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SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

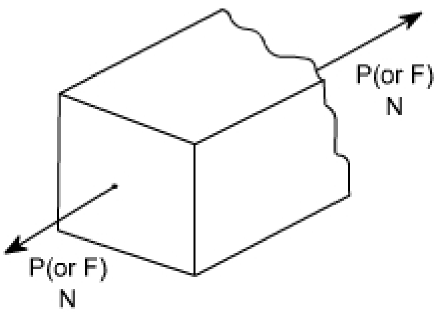
UNIT 1-INTRODUCTION TO AIRCRAFT STRUCTURES-SAEA1305

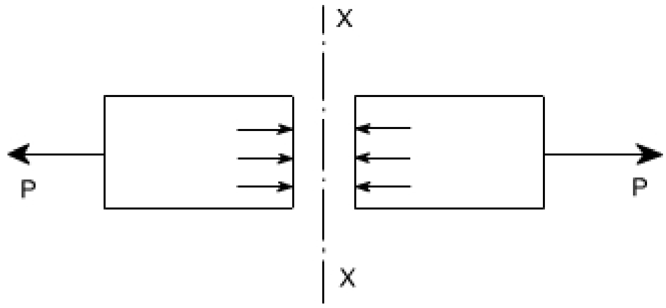
Module 1

Lecture 1

Stress

Stress is the internal resistance offered by the body to the external load applied to it per unit cross sectional area. Stresses are normal to the plane to which they act and are tensile or compressive in nature.





As we know that in mechanics of deformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion. These internal forces give rise to a concept of stress. Consider a rectangular rod subjected to axial pull P. Let us imagine that the same rectangular bar is assumed to be cut into two halves at section *XX*. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section *XX* has been shown.

Now stress is defined as the force intensity or force per unit area. Here we use a symbol σ to represent the stress.

$$\sigma = \frac{P}{A}$$

Where A is the area of the *X –X* section

Here we are using an assumption that the total force or total load carried by the rectangular bar is uniformly distributed over its cross – section. But the stress distributions may be for from uniform, with local regions of high stress known as stress concentrations. If the force carried by a component is not uniformly distributed over its cross – sectional area, A, we must consider a small area, 'δA' which carries a small load 'δP', of the total force 'P', Then definition of stress is

$$\boxed{\sigma = \frac{\delta F}{\delta A}}$$

As a particular stress generally holds true only at a point, therefore it is defined mathematically as

$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}}$$

Units :

The basic units of stress in S.I units i.e. (International system) are N / m^2 (or Pa)

MPa = 10^6 Pa

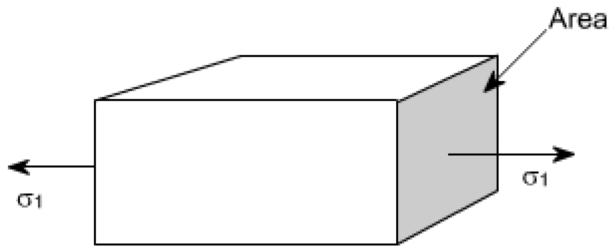
GPa = 10^9 Pa

KPa = 10^3 Pa

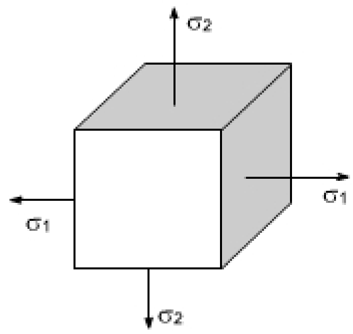
Sometimes N / mm^2 units are also used, because this is an equivalent to MPa. While US customary unit is pound per square inch psi.

TYPES OF STRESSES : Only two basic stresses exists : (1) normal stress and (2) shear stress. Other stresses either are similar to these basic stresses or are a combination of this e.g. bending stress is a combination tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress. Let us define the normal stresses and shear stresses in the following sections.

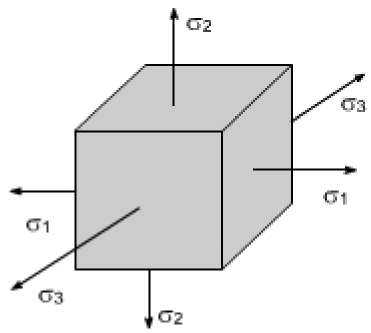
Normal stresses : We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally denoted by a Greek letter (σ)



This is also known as uniaxial state of stress, because the stresses acts only in one direction however, such a state rarely exists, therefore we have biaxial and triaxial state of stresses where either the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses acts as shown in the figures below :



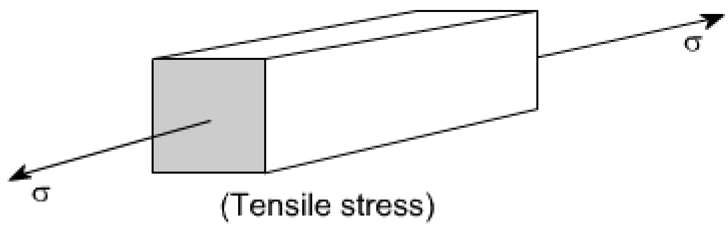
(Biaxial state of stress)



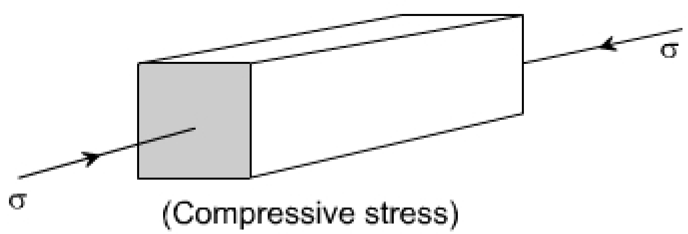
(Triaxial state of stress)

Tensile or compressive Stresses:

The normal stresses can be either tensile or compressive whether the stresses acts out of the area or into the area

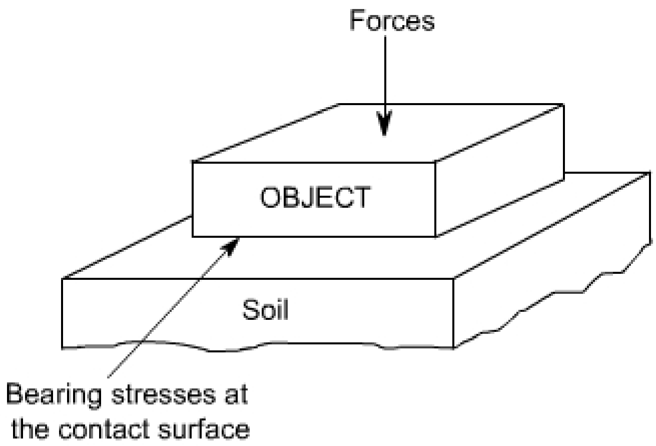


(Tensile stress)



(Compressive stress)

Bearing Stress: When one object presses against another, it is referred to a bearing stress (They are in fact the compressive stresses).



<u>Sign convections for Normal stress</u>

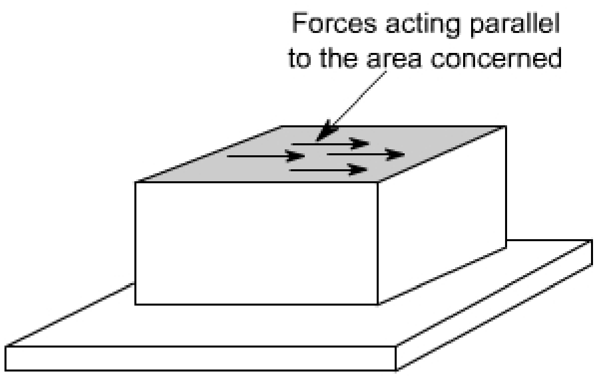
Direct stresses or normal stresses

- tensile +ve

- compressive –ve

<u>Shear Stresses:</u>

Let us consider now the situation, where the cross – sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces. The resulting stress is known as shear stress.



The resulting force intensities are known as shear stresses, the mean shear stress being equal to

$$\tau = \frac{P}{A}$$

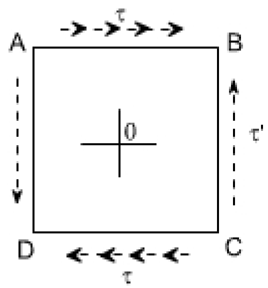
Where P is the total force and A the area over which it acts. As we know that the particular stress generally holds good only at a point therefore we can define shear stress at a point as

$$\tau = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

The Greek symbol τ (tau, suggesting tangential) is used to denote shear stress.

Complementary shear stresses:

The existence of shear stresses on any two sides of the element induces complementary shear stresses on the other two sides of the element to maintain equilibrium. As shown in the figure the shear stress τ in sides AB and CD induces a complimentary shear stress τ' in sides AD and BC.



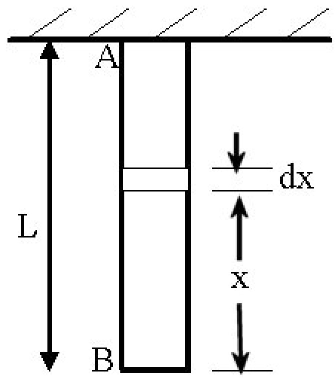
Sign convections for shear stresses:

- tending to turn the element C.W +ve.

- tending to turn the element C.C.W – ve.

Deformation of a Body due to Self Weight

Consider a bar AB hanging freely under its own weight as shown in the figure.



Let

L= length of the bar

A= cross-sectional area of the bar

E= Young's modulus of the bar material

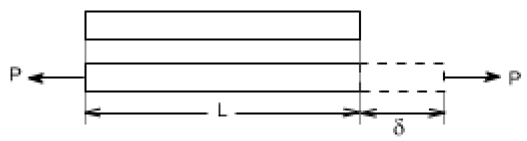
w= specific weight of the bar material

Then deformation due to the self-weight of the bar is $\boxed{\delta L = \frac{WL}{2E}}$

Members in Uni – axial state of stress

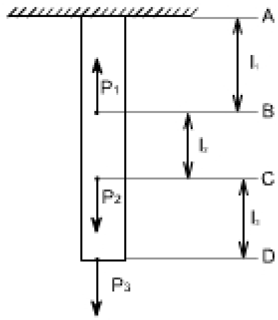
Introduction: [For members subjected to uniaxial state of stress]

For a prismatic bar loaded in tension by an axial force P, the elongation of the bar can be determined as



$$\delta = \frac{PL}{AE} \qquad \text{———— (1)}$$

Suppose the bar is loaded at one or more intermediate positions, then equation (1) can be readily adapted to handle this situation, i.e. we can determine the axial force in each part of the bar i.e. parts AB, BC, CD, and calculate the elongation or shortening of each part separately, finally, these changes in lengths can be added algebraically to obtain the total charge in length of the entire bar.



When either the axial force or the cross – sectional area varies continuosly along the axis of the bar, then equation (1) is no longer suitable. Instead, the elongation can be found by considering a deferential element of a bar and then the equation (1) becomes

$$d\delta = \frac{P_x\,dx}{E.A_x}$$

$$\delta = \int_0^l \frac{P_x\,dx}{E.A_x}$$

i.e. the axial force $P_x$and area of the cross – section A_x must be expressed as functions of x. If the expressions for $P_x$and A_x are not too complicated, the integral can be evaluated analytically, otherwise Numerical methods or techniques can be used to evaluate these integrals.

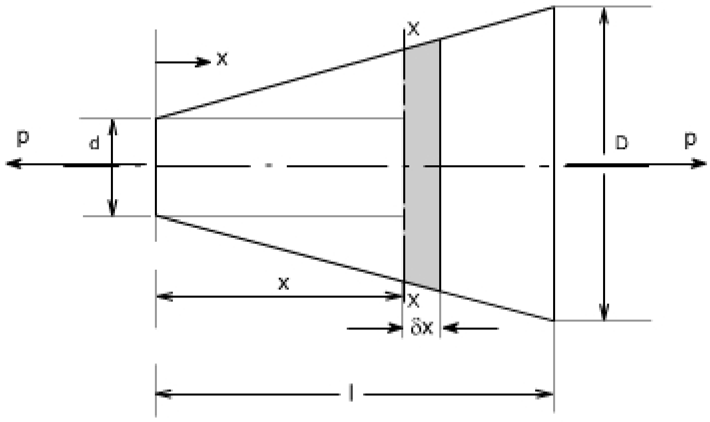
Principle of Superposition

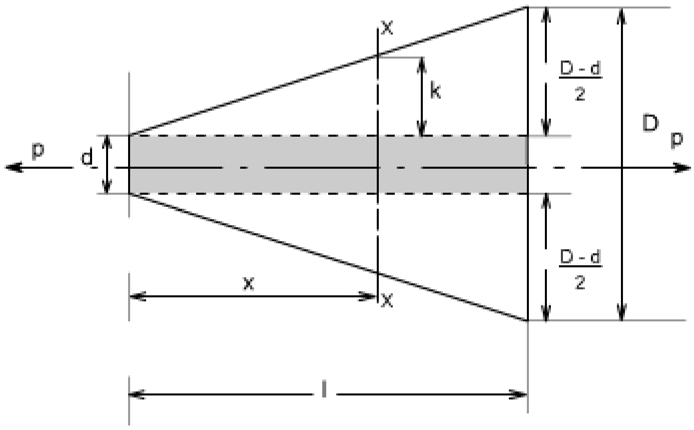
The principle of superposition states that when there are numbers of loads are acting together on an elastic material, the resultant strain will be the sum of individual strains caused by each load acting separately.

Module 1

Lecture 2: Numerical Problems on stress, shear stress in axially loaded members.

Example 1: Now let us for example take a case when the bar tapers uniformly from d at x = 0 to D at x = l





In order to compute the value of diameter of a bar at a chosen location let us determine the value of dimension k, from similar triangles

$$\frac{(D-d)/2}{l} = \frac{k}{x}$$

$$\text{Thus, } k = \frac{(D-d)x}{2l}$$

therefore, the diameter 'y' at the X-section is

or = d + 2k

$$y = d + \frac{(D-d)x}{l}$$

Hence the cross –section area at section X- X will be

$$A_x \text{ or } a = \frac{\pi}{4}y^2$$
$$= \frac{\pi}{4}\left[d + (D - d)\frac{x}{l}\right]^2$$

hence the total extension of the bar will be given by expression

$$= \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

subsitituting the value of 'a' to get the total extention of the bar

$$= \frac{\pi P}{4E}\int_0^l \frac{\delta x}{\left[d + (D - d)\frac{x}{l}\right]^2}$$

after carrying out the intergration we get

$$= -\frac{4.P.l}{\pi E}\left[\frac{1}{D} - \frac{1}{d}\right]$$

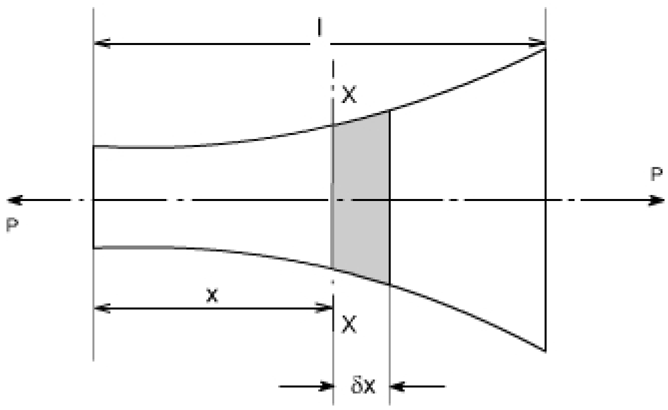
$$= \frac{4.P.l}{\pi E D.d}$$

hence the total strain in the bar $= \mathbf{\frac{4.P.l}{\pi E\, D.d}}$

An interesting problem is to determine the shape of a bar which would have a uniform stress in it under the action of its own weight and a load P.

Example 2: stresses in Non – Uniform bars

Consider a bar of varying cross section subjected to a tensile force P as shown below.



Let

a = cross sectional area of the bar at a chosen section XX

then

Stress σ = p / a

If E = Young's modulus of bar then the strain at the section XX can be calculated

□ = □ / E

Then the extension of the short element □ x. =□□ .original length = □ / E. □^{x}

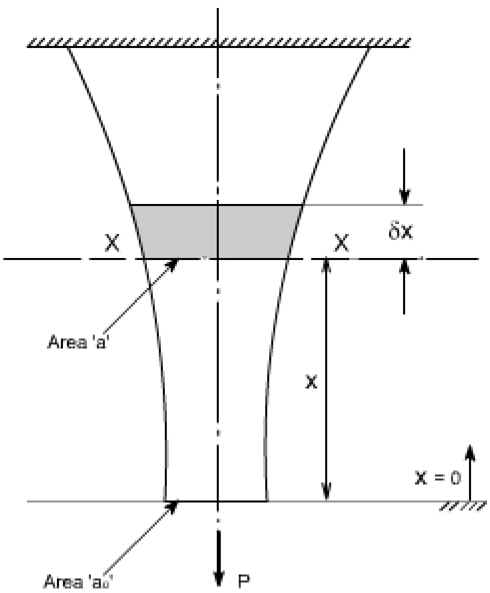
$$= \frac{P}{E.}\frac{\delta x}{a}$$

Thus, the extension for the entire bar is

$$\delta = \int_0^l \frac{P}{E}\frac{\delta x}{a}$$

$$\text{or total extension} = \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

let us consider such a bar as shown in the figure below:



The weight of the bar being supported under section XX is

$$= \int_0^x \rho g a \, dx$$

where ρ is density of the bar.

thus the stress at XX is

$$\sigma = \frac{P + \int_0^x \rho g a \, dx}{a}$$

$$\text{or } \sigma.a = P + \int_0^x \rho . g . a \, dx$$

Differentiating the above equation with respect to x we get

$$\sigma . \frac{da}{dx} = \rho . g . a$$

$$\frac{da}{a} = \frac{\rho . g}{\sigma} . dx$$

intergrating the above equation we get

$$\int \frac{da}{a} = \int \frac{\rho . g}{\sigma} dx$$

$$\log_e^a = \frac{\rho . g . x}{\sigma} + \text{constant}$$

In order to determine the constant of integration

let us apply the boundary conditions

at $x = 0; a = a_0$

thus, constant $= \log_e^{a_0}$

or

$$\log_e^a = \frac{\rho . g . x}{\sigma} + \log_e^{a_0}$$

$$\log_e \left(\frac{a}{a_0} \right) = \frac{\rho . g . x}{\sigma}$$

$$\text{or } \boxed{e^{\frac{\rho . g . x}{\sigma}} = \frac{a}{a_0}}$$

also at $x = 0$

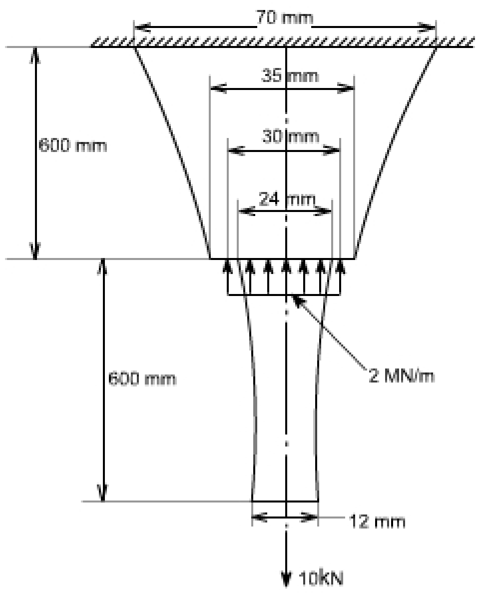
$$\sigma = \frac{P}{a_0}$$

Thus,

$$\frac{a}{a_0} = e^{\frac{\rho . g . x . a_0}{P}}$$

Example 1: Calculate the overall change in length of the tapered rod as shown in figure below. It carries a tensile load of 10kN at the free end and at the step change in section a compressive load of 2 MN/m evenly distributed around a circle of 30 mm diameter take the value of E = 208 GN / m^2.

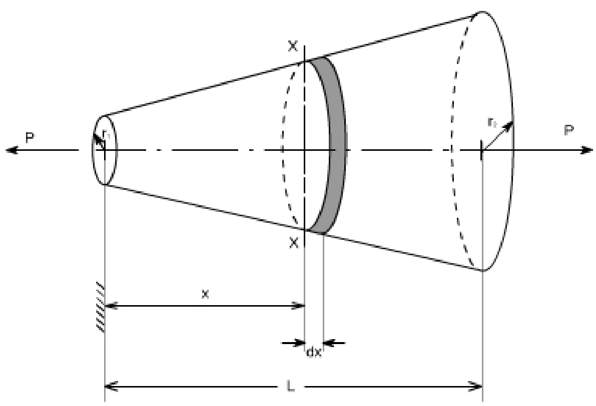
This problem may be solved using the procedure as discussed earlier in this section



Example 2: A round bar, of length L, tapers uniformly from radius r_1 at one end to radius r_2 at the other. Show that the extension produced by a tensile axial load P is $\frac{PL}{2\pi E r_1^2}$

If $r_2 = 2r_1$, compare this extension with that of a uniform cylindrical bar having a radius equal to the mean radius of the tapered bar.

Solution:



consider the above figure let r_1 be the radius at the smaller end. Then at a X crosssection XX located at a distance x from the smaller end, the value of radius is equal to

$$= r_1 + \frac{x}{L}(r_2 - r_1)$$
$$= r_1(1 + kx)$$
$$\text{where } k = \left(\frac{r_2 - r_1}{L}\right).\frac{1}{r_1}$$
$$\text{stress at section XX} = \frac{\text{load}}{\text{area}}$$
$$= \frac{P}{\pi r_1^2 (1 + kx)^2}$$
$$\text{hence strain at this section} = \frac{\text{stress}}{E}$$
$$= \frac{P}{E.\pi r_1^2 (1 + kx)^2}$$

Thus, for a small length dx of the bar at this section the extention is $\frac{P.dx}{E\pi r_1^2 (1 + kx)^2}$

Total extension of the bar can be found by integrating the above expression within the limits from x=0 to x=L

$$\text{Extension} = \int_0^L \frac{P.dx}{E.\pi r_1^2 (1 + k.x)^2}$$
$$= \frac{P}{E.\pi r_1^2}\int_0^L (1 + k.x)^{-2} dx$$
$$= \frac{P}{E.\pi r_1^2}\left[\frac{(1+kx)^{-1}}{-k}\right]_0^L$$
$$= \frac{P}{E.\pi r_1^2}\left[\frac{(1 + kx)^{-1}}{-k} - \frac{1}{-k}\right]$$
$$= \frac{P}{E.\pi r_1^2.k}\left[1 - \frac{1}{1 + kL}\right]$$
$$= \frac{PL}{E.\pi r_1^2 (1 + kL)}$$
$$\text{since } k = \frac{(r_2 - r_1)}{r_1 L}$$
$$\text{Thus, } 1 + kL = r_2/r_1$$
$$\text{Therefore, the extension} = \frac{PL}{\pi E r_1 r_2}$$

Comparing of extensions

For the case when $r_2 = 2.r_1$, the value of computed extension as above becomes equal to $\frac{PL}{2\pi E r_1^2}$

The mean radius of taper bar

= 1 / 2(r_1 + r_2)

= 1 / 2(r_1 +2 r_2)

= 3 / 2 .r_1

Therefore, the extension of uniform bar

= Orginal length . strain

$$= L.\frac{\sigma}{E}$$

$$= \frac{L}{E}.\frac{P}{\pi\left(\frac{3}{2}r_1\right)^2}$$

$$= \frac{4PL}{g\pi E\pi r_1^2}$$

hence the

$$\frac{\text{Extension of uniform}}{\text{Extension of tapered}} = \left(\frac{4PL}{g\pi E\pi r_1^2}\right) \bigg/ \frac{PL}{2\pi E r_1^2}$$

$$= \frac{8}{g}$$

Module 1

Lecture 3:

Strain:

When a single force or a system force acts on a body, it undergoes some deformation. This deformation per unit length is known as strain. Mathematically strain may be defined as deformation per unit length.

So,

Strain=Elongation/Original length

Or, $\varepsilon = \frac{\delta l}{l}$

Elasticity;

The property of material by virtue of which it returns to its original shape and size upon removal of load is known as elasticity.

Hooks Law

It states that within elastic limit stress is proportional to strain. Mathematically

E=$\frac{Stress}{Strain}$

Where E = Young's Modulus

Hooks law holds good equally for tension and compression.

Poisson's Ratio;

The ratio lateral strain to longitudinal strain produced by a single stress is known as Poisson's ratio. Symbol used for poisson's ratio is μ or $1/m$.

Modulus of Elasticity (or Young's Modulus)

Young's modulus is defined as the ratio of stress to strain within elastic limit.

