head is impressed which enables it to rise to a higher level.

Working operation of a centrifugal pump is explained in the following steps:

1. Close the delivery valve and prime the pump.
2. Start the motor connected to the pump shaft, this causes an increase in the impeller pressure.
3. Open the delivery valve gradually, so that the liquid starts flowing into the deliver pipe.
4. A partial vacuum is created at the eye of the centrifugal action, the liquid rushed from the sump to the pump due to pressure difference at the two ends of the suction pipe.
5. As the impeller continues to run, move & more liquid are made available to the pump at its eye. Therefore impeller increases the energy of the liquid and delivers it to the reservoir.
6. While stopping the pump, the delivery valve should be closed first; otherwise there may be back flow from the reservoir.

It may be noted that a uniform velocity of flow is maintained in the delivery pipe. This is due to the special design of the casing. As the flow proceeds from the tongue of the casing to the delivery pipe, the area of the casing increases. There is a corresponding change in the quantity of the liquid from the impeller. Thus a uniform flow occurs in the delivery pipe.

Centrifugal pump converts rotational energy, often from a motor, to energy in a moving fluid. A portion of the energy goes into kinetic energy of the fluid. Fluid enters axially through eye of the casing, is caught up in the impeller blades, and is whirled tangentially and radially outward until it leaves through all circumferential parts of the impeller into the diffuser part of the casing. The fluid gains both velocity and pressure while passing through the impeller. The doughnut-shaped diffuser, or scroll, section of the casing decelerates the flow and further increases the pressure. The negative pressure at the eye of the impeller helps to maintain the flow in the system. If no water is present initially, the negative pressure developed by the rotating air, at the eye will be negligibly small to suck fresh stream of water. As a result the impeller will rotate without sucking and discharging any water content. So the pump should be initially filled with water before starting it. This process is known as priming.

Use of the Casing

From the illustrations of the pump so far, one speciality of the casing is clear. It has an increasing area along the flow direction. Such increasing area will help to accommodate newly added water stream, and will also help to reduce the exit flow velocity. Reduction in the flow velocity will result in increase in the static pressure, which is required to overcome the resistance of pumping system.

NPSH - Overcoming the problem of Cavitation

If pressure at the suction side of impeller goes below vapour pressure of the water, a dangerous phenomenon could happen. Water will start to boil forming vapour bubbles. These bubbles will move along with the flow and will break in a high pressure region. Upon breaking the bubbles will send high impulsive shock waves and spoil impeller material overtime. This phenomenon is known as cavitation. More the suction head, lesser should be the pressure at suction side to lift the water. This fact puts a limit to the maximum suction head a pump can have. However Cavitation can be completely avoided by careful pump selection. The term NPSH (Net Positive Suction Head) helps the designer to choose the right pump which will completely avoid Cavitation. NPSH is defined as follows:

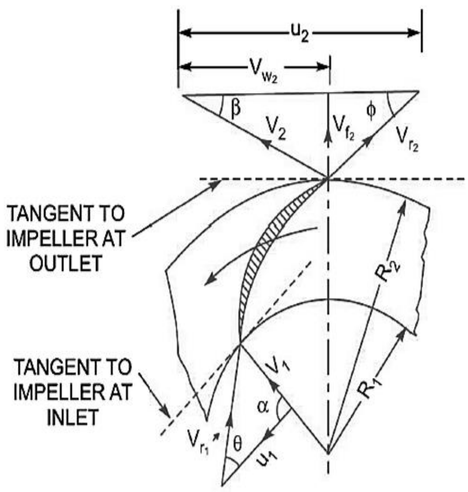
$$NPSH = \left(\frac{P}{\rho g} + \frac{V^2}{2g}\right)_{Suction} - \frac{P_v}{\rho g}$$

Where P_v is vapour pressure of water

V is speed of water at suction side

Work done by the centrifugal pump (or by impeller) on water

Velocity triangles at inlet and outlet



Let,

D_1 : *Diameter of impeller at inlet* $= 2 \times R_1$

D_2 : *Diameter of impeller at outlet* $= 2 \times R_2$

N : *Speed ofimpeller in rpm*

u_1 : *Tangential blade velocity at inlet* $= wR_1 = (\frac{2\pi N}{60})R_1$

u_2 : *Tangential blade velocity at outlet* $= wR_2 = (\frac{2\pi N}{60})R_2$

V : *Absolute velocity*

V_r : *Relative velocity*

V_f : *Velocity of flow*

V_w; *Velocity of whirl*

α_1 : *Angle mode by absolute velocity V_1 at inlet*

θ : *Inlet angle of vane*

ϕ : *Outlet angle of vane*

β : *Discharge angle of absolute velocity at outlet*

$$Angular\ momentum = mass \times tangentialvelocity \times Radius$$

$$Angular\ momentum\ entering\ the\ impeller\ per\ sec = m.V_{w1}.R_1$$

$$Angular\ momentum\ leaving\ the\ impeller\ per\ sec = m.V_{w2}.R_2$$

$$Torque\ transmitted = rate\ of\ change\ of\ angular\ momentum$$

$$= m.V_{w2}.R_2 - m.V_{w1}.R_1$$

$$= \frac{w}{g}(V_{w2}.R_2 - V_{w1}.R_1)$$

Since the work done in unit time is given by the product of torque and angular velocity

W.D per sec = Torque x W

$$= \frac{w}{g}(V_{w2}.R_2w - V_{w1}.R_1w)$$

But $R_2w = u_2$ and $R_1w = u_1$

W.D per sec = $\frac{w}{g}(V_{w2}u_2.V_{w1}u_1)$

Work done by impeller per N weight of liquid per sec,

W.D = $\frac{1}{g}(V_{w2}u_2 - V_{w1}u_1)$

But $V_{w1} = 0$ since entry is radial

W.D per N weight per sec = $\frac{V_{w2}.u_2}{g}$

Definitions of Heads and Efficiencies of a centrifugal pump

1. Suction Head (h_s). It is the vertical height of the centre line of the centrifugal pump above the water surface in the tank or pump from which water is to be lifted as shown in Fig. This height is also called suction lift and is denoted by 'h_s'.

2. Delivery Head (h_d). The vertical distance between the centre line of the pump and the water surface in the tank to which water is delivered is known as delivery head. This is denoted by 'h_d'.

3. Static Head (H_s). The sum of suction head and delivery head is known as static head. This is represented by 'H_s' and is written as

$$H_s = h_s + h_d.$$

4. Manometric Head (H_m). The manometric head is defined as the head against which a centrifugal pump has to work. It is denoted by 'H_m'. It is given by the following expressions :

(*a*) H_m = Head imparted by the impeller to the water – Loss of head in the pump

$$= \frac{V_{w_2} u_2}{g} - \text{Loss of head in impeller and casing}$$

$$= \frac{V_{w_2} u_2}{g} \quad \text{...if loss of pump is zero}$$

(*b*) H_m = Total head at outlet of the pump – Total head at the inlet of the pump

$$= \left(\frac{P_o}{\rho g} + \frac{V_o^2}{2g} + Z_o\right) - \left(\frac{p_i}{\rho g} + \frac{V_i^2}{2g} + Z_i\right)$$

(*c*) $$H_m = h_s + h_d + h_{f_s} + h_{f_d} + \frac{V_d^2}{2g}$$

where h_s = Suction head, h_d = Delivery head,
h_{f_s} = Frictional head loss in suction pipe, h_{f_d} = Frictional head loss in delivery pipe,
V_d = Velocity of water in delivery pipe.

Efficiencies of a Centrifugal Pump.

(*a*) **Manometric Efficiency (η_{man}).**

$$\eta_{man} = \frac{\text{Manometric head}}{\text{Head imparted by impeller to water}}$$

$$= \frac{H_m}{\left(\frac{V_{w_2} u_2}{g}\right)} = \frac{gH_m}{V_{w_2} u_2}$$

The power at the impeller of the pump is more than the power given to the water at outlet of the pump. The ratio of the power given to water at outlet of the pump to the power available at the impeller, is known as manometric efficiency.

(*b*) **Mechanical Efficiency (η_m).**

$$\eta_m = \frac{\text{Power at the impeller}}{\text{Power at the shaft}}$$

The power at the impeller in kW $= \dfrac{\text{Work done by impeller per second}}{1000}$

$$= \frac{W}{g} \times \frac{V_{w_2} u_2}{1000}$$

$$\eta_m = \frac{\frac{W}{g}\left(\frac{V_{w_2} u_2}{1000}\right)}{\text{S.P.}}$$

where S.P. = Shaft power.

(*c*) **Overall Efficiency (η_o).** It is defined as ratio of power output of the pump to the power input to the pump. The power output of the pump in kW

$$= \frac{\text{Weight of water lifted} \times H_m}{1000} = \frac{WH_m}{1000}$$

Power input to the pump $= \text{Power supplied by the electric motor}$

$= \text{S.P. of the pump.}$

$$\therefore \qquad \eta_o = \frac{\left(\dfrac{WH_m}{1000}\right)}{\text{S.P.}}$$

Also $\qquad \eta_o = \eta_{man} \times \eta_m.$

PRIMING OF A CENTRIFUGAL PUMP

Priming of a centrifugal pump is defined as the operation in which the suction pipe, casing of the pump and a portion of the delivery pipe upto the delivery valve is completely filled up from outside source with the liquid to be raised by the pump before starting the pump. Thus the air from these parts of the pump is removed and these parts are filled with the liquid to be pumped.

CAVITATION

Cavitation includes formation of vapour bubbles of the flowing liquid and collapsing of the vapour bubbles. Formation of vapour bubbles of the flowing liquid take place only whenever the pressure in any region falls below vapour pressure. When the pressure of the flowing liquid is less than its vapour pressure, the liquid starts boiling and vapour bubbles are formed. These vapour bubbles are carried along with the flowing liquid to higher pressure zones where these vapours condense and bubbles collapse. Due to sudden collapsing of the bubbles on the metallic surface, high pressure is produced and metallic surfaces are subjected to high local stresses. Thus the surfaces are damaged.

Cavitation in Centrifugal Pumps. In centrifugal pumps the cavitation may occur at the inlet of the impeller of the pump, or at the suction side of the pumps, where the pressure is considerably reduced. Hence if the pressure at the suction side of the pump drops below the vapour pressure of the liquid then the cavitation may occur. The cavitation in a pump can be noted by a sudden drop in efficiency and head. In order to determine whether cavitation will occur in any portion of the suction side of the pump, the critical value of Thoma's cavitation factor (σ) is calculated.

Precaution Against Cavitation.

(*i*) The pressure of the flowing liquid in any part of the hydraulic system should not be allowed to fall below its vapour pressure. If the flowing liquid is water, then the absolute pressure head should not be below 2.5 m of water.

(*ii*) The special materials or coatings such as aluminium-bronze and stainless steel, which are cavitation resistant materials, should be used.

Effects of Cavitation.

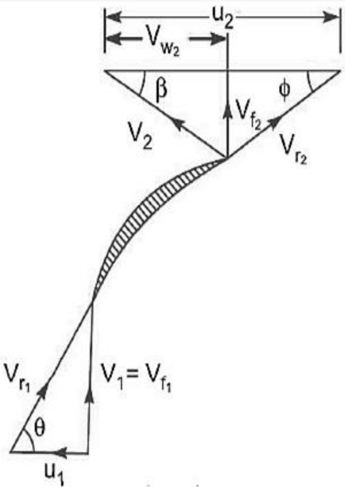
(*i*) The metallic surfaces are damaged and cavities are formed on the surfaces.

(*ii*) Due to sudden collapse of vapour bubble, considerable noise and vibrations are produced.

(*iii*) The efficiency of a turbine decreases due to cavitation. Due to pitting action, the surface of the turbine blades becomes rough and the force exerted by water on the turbine blades decreases. Hence, the work done by water or output horse power becomes less and thus efficiency decreases.

Example *The internal and external diameters of the impeller of a centrifugal pump are 200 and 400 mm respectively. The pump is running at 1200 rpm. The vane angles of the impeller at inlet and outlet are 20 and 30 respectively. The water enters the impeller radially and velocity of flow is constant. Determine the work done by the impeller per unit weight of water.*

Given:



Internal diameter of impeller, $D_1 = 200$ mm $= 0.20$ m

External diameter of impeller, $D_2 = 400$ mm $= 0.40$ m

Speed, $N = 1200$ r.p.m.

Vane angle at inlet, $\theta = 20°$

Vane angle at outlet, $\phi = 30°$

Water enters radially* means, $\alpha = 90°$ and $V_{w_1} = 0$

Velocity of flow, $V_{f_1} = V_{f_2}$

Tangential velocity of impeller at inlet and outlet are,

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.20 \times 1200}{60} = 12.56 \text{ m/s}$$

d

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.4 \times 1200}{60} = 25.13 \text{ m/s.}$$

From inlet velocity triangle, $\tan\theta = \frac{V_{f_1}}{u_1} = \frac{V_{f_1}}{12.56}$

$\therefore$ $V_{f_1} = 12.56 \tan\theta = 12.56 \times \tan 20° = 4.57$ m/s

$\therefore$ $V_{f_2} = V_{f_1} = 4.57$ m/s.

From outlet velocity triangle, $\tan\phi = \frac{V_{f_2}}{u_2 - V_{w_2}} = \frac{4.57}{25.13 - V_{w_2}}$

$$25.13 - V_{w_2} = \frac{4.57}{\tan\phi} = \frac{4.57}{\tan 30°} = 7.915$$

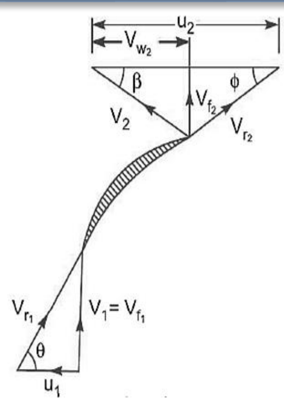
$\therefore$ $V_{w_2} = 25.13 - 7.915 = 17.215$ m/s.

The work done by impeller per kg of water per second is given by equation (

$$= \frac{1}{g} V_{w_2} u_2 = \frac{17.215 \times 25.13}{9.81} = \mathbf{44.1\ Nm/N}.$$

Example *A centrifugal pump is to discharge 0.118 m^3/s at a speed of 1450 rpm against a head of 25m. the impeller diameter is 250 mm, its width at outlet is 50 mm and manometric efficiency is 75%. Determine the vane angle at the outer periphery of the impeller.*

Given:



Discharge, $Q = 0.118\ m^3/s$

Speed, $N = 1450$ r.p.m.

Head, $H_m = 25$ m

Diameter at outlet, $D_2 = 250$ mm $= 0.25$ m

Width at outlet, $B_2 = 50$ mm $= 0.05$ m

Manometric efficiency, $\eta_{man} = 75\% = 0.75$.

