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SCHOOL OF MECHANICAL ENGINEERING

DEPARTMENT OF AUTOMOBILE ENGINEERING

SAU1304 VEHICLE DYNAMICS

UNIT I INTRODUCTION

UNIT I CONCEPT OF VIBRATION

Classification of vibrations- mechanical vibrating systems, single degree of freedom, two degree of freedom, multi degree of freedom, free, forced and damped vibrations, modelling and simulation studies, model of an automobile, magnification factor, transmissibility, vibration absorber.

VIBRATION

- It is defined as any motion that repeats itself after an interval of time.
- It involves transfer of potential energy to kinetic energy and vice versa.
- Vibration is the motion of a particle or a body or system of connected bodies displaced from a position of equilibrium
- Aim of vibration analysis:
- Why:
  - Vibrations can lead to excessive deflections and failure on the machines and structures.
  - To reduce vibration through proper design of machines and their mountings.
  - To utilize profitably in several consumer and industrial applications.
  - To improve the efficiency of certain machining, casting, forging & welding processes.
  - Cause rapid wear.
  - Create excessive noise.
  - Leads to poor surface finish (eg: in metal cutting process, vibration cause chatter).
  - Resonance – natural frequency of vibration of a machine/structure coincide with the frequency of the external excitation.
  - To stimulate earthquakes for geological research and conduct studies in design of nuclear reactors.

CLASSIFICATION OF VIBRATION

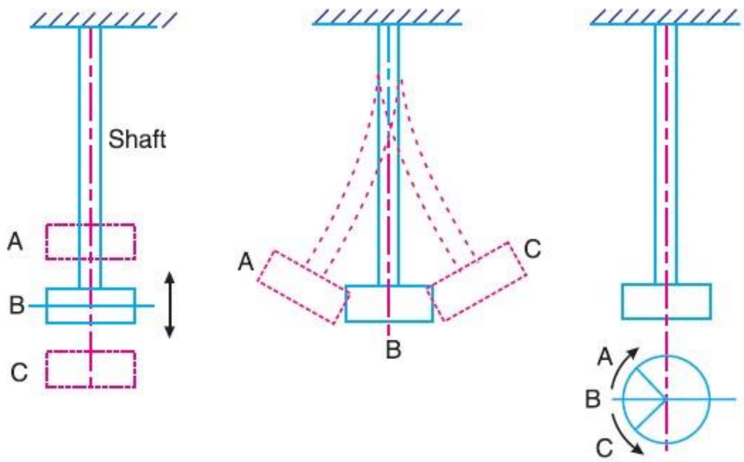
Free vibration:

- When no external force acts on the body, after giving it an initial displacement, then the body is said to be under free or natural vibrations.

- The frequency of the free vibration is called free or natural frequency.

The following three types of free vibrations are important from the subject point of view :

1. Longitudinal vibrations,
2. Transverse vibrations,
3. Torsional vibrations.



B = Mean position ; A and C = Extreme positions.

(a) Longitudinal vibrations. (b) Transverse vibrations. (c) Torsional vibrations.

Figure 1: Types of free vibration.

Forced vibration:

- When the body vibrates under the influence of external force, then the body is said to be under forced vibrations.
- The external force applied to the body is a periodic disturbing force created by unbalance.
- The vibrations have the same frequency as the applied force.
- When the frequency of the external force is same as that of the natural vibrations, resonance takes place.
- *Resonance* occurs when the frequency of the external force coincides with one of the natural frequencies of the system

Damped vibration:

- When there is a reduction in amplitude over every cycle vibration, the motion is said to be damped vibration.
- This is due to the fact that a certain amount of energy possessed by the vibration system is always dissipated in overcome friction resistances to the motion.

Linear Vibration:

- When ***all*** basic components of a vibratory system, i.e. the spring, the mass and the damper behave linearly

- Nonlinear Vibration:

  If any one of the components behave non linearly

- Deterministic Vibration:

  If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time

Nondeterministic or random Vibration:

When the value of the excitation at a given time cannot be predicted

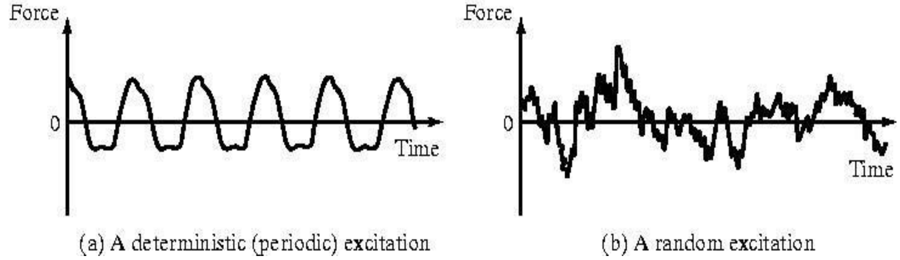


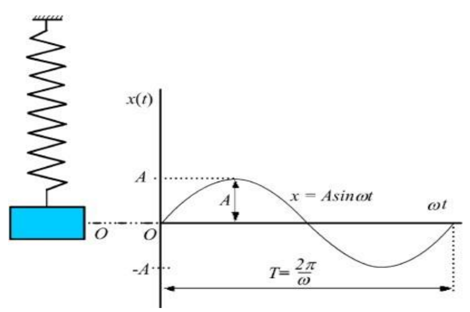
Figure 2: examples of deterministic and random vibration.

BASIC TERMS OF VIBRATION

Oscillatory motion: repeats itself regularly.

Cycle: It is the motion completed during one time period.
Periodic motion: This motion repeats at equal interval of time T

Period : the time taken for one repetition. ***Period of vibration or time period.*** It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds



$$\text{Period } \tau = \frac{2\pi}{\omega} \text{ s/cycle}$$

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi} \text{ cycles/s, or Hz}$$

Figure 3: Periodic Motion

Time period: $tp = 2\pi/\ \omega$

Frequency: (1/ tp) it is the reciprocal of time period: ***Frequency.*** It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.
Natural frequency:

Damped vibration.

When there is reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration.

Damping.

Damping is the dissipation of energy with time or distance

Viscous damping

The damping provided by fluid friction is known as viscous.

Critical damping.

It is the minimum viscous damping that will allow a displaced system to return to its initial position without oscillation.

Degree of freedom:

- The minimum number of independent co-ordinates required to define completely the position of all parts of the system at any instance of time.
- How many mass or masses will be there in a system.

Single degree-of-freedom systems:

The number of degree of freedom of a mechanical system is equal to the minimum number of independent co-ordinates required to define completely the positions of all parts of the system at any instance of time.

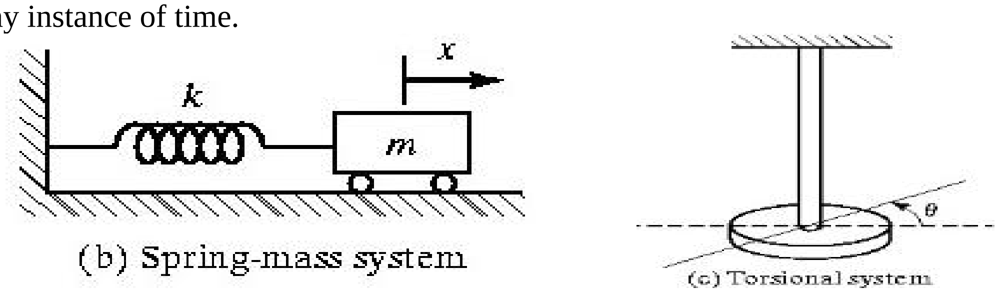


Figure 4: Examples of single degree of freedom.

Two degree-of-freedom systems:

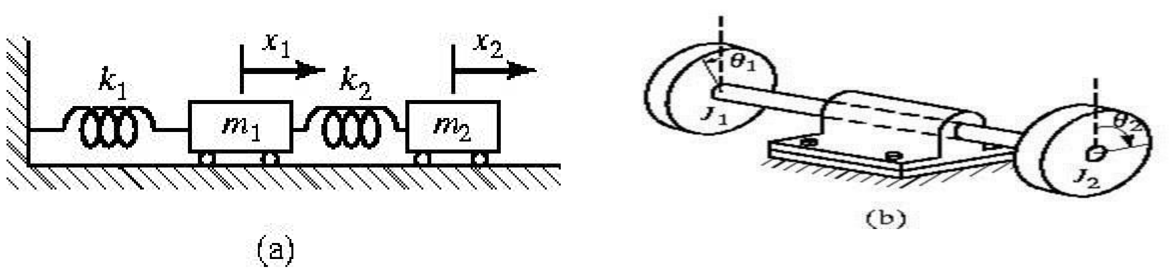


Figure 5: Examples of two degree of freedom.

Three degree of freedom systems:

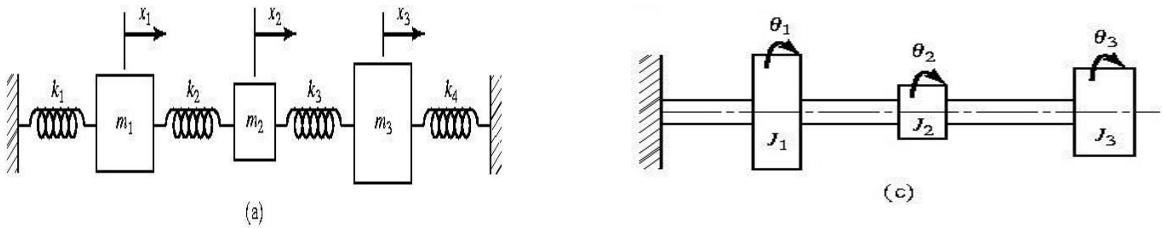


Figure 6: Examples of three degree of freedom.

Multi-degree of freedom:

- Infinite number of degrees of freedom system
- For which 2 or 3 co-ordinates are required to define completely the position of the system at any instance of time.

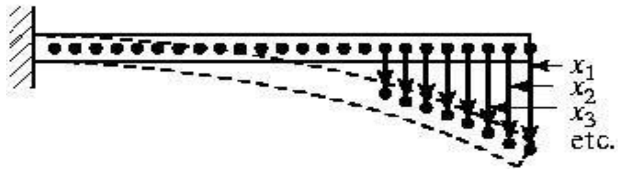


Figure 7: Examples of multi degree of freedom.

COMPONENTS OF MECHANICAL VIBRATION SYSTEMS

It consist of mass, spring and damper.

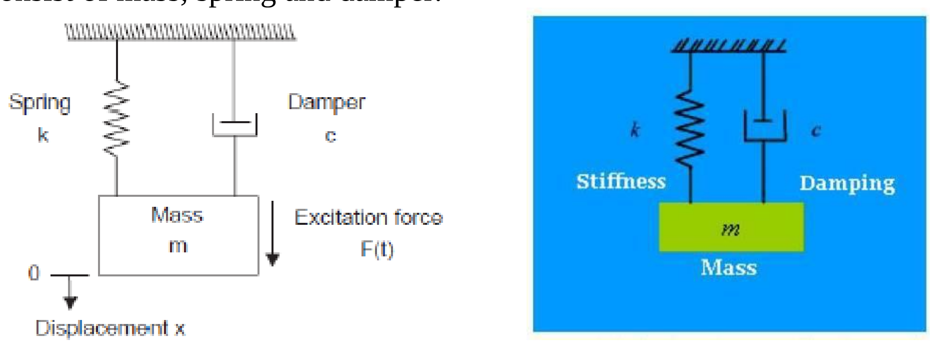


Figure 8: Mechanical vibrating system

Mass Element:
The mass provides inertia force to the system, spring provides the restoring force and the damper provides the resistance.
Spring Elements:
Linear spring is a type of mechanical link that is generally assumed to have negligible mass and damping.
Spring force is given by: F = k.x
F = spring force, *k* = spring stiffness or spring constant, and *x* = deformation (displacement of one end with respect to the other)
Combination of Springs:

Springs in parallel – if we have *n* spring constants *k*1, *k*2, …, *kn* in *parallel*, then the equivalent spring constant *k*eq is: $k_{eq} = k_1 + k_2 + \ldots + k_n$

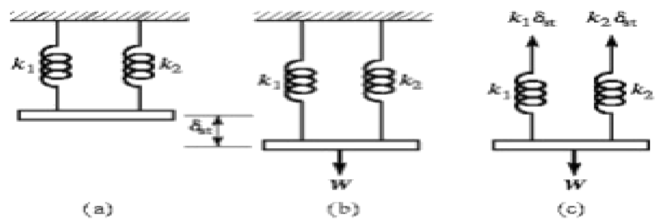


Figure 9: Springs in parallel position

Springs in series – if we have *n* spring constants *k*1, *k*2, …, *kn* in *series*, then the equivalent spring constant keq is:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}$$

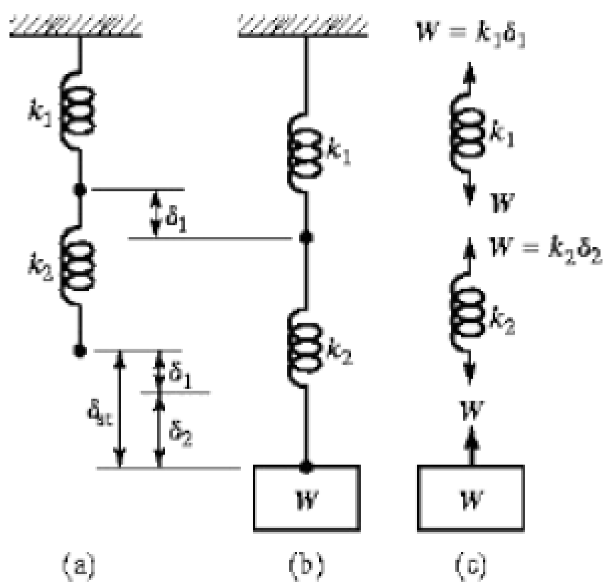


Figure 10: Springs in series position

Damping Element

- The process of energy dissipation is referred to in the study of vibration as *damping*. A damper is considered to have neither mass nor elasticity.
- The three main forms of damping are *viscous damping*, *Coulomb* or *dry-friction damping*,

and *hysteresis damping*. The most common type of energy-dissipating element used in vibrations study is the *viscous damper*, which is also referred to as a *dashpot*.

- In viscous damping, the damping force is proportional to the velocity of the body. Coulomb or dry-friction damping occurs when sliding contact that exists between surfaces in contact are dry or have insufficient lubrication. In this case, the damping force is constant in magnitude but opposite in direction to that of the motion. In dry-friction damping energy is dissipated as heat.

NATURAL FREQUENCY

1. Vehicle vibration with SINGLE DEGREE of freedom:

The natural frequency of the free longitudinal vibrations may be determined by the following three methods

1. Equilibrium Method
2. Energy method (summation of kinetic energy and potential energy must be a constant quantity which is same at all the times.)

$$\frac{d}{dt}(K.E. + P.E.) = 0$$

3. Rayleigh's method (the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position)

(a) **FREQUENCY OF FREE-UNDAMPED VIBRATIONS:**
Consider a constraint (i.e. spring) of negligible mass in an unstrained position

s = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.
m = Mass of the body suspended from the constraint in kg,
W = Weight of the body in newton = $m.g$,
δ = Static deflection of the spring in meters due to weight W newton, and
x = Displacement given to the body by the external force, in meters

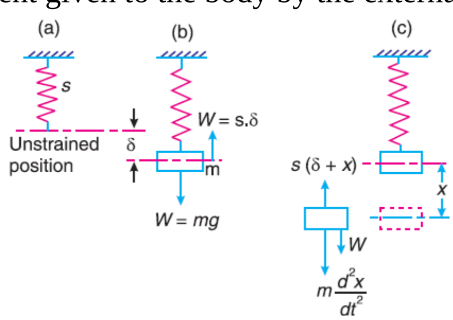


Figure 11: Free undamped vibration system

In the equilibrium position, the gravitational pull $W = m.g$, is balanced by a force of spring, such that $W = s.\delta$ .Since the mass is now displaced from its equilibrium position by a distance x, and is then released, therefore after time t,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x = s.\delta - s.\delta - s.x = -s.x$ *(Taking upward force as negative)*

Accelerating force = Mass × Acceleration

$$= m \times \frac{d^2x}{dt^2}$$

$$m \times \frac{d^2x}{dt^2} = -s.x$$

$$m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2.x = 0$$

$$\omega = \sqrt{\frac{s}{m}}$$

Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$

natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ *(m.g = s.δ)*

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{Hz}$$

(b) **FREQUENCY OF FREE DAMPED VIBRATIONS (VISCOUS DAMPING)**

- The motion of a body is resisted by frictional forces, the effect of friction is referred to as damping.
- The damping provided by fluid resistance is known as ***viscous damping*.**
- In damped vibrations, the amplitude of the resulting vibration gradually diminishes.

- Certain amount of energy is always dissipated to overcome the frictional resistance. The resistance to the motion of the body is provided partly by the medium in which the vibration takes place and partly by the internal friction

Let *m* = Mass suspended from the spring,

s = Stiffness of the spring,

x = Displacement of the mass from the mean position at time *t*,

δ = Static deflection of the spring = *m.g/s*, and

c = Damping coefficient or the damping force per unit velocity

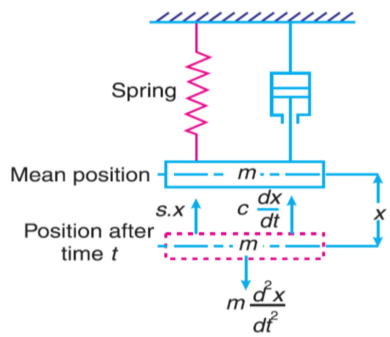


Figure 12: Free damped vibration system

Damping force or frictional force on the mass acting in ***opposite*** direction to the motion of the mass

$$= c \times \frac{dx}{dt}$$

Accelerating force on the mass, acting ***along*** the motion of the mass

$$= m \times \frac{d^2x}{dt^2}$$

Spring force on the mass, acting in ***opposite*** direction to the motion of the mass,

$$= s.x$$

Therefore the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -\left(c \times \frac{dx}{dt} + s.x \right)$$

$$\frac{d^2x}{dt^2} + \frac{c}{m} \times \frac{dx}{dt} + \frac{s}{m} \times x = 0$$

This is a differential equation of the second order.

Assuming a solution of the form $x = e^{kt}$ where k is a constant to be determined

$$k^2 . e^{kt} + \frac{c}{m} \times k . e^{kt} + \frac{s}{m} \times e^{kt} = 0 \qquad \left[\because \ \frac{dx}{dt} = k e^{kt}, \text{ and } \frac{d^2x}{dt^2} = k^2 . e^{kt} \right]$$

$$k^2 + \frac{c}{m} \times k + \frac{s}{m} = 0$$

$$k = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4 \times \frac{s}{m}}}{2}$$

$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

The two roots of the equation are:

$$k_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

$$k_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

The most general solution of the differential equation, with its right hand side equal to zero has

$$x = C_1 e^{k_1 t} + C_2 \, e^{k_2 t}$$

only complementary function and it is given by

The roots $k1$ and $k2$ may be real, complex conjugate (imaginary) or equal.

1. ***When the roots are real (over damping)***

If $\left(\frac{c}{2m}\right)^2 > \frac{s}{m}$ then the roots $k1$ and $k2$ are real but negative. This is a case of ***overdamping*** or ***large damping*** and the mass moves slowly to the equilibrium position. This motion is known as ***aperiodic.*** When the roots are real, the most general solution of the differential equation is

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

$$= C_1\, e^{\left[-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t} + C_2\, e^{\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t}$$

In actual practice, the over damped vibrations are avoided

2. *When the roots are complex conjugate (underdamping)*

If $\frac{s}{m} > \left(\frac{c}{2m}\right)^2$ then the radical (*i.e.* the term under the square root) becomes negative. The two roots $k1$ and $k2$ are then known as complex conjugate. This is a most practical case of damping and it is known as ***underdamping*** or ***small damping*****.** The two roots are

$$k_1 = -\frac{c}{2m} + i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

$$k_2 = -\frac{c}{2m} - i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

For the sake of mathematical calculations, let

$$\frac{c}{2m} = a;\ \frac{s}{m} = (\omega_n)^2;\ \text{and}\ \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \omega_d = \sqrt{(\omega_n)^2 - a^2}$$

Therefore the two roots may be written as

$$k_1 = -a + i\,\omega_d\ ;\qquad \text{and}\qquad k_2 = -a - i\,\omega_d$$

We know that the general solution of a differential equation is

$$x = C_1\, e^{k_1 t} + C_2\, e^{k_2 t} = C_1\, e^{(-a + i\omega_d)t} + C_2\, e^{(-a - i\omega_d)t}$$

$$= e^{-at}(C_1\, e^{i\omega_d . t} + C_2\, e^{-i\omega_d t}) \quad \ldots(\text{Using } e^{m+n} = e^m \times e^n) \ldots($$

Now according to Euler's theorem

$$e^{+i\theta} = \cos\theta + i\sin\theta\ ;\ \text{and}\ e^{-i\theta} = \cos\theta - i\sin\theta$$

Therefore the equation ***(iii)*** may be written as

$$x = e^{-at}\left[C_1(\cos\omega_d .t + i\sin\omega_d .t) + C_2(\cos\omega_d .t - i\sin\omega_d .t)\right]$$

$$= e^{-at}\left[(C_1 + C_2)\cos\omega_d .t + i\,(C_1 - C_2)\sin\omega_d .t)\right]$$

$$C_1 + C_2 = A,\ \text{and}\ i\,(C_1 - C_2) = B$$

$$\therefore \qquad x = e^{-at}\,(A\cos\omega_d.t + B\sin\omega_d.t) \qquad \ldots (iv)$$

Again, let $A = C\cos\theta$, and $B = C\sin\theta$, therefore

$$C = \sqrt{A^2 + B^2}\ , \text{ and } \quad \tan\theta = \frac{B}{A}$$

Now the equation (*iv*) becomes

$$x = e^{-at}(C\cos\theta\cos\omega_d.t + C\sin\theta\sin\omega_d.t)$$

$$= Ce^{-at}\cos(\omega_d.t - \theta) \qquad \ldots (v)$$

If t is measured from the instant at which the mass m is released after an initial displacement A, then

$$A = C\cos\theta \qquad \ldots \text{[Substituting } x = A \text{ and } t = 0 \text{ in equation } (v)\text{]}$$

and when $\theta = 0$, then $A = C$

$\therefore$ The equation (*v*) may be written as

$$x = Ae^{-at}\cos\omega_d.t \qquad \ldots (vi)$$

where $$\omega_d = \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{(\omega_n)^2 - a^2}\ ; \text{ and } \ a = \frac{c}{2m}$$

The motion of the mass is simple harmonic whose circular damped frequency is and ω_d the amplitude is Ae^{-at} which diminishes exponentially with time as shown in Figure below.Though the mass eventually returns to its equilibrium position because of its inertia, yet it overshoots and the oscillations may take some considerable time to die away.

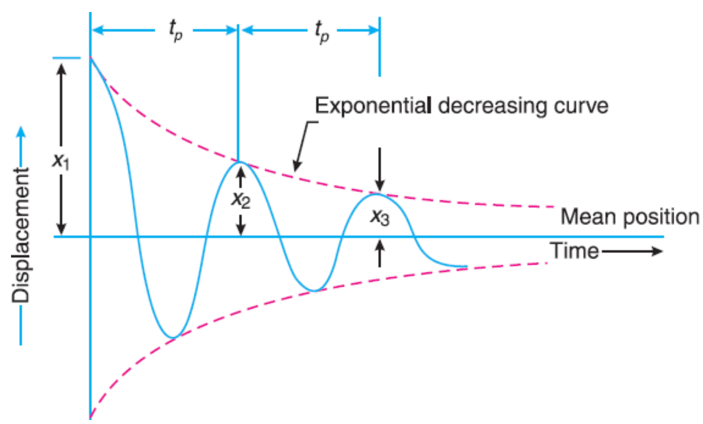


Figure 13: Condition of Logarithmic Decrement

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$f_d = \frac{1}{t_p} = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2} = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

3. **When the roots are equal (critical damping)**

If $\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$ then the radical becomes zero and the two roots $k1$ and $k2$ are equal. This is acase of **critical damping.** In other words, the critical damping is said to occur when frequency of damped vibration (*fd*) is zero (*i.e.*motion is aperiodic). This type of damping is also avoided

$$x = (C_1 + C_2)\, e^{-\frac{c}{2m}t} = (C_1 + C_2)\, e^{-\omega_n t}$$

because the mass moves back rapidly to its equilibrium position, in the shortest possible time

Thus the motion is again aperiodic. The critical damping coefficient (*cc*) may be obtained by substituting *cc* for *c* in the condition for critical damping,

$$\left(\frac{c_c}{2m}\right)^2 = \frac{s}{m} \qquad \text{or} \qquad c_c = 2m\sqrt{\frac{s}{m}} = 2m \times \omega_n$$

The critical damping coefficient is the amount of damping required for a system to be critically damped

Damping Factor or Damping Ratio

The ratio of the actual damping coefficient (c) to the critical damping coefficient (*cc*) is known as ***damping factor*** **or** ***damping ratio*****.** Mathematically

$$\text{Damping factor} = \frac{c}{c_c} = \frac{c}{2m.\omega_n}$$

Logarithmic Decrement

It is defined as the natural logarithm of the amplitude reduction factor. The amplitude reduction factor is the ratio of any two successive amplitudes on the same side of the mean position. If $x1$ and $x2$ are successive values of the amplitude on the same side of the mean position, as shown in Fig. 23.18, then amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{Ae^{-at}}{Ae^{-a(t+t_p)}} = e^{at_p} = \text{constant}$$

$$\delta = \log\left(\frac{x_1}{x_2}\right) = \log e^{at_p}$$

where *tp* is the period of forced oscillation or the time difference between two consecutive amplitudes. As per definition, logarithmic decrement

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = a.t_p = a \times \frac{2\pi}{\omega_d} = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\sqrt{(\omega_n)^2 - \left(\frac{c}{2m}\right)^2}}$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\omega_n \sqrt{1 - \left(\frac{c}{2m.\omega_n}\right)^2}} = \frac{c \times 2\pi}{c_c \sqrt{1 - \left(\frac{c}{c_c}\right)^2}}$$

$$= \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}}$$

In general, amplitude reduction factor

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots = \frac{x_n}{x_{n+1}} = e^{at_p} = \text{constant}$$

Logarithmic decrement

$$\delta = \log_e\left(\frac{x_n}{x_{n+1}}\right) = a.t_p = \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}}$$

Forced damped vibration system: (Frequency of Under Damped Forced Vibrations)

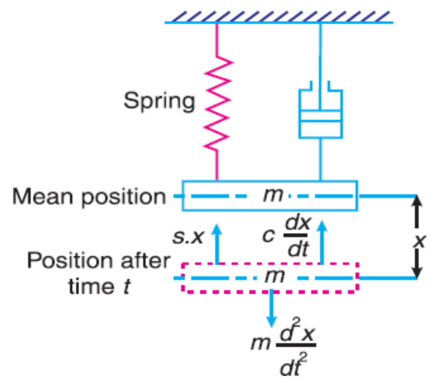


Figure 14: Forced damped vibration system

Consider a system consisting of spring, mass and damper as shown in Figure above. Let the system is acted upon by an external periodic (*i.e.* simple harmonic) disturbing force,

$$F_x = F\cos\omega.t$$

F = Static force, and

ω = Angular velocity of the periodic disturbing force.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime *t*, the mass is displaced downwards through a distance *x* from its mean position.

the equation of motion may be written as

$$m\times\frac{d^2x}{dt^2} = -c\times\frac{dx}{dt} - s.x + F\cos\omega.t$$

$$m\times\frac{d^2x}{dt^2} + c\times\frac{dx}{dt} + s.x = F\cos\omega.t$$

This equation of motion may be solved either by differential equation method or by graphical method as discussed below :

1. Differential equation method

The equation ***(i)*** is a differential equation of the second degree whose right hand side is some function in *t*. The solution of such type of differential equation consists of two parts ; one part is the complementary function and the second is particular integral. Therefore the solution may be written as

$$x = x_1 + x_2$$

$x1$ = Complementary function, and

$x2$ = Particular integral.

The complementary function is same as discussed in the previous article,

$$x_1 = Ce^{-at} \cos(\omega_d t - \theta)$$

$$c.\omega = X \sin\phi; \text{ and } s - m.\omega^2 = X \cos\phi$$

$$X = \sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}$$

$$x_2 = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

In actual practice, the value of the complementary function $x1$ at any time t is much smaller as

$$\phi = \tan^{-1}\left(\frac{c.\omega}{s - m.\omega^2}\right)$$

compared to particular integral $x2$. Therefore, the displacement x, at any time t, is given by the particular integral $x2$ only

$$x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

The equations ***(vii)*** and ***(viii)*** hold good when steady vibrations of constant amplitude takes

Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$x_{max} = \frac{F / s}{\sqrt{\frac{c^2.\omega^2}{s^2} + \frac{(s - m.\omega^2)^2}{s^2}}} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{m.\omega^2}{s}\right)^2}}$$

where xo is the deflection of the system under the static force F.

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

At resonance $\omega = \omega_n$ Therefore the angular speed at which the resonance occurs is

$$\omega = \omega_n = \sqrt{\frac{s}{m}} \text{ rad/s}$$

$$x_{max} = x_o \times \frac{s}{c.\omega_n}$$

Magnification Factor or Dynamic Magnifier

It is the ratio of ***maximum displacement of the forced vibration (xmax) to the deflection due to the static force F(xo).*** We have proved in the previous article that the maximum displacement or the amplitude of forced vibration, Magnification factor:

- It is the ratio between the maximum actual amplitude of the body and the maximum actual amplitude of the road.