Deformation of a body due to load acting on it

We know that young's modulus E=$\frac{Stress}{Strain}$,

Or, strain, $\varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$

Now, strain, $\varepsilon = \frac{\delta l}{l}$

So, deformation $\boxed{\delta l = \frac{Pl}{AE}}$

Module 1

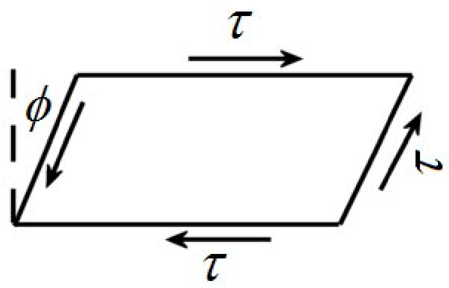
Lecture 4: Numerical problems on Stress-strain relationship, Hooke's law, Poisson's ratio, shear stress

Module 1

Lecture 5: Shear strain, modulus of rigidity, bulk modulus. Relationship between material properties of isotropic materials.

Shear Strain

The distortion produced by shear stress on an element or rectangular block is shown in the figure. The shear strain or 'slide' is expressed by angle ϕ and it can be defined as the change in the right angle. It is measured in radians and is dimensionless in nature.



Modulus of Rigidity

For elastic materials it is found that shear stress is proportional to the shear strain within elastic limit. The ratio is called modulus rigidity. It is denoted by the symbol 'G' or 'C'.

$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\phi} \text{ N/mm}^2$$

Bulk modulus (K): It is defined as the ratio of uniform stress intensity to the volumetric strain. It is denoted by the symbol K.

$$K = \frac{\text{stress intensity}}{\text{volumetric strain}} = \frac{\sigma}{\varepsilon_v}$$

Relation between elastic constants:

Elastic constants: These are the relations which determine the deformations produced by a given stress system acting on a particular material. These factors are constant within elastic limit, and known as modulus of elasticity *E*, modulus of rigidity *G*, Bulk modulus *K* and Poisson's ratio *μ.*

Relationship between modulus of elasticity (E) and bulk modulus (K):

$$\boxed{E = 3K(1-2\mu)}$$

Relationship between modulus of elasticity (E) and modulus of rigidity (G):

$$\boxed{E = 2G(1+\mu)}$$

Relation among three elastic constants:

$$\boxed{E = \frac{9KG}{G+3K}}$$

Module 1:

Lecture 6:

Numerical problems on, relation between elastic constants.

Module 1:

Lecture 7: Stress-strain diagram for uniaxial loading of ductile and brittle materials.

Stress – Strain Relationship

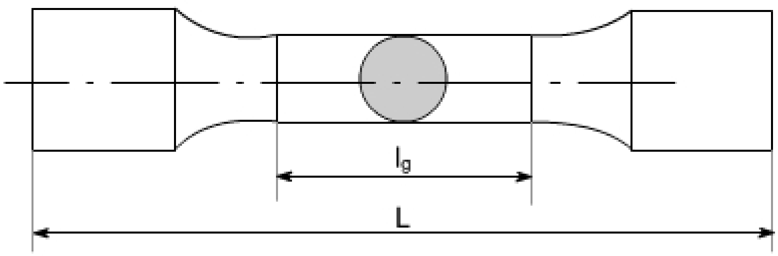
Stress – strain diagram for mild steel

Standard specimen are used for the tension test.

There are two types of standard specimen's which are generally used for this purpose, which have been shown below:

Specimen I:

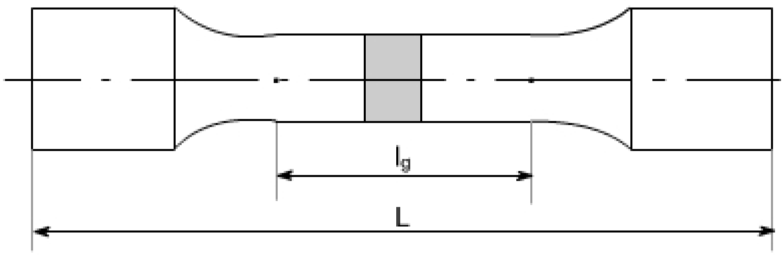
This specimen utilizes a circular X-section.



[specimen with circular X-section]

Specimen II:

This specimen utilizes a rectangular X-section.



[specimen with rectangular X-section]

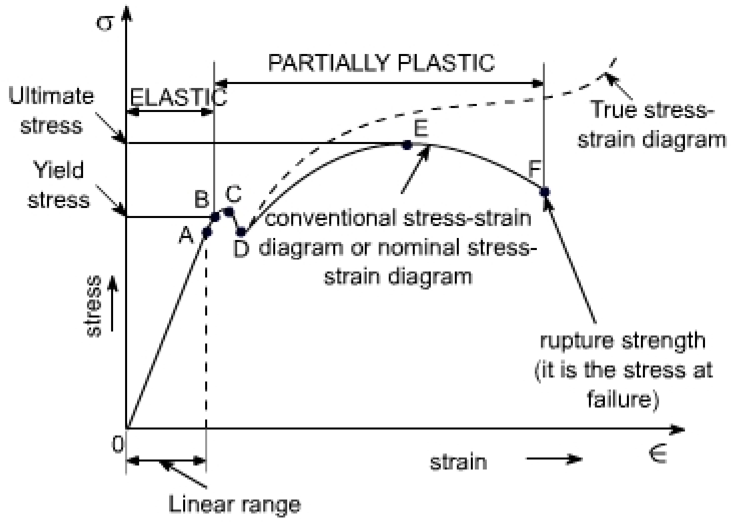
l_g = gauge length i.e. length of the specimen on which we want to determine the mechanical properties.The uniaxial tension test is carried out on tensile testing machine and the following steps are performed to conduct this test.

(i) The ends of the specimen are secured in the grips of the testing machine.

(ii) There is a unit for applying a load to the specimen with a hydraulic or mechanical drive.

(iii) There must be some recording device by which you should be able to measure the final output in the form of Load or stress. So the testing machines are often equipped with the pendulum type lever, pressure gauge and hydraulic capsule and the stress Vs strain diagram is plotted which has the following shape.

A typical tensile test curve for the mild steel has been shown below



SALIENT POINTS OF THE GRAPH:

(A) So it is evident form the graph that the strain is proportional to strain or elongation is proportional to the load giving a st.line relationship. This law of proportionality is valid upto a point A.

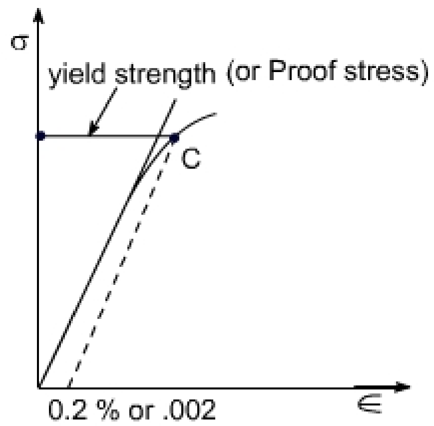
or we can say that point A is some ultimate point when the linear nature of the graph ceases or there is a deviation from the linear nature. This point is known as **the limit of proportionality or the proportionality limit**.

(B) For a short period beyond the point A, the material may still be elastic in the sense that the deformations are completely recovered when the load is removed. The limiting point B is termed as **Elastic Limit** .

(C) and (D) - Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. There will be thus permanent deformation or permanent set when load is removed. These two points are termed as upper and lower yield points respectively. The stress at the yield point is called the yield strength.

A study a stress – strain diagrams shows that the yield point is so near the proportional limit that for most purpose the two may be taken as one. However, it is much easier to locate the former. For material which do not posses a well define yield points, In order to find the yield point or yield strength, an offset method is applied.

In this method a line is drawn parallel to the straight line portion of initial stress diagram by off setting this by an amount equal to 0.2% of the strain as shown as below and this happens especially for the low carbon steel.



(E) A further increase in the load will cause marked deformation in the whole volume of the metal. The maximum load which the specimen can with stand without failure is called the load at the ultimate strength.

The highest point 'E' of the diagram corresponds to the ultimate strength of a material.

s_u = Stress which the specimen can with stand without failure & is known as Ultimate Strength or Tensile Strength.

s_u is equal to load at E divided by the original cross-sectional area of the bar.

(F) Beyond point E, the bar begins to forms neck. The load falling from the maximum until fracture occurs at F. Beyond point E, the cross-sectional area of the specimen begins to reduce rapidly over a relatively small length of bar and the bar is said to form a neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point F.

Nominal stress – Strain OR Conventional Stress – Strain diagrams:

Stresses are usually computed on the basis of the original area of the specimen; such stresses are often referred to as conventional or nominal stresses.

True stress – Strain Diagram:

Since when a material is subjected to a uniaxial load, some contraction or expansion always takes place. Thus, dividing the applied force by the corresponding actual area of the specimen at the same instant gives the so called true stress.

Percentage Elongation: 'd ':

The ductility of a material in tension can be characterized by its elongation and by the reduction in area at the cross section where fracture occurs.

It is the ratio of the extension in length of the specimen after fracture to its initial gauge length, expressed in percentage.

$$\delta = \frac{(l_1 - l_g)}{l_1} \times 100$$

l_l = gauge length of specimen after fracture(or the distance between the gage marks at fracture)

l_g= gauge length before fracture(i.e. initial gauge length)

For 50 mm gage length, steel may here a % elongation d of the order of 10% to 40%.

Ductile and Brittle Materials:

Based on this behaviour, the materials may be classified as ductile or brittle materials

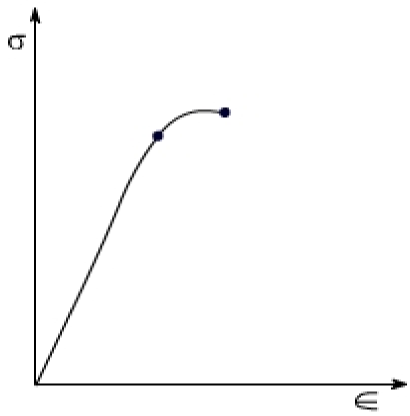
Ductile Materials:

It we just examine the earlier tension curve one can notice that the extension of the materials over the plastic range is considerably in excess of that associated with elastic loading. The Capacity of materials to allow these large deformations or large extensions without failure is termed as ductility. The materials with high ductility are termed as ductile materials.

Brittle Materials:

A brittle material is one which exhibits a relatively small extensions or deformations to fracture, so that the partially plastic region of the tensile test graph is much reduced.

This type of graph is shown by the cast iron or steels with high carbon contents or concrete.



Module 1:

Lecture 8: Introduction to mechanical properties of metals-hardness, impact

Mechanical Properties of material:

Elasticity: Property of material by virtue of which it can regain its shape after removal of external load

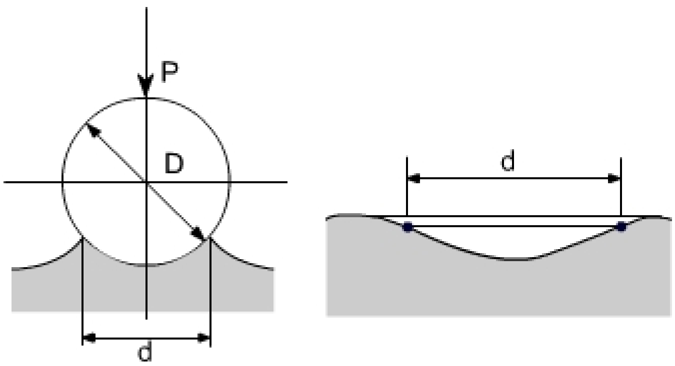
Plasticity: Property of material by virtue of which, it will be in a state of permanent deformation even after removal of external load.

Ductility: Property of material by virtue of which, the material can be drawn into wires.

Hardness: Property of material by virtue of which the material will offer resistance to penetration or indentation.

Ball indentation Tests:

iThis method consists in pressing a hardened steel ball under a constant load P into a specially prepared flat surface on the test specimen as indicated in the figures below :



After removing the load an indentation remains on the surface of the test specimen. If area of the spherical surface in the indentation is denoted as F sq. mm. Brinell Hardness number is defined as :

BHN = P / F

F is expressed in terms of D and d

D = ball diameter

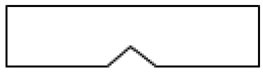
d = diametric of indentation and Brinell Hardness number is given by

$$BHN = \frac{2P}{\pi D(\mathrm{D}-\sqrt{D^2-d^2})}$$

Then is there is also **Vicker's Hardness Number** in which the ball is of conical shape.

IMPACT STRENGTH

Static tension tests of the unnotched specimen's do not always reveal the susceptibility of metal to brittle fracture. This important factor is determined in impact tests. In impact tests we use the notched specimen's



this specimen is placed on its supports on anvil so that blow of the striker is opposite to the notch the impact strength is defined as the energy A, required to rupture the specimen,

Impact Strength = A / f

Where f = It is the cross – section area of the specimen in cm^2 at fracture & obviously at notch.

The impact strength is a complex characteristic which takes into account both toughness and strength of a material. The main purpose of notched – bar tests is to study the simultaneous effect of stress concentration and high velocity load application

Impact test are of the severest type and facilitate brittle friction. Impact strength values can not be as yet be used for design calculations but these tests as rule provided for in specifications for carbon & alloy steels.Futher, it may be noted that in impact tests fracture may be either brittle or ductile. In the case of brittle fracture, fracture occurs by separation and is not accompanied by noticeable plastic deformation as occurs in the case of ductile fracture.

Impact loads:

Considering a weight falling from a height h, on to a collar attached at the end as shown in the figure.

Let P= equivalent static or gradually applied load which will produce the same extension x as that of the impact load W

Neglecting loss of energy due to impact, we can have:

Loss of potential energy= gain of strain energy of the bar

$$W(h+x)=\frac{1}{2}Px$$

Now we have extension $x = \frac{Pl}{AE}$

Substituting the value of x in the above equation we have:

$$W(\mathrm{h}+\frac{Pl}{AE})=\frac{1}{2}(P^2l/AE)$$

Solving the above equation we can have the following relation:

$$P=W[1+\sqrt{1+2hAE/Wl}]$$

Important Case: for a particular case i.e. for h=0, for a suddenly applied load P=2W, i.e. the stress produced by a suddenly applied load is twice that of the static stress.

Numerical examples:

1. Referring to the following figure let a mass of 100 kg fall 4cm on to a collar attached to a bar of steel 2cm diameter, 3m long. Find the maximum stress set up. Take E= 205,000 N/mm^2.

Applying the relation:

$$\sigma=\frac{P}{A}$$

$$=W[1+\sqrt{1+2hAE/Wl}]/\mathrm{A}$$

$$=\frac{981}{100\pi}\left[1+\sqrt{1+\frac{2\times 40\times\pi\times 100\times 205,000}{981\times 3\times 1000}}\right]$$

$=134$ M/mm^2

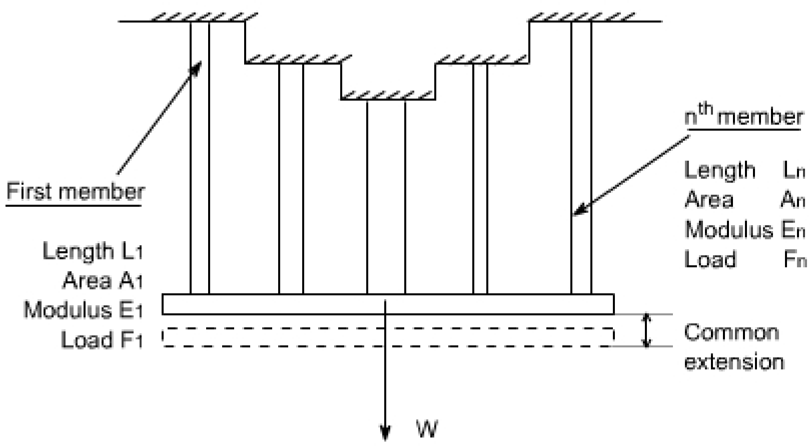
Module 1:

Lecture 9: Composite Bars In Tension & Compression:-Temperature stresses in composite rods statically indeterminate problem.

Thermal stresses, Bars subjected to tension and Compression

Compound bar: In certain application it is necessary to use a combination of elements or bars made from different materials, each material performing a different function. In over head electric cables or Transmission Lines for example it is often convenient to carry the current in a set of copper wires surrounding steel wires. The later being designed to support the weight of the cable over large spans. Such a combination of materials is generally termed compound bars.

Consider therefore, a compound bar consisting of n members, each having a different length and cross sectional area and each being of a different material. Let all member have a common extension 'x' i.e. the load is positioned to produce the same extension in each member.



For the 'n' the members

$$\frac{\text{stress}}{\text{strain}} = E_n = \frac{F_n/A_n}{x_n/L_n}$$

$$= \frac{F_n \cdot L_n}{A_n \cdot x_n}$$

$$\text{or} \qquad F_n = \frac{E_n A_n x_n}{L_n} = \frac{E_n A_n x}{L_n} \qquad \ldots\ldots(1)$$

Where F_n is the force in the nth member and A_n and L_n are its cross - sectional area and length.

Let W be the total load, the total load carried will be the sum of all loads for all the members.

$$W = \sum \frac{E_n.A_n.x}{L_n}$$

$$= x.\sum \frac{E_n.A_n}{L_n} \qquad ....(2)$$

From equation (1), force in member 1 is given as

$$F_1 = \frac{E_1.A_1.x}{L_1}$$

from equation (2)

$$x = \frac{W}{\sum \frac{E_n.A_n}{L_n}}$$

$$\text{Thus}, F_1 = \frac{E_1.A_1}{L_1}.\frac{W}{\sum\left(\frac{E_n.A_n}{L_n}\right)}$$

Therefore, each member carries a portion of the total load W proportional of EA / L value.

The above expression may be writen as $F_1 = \frac{\frac{E_1.A_1}{L_1}}{\sum \frac{E_n.A_n}{L_n}}.W$

if the length of each individual member in same then, we may write $F_1 = \frac{E_1.A_1}{\sum E.A}.W$

Thus, the stress in member '1' may be determined as $\sigma_1 = F_1 / A_1$

Determination of common extension of compound bars: In order to determine the common extension of a compound bar it is convenient to consider it as a single bar of an imaginary material with an equivalent or combined modulus E_c.

Assumption: Here it is necessary to assume that both the extension and original lengths of the individual members of the compound bar are the same, the strains in all members will than be equal.

Total load on compound bar = $F_1 + F_2 + F_3 + \ldots\ldots\ldots + F_n$

where F_1 , F_2 ,….,etc are the loads in members 1,2 etc

But force = stress . area,therefore

$\sigma\ (A_1 + A_2 + \ldots\ldots + A_n) = \sigma_1 A_1 + \sigma_2 A_2 + \ldots\ldots\ldots + \sigma_n A_n$

Where σ is the stress in the equivalent single bar

Dividing throughout by the common strain□□ .

$$\frac{\sigma}{\in}(A_1 + A_2 + ... + A_n) = \frac{\sigma_1}{\in}A_1 + \frac{\sigma_2}{\in}A_2 + ... \frac{\sigma_n}{\in}A_n$$

$$\text{i.e } E_c(A_1 + A_2 + ... + A_n) = E_1A_1 + E_2A_2 + ...E_n.A_n$$

$$\text{or } E_c = \frac{E_1A_1 + E_2A_2 + ...E_n.A_n}{A_1 + A_2 + ... + A_n}$$

$$\text{or } E_c = \frac{\sum EA}{\sum A}$$

with an external load W applied stress in the equivalent bar may be computed as

$$\text{stress} = \frac{W}{\sum A}$$

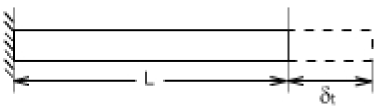
$$\text{strain inthe equivalent bar} = \frac{x}{L} = \frac{W}{\sum AE_c}$$

$$\text{hence commen extension } x = \frac{WL}{E_c \sum A}$$

Compound bars subjected to Temp. Change : Ordinary materials expand when heated and contract when cooled, hence , an increase in temperature produce a positive thermal strain. Thermal strains usually are reversible in a sense that the member returns to its original shape when the temperature return to its original value. However, there here are some materials which do not behave in this manner. These metals differs from ordinary materials in a sence that the strains are related non linearly to temperature and some times are irreversible .when a material is subjected to a change in temp. is a length will change by an amount.

$\varepsilon_t = \alpha$ .L.t

Or σ_t = E.α.t



α = coefficient of linear expansion for the material

L = original Length

t = temp. change

Thus an increase in temperature produces an increase in length and a decrease in temperature results in a decrease in length except in very special cases of materials with zero or negative coefficients of expansion which need not to be considered here.

If however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. They stress is equal in magnitude to that which would be produced in the bar by initially allowing the bar to its free length and then applying sufficient force to return the bar to its original length.

Change in Length = α L t

Therefore, strain = α L t / L

= α t

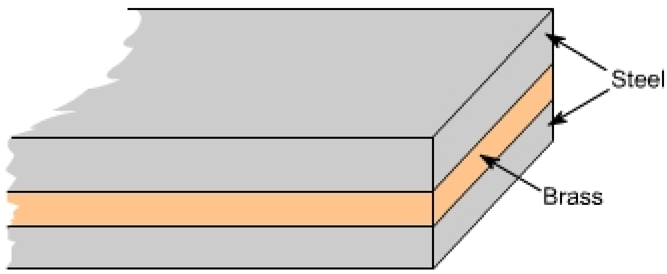
Therefore, the stress generated in the material by the application of sufficient force to remove this strain

= strain x E

or Stress = E α t

Consider now a compound bar constructed from two different materials rigidly joined together, for simplicity.

Let us consider that the materials in this case are steel and brass.



If we have both applied stresses and a temp. change, thermal strains may be added to those given by generalized hook's law equation –e.g.

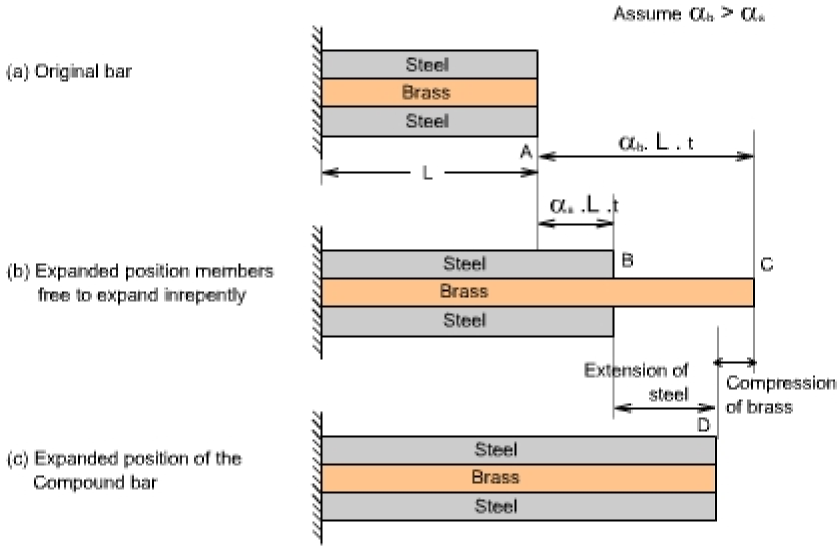
$$\in_x = \frac{1}{E}\left[\sigma_x - \gamma(\sigma_y + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_y - \gamma(\sigma_x + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_z - \gamma(\sigma_x + \sigma_y)\right] + \alpha\Delta t$$

While the normal strains a body are affected by changes in temperatures, shear strains are not. Because if the temp. of any block or element changes, then its size changes not its shape therefore shear strains do not change.

In general, the coefficients of expansion of the two materials forming the compound bar will be different so that as the temp. rises each material will attempt to expand by different amounts. Figure below shows the positions to which the individual materials will expand if they are completely free to expand (i.e not joined rigidly together as a compound bar). The extension of any Length L is given by α L t



In general, changes in lengths due to thermal strains may be calculated form equation δ_t = α Lt, provided that the members are able to expand or contract freely, a situation that exists in statically determinates structures. As a consequence no stresses are generated in a statically determinate structure when one or more members undergo a uniform temperature change. If in a structure (or a compound bar), the free expansion or contraction is not allowed then the member becomes s statically indeterminate, which is just being discussed as an example of the compound bar and thermal stresses would be generated.

If the two materials are now rigidly joined as a compound bar and subjected to the same temp. rise, each materials will attempt to expand to its free length position but each will be affected by the movement of the other. The higher coefficient of expansion material (brass) will therefore, seek to pull the steel up to its free length position and conversely, the lower coefficient of expansion martial (steel) will try to hold the brass back. In practice a compromised is reached, the compound bar extending to the position shown in fig (c), resulting in an effective compression of the brass from its free length position and an effective extension of steel from its free length position.