Let vane angle at outlet $= \phi$

Tangential velocity of impeller at outlet,

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.25 \times 1450}{60} = 18.98 \text{ m/s}$$

Discharge is given by $Q = \pi D_2 B_2 \times V_{f_2}$

$\therefore$ $$V_{f_2} = \frac{Q}{\pi D_2 B_2} = \frac{0.118}{\pi \times 0.25 \times .05} = 3.0 \text{ m/s.}$$

$$\eta_{man} = \frac{gH_m}{V_{w_2} u_2} = \frac{9.81 \times 25}{V_{w_2} \times 18.98}$$

$$V_{w_2} = \frac{9.81 \times 25}{\eta_{man} \times 18.98} = \frac{9.81 \times 25}{0.75 \times 18.98} = 17.23.$$

From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{(u_2 - V_{w_2})} = \frac{3.0}{(18.98 - 17.23)} = 1.7143$$

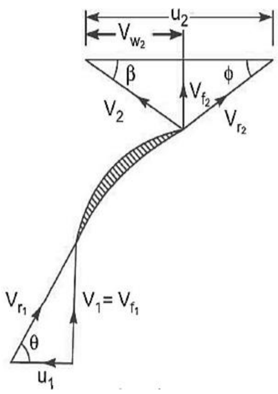
$\therefore$ $\phi = \tan^{-1} 1.7143 = 59.74°$ or **59° 44′. Ans.**

Example *A centrifugal pump delivers water against a net head of 14.5 m and a design speed of 1000 rpm. The vanes are curved back at an angle of 30° with the periphery. The impeller diameter is 300 mm and outlet width is 50 mm. determine the discharge of the pump if manometric efficiency is 95%.*

Given:

Net head, $H_m = 14.5$ m
Speed, $N = 1000$ r.p.m.
Vane angle at outlet, $\phi = 30°$
Impeller diameter means the diameter of the impeller at outlet
$\therefore$ Diameter, $D_2 = 300$ mm $= 0.30$ m
Outlet width, $B_2 = 50$ mm $= 0.05$ m
Manometric efficiency, $\eta_{man} = 95\% = 0.95$
Tangential velocity of impeller at outlet,

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.30 \times 1000}{60} = 15.70 \text{ m/s.}$$



$$\eta_{man} = \frac{gH_m}{V_{w_2} \times u_2}$$

$$0.95 = \frac{9.81 \times 14.5}{V_{w_2} \times 15.70}$$

$$V_{w_2} = \frac{0.95 \times 14.5}{0.95 \times 15.70} = 9.54 \text{ m/s.}$$

From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{(u_2 - V_{w_2})} \text{ or } \tan 30° = \frac{V_{f_2}}{(15.70 - 9.54)} = \frac{V_{f_2}}{6.16}$$

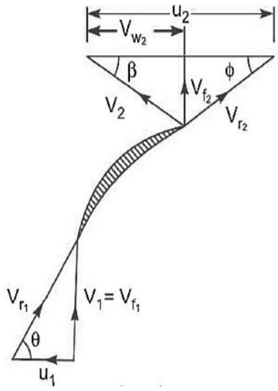
$$V_{f_2} = 6.16 \times \tan 30° = 3.556 \text{ m/s.}$$

$$Q = \pi D_2 B_2 \times V_{f_2}$$

$$= \pi \times 0.30 \times 0.05 \times 3.556 \text{ m}^3\text{/s} = \mathbf{0.1675 \text{ m}^3\text{/s.}} \textbf{ Ans.}$$

Example *A centrifugal pump having outer diameter equal to two times the inner diameter and running at 1000 rpm works against a total head of 40 m. the velocity of flow through the impeller is constant and equal to 2.5 m/s. the vanes are set back at an angle of 40° at outlet. If the outer diameter of the impeller is 500 mm and width at the outlet is 50 mm, determine: i) Vane angle at inlet, ii) work done by impeller on water per second iii) manometric efficiency*

Given:



Speed, $N = 1000$ r.p.m.

Head, $H_m = 40$ m

Velocity of flow, $V_{f_1} = V_{f_2} = 2.5$ m/s

Vane angle at outlet, $\phi = 40°$

Outer dia. of impeller, $D_2 = 500$ mm $= 0.50$ m

Inner dia. of impeller, $D_1 = \dfrac{D_2}{2} = \dfrac{0.50}{2} = 0.25$ m

Width at outlet, $B_2 = 50$ mm $= 0.05$ m

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.25 \times 1000}{60} = 13.09 \text{ m/s}$$

and
$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.50 \times 1000}{60} = 26.18 \text{ m/s.}$$

Discharge is given by, $Q = \pi D_2 B_2 \times V_{f_2} = \pi \times 0.50 \times .05 \times 2.5 = 0.1963 \text{ m}^3\text{/s.}$

(*i*) Vane angle at inlet (θ).

From inlet velocity triangle $\tan\theta = \dfrac{V_{f_1}}{u_1} = \dfrac{2.5}{13.09} = 0.191$

$\therefore$ $\theta = \tan^{-1} .191 = 10.81°$ or $\mathbf{10° 48'}$.

(*ii*) Work done by impeller on water per second is given by equation

$$= \frac{W}{g} \times V_{w_2} u_2 = \frac{\rho \times g \times Q}{g} \times V_{w_2} \times u_2$$

$$= \frac{1000 \times 9.81 \times 0.1963}{9.81} \times V_{w_2} \times 26.18$$

But from outlet velocity triangle, we have

$$\tan\phi = \frac{V_{f_2}}{u_2 - V_{w_2}} = \frac{2.5}{(26.18 - V_{w_2})}$$

$$\therefore \quad 26.18 - V_{w_2} = \frac{2.5}{\tan\phi} = \frac{2.5}{\tan 40°} = 2.979$$

$$\therefore \quad V_{w_2} = 26.18 - 2.979 = 23.2 \text{ m/s.}$$

Substituting this value of V_{w_2} in equation (*i*), we get the work done by impeller as

$$= \frac{1000 \times 9.81 \times 0.1963}{9.81} \times 23.2 \times 26.18$$

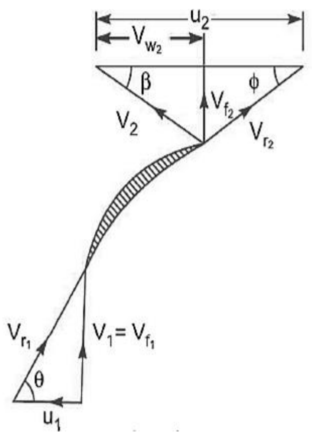
= 119227.9 Nm/s. Ans.

(*iii*) **Manometric efficiency (η_{man}).** Using equation (19.8), we have

$$\eta_{man} = \frac{gH_m}{V_{w_2} u_2} = \frac{9.81 \times 40}{23.2 \times 26.18} = 0.646 = \mathbf{64.4\%}.$$

Example *The outer diameter of an impeller of a centrifugal pump is 400 mm and outlet width is 50 mm. the pump is running at 800 rpm and is working against a total head of 15 m. the vanes angle at outlet is 40° and manometric efficiency is 75%. Determine: i) Velocity of flow at outlet, ii) velocity of water leaving the vane, iii) angle made by the absolute velocity at outlet with the direction of motion at outlet and iv) discharge*

Given:



Outer diameter, $D_2 = 400$ mm$= 0.4$ m

Width at outlet, $B_2 = 50$ mm $= 0.05$ m

Speed, $N = 800$ r.p.m.

Head, $H_m = 15$ m

Vane angle at outlet, $\phi = 40°$

Manometric efficiency, $\eta_{man} = 75\% = 0.75$

Tangential velocity of impeller at outlet,

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.4 \times 800}{60} = 16.75 \text{ m/s.}$$

$$\eta_{man} = \frac{gH_m}{V_{w_2} u_2}$$

$$0.75 = \frac{9.81 \times 15}{V_{w_2} \times 16.75}$$

$$V_{w_2} = \frac{9.81 \times 15}{0.75 \times 16.75} = 11.71 \text{ m/s.}$$

From the outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}} = \frac{V_{f_2}}{(16.75 - 11.71)} = \frac{V_{f_2}}{5.04}$$

(*i*) ∴ $V_{f_2} = 5.04 \tan \phi = 5.04 \times \tan 40° = \mathbf{4.23\ m/s.}$

(*ii*) **Velocity of water leaving the vane (V_2).**

$$V_2 = \sqrt{V_{f_2}^2 + V_{w_2}^2} = \sqrt{4.23^2 + 11.71^2}$$

$$= \sqrt{17.89 + 137.12} = \mathbf{12.45\ m/s.}$$

(*iii*) **Angle made by absolute velocity at outlet** (β),

$$\tan \beta = \frac{V_{f_2}}{V_{w_2}} = \frac{4.23}{11.71} = 0.36$$

∴ $\beta = \tan^{-1} 0.36 = 19.80°$ or $\mathbf{19° \ 48'}$.

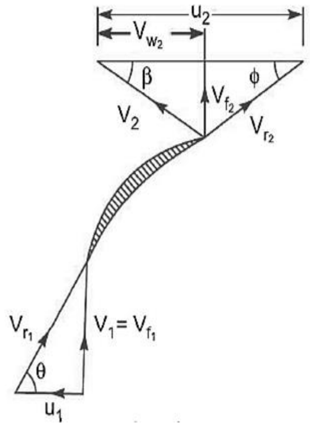
(*iv*) **Discharge** through pump is given by,

$$Q = \pi D_2 B_2 \times V_{f_2} = \pi \times 0.4 \times 0.05 \times 4.23 = \mathbf{0.265\ m^3/s.}$$

Example *The internal diameter and external diameter of an impeller of a centrifugal pump which is running at 1000 rpm are 200 and 40 mm respectively. The discharge through pump is 0.04 m3/s and velocity of flow is constant and equal to 2.0 m/s. the diameter of the suction and delivery pipes are 150 and 100 mm respectively and suction and delivery heads are 6 m (abs.) and 30 m (abs.) of water respectively. If the outlet vane angle is 45° and power required to drive the pump is 16.168 kW, determine: i) Vane angle of the impeller at inlet, ii) the overall efficiency of the pump and iii) manometric efficiency of the pump*

Given:

Speed,	$N = 1000$ r.p.m.
Internal dia.,	$D_1 = 200$ mm $= 0.2$ m
External dia.,	$D_2 = 400$ mm $= 0.4$ m
Discharge,	$Q = 0.04$ m^3/s
Velocity of flow,	$V_{f_1} = V_{f_2} = 2.0$ m/s
Dia. of suction pipe,	$D_s = 150$ mm $= 0.15$ m
Dia. of delivery pipe,	$D_d = 100$ mm $= 0.10$ m
Suction head,	$h_s = 6$ m (abs.)
Delivery head,	$h_d = 30$ m (abs.)
Outlet vane angle,	$\phi = 45°$
Power required to drive the pump,	$P = 16.186/$ kW



From inlet velocity, we have $\tan\theta = \dfrac{V_{f_1}}{u_1} = \dfrac{2.0}{u_1}$, where $u_1 = \dfrac{\pi D_1 N}{60} = \dfrac{\pi \times 0.2 \times 1000}{60} = 10.47$ m/s

$$\therefore \qquad \tan\theta = \frac{2.0}{10.47} = 0.191 \text{ or } \theta = \tan^{-1} .191 = \mathbf{10° \; 48'. \; Ans.}$$

(*ii*) **Overall efficiency of the pump (η_o).**

Using equation (19.10), we have $\eta_o = \dfrac{\left(\dfrac{WH_m}{1000}\right)}{\text{S.P.}}$

where S.P. = Power required to drive the pump and equal to P here.

$$\eta_o = \frac{\left(\dfrac{\rho \times g \times Q \times H_m}{1000}\right)}{P} = \frac{\rho g \times Q \times H_m}{1000 \times P}$$

$$= \frac{1000 \times 9.81 \times .04 \times H_m}{1000 \times 16.186} = 0.02424\, H_m \qquad \ldots(i)$$

Now H_m is given by equation (19.6) as

$$H_m = \left(\frac{p_o}{\rho g} + \frac{V_o^2}{2g} + Z_o\right) - \left(\frac{p_i}{\rho g} + \frac{V_i^2}{2g} + Z_i\right) \qquad \ldots(ii)$$

$$H_m = \left(30 + \frac{V_d^2}{2g}\right) - \left(6 + \frac{V_s^2}{2g}\right) \qquad \ldots(iii)$$

$$V_d = \frac{\text{Discharge}}{\text{Area of delivery pipe}} = \frac{0.04}{\frac{\pi}{4}(D_d)^2} = \frac{.04}{\frac{\pi}{4} \times .1^2} = 5.09 \text{ m/s}$$

$$V_s = \frac{.04}{\text{Area of suction pipe}} = \frac{.04}{\frac{\pi}{4} D_s^2} = \frac{.04}{\frac{\pi}{4} \times .15^2} = 2.26 \text{ m/s.}$$

$$H_m = \left(30 + \frac{5.09^2}{2 \times 9.81}\right) - \left(6 + \frac{2.26^2}{2 \times 9.81}\right)$$

$$= (30 + 1.32) - (6 + .26) = 31.32 - 6.26 = 25.06 \text{ m.}$$

Substituting the value of 'H_m' in equation (*i*), we get

$$\eta_o = .02424 \times 25.06 = 0.6074 = \mathbf{60.74\%.}$$

(*iii*) **Manometric efficiency of the pump (η_{man}).**

Tangential velocity at outlet is given by

$$u_2 = \frac{\pi D_2 \times N}{60} = \frac{\pi \times 0.4 \times 1000}{60} = 20.94 \text{ m/s.}$$

From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}} = \frac{2.0}{20.94 - V_{w_2}}$$

$$\therefore \quad 20.94 - V_{w_2} = \frac{2.0}{\tan \phi} = \frac{2.0}{\tan 45} = 2.0$$

$$\therefore \quad V_{w_2} = 20.94 - 2.0 = 18.94.$$

$$\eta_{man} = \frac{gH_m}{V_{w_2} u_2} = \frac{9.81 \times 25.06}{18.94 \times 20.94} = 0.6198 = \mathbf{61.98\%}$$

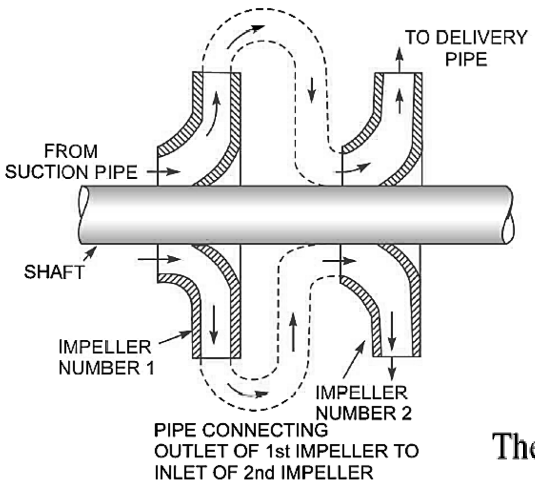
MULTISTAGE CENTRIFUGAL PUMPS

If a centrifugal pump consists of two or more impellers, the pump is called a multistage centrifugal pump. The impellers may be mounted on the same shaft or on different shafts. A multistage pump is having the following two important functions :

1. To produce a high head, and
2. To discharge a large quantity of liquid.

If a high head is to be developed, the impellers are connected in series (or on the same shaft) while for discharging large quantity of liquid, the impellers (or pumps) are connected in parallel.

Multistage Centrifugal Pumps for High Heads.



Then total head developed

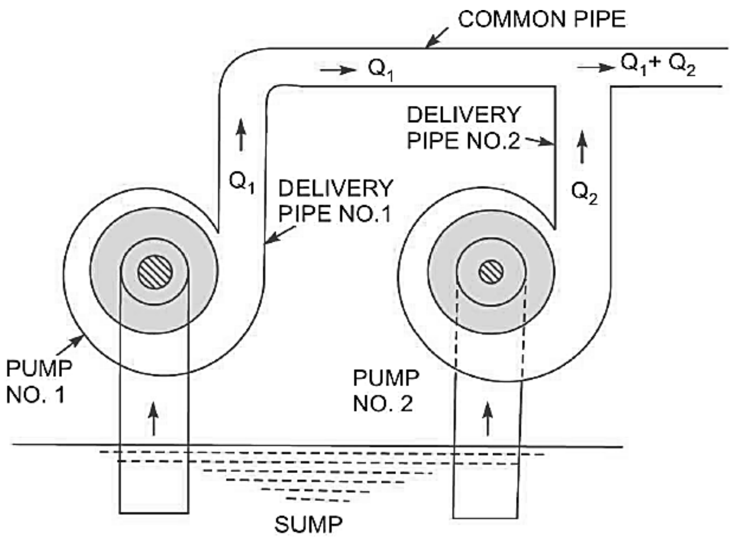
$$= n \times H_m$$

The discharge passing through each impeller is same

n = Number of identical impellers mounted on the same shaft,

H_m = Head developed by each impeller.

Multistage Centrifugal Pumps for High Discharge.



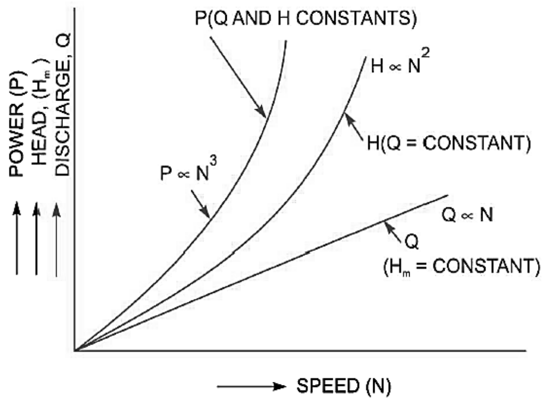
Let n = Number of identical pumps arranged in parallel.

Q = Discharge from one pump.