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

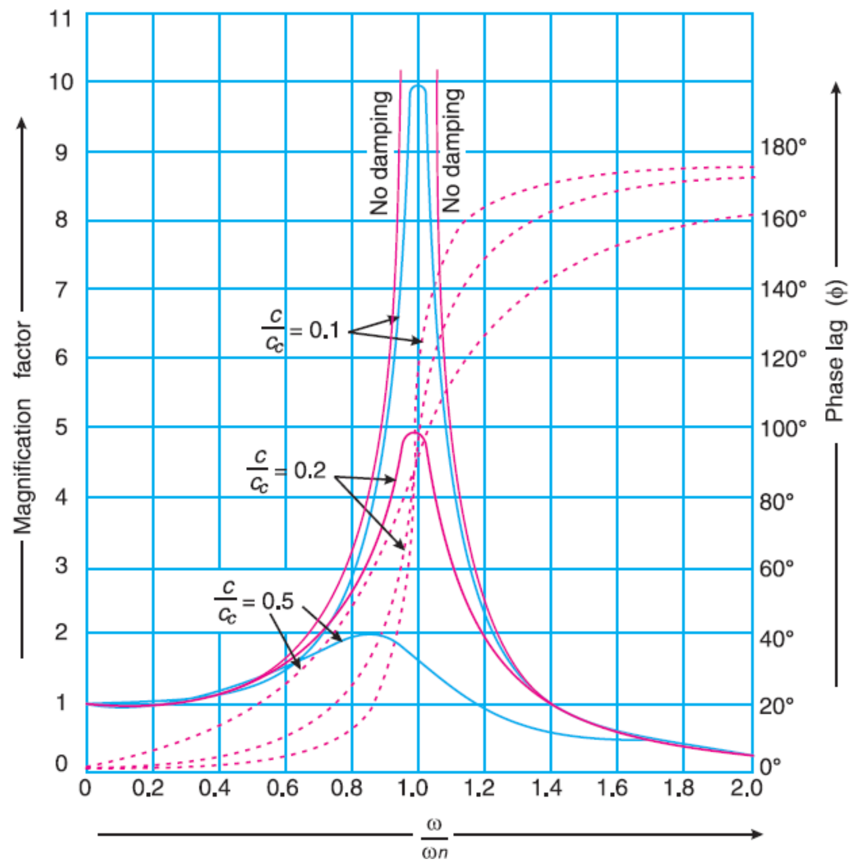


Figure 15: Magnification factor Vs damping ratio.

Magnification factor or dynamic magnifier,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$= \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

The magnification factor or dynamic magnifier gives the factor by which the static deflection produced by a force *F* (*i.e. xo*) must be multiplied in order to obtain the maximum amplitude of the forced vibration (*i.e. xmax*) by the harmonic force *F* cosw*t*

$$x_{max} = x_o \times D$$

If there is no damping (*i.e.* if the vibration is undamped), then $c = 0$. In that case, magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} = \frac{(\omega_n)^2}{(\omega_n)^2 - \omega^2}$$

At resonance, $\omega = \omega_n$ Therefore magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{s}{c.\omega_n}$$

Vibration Isolation and Transmissibility

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimize the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig. 16. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only.

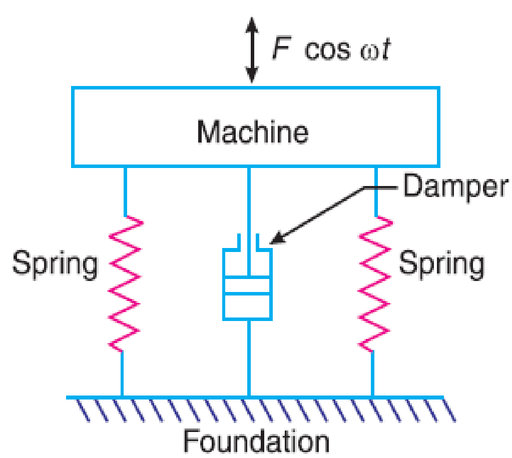


Figure 16: Condition for vibration isolation.

It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega t$ is applied to a machine of mass m supported by a spring of stiffness s, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation. The ratio of the force transmitted (FT) to the force applied (F) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

We have discussed above that the force transmitted to the foundation consists
of the following two forces :

1. Spring force or elastic force which is equal to s. *xmax*, and

2. Damping force which is equal to c.□.*xmax*.

Since these two forces are perpendicular to one another, as shown in Fig.23.23, therefore the

$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max}\sqrt{s^2 + c^2.\omega^2}$$

force transmitted,

Transmissibility ratio,

$$x_{max} = x_o \times D = \frac{F}{s} \times D$$

$$\varepsilon = \frac{D}{s}\sqrt{s^2 + c^2.\omega^2} = D\sqrt{1 + \frac{c^2.\omega^2}{s^2}}$$

$$= D\sqrt{1 + \left(\frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)^2}$$

the magnification factor,

$$D = \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1 - (\omega / \omega_n)^2}$$

From above, we see that when $\omega/\omega n > 1$, ∑ is negative. This means that there is a phase difference of 180° between the transmitted force and the disturbing force ($F \cos\omega.t$) . The value of $\omega/\omega n$ must be greater than 2 if ∑ is to be less than 1 and it is the numerical value of ∑ , independent of any phase difference between the forces that may exist which is important. It is therefore more convenient to use equation ***(ii)*** in the following form, *i.e.*

$$\varepsilon = \frac{1}{(\omega / \omega_n)^2 - 1}$$

l Fig. 17 is the graph for different values of damping factor c/cc to show the variation of transmissibility ratio (∑) against the ratio $\omega/\omega n$.

1. When $\omega/\omega n = 2$, then all the curves pass through the point $\sum = 1$ for all values of damping factor c/cc

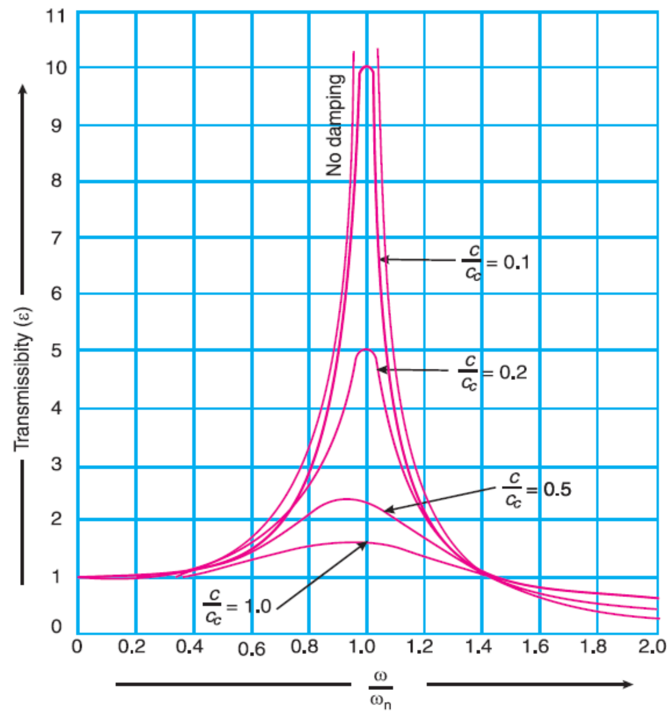


Figure 17: Transmissibility Vs Damping ratio.

2. When $\omega/\omega n < 2$, then $\sum > 1$ for all values of damping factor c/cc. This means that the force transmitted to the foundation through elastic support is greater than the force applied.

2. When $\omega/\omega n > 2$, then $\sum < 1$ for all values of damping factor c/cc. This shows that the force transmitted through elastic support is less than the applied force. Thus vibration isolation is possible only in the range of $\omega/\omega n > 2$.We also see from the curves in Fig. 23.24 that the damping is detrimental beyond $\omega/\omega n > 2$ and advantageous only in the region $\omega/\omega n > 2$. It is thus concluded that for the vibration isolation, dampers need not to be provided but in order to limit resonance amplitude,stops may be provided.

Transmissibility:

- It is the ratio between the force transmitted to the body and force acting on the road.
- It is the ratio between the force transmitted to the body and force acting on the road. Transmissibility is the non dimensional ratio of the response amplitude of a system in steady state forced vibration on the excitation amplitude. The ratio may be one for forces, displacement, velocities or accelerations.

Vibration absorber:

- It is an additional spring mass system used to make the amplitude values of vibration equal to zero.

Vehicle dynamics and its classification.

- Vehicle dynamics has been a pivotal domain in the field of automotive engineering. It is primarily divided into three subgroups: Performance, Ride and Handling. Performance mainly deals with the efficiency and effectiveness of the vehicle in its ability to accelerate, brake and overcome obstacles. Ride is related to the vibration of the vehicle due to road excitations and its effect on occupants and cargo. Handling is concerned with the overall behaviour or response of the vehicle to driver inputs.

Generalized co-ordinates

Rolling: Angular oscillation of vehicle about longitudinal axis. Pitching: Angular oscillation of vehicle about transverse axis. Yawing: Angular oscillation of vehicle about vertical axis.

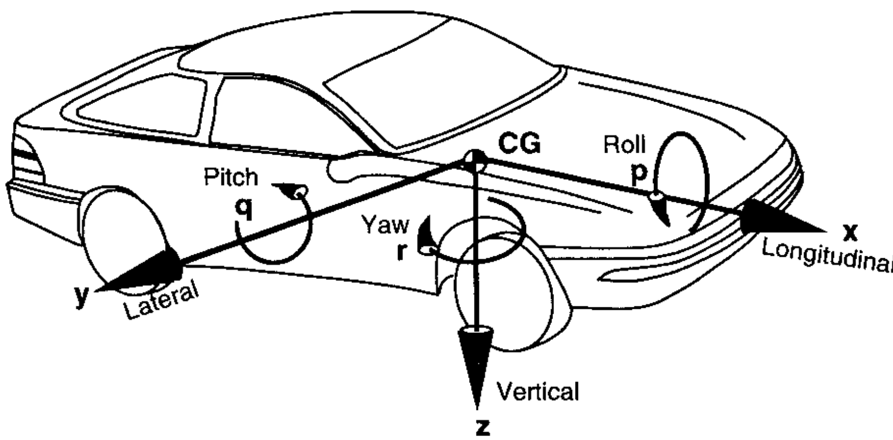


Fig. 1.4 SAE Vehicle Axis System.

Figure 18: Vehicle axis system

The source of vibration of the vehicle may be
due to The road roughness
The unbalance of the engine
The whirling of shafts
The cam forces
Torsional fluctuations

What is Modeling and simulation?

- Writing the equation corresponding to a physical system.
- This equation may be algebraic or differential equation.
- Writing the program and then fed in to the computer.

- Solving the problem by use of computer is called simulation

Model of an automobile used in vehicle dynamics analysis.

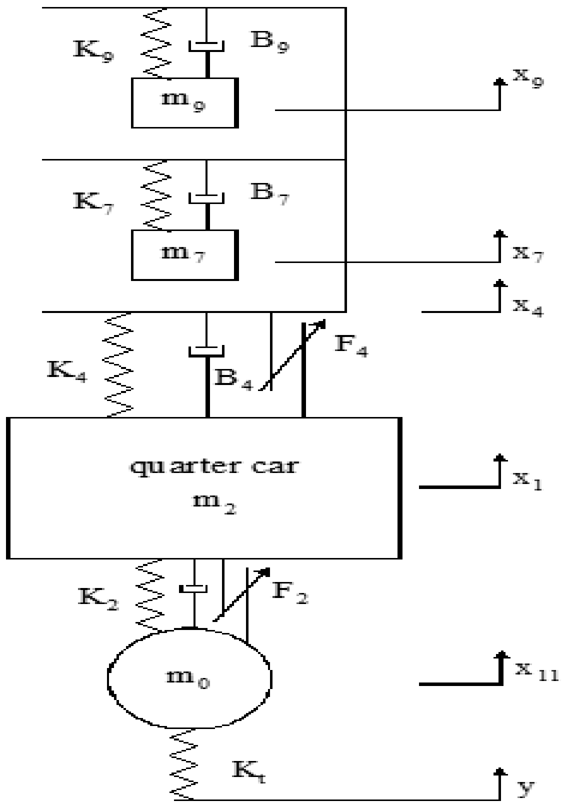


Figure 1 Model of vehicle and driver

Figure 19: Model of an automobile

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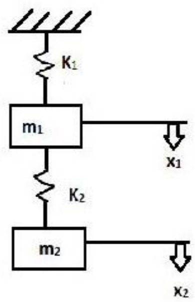
UNIT II APPLIED NUMERICAL METHODS

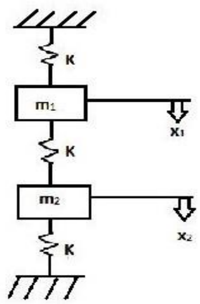
UNIT 2 APPLIED NUMERICAL METHODS

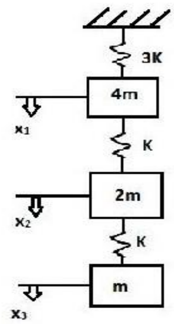
Close coupled and far coupled systems, determination of mass and stiffness matrices- eigen value problems, orthogonality of mode shapes, modal analysis, approximate methods for determining functional frequency. Dunkerley's lower bound, Reyleigh upper bound, Holzer methods for closed coupled system and branched system.

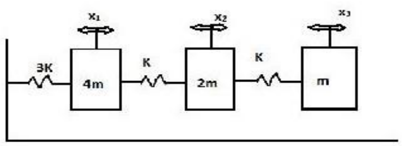
MULTIPLE DEGREES OF FREEDOM (MDOF):

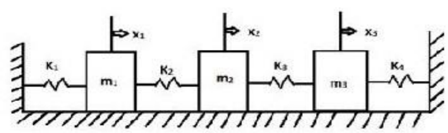
- A system having more than one degree of freedom system. Various methods are employed to determine the natural frequencies, mode shapes.











PRINCIPAL MODE OF VIBRATION OR NORMAL MODE OF VIBRATION:

- **When the** mass of a system are oscillating in such a manner that they reach maximum amplitude simultaneously and pass their equilibrium points simultaneously or all the moving parts of the system are oscillating in the same frequency and phase, such mode of vibration is called **principal mode of vibration**
- If at the principle mode of vibration, the amplitude of one of the masses is considered equal to unity, the mode of vibration is called **normal mode of vibration**
- Normal mode vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed
- In case of two degree of freedom system, mass will vibrate in two different modes
    - First principal mode and second principal mode

ORTHOGONALITY PRINCIPLE:

- The principal mode or normal modes of vibration for system having two or more degrees of freedom are orthogonal. This is known as **Orthogonality Principle**
- It is an important property while finding the natural frequency.
- It states that the principal nodes are orthogonal to each other. In other words, expansion theorem applied to vibration problems indicates that any general motion (x) for n masses

may be broke into components each of which corresponds to a principal node. This forms the basis of obtaining the response of vibration systems and is called modal analysis.

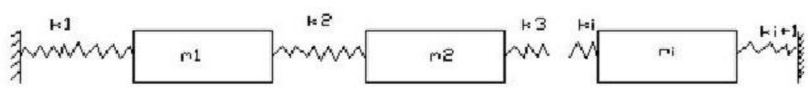
Types of coupling:

- n-degree of freedom system requires **n-**independent coordinates to specify the system completely at any instant.
- When these coordinates are independent of each other and equal in number of degree of freedom of the system, they are called as **generalized coordinates**
- To represent the motion of system one may use a number of generalized coordinate system. While using these coordinates the mass and stiffness matrices may be coupled or uncoupled.
- When the mass matrix is coupled, the system is said to be **dynamically coupled** and the stiffness matrix is coupled, the system is known as **statically coupled**.
- Static coupling due to static displacement. It contains terms as functions of coordinates like spring force coupling
- Dynamic coupling due to inertia force. It contains coupling terms as function of time derivatives of coordinates, like inertia coupling, damping.

PRINCIPAL COORDINATES:

An n degree freedom system requires 'n' independent coordinates and there will be 'n' number of differential equation of motion. It is always possible to fine a particular set of coordinates such that each equation of motion contains only one unknown quantity. Then the equation of motion can be solved independently and the unknown quantity can be found out. Such a particular set of coordinates is called **Principal Coordinates:**

CLOSED COUPLED SYSTEM:



In fig, an n degree translational and rotational close coupled system is given. For free vibration, the following equations can be written:

$$x=e$$
$$m_1\ddot{e}_1+K_1e_1+K_2(e_1-e_2)=0$$
$$m_1\ddot{e}_2+K_2(e_2-e_1)+K_3(e_2-e_3)=0$$

$$m_i\ddot{e}_i+K_i(e_i-e_{i-1})+K_{i+1}(e_i-e_{i+1})=0$$

We can ignore damping for free vibration, as the natural frequencies are not significantly affected by presence of damping. The above equations can also be used directly for torsion with appropriate alternation.

$$[M]\{\ddot{e}\}+[K]\{e\}=0$$

[M] is a square mass matrix
[K] is a stiffness matrix

In above equation, [K]{e} defines local force under static condition. The method of writing equation in this form called stiffness method using the displacements also called displacement method. In fact, for close coupled systems, the finite element method yields identical equations as above.

For free vibrations, the above solutions of can written as

$$\{x\}=\{X\} \cos \omega t$$

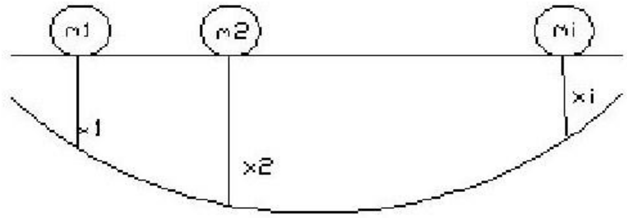
Where {X} represents the amplitudes of all the masses and ω is the natural frequency. These above equations reduce to

$$\{[K]- \omega 2[M]\}\{X\} = 0$$

The above equations known as Eigen value problem in matrix algebra and it can be solved with the aid of computer for a fairly large no of masses. ω2 is called as the characteristic value of equation.

There will be **n** such values of an **n** degree system. For each of the Eigen value, there exists a corresponding Eigen vector{x}, also called as a characteristic vector. This Eigen vector will represent the mode shape for a given frequency of the system

FAR COUPLED SYSTEM:



In fig an **n** degree far coupled system is shown with masses m1,m2,….,mi,at station 1,2,…..,i respectively. For a freely vibrating beam,the only external load is the inertia load due to masses m1,m2….. Following the influence coefficient procedure discussed two degree of freedom, we can write

$$x=e$$

$$\alpha_{11}m_1\ddot{e}_1+ \alpha_{12}m_2\ddot{e}_2+\ldots\ldots\ldots.+ \alpha_{1i}m_i\ddot{e}_i+e_1=0$$
$$\alpha_{21}m_1\ddot{e}_1+ \alpha_{22}m_2\ddot{e}_2+\ldots\ldots\ldots.+ \alpha_{2i}m_i\ddot{e}_i+e_2=0$$

$$\alpha_{i1}m_1\ddot{e}_1+ \alpha_{i2}m_2\ddot{e}_2+\ldots\ldots\ldots.+ \alpha_{ii}m_i\ddot{e}_i+e_i=0$$

In arriving at the above equations, the forces are treated as unknowns and expressed in terms of flexibility factor or influences coefficients, hence this approach is called force or flexibility method, contrary to the method adopted for close coupled systems where displacement are considered unknowns and the stiffness matrix setup.
Denoting [α] as the influences coefficient matrix, the above equation can be written as

$$[D]\ \{\ddot{e}\} + [I]\ \{e\} = 0$$

Or

$$\{\ddot{e}\}+ [\alpha]^{-1}\{e\} = 0$$

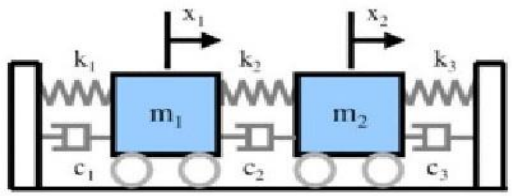
[D] is the dynamic matrix
For a four degree of freedom, the natural frequencies are obtained by setting the determinant equation to zero.

$$\left[[D] - \frac{1}{p^2}[I]\right]\{X\} = 0$$

DETERMINATION OF MASS & STIFFNESS MATRIX- EIGEN VALUE PROBLEMS

Multiple degrees of freedom systems and mode shapes

- The simple mass–spring damper model is the foundation of vibration analysis, but what about more complex systems? The mass–spring–damper model described above is called a single degree of freedom (SDOF) model since the mass is assumed to only move up and down. In the case of more complex systems the system must be discretized into more masses which are allowed to move in more than one direction – adding degrees of freedom.
- The major concepts of multiple degrees of freedom (MDOF) can be understood by looking at just a 2 degree of freedom model as shown in the figure.



The equations of motion of the 2DOF system are found to be:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1,$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2.$$

This can be rewritten in matrix format

:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}.$$

A more compact form of this matrix equation can be written as:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f\}$$

where $[M]$, $[C]$, and $[K]$ are symmetric matrices referred respectively as the mass, damping, and stiffness matrices. The matrices are NxN square matrices where N is the number of degrees of freedom of the system.

In the following analysis involves the case where there is no damping and no applied forces (i.e. free vibration). The solution of a viscously damped system is somewhat more complicated.[7]

$$[M]\{\ddot{x}\} + [K]\{x\} = 0.$$

This differential equation can be solved by assuming the following type of solution:

$$\{x\} = \{X\}e^{i\omega t}.$$

Note: Using the exponential solution of $\{X\}e^{i\omega t}$ is a mathematical trick used to solve linear differential equations. Using Euler's formula and taking only the real part of the solution it is the same cosine solution for the 1 DOF system. The exponential solution is only used because it is easier to manipulate mathematically.

The equation then becomes:

$$\left[-\omega^2[M] + [K]\right]\{X\}e^{i\omega t} = 0.$$

Since $e^{i\omega t}$ cannot equal zero the equation reduces to the following.

$$\left[[K] - \omega^2[M]\right]\{X\} = 0.$$

EIGENVALUE PROBLEM

This is referred to an eigenvalue problem in mathematics and can be put in the standard format by pre-multiplying the equation by $[M]^{-1}$

$$\left[[M]^{-1}[K] - \omega^2[M]^{-1}[M]\right]\{X\} = 0$$

and if: $[M]^{-1}[K] = [A]$ and $\lambda = \omega^2$

$$\left[[A] - \lambda[I]\right]\{X\} = 0.$$

- The solution to the problem results in N **eigenvalues** (i.e. $\omega_1^2, \omega_2^2, \cdots \omega_N^2$), where N corresponds to the number of degrees of freedom.
- The eigenvalues provide the natural frequencies of the system. When these eigenvalues are substituted back into the original set of equations, the values of $\{X\}$ that correspond to each eigenvalue are called the **Eigenvectors**.
- These eigenvectors represent the mode shapes of the system. The solution of an eigenvalue problem can be quite cumbersome (especially for problems with many degrees of freedom), but fortunately most math analysis programs have eigenvalue routines.

The eigenvalues and eigenvectors are often written in the following matrix format and describe the model of the system:

$$\left[\diagdown \omega_r^2 \diagdown\right] = \begin{bmatrix} \omega_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_N^2 \end{bmatrix}$$ and $[\Psi] = \left[\{\psi_1\}\{\psi_2\}\cdots\{\psi_N\}\right].$

Eigen value problem:

ORTHOGONALITY OF MODE SHAPES:

The mode shapes of a dynamic system exhibit orthogonality property, which is very useful in simplifying the analysis for forced and transient vibrations.

In general, the differential equation of motion for any vibrating system can be written in matrix form as follow $[M]\{\ddot{x}\} + [K]\{x\} = 0.$

Let [M] and [K] represent the mass and stiffness properties of a system,
Assuming harmonic motion i.e

$$x = X \sin \omega t$$

$\lambda = \omega^2$

The equation for the *i*th mode be

$$KX_i = \lambda_i MX_i$$

Premultiplying by the transpose of mode j,

$$X_j{}'KX_i = X_j{}'\lambda_i MX_i = \lambda_i(X_j MX_i) \qquad (1)$$

Now start with the equation for the jth mode and premultiplying by X_i' to obtain,

$$X_i{}'KX_j = \lambda_i(X_i{}'MX_j) \qquad (2)$$

Since *K* and *M* are symmetric matrices

$$X_j{}'MX_i = X_i{}'MX_j \text{ and } X_j{}'KX_i = X_i{}'KX_j$$

Subtracting (2) from (1)

$$(\lambda_i - \lambda_j)X_2{}'MX = 0$$

As $\lambda_i \neq \lambda_j$

$$X_2{}'MX_j = 0 \text{ and } X_2{}'KX_j = 0$$

The above equation shows the orthogonal character of the normal modes.

If i=j,

$$X_2{}'MX_i = M_i$$

$$X_2 KX_i = K_i$$

M*i* and Ki are known as the generalized mass and generalized stiffness of the ith mode

Modal analysis:

- An engineering system, when given an initial disturbance and allowed to execute free vibrations without a subsequent forcing excitation, will tend to do so at a particular "preferred" frequency and maintaining a particular "preferred" geometric shape.
- This frequency is termed a "natural frequency" of the system, and the corresponding shape (or motion ratio) of the moving parts of the system is termed a "mode shape." Any arbitrary motion of a vibrating system can be represented in terms of its natural frequencies and mode shapes.
- The subject of modal analysis primarily concerns determination of natural frequencies and mode shapes of a dynamic system.
- Once the modes are determined, they can be used in understanding the dynamic nature of the systems, and also in design and control.
- Modal analysis is extremely important in vibration engineering. Natural frequencies and mode shapes of a vibrating system can be determined experimentally through procedures of modal testing.
- The subject of modal testing, experimental modeling (or model identification), and associated analysis and design is known as *experimental modal analysis*

Dunkerlyes equation.

It relates the fundamental frequency of a composite system to the frequencies of its component parts. It is based on the fact that modal frequencies of most system for higher modes are high wth respect to their fundamental frequency. It is an approximate equation and can be derived from an algebraic rule.

Damped Natural frequency.

The damped natural frequency is that frequency of free vibration of a damped linear system. The free vibration of a damped system may be considered periodic in the limited sense that the time interval between zero crossings in the same direction is constant. Even though successive amplitudes decrease progressively.

Dunkerleys lower bound method to determine the frequency of a system.

Consider the eigen value problem in the form of for n degree of freedom system , the frequency equation will be

$$Z^n - C_{n-1} Z^{n-1} + \ldots + C_0 = 0 \tag{1.1}$$

Where C_{n-1} coefficient from theory of questions represents sum of all the roots of the equation from (6.24), we can write

$$C_{n-1} = \alpha_{11}m_1 + \alpha_{22}m_2 + \ldots + \alpha_{nn} m_r$$

$$= 1/p_1^2 + p_2^2 + \ldots + 1/p_n^2$$

(1.2)

Since influence coefficient $\alpha_{ii} = 1/K_{ii}$ we define

$$p_n^2 = 1/ \alpha_{11}m_1 \tag{1.3}$$

Where p_{ii} represents the natural frequency of the system with only the i^{th} mass considered

Equation (1.2) now becomes

$$1/p_1^2 + p_2^2 + \ldots + 1/p_n^2 = 1/p_{11}^2 + p_{22}^2 + \ldots + 1/p_{nn}^2$$

(1.4)

Since $p_1 < p_2 < \ldots < p_n$, we can write the above as

$$1/p_n^2 = \sum 1/ p_{ii}^2 \tag{1.5}$$

The above is Dunkereley's formula and because of the removal of p_2 ,……….., p_n terms of the left hand side of equation (1.4), p_1 estimated by (1.5) is always less than the exact value . The simplicity of the method lies in the fact that p_1 can be estimated by considering several single freedom systems with masses m1, m2 etc., considered individually, thus reducing the multi degree of freedom system calculations to single degree of freedom system calculations.

Rayleighs upper bound method to determine the frequency of a system.

Consider the multi degree of freedom system with [M] and [K] representing its mass and stiffness matrices as in equation (6.7). Let [X] be a modal vector (as in equation (6.11) with its column representing ith mode shape corresponding to its natural frequency pi) and for the case of harmonic motion with a frequency ω, the maximum kinetic and potential energies are

$$\check{T} = \tfrac{1}{2}\,\omega^2 \{X\}^T \{M\} \{X\}$$
$$\check{U} = \tfrac{1}{2} \{X\}^T \{K\} \{X\} \tag{1.6}$$

So,

$$\omega^2 = \{X\}^T \{K\} \{X\}/\{X\}^T \{M\} \{X\} \tag{1.7}$$

The above equation is known as Rayleigh's quotient. If ω is a natural frequency and {X} is corresponding modal vector, (1.7) will be exactly satisfied. However, neither of them is known at this stage of calculations. Let us assume a modal vector {X} consistent with the kinematics boundary conditions of the system. As in modal expansion, let {X} be expressed in terms of orthonormal modal vectors

$$\{\overline{X}\} = \overline{\{X^1\}} + \overline{C}_2\{X^2\} \overline{+} C_3\{X^3\}+\ldots \tag{1.8}$$

Substituting the above for {X} in (1.7) and noting that

$$\omega^2 = p_1^2 + C_2^2\, p_2^2 + \ldots /1+ C_2^{2+\ldots}$$

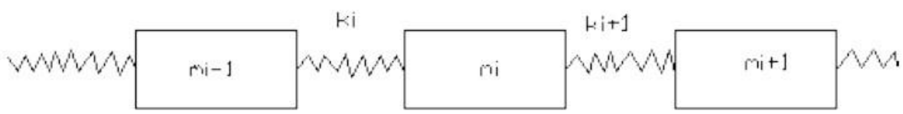
If $\overline{\{X\}}$ is close $\overline{\text{to}}$ $\{X^1\}$, then $C_2 \ll 1$, $C_3 < C_2$, then

$$\omega^2 = p_1^2\,(1+ C_2^2\, p_2^2/ p_1^2+\ldots)$$

$$= p_1^2 \tag{1.9}$$

ω^2 determined from (1.9) is always greater than the exact value p_1^2. Since Dunkerleys lower bound method gives a lower bound value, the exact frequency of a system lies in between Dunkereley's and Rayleighs approximations.

Holzer method for closed coupled system.