Module 2:

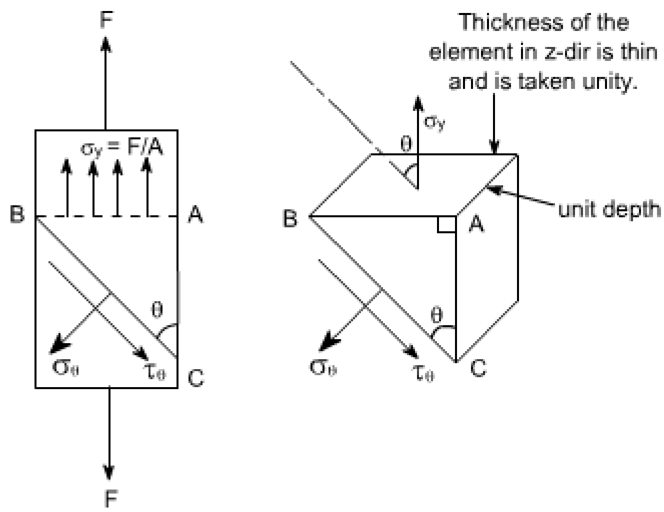
Lecture 1-5:

Two Dimensional State of Stress and Strain: Principal stresses. Numerical examples

Stresses on oblique plane: Till now we have dealt with either pure normal direct stress or pure shear stress. In many instances, however both direct and shear stresses acts and the resultant stress across any section will be neither normal nor tangential to the plane. A plane stse of stress is a 2 dimensional stae of stress in a sense that the stress components in one direction are all zero i.e

$\sigma_z = \tau_{yz} = \tau_{zx} = 0$

Examples of plane state of stress include plates and shells. Consider the general case of a bar under direct load F giving rise to a stress σ_y vertically



The stress acting at a point is represented by the stresses acting on the faces of the element enclosing the point. The stresses change with the inclination of the planes passing through that point i.e. the stress on the faces of the element vary as the angular position of the element changes. Let the block be of unit depth now considering the equilibrium of forces on the triangle portion ABC. Resolving forces perpendicular to BC, gives

σ_θ.BC.1 = $\sigma_y \sin\theta$. AB.1

but AB/BC = $\sin\theta$ or AB = BC $\sin\theta$

Substituting this value in the above equation, we get

σ_θ.BC.1 = σ_y sin θ. BC sinθ . 1 or $\sigma_\theta = \sigma_y \sin^2 2\theta$ (1)

Now resolving the forces parallel to BC

τ_θ .BC.1 = σ_y cos θ. AB sin. 1

again AB = BC cos θ

σ_θ.BC.1 = σ_y cos θ. BC sin θ.1 or $\sigma_\theta = \sigma_y \sin\theta \cos\theta$

$\tau_\theta = \frac{1}{2}.\sigma_y \sin 2\theta$ (2)

If θ = 90^0 the BC will be parallel to AB and τ_θ= 0, i.e. there will be only direct stress or normal stress.

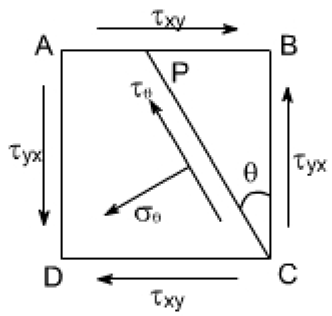
By examining the equations (1) and (2), the following conclusions may be drawn

(i) The value of direct stress σ_θ is maximum and is equal to σ_y when v= 90^0.

(ii) The shear stress τ_θ has a maximum value of 0.5 σ_y when θ= 45^0

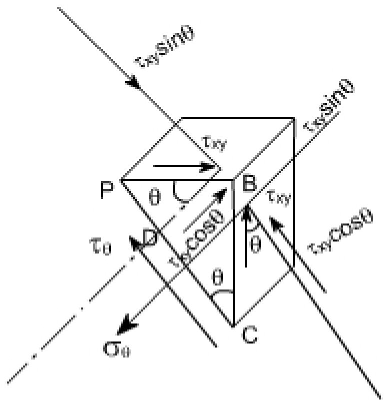
<u>Material subjected to pure shear:</u>

Consider the element shown to which shear stresses have been applied to the sides AB and DC



Complementary shear stresses of equal value but of opposite effect are then set up on the sides AD and BC in order to prevent the rotation of the element. Since the applied and complementary shear stresses are of equal value on the x and y planes. Therefore, they are both represented by the symbol τ_{xy}.

Now consider the equilibrium of portion of PBC



Assuming unit depth and resolving normal to PC or in the direction of σ_θ

σ_θ.PC.1 = τ_{xy}.PB.cosθ.1+ τ_{xy}.BC.sinθ.1

= τ_{xy}.PB.cosθ + τ_{xy}.BC.sinθ

Now writing PB and BC in terms of PC so that it cancels out from the two sides

PB/PC = sinθ BC/PC = cosθ

σ_θ.PC.1 = τ_{xy}.cosθsinθPC+ τ_{xy}.cosθ.sinθ.PC

σ_θ = 2 τ_{xy} sinθcosθ

Or, $\sigma_\theta = 2\tau_{xy}\sin 2\theta$ (1)

Now resolving forces parallel to PC or in the direction of σ_θ .then τ_{xy} PC.1

= τ_{xy} . PB sinθ - τ_{xy} BC cosθ

-ve sign has been put because this component is in the same direction as that of τ_θ .

again converting the various quantities in terms of PC we have

τ_{xy} PC. 1 = τ_{xy} . PB.sin^2 θ τ_{xy} - τ_{xy} PCcos$^2\theta$

= - τ_{xy} [cos$^2\theta$ - sin$^2\theta$]

= - τ_{xy} cos2θ (2)

the negative sign means that the sense of τ_θ is opposite to that of assumed one. Let us examine the equations (1) and (2) respectively

From equation (1) i.e,

σ_θ = τ_{xy} sin2θ

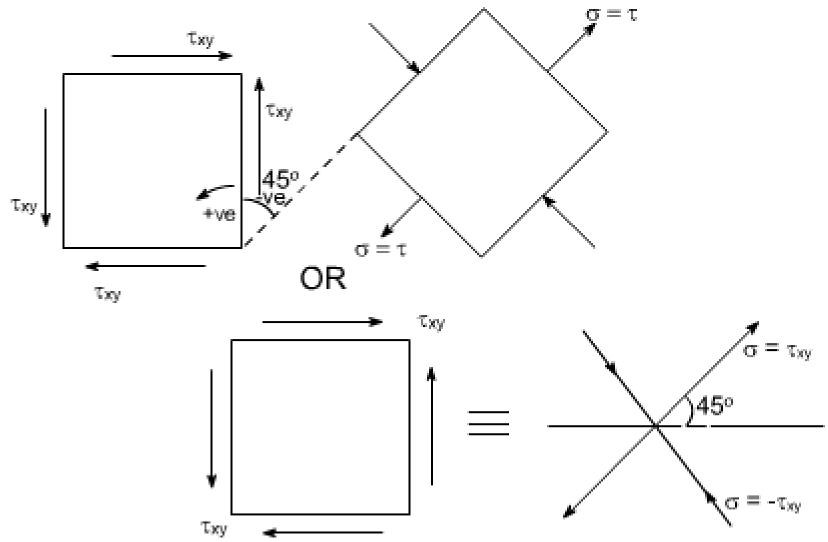
The equation (1) represents that the maximum value of σ_θ is τ_{xy} when θ = 45^0.Let us take into consideration the equation (2) which states that

σ_θ = - τ_{xy} cos2θ

It indicates that the maximum value of σ_θ is τ_{xy} when θ = 0^0 or 90^0. it has a value zero when θ = 45^0.

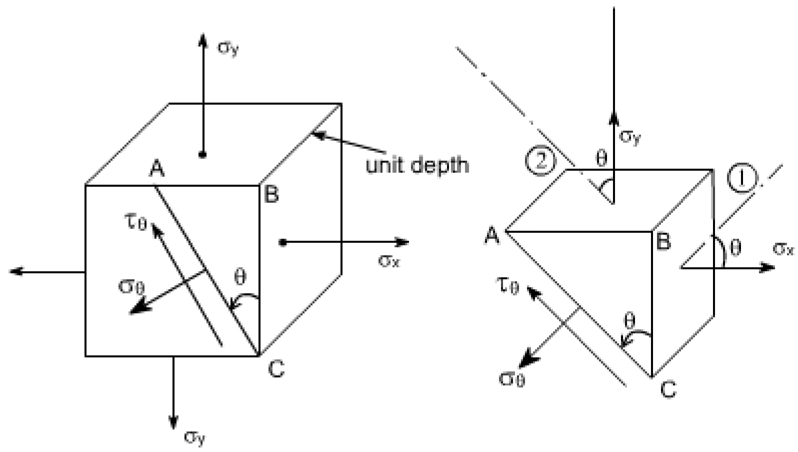
From equation (1) it may be noticed that the normal component □$_{□}$ has maximum and minimum values of +□$_{xy}$ (tension) and □□$_{xy}$(compression) on plane at ± 45^0 to the applied shear and on these planes the tangential component □$_{□}$ is zero.

Hence the system of pure shear stresses produces and equivalent direct stress system, one set compressive and one tensile each located at 45^0 to the original shear directions as depicted in the figure below:



<u>Material subjected to two mutually perpendicular direct stresses:</u>

Now consider a rectangular element of unit depth, subjected to a system of two direct stresses both tensile, □$_x$ and □$_y$acting right angles to each other.



for equilibrium of the portion ABC, resolving perpendicular to AC

σ_θ . AC.1 = σ_y sin θ. AB.1 + σ_x cos θ. BC.1

converting AB and BC in terms of AC so that AC cancels out from the sides

$\sigma_\theta = \sigma_y \sin^2 \theta + \sigma_x \cos^2 \theta$

Futher, recalling that $\cos^2\theta - \sin^2\theta = \cos 2\theta$ or $(1 - \cos 2\theta)/2 = \sin^2\theta$

Similarly $(1 + \cos 2\theta)/2 = \cos^2 q$

Hence by these transformations the expression for □□ reduces to

= 1/2□$_y$ (1 □ cos2□) + 1/2□$_x$ (1 + cos2□)

On rearranging the various terms we get

$$\boxed{\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta} \qquad (3)$$

Now resolving parallal to AC

s_q.AC.1= □□$_{xy}$..cos□.AB.1+□□$_{xy}$.BC.sin□.1

The – ve sign appears because this component is in the same direction as that of AC.

Again converting the various quantities in terms of AC so that the AC cancels out from the two sides.

$$\tau_\theta .\mathrm{AC}.1 = [\tau_x \cos\theta \sin\theta - \sigma_y \sin\theta \cos\theta]\mathrm{AC}$$

$$\tau_\theta = (\sigma_x - \sigma_y)\sin\theta\cos\theta$$

$$= \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

$$\text{or} \quad \boxed{\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta} \qquad (4)$$

Conclusions :

The following conclusions may be drawn from equation (3) and (4)

(i) The maximum direct stress would be equal to □$_x$ or □$_y$ which ever is the greater, when □ = 0^0 or 90^0

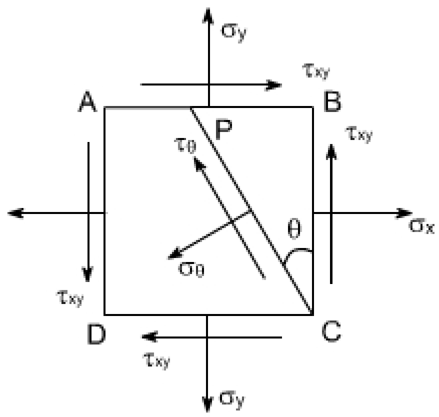
(ii) The maximum shear stress in the plane of the applied stresses occurs when □□= 45^0

$$\tau_{max} = \frac{(\sigma_x - \sigma_y)}{2}$$

<u>Material subjected to combined direct and shear stresses</u>:

Now consider a complex stress system shown below, acting on an element of material.

The stresses σ_x and σ_y may be compressive or tensile and may be the result of direct forces or as a result of bending.The shear stresses may be as shown or completely reversed and occur as a result of either shear force or torsion as shown in the figure below:



As per the double subscript notation the shear stress on the face BC should be notified as τ_{yx} , however, we have already seen that for a pair of shear stresses there is a set of complementary shear stresses generated such that $\tau_{yx} = \tau_{xy}$

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material subjected to pure stae of stress shear. In this case the various formulas deserved are as follows

$\sigma_\theta = \tau_{yx} \sin 2\theta$

$\tau_\theta = -\tau_{yx} \cos 2\theta$

(ii) Material subjected to two mutually perpendicular direct stresses. In this case the various formula's derived are as follows.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

To get the required equations for the case under consideration,let us add the respective equations for the above two cases such that

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

These are the equilibrium equations for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic behaviour

This eqn gives two values of 2θ that differ by 180^0 .Hence the planes on which maximum and minimum normal stresses occurate 90^0apart.

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

$$= 0$$

$$\text{i.e.} -(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$$

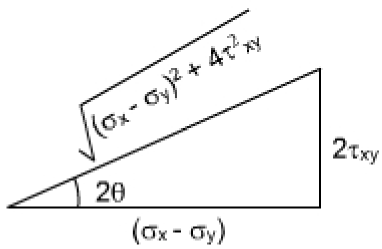
$$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$$

Thus, $\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$

From the triangle it may be determined

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$



Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}.\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}.\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{1}{2}\frac{4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}.\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}.\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = 0$$

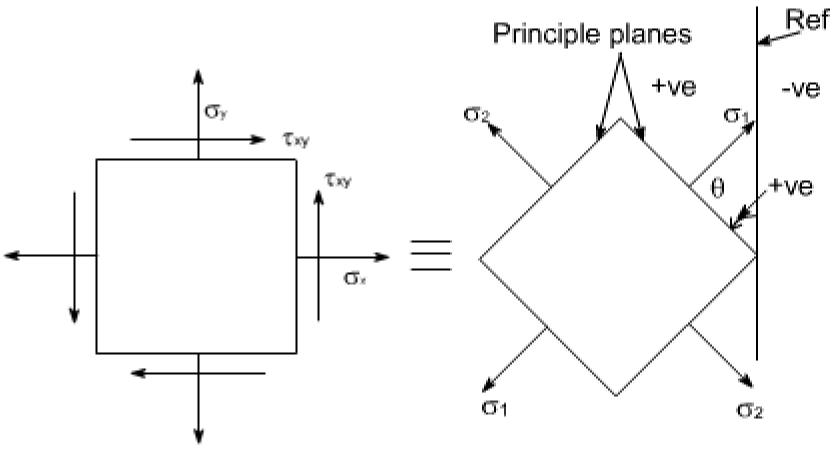
This shows that the values oshear stress is zero on the principal planes.

Hence the maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

will yield two values of 2□ separated by 180^0 i.e. two values of □ separated by 90^0 .Thus the two principal stresses occur on mutually perpendicular planes termed principal planes.

Therefore the two – dimensional complex stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a material subjected to direct stresses the value of maximum shear stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y)$ at $\theta = 45^0$,Thus, for a 2-dimensional state of stress,subjected to principle stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$, on substituting the values if σ_1 and σ_2,we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$
$$= 0$$

or $(\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p . \tan 2\theta_s = 1}$$

Therefore,it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double angle of equation (2) are 90^0 away from the corresponding angle of equation (1).

This means that the angles that angles that locate the plane of maximum or minimum shearing stresses form angles of 45^0 with the planes of principal stresses.

Futher, by making the triangle we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

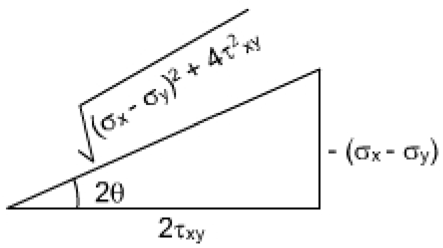
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}. - \frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$= -\frac{1}{2}.\frac{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$



Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.

The largest stress regard less of sign is always know as maximum shear stress.

Principal plane inclination in terms of associated principal stress:

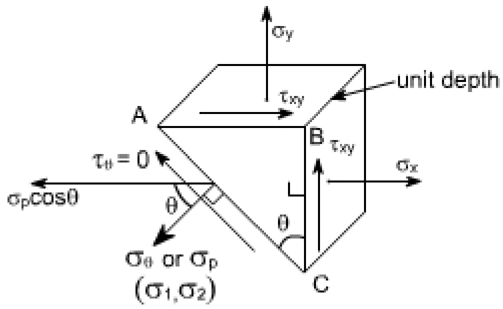
We know that the equation $\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$

yields two values of q i.e. the inclination of the two principal planes on which the principal stresses s_1 and s_2 act. It is uncertain,however, which stress acts on which plane unless equation.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

is used and observing which one of the two principal stresses is obtained.

Alternatively we can also find the answer to this problem in the following manner



Consider once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses σ_p acts, and the shear stress is zero.

Resolving the forces horizontally we get:

σ_x .BC . 1 + τ_{xy} .AB . 1 = σ_p . cosθ . AC dividing the above equation through by BC we get

$$\sigma_x . + \tau_{xy}\frac{AB}{BC} = \sigma_p . \cos\theta . \frac{AC}{BC}$$

or

$$\sigma_x + \tau_{xy}\tan\theta = \sigma_p$$

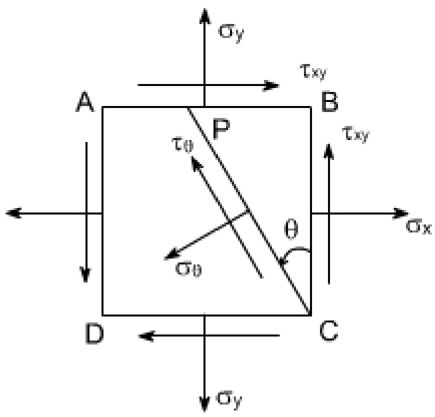
Thus

$$\boxed{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}}$$

GRAPHICAL SOLUTION – MOHR'S STRESS CIRCLE

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This grapical representation is very useful in depending the relationships between normal and shear stresses acting on any inclined plane at a point in a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure



The above system represents a complete stress system for any condition of applied load in two dimensions

The Mohr's stress circle is used to find out graphically the direct stress σ and sheer stress τ on any plane inclined at θ to the plane on which σ_x acts.The direction of θ here is taken in anticlockwise direction from the BC.

STEPS:

In order to do achieve the desired objective we proceed in the following manner

(i) Label the Block ABCD.

(ii) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(iii) Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.

Direct stresses σ tensile positive; compressive, negative

Shear stresses – tending to turn block clockwise, positive

– tending to turn block counter clockwise, negative

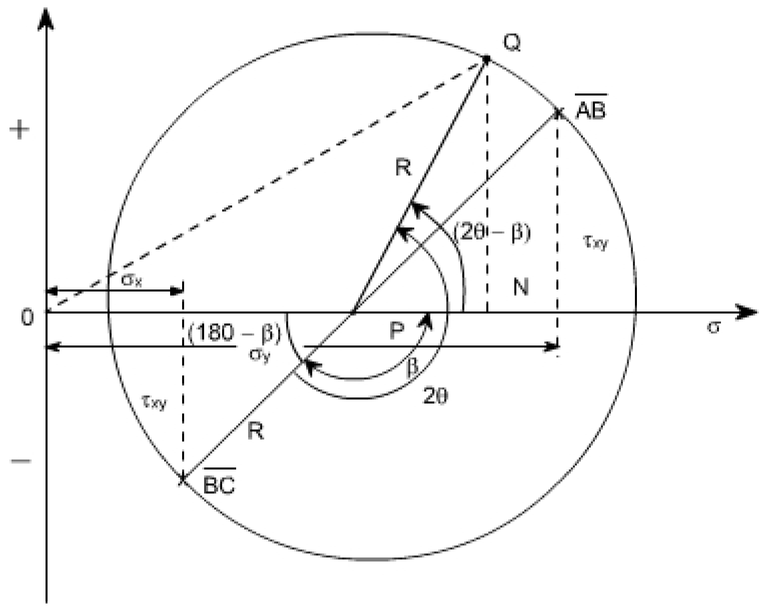
[i.e shearing stresses are +ve when its movement about the centre of the element is clockwise]

This gives two points on the graph which may than be labeled as $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

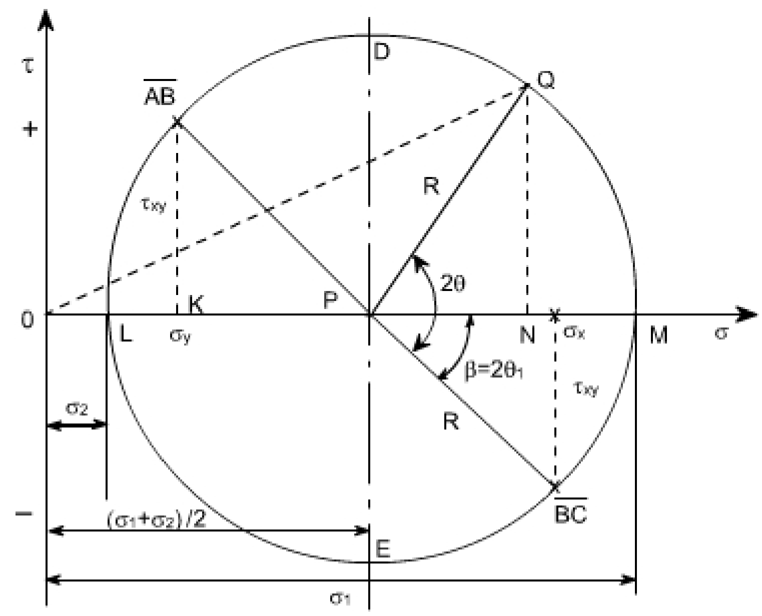
(iv) Join $\overline{AB}$ and $\overline{BC}$.

(v) The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining $\overline{AB}$ and $\overline{BC}$ is diameter. Therefore the circle can now be drawn.

Now every point on the circle then represents a state of stress on some plane through C.



Proof:



Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θ with BC, and drop a perpendicular from Q to meet the s axis at N.Then OQ represents the resultant stress on the plane an angle θ to BC. Here we have assumed that $\sigma_x > \sigma_y$

Now let us find out the coordinates of point Q. These are ON and QN.

From the figure drawn earlier

$$ON = OP + PN$$

$$OP = OK + KP$$

$$OP = \sigma_y + 1/2\,(\sigma_x - \sigma_y)$$

$$= \sigma_y / 2 + \sigma_y / 2 + \sigma_x / 2 + \sigma_y / 2$$

$$= (\sigma_x + \sigma_y) / 2$$

$$PN = R\cos(2\theta - \beta)$$

hence ON = OP + PN

$$= (\sigma_x + \sigma_y) / 2 + R\cos(2\theta - \beta)$$

$$= (\sigma_x + \sigma_y) / 2 + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$$

now make the substitutions for $R\cos\beta$ and $R\sin\beta$.

$$R\cos\beta = \frac{(\sigma_x - \sigma_y)}{2};\ R\sin\beta = \tau_{xy}$$

Thus,

$$ON = 1/2\,(\sigma_x + \sigma_y) + 1/2\,(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

Similarly $QM = R\sin(2\theta - \beta)$

$$= R\sin 2\theta\cos\beta - R\cos 2\theta\sin\beta$$

Thus, substituting the values of $R\cos\beta$ and $R\sin\beta$, we get

$$QM = 1/2\,(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta \qquad (2)$$

If we examine the equation (1) and (2), we see that this is the same equation which we have already derived analytically

Thus the co-ordinates of Q are the normal and shear stresses on the plane inclined at θ to BC in the original stress system.

N.B: Since angle $\overline{BC}$ PQ is 2θ on Mohr's circle and not θ it becomes obvious that angles are doubled on Mohr's circle. This is the only difference, however, as They are measured in the same direction and from the same plane in both figures.

Further points to be noted are :

(1) The direct stress is maximum when Q is at M and at this point obviously the sheer stress is zero, hence by definition OM is the length representing the maximum principal stresses $\square_1$ and $2\square_1$ gives the angle of the plane $\square_1$ from BC. Similar OL is the other principal stress and is represented by $\square_2$

(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle.