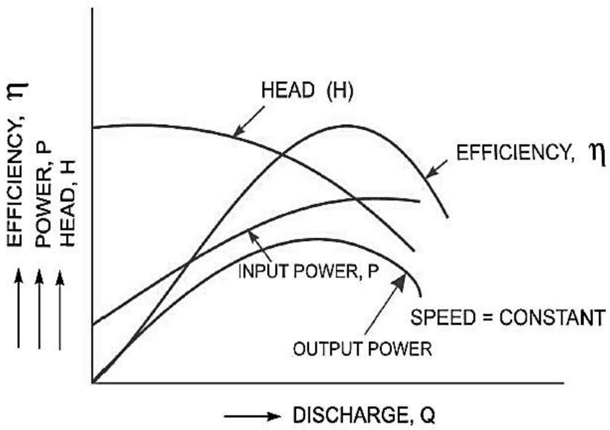
$\therefore$ Total discharge $= n \times Q$

CHARACTERISTIC CURVES OF CENTRIFUGAL PUMPS

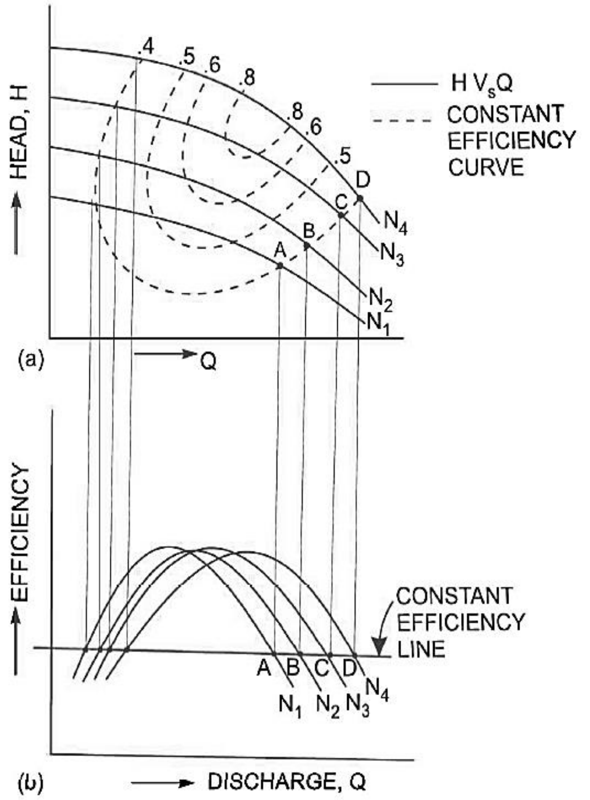
Main Characteristic Curves.



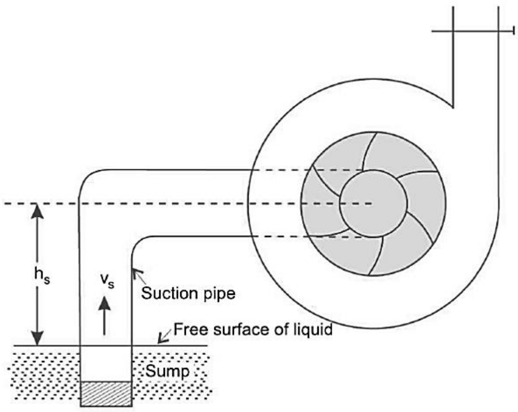
Operating Characteristic Curves.



Constant Efficiency Curves.



MAXIMUM SUCTION LIFT (or SUCTION HEIGHT)



Applying Bernoulli's equation at the free surface of liquid in the sump and section 1 in the suction pipe just at the inlet of the pump and taking the free surface of liquid as datum line, we get

$$\frac{p_a}{\rho g} + \frac{V_a^2}{2g} + Z_a = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 + h_L \quad \text{...}(i)$$

$$\frac{p_a}{\rho g} + 0 + 0 = \frac{p_1}{\rho g} + \frac{v_s^2}{2g} + h_s + h_{f_s}$$

$$\frac{p_a}{\rho g} = \frac{p_1}{\rho g} + \frac{v_s^2}{2g} + h_s + h_{f_s}$$

$$\frac{p_1}{\rho g} = \frac{p_a}{\rho g} - \left(\frac{v_s^2}{2g} + h_s + h_{f_s}\right) \quad \text{...}(ii)$$

For finding the maximum suction lift, the pressure at the inlet of the pump should not be less than the vapour pressure of the liquid. Hence for the limiting case, taking the pressure at the inlet of pump equal to vapour pressure of the liquid, we get

$p_1 = p_v$, where p_v = vapour pressure of the liquid in absolute units.

Now the equation (*ii*) becomes as

$$\frac{p_v}{\rho g} = \frac{p_a}{\rho g} - \left(\frac{v_s^2}{2g} + h_s + h_{f_s}\right)$$

$$\frac{p_a}{\rho g} = \frac{p_v}{\rho g} + \frac{v_s^2}{2g} + h_s + h_{f_s} \qquad (\because \quad p_1 = p_v) \quad ...(iii)$$

$$\frac{p_a}{\rho g} = \text{Atmospheric pressure head } = H_a \text{ (meter of liquid)}$$

$$\frac{p_v}{\rho g} = \text{Vapour pressure head} = H_v \text{ (meter of liquid)}$$

Now, equation (*iii*) becomes as

$$H_a = H_v + \frac{v_s^2}{2g} + h_s + h_{f_s}$$

$$h_s = H_a - H_v - \frac{v_s^2}{2g} - h_{f_s}$$

Equation (19.31) gives the value of maximum suction lift (or maximum suction height) for a centrifugal pump. Hence, the suction height of any pump should not be more than that given by equation (19.31). If the suction height of the pump is more, then vaporization of liquid at inlet of pump will take place and there will be a possibility of cavitation.

NET POSITIVE SUCTION HEAD (NPSH)

The term NPSH (Net Positive Suction Head) is very commonly used in the pump industry. Actually the minimum suction conditions are more frequently specified in terms of NPSH.

The net positive suction head (NPSH) is defined as the *absolute* pressure head at the inlet to the pump, minus the vapour pressure head (in absolute units) plus the velocity head.

$\therefore$ NPSH = Absolute pressure head at inlet of the pump – vapour pressure head (absolute units) + velocity head

$$= \frac{p_1}{\rho g} - \frac{p_v}{\rho g} + \frac{v_s^2}{2g} \qquad (\because \text{ Absolute pressure at inlet of pump} = p_1) .$$

the absolute pressure head at inlet of the pump is given by as

$$\frac{p_1}{\rho g} = \frac{p_a}{\rho g} - \left(\frac{v_s^2}{2g} + h_s + h_{f_s}\right)$$

$$\text{NPSH} = \left[\frac{p_a}{\rho g} - \left(\frac{v_s^2}{2g} + h_s + h_{f_s}\right)\right] - \frac{p_v}{\rho g} + \frac{v_s^2}{2g}$$

$$= \frac{p_a}{\rho g} - \frac{p_v}{\rho g} - h_s - h_{f_s}$$

$$= H_a - H_v - h_s - h_{f_s}$$

RECIPROCATING PUMP

If the mechanical energy is converted into hydraulic energy by sucking the liquid into a cylinder in which a piston is reciprocating, which exerts the thrust on the liquid and increases its hydraulic energy is known as reciprocating pump. A reciprocating pump is a positive displacement pump. It is often used where relatively small quantity of liquid is to be handled and where delivery pressure is quite large.

Reciprocating pump consists of following parts.

1. A cylinder with a piston	5. suction pipe
2. piston rod	6. delivery pipe
3. connecting rod	7. suction valve
4. crank	8. delivery valve

WORKING OF A SINGLE-ACTING RECIPROCATING PUMP

Single acting reciprocating pump:-

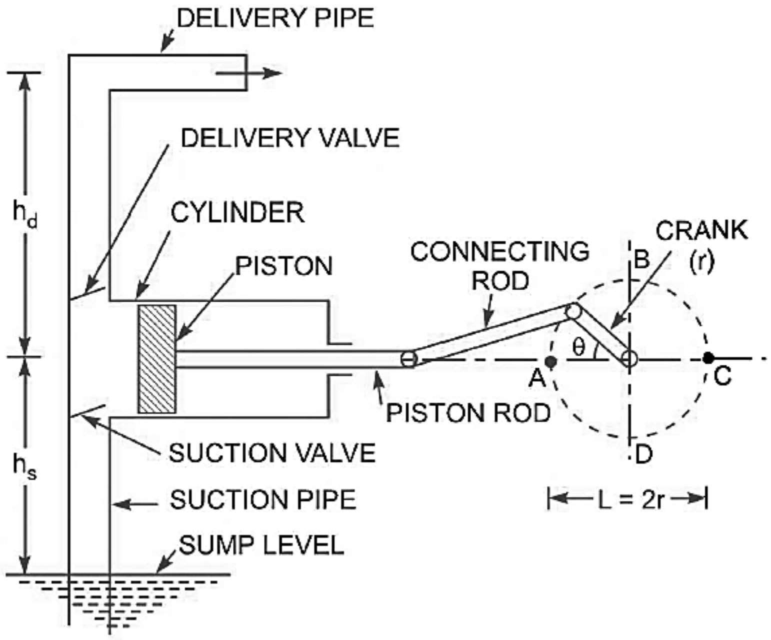
A single acting reciprocating pump, which consists of a piston which moves forwards and backwards in a close fitting cylinder. The movement of the piston is obtained by connecting the piston rod to crank by means of a connecting rod. The crank is rotated by means of an electric motor. Suction and delivery pipes with suction valve and delivery valve are connected to the cylinder. The suction and delivery valves are one way valves or non-return valves, which allow the water to flow in one direction only. Suction valve allows water from suction pipe to the cylinder which delivery valve allows water from cylinder to delivery pipe.

The rotation of the crank brings about an outward and inward movement of the piston in the cylinder. During the suction stroke the piston is moving towards right in the cylinder, this movement of piston causes vacuum in the cylinder. The pressure of the atmosphere acting on the sump water surface forces the water up in the suction pipe. The forced water opens the suction valve and the water enters the cylinder. The piston from its extreme right position starts moving towards left in the cylinder. The movement of the piston towards left increases the pressure of the liquid inside the cylinder more than atmospheric pressure. Hence suction valve closes and delivery valve opens. The liquid is forced into the delivery pipe and is raised to a required height.

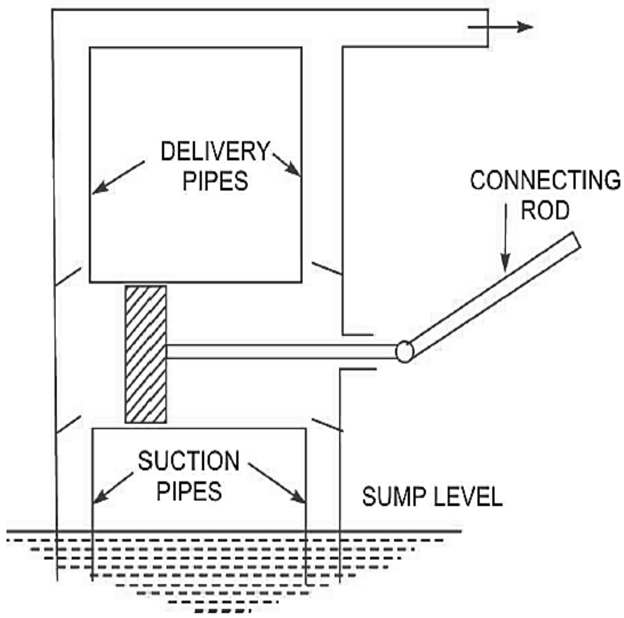
For one revolution of the crank, the quantity of water raised up in the delivery pipe is equal to the stroke volume in the cylinder in the single acting pump and twice this volume in the double acting pump. Discharge through a single acting reciprocating pump.

D = diameter of the cylinder
A = cross section are of the piston or cylinder
r = radius of crank
N = r.p.m of the crank
L = Length of the stroke = 2 x r
h_s = Suction head or height of axis of the cylinder from water surface in sump.
h_d = Delivery head or height of the delivery outlet above the cylinder axis.
Discharge of water in one revolution = Area x Length of stroke
= A x L
Number of revolution per second = N/60
Discharge of the pump per second
Q = Discharge in one revolution x No.of revolution per second

$$= A \times L \times \frac{N}{60} = \frac{ALN}{60} \text{ m}^3/\text{sec}$$



Double acting reciprocating pump



Discharge Through a Reciprocating Pump.

Let D = Diameter of the cylinder

A = Cross-sectional area of the piston or cylinder

$$= \frac{\pi}{4} D^2$$

r = Radius of crank

N = r.p.m. of the crank

L = Length of the stroke = $2 \times r$

h_s = Height of the axis of the cylinder from water surface in sump.

h_d = Height of delivery outlet above the cylinder axis (also called delivery head)

Volume of water delivered in one revolution or discharge of water in one revolution

$$= \text{Area} \times \text{Length of stroke} = A \times L$$

Number of revolution per second, $= \dfrac{N}{60}$

$\therefore$ Discharge of the pump per second,

$$Q = \text{Discharge in one revolution} \times \text{No. of revolution per second}$$

$$= A \times L \times \frac{N}{60} = \frac{ALN}{60} \qquad ...(20.1)$$

Weight of water delivered per second,

$$W = \rho \times g \times Q = \frac{\rho g ALN}{60}.$$

Work done by Reciprocating Pump.

Work done per second = Weight of water lifted per second × Total height through which water is lifted

$$= W \times (h_s + h_d) \qquad ...(i)$$

where $(h_s + h_d)$ = Total height through which water is lifted.

From equation (20.2), Weight, W, is given by

$$W = \frac{\rho g \times ALN}{60}.$$

Substituting the value of W in equation (i), we get

$$\text{Work done per second} = \frac{\rho g \times ALN}{60} \times (h_s + h_d) \qquad ...(20.3)$$

$\therefore$ Power required to drive the pump, in kW

$$P = \frac{\text{Work done per second}}{1000} = \frac{\rho g \times ALN \times (h_s + h_d)}{60 \times 1000}$$

$$= \frac{\rho g \times ALN \times (h_s + h_d)}{60{,}000} \text{ kW} \qquad ...(20.4)$$

Discharge, Work done and Power Required to Drive a Double-acting Pump.

Let D = Diameter of the piston,

d = Diameter of the piston rod

$\therefore$ Area on one side of the piston,

$$A = \frac{\pi}{4} D^2$$

Area on the other side of the piston, where piston rod is connected to the piston,

$$A_1 = \frac{\pi}{4} D^2 - \frac{\pi}{4} d^2 = \frac{\pi}{4} (D^2 - d^2).$$

$\therefore$ Volume of water delivered in one revolution of crank

$$= A \times \text{Length of stroke} + A_1 \times \text{Length of stroke}$$

$$= AL + A_1L = (A + A_1)L = \left[\frac{\pi}{4}D^2 + \frac{\pi}{4}\left(D^2 - d^2\right)\right] \times L$$

$\therefore$ Discharge of pump per second

= Volume of water delivered in one revolution × No. of revolution per second

$$= \left[\frac{\pi}{4}D^2 + \frac{\pi}{4}\left(D^2 - d^2\right)\right] \times L \times \frac{N}{60}$$

If 'd' the diameter of the piston rod is very small as compared to the diameter of the piston, then it can be neglected and discharge of pump per second,

$$Q = \left(\frac{\pi}{4}D^2 + \frac{\pi}{4}D^2\right) \times \frac{L \times N}{60} = 2 \times \frac{\pi}{4}D^2 \times \frac{L \times N}{60} = \frac{2ALN}{60} \quad ...(20.5)$$

Work done by double-acting reciprocating pump

Work done per second = Weight of water delivered × Total height

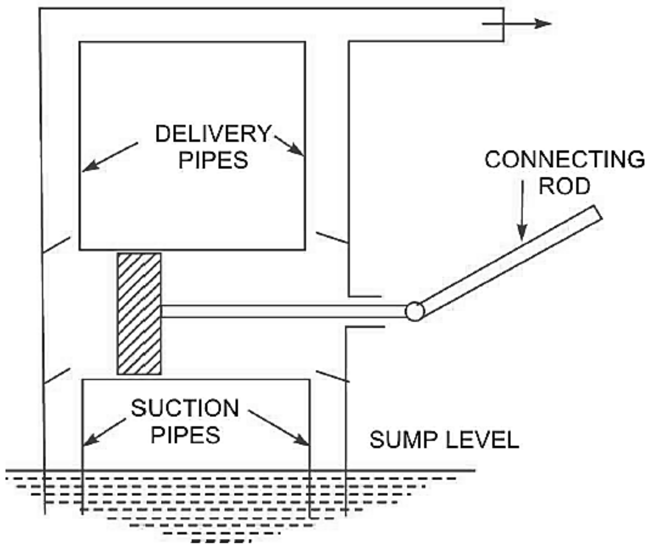
$$= \rho g \times \text{Discharge per second} \times \text{Total height}$$

$$= \rho g \times \frac{2ALN}{60} \times (h_s + h_d) = 2\rho g \times \frac{ALN}{60} \times (h_s + h_d)$$

$\therefore$ Power required to drive the double-acting pump in kW,

$$P = \frac{\text{Work done per second}}{1000} = 2\rho g \times \frac{ALN}{60} \times \frac{\left(h_s + h_d\right)}{1000}$$

$$= \frac{2\rho g \times ALN \times \left(h_s + h_d\right)}{60{,}000}$$



SLIP OF RECIPROCATING PUMP

The actual discharge of the pump is always less than theoretical discharge. The difference between theoretical discharge and actual discharge is known as Slip of the reciprocating pump

$$\text{Slip} = Q_{th} - Q_{act}$$

But slip is mostly expressed as percentage slip which is given by,

$$\text{Percentage slip} = \frac{Q_{th} - Q_{act}}{Q_{th}} \times 100 = \left(1 - \frac{Q_{act}}{Q_{th}}\right) \times 100$$

$$= (1 - C_d) \times 100 \qquad \left(\because \frac{Q_{act}}{Q_{th}} = C_d\right)$$

where C_d = Co-efficient of discharge.