Consider the close coupled system of fig. again, the i^{th} mass and neighboring elements of which are shown. We measure the displacement, velocity and acceleration positive along the outward normal. The displacement X and the force F will define the state vector.

$$\{S\} = \begin{matrix} \{X\} \\ \{F\} \end{matrix}$$

We use a suffix to denote the station number and a superscript R or L to denote the quantities to the right or left of a station respectively. The equation of motion for i^{th} mass is

$$m_i\overline{x_i} = Fi^R - Fi^L$$

The displacement of mass m_i is

$$Xi^R = Xi^L = Xi$$

We combine the equations

$$\{S\}i^R = [P]i\ \{S\}\ i^L$$

[P] is the point matrix which defines the transfer function to obtain the state vector to the right of a station in terms of the state vector to the left of a station.

The point matrix is a function of the mass of the station and the harmonic frequency ω. For an assumed value of ω, the point matrices for all station can be set up.

Now we consider the force field of the spring Ki and observe that

$$\{S\}i^L = [F]_i\ \{S\}_{i-1}{}^R$$

[F] is the field matrix which define the transfer function across field. The field matrix is a function of the stiffness of the system only and can be set up for all stations i.

$$\{S\}i^R = [T]_i\ \{S\}_{i-1}{}^R$$

[T] is the transfer matrix.

It is important to maintain the order of matrix multiplication and the station numbering in using the above transfer matrix. We can use equation for transfer of state vector from station successively to obtain the overall transfer matrix of the systems.

$\{S\}_1^R = [T]_1 \{S\}_0$

$\{S\}_{n+1} = [U] \{S\}_0$

[U] is the overall transfer matrix of the system.

Procedure to find determine a natural frequency, we adopt the following procedure.

1. Assume a value of ω2 representing the desired natural frequency. This may be obtained by making a crude model with few stations or by experience.

2. Set up transfer matrices as in equation for all stations. At the end points determine the required point or field matrices.

3. Determine the overall transfer matrix as in equation.

4. Change ω^2 by a suitable increment and repeat steps 1 to 3.

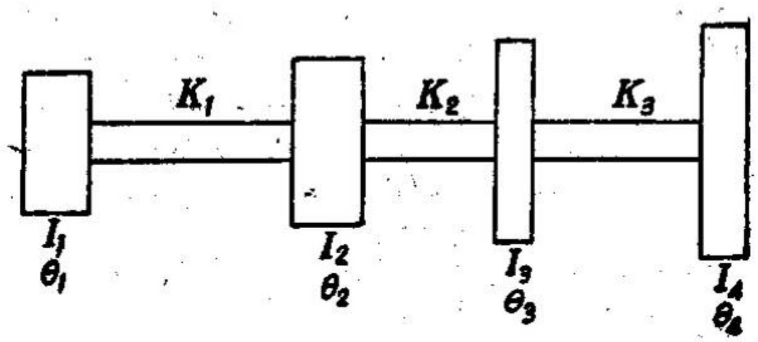
5. Plot u_{12} vs ω^2 and find the value ω^2 for which u_{12} is zero. This value of ω^2 is a natural frequency.

Holzer method for branched system.

In several mechanical systems, like ship propulsion systems, strip steel mill stands, machine tool drives etc.., there may be one or two branch points, as the employ one or two drivers driving one or two driven members.

The Holzer's Method

Consider the system shown in fig.
Equations of motion are



$$I_1\theta + K_1(\theta_1 - \theta_2) = 0$$

$$I_2\theta_2 + K_1(\theta_2 - \theta_1) - K_2(\theta_3 - \theta_2) = 0 \qquad \rightarrow (1)$$

$$I_3\theta_3 + K_2(\theta_3 - \theta_2) - K_3(\theta_4 - \theta_3) = 0$$

$$I_4\theta_4 + K_3(\theta_4 - \theta_3) = 0$$

Assuming

$$\theta_1 = \gamma_1 \cos\ \omega_n t$$

$$\theta_2 = \gamma_2 \cos\ \omega_n t \qquad \rightarrow (2)$$

$$\theta_3 = \gamma_3 \cos\ \omega_n t$$

$$\theta_4 = \gamma_4 \cos\ \omega_n t$$

and substituting in equation (1) and rearranging, we get

$$-I_1\omega_n 2\gamma_1 = K_1(\gamma_2 - \gamma_1)$$

$$-I_2\omega_n 2\gamma_2 = -K_1(\gamma_2 - \gamma_1) + K_2(\gamma_3 - \gamma_2) \qquad \rightarrow (3)$$

$$-I_3\omega_n 2\gamma_3 = -K_2(\gamma_3 - \gamma_2) + K_3(\gamma_4 - \gamma_3)$$

$$-I_4\omega_n 2\gamma_4 = -K_3(\gamma_4 - \gamma_3)$$

Adding these, equation (3) results in the right hand side being zero.

So in general

$$\sum I\omega_n 2\gamma_n = 0 \quad \rightarrow (4)$$

Where summation is for all the masses.
This means that we can find the natural frequency by trial till equation (4) is satisfied. This is the basis Holzer's Method.

Further assuming that γ_1 = 1 radian (since we are interested only in the relative amplitudes) we get

$$\gamma_2 = \gamma_1 - \frac{I_1\omega_n 2\gamma_1}{K_1}$$

$$\gamma_3 = \gamma_2 - \frac{I_1\omega_n 2\gamma_1 + I_2\omega_n 2\gamma_2}{K_2} \qquad \rightarrow (5)$$

$$\gamma_4 = \gamma_3 - \frac{I_1\omega_n 2\gamma_1 + I_2\omega_n 2\gamma_2 + I_3\omega_n 2\gamma_3}{K_a}$$

Or in general,

$$\gamma_4 = \left(\frac{1}{K_i - 1}\right)\left[K_{i-1}\gamma_{i-1} - \omega_n 2 \sum_{1}^{i-1} I\,\gamma_3\right] \quad \rightarrow (6)$$

The Holzer's method then consists in assuming value for the natural frequency and displacement of one of the rotors. Equation (6) then may be used to find the displacements of any other rotor and the sum of the inertia forces. If the system is free at the ends, equation (4) must hold. If it is fixed at same point, equation (6) which can be used to obtain the displacement of that point should yield zero displacement. If it is not zero, another trial must be made with another frequency. Thus a graph may be plotted for displacement vs assumed frequency. The frequency for zero displacement is then the natural frequency. The mode shapes may then be obtained with the help of equation (6).

The method is equally applicable to translational systems.

Holzer's method can be applied to branched systems. Any end rotor could be given a unit displacement to state with without affecting the final result. All amplitudes and moments have to be proportional to this initially assumed displacements. It may be further seen that the joint must be equal, and that the total; moment at the joint including its inertia moment must equal zero.

A single degree of freedom spring mass system has a natural frequency of 10 cycles per second. Another single degree of freedom spring mass system is attached to it. The latter had a natural frequency of 20 cycles/second. What is the approximate fundamental frequency of composite system?

Solution : **from Dunkerley's equation**

$$\frac{1}{\omega_{1n^2}} \approx \frac{1}{\omega_{11^2}} + \frac{1}{\omega_{22^2}}$$

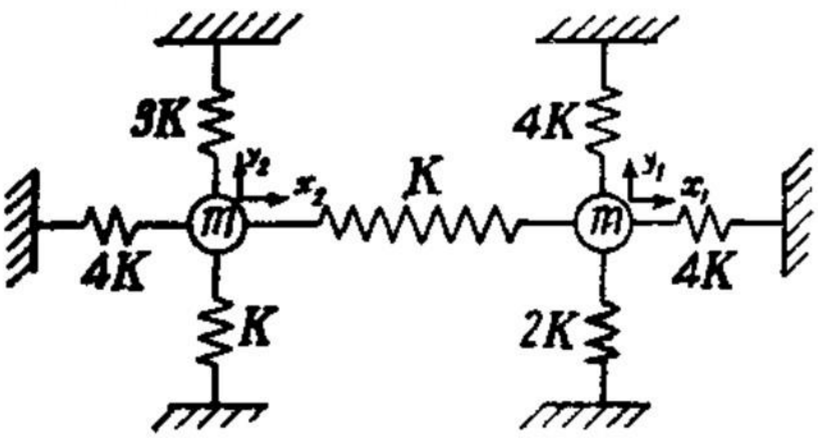
$$= \frac{1}{100} + \frac{1}{400}$$

$$= \frac{5}{400} = \frac{1}{80}$$

or $\omega_{1n^2} = 80$

therefore $\omega = \sqrt{80} \approx 8.95$ cycle / second.

6. Find the principal modes of the system shown in fig.



Solution :

Inertia or mass matrix is

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

and the stiffness matrix is K

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

So dynamic matrix C is

$$[m]^{-1}[K]$$

$$= \frac{K}{m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{K}{m}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

Eigen values of this matrix are given as

$$\left(\lambda - \frac{5K}{m}\right)^2 - \left(\frac{K}{m}\right)^2 = 0$$

$$\text{or} \qquad \begin{vmatrix} \lambda\text{-}\frac{5K}{m} & \frac{K}{m} \\ \frac{K}{m} & \lambda - \frac{5K}{m} \end{vmatrix} = 0 \qquad \rightarrow (2)$$

$$\text{so} \qquad \lambda = \frac{6K}{m} \text{ or } \frac{4K}{m}$$

Hence

$$\omega_n = \sqrt{\frac{6K}{m}} \text{ and } \sqrt{\frac{4K}{m}}$$

The adjoint of (2) is

$$\begin{bmatrix} \lambda - \frac{5K}{m} & -\frac{K}{m} \\ -\frac{K}{m} & \lambda - \frac{5K}{m} \end{bmatrix}$$

and hence the two principal modes are

$$\begin{Bmatrix} - \\ -1 \end{Bmatrix} \text{ and } \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

For the two cases,

For displacements in vertical direction

$$my_1+6Ky_1 = 0$$

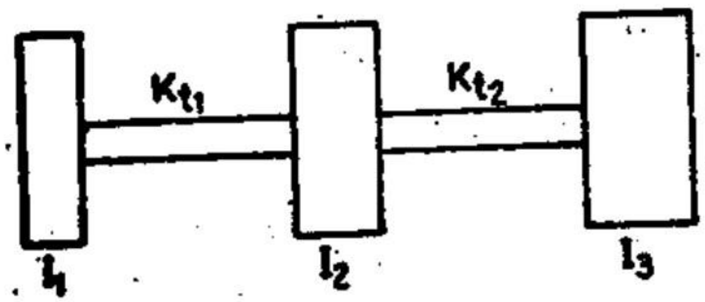
and $$my_2+4Ky_2 = 0 \qquad \rightarrow(3)$$

and hence the natural frequencies are

$$\sqrt{\frac{6K}{m}} \text{ and } \sqrt{\frac{4K}{m}}$$

in vertical direction. This is a system which has same natural frequencies in both the directions.

Find one natural frequency of the system shown in fig. (a) by the Holzer's Method.



$$I_1 = I_2 = I_a$$

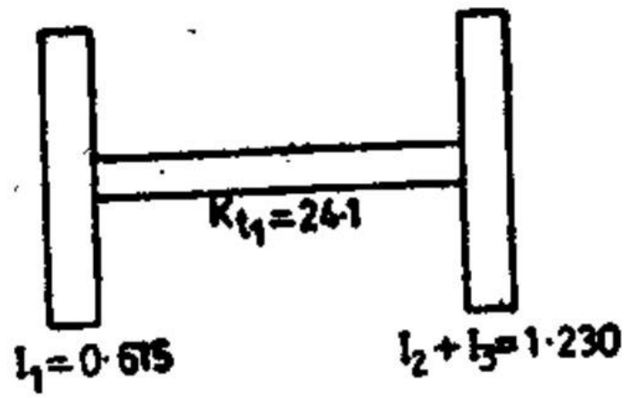
$$= 0.615 \text{ kg cm/sec}^2$$

$$K_{t1} = 24.1 \text{ kg cm/radian}$$

$$K_{t2} = 25.84 \text{ kg cm/radian}$$

Solution : Let us first find approximate natural frequency of the system. For this we will be grouping together discs that have shafts with high relative stiffness between them.

Since here K_{i1} is comparatively greater than K_{t1}, we group together discs 2 and 3, thus the system reduces to a two degree system shown in fig.



$$\omega_n = \sqrt{K_{t1}\left(\frac{1}{I_1} + \frac{1}{I_2}\right)}$$

$$= \sqrt{24.1\left(\frac{1}{0.615} + \frac{1}{1.230}\right)}$$

$$= \sqrt{59}$$

$$\omega^2 = 59$$

So let us start with $\omega^2 = 59$

Trial 1
$\omega^2 = 59$

S.No	I	γ	$I\gamma\omega_{n^2}$	$\sum I\gamma\omega_{n^2}$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1	36.30	36.30	24.1	1.535
2	0.615	-0.535	-18.15	17.15	25.84	0.664
3	0.615	-1.199	-42.5	-25.35		

The external Torque $\sum I\gamma\omega^2$ should be zero, so choose next approximation so as to achieve this.

Trial 2
$\omega_n^2 = 40.3$

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	24.8	24.8	24.1	1.03

2	0.615	-0.03	-0.745	24.055	24.84	0.93
3	0.615	-0.96	-23.1	0.955		

Trial 3
$\omega^2 = 42$

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	25.81	25.84	24.1	1.072
2	0.615	-0.072	-1.84	23.94	25.84	0.925
3	0.615	-0.997	-25.7	-1.76		

Trial 4
$\omega^2 = 41.0$

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	25.2	25.2	24.1	1.04
2	0.615	-0.047	-1.185	24.015	25.84	0.93
3	0.615	-0.977	-24.6	-0.585		

Trial 5
$\omega^2 = 40.8$

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	25.1	25.1	24.1	1.042
2	0.615	-0.042	-1.055	24.045	25.81	0.93
3	0.615	-0.972	-24.39	-0.345		

Trial 6
$\omega^2 = 40.5$

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	24.9	24.9	24.1	1.035
2	0.615	-0.035	-0.872	24.028	25.84	0.93
3	0.615	-0.965	-24.028	0		

Hence ω^2 40.5

So exact natural frequency of the system is

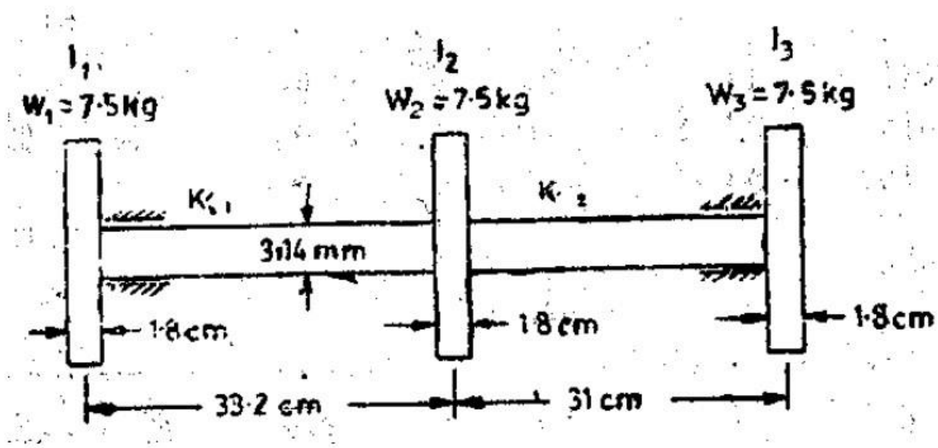
$$\omega_{n^2} = 40.5$$

$$\text{i.e} \quad \omega_n = \sqrt{40.5}$$

$$=6.37 \text{ rad/sec.}$$

Find the natural frequencies and mode shapes of the torsional three rotor system show in fig. diameter of each rotor is 25.4 cm. weight of each rotor and its thickness are shown in the diagram. G for the material = 8.5×10^5 kg/cm^2.

Solution : If θ_1, θ_2, θ_3 are the displacements of the three rotors, the equations of motion of the three rotors are,



$$I_1\theta_1 = -K_{t1}(\theta_1 - \theta_2)$$

$$I_2\theta_2 = -K_{t1}(\theta_2 - \theta_1) - K_{t2}(\theta_2 - \theta_3) \qquad \rightarrow (1)$$

$$I_a\theta_3 = -K_{t2}(\theta_3 - \theta_2)$$

Assuming simple harmonic motion with γ_1, γ_2, γ_3, as amplitudes and frequency ω_n.

$$\theta_1 = \gamma_1 \text{ Sin } \omega_{nt} \qquad \theta_1 = -\gamma_1\omega_{n^2} \text{ Sin } \omega_{nt}$$

$$\theta_2 = \gamma_2 \text{ Sin } \omega_{nt} \qquad \theta_2 = -\gamma_2\omega_{n^2} \text{ Sin } \omega_{nt} \qquad \rightarrow (2)$$

$$\theta_3 = \gamma_3 \text{ Sin } \omega_{nt} \qquad \theta_3 = -\gamma_3\omega_{n^2} \text{ Sin } \omega_{nt}$$

From (1) and (2)

$$\text{-}I_1\omega_{n^2}\gamma_1 + K_{t1}(\gamma_1 - \gamma_2)$$

or $$\gamma_1\{K_{t1} - I_1\omega_n\} - K_{t1}\theta_2 = 0 \qquad \rightarrow (a)$$

$$\text{-}I_2\omega n^2\gamma_2 + K_{t1}(\gamma_2 - \gamma_1) + K_{t2}(\gamma_2 - \gamma_3) = 0$$

or $$K_{t1}\gamma_1 + \{K_{t1} + K_{t2} - I_2\omega_{n^2}\}\gamma_2 - K_{t2}\gamma_3 = 0 \qquad \rightarrow (b)$$

and $$\text{-}I_3\omega_{n^2}\gamma_3 + K_{t2}(\gamma_3 - \gamma_2) = 0$$

or $$\text{-}\gamma_2 K_{t2} + \gamma_3(K_{t2} - I_3\omega_{n^3}) = 0 \qquad \rightarrow (c)$$

This is a set of homogeneous equations. It will have a non-zero solution only if determinant formed out of coefficients of γ_1, γ_2 and γ_3 vanishers.

Or

$$\begin{vmatrix} (K_{t1} - I_1\omega_{n^2}) & -K_{t1} & 0 \\ K_{t1} & (K_{t1} + K_{t2} - I_2\omega_{n^2}) & -K_{t2} \\ 0 & -K_{t2} & (K_{t2} - I_3\omega_{n^2} \end{vmatrix} = 0$$

on expanding

$$\left(K_{t1} - I_1\omega_{n^2}\right)\left\{\left(K_{t1} + K_{t2} - I_2\omega_{n^2}\right)\left(k_{t2} - I_3\omega_{n^2}\right) - K_{t2}\right\} + K_{t1}\left\{-K_{t1}\left(K_{t2} - I_3\omega_{n^2}\right)\right\}$$

or

$$\omega_{n^2}\left\{w_{n^4} - \left[K_{t1}\left(\frac{1}{I_1} + \frac{1}{I_2}\right) + K_{t2}\left(\frac{1}{I_2} + \frac{1}{I_3}\right)\right]\omega^3 + K_{t1}K_{t2}\frac{I_1 + I_2 + I_3}{I_1 I_2 I_3}\right\} = 0 \qquad \rightarrow (3)$$

Mode Shapes

The amplitude ratio of principal modes of vibration can be obtained from equation I and are found to be

$$\frac{\gamma_1}{\gamma_2} = \frac{K_{t1}}{K_{t1} - I_1\omega_{n^2}}$$

and $$\frac{\gamma_2}{\gamma_3} = \frac{K_{t2} - I_3\omega_{n^2}}{K_{t2}}$$

When ω_n^2 is zero, amplitude ratios of the discs are

$$\frac{\gamma_1}{\gamma_2} = \frac{\gamma_2}{\gamma_3} = 1$$

This indicates that whole assembly rotates as a rigid body when

$$\omega_n = 0$$

Since one of the frequencies of this system is zero, this system is a semidefinite system.

Equation (3) is cubic in ω_n^2. One root may be $\omega_{n1}^2 = 0$. The two other natural frequencies can be obtained by solving fourth power equation in ω_n in equation (3)

$$\omega_{n2^2},\ \omega_{n3^2} = \frac{1}{2}\left[\left(\frac{K_{t1}}{I_1} + \frac{K_{t1}+K_{t2}}{I_2} + \frac{K_{t2}}{I_3}\right)\right]$$

$$\pm\sqrt{\left[\left(\frac{K_{t1}}{I_1} + \frac{K_{t1}+K_{t2}}{I_2} + \frac{K_{t2}}{I_3}\right) - 4K_{t1}K_{t2}\frac{I_1+I_2+I_3}{I_1I_2I_3}\right]}$$

$$I_1 = \frac{1}{2}\ mr^2 = \frac{1}{2}\ \frac{w}{g}\left(\frac{25.4}{4}\right)^2 = \frac{1}{2}.\frac{7.5}{981}(12.7)^2$$

$$=0.616\ \text{kg cm sec}^2$$

$$K_{t1} = \frac{\pi d_4 G}{32 l_1} = \frac{8.5\times10^4\times\pi\times(0.314)^4}{32\times33.2} = 24.7\ \text{kg cm.rad}$$

$$K_{t2} = \frac{8.5\times10^5\times\pi\times\left(\frac{3.14}{10}\right)^4}{}$$

$$=26.5\ \text{kg cm/rad}$$

$$\omega_{n_2^2}, \omega_{n_3^2} = \frac{1}{2}\left[\left(\frac{24.7}{0.616} + \frac{24.7 \times 26.5}{0.616} + \frac{26.5}{0.616}\right)\right]$$

$$\pm\sqrt{\left(\frac{24.7}{0.616} + \frac{24.7 + 26.5}{0.616} + \frac{26.5}{0.616}\right)^2 - \frac{4 \times 24.7 \times 26.5(0.616 \times 3)}{0.616 \times 0.616 \times 0.616}}$$

$$= \frac{1}{2}[166 \pm 83.8]$$

$$\omega_{r2^2} = \frac{1}{2}[166 - 83.8]41.4$$

or $\quad \omega_{n2} = \sqrt{41.4} = 6.4$ rad/sec

$$\omega_{n3^2} = \frac{1}{2}(166 + 83.8) = 124.9$$

or $\quad \omega_{n3} = \sqrt{124.9} = 11.2$ rad/sec

a) Mode shape for $\omega_{n1} = 0$

$$\frac{\gamma_1}{\gamma_2} = \frac{K_{t1}}{K_{t1} - I_1\omega_{n1}^2} = \frac{K_{t1}}{K_{t2}} = 1$$

$$\frac{\gamma_2}{\gamma_3} = \frac{K_{t2} - I_3\omega_{n1^2}}{K_{t2}} = \frac{K_{t2}}{K_{t2}} = 1$$

b) Mode shape for $\omega_{n2} = 6.4$ rad/sec

$$\frac{\gamma_1}{\gamma_2} = \frac{24.7}{24.7 - 0.616 \times 41.1} = -38$$

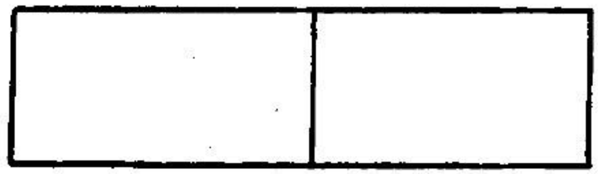
$$\frac{\gamma_2}{\gamma_3} = \frac{26.5 - 0.616 \times 41.1}{26.5} = 0.0434$$

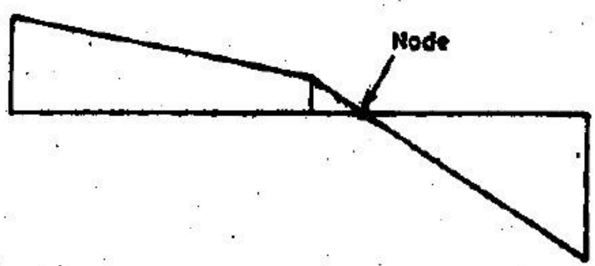
c) Mode shape for $\omega_{n2} = 11.2$ rad/sec

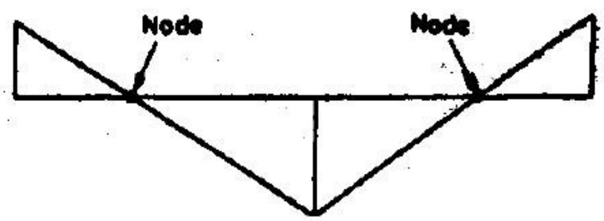
$$\frac{\gamma_1}{\gamma_2} = \frac{24.7}{24.7 - 0.616 \times 124.9} = -0.472$$

$$\frac{\gamma_2}{\gamma_3} = \frac{26.5 - 0.616 \times 124.9}{26.5} = -1.905$$

Mode shapes are plotted in fig.







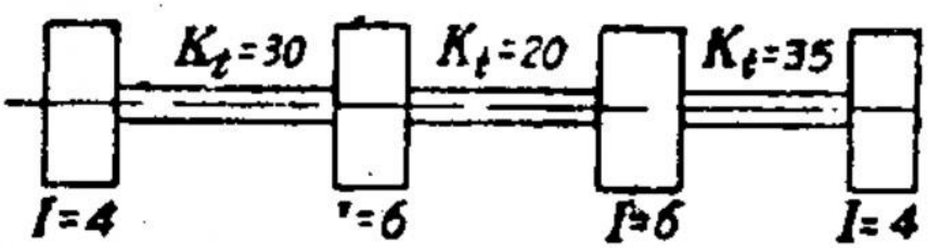
(b)

(c)

(d)

Fig. E-4·16

Natural frequencies of the torsional system shown in fig.



Solution :

$$\gamma_2 = \gamma_1\left(1 - \frac{4\omega_{n^2}}{30}\right)$$

$$= \left(1 - \frac{\omega_{n^2}}{7.5}\right)\gamma_1$$

$$\gamma_3 = \gamma_2 - \frac{4\gamma_1\omega_{n^2} + 6\gamma_2\omega_{n^2}}{20}$$

$$= \left(1 - \frac{\omega_{n^2}}{7.5}\right)\gamma_1 - \frac{4\omega_{n^2} + 6\left(1 - \frac{\omega_{n^2}}{7.5}\right)\gamma_1\omega_{n^2}}{20}$$

$$= \left(1 - \frac{\omega_{n^2}}{7.5} - \frac{\omega_{n^2}}{7.5} + \frac{\omega_{n^4}}{25}\right)\gamma_1$$

$$\gamma_4 = \left(\frac{\omega_{n^2}}{2} + \frac{\omega_{n^4}}{25} - \frac{\omega_{n^2}}{7.5} + 1\right)\gamma_1 - \frac{(10\omega_{n^2} - 8\omega_{n^4})\gamma_1}{35} + \frac{6\omega_{n^2}\left(1 - \frac{\omega_{n^2}}{2} + \frac{\omega_{n^4}}{25} - \frac{\omega_{n^2}}{7}\right)}{35}$$

$$= 0.5 + 0.04\omega_{n^4} - 0.505\omega_{n^2} - 0.0066\omega_{n^6})\gamma_1$$

Let us now construct a Holzer's table

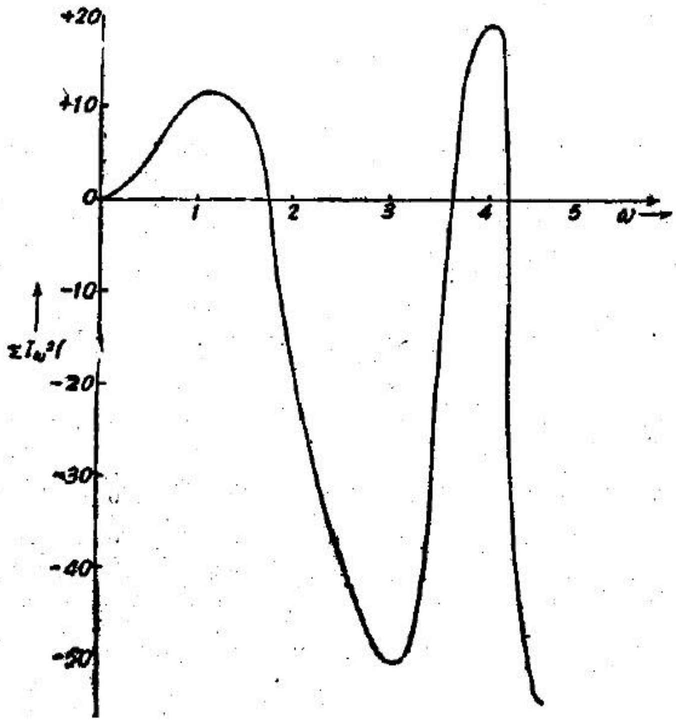
	N_0	I	γ_n	$I\omega^2\gamma_n$	$\Sigma I\omega^2\gamma_n$
ω_n=2.1	1	4	1	17.64	17.64
	2	6	0.413	10.9	28.54
	3	6	-1.012	-26.7	2.84
	4	4	-0.96	-16.9	-1500
ω_n=1	1	4	1	4	4
	2	6	0.867	5.2	9.2
	3	6	0.407	2.442	11.64
	4	4	0.074	0.296	11.94
ω_n=1.5	1	4	1	9	9

	2	6	0.7	9.45	18.45
	3	6	-0.225	-3.04	15.41
	4	4	-0.666	-6.00	9.41
ω_n=1.9	1	4	1	15.24	15.24
	2	6	0.492	11.29	26.49
	3	6	-0.833	-19.1	7.39
	4	4	-1.4	-21.3	-13.97
ω_n=1.7	1	4	1	11.55	11.55
	2	6	0.615	10.70	22.25
	3	6	-0.50	-8.67	13.58
	4	4	-0.888	-10.25	+2.33
ω_n=3	1	4	1	36	36
	2	6	-0.2	-10.8	25.2
	3	6	-1.46	-78.8	-53.6
	4	4	-0.07	2.52	-51.08
ω_n=4	1	4	1	64	64
	2	6	-1.13	-108.5	-44.5
	3	6	1.095	105	60.5
	4	4	-0.63	-44.2	19.3
ω_n=3.8	1	4	1	57.75	57.75
	2	6	-0.925	-80.0	-22.25
	3	6	0.187	16.1	-6.25
	4	4	0.357	20.6	14.35
ω_n=3.6	1	4	1	52	52
	2	6	-0.733	-57.2	-5.2
	3	6	0.473	-36.9	-42.1
	4	4	0.727	37.8	-4.3
ω_n=5	1	4	1	100	100
	2	6	-2.33	-350	-250

	3	6	10.17	1520	1270
	4	4	-26.33	-2613	-1343
ω_n=4.5	1	4	1	81	81
	2	6	-1.7	-206.5	-125
	3	6	4.57	544.0	419.0
	4	4	17.33	-583.0	-164.0

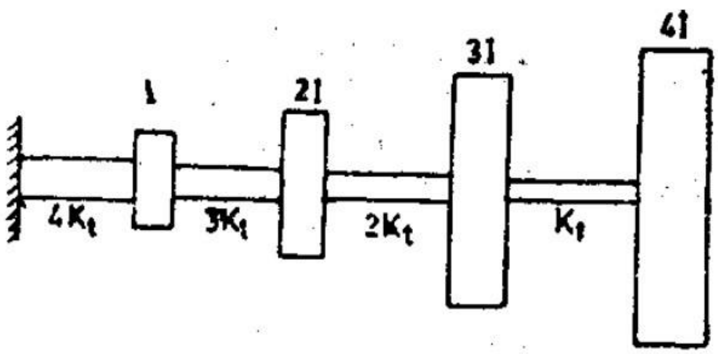
ω_n=4.2	1	4	1	81	81
	2	6	-1.7	-206.5	-125
	3	6	4.57	544.0	419.0
	4	4	-17.33	-583.0	-164.0
ω_n=4.2	1	4	1	70.4	70.4
	2	6	-1.35	-143	-72.6
	3	6	2.28	242	169.4
	4	4	-2.55	-180	-10.6

Now we can plot a graph between ω and $\sum I\omega\gamma$. fig (b) shows this graph. From this natural frequencies re found to be 1.73, 3.64, 4.17.