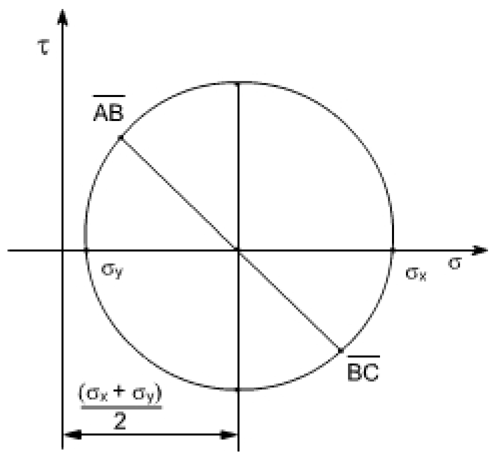
This follows that since shear stresses and complimentary sheer stresses have the same value; therefore the centre of the circle will always lie on the s axis midway between $\square_x$ and $\square_y$. [since $+\square_{xy}$ & $\square\square_{xy}$ are shear stress & complimentary shear stress so they are same in magnitude but different in sign.]

(3) From the above point the maximum sheer stress i.e. the Radius of the Mohr's stress circle would be

$$\frac{(\sigma_x - \sigma_y)}{2}$$

While the direct stress on the plane of maximum shear must be mid – may between $\square_x$ and $\square_y$ i.e

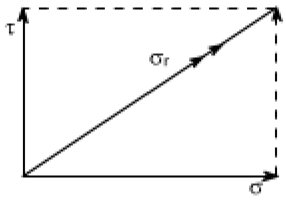
$$\frac{(\sigma_x + \sigma_y)}{2}$$



(4) As already defined the principal planes are the planes on which the shear components are zero.

Therefore are conclude that on principal plane the sheer stress is zero.

(5) Since the resultant of two stress at 90^0 can be found from the parallogram of vectors as shown in the diagram.Thus, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.



(6) The graphical method of solution for a complex stress problems using Mohr's circle is a very powerful technique, since all the information relating to any plane within the stressed element is contained in the single construction. It thus, provides a convenient and rapid means of solution. Which is less prone to arithmetical errors and is highly recommended.

Numericals:

Let us discuss few representative problems dealing with complex state of stress to be solved either analytically or graphically.

Q 1: A circular bar 40 mm diameter carries an axial tensile load of 105 kN. What is the Value of shear stress on the planes on which the normal stress has a value of 50 MN/m^2 tensile.

Solution:

Tensile stress □$_y$= F / A = 105 x 10^3 / □ x $(0.02)^2$

= 83.55 MN/m^2

Now the normal stress on an obliqe plane is given by the relation

□$_{□□□}$= □$_y$sin^2□

50 x 10^6 = 83.55 MN/m^2 x 10^6sin^2□

□ = $50^0$68'

The shear stress on the oblique plane is then given by

□$_{□□}$= 1/2 □$_y$sin2□

= 1/2 x 83.55 x 10^6 x sin 101.36

= 40.96 MN/m^2

Therefore the required shear stress is 40.96 MN/m^2

Q2:

For a given loading conditions the state of stress in the wall of a cylinder is expressed as follows:

(a) 85 MN/m^2 tensile

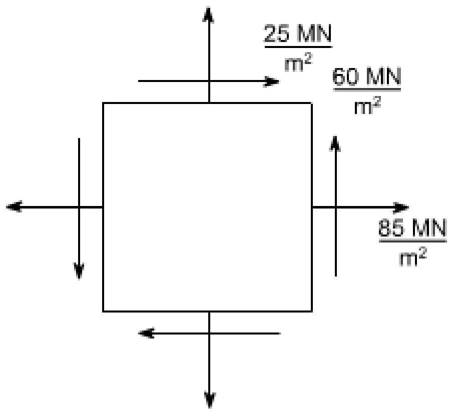
(b) 25 MN/m^2 tensile at right angles to (a)

(c) Shear stresses of 60 MN/m^2 on the planes on which the stresses (a) and (b) act; the sheer couple acting on planes carrying the 25 MN/m^2 stress is clockwise in effect.

Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged

Solution:

The problem may be attempted both analytically as well as graphically. Let us first obtain the analytical solution



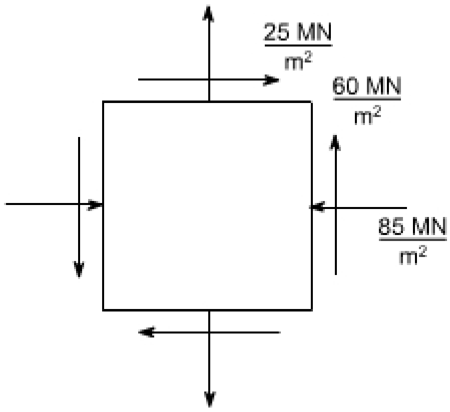
The principle stresses are given by the formula

$$\sigma_1 \text{ and } \sigma_2$$
$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$
$$= \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 + 25)^2 + (4 \times 60^2)}$$
$$= 55 \pm \frac{1}{2}.60\sqrt{5} = 55 \pm 67$$
$$\Rightarrow \sigma_1 = 122 \text{ MN/m}^2$$
$$\sigma_2 = -12 \text{ MN/m}^2 \text{(compressive)}$$

For finding out the planes on which the principle stresses act us the equation $\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$

The solution of this equation will yeild two values θ i.e they θ_1 and θ_2 giving θ_1= $31^0 71'$ & θ_2= $121^0 71'$

(b) In this case only the loading (a) is changed i.e. its direction had been changed. While the other stresses remains unchanged hence now the block diagram becomes.



Again the principal stresses would be given by the equation.

$$\sigma_1 \,\&\, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{1}{2}(-85 + 25) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= \frac{1}{2}(-60) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= -30 \pm \frac{1}{2}\sqrt{12100 + 14400}$$

$$= -30 \pm 81.4$$

$$\sigma_1 = 51.4 \text{ MN/m}^2; \ \sigma_2 = -111.4 \text{ MN/m}^2$$

Again for finding out the angles use the following equation.

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

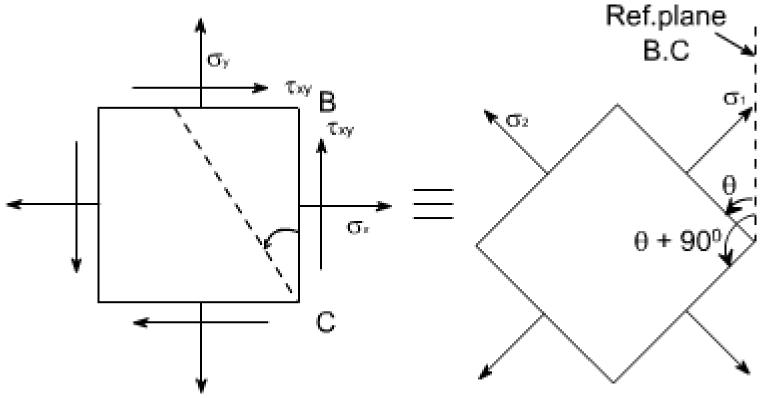
$$= \frac{2 \times 60}{-85 - 25} = +\frac{120}{-110}$$

$$= -\frac{12}{11}$$

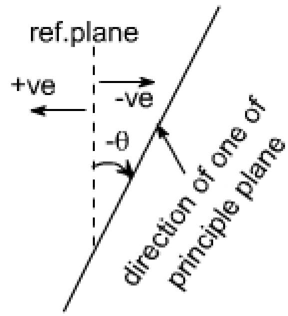
$$2\theta = \tan\left(-\frac{12}{11}\right)$$

$$\Rightarrow \quad \theta = -23.74^0$$

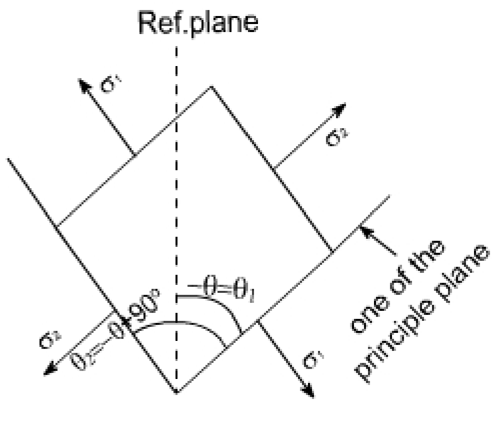
Thus, the two principle stresses acting on the two mutually perpendicular planes i.e principle planes may be depicted on the element as shown below:



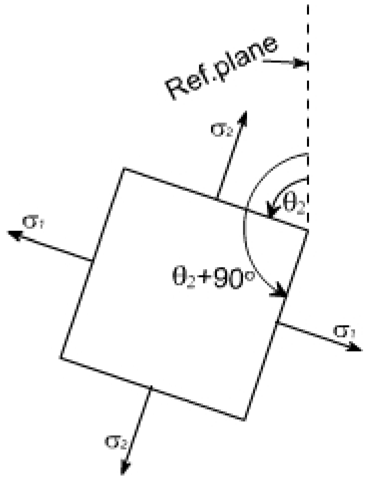
So this is the direction of one principle plane & the principle stresses acting on this would be □$_1$ when is acting normal to this plane, now the direction of other principal plane would be 90^0 + □ because the principal planes are the two mutually perpendicular plane, hence rotate the another plane □ + 90^0 in the same direction to get the another plane, now complete the material element if □ is negative that means we are measuring the angles in the opposite direction to the reference plane BC .



Therefore the direction of other principal planes would be {□□ + 90} since the angle □□ is always less in magnitude then 90 hence the quantity (□□□ + 90) would be positive therefore the Inclination of other plane with reference plane would be positive therefore if just complete the Block. It would appear as



If we just want to measure the angles from the reference plane, than rotate this block through 180^0 so as to have the following appearance.



So whenever one of the angles comes negative to get the positive value,

first Add 90^0 to the value and again add 90^0 as in this case □ = □$23^0 74'$

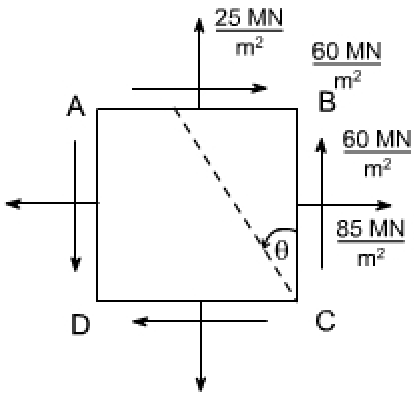
so □$_1$ = □$23^0 74'$ + 90^0 = $66^0 26'$ .Again adding 90^0 also gives the direction of other principle planes

i.e □$_2$ = $66^0 26'$ + 90^0 = $156^0 26'$

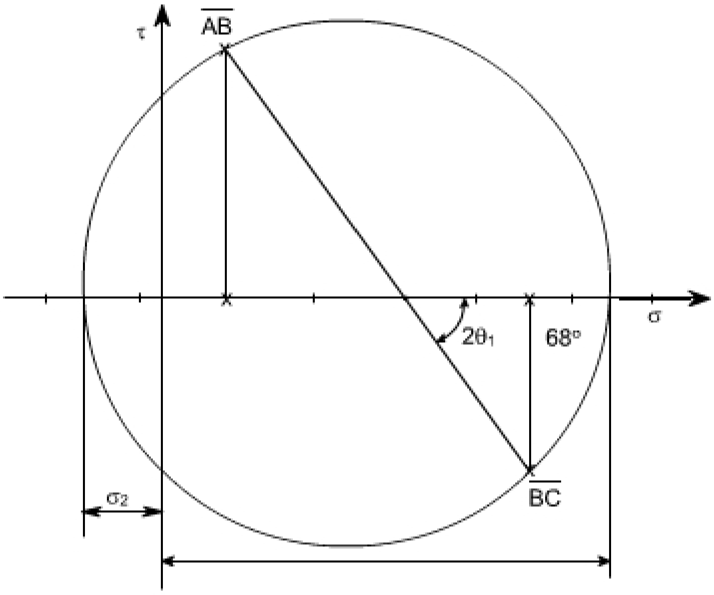
This is how we can show the angular position of these planes clearly.

GRAPHICAL SOLUTION:

Mohr's Circle solution: The same solution can be obtained using the graphical solution i.e the Mohr's stress circle,for the first part, the block diagram becomes



Construct the graphical construction as per the steps given earlier.



Taking the measurements from the Mohr's stress circle, the various quantities computed are

σ_1 = 120 MN/m^2 tensile

σ_2 = 10 MN/m^2 compressive

θ_1 = 34^0 counter clockwise from BC

θ_2 = 34^0 + 90 = 124^0 counter clockwise from BC

Part Second : The required configuration i.e the block diagram for this case is shown along with the stress circle.

By taking the measurements, the various quantites computed are given as

σ_1 = 56.5 MN/m^2 tensile

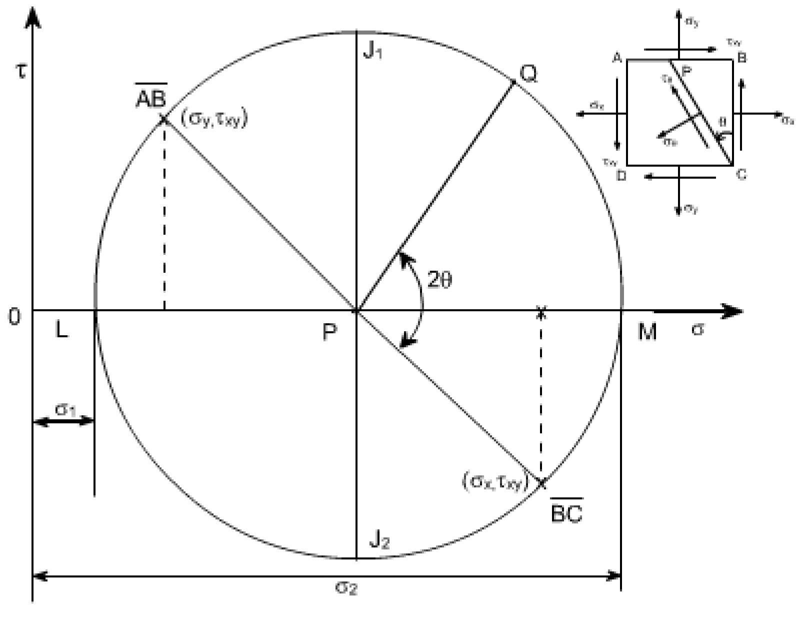
□$_2$ = 106 MN/m^2 compressive

□$_1$ = 66^{0}15' counter clockwise from BC

□$_2$ = 156^{0}15' counter clockwise from BC

Salient points of Mohr's stress circle:

1. complementary shear stresses (on planes 90^0 apart on the circle) are equal in magnitude
2. The principal planes are orthogonal: points L and M are 180^0 apart on the circle (90^0 apart in material)
3. There are no shear stresses on principal planes: point L and M lie on normal stress axis.
4. The planes of maximum shear are 45^0 from the principal points D and E are 90^0 , measured round the circle from points L and M.
5. The maximum shear stresses are equal in magnitude and given by points D and E
6. The normal stresses on the planes of maximum shear stress are equal i.e. points D and E both have normal stress co-ordinate which is equal to the two principal stresses.



As we know that the circle represents all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co-ordinates of the point ‘Q' are seen to be the same as those derived from equilibrium of the element. i.e. the normal and shear stress components on any plane passing through the point can be found using Mohr's circle. Worthy of note:

1. The sides AB and BC of the element ABCD, which are 90^0 apart, are represented on the circle by $\overline{AB}$ P and $\overline{BC}$ P and they are 180^0 apart.

2. It has been shown that Mohr's circle represents all possible states at a point. Thus, it can be seen at a point. Thus, it, can be seen that two planes LP and PM, 180^0 apart on the diagram and therefore 90^0 apart in the material, on which shear stress $\square_{\square}$ is zero. These planes are termed as principal planes and normal stresses acting on them are known as principal stresses.

Thus , $\square_1$ = OL

$\square_2$ = OM

3. The maximum shear stress in an element is given by the top and bottom points of the circle i.e by points J_1 and J_2 ,Thus the maximum shear stress would be equal to the radius of i.e. $\square_{max}$= 1/2($\square\square_1\square\square\square_2$),the corresponding normal stress is obviously the distance OP = 1/2 ($\square\square_x$+ $\square_y$) , Further it can also be seen that the planes on which the shear stress is maximum are situated 90^0 from the principal planes (on circle), and 45^0 in the material.

4.The minimum normal stress is just as important as the maximum. The algebraic minimum stress could have a magnitude greater than that of the maximum principal stress if the state of stress were such that the centre of the circle is to the left of orgin.

i.e. if $\square_1$ = 20 MN/m^2 (say)

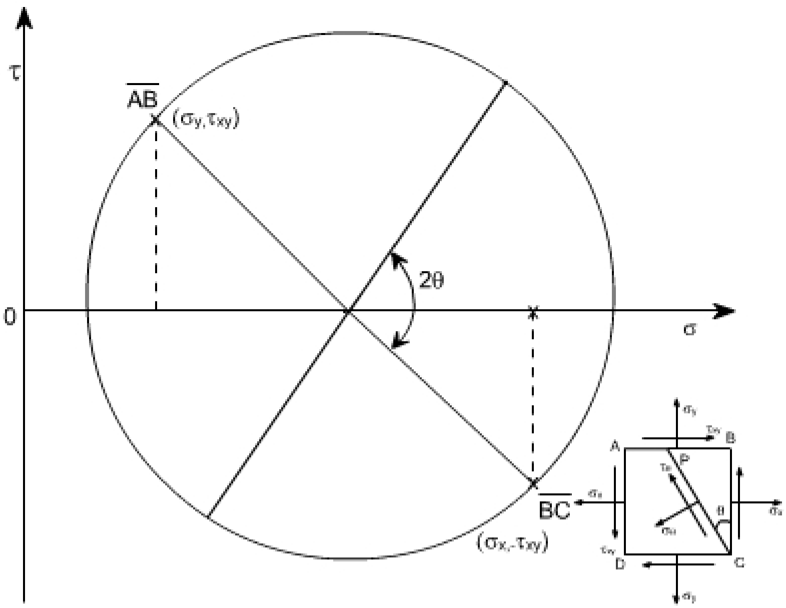
$\square_2$ = $\square$80 MN/m^2 (say)

Then $\square_{max}{}^m$ = ($\square_1$ $\square\square\square_2$ / 2) = 50 MN/m^2

If should be noted that the principal stresses are considered a maximum or minimum mathematically e.g. a compressive or negative stress is less than a positive stress, irrespective or numerical value.

5. Since the stresses on perpendular faces of any element are given by the co-ordinates of two diametrically opposite points on the circle, thus, the sum of the two normal stresses for any and all orientations of the element is constant, i.e. Thus sum is an invariant for any particular state of stress.

Sum of the two normal stress components acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the orientation of these planes.



This can be also understand from the circle Since AB and BC are diametrically opposite thus, what ever may be their orientation, they will always lie on the diametre or we can say that their sum won't change, it can also be seen from analytical relations

We know $\sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

on plane BC; $\square = 0$

$\square_{n1} = \square_x$

on plane AB; $\square = 270^0$

$\square_{n2} = \square_y$

Thus $\square_{n1} + \square_{n2} = \square_x + \square_y$

6. If $\square_1 = \square_2$, the Mohr's stress circle degenerates into a point and no shearing stresses are developed on xy plane.

7. If $\square_x + \square_y = 0$, then the center of Mohr's circle coincides with the origin of □□□□□ co-ordinates.

SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT 2-INTRODUCTION TO AIRCRAFT STRUCTURES-SAEA1305

Module 3

Lecture 1- 4: Shear Force and Bending Moment

Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms

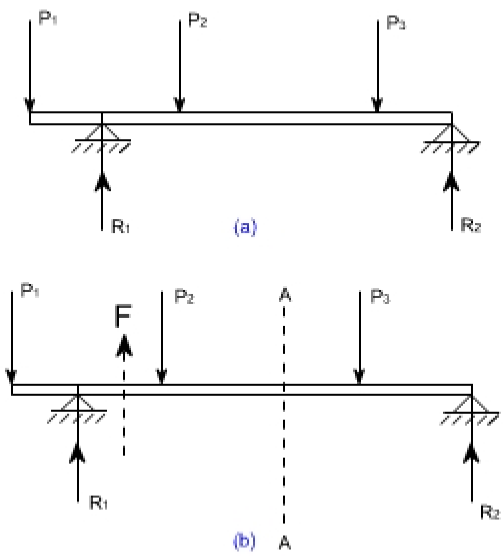


Fig 1

Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is 'F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces 'F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force 'F' to as follows:

At any x-section of a beam, the shear force 'F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.

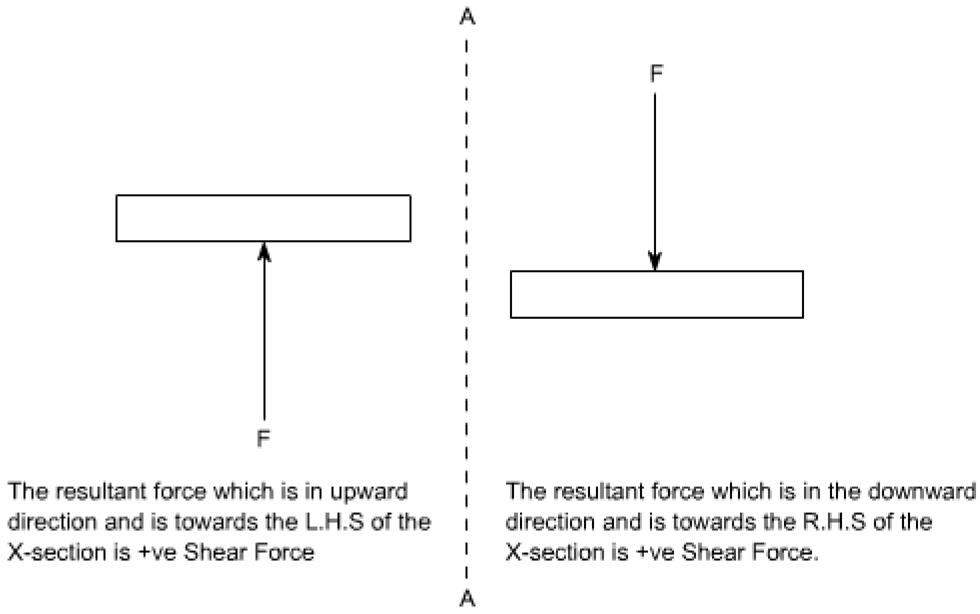


Fig 2: Positive Shear Force

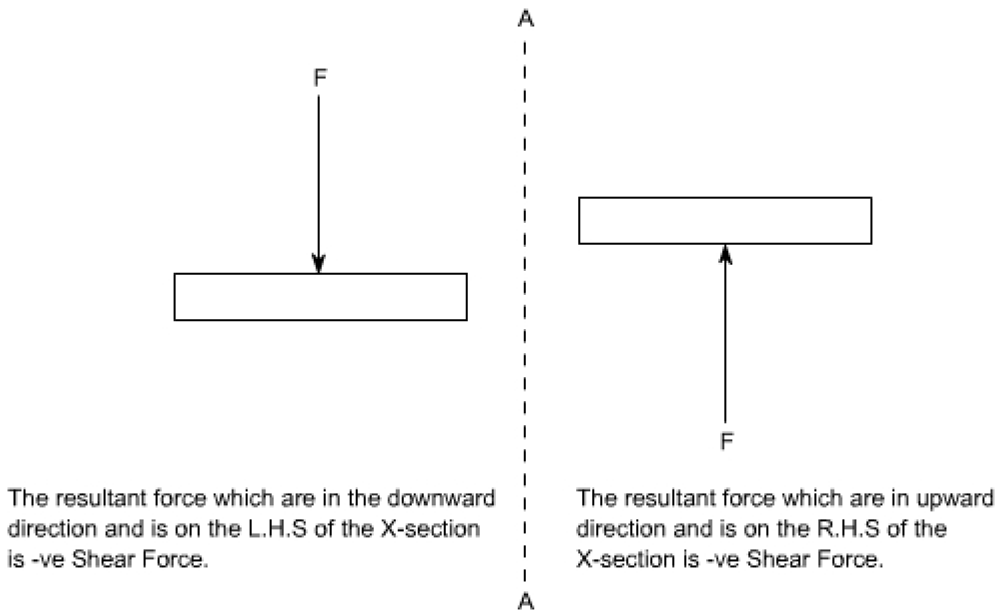


Fig 3: Negative Shear Force

Bending Moment:

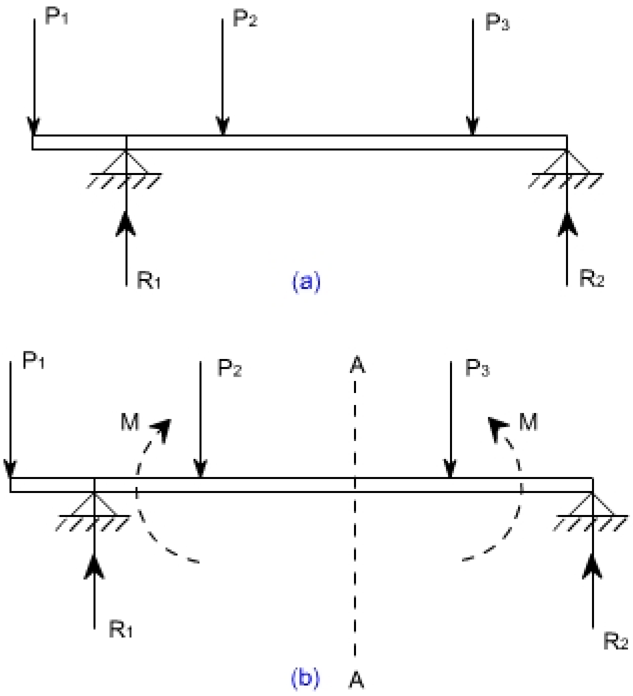


Fig 4

Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be ‘M' in C.C.W. Then ‘M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as the algebraic sum of the moments about an x-section of all the forces acting on either side of the section

Sign Conventions for the Bending Moment:

For the bending moment, following sign conventions may be adopted as indicated in Fig 5 and Fig 6.