Negative Slip of the Reciprocating Pump.

Negative Slip of the Reciprocating Pump. Slip is equal to the difference of theoretical discharge and actual discharge. If actual discharge is more than the theoretical discharge, the slip of the pump will become –ve. In that case, the slip of the pump is known as negative slip.

Negative slip occurs when delivery pipe is short, suction pipe is long and pump is running at high speed.

Example *A single acting reciprocating pump, running at 50 rpm, delivers 0.01m3/s of water. The diameter of the piston is 200 mm and stroke length 400 m. Determine: i) theoretical discharge of the pump ii) Co – efficient of discharge and iii) Slip and the percentage of slip of the pump.*

Given:

Solution. Given :

Speed of the pump, $N = 50$ r.p.m.

Actual discharge, $Q_{act} = .01 \text{ m}^3/\text{s}$

Dia. of piston, $D = 200 \text{ mm} = .20 \text{ m}$

$\therefore$ Area, $A = \frac{\pi}{4}(.2)^2 = .031416 \text{ m}^2$

Stroke, $L = 400 \text{ mm} = 0.40 \text{ m}$.

(*i*) Theoretical discharge for single-acting reciprocating pump is given by equation (20.1) as

$$Q_{th} = \frac{A \times L \times N}{60} = \frac{.031416 \times .40 \times 50}{60} = \mathbf{0.01047\ m^3/s.\ Ans.}$$

(*ii*) Co-efficient of discharge is given by

$$C_d = \frac{Q_{act}}{Q_{th}} = \frac{0.01}{.01047} = \mathbf{0.955.\ Ans.}$$

(*iii*) Using equation (20.8), we get

$$\text{Slip} = Q_{th} - Q_{act} = .01047 - .01 = \mathbf{0.00047\ m^3/s.\ Ans.}$$

And percentage slip

$$= \frac{(Q_{th} - Q_{act})}{Q_{th}} \times 100 = \frac{(.01047 - .01)}{.01047} \times 100$$

$$= \frac{.00047}{.01047} \times 100 = \mathbf{4.489\%.\ Ans.}$$

Example *A double-acting reciprocating pump, running at 40 r.p.m., is discharging 1.0 m^3 of water per minute. The pump has a stroke of 400 mm. The diameter of the piston is 200 mm. The delivery and suction head are 20 m and 5 m respectively. Find the slip of the pump and power required to drive the pump.*

Speed of pump, $N = 40$ r.p.m.

Actual discharge, $Q_{act} = 1.0\ m^3/min = \frac{1.0}{60}\ m^3/s = 0.01666\ m^3/s$

Stroke, $L = 400$ mm $= 0.40$ m

Diameter of piston, $D = 200$ mm $= 0.20$ m

$\therefore$ Area, $A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (.2)^2 = 0.031416\ m^2$

Suction head, $h_s = 5$ m

Delivery head, $h_d = 20$ m.

Theoretical discharge for double-acting pump is given by equation (20.5) as,

$$Q_{th} = \frac{2ALN}{60} = \frac{2 \times .031416 \times 0.4 \times 40}{60} = .01675\ m^3/s.$$

Using equation (20.8), Slip $= Q_{th} - Q_{act} = .01675 - .01666 =$ **.00009 m^3/s. Ans.**

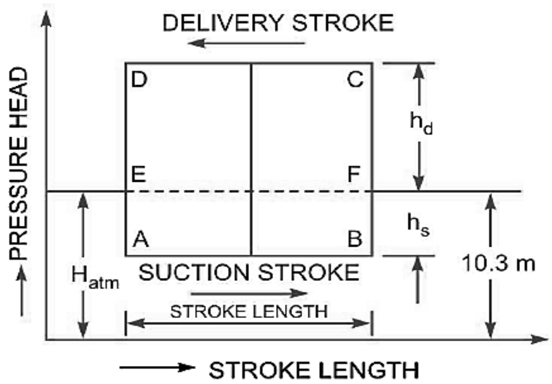
Power required to drive the double-acting pump is given by equation (20.7) as,

$$P = \frac{2 \times \rho g \times ALN \times (h_s + h_d)}{60{,}000} = \frac{2 \times 1000 \times 9.81 \times .031416 \times .4 \times 40 \times (5 + 20)}{60{,}000}$$

$= $ **4.109 kW. Ans.**

INDICATOR DIAGRAM

indicator diagram is a graph between pressure head and stroke length of the piston for one complete revolution. The pressure head is taken as ordinate and stroke length as abscissa.



Let H_{atm} = Atmospheric pressure head = 10.3 m of water,

L = Length of the stroke,

h_s = Suction head, and

h_d = Delivery head.

Fig. *Ideal indicator diagram.*

we know that the work done by the pump per second

$$= \frac{\rho \times g \times ALN}{60} \times (h_s + h_d)$$

$$= K \times L(h_s + h_d) \qquad \left(\text{where } K = \frac{\rho g AN}{60} = \text{Constant}\right)$$

$$\propto L \times (h_s + h_d) \qquad ...(i)$$

Work done by pump $\propto$ Area of indicator diagram.

SEPARATION OF LIQUID

If the pressure in the cylinder is below the vapour pressure, dissolved gasses will be liberated from the liquid and cavitation will takes place. The continuous flow of liquid will not exist which means separation of liquid takes place. The pressure at which separation takes place is called separation pressure and head corresponding to the separation pressure is called separation pressure head.

The ways to avoid cavitation in reciprocating pumps:

1. **Design:** Ensure that there are no sharp corners or curvatures of flow in the system while designing the pump.
2. **Material:** Cavitation resistant materials like Bronze or Nickel can be used.
3. **Model Testing:** Before manufacturing, a scaled down model should be tested.
4. **Admission of air:** High pressure air can be injected into the low pressure zones of flowing liquid to prevent bubble formation.

AIR VESSELS

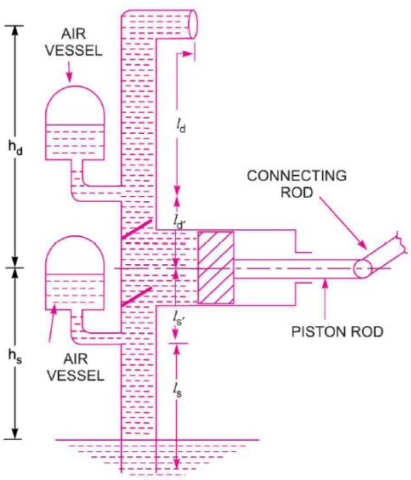
An air vessel is a closed chamber containing compressed air in the top portion and liquid (or water) at the bottom of the chamber. At the base of the chamber there is an opening through which the liquid (or water) may flow into the vessel or out from the vessel. When the liquid enters the air vessel, the air gets compressed further and when the liquid flows out the vessel, the air will expand in the chamber.

An air vessel is fitted to the suction pipe and to the delivery pipe at a point close to the cylinder of a single-acting reciprocating pump :

(*i*) to obtain a continuous supply of liquid at a uniform rate,

(*ii*) to save a considerable amount of work in overcoming **friction in suction and delivery pipes**

(*iii*) to run the pump at a high speed without separation.



COMPARISON BETWEEN CENTRIFUGAL PUMPS AND RECIPROCATING PUMPS

Centrifugal pumps	*Reciprocating pumps*
1. The discharge is continuous and smooth.	1. The discharge is fluctuating and pulsating.
2. It can handle large quantity of liquid.	2. It handles small quantity of liquid only.
3. It can be used for lifting highly viscous liquids.	3. It is used only for lifting pure water or less viscous liquids.
4. It is used for large discharge through smaller heads.	4. It is meant for small discharge and high heads.
5. Cost of centrifugal pump is less as compared to reciprocating pump.	5. Cost of reciprocating pump is approximately four times the cost of centrifugal pump.
6. Centrifugal pump runs at high speed. They can be coupled to electric motor.	6. Reciprocating pump runs at low speed. Speed is limited due to consideration of separation and cavitation.
7. The operation of centrifugal pump is smooth and without much noise. The maintenance cost is low.	7. The operation of reciprocating pump is complicated and with much noise. The maintenance cost is high.
8. Centrifugal pump needs smaller floor area and installation cost is low.	8. Reciprocating pump requires large floor area and installation cost is high.
9. Efficiency is high.	9. Efficiency is low.

TURBINES

Hydraulic machines are defined as those machines which convert either hydraulic energy (energy possessed by water) into mechanical energy (which is further converted into electrical energy) or mechanical energy into hydraulic energy. The hydraulic machines, which convert the hydraulic energy into mechanical energy, are called turbines while the hydraulic machines which convert the mechanical energy into hydraulic energy. The study of hydraulic machines consists of turbines and pumps.

Turbines are defined as the hydraulic machines which convert hydraulic energy into mechanical energy. This, mechanical energy is used in running an electric generator which is directly coupled to the shaft of the turbine. Thus the mechanical energy is converted into electrical energy. The electric power which is obtained from the hydraulic energy (energy of water) is known as Hydroelectric power. At present the generation of hydroelectric power is the cheapest as compared by the power generated by other sources such as oil, coal etc.

General Layout of a Hydroelectric Power Plant

1. A dam constructed across a river to store water.
2. Pipes of large diameters called penstocks, which carry water under pressure from the storage reservoir to the turbines. These pipes are made of steel or reinforced concrete.
3. Turbines having different types of vanes fitted to the wheels.
4. Tail race, which is a channel which carries water away from the turbines after the water has worked on the turbines. The surface of water in the tail race channel is also known as tail race.

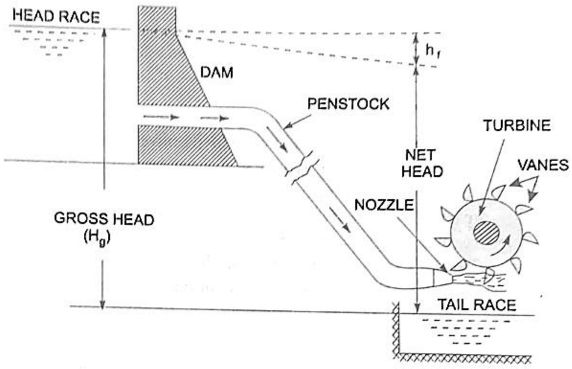


Fig. Layout of hydroelectric power plant

Definitions of Heads and Efficiencies of a Turbine

1. Gross Head. The difference between the head race level and tail race level when no water is flowing is known as Gross Head. It is denoted by 'H_g".
2. Net Head. It is also called effective head and is defined as the head available at the inlet of the turbine, when water is flowing from head race to the turbine, a loss of head due to friction between water and penstock occurs. Though there are other losses also such as loss due to bend, Pipes, fittings, loss at the entrance of penstock etc., yet they are having small magnitude as compared to head loss due to friction. In 'h_f' is the head loss due to friction between penstocks and water then net heat on turbine is given by

$$H = H_g - h_f$$

where H_g = Gross head, $h_f = \dfrac{4 \times f \times L \times V^2}{D \times 2g}$,

in which V = Velocity of flow in penstock,
L = Length of penstock,
D = Diameter of penstock.

Efficiencies of a Turbine.

(*a*) **Hydraulic Efficiency (η_h).**

$$\eta_h = \frac{\text{Power delivered to runner}}{\text{Power supplied at inlet}} = \frac{\text{R.P.}}{\text{W.P.}}$$

Power supplied at the inlet of turbine in S.I.units is known as water power. It is given by

$$\text{W.P.} = \frac{\rho \times g \times Q \times H}{1000} \text{ kW}$$

R.P. = Power delivered to runner *i.e.*, runner power

$$= \frac{W}{g}\frac{\left[V_{w_1} \pm V_{w_2}\right] \times u}{1000} \text{ kW} \qquad \text{...for Pelton Turbine}$$

$$= \frac{W}{g}\frac{\left[V_{w_1} u_1 \pm V_{w_2} u_2\right]}{1000} \text{ kW} \qquad \text{...for a radial flow turbine}$$

(*b*) **Mechanical Efficiency (η_m).**

$$\eta_m = \frac{\text{Power at the shaft of the turbine}}{\text{Power delivered by water to the runner}} = \frac{\text{S.P.}}{\text{R.P.}}$$

(*c*) **Volumetric Efficiency (η_v).**

$$\eta_v = \frac{\text{Volume of water actually striking the runner}}{\text{Volume of water supplied to the turbine}}$$

(*d*) **Overall Efficiency (η_o)**

$$\eta_o = \frac{\text{Volume available at the shaft of the turbine}}{\text{Power supplied at the inlet of the turbine}} = \frac{\text{Shaft power}}{\text{Water power}}$$

$$= \frac{\text{S.P.}}{\text{W.P.}}$$

$$= \eta_m \times \eta_h$$

CLASSIFICATION OF HYDRAULIC TURBINES

1. According to the type of energy at inlet :
   (*a*) Impulse turbine, and (*b*) Reaction turbine.
2. According to the direction of flow through runner :
   (*a*) Tangential flow turbine, (*b*) Radial flow turbine,
   (*c*) Axial flow turbine, and (*d*) Mixed flow turbine.
3. According to the head at the inlet of turbine :
   (*a*) High head turbine, (*b*) Medium head turbine, and
   (*c*) Low head turbine.
4. According to the specific speed of the turbine :
   (*a*) Low specific speed turbine, (*b*) Medium specific speed turbine, and
   (*c*) High specific speed turbine.

Impulse Turbine	Reaction Turbine
1. All the available energy of the fluid is converted into kinetic energy by an efficient nozzle that forms a free jet.	1. Only a portion of the fluid energy is transformed into kinetic energy before the fluid enters the turbine runner.
2. The jet is unconfined and at atmospheric pressure throughout the action of water on the runner, and during its subsequent flow to the tail race.	2. Water enters the runner with an excess pressure, and then both the velocity and pressure change as water passes through the runner.
3. Blades are only in action when they are in front of the nozzle.	3. Blades are in action all the time.
4. Water may be allowed to enter a part or whole of the wheel circumference.	4. Water is admitted over the circumference of the wheel.
5. The wheel does not run full and air has free access to the buckets.	5. Water completely fills the vane passages throughout the operation of the turbine.
6. Casing has no hydraulic function to perform; it only serves to prevent splashing and to guide the water to the tail race.	6. Pressure at inlet to the turbine is much higher than the pressure at outlet ; unit has to be sealed from atmospheric conditions and, therefore, casing is absolutely essential.
7. Unit is installed above the tail race.	7. Unit is kept entirely submerged in water below the tail race.
8. Flow regulation is possible without loss.	8. Flow regulation is always accompanied by loss.
9. When water glides over the moving blades, its relative velocity either remains constant or reduces slightly due to friction.	9. Since there is continuous drop in pressure during flow through the blade passages, the relative velocity does increase.

PELTON WHEEL (OR TURBINE)

The Pelton wheel or Pelton turbine is a tangential flow impulse turbine. The water strikes the bucket along the tangent of the runner. The energy available at the inlet of the turbine is only kinetic energy. The pressure at the inlet and outlet of the turbine is atmospheric. This turbine is used for high heads and is named after L.A. Pelton, an American Engineer.