Using Holzer's method, determine the natural frequencies of the system shown in fig.

K_t = 1 kg-cm/radian
I = 1 kg-cm^2
Both in consistent S.I units.



Solution : In this example we shall use the Holzer's method but the criterion applied to determine the natural frequency will be that when the trial is made with the natural frequency, the displacement at the left hand support will be zero.

One can estimate first natural frequency by Dunkerley's equation but the trials are made without that help.

Different trials are recorded in the following table.

Item	I	$I\omega^2$	γ	$I\omega^2\gamma$	$\Sigma I\omega^2\gamma$	K_t	$\frac{\sum I\omega^{2\gamma}}{K_t}$
Trial with $\omega = 0.2$							
1	4	0.16	1	0.16	0.16	1	0.16
2	3	0.12	0.84	0.101	0.261	2	0.13
3	2	0.08	0.71	0.056	0.317	3	0.105
4	1	0.04	0.605	0.025	0.342	4	0.0855
5	∞	∞	0.5195				
Trial with $\omega = 0.3$							
1	4	0.36	1	0.36	0.36	1	0.36
2	3	0.27	0.64	0.173	0.533	2	0.267
3	2	0.18	0.373	0.067	0.600	3	0.200
4	1	0.09	0.173	0.0155	0.6155	4	0.1539
5	∞	∞	0.0192				
Trial with $\omega = 0.4$							
1	4	0.64	1	0.64	0.64	1	0.64
2	3	0.48	0.36	0.173	0.813	2	0.406
3	2	0.32	-0.046	0.0147	0.798	3	0.266
4	1	0.16	-0.312	0.049	0.748	4	0.187
5	∞	∞	-0.499				
Trial with $\omega = 0.6$							
1	4	1.44	1	1.41	1.44	1	1.44
2	3	1.08	-0.44	-0.475	0.965	2	0.482
3	2	0.72	-0.922	-0.664	0.301	3	0.100

4	1	0.36	-1.023	-0.368	-0.067	4	0.017
5	∞	∞	-1.006				

Trial with ω = 0.8

1	4	2.56	1	2.56	2.56	1	2.56
2	3	1.92	-1.56	-3.00	0.44	2	-0.22
3	2	1.28	-1.34	-1.72	2.16	3	-0.73
4	1	0.64	0.61	0.39	2.55	4	0.64
5	∞	∞	0.03				

Trial with ω = 1.0

1	4	4	1	4	4	1	4
2	3	3	-3	9	-5	2	-2.5
3	2	2	-0.5	-1	-6	3	-2.0
4	1	1	1.5	1.5	4.5	4	-1.13
5	∞	∞	2.63				

Trial with ω = 1.5

1	4	9	1	9	9	1	9
2	3	6.75	-8	-54	-45	2	-22.5
3	2	4.5	14.5	65.3	20.3	3	6.77
4	1	2.25	7.73	17.4	37.7	4	9.43
5	∞	∞	-1.70				

Trial with ω = 1.8

1	4	12.96	1	12.96	12.96	1	12.96
2	3	9.72	-11.96	-116.4	-103.44	2	-51.72
3	2	6.48	-39.76	257.7	134.26	3	51.42
4	1	3.24	-11.66	-37.8	116.46	4	29.12
5	∞	∞	-40.78				

Trial with ω = 2.0

1	4	16	1	16	16	1	16
2	3	12	-15	-180	-164	2	-82
3	2	8	+67	536	372	3	124
4	1	4	-57	-228	144	4	36
5	∞	∞	-93				

Trial with ω = 2.5

1	4	25	1	25	25	1	25
2	3	18.75	-25	-450	-425	2	-212.5
3	2	12.5	188.5	2360	1935	3	645
4	1	6.25	-456.5	-2860	-1925	4	-231
5	∞	∞	-225.5				

Trial with ω = 3.0

1	4	36	1	36	36	1	36
2	3	27	-35	-945	-909	2	-455
3	2	18	420	7560	6651	3	2220
4	1	9	-1800	-16200	-9550	4	-2388
5	∞	∞	588				

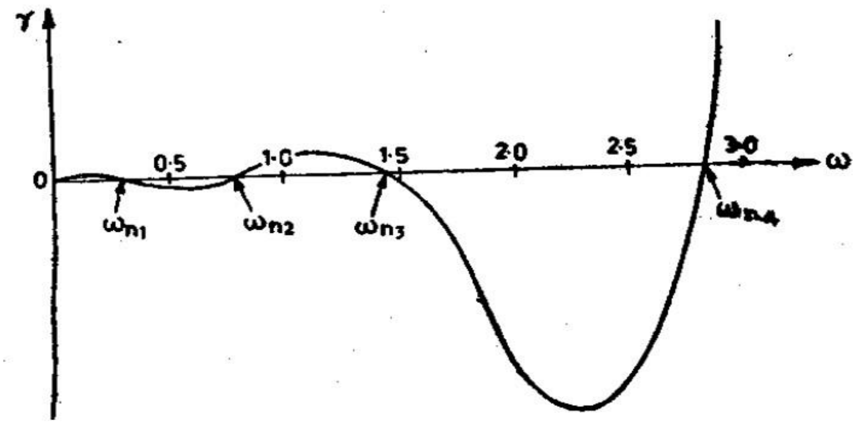


Fig.

Now plotting displacement of the fixed end γ versus ω we get natural frequencies where the curve intersects the x-axis. Natural frequencies are the frequencies which make the displacement of the support zero. They are red off the graph fig.(b) as

ω_{n1} = 0.30 rad./sec

ω_{n2} = 0.81 rsd/sec

ω_{n3} = 1.45 rad./sec

ω_{n4} = 2.83 rad./sec

Student should check the validity of the Dunkerley's equation.

Find the fundamental natural frequency of the system shown in fig(a). The gear ratio for both the branches is $\sqrt{10}$.

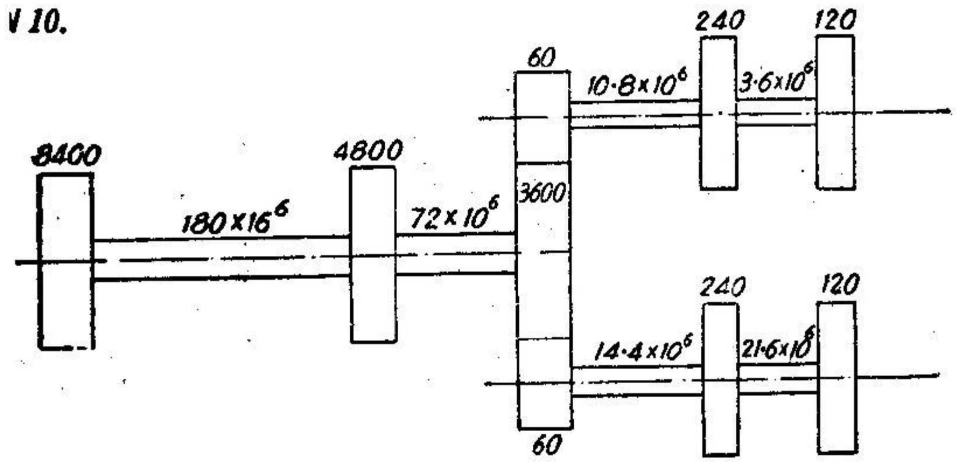


Fig.

All in consistent SI units.

Solution : The equivalent system is given in fig.(b).

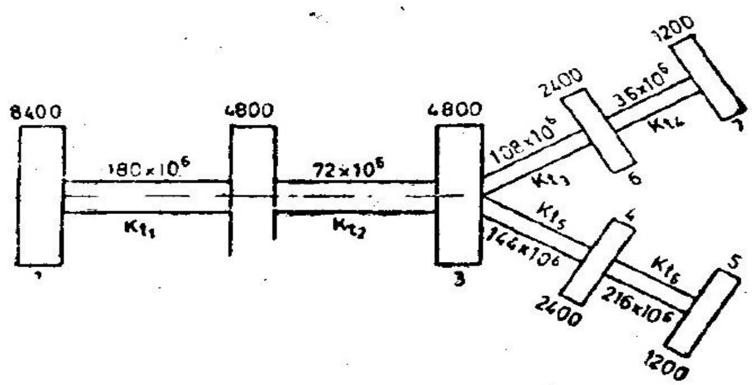


Fig.

Let us take 1200 as a unit for inertia and 36×10^6 as a unit for K_t. since K_{t3}, K_{t4}, and K_{t6} are all too large, we can lump

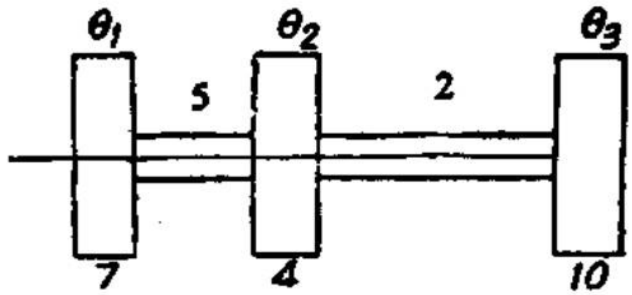


Fig.

3, 4, 5, 6, 7 together. So our equivalent approximate system becomes fig. (c)

We can use $7\theta_1+5(\theta_1-\theta_3)=0$

$4\theta_2+5(\theta_2-\theta_1)+4(\theta_2-\theta_3)=0$

$10\theta_3+2(\theta_3-\theta_2=0$

$$\omega^2\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}=\begin{bmatrix}\frac{5}{7}-\frac{5}{7} & & 0\\ -\frac{5}{4} & \frac{7}{4} & -0.5\\ 0 & -0.2 & 0.2\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}$$

The above model however is a crude one. We shall therefore treat the problem as that of a branched system as shown in fig.(b). we shall use the Holzer's method to obtain the natural frequency.

Let us assume a unit displacement for mass 7. the displacement of masses 6 and 3 can than be determined from equation, If unit displacement for mass 3 will result. These two values of displacement for mass 3 must be made the same by suitably modifying proportionately the displacement of one of the branches. After this is done, displacement of masses 2 and 1 are calculated and sum of all inertias is found. This process is repeated for each assumed frequency. Frequency corresponding to which the sum zero is naturally frequency. Let us start with $\omega = 1$.

Item	**I**	**Iω^2**	γ	**I$\omega^2\gamma$**	**ΣI$\omega^2\gamma$**	**K$_t$**	$\frac{\sum I\omega^2\gamma}{K_t}$
7	1	1	1	1	1	1	1
6	2	2	0	0	1	3	0.33
3	4	4	-0.33	-1.33	-0.33		
5	1	1	1	1	1	6	0.167
4	2	2	0.833	1.67	2.67	4	0.67

3	4	4	+0.167

Hence $$\gamma_5 = \frac{-0.33}{0.167} = -2$$

and we can re-construct the table for masses 5, 4, 3 and add for rotors 1 and 2.

Item	**I**	**Iω^2**	**γ**	**I$\omega^2\gamma$**	**ΣI$\omega^2\gamma$**	**K$_t$**	$\frac{\sum I\omega^2\gamma}{K_t}$
5	1	1	-2	-2	-2	6	-0.333
4	2	2	-1.67	-3.33	-5.33	4	-1.33
3	4	4	-0.33	-1.33	-6.66	2	-2.83
Torque acting on mass K$_{t2}$ = 1-5.30-1.33=-5.66							
2	4	4	2.5	10.0	4.34	5	0.87
1	7	7	1.63	11.41	15.75		

Trial with ω = 1.5

Item	**I**	**Iω^2**	**γ**	**I$\omega^2\gamma$**	**ΣI$\omega^2\gamma$**	**K$_t$**	$\frac{\sum I\omega^2\gamma}{K_t}$
7	1	2.25	1	2.25	2.25	1	2.25
6	2	1.5	-1.25	-5.6	-3.35	3	-1.12
3	1	9.0	-0.13	-1.17	-4.52		
5	1	2.25	1	2.25	2.25	6	0.39
4	2	4.5	0.61	2.75	5.0	4	1.25
3	4	9.0	-0.64				

$$\gamma_5 = \left(\frac{0.64}{0.13}\right)^{-1} = 0.2$$

Let us reconstruct the table for masses 5, 4, 3 and add for masses 1 and 2.

Item	**I**	**Iω^2**	**γ**	**I$\omega^2\gamma$**	**ΣI$\omega^2\gamma$**	**K$_t$**	$\frac{\sum I\omega^2\gamma}{K_t}$
5	1	2.25	0.2	0.450	0.45	6	0.0715
4	2	4.5	0.125	0.56	1.01	4	0.25
2	4	9.0	-0.13	-1.17	-3.51	2	-1.75

Torque acting in K_{t2} = -3.35+1.01-1.17 = -3.51							
2	4	9.0	1.62	13.58	10.07	5	2.01
1	7	15.8	-0.39	-6.15	3.82		

Trial with ω = 2.0

Item	I	$I\omega^2$	γ	$I\omega^2\gamma$	$\Sigma I\omega^2\gamma$	K_t	$\frac{\sum I\omega^2\gamma}{K_t}$
7	1	4	1	4	4	1	4
6	2	8	-3	-24	-20	3	-6.67
3	4	16	3.37				
5	1	4	1	4	4	6	0.67
4	2	8	0.33	2.64	6.64	4	1.66
3	4	16	-133				

$$\gamma_5 = \frac{-3.37}{1.33} = -2.53$$

Let us now re-construct table for masses 5, 4, 3 and add for masses 1 and 2.

Item	I	$I\omega^2$	γ	$I\omega^2\gamma$	$\Sigma I\omega^2\gamma$	K_t	$\frac{\sum I\omega^2\gamma}{K_t}$
5	1	4	-2.53	-10.12	-10.12	6	-1.68
4	2	8	-0.85	-6.75	-17	4	-4.21
3	4	16	3.37	53.9	16.9*	2	8.45
2	4	16	-5.03	-80.48	63.6	5	-12.7
1	7	28	-7.67	-2.5	-278.6		

Now we plot a graph between $\Sigma I\omega^2\gamma$ and ω. Natural frequency equals ω when curve intersects the ω axis.

- Torque acting in K_{t2} = -20-17+53.9 = 16.9

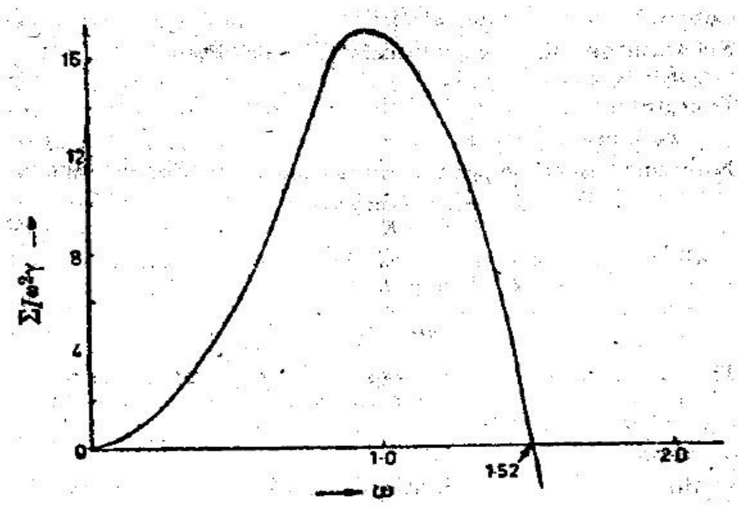


Fig.

From graph fig. $\omega_n = 1.52$, But this must be modified because of our units for inertias and stiffnesses taken to facilitates calculations. Thus ω_n, the natural frequency is given by

$$\omega_n = 1.52 \times \sqrt{\frac{36 \times 10^6}{1200}}$$

$$= 283 \text{ radians/sec.}$$

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UNIT III SUSPENSION SYSTEM

UNIT 3 SUSPENSION SYSTEM

Vehicle dynamics and suspension requirements, natural spring frequencies, force acting on a semi-elliptic suspension, design of laminated spring, design of coil spring and torsion bar, spring characteristics, mechanics of an independent suspension system, roll axis and the vehicle under the action of side forces, coupled front and rear suspension effects, relative pitch and bounce frequencies, anti-roll rates, roll angles in cornering, attributes changes due to brakes, traction and independent suspension.

Function of suspension system

- It must provide a high degree of isolation between a passenger/ goods of the vehicle from irregularities of the road surface, forces due to non-balancing of the wheel.
- It has to maintain the close contact of the wheel with the road surface to get adequate adhesion for acceleration, braking and cornering.

Requirements of suspension system:

- The main requirement of the good spring system is that it should allow the wheels to have a vertical movement.
- It should resist the roll of chassis
- It must keep the tires in contact with the road with the minimum road variations
- It must provide vertical compliance so that wheels can follow the even road, isolating the chassis form the roughness in the road
- It maintain the wheel in the proper steer
- Stiffness/Displacement bound
- Compatibility
- Minimum wear
- Low Maintenance
- Low Initial cost
- Suspension terminology

Sprung mass frequency

It is the resonant condition of the sprung mass acting on the suspension spring in series with the tire spring with the interposed unsprung mass.

$$\omega = 2\pi f$$
$$f = \omega / 2\pi$$

Wheel hop

The vertical oscillating motion of the wheel between the road surface and the sprung mass of the vehicle.

Contact area of the tire is very less at the time of wheel hop.

Wheel hop occurs

1. Road irregularities
2. Wheel out of round balance

Wheel wobble

A self- excited oscillation of steerable wheels about their steering axes, occurring without appreciable tramp

Horizontal vibration of front axle assembly around the longitudinal axis

Front axle is mounted between two springs namely 1. Chassis spring 2. Tire spring when there is wheel wobble chassis spring on one side tire on the other side compresses simultaneously while other spring on the tire rebumb this condition called wheel wobble

Wheel shimmy:

A self-excited oscillation of a pair of steerable wheel about their steering axes, accompanied by appreciable tramp.

A violent front wheel shake caused by over corrective action

Spring Rate: The change of load of a spring per unit deflection, taken as a mean between loading and unloading at a specified load.

Sprung Weight: All weight which is supported by the suspension, including portions of the weight of the suspension members

Unsprung Weight: All weight which is not carried by the suspension system, but is supported directly by the tire or wheel and considered to move with it

Suspension Rate(Wheel Rate): the change of wheel load, at the center of tire contact, per unit vertical displacement of the sprung mass relative to the wheel at a specified load

Tire Rate:(static) the static rate measured by the change of wheel load per unit vertical displcement of the wheel load per unit vertical displacement of the wheel relative to the ground at a specified load and inflation pressure

Ride Rate: the change of wheel load, at the centre of tire contact per unit vertical displacement of the sprung mass relative to the ground at a specified load.

Caster angle: the angle in side elevation between the steeing axis and the vertical. It is considered positive when the steering axis is inclined rearward and negative when the steering axis is inclined forward

Jounce: the condition of the suspension which causes spring compression

Rebound:

- An expansion of a suspension spring after it has been compressed as a result of jounce.
- the relative displacement of the sprung and unsprung masses in a suspension system in which the distance between the masses increases from that at static condition.

Camber angle: the inclination of the wheel plane to the vertical. It is considered positive when the wheel leans outward at the top and negative when it leands inward.

Choice of suspension spring rate

$$\omega_{nat} = \frac{187.8}{\sqrt{d}}$$

Where the **d** is the static deflection in inch when spring rate is linear
Let us consider the system having:
Unladen mass = m
Laden mass= m+L
Change in deflection= x
Then

$$S = \frac{laden\ mass - unladen\ mass}{change\ in\ deflection}$$
$$= \frac{M + L - M}{x}$$
$$= \frac{L}{x}$$

Sprung mass frequency,

$$\omega_{nat} = \frac{1}{2\pi}\left(\frac{\sqrt{gS}}{m}\right)$$

$$= \frac{1}{2\pi}\left(\frac{\sqrt{g\frac{L}{x}}}{m}\right)$$

To provide maximum isolation of the vehicle from irregularities the minimum possible spring rate is the best.

CALCULATION OF EFFECTIVE SPRING RATE

Let W be the load acting on the tire and L be the load acting on the spring.

Due to some disturbance on the wheel due to road irregularities there will be some displacement in wheel and spring.

The corresponding values assume to be dx and dy respectively.
Spring rate = (Effective spring rate) X (Installation ratio)2
Installation ratio = dx/dy
Effective spring rate = dW/dx
Actual spring rate = dL/dx

$$\frac{dL}{dy} = \frac{dW}{dx} + \left(\frac{dx}{dy}\right)^2$$

We know that

$$W.dx = Ldy$$

$$W = L\frac{dy}{dx}$$

Differentiating the above, we get

$$\frac{dW}{dx} = \frac{d}{dx}\left(L\frac{dy}{dx}\right)$$

$$= \frac{dL}{dy}\frac{dy}{dx} + L\frac{d^2y}{dx^2}$$

Multiplying and dividing by dy in first term

$$= \frac{dL}{dy}\frac{dy}{dx}\frac{dy}{dx} + L\frac{d^2y}{dx^2}$$

$$= \frac{dL}{dy}\left(\frac{dy}{dx}\right)^2 + L\frac{d^2y}{dx^2}$$

Neglecting the second order terms

$$\frac{dW}{dx} = \frac{dL}{dy}\left(\frac{dy}{dx}\right)^2$$

Now as per the law if conservation of energy, the work done by the movement of wheels must be equal to the potential energy stored in the spring

$$W.dx=L.dy$$

The ratio between the small displacements at the wheel corresponding small displacement of spring is called as installation ratio. Installation ratio is assumed to be a constant value

Vehicle suspension in fore and aft,
Roll axis and vehicle under the action of side forces, the dynamics,

- To describe the characteristics of a tire and the forces and moments acting on it, it is necessary to define an axis system that serves as a reference for the definition of various parameters. One of the commonly used axis systems recommended by the Society of Automotive Engineers is shown
- The origin of the axis system is the center of tire contact.
- The X axis is the intersection of the wheel plane and the ground plane with a positive direction forward.
- The Z axis is perpendicular to the ground plane with a positive direction downward.
- The Y axis is in the ground plane, and its direction is chosen to make the axis system orthogonal and right hand
- There are three forces and three moments acting on the tire from the ground.
- Tractive force (or longitudinal force) *F*, is the component in the X direction of the resultant force exerted on the tire by the road.
- Lateral force *F*, is the component in the Y direction, and normal force *F*, is the component in the Z direction. Overturning moment ***M,*** is the moment about the X axis exerted on the tire by the road.
- Rolling resistance moment ***My*** is the moment about the Y axis, and aligning torque ***M,*** is the moment about the Z axis
- The longitudinal shift of the center of normal pressure is determined by the ratio of the rolling resistance moment to the normal load.
- The lateral shift of the center of normal pressure is defined by the ratio of the overturning moment to the normal load.
- The integration of longitudinal shear stresses over the entire contact patch represents the tractive or braking force.
- A driving torque about the axis of rotation of the tire produces a force for accelerating the vehicle, and a braking torque produces a force for decelerating the vehicle.

- Slip angle *a* is the angle formed between the direction of wheel travel and the line of intersection of the wheel plane with the road surface.
- Camber angle *y* is the angle formed between the *XZ* plane and the wheel plane. The lateral force at the tire-ground contact patch is a function of both the slip angle and the camber angle.

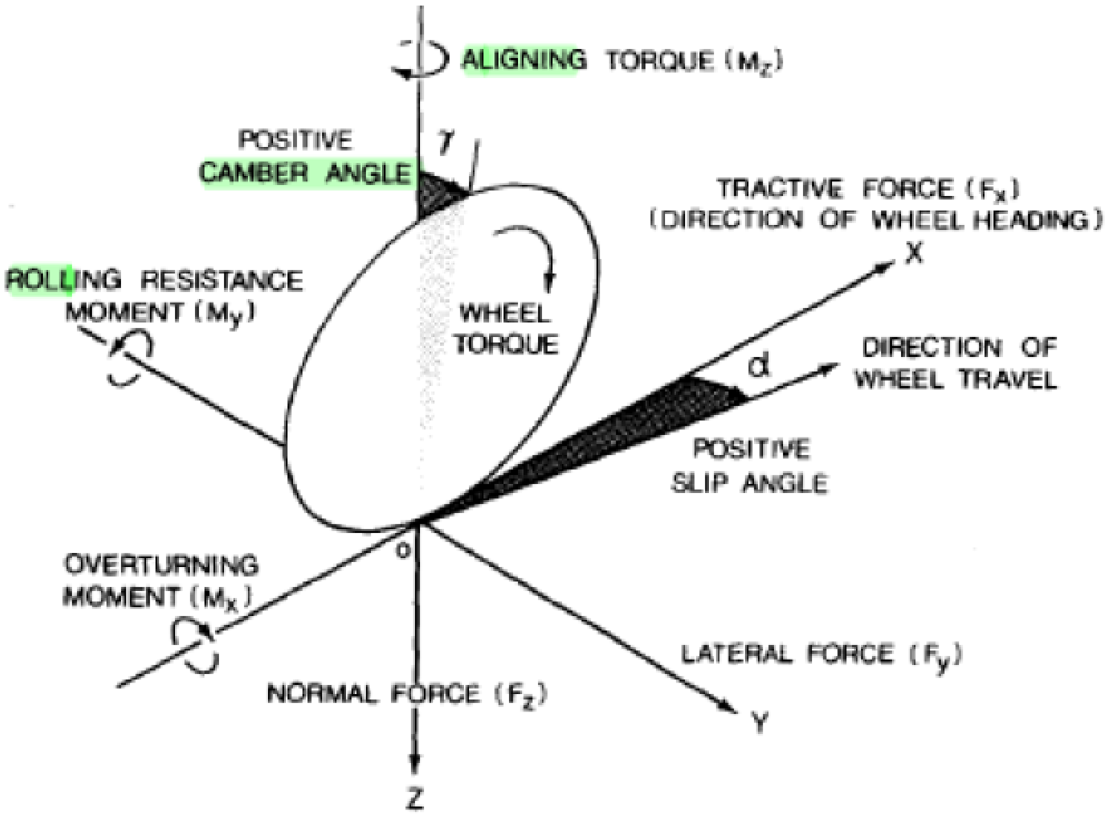


Figure 1: Forces and moments acting on tire

DESIGN OF LEAF SPRING

There are several geometric variations in leaf spring-. Semi-elliptic single stage multi-leaf spring is the most commonly used in light passenger vehicles, which is shown in Fig.1 below The unloaded spring is cambered, the magnitude of the camber being such that the spring is approximately straight under the full load. The individual leaves of steel leaf spring are in a sliding contact that allows the spring to deflect more easily. The dimensional details are listed in Table1. According to Indian standards the recommended

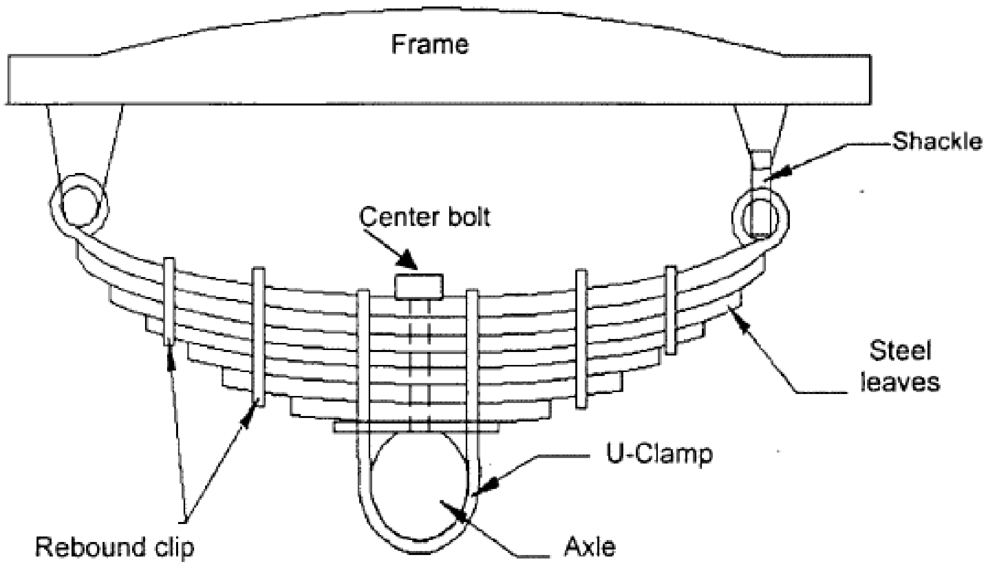


Figure 2: Leaf suspension system

Table 1: Leaf suspension system dimension (Passenger car)

Parameter	Value
Straight length	1219 mm
Leaf thickness	6.45 mm
Leaf width	37.8 mm
Camber	160.5 mm
Number of leaves	7

Table 3.2 Material properties of steel leaf spring

Parameter	Value
UTS	1820 - 2060 N/mm^2
Tensile Strength	1680 - 1920 N/mm^2
Young's Modulus (E)	$210X10^3$ N/mm^2
Poisson's ratio (ν)	0.3
Density (ρ)	7800 Kg/m^3

THEORETICAL ANALYSIS OF STEEL LEAF SPRING

An accurate analysis of multi-leaf steel spring is complicated because of the interaction-taking place between the leaves. Based on the following simplified theory, the leaf spring is assumed to be statically equivalent to a flat trapezoidal beam of uniform thickness. Two symmetrical trapezoids linked together are shown in Fig. 2. By using the symbols shown in the Fig.2, maximum deflection (δ) and maximum bending stress (σ_b) can be determined using the equation (1) and equation (2).

$$\delta = \frac{K_1 F L^3 (1 - v^2)}{3EI} \qquad \text{----------------(1)}$$

$$\sigma_b = \frac{6FL}{bt^2} \qquad \text{-----------------(2)}$$

Maximum width of the trapezoidal beam b = Nb’
Where N= Number of leaves. The moment of inertia 1 of the cross section at the middle section is l=bt3/12 and Ki is a function of ratio b’/b. Fig. 3.3 shows the variation of with respect to ratio b’/b.