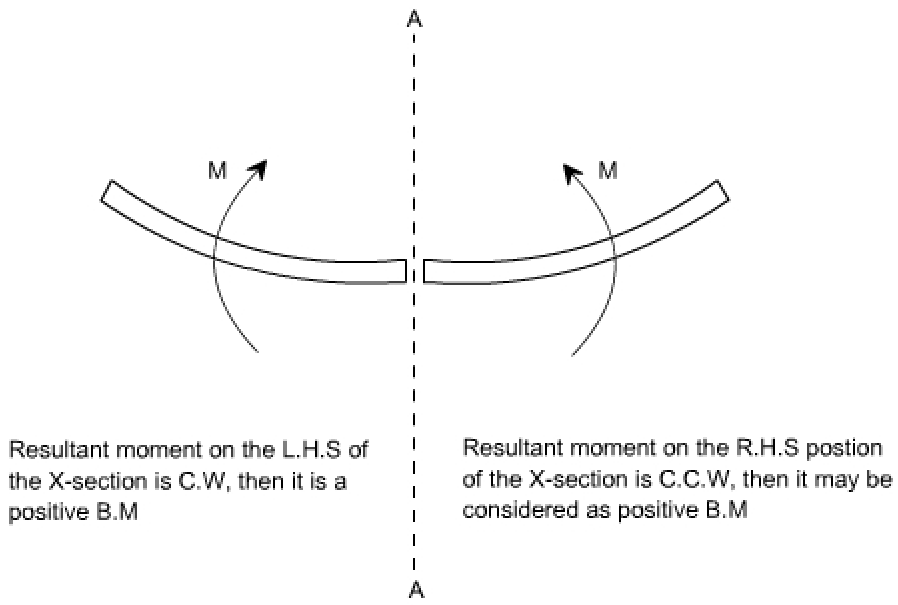


Fig 5: Positive Bending Moment

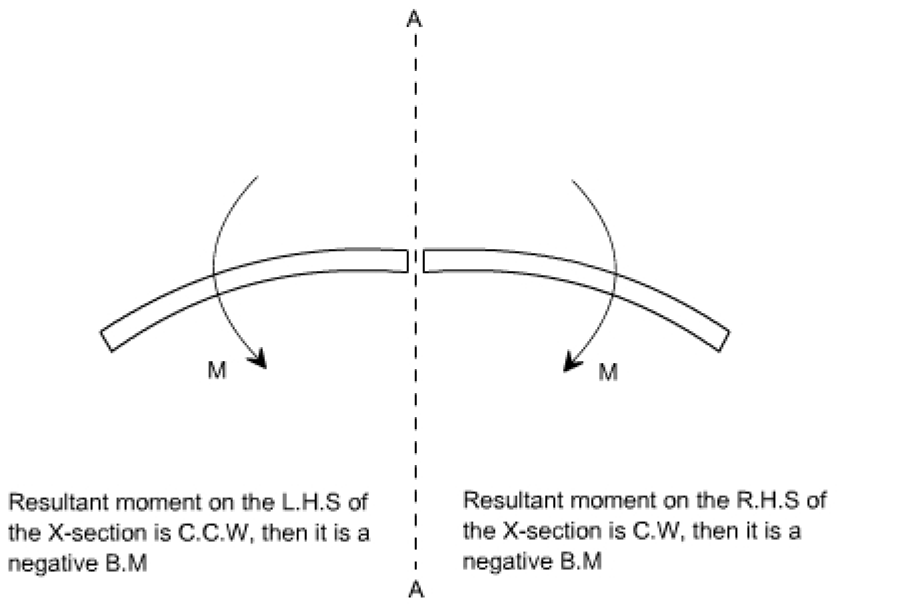


Fig 6: Negative Bending Moment

Some times, the terms 'Sagging' and Hogging are generally used for the positive and negative bending moments respectively.

Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further.

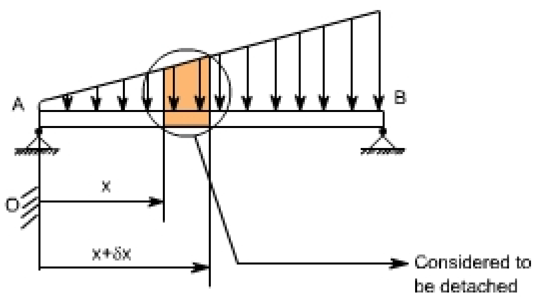
Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force ‘F' varies along the length of beam. If x dentotes the length of the beam, then F is function x i.e. F(x).

Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment ‘M' varies along the length of the beam. Again M is a function x i.e. M(x).

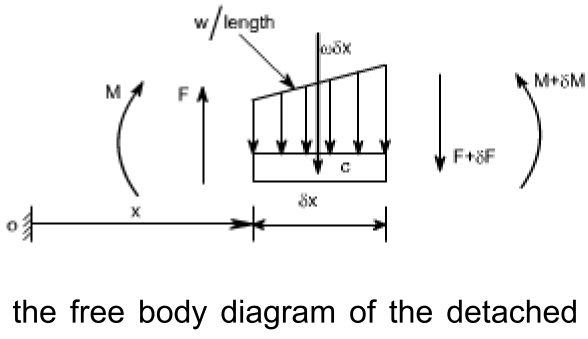
Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established.

Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance ‘x' from the origin ‘0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ δF at the section x and x + δx respectively.
- The bending moment at the sections x and x + δx be M and M + dM respectively.

• Force due to external loading, if 'w' is the mean rate of loading per unit length then the total loading on this slice of length δx is w. δx, which is approximately acting through the centre 'c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre 'c'.

This small element must be in equilibrium under the action of these forces and couples.

Now let us take the moments at the point 'c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \quad [\text{Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities}]$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \to 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \to 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots(2)$$

Conclusions: From the above relations,the following important conclusions may be drawn

• From Equation (1), the area of the shear force diagram between any two points, from the basic calculus is the bending moment diagram

$$M = \int F.dx$$

• The slope of bending moment diagram is the shear force, thus

$$F = \frac{dM}{dx}$$

Thus, if F=0; the slope of the bending moment diagram is zero and the bending moment is therefore constant.'

- The maximum or minimum Bending moment occurs where $\frac{dM}{dx} = 0.$

The slope of the shear force diagram is equal to the magnitude of the intensity of the distributed loading at any position along the beam. The –ve sign is as a consequence of our particular choice of sign conventions

Procedure for drawing shear force and bending moment diagram:

Preamble:

The advantage of plotting a variation of shear force F and bending moment M in a beam as a function of ‘x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment.

Further, the determination of value of M as a function of ‘x' becomes of paramount importance so as to determine the value of deflection of beam subjected to a given loading.

Construction of shear force and bending moment diagrams:

A shear force diagram can be constructed from the loading diagram of the beam. In order to draw this, first the reactions must be determined always. Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.

When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam. No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations.

The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams giving due regard to sign. The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

It may also be observed that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. It may also further observe that dm/dx= F therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero. In order to check the validity of the bending moment diagram, the terminal conditions for the moment must be satisfied. If the end is free or pinned, the computed sum must be equal to zero. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction. These conditions must always be satisfied.

Illustrative problems:

In the following sections some illustrative problems have been discussed so as to illustrate the procedure for drawing the shear force and bending moment diagrams

1. A cantilever of length carries a concentrated load 'W' at its free end.

Draw shear force and bending moment.

Solution:

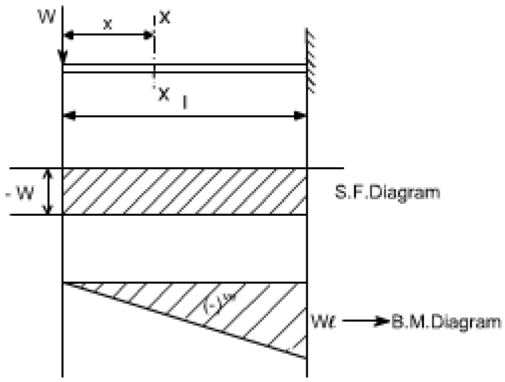
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative

Taking moments about the section gives (obviously to the left of the section)

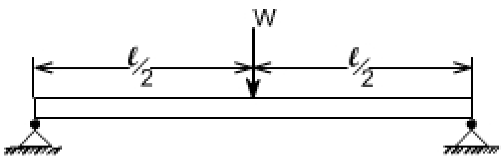
M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention)

so that the maximum bending moment occurs at the fixed end i.e. M = -W l

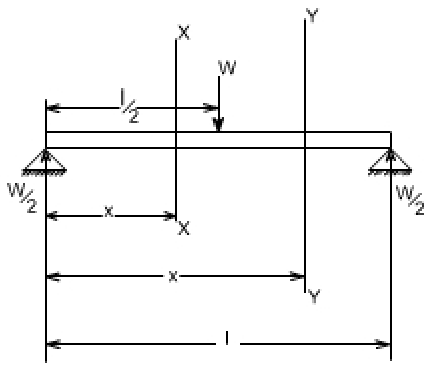
From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



2. Simply supported beam subjected to a central load (i.e. load acting at the mid-way)



By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.

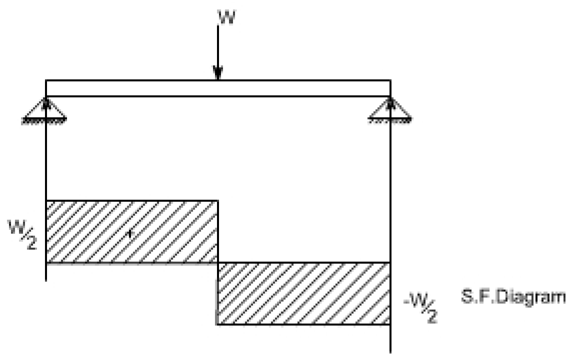


.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

If we consider another section Y-Y which is beyond l/2 then

$S.F_{Y\text{-}Y} = \frac{W}{2} - W = \frac{-W}{2}$ for all values greater = l/2

Hence S.F diagram can be plotted as,



.For B.M diagram:

If we just take the moments to the left of the cross-section,

$$B.M_{x\text{-}x} = \frac{W}{2}\,x \text{ for } x \text{ lies between } 0 \text{ and } l/2$$

$$B.M_{\text{at } x = \frac{l}{2}} = \frac{W}{2}\,\frac{l}{2} \text{ i.e } B.M \text{ at } x = 0$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}\,x - W\left(x - \frac{l}{2}\right)$$

Again

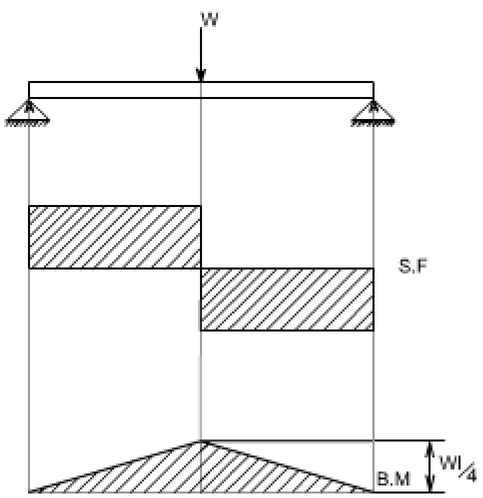
$$= \frac{W}{2}\,x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}\,x + \frac{Wl}{2}$$

$$B.M_{\text{at } x = l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

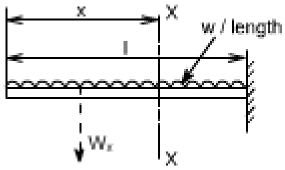
$$= 0$$

Which when plotted will give a straight relation i.e.



It may be observed that at the point of application of load there is an abrupt change in the shear force, at this point the B.M is maximum.

3. A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$S.F_{xx} = -Wx$ for all values of 'x'. **---------- (1)**

$S.F_{xx} = 0$

$S.F_{xx \text{ at } x=1} = -Wl$

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

Therefore, the bending moment at any cross-section X-X is

$$B.M_{X\text{-}X} = -\,W x\,\frac{x}{2}$$

$$= -\,W\frac{x^2}{2}$$

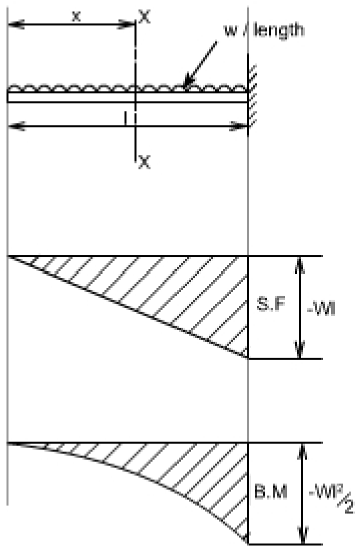
The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

The extreme values of this would be at x = 0 and x = l

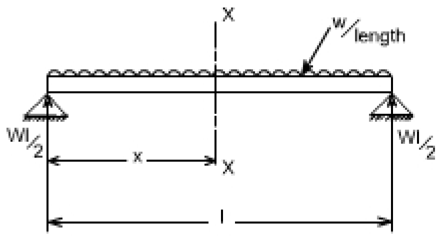
$$B.M_{\text{at } x = l} = -\frac{Wl^2}{2}$$

$$= \frac{Wl}{2} - Wx$$

Hence S.F and B.M diagram can be plotted as follows:



4. Simply supported beam subjected to a uniformly distributed load [U.D.L].



The total load carried by the span would be

= intensity of loading x length

= w x l

By symmetry the reactions at the end supports are each wl/2

If x is the distance of the section considered from the left hand end of the beam.

S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$

$$= W\left(\frac{l}{2} - x\right)$$

Giving a straight relation, having a slope equal to the rate of loading or intensity of the loading.

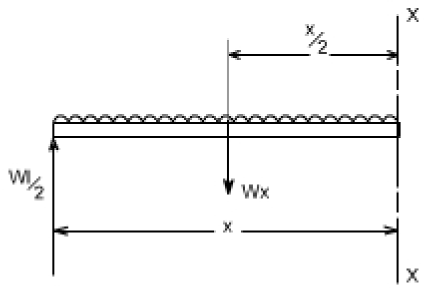
$$S.F_{at\ x=0} = \frac{wl}{2} - wx$$

so at

$$S.F_{at\ x=\frac{l}{2}} = 0 \text{ hence the S.F is zero at the centre}$$

$$S.F_{at\ x=l} = -\frac{Wl}{2}$$

The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



$$B.M_{x\text{-}x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

so the

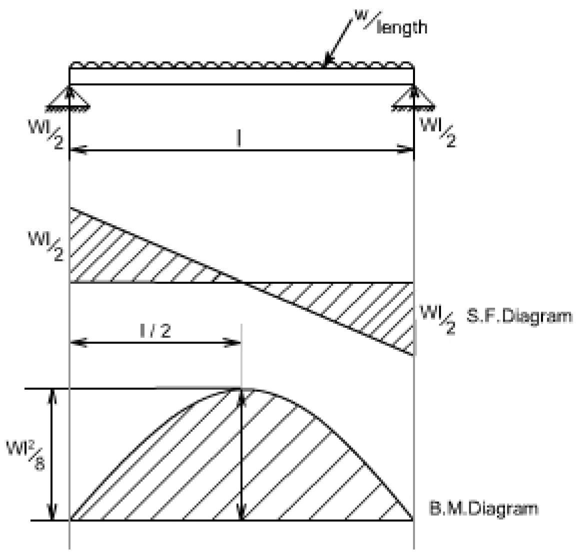
$$= W.\frac{x}{2}(l - 2) \quad ....(2)$$

$$B.M_{at\ x=0} = 0$$

$$B.M_{at\ x=l} = 0$$

$$B.M\Big|_{at\ x=l} = -\frac{Wl^2}{8}$$

So the equation (2) when plotted against x gives rise to a parabolic curve and the shear force and bending moment can be drawn in the following way will appear as follows:



Module 3

Lecture 5-8: Pure Bending

Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

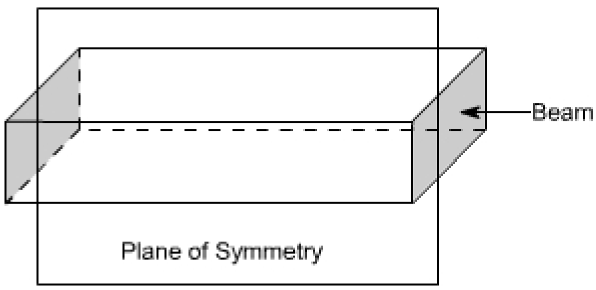


Fig (1)

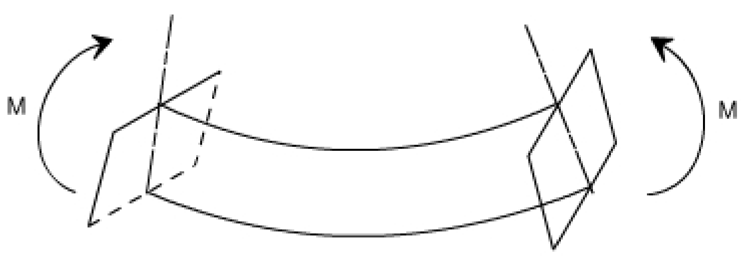
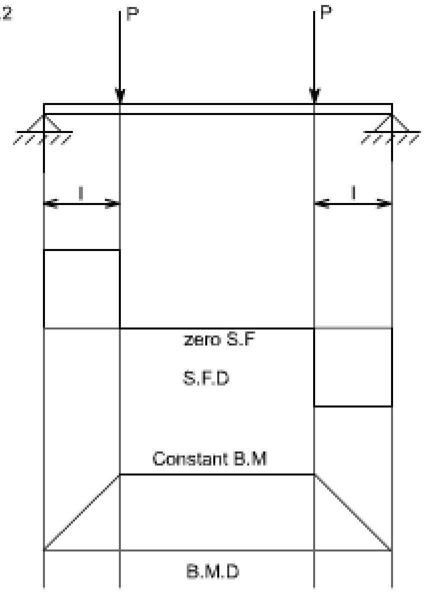


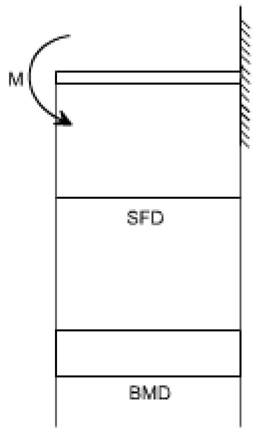
Fig (2)

When a member is loaded in such a fashion it is said to be in **pure bending.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below :

EX .2



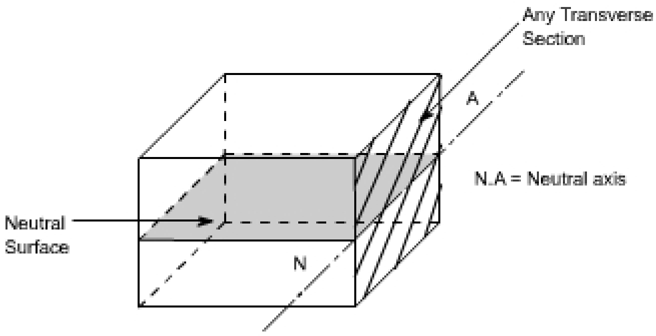
EX. 1



When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,

1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross-section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.

2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.

The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axisNeutral axis **(N A)** .

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam**HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e.**H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle □.

Consider now fiber AB in the material, at adistance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

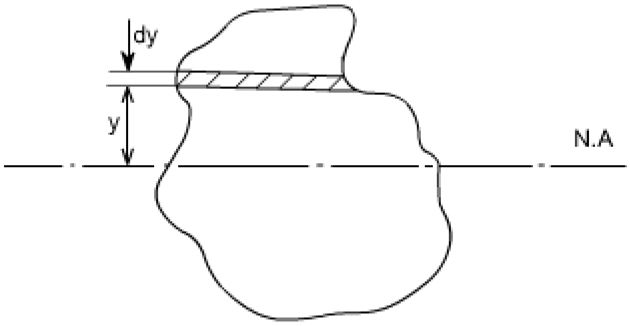
Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$



Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance ‘y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area 'dA'

then the force on the strip is

$$F = \sigma\, \delta A = \frac{E}{R} y\, \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \Sigma \frac{E}{R} y^2\ \delta A = \frac{E}{R} \Sigma y^2 \delta A$$

Now the term $\Sigma y^2 \delta A$ is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation. The above proof has involved the assumption of pure bending without any shear force being present.

Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max}{}^{m} = \frac{M}{I} y_{max}{}^{m}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max}{}^{m}} \sigma_{max}{}^{m}$$

For ready comparison of the strength of various beam cross-section this relationship is some times written in the form

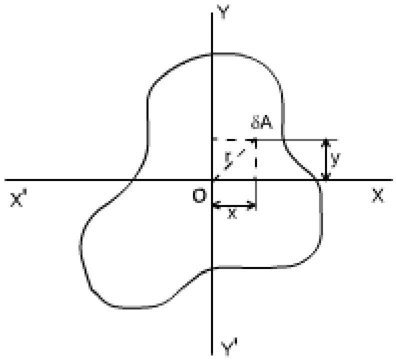
$$M = Z \sigma_{max}{}^{m} \text{ where } Z = \frac{I}{y_{max}{}^{m}}$$ **Is termed as section modulus**

The higher value of Z for a particular cross-section, the higher the bending moment which it can withstand for a given maximum stress.

Theorems to determine second moment of area: There are two theorems which are helpful to determine the value of second moment of area, which is required to be used while solving the simple bending theory equation.

Second Moment of Area :

Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis. (This property arised while we were driving bending theory equation). This is also known as the moment of inertia. An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a given axis and the second moment being the square of the distance or $\int y^2 \, dA$.



Consider any cross-section having small element of area d A then by the definition

I_x(Mass Moment of Inertia about x-axis) = $\int y^2 \, dA$ and I_y(Mass Moment of Inertia about y-axis) = $\int x^2 \, dA$

Now the moment of inertia about an axis through 'O' and perpendicular to the plane of figure is called the polar moment of inertia. (The polar moment of inertia is also the area moment of inertia).

i.e,

J = polar moment of inertia

$$
\begin{aligned}
&= \int r^2 dA \\
&= \int (x^2 + y^2) dA \\
&= \int x^2 dA + \int y^2 dA \\
&= I_X + I_Y
\end{aligned}
$$

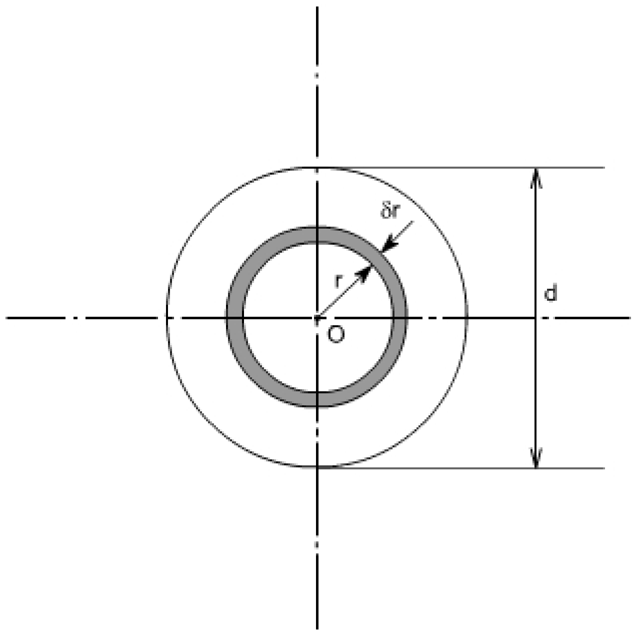
$$\text{or } J = I_X + I_Y \qquad \text{..........} \; (1)$$

The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION :

For a circular x-section, the polar moment of inertia may be computed in the following manner



Consider any circular strip of thickness δr located at a radius 'r'.