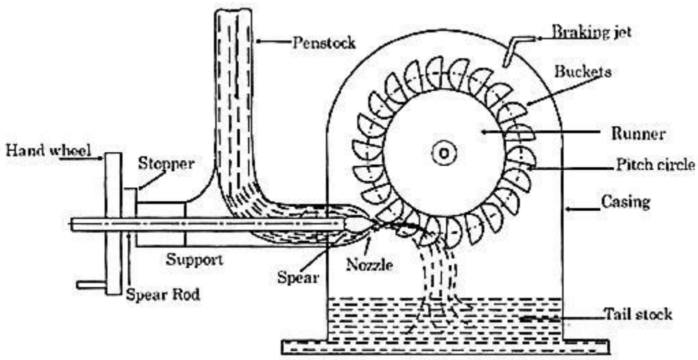
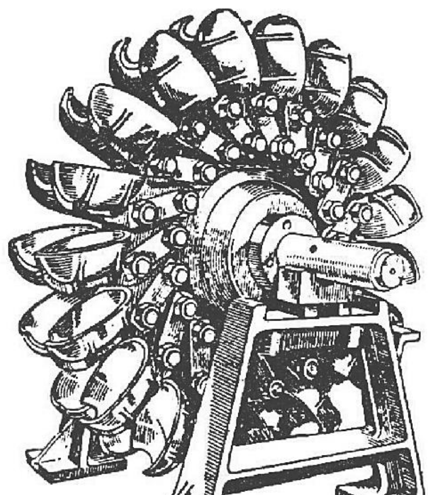


Fig. : Pelton Turbine



Main parts of Pelton Wheel

1. Nozzle and Flow Regulating Arrangement. The amount of water striking the buckets (vanes) of the runner is controlled by providing a spear in the nozzle as shown in Fig. 18.2. The spear is a conical needle which is operated either by a hand wheel or automatically in an axial direction depending upon the size of the unit. When the spear is pushed forward into the nozzle the amount of water striking the runner is reduced. On the other hand, if the spear is pushed back, the amount of water striking the runner increases.

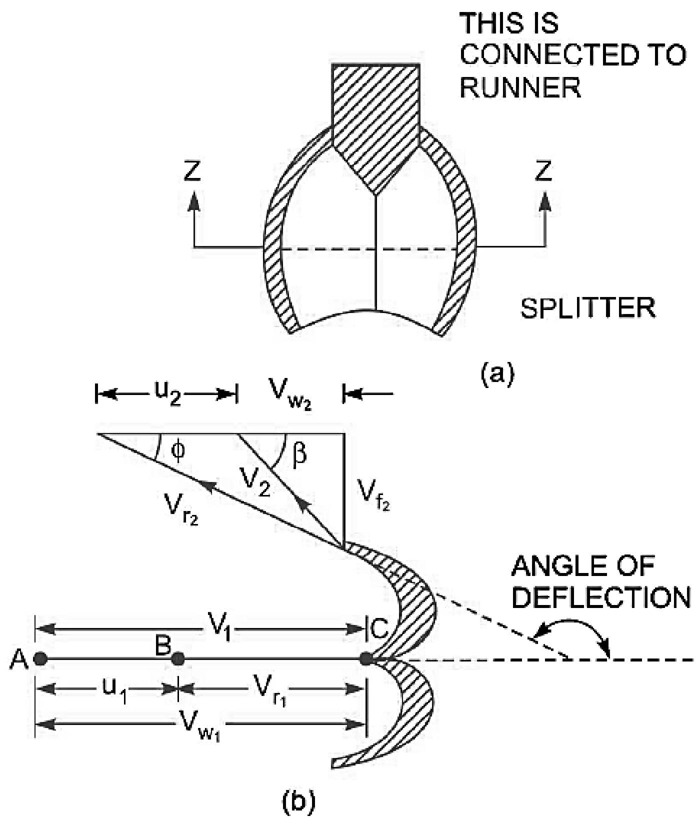
2. Runner with Buckets. Fig. 18.3 shows the runner of a Pelton wheel. It consists of a circular disc on the periphery of which a number of buckets evenly spaced are fixed. The shape of the buckets is of a double hemispherical cup or bowl. Each bucket is divided into two symmetrical parts by a dividing wall which is known as splitter.

The jet of water strikes on the splitter. The splitter divides the jet into two equal parts and the jet comes out at the outer edge of the bucket. The buckets are shaped in such a way that the jet gets deflected through 160° or 170°. The buckets are made of cast iron, cast steel bronze or stainless steel depending upon the head at the inlet of the turbine.

3. Casing. Fig. 18.4 shows a Pelton turbine with a casing. The function of the casing is to prevent the splashing of the water and to discharge water to tail race. It also acts as safeguard against accidents. It is made of cast iron or fabricated steel plates. The casing of the Pelton wheel does not perform any hydraulic function.

4. Breaking Jet.

Velocity Triangles and Work done for Pelton Wheel.



Let H = Net head acting on the Pelton wheel

$$= H_g - h_f$$

where H_g = Gross head and $h_f = \dfrac{4fLV^2}{D^* \times 2g}$

where D^* = Dia. of Penstock, N = Speed of the wheel in r.p.m.,
D = Diameter of the wheel, d = Diameter of the jet.

Then V_1 = Velocity of jet at inlet = $\sqrt{2gH}$

$$u = u_1 = u_2 = \frac{\pi DN}{60}.$$

The velocity triangle at inlet will be a straight line where

$$V_{r_1} = V_1 - u_1 = V_1 - u$$

$$V_{w_1} = V_1$$

$$\alpha = 0^\circ \text{ and } \theta = 0^\circ$$

From the velocity triangle at outlet, we have

$$V_{r_2} = V_{r_1} \text{ and } V_{w_2} = V_{r_2} \cos\phi - u_2.$$

The force exerted by the jet of water in the direction of motion is given by equation

$$F_x = \rho a V_1 \, [V_{w_1} + V_{w_2}]$$

As the angle β is an acute angle, +ve sign should be taken.

$$a = \text{Area of jet} = \frac{\pi}{4} d^2.$$

Now work done by the jet on the runner per second

$$= F_x \times u = \rho a V_1 \, [V_{w_1} + V_{w_2}] \times u \text{ Nm/s}$$

Power given to the runner by the jet

$$= \frac{\rho a V_1 \left[V_{w_1} + V_{w_2}\right] \times u}{1000} \text{ kW}$$

Work done/s per unit weight of water striking/s

$$= \frac{\rho a V_1 \left[V_{w_1} + V_{w_2}\right] \times u}{\text{Weight of water striking/s}}$$

$$= \frac{\rho a V_1 \left[V_{w_1} + V_{w_2}\right] \times u}{\rho a V_1 \times g} = \frac{1}{g}\left[V_{w_1} + V_{w_2}\right] \times u$$

The energy supplied to the jet at inlet is in the form of kinetic energy and is equal to $\frac{1}{2} m V^2$

$\therefore$ K.E. of jet per second $= \dfrac{1}{2} (\rho a V_1) \times V_1^2$

$\therefore$ **Hydraulic efficiency**, $\eta_h = \dfrac{\text{Work done per second}}{\text{K.E. of jet per second}}$

$$= \frac{\rho a V_1 \left[V_{w_1} + V_{w_2}\right] \times u}{\frac{1}{2}(\rho a V_1) \times V_1^2} = \frac{2\left[V_{w_1} + V_{w_2}\right] \times u}{V_1^2}$$

Now $\quad V_{w_1} = V_1,\ V_{r_1} = V_1 - u_1 = (V_1 - u)$

$\therefore \quad V_{r_2} = (V_1 - u)$

and $\quad V_{w_2} = V_{r_2} \cos \phi - u_2 = V_{r_2} \cos \phi - u = (V_1 - u) \cos \phi - u$

Substituting the values of V_{w_1} and V_{w_2} in equation

$$\eta_h = \frac{2\left[V_1 + (V_1 - u) \cos \phi - u\right] \times u}{V_1^2}$$

$$= \frac{2\left[V_1 - u + (V_1 - u) \cos \phi\right] \times u}{V_1^2} = \frac{2(V_1 - u)\,[1 + \cos \phi]\, u}{V_1^2}.$$

The efficiency will be maximum for a given value of V_1 when

$$\frac{d}{du}(\eta_h) = 0 \quad \text{or} \quad \frac{d}{du}\left[\frac{2u(V_1 - u)(1 + \cos \phi)}{V_1^2}\right] = 0$$

or $\quad \dfrac{(1 + \cos \phi)}{V_1^2} \dfrac{d}{du} (2uV_1 - 2u^2) = 0 \quad$ or $\quad \dfrac{d}{du} [2uV_1 - 2u^2] = 0 \quad \left(\because \dfrac{1 + \cos \phi}{V_1^2} \neq 0\right)$

$$2V_1 - 4u = 0 \quad \text{or} \quad u = \frac{V_1}{2}$$

substituting the value of $u = \dfrac{V_1}{2}$

$$\text{Max. } \eta_h = \frac{2\left(V_1 - \frac{V_1}{2}\right)(1 + \cos \phi) \times \frac{V_1}{2}}{V_1^2}$$

$$= \frac{2 \times \frac{V_1}{2}(1 + \cos \phi)\,\frac{V_1}{2}}{V_1^2} = \frac{(1 + \cos \phi)}{2}.$$

Points to be Remembered for Pelton Wheel

(*i*) The velocity of the jet at inlet is given by $V_1 = C_v\sqrt{2gH}$

where C_v = Co-efficient of velocity = 0.98 or 0.99

H = Net head on turbine

(*ii*) The velocity of wheel (u) is given by $u = \phi\sqrt{2gH}$

where ϕ = Speed ratio. The value of speed ratio varies from 0.43 to 0.48.

(*iii*) The angle of deflection of the jet through buckets is taken at 165° if no angle of deflection is given.

(*iv*) The mean diameter or the pitch diameter D of the Pelton wheel is given by

$$u = \frac{\pi D N}{60} \text{ or } D = \frac{60u}{\pi N}.$$

(*v*) **Jet Ratio.** It is defined as the ratio of the pitch diameter (D) of the Pelton wheel to the diameter of the jet (d). It is denoted by 'm' and is given as

$$m = \frac{D}{d} \text{ (= 12 for most cases)}$$

(*vi*) Number of buckets on a runner is given by

$$Z = 15 + \frac{D}{2d} = 15 + 0.5 \text{ m}$$

where m = Jet ratio

(*vii*) **Number of Jets.** It is obtained by dividing the total rate of flow through the turbine by the rate of flow of water through a single jet.

Example *A Pelton wheel has a mean bucket speed of 10 metres per second with a jet of water flowing at the rate of 700 litres/s under a head of 30 metres. The buckets deflect the jet through an angle of 160°. Calculate the power given by water to the runner and the hydraulic efficiency of the turbine. Assume co-efficient of velocity as 0.98.*

Speed of bucket, $u = u_1 = u_2 = 10$ m/s

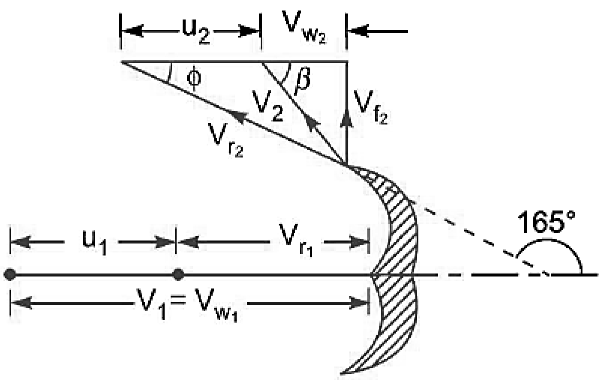
Discharge, $Q = 700$ litres/s $= 0.7$ m^3/s, Head of water, $H = 30$ m

Angle of deflection $= 160°$

$\therefore$ Angle, $\phi = 180° - 160° = 20°$

Co-efficient of velocity, $C_v = 0.98$.

The velocity of jet, $V_1 = C_v\sqrt{2gH} = 0.98\sqrt{2 \times 9.81 \times 30} = 23.77$ m/s



$$\therefore \quad V_{r_1} = V_1 - u_1 = 23.77 - 10$$

$$= 13.77 \text{ m/s}$$

$$V_{w_1} = V_1 = 23.77 \text{ m/s}$$

From outlet velocity triangle,

$$V_{r_2} = V_{r_1} = 13.77 \text{ m/s}$$

$$V_{w_2} = V_{r_2} \cos\phi - u_2$$

$$= 13.77 \cos 20° - 10.0 = 2.94 \text{ m/s}$$

Work done by the jet per second on the runner is given by equation (18.9) as

$$= \rho a V_1 \left[V_{w_1} + V_{w_2}\right] \times u$$

$$= 1000 \times 0.7 \times [23.77 + 2.94] \times 10 \qquad (\because \quad aV_1 = Q = 0.7 \text{ m}^3/\text{s})$$

$$= 186970 \text{ Nm/s}$$

$\therefore$ Power given to turbine $= \dfrac{186970}{1000} =$ **186.97 kW. Ans.**

The hydraulic efficiency of the turbine is given by equation (18.12) as

$$\eta_h = \frac{2\left[V_{w_1} + V_{w_2}\right] \times u}{V_1^2} = \frac{2[23.77 + 2.94] \times 10}{23.77 \times 23.77}$$

$= \mathbf{0.9454}$ or **94.54%. Ans.**

Example *A Pelton wheel is to be designed for the following specifications : Shaft power = 11,772 kW ; Head = 380 metres ; Speed = 750 r.p.m. ; Overall efficiency = 86% ; Jet diameter is not to exceed one-sixth of the wheel diameter. Determine :*

(i) The wheel diameter, *(ii) The number of jets required, and*

(iii) Diameter of the jet.

Take $K_{v_1} = 0.985$ *and* $K_{u_1} = 0.45$

Shaft power, S.P. = 11,772 kW

Head, $H = 380$ m

Speed, $N = 750$ r.p.m.

Overall efficiency, $\eta_0 = 86\%$ or 0.86

Ratio of jet dia. to wheel dia. $= \dfrac{d}{D} = \dfrac{1}{6}$

Co-efficient of velocity, $K_{v_1} = C_v = 0.985$

Speed ratio, $K_{u_1} = 0.45$

The velocity of wheel, $u = u_1 = u_2$

$$= \text{Speed ratio} \times \sqrt{2gH} = 0.45 \times \sqrt{2 \times 9.81 \times 380} = 38.85 \text{ m/s}$$

$$u = \frac{\pi DN}{60} \quad \therefore \quad 38.85 = \frac{\pi DN}{60}$$

$$D = \frac{60 \times 38.85}{\pi \times N} = \frac{60 \times 38.85}{\pi \times 750} = \mathbf{0.989 \text{ m.}}$$

But $\dfrac{d}{D} = \dfrac{1}{6}$

$\therefore$ Dia. of jet, $d = \dfrac{1}{6} \times D = \dfrac{0.989}{6} =$ **0.165 m. Ans.**

Discharge of one jet, q = Area of jet $\times$ Velocity of jet

$$= \frac{\pi}{4} d^2 \times V_1 = \frac{\pi}{4}(.165) \times 85.05 \text{ m}^3/\text{s} = 1.818 \text{ m}^3/\text{s}$$

Now
$$\eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \frac{11772}{\dfrac{\rho g \times Q \times H}{1000}}$$

$$0.86 = \frac{11772 \times 1000}{1000 \times 9.81 \times Q \times 380}, \text{ where } Q = \text{Total discharge}$$

∴ Total discharge,
$$Q = \frac{11772 \times 1000}{1000 \times 9.81 \times 380 \times 0.86} = 3.672 \text{ m}^3/\text{s}$$

∴ Number of jets
$$= \frac{\text{Total discharge}}{\text{Discharge of one jet}} = \frac{Q}{q} = \frac{3.672}{1.818} = \mathbf{2 \text{ jets. Ans.}}$$

Example *The penstock supplies water from a reservoir to the Pelton wheel with a gross head of 500 m. One third of the gross head is lost in friction in the penstock. The rate of flow of water through the nozzle fitted at the end of the penstock is 2.0 m*^{3}*/s. The angle of deflection of the jet is 165°. Determine the power given by the water to the runner and also hydraulic efficiency of the Pelton wheel. Take speed ratio = 0.45 and* $C_v = 1.0$.