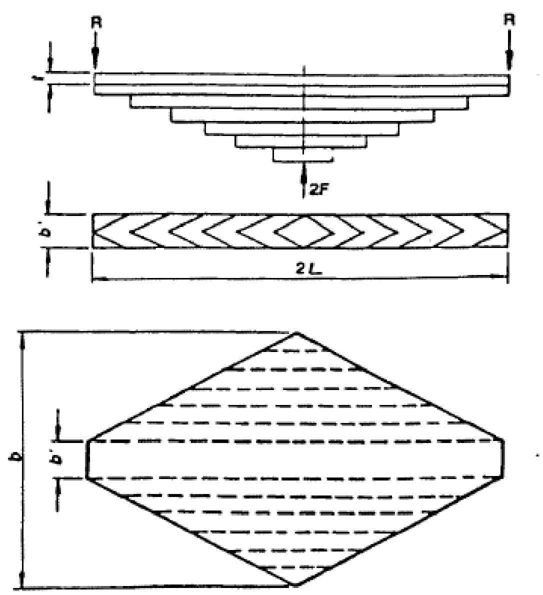


Figure 3: Analytical view of Leaf suspension system.

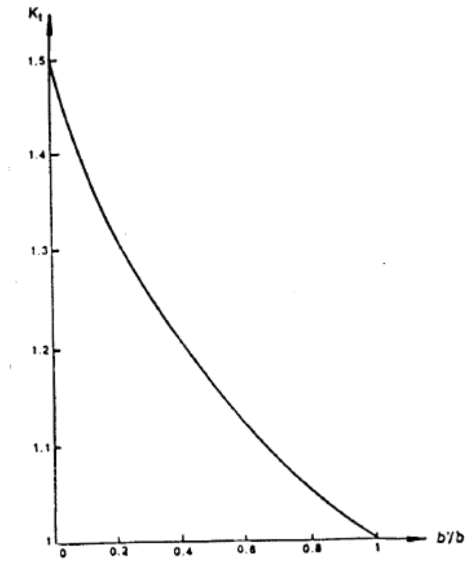


Fig. 3.3 Variation of factor K_1 of trapezoidal beam with ratio b'/b

Figure 4: Variation of K of trapezoidal beam

Selection of Design Method

The leaf springs behave like a simply supported beam with three point bending and hence the flexural analysis is done considering it as a simply supported beam. The simply supported beam is subjected to both bending stress and transverse shear stress. Among these stresses the main stress is the tensile stress acting on one side of the neutral axis and the compressive stress on the other side of the neutral axis. The magnitude of the other stress namely the shear stress is less and hence neglected. Flexural rigidity is an important parameter in the leaf spring design and it should increase from two ends to the center. This concept gives the following three design possibilities.

I. Constant thickness, varying width design

In this design the thickness is kept constant over the entire length of the leaf spring, while the width varies from a minimum at the two ends to a maximum at the center.

II. Constant width, varying thickness design

In this design the width is kept constant, while the thickness varies from a minimum at the two ends to a maximum at the center.

III. Constant cross section design

In this design, both the thickness and width are varied through out the leaf spring such that the cross section area remains constant along the length of the leaf spring.

DESIGN OF COIL SPRINGS

The design of a new spring involves the following considerations:-Space into which the spring must fit and operate. -Values of working forces and deflections. -Accuracy and reliability needed

Design Consideration

The primary consideration in the design of the coil springs are that the induced stresses are below the permissible limits while subjected to or exerting the external force F capable of providing the needed deflection or maintaining the spring rate desired

Stresses In Helical Springs

The flexing of a helical spring creates torsion in the wire and the force applied induces a direct stress. The maximum stress in the wire may be computed by super position. The result is:

$$t_{max} = +\frac{T_r}{J} + \frac{F}{A}$$

Replacing the terms,

$$T = \frac{FD}{2}, r = \frac{d}{2}, J = \frac{\pi d^4}{32} \text{ and } A = \frac{\pi d^4}{4}$$

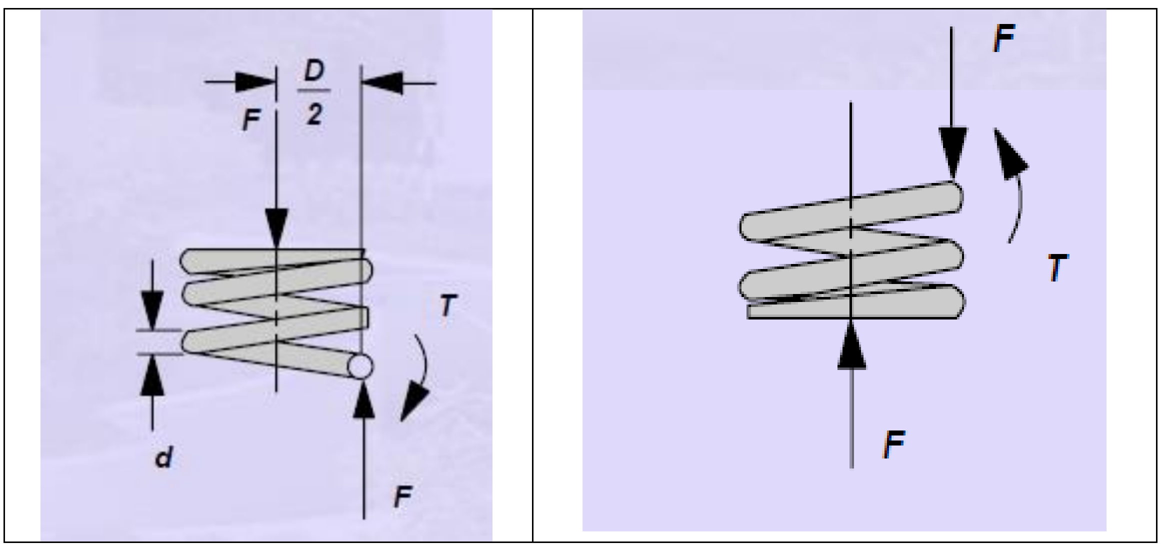


Figure 5: Analysis of coil spring

And re-arranging,

$$\tau = K_s \frac{8FD}{\pi d^3} \quad \text{or} \quad \tau = K_s \frac{8FC}{\pi d^2}$$

Where K_s is the shear-stress correction factor and is defined by the equation:

$$K_s = \frac{2C+1}{2C}$$

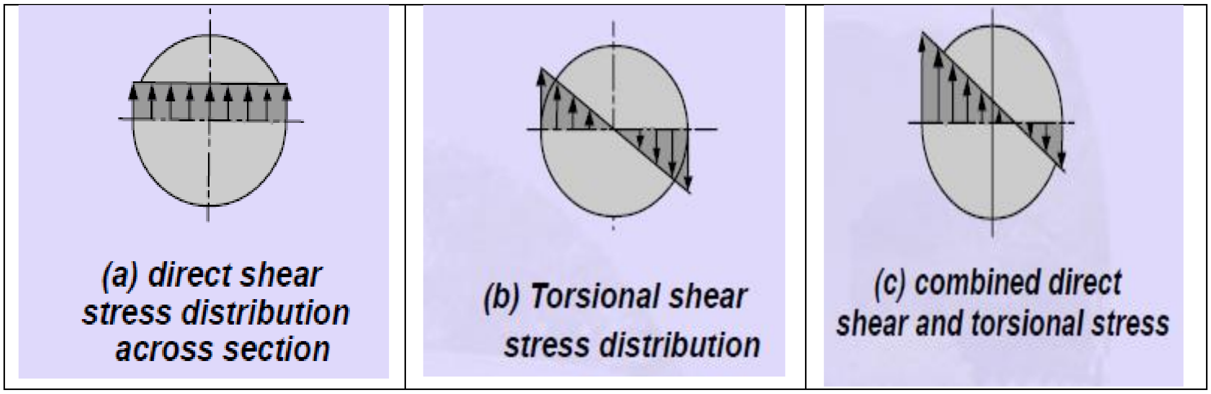


Figure 6: Stress analysis on coil spring

Whal's correction factor

The combined effect of direct shear and curvature correction is accounted by Wahl's correction factor and is given as:

$$K_W = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

Deflection and Stiffness of the spring

A systems strain energy is related to its force deflection behaviour and using the Castigliano's theorem the deflection of a spring can be estimated using the strain energy stored in it. The total strain energy for a helical spring is composed of torsional component and a shear component. The shear component is quite negligible, and the final equation is,

$$u = \frac{T^2}{2G}\frac{1}{J} + \frac{F^2}{2A}\frac{1}{G}$$

The spring rate and hence,

$$T = F\frac{D}{2};\ 1 = \pi.D.N;\ J = \pi.\frac{d^4}{32}\ \text{and}\ A = \frac{\pi d^2}{4}$$

$$U = \frac{4F^2D^3N}{G.d^4} + \frac{F^2DN}{Gd^2}$$

Where N is the number of active coils. The deflection in the spring, using Castigliano's theorem

$$y = \frac{\partial U}{\partial F} = \frac{8FD^3N}{Gd^4} + \frac{4FDN}{Gd^2}$$

Substituting C=D/d and rearranging

$$y = \frac{8FD^3N}{Gd^4}\left(1 + \frac{1}{2C^2}\right)$$

For normal range of C, the term within bracket (contribution of direct shear) is so negligible we can write

$$y = \frac{8FD^3N}{Gd^4} \quad \text{or} \quad \frac{8FC^3N}{Gd}$$

$$k = \frac{F}{y} = \frac{G.d}{8C^3N} = \frac{Gd^4}{8D^3N}$$

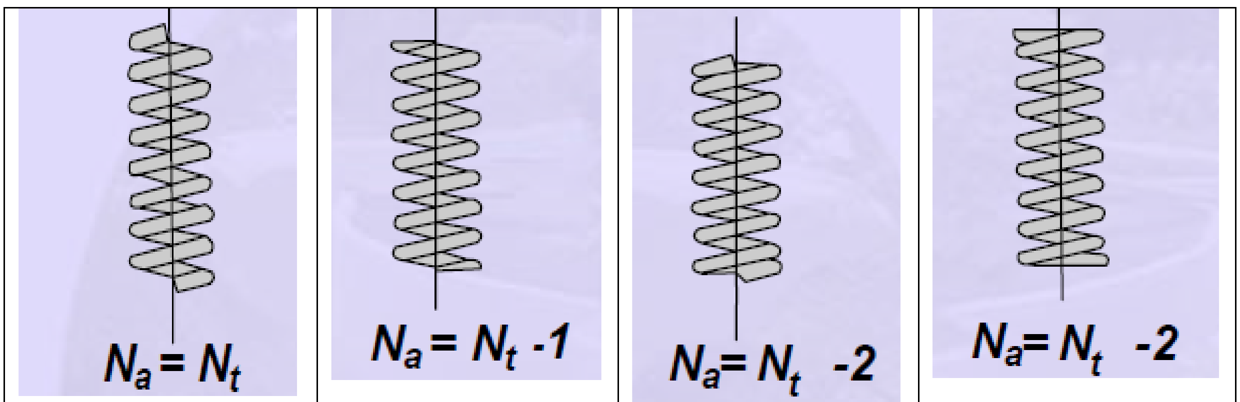
The spring stiffness or springs rate,

$$k = \frac{F}{y} = \frac{G.d}{8C^3N} = \frac{Gd^4}{8D^3N}$$

Using the equation the number of active coils needed to maintain the desired deflection or spring stiffness will be determined. In order to maintain proper contact and align the force along the spring axis the ends are to be properly shaped.

End Construction

Coil compression springs generally use four different types of ends. These are illustrated in Fig. below. and Table shows how the type of end used affects the number of coils and the spring length. Foe important applications the ends of springs should always be of both squared and ground, because a better or even transfer of the load is obtained.



Plain ends	Plain ground ends	Squared ends	Squared ground ends

Figure 7: End construction of coil spring.

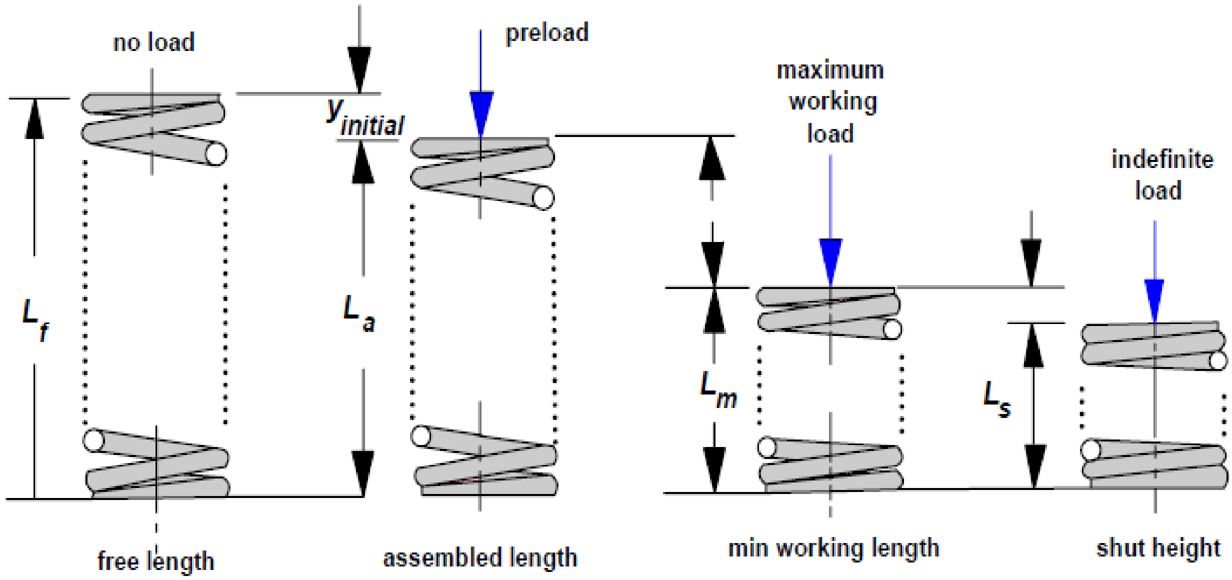


Figure 8: Deflection analysis on coil spring.

Analysis of Suspension Mechanisms

- 3D mechanisms
- Compliant bushes create variable link lengths
- 2D approximations used for analysis

Requirement

- Guide the wheel along a vertical path
- Without change in camber
- Suspension mechanism has various SDOF mechanisms

The mobility of suspension mechanisms

Guide motion of each wheel along (unique) vertical path relative to the vehicle body without significant change in camber.

Mobility (DOF) analysis is useful for checking for the appropriate number of degrees of freedom,

Does not help in synthesis to provide the desired motion

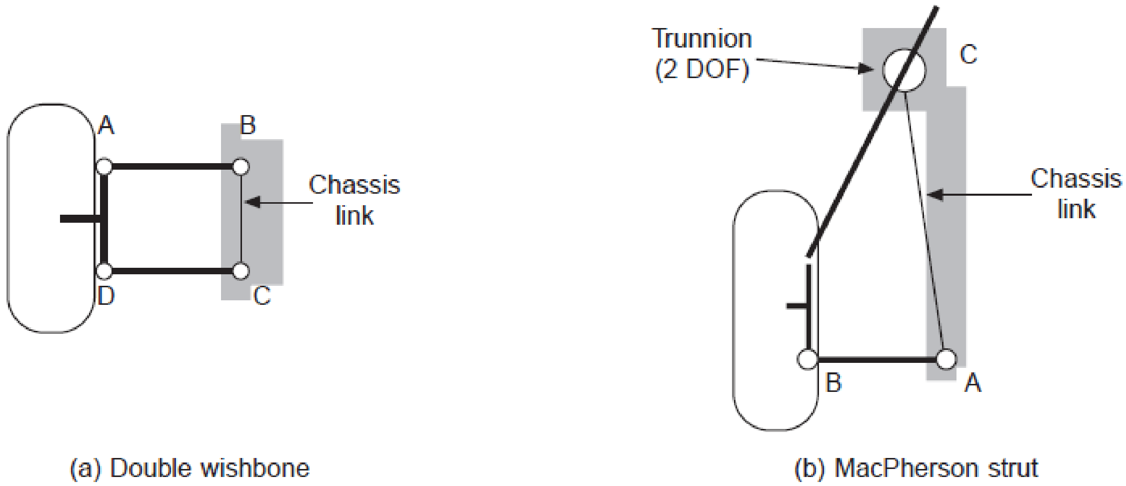


Figure 9: Two-dimensional kinematics of common suspension mechanisms

$$\mathbf{M = 3(n - 1) - jh - 2jl}$$

Semi-dependent Suspension

- The rigid connection between pairs of wheels is replaced by a compliant link.
- A beam which can bend and flex providing both positional control of the wheels as well as compliance.
- Tend to be simple in construction but lack scope for design flexibility
- Additional compliance can be provided by rubber or hydro-elastic springs.
- Wheel camber is, in this case, the same as body roll

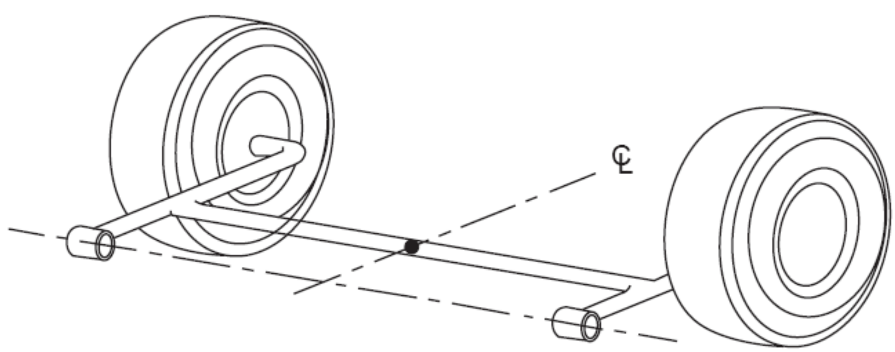


Figure 10: Trailing twist axle suspension

Suspension Types – Independent

- Motion of wheel pairs is independent, so that a disturbance at one wheel is not directly transmitted to its partner
- Better ride and handling

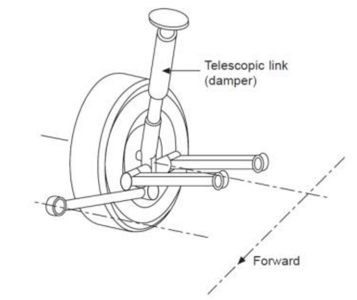
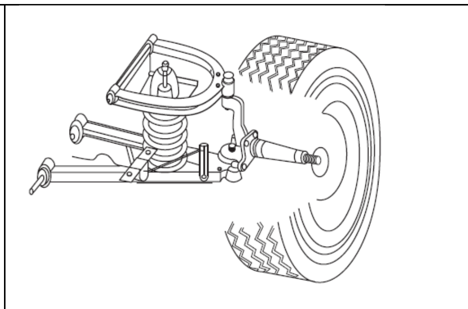
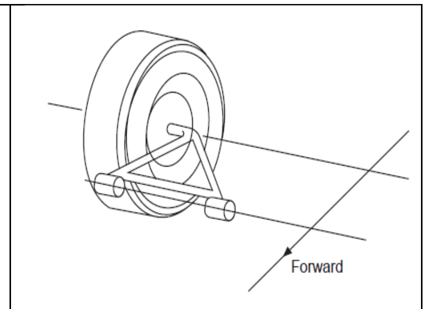
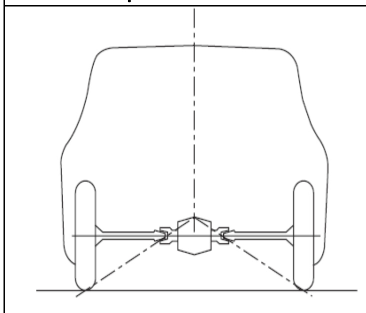
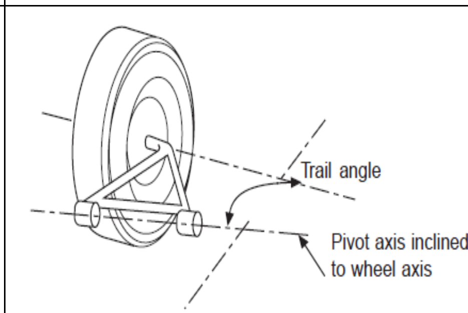
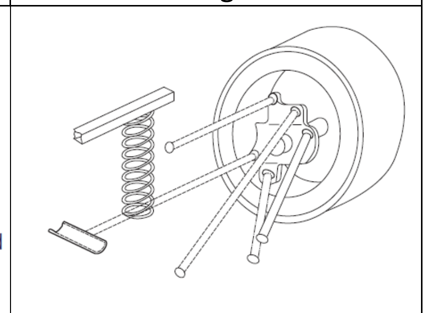
		
Macpherson Strut	Double wishbone	Trailing arm
		
Swing axle	Semi trailing arm	Multi-link

Figure 11: different types of independent suspension system

Roll centre analysis

- A point in the transverse plane through any pair of wheels at which a transverse force may be applied to the sprung mass without causing it to roll
- Kinematics : the roll centre is the point about which the body can roll without any lateral movement at either of the wheel contact areas

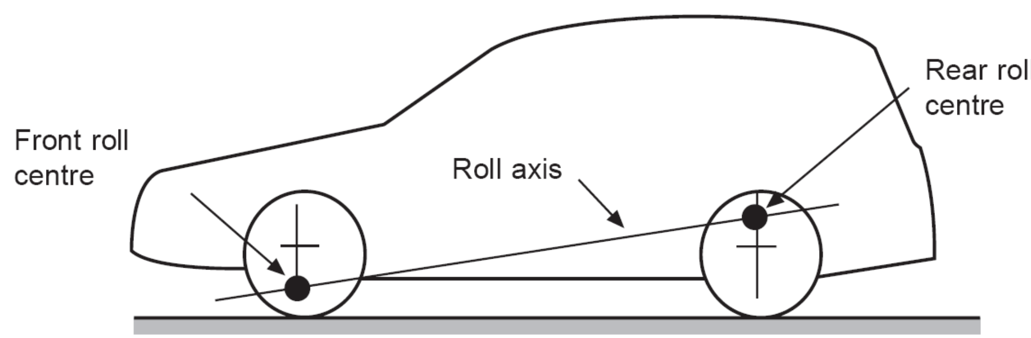


Figure 10.16 Roll axis location

Limitations of Roll Centre Analysis

- As roll of the sprung mass takes place, the suspension geometry changes, symmetry of the suspension across the vehicle is lost and the definition of roll centre becomes invalid.
- It relates to the non-rolled vehicle condition and can therefore only be used for approximations involving small angles of roll .
- Assumes no change in vehicle track as a result of small angles of roll.

Force Analysis - spring and wheel rates

- Relationship between spring deflections and wheel displacements in suspensions is non-linear
- Desired wheel-rate (related to suspension natural frequency) has to be interpreted into a spring-rate

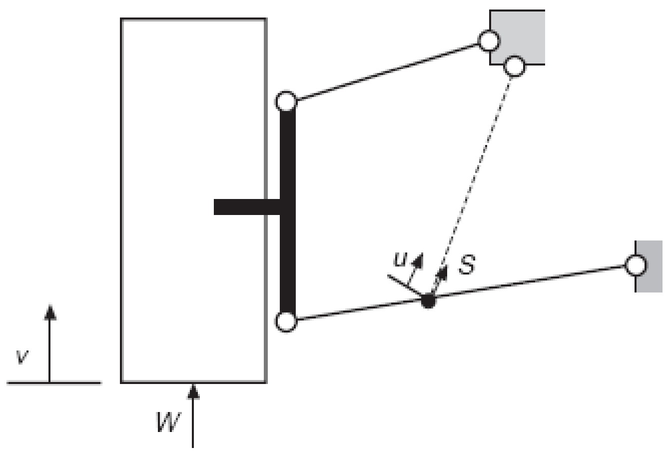


Figure 12: Force analysis on independent suspension system

- W and S are the wheel and spring forces respectively
- v and u are the corresponding deflections
- Notation for analysing spring and wheel rates in a double wishbone suspension

Spring and wheel rates

Begin by defining the suspension ratio as: $R = \frac{S}{W}$

The spring stiffness is: $k_s = \frac{dS}{du} = d(RW) = R\frac{dW}{dv}\frac{dv}{du} + W\frac{dR}{dv}\frac{dv}{du}$

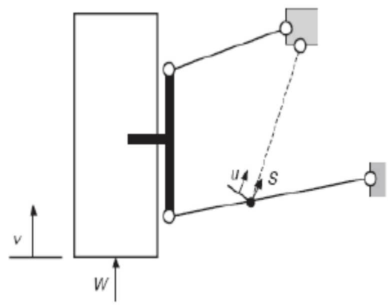
From principle of virtual work

$$S\,du = W\,dv$$

$$R = \frac{S}{W} = \frac{dv}{du}$$

Wheel rate

$$k_w = \frac{dW}{dv}$$



Combined Equation is

$$k_s = k_w R^2 + S\frac{dR}{dv}$$

Similarly can be derived for other suspension geometries

Wheel-rate for constant natural frequency with variable payload

Simplest representation of undamped vibration

$$\omega_n = \sqrt{\frac{k_w}{m_s}}$$

k_w – wheel rate

m_s – proportion of un-sprung mass

Change in wheel rate required for change in payload.

Static displacement

$$\delta_s = \frac{m_s g}{k_w}$$

To maintain w_n constant, the static deflection needs to be constant. Combining both equations

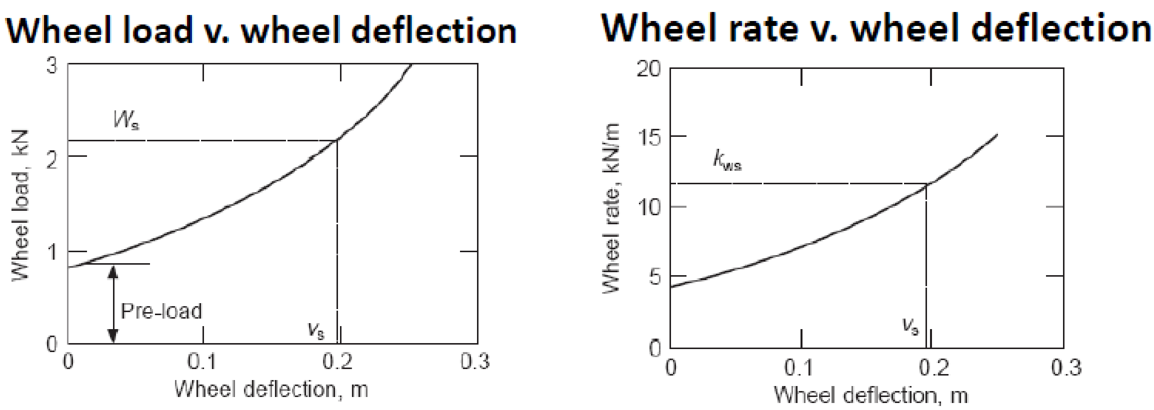
$$\frac{W}{dW/dv} = \delta_s = \text{constant, or } \frac{dW}{W} = \frac{dv}{\delta_s}$$

Integrating the equation and substituting with initial conditions provides the following expression

$$W = W_s e^{\frac{v-v_s}{\delta_s}}$$

Substituting back , we obtain

$$k_w = \frac{dW}{dv} = \frac{W_s}{\delta_s} e^{\frac{v-v_s}{\delta_s}}$$



Typical wheel load and wheel rate as functions of wheel displacement

R and dR/dv are known from geometric analysis

$$k_s = k_w R^2 + S\frac{dR}{dv}$$ gives k_s if k_w is known from the above graphs

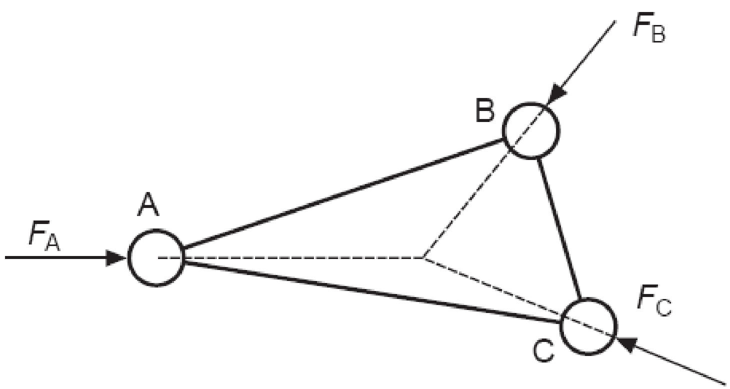
k_s can be obtained as a function of v and u so as to get constant frequency

Forces in suspension members – Basics

- Mass of the members is negligible compared to that of the applied loading.
- Friction and compliance at the joints assumed negligible and the spring or wheel rate needs to be known
- Familiar with the use of free-body diagrams for determining internal forces in structures
- Conditions for equilibrium

For equilibrium: F_A = F_B and forces must be collinear

(a)



For equilibrium
$\sum F = 0$

Equilibrium of two and three force members, (a) Requirements for equilibrium of a two force member (b) Requirements for equilibrium of a three-force member

Anti-squat / Anti-dive

- During braking there is a tendency for the sprung mass to "dive" (nose down) and
- During acceleration the reverse occurs, with the nose lifting and the rear end "squatting"

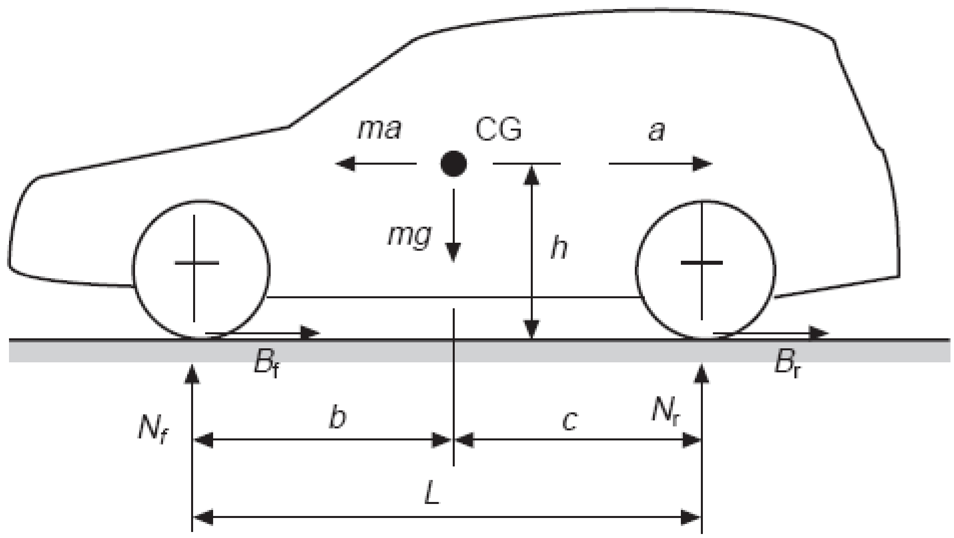


Figure 13: Free body diagram of a vehicle during braking

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UNIT IV PERFORMANCE OF VEHICLE

PERFORMANCE OF AUTOMOBILE

Forces and couples on the wheels, tractive and breaking properties of tires, cornering properties of the tires, slip angle, cornering force, camber thrust, deformation of wheel and ground, power of propulsion, road performance curves, resistances- air, rolling and grade, deformation of center of gravity of vehicle, load distribution, stability on a curved track slope and banked road. Calculation of tractive effort and reactions for different drive.