Than the area of the circular strip would be dA = 2πr. δr

$$J = \int r^2 \, dA$$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 \, 2\pi r \, \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \, \delta r$$

$$J = 2\pi \left[\frac{r^4}{4} \right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however, by perpendicular axis theorem

$$J = I_x + I_y$$

But for the circular cross-section, the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

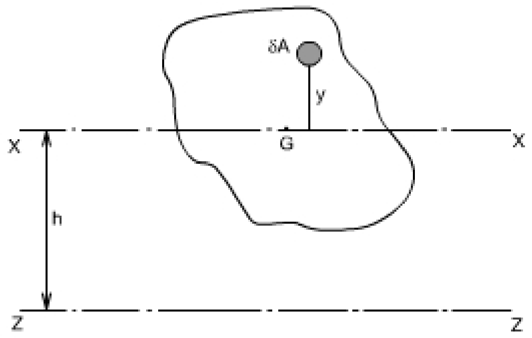
for a hollow circular section of diameter D and d, the values of J and I are defined as

$$J = \frac{\pi \left(D^4 - d^4 \right)}{32}$$

Thus $$I = \frac{\pi \left(D^4 - d^4 \right)}{64}$$

Parallel Axis Theorem:

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the area times the square of the distance between the axes.



If 'ZZ' is any axis in the plane of cross-section and 'XX' is a parallel axis through the centroid G, of the cross-section, then

$I_z = \int (y+h)^2 \, dA$ by definition (moment of inertia about an axis ZZ)

$$= \int \left(+ 2yh + h^2 \right) dA$$

$$= \int y^2 dA + h^2 \int dA + 2h \int y dA$$

Since $\int y dA = 0$

$$= \int y^2 dA + h^2 \int dA$$

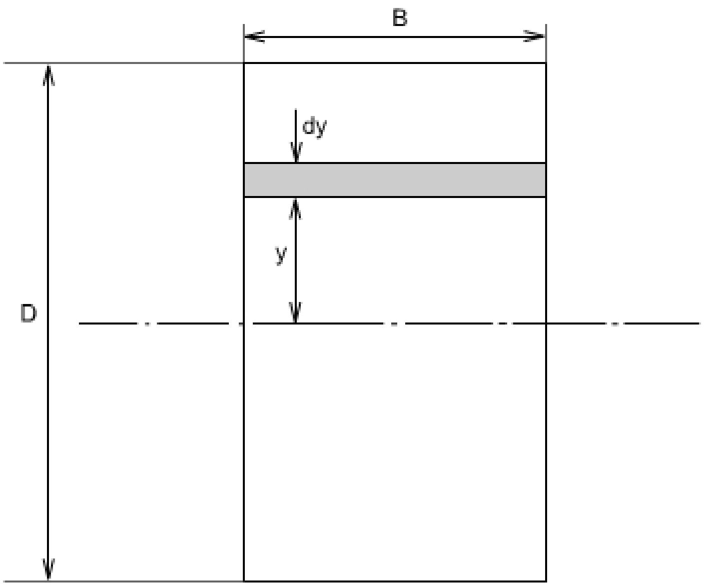
$$= \int y^2 dA + h^2 A$$

$I_z = I_x + Ah^2 \quad I_x = I_G$ (since cross-section axes also pass through G)

Where A = Total area of the section

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed as below :



Consider the rectangular beam cross-section as shown above and an element of area **dA** , thickness **dy** , breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

$$I_{N.A} = \int y^2 \, dA$$

Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

$$I_{N.A} = \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 \left(B\, dy\right)$$

$$= B \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 \, dy$$

$$= B \left[\frac{y^3}{3}\right]_{\frac{D}{2}}^{\frac{-D}{2}}$$

$$= \frac{B}{3}\left[\frac{D^3}{8} - \left(\frac{-D^3}{8}\right)\right]$$

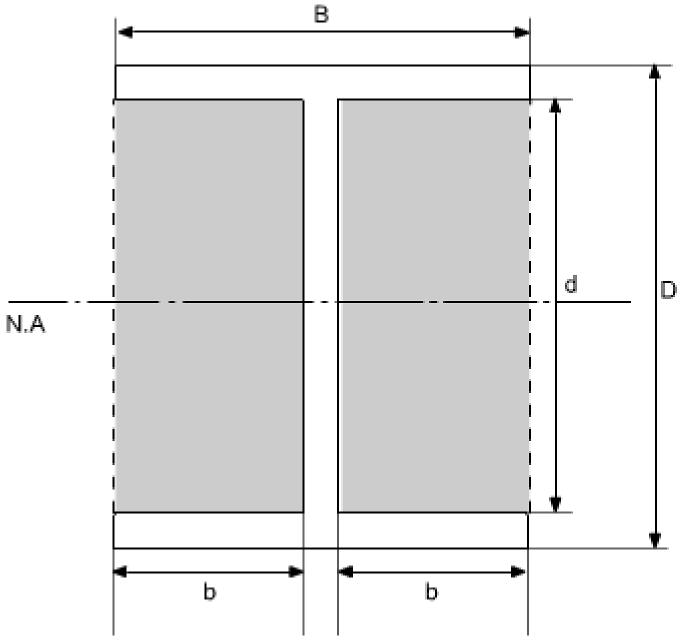
$$= \frac{B}{3}\left[\frac{D^3}{8} + \frac{D^3}{8}\right]$$

$$I_{N.A} = \frac{BD^3}{12}$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of **0** to **D** .

Therefore $I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3}$

These standards formulas prove very convenient in the determination of I_{NA} for build up sections which can be conveniently divided into rectangles. For instance if we just want to find out the Moment of Inertia of an I - section, then we can use the above relation.



$$I_{N.A} = I_{\text{of dotted rectangle}} - I_{\text{of shaded portion}}$$

$$\therefore\ I_{N.A} = \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right)$$

$$I_{N.A} = \frac{BD^3}{12} - \frac{bd^3}{6}$$

Use of Flexure Formula:

Illustrative Problems:

An I - section girder, 200mm wide by 300 mm depth flange and web of thickness is 20 mm is used as simply supported beam for a span of 7 m. The girder carries a distributed load of 5 KN /m and a concentrated load of 20 KN at mid-span.

Determine the

(i). The second moment of area of the cross-section of the girder (ii). The maximum stress set up.

Solution:

The second moment of area of the cross-section can be determained as follows :

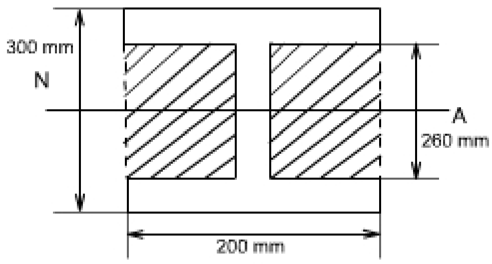
For sections with symmetry about the neutral axis, use can be made of standard I value for a rectangle about an axis through centroid i.e. (bd 3)/12. The section can thus be divided into convenient rectangles for each of which the neutral axis passes through the centroid. Example in the case enclosing the girder by a rectangle

$$I_{girder} = I_{rectangle} - I_{shaded\ portion}$$

$$= \left[\frac{200 \times 300^3}{12}\right] 10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right] 10^{-12}$$

$$= (4.5 - 2.64)10^{-4}$$

$$= 1.86 \times 10^{-4}\ m^4$$



The maximum stress may be found from the simple bending theory by equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

i.e.

$$\sigma_{max^m} = \frac{M_{max^m}}{I}\ y_{max^m}$$

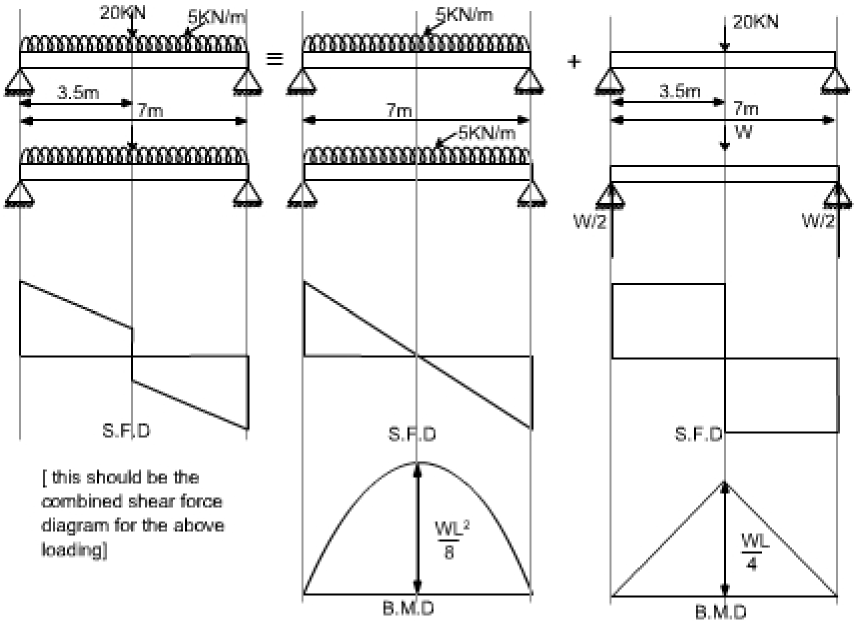
<u>Computation of Bending Moment:</u>

In this case the loading of the beam is of two types

(a) Uniformly distributed load

(b) Concentrated Load

In order to obtain the maximum bending moment the technique will be to consider each loading on the beam separately and get the bending moment due to it as if no other forces acting on the structure and then superimpose the two results.



Hence

$$M_{max^m} = \frac{wL}{4} + \frac{wL^2}{8}$$
$$= \frac{20 \times 10^3 \times 7}{4} + \frac{5 \times 10^3 \times 7^2}{8}$$
$$= (35.0 + 30.63)10^3$$
$$= 65.63 \text{ kNm}$$
$$\sigma_{max^m} = \frac{M_{max^m}}{I} y_{max^m}$$
$$= \frac{65.63 \times 10^3 \times 150 \times 10^3}{1.06 \times 10^{-4}}$$
$$\sigma_{max^m} = 51.8 \text{ MN/m}^2$$

Shearing Stresses in Beams

All the theory which has been discussed earlier, while we discussed the bending stresses in beams was for the case of pure bending i.e. constant bending moment acts along the entire length of the beam.

SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT 3-INTRODUCTION TO AIRCRAFT STRUCTURES-SAEA1305

<u>Module 3</u>

<u>Lecture 9- 12:</u> Deflection of Beams

Deflection of Beams

Introduction:

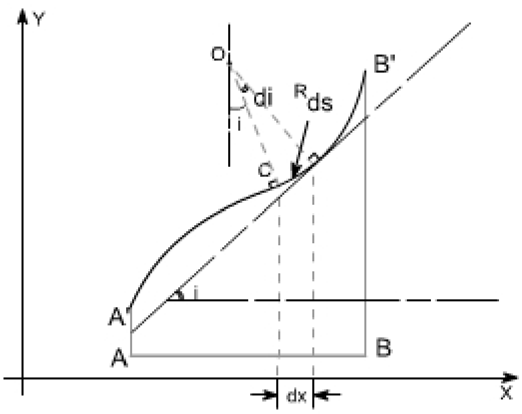
In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.
2. The curvature is always small.
3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di

But for the deflected shape of the beam the slope i at any point C is defined,

$\tan i = \frac{dy}{dx}$ ......(1) or $i = \frac{dy}{dx}$ Assuming $\tan i = i$

Futher

$ds = R\,di$

however,

$ds = dx$ [usually for small curvature]

Hence

$ds = dx = R\,di$

or $\boxed{\frac{di}{dx} = \frac{1}{R}}$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force,bending moment and deflection of the beam may be obtained as

Differentiating the equation as derived

$$\frac{dM}{dx}=EI\frac{d^3y}{dx^3} \quad \text{Re calling } \frac{dM}{dx}=F$$

Thus,

$$F=EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{slope = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods.The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A \text{ - - - - this equation gives the slope of the loaded beam.}$$

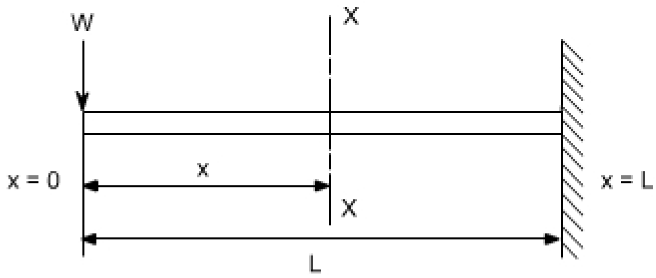
Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples : let us consider few illustrative examples to have a familiarty with the direct integration method

Case 1: Cantilever Beam with Concentrated Load at the end:- A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force abd the bending moment

$S.F|_{x-x} = -W$

$B.M|_{x-x} = -W.x$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}\,dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}\,dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

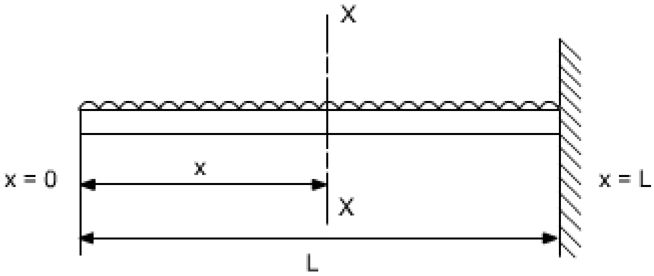
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max}m = +\frac{WL^2}{2EI}}$$

<u>**Case 2:** A Cantilever with Uniformly distributed Loads:-</u> In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length.The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int\frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}\,dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int\frac{dy}{dx} = \int -\frac{wx^3}{6EI}\,dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0
2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

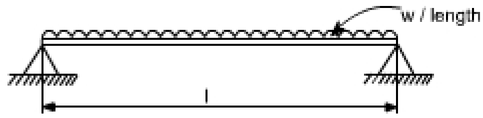
$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

So $y_{max}m$ will be at $x = 0$

$$\boxed{y_{max}m = -\frac{wL^4}{8EI}}$$

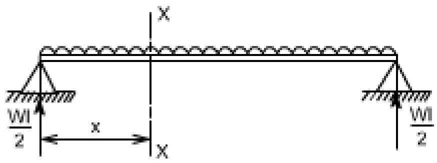
$$\boxed{\left(\frac{dy}{dx}\right)_{max}m = \frac{wL^3}{6EI}}$$

Case 3: <u>Simply Supported beam with uniformly distributed Loads:-</u> In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



$\equiv$

In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----}(1)$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

Then $$y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

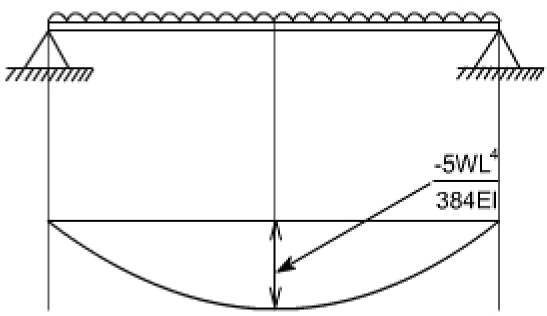
(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) Thevalue of maximum deflection would be at the centre i.e. at x = L/2.

The final equation which is governs the deflection of the loaded beam in this case is

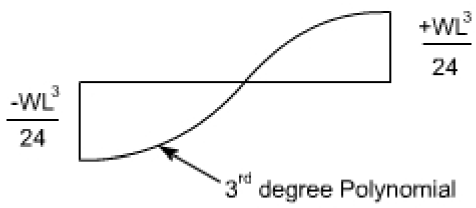
$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.



Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

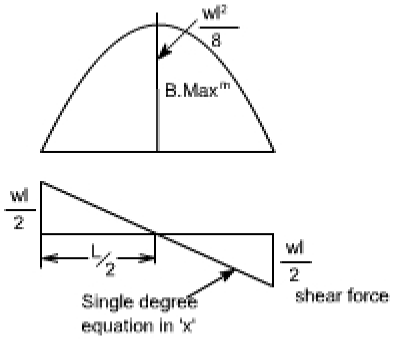


Slope (dy/dx)

$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$

Bending Moment So the bending moment diagram would

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$ be



Shear Force

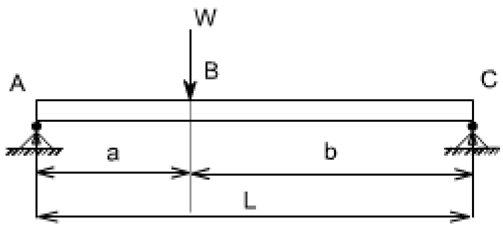
Shear force is obtained by taking third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

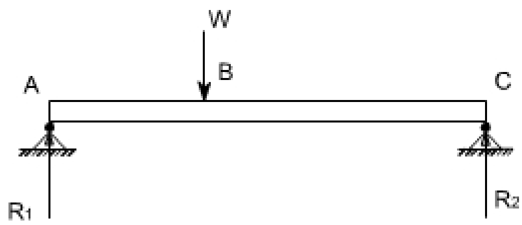
Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$$M|_{AB} = R_1.x \quad 0 \le x \le a$$

B.M for the portion BC

$$M|_{BC} = R_1.x - W(x-a) \quad a \le x \le l$$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1 x$$

$$EI\frac{d^2y}{dx^2} = R_1 x - W(x-a)$$

These two equations can be integrated in the usual way to find ‘y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at $x = a$. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at $x = 0$; $y = 0$ in the portion AB i.e. $0 \le x \le a$

(b) at $x = l$; $y = 0$ in the portion BC i.e. $a \le x \le l$

(c) at $x = a$; dy/dx, the slope is same for both portion

(d) at $x = a$; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{--------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx}=\frac{Wb}{2(a+b)}x^2+k \qquad 0\le x\le a------(3)$$

$$EI\frac{dy}{dx}=\frac{Wb}{2(a+b)}x^2-\frac{W(x-a)^2}{2}+k \qquad a\le x\le l-------(4)$$

Integrating agian equation (3) and (4) we get

$$EIy=\frac{Wb}{6(a+b)}x^3+kx+k_3 \qquad 0\le x\le a------(5)$$

$$EIy=\frac{Wb}{6(a+b)}x^3-\frac{W(x-a)^3}{6}+kx+k_4 \qquad a\le x\le l------(6)$$

Utilizing condition (a) in equation (5) yields

$$k_3=0$$

Utilizing condition (b) in equation (6) yields

$$0=\frac{Wb}{6(a+b)}l^3-\frac{W(l-a)^3}{6}+kl+k_4$$

$$k_4=-\frac{Wb}{6(a+b)}l^3+\frac{W(l-a)^3}{6}-kl$$

But $a+b=l$,

Thus,

$$k_4=-\frac{Wb(a+b)^2}{6}+\frac{Wb^3}{6}-k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y\big|_{\text{from equation 5}} = y\big|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{----}\mathbf{for\ 0 \le x \le a}\text{-----(7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k'in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{For } \mathbf{for\ a \le x \le l}\text{-----(8)}$$

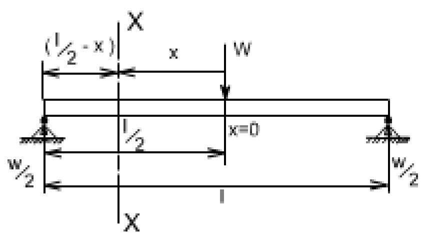
so either of the equation (7) or (8) may be used to find the deflection at x = a
hence substituting x = a in either of the equation we get

$$Y\big|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

$$\boxed{Y_{\max^m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD: There is also an alternative way to attempt this problem in a more simpler way. Let us considering the origin at the point of application of the load,



$$S.F|_{xx} = \frac{W}{2}$$

$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$

$$B = -\frac{WL^3}{48}$$

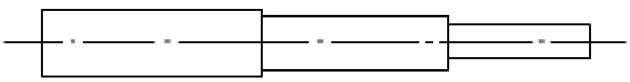
Hence he equation which governs the deflection would be

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$Y_{max^m}\Big\vert_{at\,x=0}$	$= -\frac{WL^3}{48EI}$	At the centre
$\left(\frac{dy}{dx}\right)_{max^m}\Big\vert_{at\,x=\pm\frac{L}{2}}$	$= \pm\frac{WL^2}{16EI}$	At the ends

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,



i.e. it is having different cross-section then this method also fails.

So there are other methods by which we find the deflection like

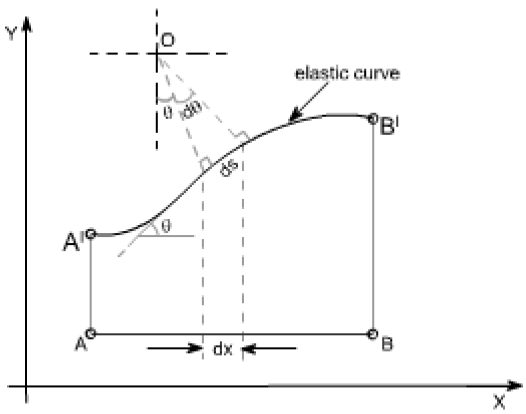
1. Macaulay's method in which we can write the different equation for bending moment for different sections.

2. Area moment methods

MOMENT-AREA METHODS:

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that d□ is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.

We can assume,

ds = dx [since the curvature is small]

hence, R d□ = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

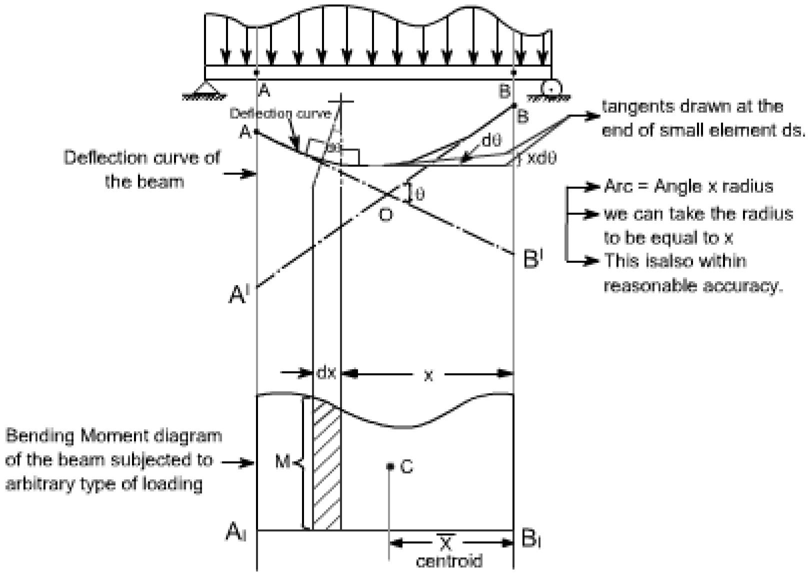
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle, slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$, hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting $ds \approx dx$]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \text{ - - - - - (1)}$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and $A_1B_1$is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are futher understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be d□. Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.

From the above relationship the total angle □ between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or }\theta\\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI}\times\text{area of B.M diagram between}\\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x d□□[each of this intercept may be considered as the arc of a circle of radius x subtended by the angle □]

Hence the total distance B'B becomes $\delta = \int_A^B x\,d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of d□ = M dx / EI as derived earlier

$\delta = \int_A^B x\frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$ [This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Therefore,we are in a position to state the above conclusion in the form of theorem as follows:

Theorem II:

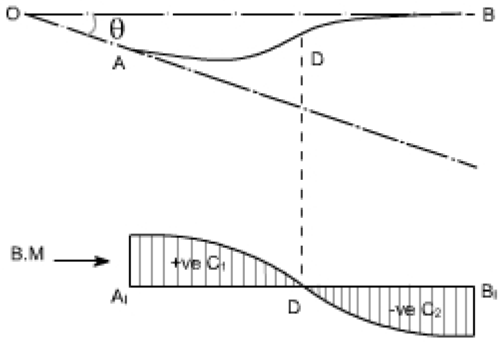
Deflection of point 'B' relative to point A $= \frac{1}{EI}\times\left\{\begin{matrix}\text{first moment of area with respect}\\ \text{to point B, of the total B.M diagram}\end{matrix}\right\}$

Futher, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $\boxed{\delta_A = \frac{1}{EI}A\bar{X}}$

Therefore,the first moment of area may be obtained simply as a product of the total area of the B.M diagram betweenthe points A and B multiplied by the distance $\bar{x}$ to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids $C_1$and C_2. Then to find an angle between the tangentsat the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

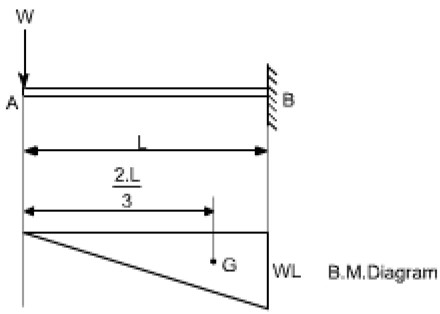
$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Illustrative Examples: Let us study few illustrative examples, pertaining to the use of these theorems

Example 1:

1. A cantilever is subjected to a concentrated load at the free end.It is required to find out the deflection at the free end.