Solution. Given :

Gross head, $H_g = 500$ m

Head lost in friction, $h_f = \frac{H_g}{3} = \frac{500}{3} = 166.7$ m

∴ Net head, $H = H_g - h_f = 500 - 166.7 = 333.30$ m

Discharge, $Q = 2.0 \text{ m}^3/\text{s}$

Angle of deflection $= 165°$

∴ Angle, $\phi = 180° - 165° = 15°$

Speed ratio $= 0.45$

Co-efficient of velocity, $C_v = 1.0$

Velocity of jet, $V_1 = C_v\sqrt{2gH} = 1.0 \times \sqrt{2 \times 9.81 \times 333.3} = 80.86$ m/s

Velocity of wheel, $u = \text{Speed ratio} \times \sqrt{2gH}$

or $u = u_1 = u_2 = 0.45 \times \sqrt{2 \times 9.81 \times 333.3} = 36.387$ m/s

∴ $V_{r_1} = V_1 - u_1 = 80.86 - 36.387$

$= 44.473$ m/s

Also $V_{w_1} = V_1 = 80.86$ m/s

From outlet velocity triangle, we have

$$V_{r_2} = V_{r_1} = 44.473$$

$$V_{r_2} \cos \phi = u_2 + V_{w_2}$$

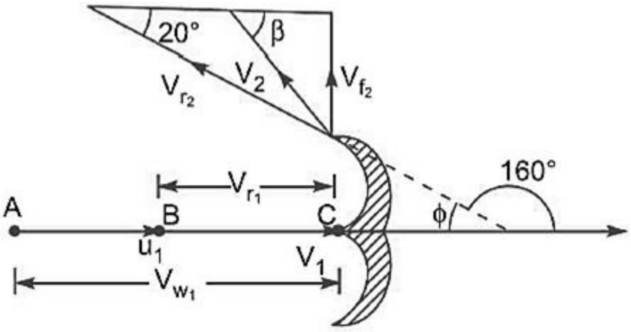
$$44.473 \cos 15° = 36.387 + V_{w_2}$$

or $$V_{w_2} = 44.473 \cos 15° - 36.387 = 6.57 \text{ m/s}.$$

Work done by the jet on the runner per second is given by equation (18.9) as

$$\rho a V_1 [V_{w_1} + V_{w_2}] \times u = \rho Q [V_{w_1} + V_{w_2}] \times u \qquad (\because \quad aV_1 = Q)$$

$$= 1000 \times 2.0 \times [80.86 + 6.57] \times 36.387 = 6362630 \text{ Nm/s}$$



∴ Power given by the water to the runner in kW

$$= \frac{\text{Work done per second}}{1000} = \frac{6362630}{1000} = \mathbf{6362.63 \text{ kW}}$$

Hydraulic efficiency of the turbine is given by equation (18.12) as

$$\eta_h = \frac{2\left[V_{w_1} + V_{w_2}\right] \times u}{V_1^2} = \frac{2[80.86 + 6.57] \times 36.387}{80.86 \times 80.86}$$

$$= \mathbf{0.9731} \text{ or } \mathbf{97.31\%. \text{ Ans.}}$$

Example *A Pelton wheel is to be designed for a head of 60 m when running at 200 r.p.m. The Pelton wheel develops 95.6475 kW shaft power. The velocity of the buckets = 0.45 times the velocity of the jet, overall efficiency = 0.85 and co-efficient of the velocity is equal to 0.98.*

Head, $H = 60$ m

Speed $N = 200$ r.p.m

Shaft power, S.P. = 95.6475 kW

Velocity of bucket, $u = 0.45 \times$ Velocity of jet

Overall efficiency, $\eta_o = 0.85$

Co-efficient of velocity, $C_v = 0.98$

(*i*) *Velocity of jet*, $$V_1 = C_v \times \sqrt{2gH} = 0.98 \times \sqrt{2 \times 9.81 \times 60} = 33.62 \text{ m/s}$$

∴ Bucket velocity, $$u = u_1 = u_2 = 0.45 \times V_1 = 0.45 \times 33.62 = 15.13 \text{ m/s}$$

But $$u = \frac{\pi D N}{60},$$ where D = Diameter of wheel

∴ $$15.13 = \frac{\pi \times D \times 200}{60} \quad \text{or} \quad D = \frac{60 \times 15.13}{\pi \times 200} = \mathbf{1.44 \text{ m. Ans.}}$$

(*ii*) *Diameter of the jet* (d)

Overall efficiency $\eta_o = 0.85$

But $\eta_o = \dfrac{\text{S.P.}}{\text{W.P.}} = \dfrac{95.6475}{\left(\dfrac{\text{W.P.}}{1000}\right)} = \dfrac{95.6475 \times 1000}{\rho \times g \times Q \times H}$ $\quad (\because \quad \text{W.P.} = \rho g Q H)$

$$= \frac{95.6475 \times 1000}{1000 \times 9.81 \times Q \times 60}$$

$$Q = \frac{95.6475 \times 1000}{\eta_o \times 1000 \times 9.81 \times 60} = \frac{95.6475 \times 1000}{0.85 \times 1000 \times 9.81 \times 60} = 0.1912 \text{ m}^3/\text{s}$$

But the discharge, Q = Area of jet × Velocity of jet

$$\therefore \quad 0.1912 = \frac{\pi}{4} d^2 \times V_1 = \frac{\pi}{4} d^2 \times 33.62$$

$$\therefore \quad d = \sqrt{\frac{4 \times 0.1912}{\pi \times 33.62}} = 0.085 \text{ m} = \mathbf{85 \text{ mm.}}$$

(*iii*) *Size of buckets*

Width of buckets $= 5 \times d = 5 \times 85 = 425$ mm

Depth of buckets $= 1.2 \times d = 1.2 \times 85 =$ **102 mm.**

(*iv*) *Number of buckets on the wheel* is given by equation (18.17) as

$$Z = 15 + \frac{D}{2d} = 15 + \frac{1.44}{2 \times .085} = 15 + 8.5 = \mathbf{23.5 \text{ say } 24.}$$

FRANCIS TURBINE

The Francis turbine is a mixed flow reaction turbine. This turbine is used for medium heads with medium discharge. Water enters the runner and flows towards the center of the wheel in the radial direction and leaves parallel to the axis of the turbine.

Turbines are subdivided into impulse and reaction machines. In the impulse turbines, the total head available is converted into the kinetic energy. In the reaction turbines, only some part of the available total head of the fluid is converted into kinetic energy so that the fluid entering the runner has pressure energy as well as kinetic energy. The pressure energy is then converted into kinetic energy in the runner.

The Francis turbine is a type of reaction turbine that was developed by James B. Francis. Francis turbines are the most common water turbine in use today. They operate in a water head from 40 to 600 m and are primarily used for electrical power production. The electric generators which most often use this type of turbine have a power output which generally ranges just a few kilowatts up to 800 MW.

Main components of Francis turbine

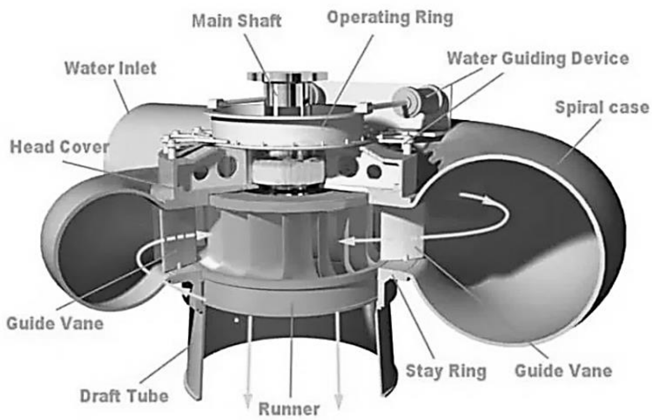
1. Spiral Casing

The water flowing from the reservoir or dam is made to pass through this pipe with high pressure. The blades of the turbines are circularly placed, which means the water striking the blades of the turbine should flow in the circular axis for efficient striking. So, the spiral casing is used, but due to the circular movement of the water, it loses its pressure.

To maintain the same pressure, the diameter of the casing is gradually reduced, to maintain the pressure uniformly, thus uniform momentum or velocity striking the runner blades.

2. Stay Vanes

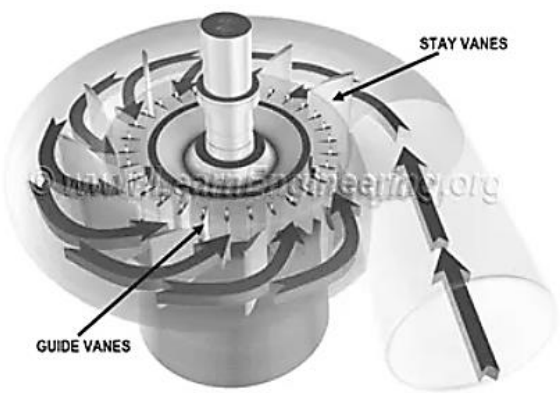
This guides the water to the runner blades. Stay vanes remain stationary at their position and reduces the swirling of water due to radial flow and as it enters the runner blades. Hence, makes the turbine more efficient.



Francis Turbine

3. Guide Vanes

Guide vanes are also known as wicket gates. The main function or usages of the guide vanes are to guide the water towards the runner and it also regulates the quantity of water supplied to runner. It also guides the water to flow at an angle and that is appropriate for the design.



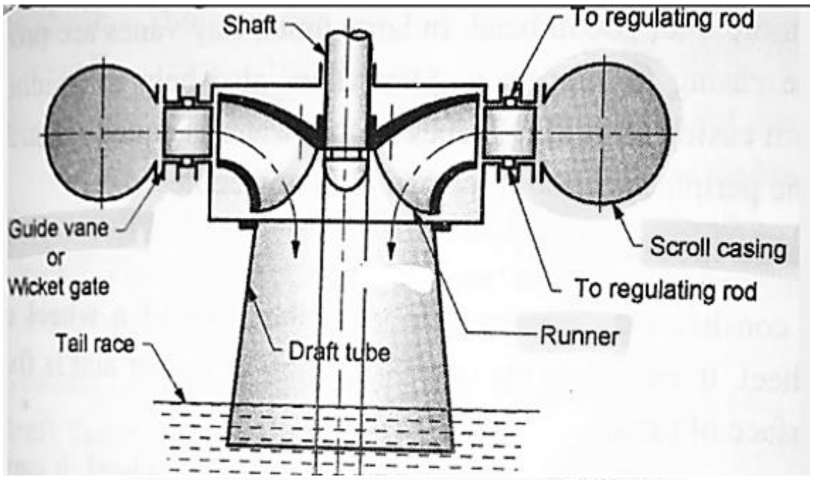
4. Runner Blades:

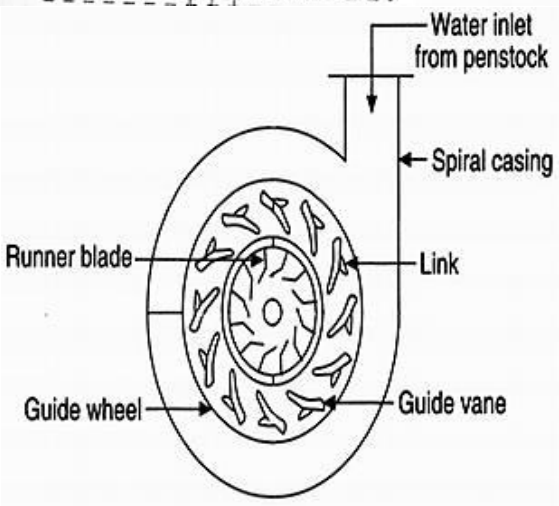
Absorbs the energy from the water and converts it to rotational motion of the main shaft. The runner blades design decides how effectively a turbine is going to perform. The runner blades are divided into two parts. The lower half is made in the shape of a small bucket so that it uses the impulse action of water to rotate the turbine.

The upper part of the blades uses the reaction force of water flowing through it. These two forces together make the runner rotate.

Draft Tube

The draft tube is an expanding tube which is used to discharge the water through the runner and next to the tailrace. The main function of the **draft tube** is to reduce the water velocity at the time of discharge. Its cross-section area increases along its length, as the water coming out of runner blades, is at considerably low pressure, so its expanding cross-section area helps it to recover the pressure as it flows towards the tailrace.





Working principles of Francis turbine

- The water is admitted to the runner through guide vanes or wicket gates. The opening between the vanes can be adjusted to vary the quantity of water admitted to the turbine. This is done to suit the load conditions.
- The water enters the runner with a low velocity but with a considerable pressure. As the water flows over the vanes the pressure head is gradually converted into velocity head.
- This kinetic energy is utilized in rotating the wheel Thus the hydraulic energy is converted into mechanical energy.
- The outgoing water enters the tailrace after passing through the draft tube. The draft tube enlarges gradually and the enlarged end is submerged deeply in the tailrace water.
- Due to this arrangement a suction head is created at the exit of the runner.

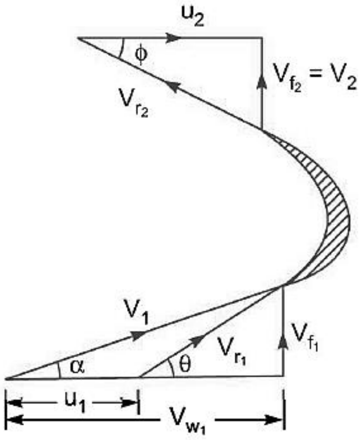
Velocity Triangle

velocity of whirl at outlet (*i.e.*, V_{w_2}) will be zero. Hence the work done by water on the runner per second will be

$$= \rho Q[V_{w_1} u_1]$$

And work done per second per unit weight of water striking/s = $\frac{1}{g}\left[V_{w_1} u_1\right]$

Hydraulic efficiency will be given by, $\eta_h = \frac{V_{w_1} u_1}{gH}$.



Example *A Francis turbine with an overall efficiency of 75% is required to produce 148.25 kW power. It is working under a head of 7.62 m. The peripheral velocity = 0.26* $\sqrt{2gH}$ *and the radial velocity of flow at inlet is 0.96*$\sqrt{2gH}$. *The wheel runs at 150 r.p.m. and the hydraulic losses in the turbine are 22% of the available energy. Assuming radial discharge, determine :*

(i) The guide blade angle, *(ii) The wheel vane angle at inlet,*
(iii) Diameter of the wheel at inlet, and *(iv) Width of the wheel at inlet.*

Overall efficiency $\eta_o = 75\% = 0.75$

Power produced, S.P. = 148.25 kW

Head, $H = 7.62$ m

Peripheral velocity, $u_1 = 0.26\sqrt{2gH} = 0.26 \times \sqrt{2 \times 9.81 \times 7.62} = 3.179$ m/s

Velocity of flow at inlet, $V_{f_1} = 0.96\sqrt{2gH} = 0.96 \times \sqrt{2 \times 9.81 \times 7.62} = 11.738$ m/s.

Speed, $N = 150$ r.p.m.

Hydraulic losses = 22% of available energy

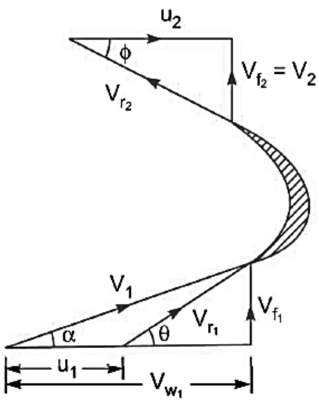
Discharge at outlet = Radial

$$V_{w_2} = 0 \text{ and } V_{f_2} = V_2$$

Hydraulic efficiency is given as

$$\eta_h = \frac{\text{Total head at inlet} - \text{Hydraulic loss}}{\text{Head at inlet}}$$

$$= \frac{H - .22\,H}{H} = \frac{0.78\,H}{H} = 0.78$$



But $$\eta_h = \frac{V_{w_1} u_1}{gH}$$

$\therefore$ $$\frac{V_{w_1} u_1}{gH} = 0.78$$

$\therefore$ $$V_{w_1} = \frac{0.78 \times g \times H}{u_1}$$

$$= \frac{0.78 \times 9.81 \times 7.62}{3.179} = 18.34 \text{ m/s.}$$

(*i*) *The guide blade angle*, *i.e.*, α. From inlet velocity triangle,

$$\tan \alpha = \frac{V_{f_1}}{V_{w_1}} = \frac{11.738}{18.34} = 0.64$$

$\therefore$ $\alpha = \tan^{-1} 0.64 = \mathbf{32.619°}$ or $\mathbf{32° \; 37'}$. **Ans.**

(*ii*) *The wheel vane angle at inlet*, *i.e.*, θ

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{11.738}{18.34 - 3.179} = 0.774$$

$\therefore$ $\theta = \tan^{-1} .774 = \mathbf{37.74}$ or $\mathbf{37° \; 44.4'}$. **Ans.**

(*iii*) *Diameter of wheel at inlet* (D_1).