TIRE

A modern tire is a mixture of steel, fabric, and rubber. The main composites of a passenger car tire with an overall mass of 8:5 kg are listed in Table below.

Table 1: Tire composites

Reinforcements: steel, rayon, nylon	16%
Rubber: natural/synthetic	38%
Compounds: carbon, silica, chalk	30%
Softener: oil, resin	10%
Vulcanization: sulphur, zinc oxide,	4%
Miscellaneous	2%

TIRE FORCES AND TORQUES

In any point of contact between the tire and the road surface normal and friction forces are transmitted. According to the tire's profile design the contact patch forms a not necessarily coherent area

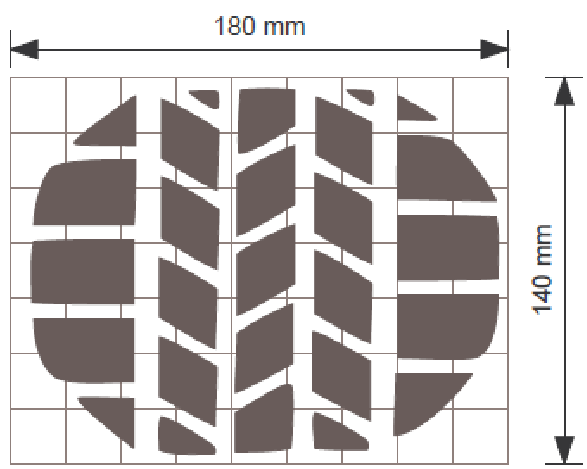


Figure 1: Footprint of a test tire of size 205/55 R16 at Fz = 4700 N and p = 2:5 bar

The effect of the contact forces can be fully described by a resulting force vector applied at a specific point of the contact patch and a torque vector. The vectors are described in a track-fixed reference frame. The z-axis is normal to the track, the x- axis is perpendicular to the z-axis and perpendicular to the wheel rotation axis eyR. Then, the demand for a right-handed reference frame also fixes the y-axis.

Fx longitudinal force
Fy lateral force
Fz vertical force or wheel load
Tx tilting torque
Ty rolling resistance torque
Tz self aligning and bore torque

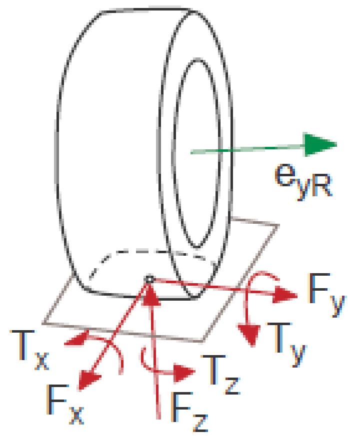


Figure 2: Contact forces and moments of tire.

The components of the contact force vector are named according to the direction of the axes, Fig. 2. A non-symmetric distribution of the forces in the contact patch causes torques around the x and y axes. A cambered tire generates a tilting torque Tx. The torque Ty includes the rolling resistance of the tire. In particular, the torque around the z-axis is important in vehicle dynamics. It consists of two parts,

$$Tz = TB + TS \text{ ------------------------1}$$

The rotation of the tire around the z-axis causes the bore torque TB. The self-aligning torque TS takes into account that, in general, the resulting lateral force is not acting in the center of the contact patch.

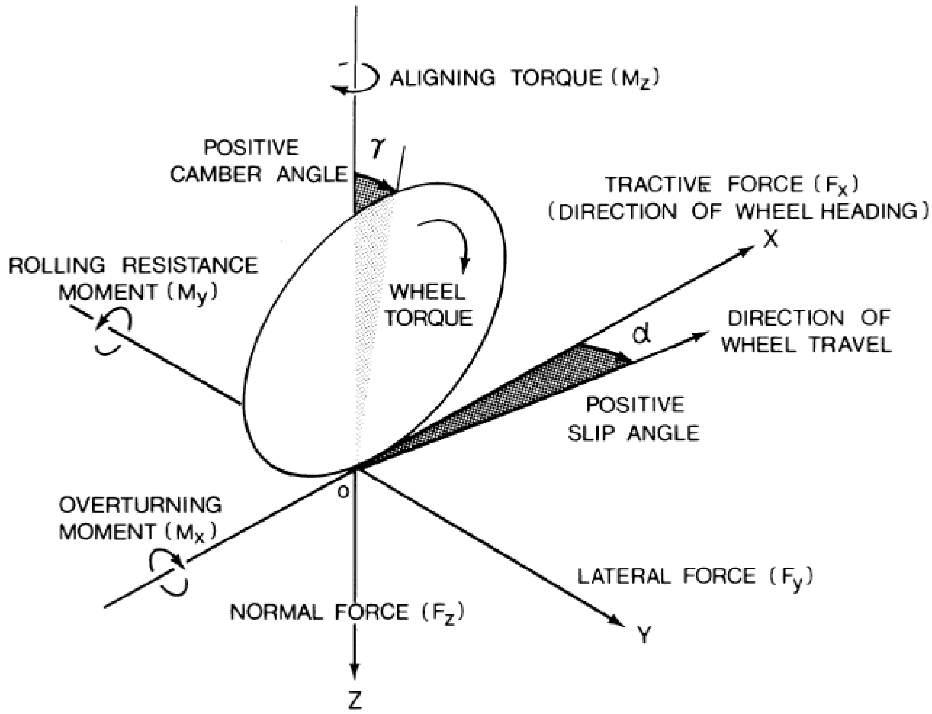


Figure 3: Tire axis system

There are three forces and three moments acting on the tire from the ground. Tractive force (or longitudinal force) *Fx* is the component in the *X* direction of the resultant force exerted on the tire by the road. Lateral force *Fy* is the component in the *Y* direction, and normal force *Fz* is the component in the *Z* direction. Overturning moment *Mx* is the moment about the *X* axis exerted on the tire by the road. Rolling resistance moment *My* is the moment about the *Y* axis, and aligning torque *Mz* is the moment about the *Z* axis With this axis system, many performance parameters of the tire can be conveniently defined. For instance, the longitudinal shift of the center of normal pressure is determined by the ratio of the rolling resistance moment to the normal load. The lateral shift of the center of normal pressure is defined by the ratio of the overturning moment to the normal load. The integration of longitudinal shear stresses over the entire contact patch represents the tractive or braking force. A driving torque about the axis of rotation of the tire produces a force for accelerating the vehicle, and a braking torque produces a force for decelerating the vehicle.

There are two important angles associated with a rolling tire: the slip angle and the camber angle. Slip angle (α)is the angle formed between the direction of wheel travel and the line of intersection of the wheel plane with the road surface. Camber angle (ϒ) is the angle formed between the *XZ* plane and the wheel plane. The lateral force at the tire–ground contact patch is a function of both the slip angle and the camber angle.

ROLLING RESISTANCE OF TIRES

The rolling resistance of tires on hard surfaces is primarily caused by the hysteresis in tire materials due to the deflection of the carcass while rolling. Friction between the tire and the road caused by sliding, the resistance due to air circulating inside the tire, and the fan effect of the rotating tire on the surrounding air also contribute to the rolling resistance of the tire, but they are of secondary importance When a tire is rolling, the carcass is deflected in the area of ground contact.

As a result of tire distortion, the normal pressure in the leading half of the contact patch is higher than that in the trailing half. The center of normal pressure is shifted in the direction of rolling. This shift produces a moment about the axis of rotation of the tire, which is the rolling resistance moment. In a free-rolling tire, the applied wheel torque is zero; therefore, a horizontal force at the tire–ground contact patch must exist to maintain equilibrium. This resultant horizontal force is generally known as the rolling resistance.
The ratio of the rolling resistance to the normal load on the tire is defined as the coefficient of rolling resistance.
A number of factors affect the rolling resistance of a pneumatic tire. They include the structure of the tire (construction and materials) and its operating conditions (surface conditions, inflation pressure, speed, temperature, etc.). Tire construction has a significant influence on its rolling resistance. Figure 1.3 shows the rolling resistance coefficient at various speeds of a range of bias-ply and radial-ply passenger car tires at rated loads and inflation pressures on a smooth road. The difference in rolling resistance coefficient between a bias-ply and a radial-ply truck tire of the same size under rated conditions is shown in Fig. 1.4. Thicker treads and sidewalls and an increased number of carcass plies tend to increase the rolling resistance because of greater hysteresis losses. Tires made of synthetic rubber compounds generally have higher rolling resistance than those made of natural rubber.

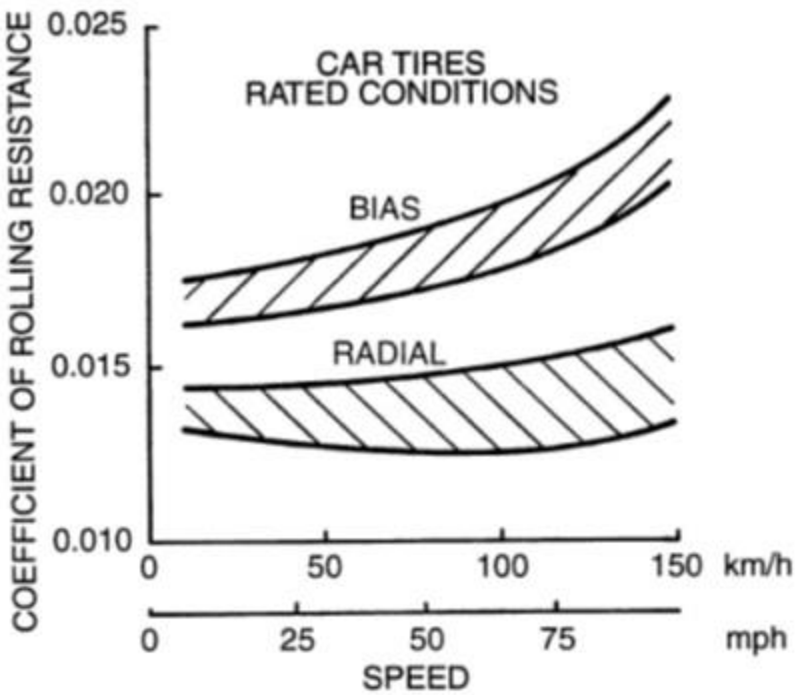


Figure 4: Variation of rolling resistance coefficient of radial ply and bias ply tire in different road condition.

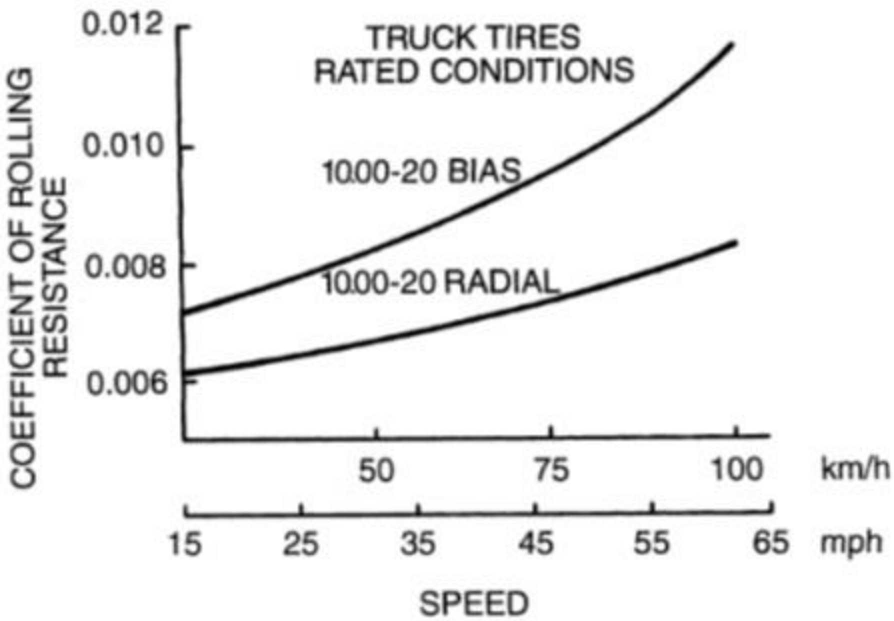


Figure 5: Variation of rolling resistance coefficient of radial ply and bias ply tire in different road condition and inflation pressure.

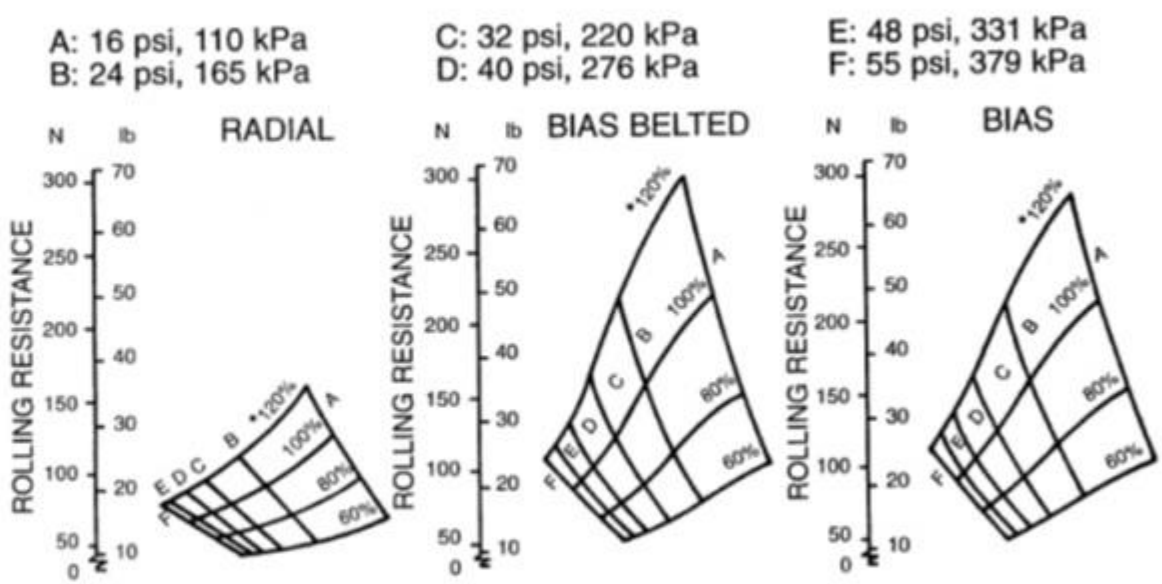


Figure 6: Variation of rolling resistance coefficient of radial ply, bias and belted bias ply tire in different road condition and inflation pressure.

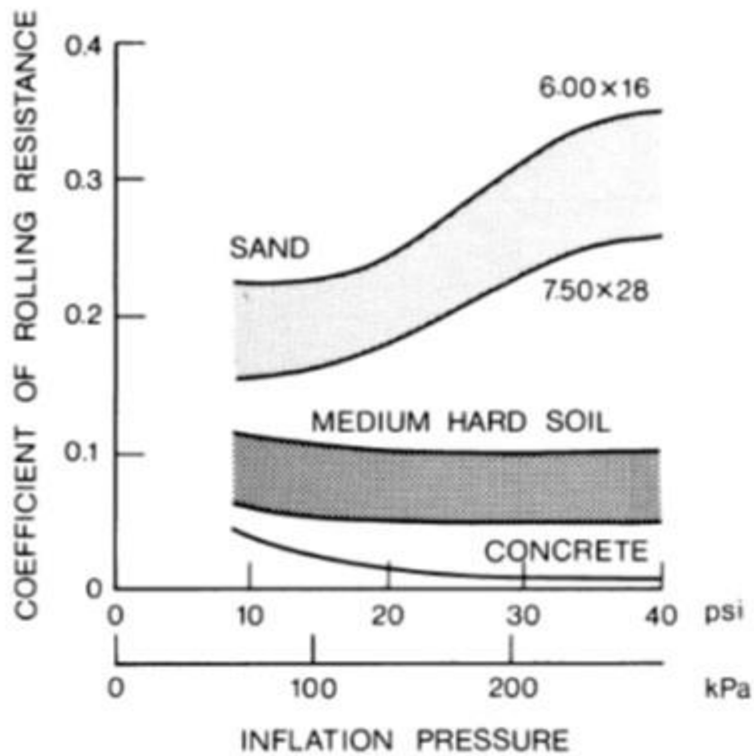


Figure 7: Variation of rolling resistance coefficient with inflation of tire pressure on various surfaces.

TRACTIVE (BRAKING) EFFORT AND LONGITUDINAL SLIP (SKID) OF TIRE

When a driving torque is applied to a pneumatic tire, a tractive force is developed at the tire–ground contact patch, as shown in Fig. 1.15 [1.6]. At the same time, the tire tread in front of and within the contact patch is subjected to compression. A corresponding shear deformation of the sidewall of the tire is also developed.

As tread elements are compressed before entering the contact region, the distance that the tire travels when subject to a driving torque will be less than that in free rolling. This phenomenon is usually referred to as longitudinal slip. The longitudinal slip of the vehicle running gear, when a driving torque is applied, is usually defined by

$$i = \left(1 - \frac{V}{r\omega}\right) \times 100\% = \left(1 - \frac{r_e}{r}\right) \times 100\%$$

where V is the linear speed of the tire center, ω is the angular speed of the tire, r is the rolling radius of the free-rolling tire, and *re* is the effective rolling radius of the tire, which is the ratio of the linear speed of the tire center to the angular speed of the tire.

TABLE 1.1 Coefficient of Rolling Resistance

Road Surface	Coefficient of Rolling Resistance
Car tires	
Concrete, asphalt	0.013
Rolled gravel	0.02
Tarmacadam	0.025
Unpaved road	0.05
Field	0.1–0.35
Truck tires	
Concrete, asphalt	0.006–0.01

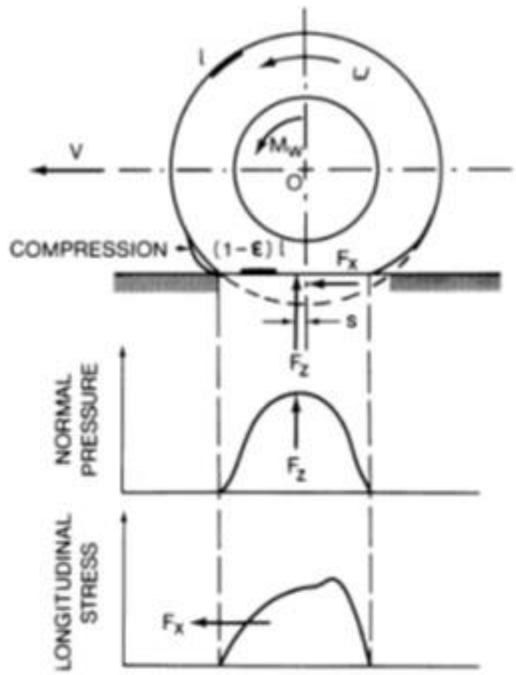


Figure 8: Behaviour of a tire under the action of a driving torque.

When a driving torque is applied, the tire rotates without the equivalent translatory progression; therefore, $r_w > V$ and a positive value for slip results. If a tire is rotating at a certain angular speed but the linear speed of the tire center is zero, then in accordance with Eq. 1.5, the longitudinal slip of the tire will be 100%. This is often observed on an icy surface, where the driven tires are spinning at high angular speeds, while the vehicle does not move forward.

Longitudinal slip is defined as ‘‘the ratio of the longitudinal slip velocity to the spin velocity of the straight free-rolling tire expressed as a percentage.’’

The longitudinal slip velocity is taken as ‘‘the difference between the spin velocity of the driven or braked tire and the spin velocity of the straight free-rolling tire

CORNERING PROPERTIES OF TIRES

Slip Angle and Cornering Force

When a pneumatic tire is not subject to any force perpendicular to the wheel plane (i.e., side force), it will move along the wheel plane. If, however, a side force *Fs* is applied to a tire, a lateral force will be developed at the contact patch, and the tire will move along a path at an angle (α) with the wheel plane, as *OA* shown in Fig.9. The angle (α) is usually referred to as the slip angle, and the phenomenon of side slip is mainly due to the lateral elasticity of the tire.

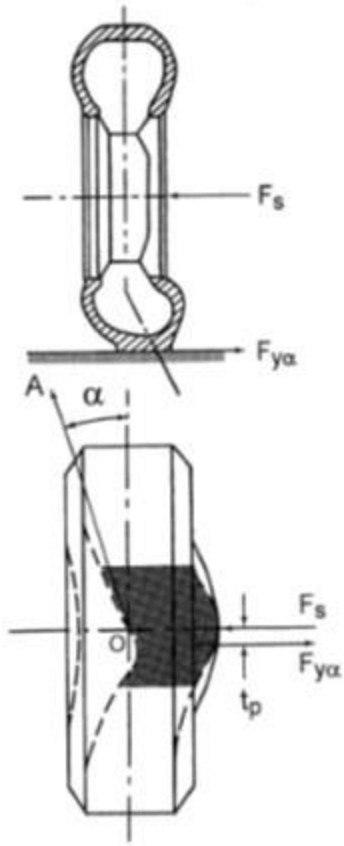


Figure 9: Behaviour of tire subjected to side force.

The lateral force developed at the tire–ground contact patch is usually called the cornering force *Fy_* when the camber angle of the wheel is zero. The relationship between the cornering force and the slip angle is of fundamental importance to the directional control and stability of road vehicles. At small slip angles, the cornering force in the ground plane is normally behind the applied side force, giving rise to a torque (or couple), which tends to align the wheel plane with the direction of motion. This torque is called the aligning or self-aligning torque, and is one of the primary restoring moments which help the steered tire return to the original position after negotiating a turn. The distance *tp* between the side force and the cornering force is called the pneumatic trail, and the product of the cornering force and the pneumatic trail determines the self-aligning torque.

Typical plots of the cornering force as a function of the slip angle for a bias-ply and a radial-ply passenger car tire are shown in Fig.9. It can be seen that for slip angles below a certain value, such as 4_ shown in Fig 10, the cornering force is approximately proportional to the slip angle. Beyond that, the cornering force increases at a lower rate with an increase of the slip angle, and it reaches a maximum value where the tire begins sliding laterally. For passenger car tires, the maximum cornering force may occur at a slip angle of about 18°, while for racing car tires, the cornering force may peak at approximately 6°. Figure 10 shows that the cornering force of a bias-ply tire increases more slowly with an increase of the slip angle than that of a radial-ply tire. These characteristics are considered to be more suited to two-wheeled vehicles, such as motorcycles. A more gradual increase of the cornering force with the slip angle enables the driver to exercise better control over a two-wheeled vehicle. This is one of the reasons why bias-ply tires are used for motorcycles.

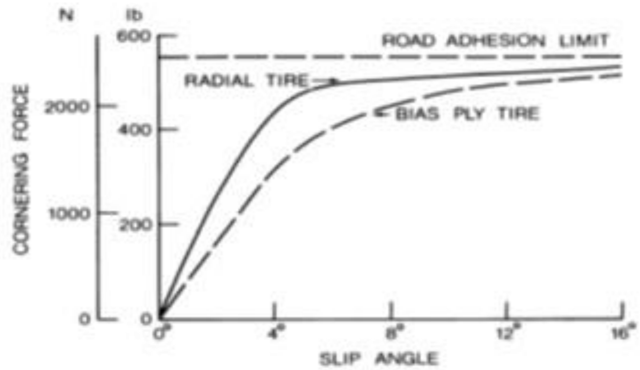


Figure 10: Cornering characteristics of a bias ply and radial ply car tire.

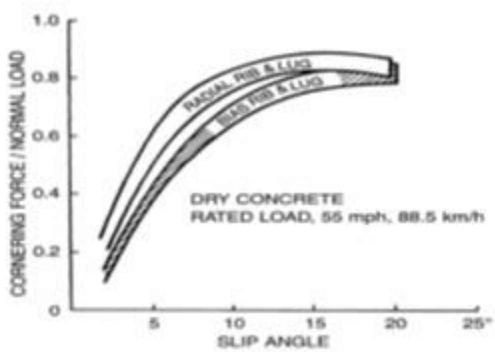


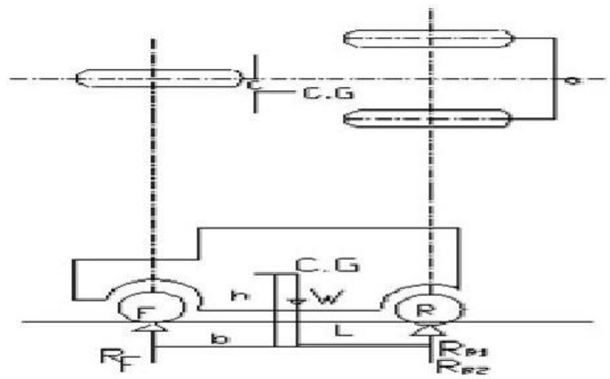
Figure 11: Cornering characteristics of a bias ply and radial ply of truck tire.

LOAD DISTRIBUTION ON VEHICLE

1. LOAD DISTRIBUTION

THREE WHEELED VEHICLE:

The forces acting on a vehicle at rest are shown in figure.



Where,

W= weight of the wheel, N,
b= wheelbase, m,
l=distance of CG from the rear axle, m,
h=height of CG from road surface, m,
c=distance of CG from the central axis, m,
a= wheel track, m,
RF = vertical reaction at front wheel, N,
RR1, RR2 = vertical reaction at the rear wheels, N

There are three unknowns which can be determined as follows:

Moment about rear axle gives

$$R_F b = Wl$$

Therefore,

$$R_F = \frac{Wl}{b}$$

Moment about central axis gives,

$$(R_{R1} + R_{R2})\frac{a}{2} = Wc$$

Therefore,

$$R_{R2} - R_{R1} = \frac{2W_c}{a}$$

Moment about central axis of the front wheel gives,

$$(R_{R1} + R_{R2})b = W(b - l)$$

Therefore

$$R_{R1} + R_{R2} = W\frac{(b-l)}{b} = W\left(1 - \frac{l}{b}\right)$$

Addition and subtraction of last two equations respectively gives,

$$R_{R2} = \frac{W}{2}\left(\frac{2c}{a} - \frac{l}{b} + 1\right)$$

And

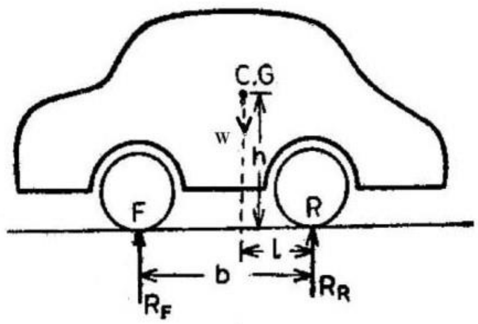
$$R_{R1} = \frac{W}{2}\left(1 - \frac{l}{b} - \frac{2c}{a}\right)$$

Also $W = R_F + R_{R1} + R_{R2}$ must be satisfied and server as an extra equation for alternative solution.