Fpr a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area - moment theorem

$$\text{slope at } A = \frac{1}{EI}\left[\text{Area of B.M diagram between the points A and B}\right]$$
$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$
$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

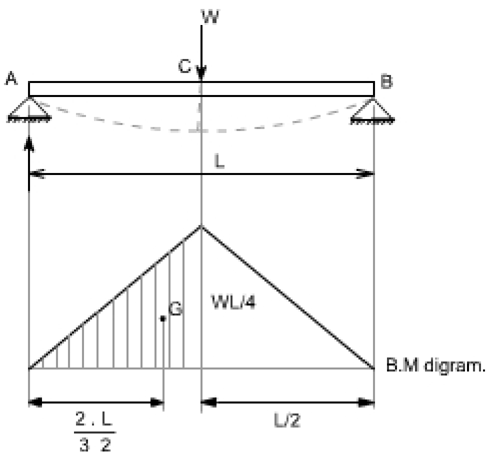
NOTE: In this case the point B is at zero slope.

Thus,

$$\delta = \frac{1}{EI}\left[\text{first moment of area of B.M diagram between A and B about A}\right]$$
$$= \frac{1}{EI}\left[A\bar{y}\right]$$
$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$
$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid span determine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.



Again working relative to the zero slope at the centre C.

$$\text{slope at } A = \frac{1}{EI}\left[\text{Area of B.M diagram between A and C}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

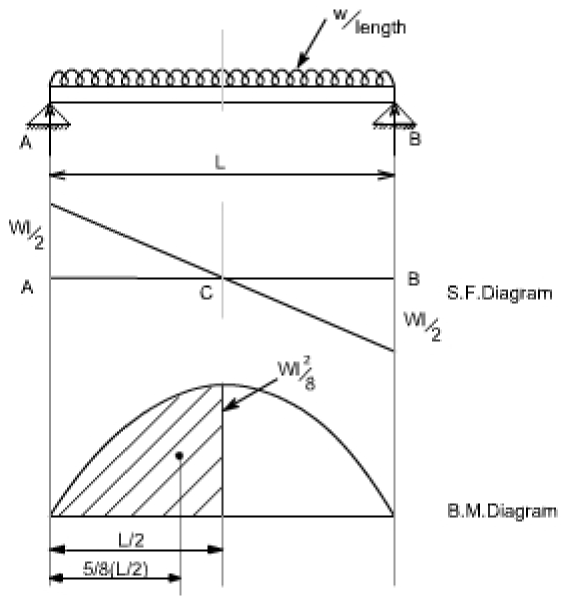
$$\delta_C = \frac{1}{EI}\left[\text{Moment of B.M diagram between points A and C about A}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, with a intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam, the value of maximum B.M is equal to Wl^2 / 8



So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}[\text{Area of B.M diagram between point A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}[A\,\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

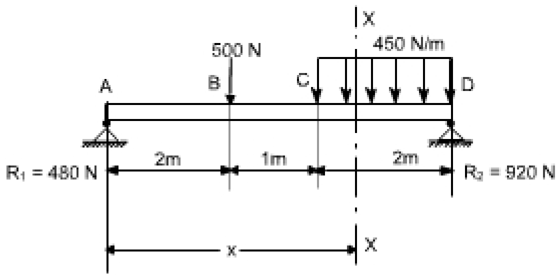
Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

For example consider the beam shown in fig below:

Let us write the general moment equation using the definition M = (∑ M)$_L$, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows

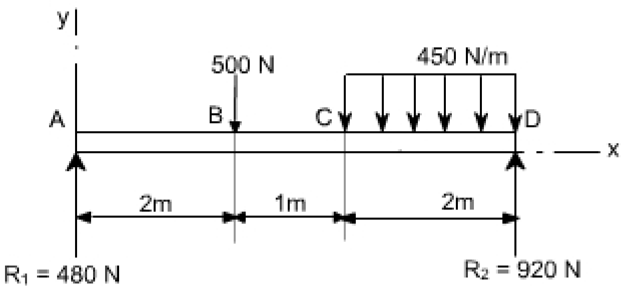


$$M_{AB} = 480\ x\ \text{N.m}$$

$$M_{BC} = \left[480\ x - 500(x-2)\right]\text{N.m}$$

$$M_{CD} = \left[480\ x - 500(x-2) - \frac{450}{2}(x-3)^2\right]\text{N.m}$$

It may be observed that the equation for M_{CD} will also be valid for both M_{AB} and M_{BC} provided that the terms (x - 2) and (x - 3)^{2}are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and (x - 3)^2 are nonexistent for values of x for which the terms in parentheses are negative.

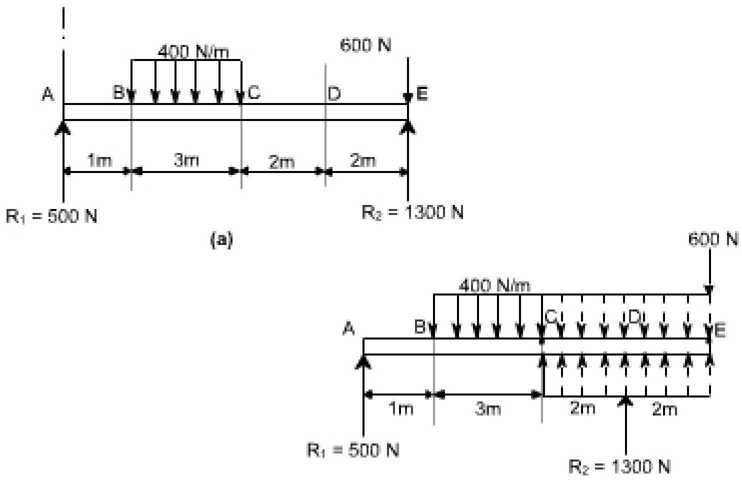


As an clear indication of these restrictions,one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

$$M = \left(480x - 500(x-2) - \frac{450}{2}(x-3)^2\right)\text{N.m}$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As an another example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward-distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:



$$M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right)\text{N.m}$$

It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratary just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

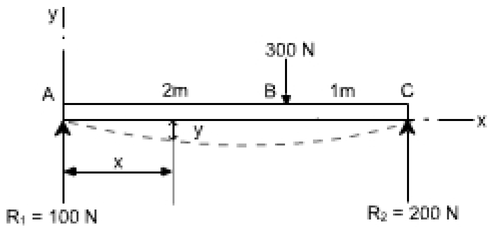
Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

Ilustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig.Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution : writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \quad ......(3)$$

To evaluate the two constants of integration. Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find C_2 = 0.keep in mind that$\langle x - 2\rangle^3$ is to be neglected for negative values.

2. At the other support where x = 3m, the value of deflection y is also zero. substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50(3-2)^3 + 3.C_1\right) \text{or } C_1 = -133N.m^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB $(0 \le x \le 2m)$

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right)N.m^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right)N.m^3 \qquad ....(5)$$

segment BC $(2m \le x \le 3m)$

$$EI\frac{dy}{dx} = \left(50x^2 - 150\left(x-2\right)^2 - 133x\right)N.m^2 ....(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\left(x-2\right)^3 - 133x\right)N.m^3 ....(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

We obtain

$50\ x^2 – 133 = 0$ or $x = 1.63$ m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max}m = -145 N.m^3 \qquad ....(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by □, the use of y may be reserved to indicate a directed value of deflection.

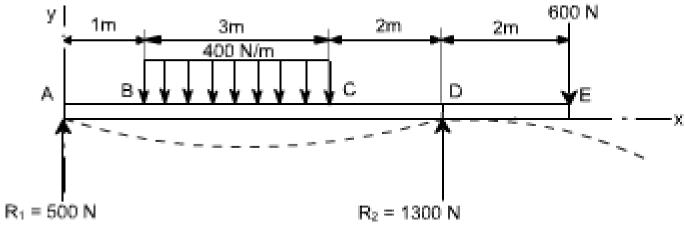
if E = 30 Gpa and I = 1.9×10^6 mm^4 = 1.9×10^{-6} m^4 , Eq. (h) becomes

$$y\big|_{max}m = \left(30 \times 10^9\right)\left(1.9 \times 10^{-6}\right)$$

Then $$= -2.54 mm$$

Example 2:

It is required to determine the value of EIy at the position midway between the supports and at the overhanging end for the beam shown in figure below.



Solution:

Writing down the moment equation which is valid for the entire span of the beam and applying the differential equation of the elastic curve, and integrating it twice, we obtain

$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right)N.m$$

$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}(x-1)^3 + \frac{200}{3}(x-4)^3 + 650(x-6)^2 + C_1\right)N.m$$

$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}(x-1)^4 + \frac{50}{3}(x-4)^4 + \frac{650}{3}(x-6)^3 + C_1x + C_2\right)N.m^3$$

To determine the value of C_2, It may be noted that EIy = 0 at x = 0,which gives C_2 = 0.Note that the negative terms in the pointed brackets are to be ignored Next,let us use the condition that EIy = 0 at the right support where x = 6m.This gives

$$0 = \frac{250}{3}(6)^3 - \frac{50}{3}(5)^4 + \frac{50}{3}(2)^4 + 6C_1 \quad \text{or} \quad C_1 = -1308 N.m^2$$

Finally, to obtain the midspan deflection, let us substitute the value of x = 3m in the deflection equation for the segment BC obtained by ignoring negative values of the bracketed terms □ x - 4 □^{4} and □ x - 6 □^{3}. We obtain

$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941 \ N.m^3$$

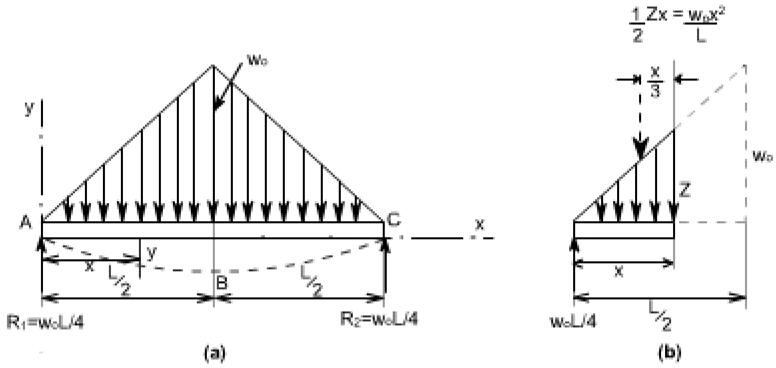
For the overhanging end where x=8 m,we have

$$EIy = \left(\frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)\right)$$

$$= -1814 N.m^3$$

Example 3:

A simply supported beam carries the triangularly distributed load as shown in figure. Determine the deflection equation and the value of the maximum deflection.



(a) (b)

Solution:

Due to symmetry, the reactionsis one half the total load of $1/2w_0L$, or $R_1 = R_2 = 1/4w_0L$.Due to the advantage of symmetry to the deflection curve from A to B is the mirror image of that from C to B. The condition of zero deflection at A and of zero slope at B do not require the use of a general moment equation. Only the moment equation for segment AB is needed, and this may be easily written with the aid of figure(b).

Taking into account the differential equation of the elastic curve for the segment AB and integrating twice, one can obtain

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{w_0L}{4}x - \frac{w_0x^2}{L}.\frac{x}{3} \qquad ......(1)$$

$$EI\frac{dy}{dx} = \frac{w_0Lx^2}{8} - \frac{w_0x^4}{12L} + C_1 \qquad ......(2)$$

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} + C_1x + C_2 ......(3)$$

In order to evaluate the constants of integration,let us apply the B.C'swe note that at the support A, y = 0 at x = 0.Hence from equation (3), we get $C_2 = 0$. Also,because of symmetry, the slope dy/dx = 0 at midspan where x = L/2.Substituting these conditions in equation (2) we get

$$0 = \frac{w_0L}{8}\left(\frac{L}{2}\right)^2 - \frac{w_0}{12L}\left(\frac{L}{2}\right)^4 + C_1C_1 = -\frac{5w_0L^3}{192}$$

Hence the deflection equation from A to B (and also from C to B because of symmetry) becomes

$$EIy = \frac{w_0 L x^3}{24} - \frac{w_0 x^5}{60L} - \frac{5 w_0 L^3 x}{192}$$

Whichreducesto

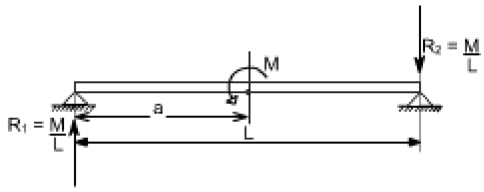
$$EIy = -\frac{w_0 x}{960L}\left(25L^4 - 40L^2x^2 + 16x^4\right)$$

The maximum deflection at midspan,where $x = L/2$ is then found to be

$$\boxed{EIy = -\frac{w_0 L^4}{120}}$$

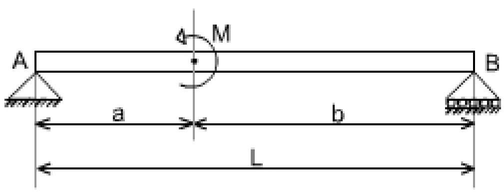
Example 4: couple acting

Consider a simply supported beam which is subjected to a couple M at adistance 'a' from the left end. It is required to determine using the Macauley's method.



To deal with couples, only thing to remember is that within the pointed brackets we have to take some quantity and this should be raised to the power zero.i.e. M□□ x - a □^{0} . We have taken the power 0 (zero) ' because ultimately the term M□□ x - a □^{0}Should have the moment units.Thus with integration the quantity□□ x - a □ becomes either □ x - a □^{1}or□□ x - a □^{2}

Or



Therefore, writing the general moment equation we get

$$M = R_1 x - M\langle x - a\rangle \text{ or } EI\frac{d^2y}{dx^2} = M$$

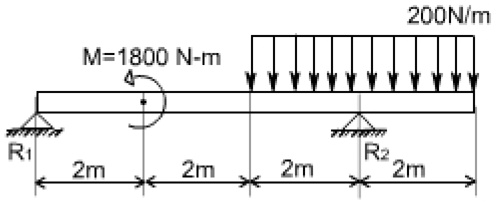
Integrating twice we get

$$EI\frac{dy}{dx} = R_1.\frac{x^2}{2} - M\langle x - a\rangle^1 + C_1$$

$$EI.y = R_1.\frac{x^3}{6} - \frac{M}{2}\langle x - a\rangle^2 + C_1 x + C_2$$

Example 5:

A simply supported beam is subjected to U.d.l in combination with couple M. It is required to determine the deflection.



This problem may be attemped in the some way. The general moment equation my be written as

$$M(x) = R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle\langle x-4\rangle}{2} + R_2\langle x-6\rangle$$

$$= R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle^2}{2} + R_2\langle x-6\rangle$$

Thus,

$$EI\frac{d^2y}{dx^2} = R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle^2}{2} + R_2\langle x-6\rangle$$

Integrate twice to get the deflection of the loaded beam.

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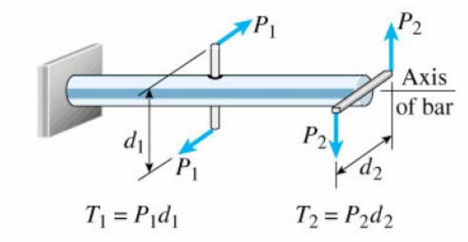
SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

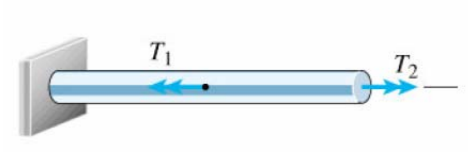
UNIT 4-INTRODUCTION TO AIRCRAFT STRUCTURES-SAEA1305

Torsion

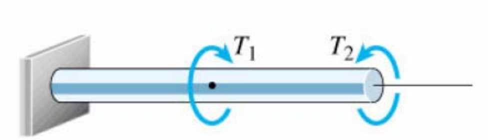
3.1 Introduction



(a)



(b)



(c)

Torsion : twisting of a structural member, when it is loaded by couples that produce rotation about its longitudinal axis

$$T_1 = P_1 d_1 \qquad T_2 = P_2 d_2$$

the couples T_1, T_2 are called torques, twisting couples or twisting moments

unit of T : N-m, lb-ft

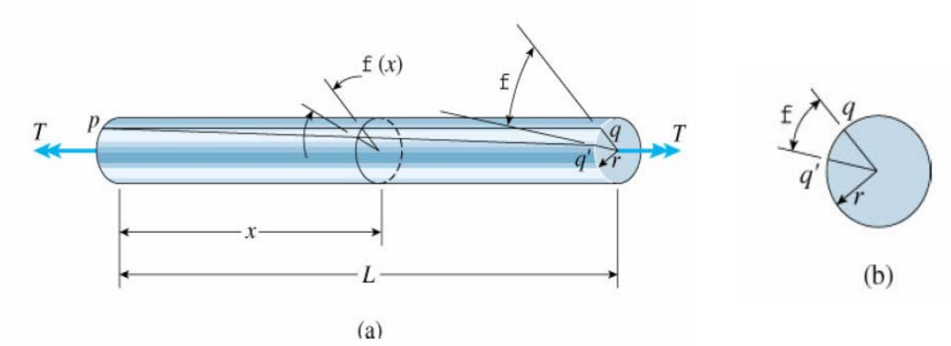
in this chapter, we will develop formulas for the stresses and deformations produced in circular bars subjected to torsion, such as drive shafts, thin-walled members

analysis of more complicated shapes required more advanced method then those presented here

this chapter cover several additional topics related to torsion, such statically indeterminate members, strain energy, thin-walled tube of noncircular section, stress concentration, and nonlinear behavior

3.2 Torsional Deformation of a Circular Bar

consider a bar or shaft of circular cross section twisted by a couple T, assume the left-hand end is fixed and the right-hand end will rotate a small angle ϕ, called angle of twist

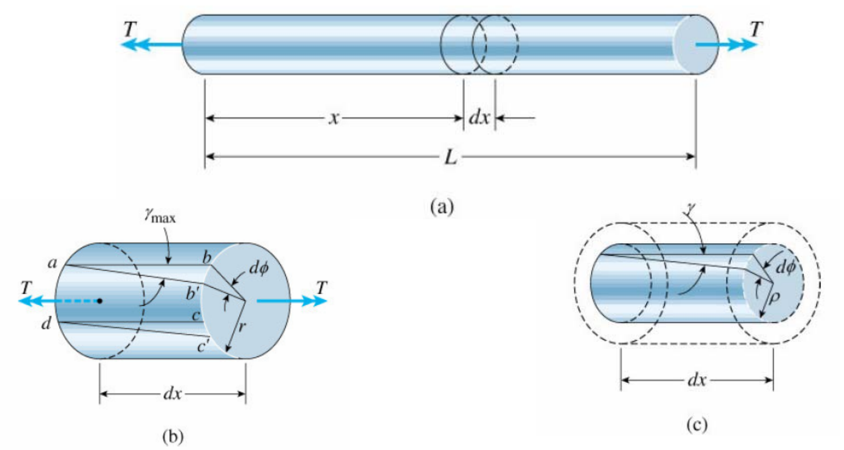


if every cross section has the same radius and subjected to the same torque, the angle $\phi(x)$ will vary linearly between ends

under twisting deformation, it is assumed

1. plane section remains plane
2. radii remaining straight and the cross sections remaining plane and circular
3. if ϕ is small, neither the length L nor its radius will change

consider an element of the bar dx, on its outer surface we choose an small element $abcd$,



during twisting the element rotate a small angle $d\phi$, the element is in a state of pure shear, and deformed into $ab'c'd$, its shear strain γ_{max} is

$$\gamma_{max} = \frac{b\,b'}{a\,b} = \frac{r\,d\phi}{dx}$$

$d\phi / dx$ represents the rate of change of the angle of twist ϕ, denote $\theta = d\phi / dx$ as the angle of twist per unit length or the rate of twist, then

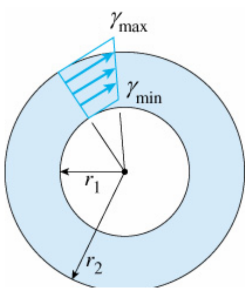
$$\gamma_{max} = r\,\theta$$

in general, ϕ and θ are function of x, in the special case of pure torsion, θ is constant along the length (every cross section is subjected to the same torque)

$$\theta = \frac{\phi}{L} \qquad \text{then} \qquad \gamma_{max} = \frac{r\,\phi}{L}$$

and the shear strain inside the bar can be obtained

$$\gamma = \rho\,\theta = \frac{\rho}{r}\,\gamma_{max}$$



for a circular tube, it can be obtained

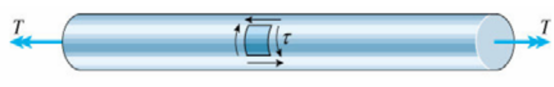
$$\gamma_{min} = \frac{r_1}{r_2}\,\gamma_{max}$$

the above relationships are based only upon geometric concepts, they are valid for a circular bar of any material, elastic or inelastic, linear or nonlinear

3.3 Circular Bars of Linearly Elastic Materials

shear stress τ in the bar of a linear elastic material is

$$\tau = G\,\gamma$$



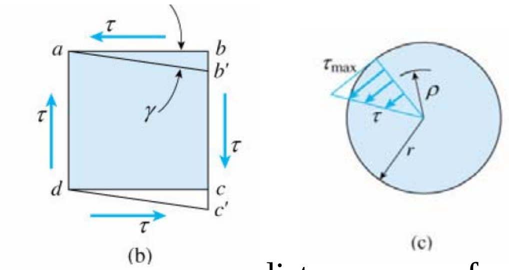
(a)

G : shear modulus of elasticity

with the geometric relation of the shear strain, it is obtained

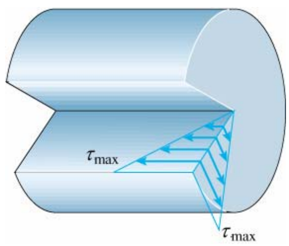
$$\tau_{max} = G\, r\, \theta$$

$$\tau = G\, \rho\, \theta = \frac{\rho}{r}\, \tau_{max}$$



τ and γ in circular bar vary linear with the radial distance ρ from the center, the maximum values τ_{max} and γ_{max} occur at the outer surface

the shear stress acting on the plane of the cross section are accompanied by shear stresses of the same magnitude acting on longitudinal plane of the bar

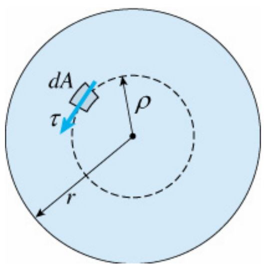


if the material is weaker in shear on longitudinal plane than on cross-sectional planes, as in the case of a circular bar made of wood, the first crack due to twisting will appear on the surface in longitudinal direction

a rectangular element with sides at 45^o to the axis of the shaft will be subjected to tensile and compressive stresses

The Torsion Formula

consider a bar subjected to pure torsion, the shear force acting on an element dA is $\tau\, dA$, the moment of this force about the axis of bar is $\tau\, \rho\, dA$



$$dM = \tau\, \rho\, dA$$

equation of moment equilibrium

$$T = \int_A dM = \int_A \tau \rho \, dA = \int_A G \theta \rho^2 \, dA = G \theta \int_A \rho^2 \, dA$$

$$= G \theta I_p \qquad [\tau = G \theta \rho]$$

in which $I_p = \int_A \rho^2 \, dA$ is the polar moment of inertia

$$I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad \text{for circular cross section}$$

the above relation can be written

$$\theta = \frac{T}{G I_p}$$

$G I_p$: torsional rigidity

the angle of twist ϕ can be expressed as

$$\phi = \theta L = \frac{T L}{G I_p} \qquad \phi \text{ is measured in radians}$$

torsional flexibility $f = \dfrac{L}{G I_p}$

torsional stiffness $k = \dfrac{G I_p}{L}$

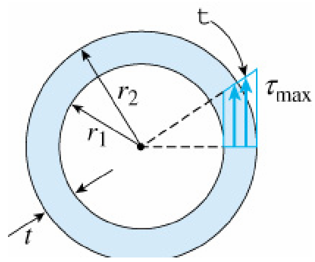
and the shear stress is

$$\tau = G \rho \theta = G \rho \frac{T}{G I_p} = \frac{T \rho}{I_p}$$

the maximum shear stress $\tau_{\max}$ at $\rho = r$ is

$$\tau_{\max} = \frac{T\,r}{I_p} = \frac{16\,T}{\pi\,d^3}$$

for a circular tube



$$I_p = \pi\,(r_2^4 - r_1^4) / 2 = \pi\,(d_2^4 - d_1^4) / 32$$

if the hollow tube is very thin

$$I_p \simeq \pi\,(r_2^2 + r_1^2)\,(r_2 + r_1)\,(r_2 - r_1) / 2$$

$$= \pi\,(2r^2)\,(2r)\,(t) = 2\,\pi\,r^3\,t = \pi\,d^3\,t / 4$$

limitations

1. bar have circular cross section (either solid or hollow)
2. material is linear elastic

note that the above equations cannot be used for bars of noncircular shapes, because their cross sections do not remain plane and their maximum stresses are not located at the farthest distances from the midpoint

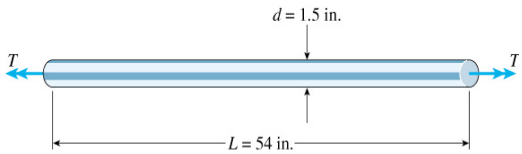
Example 3-1

a solid bar of circular cross section

d = 40 mm, L = 1.3 m, G = 80 GPa

(a) T = 340 N-m, τ_{max}, ϕ = ?