Using the relation, $$u_1 = \frac{\pi D_1 N}{60}$$

$$D_1 = \frac{60 \times u_1}{\pi \times N} = \frac{60 \times 3.179}{\pi \times 50} = \mathbf{0.4047 \text{ m.}} \textbf{ Ans.}$$

(*iv*) *Width of the wheel at inlet* (B_1)

$$\eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \frac{148.25}{\text{W.P.}}$$

But $$\text{W.P.} = \frac{\text{WH}}{1000} = \frac{\rho \times g \times Q \times H}{1000} = \frac{1000 \times 9.81 \times Q \times 7.62}{1000}$$

$\therefore$ $$\eta_o = \frac{148.25}{\dfrac{1000 \times 9.81 \times Q \times 7.62}{1000}} = \frac{148.25 \times 1000}{1000 \times 9.81 \times Q \times 7.62}$$

$$Q = \frac{148.25 \times 1000}{1000 \times 9.81 \times 7.62 \times \eta_o} = \frac{148.25 \times 1000}{1000 \times 9.81 \times 7.62 \times 0.75} = 2.644 \text{ m}^3/\text{s}$$

$$Q = \pi D_1 \times B_1 \times V_{f_1}$$

$$2.644 = \pi \times .4047 \times B_1 \times 11.738$$

$$B_1 = \frac{2.644}{\pi \times .4047 \times 11.738} = \mathbf{0.177 \text{ m.}}$$

Example *The following data is given for a Francis Turbine. Net head $H = 60$ m ; Speed $N = 700$ r.p.m.; shaft power = 294.3 kW ; $\eta_o = 84\%$; $\eta_h = 93\%$; flow ratio = 0.20 ; breadth ratio $n = 0.1$; Outer diameter of the runner = 2 × inner diameter of runner. The thickness of vanes occupy 5% of circumferential area of the runner, velocity of flow is constant at inlet and outlet and discharge is radial at outlet. Determine :*

(i) Guide blade angle,
(ii) Runner vane angles at inlet and outlet,
(iii) Diameters of runner at inlet and outlet, and
(iv) Width of wheel at inlet.

Net head, $H = 60$ m

Speed, $N = 700$ r.p.m.

Shaft power $= 294.3$ kW

Overall efficiency, $\eta_o = 84\% = 0.84$

Hydraulic efficiency, $\eta_h = 93\% = 0.93$

Flow ratio, $\dfrac{V_{f_1}}{\sqrt{2gH}} = 0.20$

$$\therefore \quad V_{f_1} = 0.20 \times \sqrt{2gH}$$
$$= 0.20 \times \sqrt{2 \times 9.81 \times 60} = 6.862 \text{ m/s}$$

Breadth ratio, $\dfrac{B_1}{D_1} = 0.1$

Outer diameter, $D_1 = 2 \times$ Inner diameter $= 2 \times D_2$

Velocity of flow, $V_{f_1} = V_{f_2} = 6.862$ m/s.

Thickness of vanes = 5% of circumferential area of runner

∴ Actual area of flow $= 0.95\, \pi D_1 \times B_1$

Discharge at outlet = Radial

$$\therefore \quad V_{w_2} = 0 \text{ and } V_{f_2} = V_2$$

Using relation, $\eta_o = \dfrac{\text{S.P.}}{\text{W.P.}}$

$$0.84 = \frac{294.3}{\text{W.P.}}$$

$$\therefore \quad \text{W.P.} = \frac{294.3}{0.84} = 350.357 \text{ kW}.$$

But $$\text{W.P.} = \frac{WH}{1000} = \frac{\rho \times g \times Q \times H}{1000} = \frac{1000 \times 9.81 \times Q \times 60}{1000}$$

$$\therefore \quad \frac{1000 \times 9.81 \times Q \times 60}{1000} = 350.357$$

$$\therefore \quad Q = \frac{350.357 \times 1000}{60 \times 1000 \times 9.81} = 0.5952 \text{ m}^3\text{/s}.$$

Using equation (18.21), Q = Actual area of flow × Velocity of flow

$$= 0.95\, \pi D_1 \times B_1 \times V_{f_1}$$
$$= 0.95 \times \pi \times D_1 \times 0.1 D_1 \times V_{f_2} \qquad (\because \quad B_1 = 0.1\, D_1)$$

or $$0.5952 = 0.95 \times \pi \times D_1 \times 0.1 \times D_1 \times 6.862 = 2.048\ D_1^2$$

$\therefore$ $$D_1 = \sqrt{\frac{0.5952}{2.048}} = 0.54 \text{ m}$$

But $$\frac{B_1}{D_1} = 0.1$$

$\therefore$ $$B_1 = 0.1 \times D_1 = 0.1 \times .54 = .054 \text{ m} = 54 \text{ mm}$$

Tangential speed of the runner at inlet,

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.54 \times 700}{60} = 19.79 \text{ m/s.}$$

Using relation for hydraulic efficiency,

$$\eta_h = \frac{V_{w_1} u_1}{gH} \text{ or } 0.93 = \frac{V_{w_1} \times 19.79}{9.81 \times 60}$$

$\therefore$ $$V_{w_1} = \frac{0.93 \times 9.81 \times 60}{19.79} = 27.66 \text{ m/s.}$$

(*i*) *Guide blade angle* (α)

From inlet velocity triangle, $\tan \alpha = \dfrac{V_{f_1}}{V_{w_1}} = \dfrac{6.862}{27.66} = 0.248$

$\therefore$ $\alpha = \tan^{-1} 0.248 = 13.928°$ or **13° 55.7′. Ans.**

(*ii*) *Runner vane angles at inlet and outlet (θ and ϕ)*

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{6.862}{27.66 - 19.79} = 0.872$$

$\therefore$ $\theta = \tan^{-1} 0.872 = 41.09°$ or **41° 5.4′. Ans.**

From outlet velocity triangle, $\tan \phi = \dfrac{V_{f_2}}{u_2} = \dfrac{V_{f_1}}{u_2} = \dfrac{6.862}{u_2}$ ...(*i*)

But $$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times D_1}{2} \times \frac{N}{60} \qquad \left(\because\ D_2 = \frac{D_1}{2} \text{ given}\right)$$

$$= \pi \times \frac{.54}{2} \times \frac{700}{60} = 9.896 \text{ m/s.}$$

Substituting the value of u_2 in equation (*i*),

$$\tan \phi = \frac{6.862}{9.896} = 0.6934$$

$\therefore$ $\phi = \tan^{-1} .6934° = 34.74$ or **34° 44.4′. Ans.**

(*iii*) *Diameters of runner at inlet and outlet*

$D_1 = 0.54$ m, $D_2 =$ **0.27 m. Ans.**

(*iv*) *Width of wheel at inlet*

$B_1 =$ **54 mm. Ans.**

AXIAL FLOW REACTION TURBINE

If the water flows parallel to the axis of the rotation of the shaft, the turbine is known as axial flow turbine. And if the head at the inlet of the turbine is the sum of pressure energy and kinetic energy and during the flow of water through runner a part of pressure energy is converted into kinetic energy, the turbine is known as reaction turbine.

For the axial flow reaction turbine, the shaft of the turbine is vertical. The lower end of the shaft is made larger which is known as 'hub' or 'boss'. The vanes are fixed on the hub and hence hub acts as a runner for axial flow reaction turbine. The following are the important type of axial flow reaction turbines :

1. Propeller Turbine, and 2. Kaplan Turbine.

When the vanes are fixed to the hub and they are not adjustable, the turbine is known as propeller turbine. But if the vanes on the hub are adjustable, the turbine is known as a *Kaplan Turbine*, after the name of V Kaplan, an Austrian Engineer. This turbine is suitable where a large quantity of water at low head is available. Fig. 18.25 shows the runner of a Kaplan turbine, which consists of a hub fixed to the shaft. On the hub, the adjustable vanes are fixed as shown in Fig. 18.25.

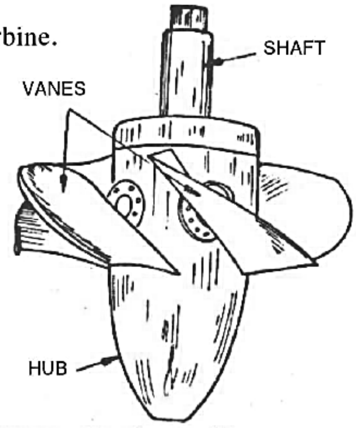
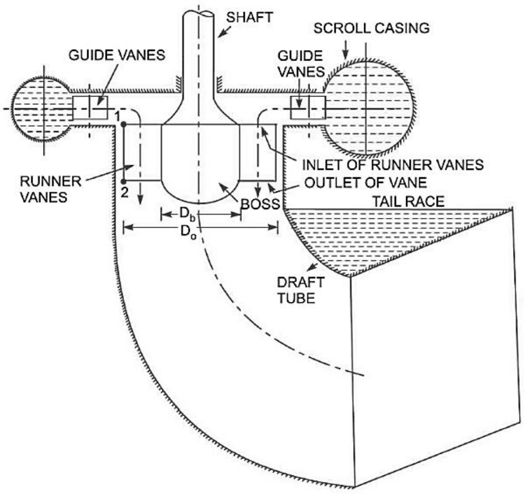


Fig. 18.25 *Kaplan turbine runner.*

The main parts of a Kaplan turbine are :

1. Scroll casing,
2. Guide vanes mechanism,
3. Hub with vanes or runner of the turbine, and
4. Draft tube.

Main components of Kaplan turbine



$$Q = \frac{\pi}{4}\left(D_o^2 - D_b^3\right) \times V_{f_1}$$

where D_o = Outer diameter of the runner,

D_b = Diameter of hub, and

V_{f_1} = Velocity of flow at inlet.

Some Important Point for Propeller (Kaplan Turbine)

1. The peripheral velocity at inlet and outlet are equal

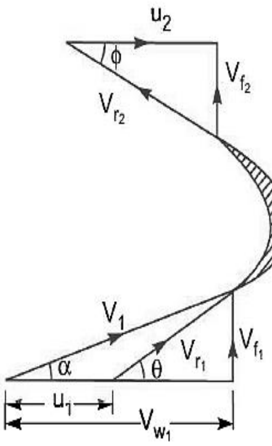
$$\therefore \qquad u_1 = u_2 = \frac{\pi D_o N}{60}, \text{ where } D_o = \text{Outer dia. of runner}$$

2. Velocity of flow at inlet and outlet are equal

$$\therefore \qquad V_{f_1} = V_{f_2}.$$

3. Area of flow at inlet = Area of flow at outlet

$$= \frac{\pi}{4}\left(D_o^2 - D_b^2\right).$$

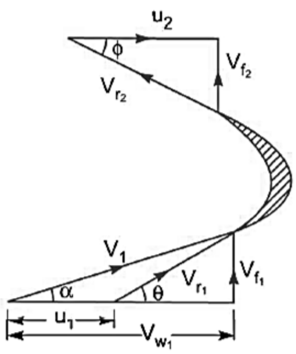


Example *A Kaplan turbine working under a head of 20 m develops 11772 kW shaft power. The outer diameter of the runner is 3.5 m and hub diameter is 1.75 m. The guide blade angle at the extreme edge of the runner is 35°. The hydraulic and overall efficiencies of the turbines are 88% and 84% respectively. If the velocity of whirl is zero at outlet, determine :*

(i) Runner vane angles at inlet and outlet at the extreme edge of the runner, and

(ii) Speed of the turbine.

Solution. Given :



Head, $H = 20$ m

Shaft power, S.P. = 11772 kW

Outer dia. of runner, $D_o = 3.5$ m

Hub diameter, $D_b = 1.75$ m

Guide blade angle, $\alpha = 35°$

Hydraulic efficiency, $\eta_h = 88\%$

Overall efficiency, $\eta_o = 84\%$

Velocity of whirl at outlet = 0.

Using the relation, $\eta_o = \dfrac{\text{S.P.}}{\text{W.P.}}$

where $\quad \text{W.P.} = \dfrac{\text{W.P.}}{1000} = \dfrac{\rho \times g \times Q \times H}{1000}$, we get

$$0.84 = \frac{11772}{\dfrac{\rho \times g \times Q \times H}{1000}}$$

$$= \frac{11772 \times 1000}{1000 \times 9.81 \times Q \times 20} \qquad (\because \ \rho = 1000)$$

$$\therefore \qquad Q = \frac{11772 \times 1000}{0.84 \times 1000 \times 9.81 \times 20} = 71.428 \text{ m}^3/\text{s}.$$

Using equation (18.25), $\quad Q = \frac{\pi}{4}\left(D_o^2 - D_b^2\right) \times V_{f_1}$

or $$71.428 = \frac{\pi}{4}(3.5^2 - 1.75^2) \times V_{f_1} = \frac{\pi}{4}(12.25 - 3.0625)\ V_{f_1}$$

$$= 7.216\ V_{f_1}$$

$$\therefore \qquad V_{f_1} = \frac{71.428}{7.216} = 9.9 \text{ m/s}.$$

From inlet velocity triangle, $\tan \alpha = \frac{V_{f_2}}{V_{w_1}}$

$$\therefore \qquad V_{w_1} = \frac{V_{f_1}}{\tan \alpha} = \frac{9.9}{\tan 35^\circ} = \frac{9.9}{.7} = 14.14 \text{ m/s}$$

Using the relation for hydraulic efficiency,

$$\eta_h = \frac{V_{w_1} u_1}{gH} \qquad (\because \ V_{w_2} = 0)$$

$$0.88 = \frac{14.14 \times u_1}{9.81 \times 20}$$

(*i*) Runner vane angles at inlet and outlet at the extreme edge of the runner are given as :

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{9.9}{(14.14 - 12.21)} = 5.13$$

$$\therefore \qquad \theta = \tan^{-1} 5.13 = \mathbf{78.97^\circ} \text{ or } \mathbf{78^\circ\ 58'}. \ \textbf{Ans.}$$

For Kaplan turbine, $\quad u_1 = u_2 = 12.21$ m/s and $V_{f_1} = V_{f_2} = 9.9$ m/s

$$\therefore \text{ From outlet velocity triangle, } \tan \phi = \frac{V_{f_2}}{u_2} = \frac{9.9}{12.21} = 0.811$$

$$\therefore \qquad \phi = \tan^{-1} .811 = \mathbf{39.035^\circ} \text{ or } \mathbf{39^\circ\ 2'}. \ \textbf{Ans.}$$

(*ii*) Speed of turbine is given by $u_1 = u_2 = \frac{\pi D_o N}{60}$

$$12.21 = \frac{\pi \times 3.5 \times N}{60}$$

$$\therefore \qquad N = \frac{60 \times 12.21}{\pi \times 3.50} = \mathbf{66.63 \ r.p.m.} \ \textbf{Ans.}$$

Example *A Kaplan turbine runner is to be designed to develop 9100 kW. The net available head is 5.6 m. If the speed ratio = 2.09, flow ratio = 0.68, overall efficiency = 86% and the diameter of the boss is 1/3 the diameter of the runner. Find the diameter of the runner, its speed and the specific speed of the turbine.*

Power, $P = 9100$ kW

Net head, $H = 5.6$ m

Speed ratio $= 2.09$

Flow ratio $= 0.68$

Overall efficiency, $\eta_o = 86\% = 0.86$

Diameter of boss $= \frac{1}{3}$ of diameter of runner

or $$D_b = \frac{1}{3} D_o$$

Now, speed ratio $= \dfrac{u_1}{\sqrt{2gH}}$

$\therefore$ $$u_1 = 2.09 \times \sqrt{2 \times 9.81 \times 5.6} = 21.95 \text{ m/s}$$

Flow ratio $= \dfrac{V_{f_1}}{\sqrt{2gH}}$

$\therefore$ $$V_{f_1} = 0.68 \times \sqrt{2 \times 9.81 \times 5.6} = 7.12 \text{ m/s}$$