FOUR-WHEELED VEHICLE:

Forces acting on a four-wheeled vehicle at rest, we can form three independent equations to take care of four unknown viz., four reactions at the wheels. Thus the problem is simplified by considering it as a two-wheeled vehicle, i.e. the reactions on both rear wheels is equal and also on both front wheels. Let R_F and R_R be vertical reactions at front and rear wheel respectively,

The,



$$\sum V = 0$$

Gives,

$$W = R_R + R_F$$

$$\sum M_R = 0$$

Gives,

$$Wl = R_F b$$

$$R_F = \frac{Wl}{b}$$

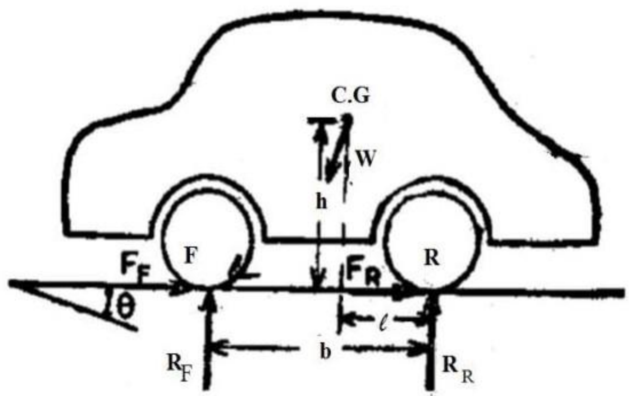
Substituting the value of R_F in the above,

$$R_R = W(1 - \frac{l}{b})$$

2. STABILITY ON A SLOPE, CURVED TRACK AND A BANKED ROAD

STABILITY OF A VEHICLE ON A SLOPE:

Let the vehicle rests on a slope of inclination Θ to the horizonta1. This alters the distribution of the weight between the front and back axle and gives rise to reaction which can have components along the perpendicular to the inclined plane as shown.



Now, resolving forces parallel and perpendicular to the slope respectively,

$$W \sin\theta = F_F + F_R$$
$$W \cos\theta = R_F + R_R$$

And $\sum M_F = 0$

$$Wh \sin\theta + R_R b = W(b - l)\cos\theta$$

Therefore,

$$R_R = \frac{W}{b}[(b - l)\cos\theta - h\sin\theta]$$

And hence

$$R_F = W\cos\theta - R_R$$

$$= Wcos\theta - Wcos\,\theta + \frac{Wl}{b}cos\theta + \frac{Wh}{b}sin\theta$$
$$= \frac{W}{b}[lcos\theta + hsin\theta]$$

If an angle Θ is gradually increased a situation arises when

(a)Either the vehicle is about to overturn or

(b)The vehicle is about to slide down the slope

The Instability due to case (a) happens at a point when RF becomes zero and hence,

$$\frac{W}{b}[(b - l)cos\theta - hsin\theta]$$

Thus the limiting angle ΘL for **overturning** is given by,

$$\mathbf{tan\,\theta_L = \left(\frac{b - l}{h}\right)}$$

This indicates that at the point of overturning, the lines of action of weights W passes through the contact point F. For the case (b) to arise, the limiting value of Θ is given b

$$W \sin\theta = F_F + F_R$$

The value of (FR+ FL) can be determined under following condition. Let the brakes be applied to prevent this situation.

Then two cases may arrive:

(i) The brakes are not efficient enough to prevent the wheels from turning before they slide. In case the limiting value of Θ can be determined by the brake Torques available.

If TF and TR are the braking Torques at front and rear wheels respectively then,

$$T_F = F_F r$$
$$T_R = F_R r$$

Where r is the radius of the wheel

Now,

$$F_F + F_R = \frac{T_F + T_R}{r} = W \sin\theta_L$$

Or

$$\sin\theta_L = \frac{T_F + T_R}{Wr}$$

(ii) The Brakes are Sufficiently powerful for the coefficient of adhesion μ to limit the sliding of the vehicle, then,

$$F_F + F_R = \mu(R_F + R_R) = \mu W \cos\theta$$
$$F_F + F_R = \mu W \sin\theta$$

But

Therefore, ΘL is given by,

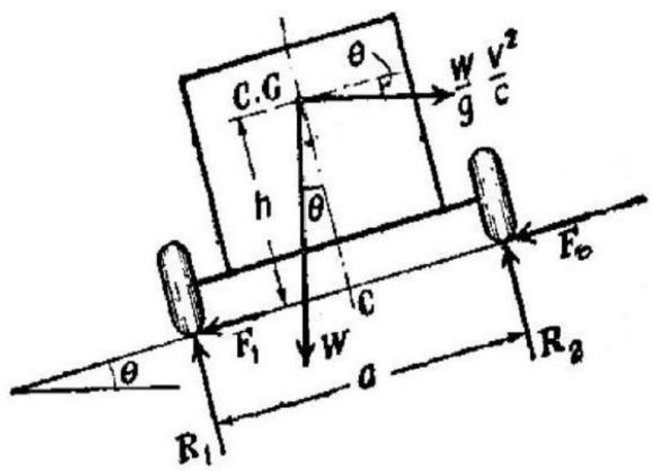
$$W \sin\theta_L = \mu W \cos\theta_L$$

Or

$$\tan\theta_L = \mu$$

It should be noted up that when the vehicle is being driven up, the angle of overturning is in,
General, smaller than in the present case and also the condition of instability becomes different from those discussed above.

STABILITY OF A VEHICLE ON A BANKED TRACK:



The force, which are acting are shown in fig.
Where,

W= weight of vehicle
V= velocity of vehicle on banked track.
C= radius of curved path measured at CG of vehicle.
R_I and R_o=normal reaction at inner and outer wheels respectively,
F_I and F_o=normal reaction at inner and outer wheels respectively,
μ= coefficient of adhesion between tyre and road surface
a = length of wheel track,
Θ= inclination of wheels axes to the horizontal,
$\frac{WV^2}{gC} = centrifugal\ force\ actign\ at\ CG$

Then,
Equating sum of vertical force to zero:

$$\sum V = 0$$

$$R_O + R_I = W\cos\theta + \frac{WV^2}{gC}\sin\theta$$

Equating sum of Horizontal force to zero:

$$\sum H = 0$$

$$F_O + F_I = \frac{WV^2}{gC}\cos\theta - W\sin\theta$$

Equating sum of moment about 'o' equal to zero:

$$\sum M_o = 0$$

$$(R_o - R_I)\frac{a}{2} = \frac{w}{g}\frac{V^2}{C}h\cos\theta - Wh\sin\theta$$

$$(R_o - R_I) = \frac{w}{g}\frac{V^2}{C}\frac{2h}{a}\cos\theta - W\frac{2h}{a}\sin\theta$$

$$2R_o = \frac{W}{g}\frac{V^2}{C}\left(\sin\theta + \frac{2h}{a}\cos\theta\right) + W\left(\cos\theta - \frac{2h}{a}\sin\theta\right)$$

And subtracting equation (iii) from equation (i)

$$2R_I = \frac{W}{g}\frac{V^2}{C}\left(\sin\theta - \frac{2h}{a}\cos\theta\right) + W\left(\cos\theta + \frac{2h}{a}\sin\theta\right)$$

If the vehicle begins to slide outward along the banking at the limiting speed, V_s, then,

$$F_1 + F_0 = \mu(R_1 + R_0)$$

Substitution of equation (i) and (ii) in equation (iii) gives,

$$\mu W\cos\theta + \mu\frac{W}{g}\frac{V_s^2}{C}\sin\theta = \frac{W}{g}\frac{V_s^2}{C}\cos\theta - W\sin\theta$$

Or,

Limiting speed: $$\boldsymbol{V_s^2 = gC\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}$$

If V_0 is the overturning speed of vehicle then putting $R_I = 0$ in equation (i) and (iii), gives

$$W\cos\theta + \frac{W}{g}\frac{V_o^2}{C}\sin\theta = \frac{W}{g}\frac{V_o^2}{C}\frac{2h}{a}\cos\theta - W\frac{2h}{a}\sin\theta$$

or,

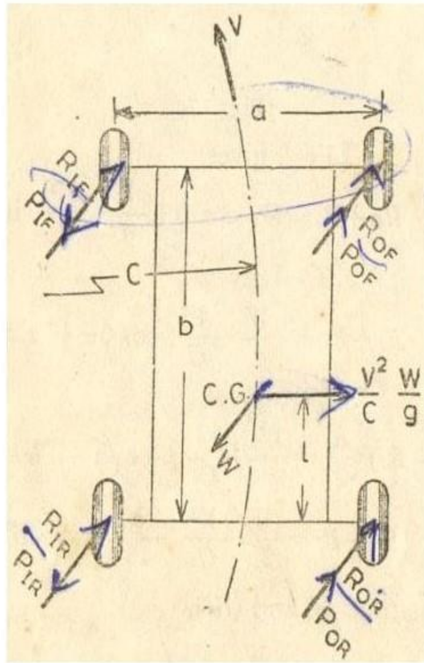
Overturning Speed:

$$\boldsymbol{V_o^2 = gc\frac{(a\,sin\theta + 2h\,cos\theta)}{2h\,cos\theta - asin\theta}}$$

CURVED TRACK

Stability of a Vehicle Taking a Turn

a) A four-wheeled vehicle



Let a vehicle take a turn to the left as shown in fig.

Where C = radius of curved path measured at C.G. of the vehicle, m
r = wheel radius, m
a = wheel track, m
b = wheel base, m
h = height of C.G of the vehicle from ground, m
l = distance of C.G. in front of rear axle axis, m
V = linear speed of the vehicle on the road, m/sec
W = weight of the vehicle, kgf.

I) Reaction at the wheels due to weight

Let R_{IF} and R_{IR} be the normal reactions at the inner front and inner rear wheels respectively and R_{OF} and R_{OR} be the normal reactions at the outer front and outer rear wheels respectively.

$$R_{IF} + R_{OF} = \frac{Wl}{b}$$

$$R_{IR} + R_{OR} = W\left(1 - \frac{l}{b}\right)$$

Since, $R_{IF} = R_{OF}$

and $R_{IR} = R_{OR}$

therefore, $R_{IF} = R_{OR} = \frac{Wl}{2b}\text{kgf}$

$$R_{IR} = R_{OR} = \frac{W}{2}\left(1 - \frac{l}{b}\right)\text{kgf}$$

ii) Reaction at the wheels due to centrifugal force

The centrifugal force acts outward through C.G of the vehicle with magnitude = $\frac{W}{g}-\frac{V^2}{C}$. This produces a horizontal reaction that constitutes a couple $\left(=\frac{W}{g}\frac{V^2}{C}h\right)$ tending to overturn the vehicle. This couple is balanced by vertical reactions at the wheels which are downward at the two inner wheels and upward at the two outer wheels, as shown in above fig.

If P_{IF} and P_{OF} are the inner and outer normal reactions at front wheels and similarly P_{IR} and P_{OR} for the rear wheels, then

$$P_{IF}+P_{IR}=P_{OF}+P_{OR}=\frac{W}{g}\frac{V^2}{C}\frac{h}{a}\text{kgf}$$

$$\text{Now}\qquad P_{IF}=P_{OF}=\frac{Wl}{2b}\frac{V^2}{gC}\frac{h}{a}\text{kgf}$$

$$\text{and}\qquad P_{IR}+P_{OR}=\frac{W}{2b}\left(1-\frac{l}{b}\right)\frac{V^2}{gC}\frac{h}{a}\text{kgf}$$

iii) Reaction at the wheels due to Gyroscopic effect

*If a body revolves about OX and if a couple, called Gyroscopic couple, is applied along OY, then the body tries to process about axis OZ. this is called Gyroscopic effect. The planes of spin, gyroscopic couple and precession are mutually perpendicular. The direction of precession is such that the axis of spin tends to place itself in line with the axis of the applied torque (i.e gyroscopic torque) and in the same sense.

The magnitude of applied gyroscopic couple = $I\omega\omega_p$
Where I = the polar moment of inertia of the body
ω = angular velocity of spin
and ω_p = angular velocity of precession

The reaction couple exerted on the body is equal in magnitude to the applied couple but opposite in direction.

When the vehicle takes a turn the gyroscopic* effect appears

a) due to the precession of rotating wheels and other parts either rotating at the engine speed or the wheel speed but parallel to the plane of rotation of the wheel.
b) due to the precession of engine parts and also others rotating either at engine or wheel speed but perpendicular to the plane of rotation of wheel. The axes of rotation in (a) and (b) are horizontal.

Let I_f = moment of inertia of the rotating parts of the engine (faster moving), kgf-m^2
I_s = moment of inertia of the slower rotating parts like wheels, kgf-m^2
ω_f, ω_s = angular velocity of the engine and the wheel respectively (velocity of spin), rad/sec

G = overall gear ration = ω_f / ω_s
N = r.p.m. of the engine = $(60/2\pi)\ \omega_f$.

Case (a) we have

$$\omega_s = \frac{V}{r}, \quad \omega_f = \pm\frac{GV}{r}$$

Then, $$\sum I\omega = I_s\frac{V}{r} \pm I_f\frac{GV}{r} = \frac{V}{r}\left(I_s \pm GI_f\right)$$

-ve sign has been incorporated to take care of situations where the direction of ω_f is opposite to that of ω_s.

The angular velocity of precession,

$$\omega_p = \frac{V}{C}\,\text{rad/sec} = \frac{2\pi}{60}\frac{Nr}{GC}\,\text{rad/sec}$$

Hence the gyroscopic couple

$$= \frac{V^2}{Crg}\left(I_s \pm GI_f\right)\text{kgf} - \text{m}$$

The magnitude of vertical reaction due to this couple is same at the outer and inner wheels and its effect (reaction couple is to roll the car in an outward manner similarly to that caused by the centrifugal effect. The situation is shown in fig.

If Q_{IF} and Q_{OF} are respectively inner and outer vertical reactions at the front wheels and Q_{IR} and Q_{OR} are similarly at the rear wheels, then

$$Q_I = Q_{IF} + Q_{IR} = Q_{OF} + Q_{OR} = Q_O$$
$$= \frac{V^2}{Crag}\,(I \pm GI_f)$$

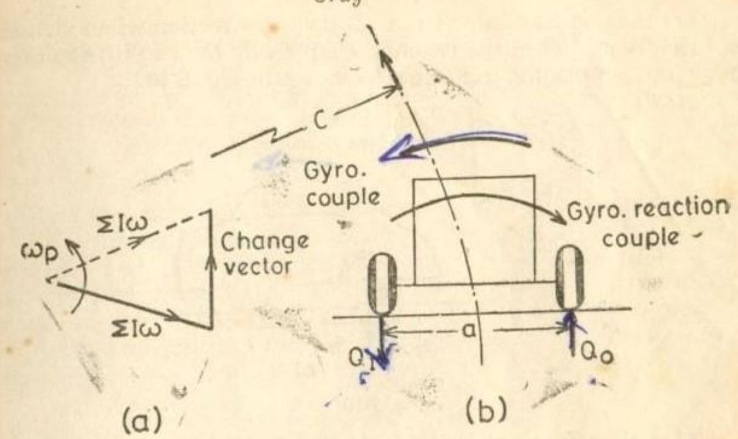


Fig. 9·9

$$\therefore \quad Q_{IF} = Q_{OF} = Q_{IR} = Q_{OR}$$
$$= \frac{V^2}{2Crag}\,(I_s + GI_f)\ \text{kg}_f$$

Where Q_{IF} and Q_{IR} act downward and Q_{OF} and Q_{OR} act upward.

In this analysis, I_s represents the moment of inertia of all the wheels taken together; and the moment of inertia of the individual wheels have been assumed to equal one another. If the moment of inertia of front and rear wheels are different then while calculating individual reactions at the wheels, the respective moment of inertia should be used; but for the calculation of the total gyroscopic torque, individual moment of inertia can be added, giving $I_s = \sum I\omega$, where $I\omega$ = moment of inertia of the wheels. This is because the velocity of spin and the velocity of precession are the same for all the wheels.

Case (b) In this case also we can write

$$\sum I\omega = I_f \frac{Gv}{r} \pm I_s \frac{V}{r}$$

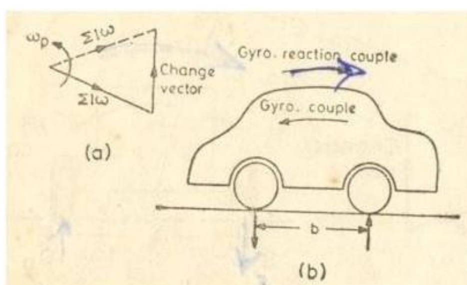
But here I_s stands for the moment of inertia of the slower moving parts rotating in a plane parallel to the plane of the rotation of the engine. The –ve sign takes care of the rotation in a direction opposite to that of the engine.
The angular velocity of precession

$$= \frac{V}{C} \text{ rad/sec}$$

Hence gyroscopic couple

$$= \frac{V^2Crg}{}\left[I_f G \pm I_s\right] \text{kgf} - \text{m}$$

Let the engine rotate in the clockwise direction when viewed from the front. then the reaction couple will tend to lift the rear wheels and depress the front ones as shown in fig.



Thus the effect is to increase the front wheel loads or decrease the rear wheel loads by

$$\frac{V}{2Crag}\left[I_f G \pm I_s\right] \text{kgf}$$

If during a left-hand turn, the engine rotates in the anti-clock-wise direction when viewed from the front, the effect of the reaction couple due to the engine gyroscopic couple will be just the reverse of the above situation, i.e the reaction couple will tend to lift the front wheels and depress the rear ones.

Now if the vehicle takes a right-hand turn, the above results for the inner and outer wheels will not be affected both in magnitude and direction but the gyroscopic torque due to the engine will be only affected in direction and will be opposite that in the above cases.

When the vehicle takes a turn, the present analysis reveals that precessional (that of wheels) and centrifugal effects are added together and act in a direction opposite to the static distribution of the load at the wheels tending to overturn it. Considering the sum total of the reactions at the inner and outer wheels it can be stated that the parameters responsible for overturning are as follows.

1. If the vehicle taking a turn at high speed. i.e ω is high.
2. If the loaded vehicle is sufficiently high over the ground making h high.
3. If the vehicle is taking a sharp turn, i.e C is small.
4. If the vehicle is overloaded, i.e W is high.

In this article, the case of the vehicle taking a turn on a level is considered. If the turning of the vehicle on a banked track is considered, then an additional parameter θ, the inclination of the wheel axes to the horizontal will appear. Art 9.10 gives partial treatment of this situation where gyroscopic effects have not been considered.

3. CALCULATION OF TRACTIVE EFFORT AND REACTIONS FOR DIFFERENT DRIVES

Calculation of Maximum Acceleration, Maximum Tractive Effort and Reactions for Different Drives:

Front Wheel Drive:

The forces acting on the vehicle and giving rise to dynamic equilibrium are shown in fig.

If

b = wheel base,
h = height of C.G from the road surface,
l = distance of C.G from rear axle,
μ = coefficient of adhesion between the tyres and road surface
RF and RR = total normal reactions at front and rear wheels respectively,
W = weight of the car.

Then the maximum tractive effort, $F_F = \mu R_F$ produces maximum forward acceleration, f and (W/g)f is the inertia force opposite to acceleration, f.

Hence ∑V = 0 gives,

$$W = R_F + R_R$$

∑H = 0 gives,

$$F_F = \mu R_F = \frac{W}{g} f$$

And

∑MR = 0 gives,

$$R_F b + \frac{W}{g} f h = wl$$

Substituting the value of RF, gives

$$\frac{W}{g} f \frac{1}{\mu} b + \frac{W}{g} f h = Wl$$

$$\frac{f}{g}\left(\frac{b}{\mu} + h\right) = l$$

Therefore,

$$\frac{f}{g} = \frac{\mu l}{b + \mu h}$$

Solving R_F and R_R become

$$R_F = \frac{l}{b+\mu h} W$$

$$R_R = \frac{b - l + \mu h}{b + \mu h} W$$

Rear Wheel Drive:

Here the tractive effort acts only on rear wheels. Hence eliminating FF and applying FR to the rear wheels in fig., maximum tractive effort becomes,

$$F_R = \mu R_R$$

Now, $\sum$V=0 gives, W=RR+RF

$\sum$H=0 gives, FR = μRR=$(W/g)f$

And

$\sum$MF=0 gives,

RR b = W$(b - l) + (W/g)fh$

Substituting the value of RR gives,

$$\frac{W}{g}\frac{f}{\mu}b = W(b-l) + \frac{W}{g}fh$$

Or,

$$\frac{f}{g}\left(\frac{b}{\mu} - h\right) = b - l.$$

Therefore,

$$\frac{f}{g} = \frac{\mu(b-l)}{b - \mu h}$$

Solving of RR and RF gives,

$$RR = \frac{b-l}{b-\mu h}W \quad \text{and,}$$

$$RF = \frac{l-\mu h}{b-\mu h}W$$

FOUR WHEEL DRIVE

This may be with or without third differential

(1) Without third differential:-in this case both FF and FR come into play. Assuming that limiting friction occurs at all the four wheels simultaneously, the maximum tractive effect,

$$F=FR+FF = \mu RR + \mu RF$$

$$\Sigma V=0,$$

Gives

$$W=RF+RR$$

$$\Sigma H=0,$$

Gives

$$(W\backslash g)\ f = \mu RR + \mu RF = \mu(RR+RF) = \mu W$$

Hence,

$$(f/g)=\mu$$

(2)With third differential:-The torque at the front and rear wheels becomes equal with the application of third differential. Slip occurs at the wheels where the normal reaction is smaller and thus limits the tractive effect. In case, the load distribution to the front and rear wheels is equal, the slip has to occur first at the front wheels because the static normal reaction at front wheels is reduced due to inertia effect.

Thus,

$$\Sigma V=0$$

Gives

$$W=RR+RF$$

$$\Sigma H=0$$

Gives,

$$(W/g)\ f = \mu RR + \mu RF$$

And μ_sRR=μRF due to application of third differential, where μ_s is the critical working coefficient of friction being less than μ, the limiting value.

Assuming slip to occur at front wheels first, RF<RR, then

$$2\mu RF = (W/g)f$$

$\Sigma MR = 0$ gives,

$$RF\ b + (W/g)\ f\ h = Wl$$

Substituting the value of

PROBLEMS 1:

A car weighing 2175 kgf has a static weight distribution on the axles of 50 : 50. The wheel base in 3 m and the height of centre of gravity above ground is 0.55 m. if the coefficient of friction on the highway is 0.6, calculate the advantage of having rear wheel drive rather than front wheel drive as far as gradiability is concerned, if engine power is not a limitation.

Solution: C.G. bisects the wheel base as distribution of static weight on the axle is 50 : 50.

Hence, $l = \frac{b}{2} = 1.5\ \text{m}.$

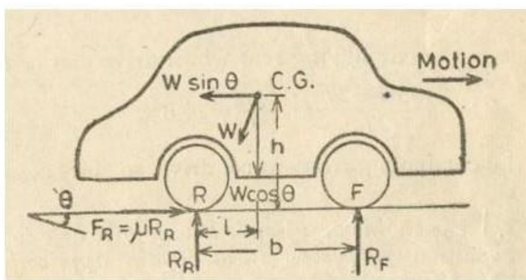
Rear Wheel Drive

Considering the car moving at constant speed up a grade of angle θ, the forces giving equilibrium are as shown in fig.

Then, we have

$$R_R + R_F = W\cos\theta$$

and $$F_R = \mu R_R = W\sin\theta$$



By division, $$\tan\theta = \frac{\mu R_R}{R_R + R_F}$$

Taking moment about G.G

$$R_F\ 1.5 + \mu R_R\ 0.55 = R_R\ 1.5$$

or $$R_F = R_R\left(1 - \frac{\mu 0.55}{1.5}\right)$$

$$= R_R\left(1 - \frac{0.6 \times 0.55}{1.5}\right)$$

$$= R_R(1 - 0.2) = 0.78_R$$

Therefore, $$\tan\theta = \frac{0.6R_R}{R_R + 0.78R_R} = \frac{0.6}{1.78} = 0.337$$

Percentage grade = -tan θ × 100 = 33.7%

Hence in case of rear wheel drive the maximum grade which the car can negotiate is 33.7%.

Front Wheel Drive

In this case also, similarly as last one, we have

$$R_R + R_F = W \cos \theta$$

$$R_F = \mu R_F = W \sin \theta.$$

Hence, $$\tan \theta = \frac{\mu R_F}{R_R + R_F}$$

$\sum M c.G. = 0$ gives,

$$R_F 1.5 + \mu R_F 0.55 = R_R 1.5$$

or, $$R_R = R_F\left(1 + \frac{0.6 \times 0.55}{1.5}\right) = 1.22\ R_F$$

Therefore, $$\tan\theta = \frac{0.6 R_F}{R_F + 1.22 R_F} = \frac{0.6}{2.22}$$

$$= 0.27 \text{ or } 27\%$$

Therefore, the car having rear wheel drive can negotiate

$$\frac{33.7 - 27}{27} = \frac{6.7}{27} = 24.8\%$$

more than the car having front wheel drive, so far as gradiability is concerned

---

PROBLEMS 2:

A motor car with wheel base 275 cm with a centre of gravity 85 cm above the ground and 115 cm behind the front axle has a coefficient of adhesion 0.6 between the tyre and the ground. Calculate the maximum possible acceleration when the vehicle, is.

a) driven on four wheels,

b) driven on the front wheels only

c) driven on the rear wheels only

Solution :

b =2.75 m

l = 3.75 – 1.15 = 3.6 m

h = 0.85 m

μ = 0.6

a) **Four wheel drive**

Since it has not been mentioned about the use of third differential we shall take up both the cases.

i) Without third differential ;

$f = \mu g = 0.6 \times 9.81 = 5.886$ m/sec².

ii) With third differential ; assuming that the slip occurs first at the front wheels, then

$$f = \frac{2\mu l\ g}{b+2\mu h} = \frac{2\times 0.6\times 1.6\times 9.81}{2.75+2\times 0.6\times 0.85}$$

$$= \frac{18.82}{2.75+1.02} = \frac{18.82}{3.77} = 5\ m/sec2$$

check $\quad R_F = \frac{l}{b+2\mu h} W = \frac{1.6}{3.77} W = 0.425\ W.$

i.e $R_F < R_R$ (=0.575 W), hence our assumption is correct.

b) **Front wheel drive**

$$f = \frac{\mu l\ g}{b+\mu h} = \frac{0.6\times 1.6\times 9.81}{2.75+0.6\times 0.85}$$

$$= \frac{9.41}{2.75+0.51} = \frac{9.41}{3.26}$$

$$= 2.89\ m/sec^2$$

c) Rear wheel drive

$$f = \frac{\mu(b-l)\ g}{b-\mu h} = \frac{0.6(2.75-1.6)9.81}{2.75-0.6\times 0.85}$$

$$= \frac{0.6\times 1.15\times 9.81}{2.75-0.51} = \frac{6.765}{2.24}$$

$$= 3.02\ m/sec^2$$

PROBLEMS 3:

A vehicle of total weight 5000 kg, is held at rest on a slope of 10°. It has a wheel base of 225 cm and its centre of gravity is 100 cm in front of the rear axle and 150 cm above the ground level.

Find:

a) What are the normal reactions at the wheels?
b) Assuming that sliding does not occur first, what will be the angle of slope so that the vehicle will overturn?
c) Assuming all the wheels are to be braked, what will be the angle of the slope so that the vehicle will begin to slide if the co-efficient of adhesion between the type and the ground is 0.35?

Solution:

a) $\quad R_F + R_R = W \cos\theta$

$$= 5000\times \cos 10 = 5000\times 0.9848$$

$$= 4924.0\ kg^f.$$

$$R_R = \frac{W}{b}\left[(b-1)\cos\ \theta - h\ \sin\theta\right]$$

$$= \frac{5000}{2.25}\left[(2.25-1.0)\times 0.9848 - 1.5\times 0.1736\right]$$

$$= \frac{5000}{2.25}[1.25 \times 0.9848 - 0.2604]$$

$$= (5000 \times 0.9706)/2.25 = 21.59 \text{ kgf.}$$

Therefore, $R_F = 4924 - 2159 = 2765$ kgf.

b) $\tan \theta_L = (b - l)/h$

$=(2.25-1.0)/1.5=0.8337$

Hence $\theta_L = 39^\circ\ 49'$.

c) $\tan \theta_L = \mu = 0.35$

Hence $\theta_L = 19^\circ\ 17'$.

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UNIT V WHEELS AND TIRES

UNIT 5 WHEELS AND TYRES

Automobile wheels, pneumatic tires- construction, properties, characteristics, tread, bite, operation and inflation pressure, wheel balancing, tire dynamics, ride characteristics, power consumed by tire, over steer, under steer, steady state cornering, effect of breaking, driving torques on steering, effect of camber, transient effects in cornering.

WHEELS

Modern vehicle wheels generally comprise a rim and nave or wheel disc. The rim is that part of the wheel on which the tire is mounted. The nave joins the rim to the wheel hub. Wheel size is primarily determined by the wheel-brake diameter and by the load-bearing capacity of the tire. The most important terms are: rim width, rim diameter, center hole, hole circle diameter, number of mounting holes, countersunk and stud design, and rim offset.

Rim designs

Rims differ from one another (depending upon the type of tire) in terms of number of parts and cross-sectional rim shape. The most important rim details are: rim flange, rim bead seat and rim base. Rims have one of the following cross-sectional shapes:

- Drop center,
- Flat base,
- 5° tapered bead seat or
- 15° tapered bead seat.

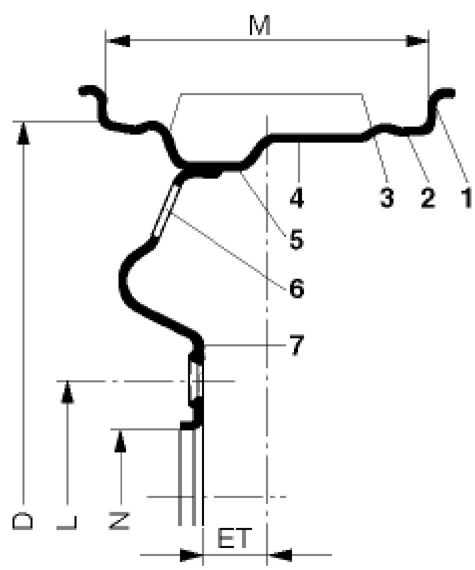


Figure 1: Cross sectional view of Disc wheel (e.g. 6J x 14 H 2).

Rim flange (e.g. "J" section), 2 5° tapered bead seat, 3 Hump (e.g. double hump H 2), 4 Rim, 5 Rim well, 6 Vent hole, 7 Wheel disc. D Rim diameter (e.g. 14"), L Hole circle diameter, M Rim width (e.g. 6"), N Center hole, ET Rim offset.

Passenger Car wheel

Mass-produced wheels are made of sheet steel, with forged or cast aluminum and (less commonly) magnesium being used primarily for special wheels and aftermarket wheels. Sheet aluminum, although used in some cases, has not become popular for reasons of cost.
The wheels in particular feature considerable potential when it comes to complying with the general trend towards lightweight construction in the automobile industry. The sheet-aluminum wheel (with the wheel cap in the form of a full cover or painted in silver as a styling variant), and the split wheel derived from the classical forged wheel, are among the lightest wheel versions while at the same time being reasonably priced considering the weight savings involved.
Whereas the manufacturing process for the two-piece sheet-aluminium wheels and for the steel wheel are practically identical, a difference is made between two different versions of 1-piece split wheel:
1. An extruded section formed by a forging process. This is rolled and split at its circumference.
2. A circular blank stamped from the coil and deep-drawn.
There are further important lightweight-wheel versions on the market although these have still not become very widespread. They include the cast-aluminium wheel with pressure-rolled rim and the structured wheel which is designed primarily for maximum force transfer and less with regard to an attractive appearance. Since wheel covers in different materials, shapes and colors can be fitted, this provides a wide range of design possibilities.

The disc and rim are welded together on sheet-steel wheels; in the case of forged and cast light-alloy wheels, these two components are usually manufactured in one piece. Multiple-piece designs, even those which are made of different materials (e.g., magnesium disc and aluminum rim), are available only in special cases and for racing vehicles. Car-wheel rims are almost always drop-centre rims with double humps H2 (rarely with flat hump FH), tapered bead seats and "J" section rim flange. The lower flange shape is frequently found on smaller vehicles; the higher flange shapes JK and K are only rarely seen, and only on higher-weight vehicles.

Tires

The tire is the connecting link between the vehicle and the road. It is at that point that the safe handling of a vehicle is ultimately decided. The tire transmits motive, braking and lateral forces within a physical environment whose parameters define the limits of the dynamic loads to which the vehicle is subjected. The decisive criteria for the assessment of tire quality are:

- ❖ Straight-running ability
- ❖ Stable cornering properties
- ❖ Ability to grip on a variety of road surfaces
- ❖ Ability to grip in a variety of weather conditions
- ❖ Steering characteristics
- ❖ Ride comfort (vibration absorption and damping, quietness)
- ❖ Durability and
- ❖ Economy

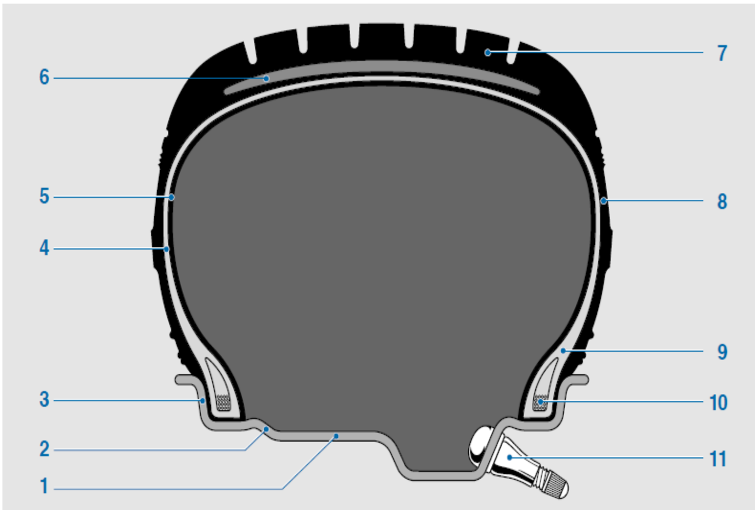
Tire Design

The design of a conventional tire is determined by the characteristics required of it in normal conditions and emergency situations.

Radial tires

In a radial tire, the type which has now become the standard for cars, the cords of the

tire-casing plies run radially, following the shortest route from bead to bead (Fig. 1).
A reinforcing belt runs around the perimeter of the relatively thin, flexible casing.



1 Rim bead seat
2 Hump
3 Rim flange
4 Casing
5 Air-tight rubber layer
6 Belt
7 Tread
8 Sidewall
9 Bead
10 Bead core
11 Valve

Figure 1: cross sectional view of a radial ply tire

Cross-ply tires
The cross-ply tire takes its name from the fact that the cords of alternate plies of the tire casing run at right angles to one another so that they cross each other. This type of tire is now only of significance for motorcycles, bicycles, and industrial and agricultural vehicles. On commercial vehicles it is increasingly being supplanted by the radial tire.

TIRE SLIP

When a wheel rotates under the effect of power transmission or braking, complex physical processes take place in the contact area between tire and road which place the rubber parts under stress and cause them to partially slide, even if the wheel does not fully lock. In other words, the elasticity of the tire causes it to deform and “flex” to a greater or lesser extent depending on the weather conditions and the nature of the road surface.
As the tire is made largely of rubber, only a proportion of the “deformation energy” is recovered as the tread moves out of the contact area. The tire heats up in the process and energy loss occurs.

Illustration of slip
The slip component of wheel rotation is referred to by λ, where $\lambda = (\upsilon F - \upsilon U)/Yf$

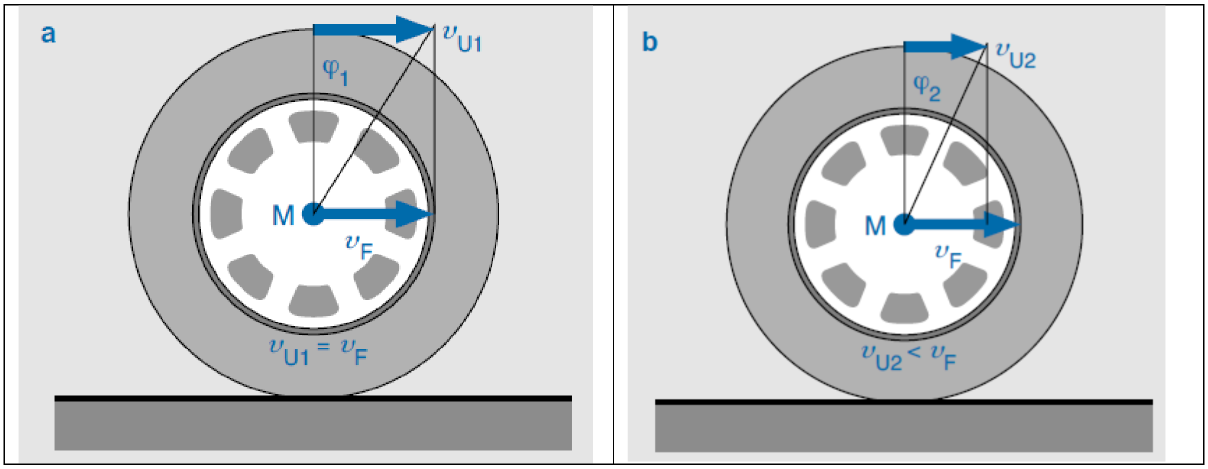


Figure 2: Effect of braking on a rolling wheel

The quantity υF is the vehicle road speed, υU is the circumferential velocity of the wheel (Fig. 3). The formula states that brake slip occurs as soon as the wheel is rotating more slowly than the vehicle road speed would normally demand. Only under that condition can braking forces or acceleration forces be transmitted.
Since the tire slip is generated as a result of the vehicle's longitudinal movement, it is also referred to as "longitudinal slip".
The slip generated during braking is usually termed "brake slip".
If a tire is subjected to other factors in addition to slip (e.g. greater weight acting on the wheels, extreme wheel positions), its force transmission and handling characteristics will be adversely affected.

FORCES ACTING ON TIRE

A motor vehicle can only be made to move or change its direction in a specific way by forces acting through the tires. Those forces are made up of the following components

Circumferential force

The circumferential force *F*U is produced by power transmission or braking. It acts on the road surface as a linear force in line with the longitudinal axis of the vehicle and enables the driver to increase the speed of the vehicle using the accelerator or slow it down with the brakes.

Vertical tire force (normal force)

The vertical force acting downwards between the tire and road surface is called the vertical tire force or normal force *F*N. It acts on the tires at all times regardless of the state of motion of the vehicle, including, therefore, when the vehicle is stationary.
The vertical force is determined by the proportion of the combined weight of vehicle and payload that is acting on the individual wheel concerned. It also depends on the degree of upward or downward gradient of the road that the vehicle is standing on.

The highest levels of vertical force occur on a level road. Other forces acting on the vehicle (e.g. heavier payload) can increase or decrease the vertical force. When cornering, the force is reduced on the inner wheels and increased on the outer wheels.

Lateral force

Lateral forces act upon the wheels when steering or when there is a crosswind, for example. They cause the vehicle to change direction.

Braking torque

When the brakes are applied, the brake shoes press against the brake drums (in the case of drum brakes) or the brake pads press against the disks (in the case of disk brakes). This generates frictional forces, the level of which can be controlled by the driver by the pressure applied to the brake pedal.

The product of the frictional forces and the distance at which they act from the axis of rotation of the wheel is the braking torque MB. That torque is effective at the circumference of the tire under braking (Fig. 1).

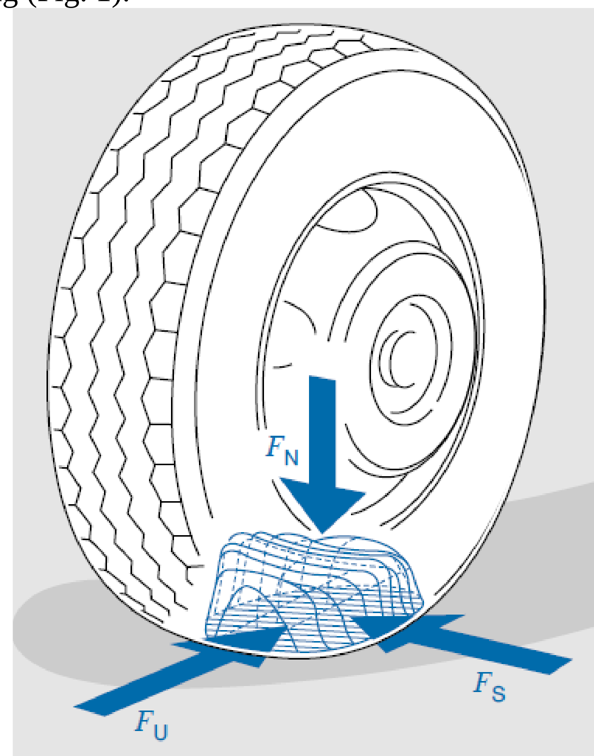


Figure 3: Components of tire force and pressure distribution over the footprint of a radial tire

Coefficient of friction

When braking torque is applied to a wheel, a braking force FB is generated between the tire and the road surface that is proportional to the braking torque under stationary conditions (no wheel acceleration). The braking force transmitted to the road (frictional force FR) is proportional to the vertical tire force FN:

$$F\text{R} = \mu\text{HF} \cdot F\text{N}$$

The factor μHF is the coefficient of friction.
It defines the frictional properties of the various possible material pairings between tire and road surface and the environmental conditions to which they are exposed.
The coefficient of friction is thus a measure of the braking force that can be transmitted.
It is dependent on

- the nature of the road surface,
- the condition of the tires,
- the vehicle's road speed, and
- the weather conditions.

The coefficient of friction ultimately determines the degree to which the braking torque is actually effective. For motor-vehicle tires, the coefficient of friction is at its highest on a clean and dry road surface; it is at its lowest on ice. Fluids (e.g. water) or dirt between the tire and the road surface reduce the coefficient of friction.

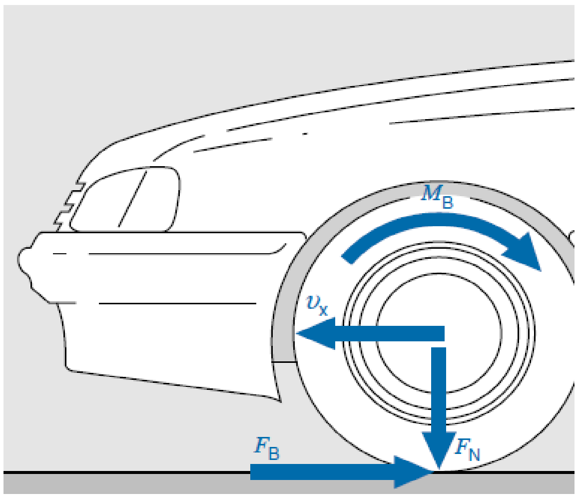


Figure 4: Linear wheel velocity, υX, with braking force, *F*B, and braking torque, *M*B

υx Linear velocity of wheel
*F*N Vertical tire force (normal force)
*F*B Braking force
*M*B Braking torque

Table 1: Coefficients of friction, μHF, for tires in various conditions of wear, on various road conditions and at various speeds

Vehicle road speed km/h	Tire condition	Dry road μ_{HF}	Wet road (depth of water 0.2 mm) μ_{HF}	Heavy rain (depth of water 1 mm) μ_{HF}	Puddles (depth of water 2 mm) μ_{HF}	Icy (black ice) μ_{HF}
50	new	0.85	0.65	0.55	0.5	0.1 and below
	worn out	1	0.5	0.4	0.25	
90	new	0.8	0.6	0.3	0.05	
	worn out	0.95	0.2	0.1	0.0	
130	new	0.75	0.55	0.2	0	
	worn out	0.9	0.2	0.1	0	

Kinetic friction

- When describing processes involving friction, a distinction is made between static friction and kinetic friction. With solid bodies, the static friction is greater than kinetic friction.
- Accordingly, for a rolling rubber tire there are circumstances in which the coefficient of friction is greater than when the wheel locks.
- Nevertheless, the tire can also slide while it is rolling, and on motor vehicles this is referred to as slip. curve represents the “stable zone” (partial braking zone), while the falling slope is the “unstable zone”.
- Most braking operations involve minimal levels of slip and take place within the stable zone so that an increase in the degree of slip simultaneously produces an increase in the usable adhesion. In the unstable zone, an increase in the amount of slip generally produces a reduction in the level of adhesion.
- When braking in such situations, the wheel can lock up within a fraction of a second, and under acceleration the excess power-transmission torque rapidly increases the wheel’s speed of rotation causing it to spin.
- When a vehicle is traveling in a straight line, ABS and TCS prevent it entering the unstable zone when braking or accelerating.

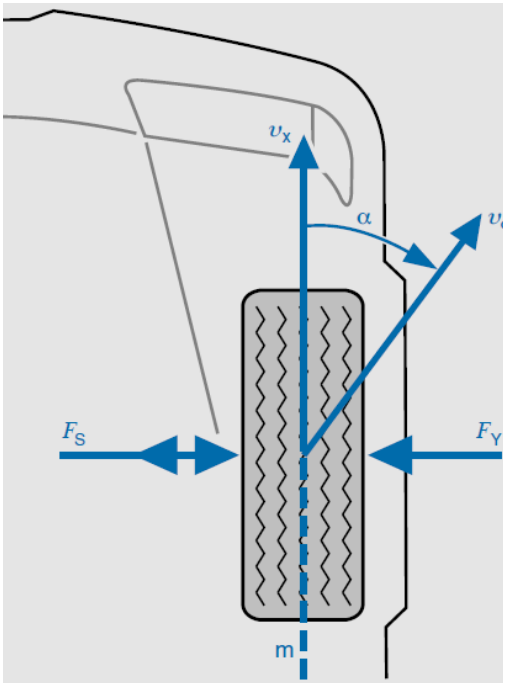


Figure 5: Lateral slip angle, α, and the effect of lateral force, *F*S, (overhead view)

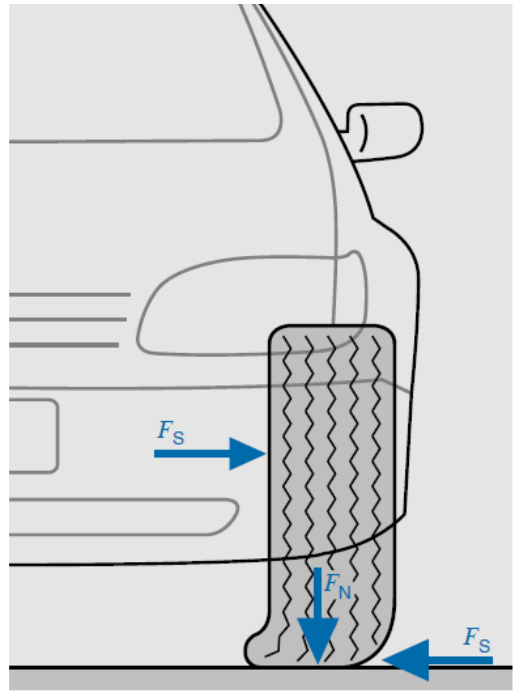


Figure 6: Position of tire contact area relative to wheel in a right-hand bend showing lateral force, *F*S, (front view)

There is a nonlinear relationship between the slip angle α and the lateral-force coefficient μS that can be described by a lateral slip curve. In contrast with the coefficient of friction μHF that occurs under acceleration and braking, the lateral-force coefficient μS is heavily dependent on the wheel contact force *F*N. This characteristic is of particular interest to vehicle manufacturers when designing suspension systems so that handling characteristics can be enhanced by stabilizers.

Effect of brake slip on lateral forces

When a vehicle is cornering, the centrifugal force acting outwards at the center of gravity must be held in equilibrium by lateral forces on all the wheels in order for the vehicle to be able to follow the curve of the road.

However, lateral forces can only be generated if the tires deform flexibly sideways so that the direction of movement of the wheel's center of gravity at the velocity, υα, diverges from the wheel center plane "m" by the lateral slip angle, α (Fig. 6).

Steady State cornering

Assume that a lateral force F_Y is applied at the center of gravity of a car driving straight

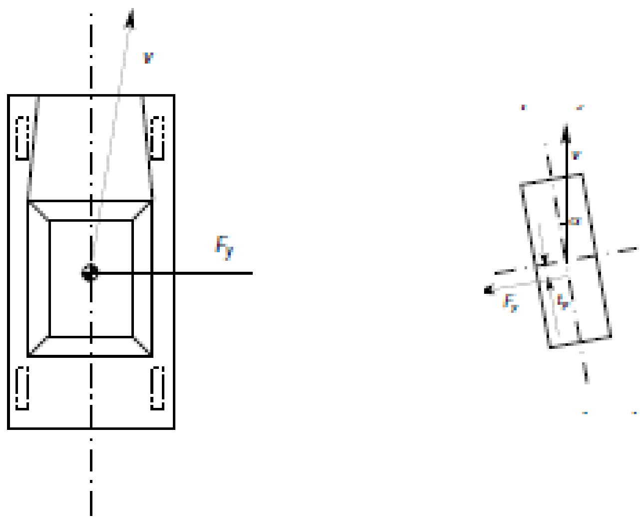
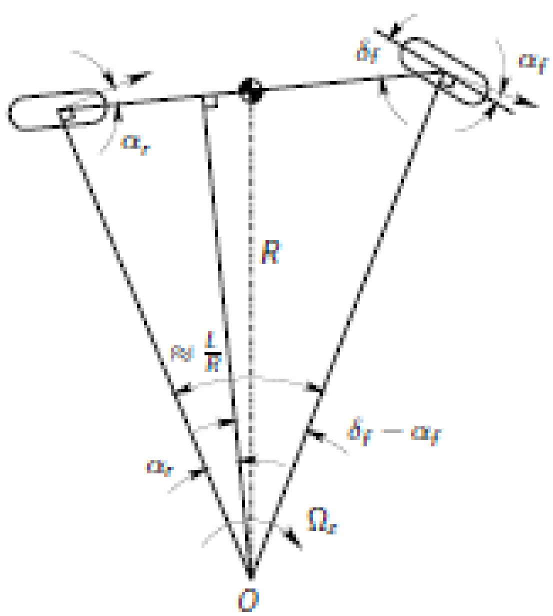


Figure 7: Vehicle behaviour on straight road condition

A wheel moving in the lateral lateral direction give rise to a cornering force.
Fy och self-aligning torque Mz = tp . Fy :

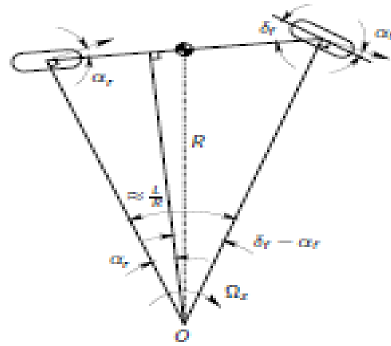
The slip angle α is the angle between the direction of the wheel and the direction of the velocity vector v.
To begin with, a simple linear model will be used to represent the relation between the slip angle α and the cornering force Fy
Fy = C_a. A

where C_a is called the cornering stiffness.

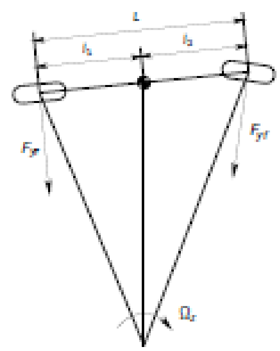


Triangle to the left: $\alpha_r + (90^o - \alpha_r) + 90^o = 180^o$
Triangle to the right: $(\delta_f - \alpha_f) + (90^o - (\delta_f - \alpha_f)) + 90^o = 180^o$

Approximation of the angle at O: $\alpha_r + (\delta_f - \alpha_f) \approx \frac{L}{R}$



$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$



Equations of motion with solutions:

$$F_{yf} + F_{yr} = ma_y = \frac{W}{g}\frac{V^2}{R}$$
$$F_{yf} l_1 - F_{yr} l_2 = I_z \dot{\Omega}_z = 0$$

$$F_{yf} = ma_y \frac{l_2}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_2}{L}$$
$$F_{yr} = ma_y \frac{l_1}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_1}{L}$$

Equations of motion with solutions

$$F_{yf} + F_{yr} = ma_y = \frac{W}{g}\frac{V^2}{R} \qquad F_{yf} = ma_y\frac{l_2}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_2}{L}$$

$$F_{yf}l_1 - F_{yr}l_2 = I_z\dot{\Omega}_z = 0 \qquad F_{yr} = ma_y\frac{l_1}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_1}{L}$$

Equations of motions with solutions

$$W_f + W_r = \frac{mg}{2} = \frac{W}{2} \qquad W_f = \frac{mg}{2}\frac{l_2}{L} = \frac{W}{2}\frac{l_2}{L}$$

$$W_f l_1 - W_r l_2 = I_z\dot{\Omega}_y = 0 \qquad W_r = \frac{mg}{2}\frac{l_1}{L} = \frac{W}{2}\frac{l_1}{L}$$

A comparison of the solutions gives the relations

$$F_{yf} = 2W_f\frac{a_y}{g}, \quad F_{yr} = 2W_r\frac{a_y}{g}$$

Using the tire models, $F_{yf} = 2C_{\alpha f}\alpha_f$ and $F_{yr} = 2C_{\alpha r}\alpha_r$, the slip angles can be written as

$$\alpha_f = \frac{F_{yf}}{2C_{\alpha f}} = \frac{W_f}{C_{\alpha f}}\frac{a_y}{g}$$

$$\alpha_r = \frac{F_{yr}}{2C_{\alpha r}} = \frac{W_r}{C_{\alpha r}}\frac{a_y}{g}$$

The steering angle is

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r = \frac{L}{R} + \frac{W_f}{C_{\alpha f}}\frac{a_y}{g} - \frac{W_r}{C_{\alpha r}}\frac{a_y}{g} = \frac{L}{R} + K_{us}\frac{a_y}{g}$$

where the understeer gradient is defined as

$$K_{us} = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}$$

$$K_{us} = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} = \frac{W}{2LC_{\alpha f}C_{\alpha r}}(C_{\alpha r}l_2 - C_{\alpha f}l_1)$$

$K_{us} > 0$: The car is said to be understeer.

$K_{us} = 0$: The car is said to be neutral steer.

$K_{us} < 0$: The car is said to be oversteer.

The steering angle is

$$\delta_f = \frac{L}{R} + K_{us}\frac{V^2}{gR} = \frac{L}{R} + K_{us}\frac{a_y}{g}$$

where the understeer gradient can be rewritten as

$$K_{us} = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} = \frac{W}{2L C_{\alpha f} C_{\alpha r}}(C_{\alpha r} l_2 - C_{\alpha f} l_1)$$

The sign of K_{us} gives the answer to Question 3:

- Positive sign: You have to turn the steering wheel clockwise to make the car stay on the same circle.
  The car turns towards the direction of the unperturbed path.
- Equal to zero: You don't have to do anything. Just lie back and relax.
- Negative sign: You have to turn the steering wheel counterclockwise to make the car stay on the same circle.

Understeer Coefficient (K_{us})

Consider the relation

$$\delta_f = \frac{L}{R} + \underbrace{\left(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\right)}_{=K_{us}}\frac{V^2}{gR}$$

Interpretation:

The understeer gradient is equal to the difference between the ratios of the load and cornering stiffness at the front and rear wheels, respectively.

The understeer gradient can be rewritten as

$$\delta_f = \frac{L}{R} + \underbrace{\frac{W}{2L C_{\alpha f} C_{\alpha r}} (C_{\alpha r} l_2 - C_{\alpha f} l_1)}_{=K_{us}} \frac{V^2}{gR}$$

Interpretation:

The sign of the understeer gradient depends on the difference between the products of the cornering stiffness and the distance to the center of gravity at the rear and front wheels, respectively.

To give a more direct interpretation we shall only consider the equilibrium of moments

$$l_1 F_{yf} - l_2 F_{yr} = 2 l_1 C_{\alpha f} \alpha_f - 2 l_2 C_{\alpha r} \alpha_r = 0,$$

which gives the relation

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r,$$

and the relation

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

between the angles.

Case of Neutral Steer

Assume that the radius of the curve R is constant and the speed V increases. In this case $l_1 C_{\alpha f} - l_2 C_{\alpha r} = 0$ and the relation

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r$$

gives $\alpha_f = \alpha_r$ and

$$\delta_f = \frac{L}{R} + \underbrace{\alpha_f - \alpha_r}_{=0}$$

does not depend on the speed.

Case of under steer

In this case $l_2 C_{\alpha r} - l_1 C_{\alpha f} > 0$ and it follows from

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r$$

that the increase of α_f has to larger than the increase of α_r, when V increases.

Hence, the steering angle

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

has to be increased.

Case of Oversteer

In this case $l_2 C_{\alpha r} - l_1 C_{\alpha f} < 0$ and it follows from

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r$$

that the increase of α_r has to larger than the increase of α_f, when V increases.

Hence, the steering angle

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

has to be increased.
Observation: If $K_{us} < 0$ and $V = V_{crit} = \sqrt{gL/-K_{us}}$ then

$$\delta_f = \frac{L}{R} + K_{us} \frac{V^2}{gR} = 0$$

and does not depend on R.

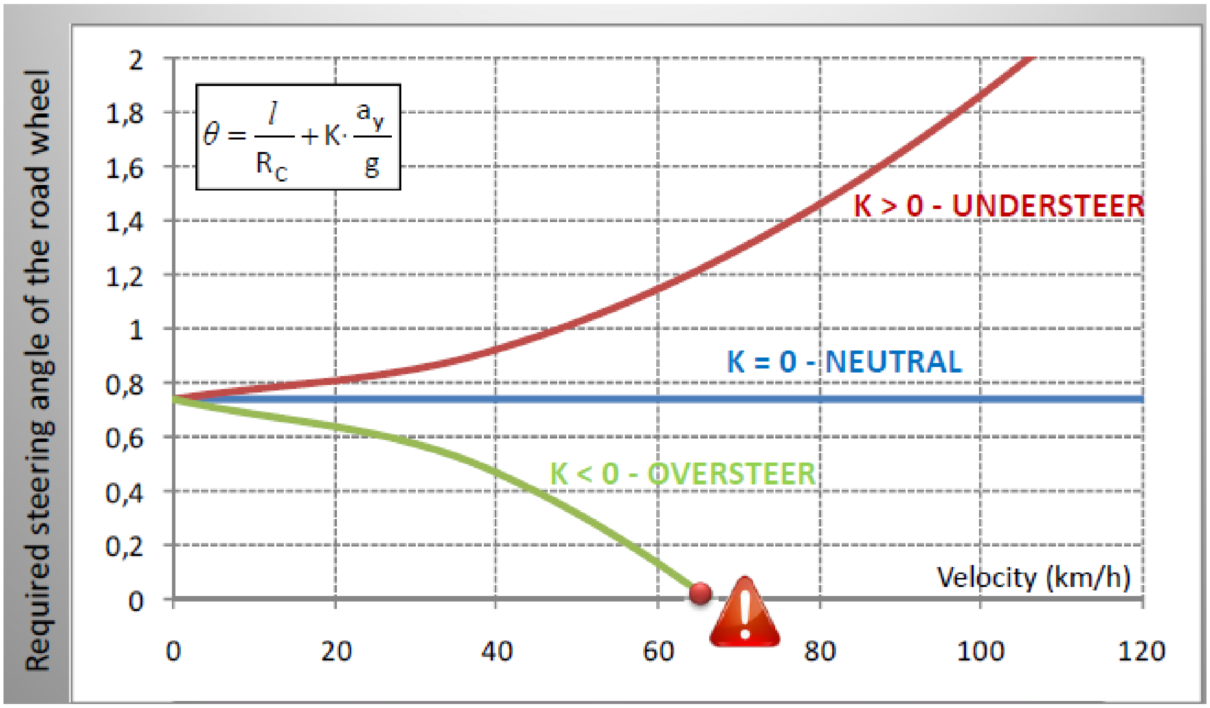


Figure 8 : Basic analysis of steering behaviour

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