The overall efficiency is given by, $\eta_o = \dfrac{P}{\left(\dfrac{\rho \times g.Q.H}{1000}\right)}$

or $$Q = \frac{P \times 1000}{\rho \times g \times H \times \eta_o} = \frac{9100 \times 1000}{1000 \times 9.81 \times 5.6 \times 0.86} \quad (\because \; \rho g = 1000 \times 9.81 \text{ N/m}^3)$$

$$= 192.5 \text{ m}^3\text{/s.}$$

The discharge through a Kaplan turbine is given by

$$Q = \frac{\pi}{4}[D_o^2 - D_b^2] \times V_{f_1}$$

or $$192.5 = \frac{\pi}{4}\left[D_o^2 - \left(\frac{D_o}{3}\right)^2\right] \times 7.12 \quad \left(\because \; D_b = \frac{D_o}{3}\right)$$

$$= \frac{\pi}{4}\left[1 - \frac{1}{9}\right] D_o^2 \times 7.12$$

$\therefore$ $$D_o = \sqrt{\frac{4 \times 192.5 \times 9}{\pi \times 8 \times 7.12}} = \mathbf{6.21 \text{ m. Ans.}}$$

The speed of turbine is given by, $u_1 = \dfrac{\pi D N}{60}$

$\therefore$ $$N = \frac{60 \times u_1}{\pi \times D} = \frac{60 \times 21.95}{\pi \times 6.21} = \mathbf{67.5 \text{ r.p.m. Ans.}}$$

The specific speed is given by, $N_s = \dfrac{N\sqrt{P}}{H^{5/4}} = \dfrac{67.5 \times \sqrt{9100}}{5.6^{5/4}} = \mathbf{746. \text{ Ans.}}$

Example *A Kaplan turbine runner is to be designed to develop 7357.5 kW shaft power. The net available head is 5.50 m. Assume that the speed ratio is 2.09 and flow ratio is 0.68, and the overall efficiency is 60%. The diameter of the boss is $\frac{1}{3}$rd of the diameter of the runner. Find the diameter of the runner, its speed and its specific speed.*

Shaft power, $P = 7357.5$ kW

Head, $H = 5.50$ m

Speed ratio $= \dfrac{u_1}{\sqrt{2gH}} = 2.09$

$\therefore$ $u_1 = 2.09 \times \sqrt{2 \times 9.81 \times 5.50} = 21.71$ m/s

Flow ratio $= \dfrac{V_{f_1}}{\sqrt{2gH}} = 0.68$

$\therefore$ $V_{f_1} = 2.68 \times \sqrt{2 \times 9.81 \times 5.50} = 7.064$ m/s

Overall efficiency, $\eta_o = 60\% = 0.60$

Diameter of boss, $D_b = \dfrac{1}{3} \times D_o$

Using relation,
$$\eta_o = \frac{\text{Shaft power}}{\text{Water power}} = \frac{7357.5}{\dfrac{\rho \times g \times Q \times H}{1000}}$$

or
$$0.60 = \frac{7357.5 \times 1000}{\rho \times g \times Q \times H} = \frac{7357.5 \times 1000}{1000 \times 9.81 \times Q \times 5.5}$$

$\therefore$
$$Q = \frac{7357.5 \times 1000}{1000 \times 9.81 \times 5.5 \times 0.60} = 227.27 \text{ m}^3\text{/s.}$$

$$Q = \frac{\pi}{4}\left(D_o^2 - D_b^2\right) \times V_{f_1}$$

$$227.27 = \frac{\pi}{4}\left[D_o^2 - \left(\frac{D_o}{3}\right)^2\right] \times 7.064 \qquad \left(\because\ D_b = \frac{D_o}{3}\right)$$

$$= \frac{\pi}{4} \times \frac{8}{9} D_o^2 \times 7.064 = 4.9316\ D_o^2$$

$\therefore$
$$D_o = \sqrt{\frac{227.27}{4.9316}} = \textbf{6.788 m. Ans.}$$

And
$$D_b = \frac{1}{3} \times 6.788 = \textbf{2.262 m. Ans.}$$

Using the relation,
$$u_1 = \frac{\pi D_o \times N}{60}$$

$\therefore$
$$N = \frac{60 \times u_1}{\pi D_o} = \frac{60 \times 21.71}{\pi \times 6.788} = \textbf{61.08 r.p.m. Ans.}$$

The specific speed (N_s) is given by,
$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{61.08 \times \sqrt{7357.5}}{5.50^{5/4}} = \textbf{622 r.p.m. Ans.}$$

Dimensional analysis

Dimensional analysis is a method of dimensions. It is a mathematical technique used in research work for design and for conducting model tests. It deals with the dimensions of the physical quantities involved in the phenomenon. All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value. Length L, mass M and time T are three fixed dimensions which are of importance in Fluid Mechanics. If in any problem of fluid mechanics, heat is involved then temperature is also taken as fixed dimension. These fixed dimensions are called fundamental dimensions or fundamental quantity.

Secondary or derived quantities are those quantities which possess more than one fundamental dimension. For example, velocity is denoted by distance per unit time (L/T), density by mass per unit volume| (M/L^3) and acceleration distance per second Square (L/T^2). Then velocity, density, deceleration become as secondary or derived quantities. The expressions (L/T), (M/L^3) and (L/T^2) are called the dimensions of velocity, density and acceleration respectively. The dimensions of mostly used physical quantities in Fluid Mechanics.

Dimensional Homogeneity

If an equation truly expresses a proper relationship among variables in a physical process, then it will be dimensionally homogeneous. The equations are correct for any system of units and consequently each group of terms in the equation must have the same dimensional representation. This is also known as the law of dimensional homogeneity.

Dimensional variables

These are the quantities, which actually vary during a given case and can be plotted against each other. Dimensional constants: These are normally held constant during a given run. But, they may vary from case to case.

Pure constants

They have no dimensions, but, while performing the mathematical manipulation, they can arise.

Buckingham pi Theorem

The dimensional analysis for the experimental data of unknown flow problems leads to some non-dimensional parameters. These dimensionless products are frequently referred as *pi terms*. Based on the concept of *dimensional homogeneity*, these dimensionless parameters may be grouped and expressed in functional forms. This idea was explored by the famous scientist Edgar Buckingham (1867-1940) and the theorem is named accordingly.

Buckingham pi theorem, states that if an equation involving k variables is dimensionally homogeneous, then it can be reduced to a relationship among $(k-r)$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variable. For a physical system, involving k variables, the functional relation of variables can be written mathematically as,

$$y = f(x_1, x_2 ...., x_k)$$

It should be ensured that the dimensions of the variables on the left side of the equation are equal to the dimensions of any term on the right side of equation. Now, it is possible to rearrange the above equation into a set of dimensionless products (*pi terms*), so that

$$\Pi_1 = \varphi(\Pi_2, \Pi_3 ...., \Pi_{k-r})$$

Here, $\varphi(\Pi_2, \Pi_3 ...., \Pi_{k-r})$ is a function of Π_2 through Π_{k-r}. The required number of *pi terms* is less than the number of original reference variables by r. These reference dimensions are usually the basic dimensions M, L and T (Mass, Length and Time).

Determination of pi Terms

Several methods can be used to form dimensionless products or *pi terms* that arise in dimensional analysis. But, there is a systematic procedure called *method of repeating variables* that allows in deciding the dimensionless and independent *pi terms*. For a given problem, following distinct steps are followed.

Step I: List out all the variables that are involved in the problem. The 'variable' is any quantity including dimensional and non-dimensional constants in a physical situation under investigation. Typically, these variables are those that are necessary to describe the "geometry" of the system (diameter, length etc.), to define fluid properties (density, viscosity etc.) and to indicate the external effects influencing the system (force, pressure etc.). All the variables must be independent in nature so as to minimize the number of variables required to describe the complete system.

Step II: Express each variable in terms of basic dimensions. Typically, for fluid mechanics problems, the basic dimensions will be either M, L and T or F, L and T. Dimensionally, these two sets are related through Newton's second law $(F = m.a)$ so that $F = MLT^{-2}$ e.g. $\rho = ML^{-3}$ or $\rho = FL^{-4}T^{2}$. It should be noted that these basic dimensions should not be mixed.

Step III: Decide the required number of *pi terms*. It can be determined by using *Buckingham pi theorem* which indicates that the number of *pi terms* is equal to $(k-r)$, where k is the number of variables in the problem (determined from Step I) and r is the number of reference dimensions required to describe these variables (determined from Step II).

Step IV: Amongst the original list of variables, select those variables that can be combined to form *pi terms*. These are called as *repeating variables*. The required number of *repeating variables is equal to the number of reference dimensions*. Each *repeating variable* must be dimensionally independent of the others, i.e. they cannot be combined themselves to form any dimensionless product. Since there is a possibility of repeating variables to appear in more than one *pi term*, so dependent variables should not be chosen as one of the repeating variable.

Step V: Essentially, the *pi terms* are formed by multiplying one of the non-repeating variables by the product of the repeating variables each raised to an exponent that will make the combination dimensionless. It usually takes the form of $x_i\ x_1^a\ x_2^b\ x_3^c$ where the exponents a, b and c are determined so that the combination is dimensionless.

Step VI: Repeat the 'Step V' for each of the remaining non-repeating variables. The resulting set of *pi terms* will correspond to the required number obtained from Step

III.

Step VII: After obtaining the required number of *pi terms*, make sure that all the *pi terms* are dimensionless. It can be checked by simply substituting the basic dimension $(M, L \text{ and } T)$ of the variables into the *pi terms*.

Step VIII: Typically, the final form of relationship among the *pi terms* can be written in the form of Eq. (6.1.2) where, Π_1 would contain the dependent variable in the numerator. The actual functional relationship among *pi terms* is determined from experiment.

Non Dimensional numbers in Fluid Dynamics

Forces encountered in flowing fluids include those due to inertia, viscosity, pressure, gravity, surface tension and compressibility. These forces can be written as follows;

$$\begin{aligned}
&\text{Inertia force: } m.a = \rho V \frac{dV}{dt} \propto \rho V^2 L^2 \\
&\text{Viscous force: } \tau A = \mu A \frac{du}{dy} \propto \mu V L \\
&\text{Pressure force: } (\Delta p) A \propto (\Delta p) L^2 \\
&\text{Gravity force: } m g \propto g \rho L^3 \\
&\text{Surface tension force: } \sigma L \\
&\text{Compressibility force: } E_v A \propto E_v L^2
\end{aligned}$$

The notations used in Eq. (6.2.1) are given in subsequent paragraph of this section. It may be noted that the ratio of any two forces will be dimensionless. Since, inertia forces are very important in fluid mechanics problems, the ratio of the inertia force to each of the other forces listed above leads to fundamental dimensionless groups. Some of them are defined as given below;

Reynolds number (Re): It is defined as the ratio of inertia force to viscous force. Mathematically,

$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

where V is the velocity of the flow, L is the characteristics length, ρ, μ and ν are the density, dynamic viscosity and kinematic viscosity of the fluid respectively. If Re is very small, there is an indication that the viscous forces are dominant compared to inertia forces. Such types of flows are commonly referred to as "creeping/viscous flows". Conversely, for large Re, viscous forces are small compared to inertial effects and such flow problems are characterized as inviscid analysis. This number is also used to study the transition between the laminar and turbulent flow regimes.

Euler number (E_u): In most of the aerodynamic model testing, the pressure data are usually expressed mathematically as,

$$E_u = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

where Δp is the difference in local pressure and free stream pressure, V is the velocity of the flow, ρ is the density of the fluid. The denominator in Eq. (6.2.3) is called "dynamic pressure". E_u is the ratio of pressure force to inertia force and many a times the pressure coefficient (c_p) is a also common name which is defined by same manner. In the study of cavitations phenomena, similar expressions are used where, Δp is the difference in liquid stream pressure and liquid-vapour pressure. This dimensional parameter is then called as "cavitation number".

Froude number (F_r): It is interpreted as the ratio of inertia force to gravity force. Mathematically, it is written as,

$$F_r = \frac{V}{\sqrt{g.L}}$$

where V is the velocity of the flow, L is the characteristics length descriptive of the flow field and g is the acceleration due to gravity. This number is very much significant for flows with free surface effects such as in case of open-channel flow. In such types of flows, the characteristics length is the depth of water. F_r less than unity indicates sub-critical flow and values greater than unity indicate super-critical flow. It is also used to study the flow of water around ships with resulting wave motion.

Weber number (W_e): It is defined as the ratio of the inertia force to surface tension force. Mathematically,

$$W_e = \frac{\rho V^2 L}{\sigma}$$

where V is the velocity of the flow, L is the characteristics length descriptive of the flow field, ρ is the density of the fluid and σ is the surface tension force. This number is taken as an index of droplet formation and flow of thin film liquids in which there is an interface between two fluids. The inertia force is dominant compared to surface tension force when, $W_e \gg 1$ (e.g. flow of water in a river).

<u>Mach number</u> (M): It is the key parameter that characterizes the compressibility effects in a fluid flow and is defined as the ratio of inertia force to compressibility force. Mathematically,

$$M = \frac{V}{c} = \frac{V}{\sqrt{\frac{dp}{d\rho}}} = \frac{V}{\sqrt{\frac{E_v}{\rho}}}$$

where V is the velocity of the flow, c is the local sonic speed, ρ is the density of the fluid and E_v is the bulk modulus. Sometimes, the square of the Mach number is called "Cauchy number" (C_a) i.e.

$$C_a = M^2 = \frac{\rho V^2}{E_v}$$

Both the numbers are predominantly used in problems in which fluid compressibility is important. When, M_a is relatively small (say, less than 0.3), the inertial forces induced by fluid motion are sufficiently small to cause significant change in fluid density. So, the compressibility of the fluid can be neglected. However, this number is most commonly used parameter in compressible fluid flow problems, particularly in the field of gas dynamics and aerodynamics.

<u>Strouhal number</u> (S_t): It is a dimensionless parameter that is likely to be important in unsteady, oscillating flow problems in which the frequency of oscillation is ω and is defined as,

$$S_t = \frac{\omega L}{V}$$

where V is the velocity of the flow and L is the characteristics length descriptive of the flow field. This number is the measure of the ratio of the inertial forces due to unsteadiness of the flow (local acceleration) to inertia forces due to changes in

Parameter	Mathematical expression	Qualitative definition	Importance
Prandtl number	$P_r = \frac{\mu c_p}{k}$	$\frac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$E_c = \frac{V^2}{c_p T_0}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific heat ratio	$\gamma = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Roughness ratio	$\frac{\varepsilon}{L}$	$\frac{\text{Wall roughness}}{\text{Body length}}$	Turbulent rough walls
Grashof number	$G_r = \frac{\beta(\Delta T) g L^3 \rho^2}{\mu^2}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural onvection
Temperature ratio	$\frac{T_w}{T_0}$	$\frac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \frac{p - p_\infty}{(1/2)\rho V^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Hydrodynamics, Aerodynamics
Lift coefficient	$C_L = \frac{L}{(1/2) A \rho V^2}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Hydrodynamics,Aero dynamics
Drag coefficient	$C_D = \frac{D}{(1/2) A \rho V^2}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$	Hydrodynamics, Aero dynamics

Flow Similarity

In order to achieve similarity between model and prototype behavior, all the corresponding pi terms must be equated to satisfy the following conditions.

Geometric similarity

A model and prototype are geometric similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio. In order to have geometric similarity between the model and prototype, the model and the prototype should be of the same shape, all the linear dimensions of the model can be related to corresponding dimensions of the prototype by a constant scale factor. Usually, one or more of these *pi terms* will involve ratios of important lengths, which are purely geometrical in nature.

Kinematic similarity

The motions of two systems are kinematically similar if homogeneous particles lie at same points at same times. In a specific sense, the velocities at corresponding points are in the same direction (i.e. same streamline patterns) and are related in magnitude by a constant scale factor.

Dynamic similarity

When two flows have force distributions such that identical types of forces are parallel and are related in magnitude by a constant scale factor at all corresponding points, then the flows are dynamic similar. For a model and prototype, the dynamic similarity exists, when both of them have same length-scale ratio, timescale ratio and force-scale (or mass-scale ratio).

In order to have complete similarity between the model and prototype, all the similarity flow conditions must be maintained. This will automatically follow if all the important variables are included in the dimensional analysis and if all the similarity requirements based on the resulting pi terms are satisfied. For example, in compressible flows, the model and prototype should have same Reynolds number, Mach number and specific heat ratio etc. If the flow is incompressible (without free surface), then same Reynolds numbers for model and prototype can satisfy the complete similarity.

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