(b) τ_{all} = 42 MPa, ϕ_{all} = 2.5°, T = ?



(a) $$\tau_{max} = \frac{16\,T}{\pi\,d^3} = \frac{16 \text{ x } 340 \text{ N-M}}{\pi\,(0.04 \text{ m})^3} = 27.1 \text{ MPa}$$

$$I_p = \pi\,d^4 / 32 = 2.51 \text{ x } 10^{-7} \text{ m}^4$$

$$\phi = \frac{T\,L}{G\,I_p} = \frac{340 \text{ N-m x } 1.3 \text{ m}}{80 \text{ GPa x } 2.51 \text{ x } 10^{-7} \text{ m}^4} = 0.02198 \text{ rad} = 1.26^{\circ}$$

(b) due to τ_{all} = 42 MPa

$T_1 = \pi\, d^3\, \tau_{all} / 16 = \pi\, (0.04 \text{ m})^3 \text{ x } 42 \text{ MPa} / 16 = 528 \text{ N-m}$

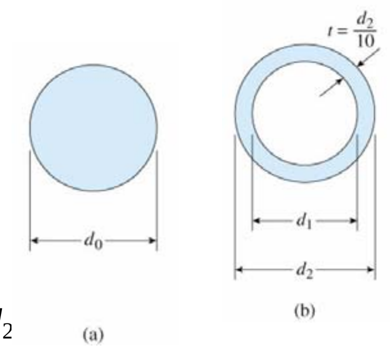
due to ϕ_{all} = 2.5^o = $2.5 \text{ x } \pi \text{ rad} / 180^o$ = 0.04363 rad

T_2 = $G\, I_p\, \phi_{all} / L$ = $80 \text{ GPa x } 2.51 \text{ x } 10^{-7} \text{ m}^4 \text{ x } 0.04363 / 1.3 \text{ m}$

= 674 N-m

thus T_{all} = min [T_1, T_2] = 528 N-m

Example 3-2



a steel shaft of either solid bar or circular tube

T = 1200 N-m, τ_{all} = 40 MPa

θ_{all} = 0.75^o / m G = 78 GPa

(a) determine d_0 of the solid bar

(b) for the hollow shaft, $t = d_2 / 10$, determine d_2

(c) determine d_2 / d_0, W_{hollow} / W_{solid}

(a) for the solid shaft, due to τ_{all} = 40 MPa

$d_0^3 = 16\, T / \pi\, \tau_{all} = 16 \text{ x } 1200 / \pi\, 40 = 152.8 \text{ x } 10^{-6} \text{ m}^3$

d_0 = 0.0535 m = 53.5 mm

due to θ_{all} = 0.75^o / m = $0.75 \text{ x } \pi \text{ rad} / 180^o / \text{m}$ = 0.01309 rad / m

$I_p = T / G\, \theta_{all} = 1200 / 78 \text{ x } 10^9 \text{ x } 0.01309 = 117.5 \text{ x } 10^{-8} \text{ m}^4$

$d_0^4 = 32\, I_p / \pi = 32 \text{ x } 117.5 \text{ x } 10^{-8} / \pi = 1197 \text{ x } 10^{-8} \text{ m}^4$

d_0 = 0.0588 m = 58.8 mm

thus, we choose d_0 = 58.8 mm [in practical design, d_0 = 60 mm]

(b) for the hollow shaft

$d_1 = d_2 - 2\,t = d_2 - 0.2\, d_2 = 0.8\, d_2$

$I_p \quad = \pi\,(d_2^4 - d_1^4)\,/\,32 \quad = \pi\,[d_2^4 - (0.8d_2)^4]\,/\,32 \quad = 0.05796\,d_2^4$

due to $\tau_{all} \quad = \quad 40$ MPa

$I_p \quad = \quad 0.05796\,d_2^4 \quad = \quad T\,r\,/\tau_{all} \quad = \quad 1200\,(d_2/2)\,/\,40$

$d_2^3 \quad = \quad 258.8 \times 10^{-6}\ \text{m}^3$

$d_2 \quad = \quad 0.0637\ \text{m} \quad = \quad 63.7\ \text{mm}$

due to $\theta_{all} \quad = \quad 0.75^{\text{o}}\ /\ \text{m} \quad = \quad 0.01309$ rad / m

$\theta_{all} \quad = 0.01309 \quad = T\,/\,G\,I_p \quad = 1200\,/\,78 \times 10^9 \times 0.05796\,d_2^4$

$d_2^4 \quad = \quad 2028 \times 10^{-8}\ \text{m}^4$

$d_2 \quad = 0.0671\ \text{m} \quad = 67.1\ \text{mm}$

thus, we choose $d_0 = 67.1$ mm [in practical design, $d_0 = 70$ mm]

(c) the ratios of hollow and solid bar are

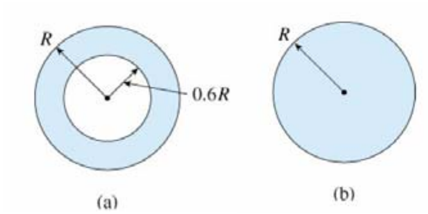
$d_2\,/\,d_0 \quad = \quad 67.1\,/\,58.8 \quad = \quad 1.14$

$$\frac{W_{\text{hollow}}}{W_{\text{solid}}} \quad = \quad \frac{A_{\text{hollow}}}{A_{\text{solid}}} \quad = \quad \frac{\pi\,(d_2^2 - \text{d}_1^2)/4}{\pi\,\text{d}_0^2/4} \quad = \quad 0.47$$

the hollow shaft has 14% greater in diameter but 53% less in weight

Example 3-3

a hollow shaft and a solid shaft has same material, same length, same outer radius R, and $r_i \quad = \quad 0.6\,R$ for the hollow shaft



(a) for same T, compare their τ, θ, and W

(b) determine the strength-to-weight ratio

(a) $\because \quad \tau \quad = \quad T\,R\,/\,I_p \qquad \theta \quad = \quad T\,L\,/\,G\,I_p$

$\therefore$ the ratio of τ or θ is the ratio of $1\,/\,I_p$

$(I_p)_H \quad = \quad \pi\,R^2\,/2 \quad - \quad \pi\,(0.6R)^2\,/2 \quad = \quad 0.4352\,\pi\,R^2$

$(I_p)_S \quad = \quad \pi R^2 / 2 \quad = \quad 0.5 \pi R^2$

$(I_p)_S / (I_p)_H \quad = \quad 0.5 / 0.4352 \quad = \quad 1.15$

thus $\quad \beta_1 \quad = \quad \tau_H / \tau_S \quad = \quad (I_p)_S / (I_p)_H \quad = \quad 1.15$

also $\quad \beta_2 \quad = \quad \phi_H / \phi_S \quad = \quad (I_p)_S / (I_p)_H \quad = \quad 1.15$

$\beta_3 \quad = W_H / W_S = A_H / A_S = \pi [R^2 - (0.6R)^2] / \pi R^2 \quad = 0.64$

the hollow shaft has 15% greater in τ and ϕ, but 36% decrease in weight

(b) strength-to-weight ratio $\quad S \quad = \quad T_{all} / W$

$T_H \quad = \tau_{\max} I_p / R \quad = \tau_{\max} (0.4352 \pi R^4) / R \quad = 0.4352 \pi R^3 \tau_{\max}$

$T_S \quad = \quad \tau_{\max} I_p / R \quad = \quad \tau_{\max} (0.5 \pi R^4) / R \quad = \quad 0.5 \pi R^3 \tau_{\max}$

$W_H \quad = \quad 0.64 \pi R^2 L \gamma \qquad\qquad W_S \quad = \pi R^2 L \gamma$

thus $\quad S_H \quad = \quad T_H / W_H \quad = \quad 0.68 \tau_{\max} R / \gamma L$

$S_S \quad = \quad T_S / W_S \quad = \quad 0.5 \tau_{\max} R / \gamma L$

S_H is 36% greater than S_S

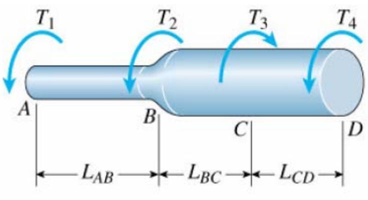
3.4 Nonuniform Torsion

(1) constant torque through each segment

$T_{CD} \quad = \quad - T_1 \quad - \quad T_2 \quad + \quad T_3$

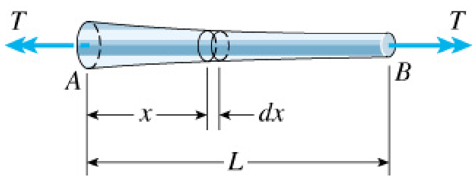
$T_{BC} \quad = \quad - T_1 \quad - T_2 \qquad T_{AB} \quad = \quad - T_1$

$$\phi \quad = \quad \sum_{i=1}^{n} \phi_i \quad = \quad \sum_{i=1}^{n} \frac{T_i L_i}{G_i I_{pi}}$$



(a)

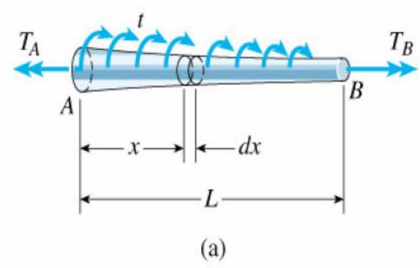
(2) constant torque with continuously varying cross section



$$d\phi = \frac{T\,dx}{G\,I_p(x)}$$

$$\phi = \int_0^L d\phi = \int_0^L \frac{T\,dx}{G\,I_p(x)}$$

(3) continuously varying cross section and continuously varying torque



$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)\,dx}{G\,I_p(x)}$$

Example 3-4

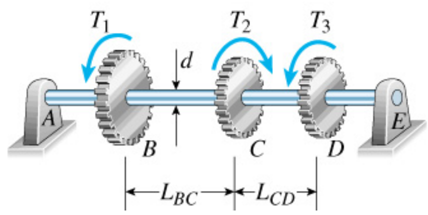
a solid steel shaft *ABCDE*, $d = 30$ mm

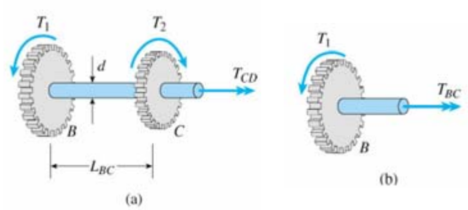
$T_1 = 275$ N-m $T_2 = 450$ N-m

$T_3 = 175$ N-m $G = 80$ GPa

$L_1 = 500$ mm $L_2 = 400$ mm

determine τ_{max} in each part and ϕ_{BD}





$$T_{CD} = T_2 - T_1 = 175 \text{ N-m}$$

$$T_{BC} = -T_1 = -275 \text{ N-m}$$

$$\tau_{BC} = \frac{16\,T_{BC}}{\pi\,d^3} = \frac{16 \times 275 \times 10^3}{\pi\,30^3} = 51.9 \text{ MPa}$$

$$\tau_{CD} = \frac{16\,T_{CD}}{\pi\,d^3} = \frac{16 \times 175 \times 10^3}{\pi\,30^3} = 33 \text{ MPa}$$

$$\phi_{BD} = \phi_{BC} + \phi_{CD}$$

$$I_p = \frac{\pi\,d^4}{32} = \frac{\pi\,30^4}{32} = 79{,}520 \text{ mm}^2$$

$$\phi_{BC} = \frac{T_{BC} L_1}{G I_p} = \frac{-275 \times 10^3 \times 500}{80 \times 10^3 \times 79{,}520} = -0.0216 \text{ rad}$$

$$\phi_{CD} = \frac{T_{CD} L_2}{G I_p} = \frac{175 \times 10^3 \times 400}{80 \times 10^3 \times 79{,}520} = 0.011 \text{ rad}$$

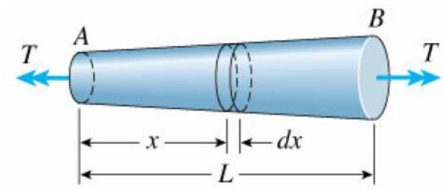
$$\phi_{BD} = \phi_{BC} + \phi_{CD} = -0.0216 + 0.011 = -0.0106 \text{ rad} = -0.61^o$$

Example 3-5

a tapered bar AB of solid circular cross section is twisted by torque T

$d = d_A$ at A, $d = d_B$ at B, $d_B \geqq d_A$

determine τ_{max} and ϕ of the bar



(a) T = constant over the length,

thus τ_{max} occurs at d_{min} [end A]

$$\tau_{max} = \frac{16\, T}{\pi\, d_A^3}$$

(b) angle of twist

$$d(x) = d_A + \frac{d_B - d_A}{L} x$$

$$I_p(x) = \frac{\pi\, d^4}{32} = \frac{\pi}{32} (d_A + \frac{d_B - d_A}{L} x)^4$$

then

$$\phi = \int_0^L \frac{T\, dx}{G\, I_p(x)} = \frac{32\, T}{\pi\, G} \int_0^L \frac{dx}{(d_A + \frac{d_B - d_A}{L} x)^4}$$

to evaluate the integral, we note that it is of the form

$$\int \frac{dx}{(a + bx)^4} = - \frac{1}{3\, b\, (a + bx)^3}$$

if we choose $a = d_A$ and $b = (d_B - d_A) / L$, then the integral of ϕ can be obtained

$$\phi = \frac{32\, T\, L}{3\pi G(d_B - d_A)} \left(\frac{1}{d_A^{\,3}} - \frac{1}{d_B^{\,3}} \right)$$

a convenient form can be written

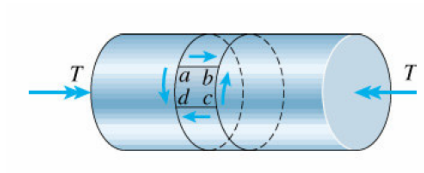
$$\phi = \frac{T\, L}{G\, I_{pA}} \left(\frac{\beta^2 + \beta + 1}{3\, \beta^3} \right)$$

where $\beta = d_B / d_A$ $I_{pA} = \pi\, d_A^{\,4} / 32$

in the special case of a prismatic bar, $\beta = 1$, then $\phi = T\, L / G\, I_p$

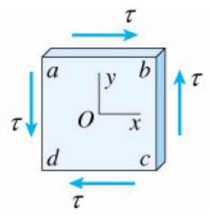
3.5 Stresses and Strains in Pure Shear

for a circular bar subjected to torsion, shear stresses act over the cross sections and on longitudinal planes



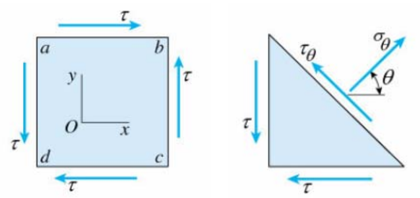
(a)

an stress element *abcd* is cut between two cross sections and between two longitudinal planes, this element is in a state of pure shear

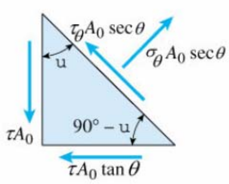


(b)

we now cut from the plane stress element to a wedge-shaped element, denote A_0 the area of the vertical side face, then the area of the bottom face is A_0 tan θ,



(a) (b)



(c)

and the area of the inclined face is A_0 sec θ

summing forces in the direction of σ_θ

$$\sigma_\theta A_0 \sec\theta \quad = \quad \tau A_0 \sin\theta \quad + \quad \tau A_0 \tan\theta \cos\theta$$

or $$\sigma_\theta \quad = \quad 2\,\tau \sin\theta \cos\theta \quad = \quad \tau \sin 2\theta$$

summing forces in the direction of τ_θ

$$\tau_\theta A_0 \sec\theta \quad = \quad \tau A_0 \cos\theta \quad - \quad \tau A_0 \tan\theta \sin\theta$$

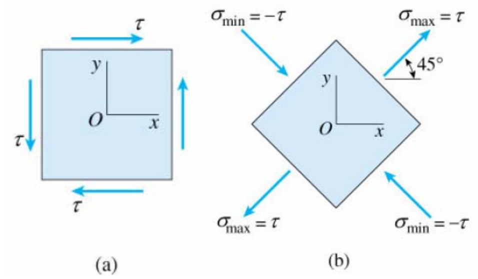
or $$\tau_\theta \quad = \quad \tau\,(\cos^2\theta \quad - \quad \sin^2\theta) \quad = \quad \tau \cos 2\theta$$

σ_θ and τ_θ vary with θ is plotted in figure

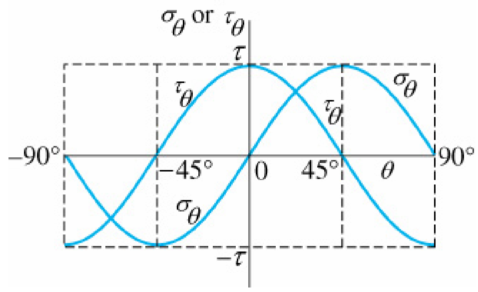
$(\tau_\theta)_{max} = \tau$ at $\theta = 0^o$

$(\tau_\theta)_{min} = -\,\tau$ at $\theta = \pm\, 90^o$

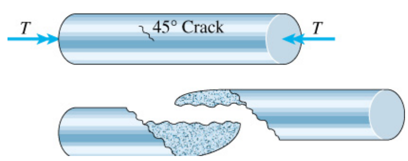
$(\sigma_\theta)_{max} = \pm\,\tau$ at $\theta = \pm\, 45^o$



the state of pure shear stress is equivalent to equal tensile and compressive stresses on an element rotation through an angle of 45^o



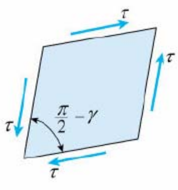
if a twisted bar is made of material that is weaker in tension than in shear, failure will occur in tension along a helix inclined at 45^o, such as chalk



Strains in pure shear

if the material is linearly elastic

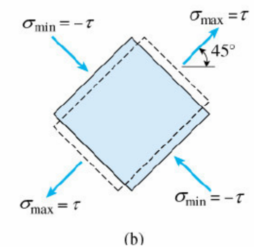
$$\gamma \quad = \quad \tau / G$$



(a)

where G is the shear modulus of elasticity

consider the strains that occur in an element oriented at $\theta = 45^o$, $\sigma_{max} = \tau$ applied at 45^o and $\sigma_{min} = - \tau$ applied at $\theta = - 45^o$



then at $\theta = 45^o$

$$\varepsilon_{max} = \frac{\sigma_{max}}{E} - \frac{\nu\,\sigma_{min}}{E} = \frac{\tau}{E} + \frac{\nu\,\tau}{E} = \frac{\tau}{E}(1 + \nu)$$

at $\theta = - 45^o$ $\quad \varepsilon = - \varepsilon_{max} = - \tau\,(1 + \nu) / E$

it will be shown in next section the following relationship

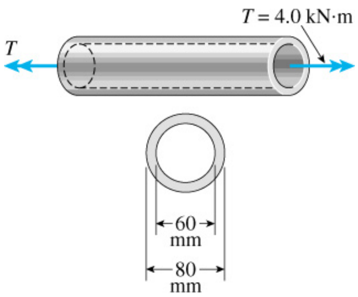
$$\varepsilon_{max} = \frac{\gamma}{2}$$

Example 3-6

a circular tube with $d_o = 80$ mm, $d_i = 60$ mm

$T = 4$ kN-m $\quad G = 27$ GPa

determine (a) maximum tensile, compressive and shear stresses (b) maximum strains



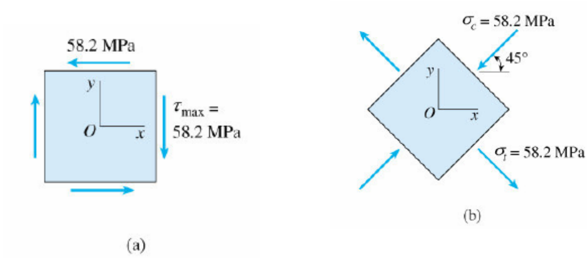
(a) the maximum shear stress is

$$\tau_{max} = \frac{T\,r}{I_p} = \frac{4000 \text{ x } 0.04}{\dfrac{\pi}{32}\,[(0.08)^4 - (0.06)^4]} = 58.2 \text{ MPa}$$

the maximum tensile and compressive stresses are

$\sigma_t = 58.2$ MPa $\qquad$ at $\theta = - 45^o$

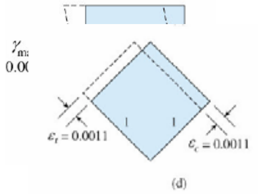
$\sigma_c = - 58.2$ MPa $\qquad$ at $\theta = 45^o$



(b) maximum strains

$$\gamma_{\max} = \tau_{max} / G = 58.2 / 27 \text{ x } 10^3 = 0.0022$$

the maximum normal strains is

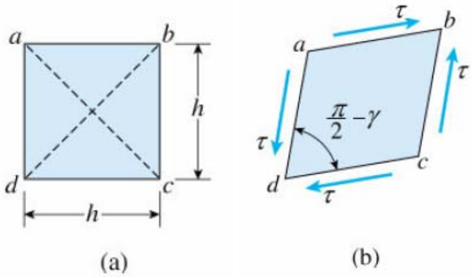


$$\varepsilon_{max} = \gamma_{max} / 2 = 0.011$$

i.e. $\varepsilon_t = 0.011 \qquad \varepsilon_c = -0.011$

3.6 Relationship Between Moduli of Elasticity E, G and ν

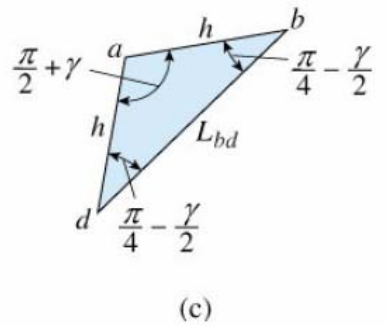
an important relationship between E, G and ν can be obtained



consider the square stress element *abcd*, with the length of each side denoted as h, subjected to pure shear stress τ, then

$$\gamma = \tau / G$$

the length of diagonal bd is $\sqrt{2}\, h$, after deformation



$$L_{bd} = \sqrt{2}\, h\, (1 + \varepsilon_{max})$$

using the law of cosines for $\triangle\, abd$

$$L_{bd}^{\,2} = h^2 + h^2 - 2\, h^2 \cos\left(\frac{\pi}{2} + \gamma\right) = 2\, h^2 \left[\, 1 - \cos\left(\frac{\pi}{2} + \gamma\right)\right]$$

then $\quad (1 + \varepsilon_{max})^2 = 1 - \cos\left(\frac{\pi}{2} + \gamma\right) = 1 + \sin\gamma$

thus $\quad 1 + 2\varepsilon_{max} + \varepsilon_{max}^2 = 1 + \sin\gamma$

$\because \quad \varepsilon_{max}$ is very small, then $\varepsilon^2_{max} \rightarrow 0$, and $\sin\gamma \rightarrow \gamma$

the resulting expression can be obtained

$$\varepsilon_{max} = \gamma / 2$$

with $\quad \varepsilon_{max} = \tau (1 + \nu) / E \qquad$ and $\qquad \gamma = \tau / G$

the following relationship can be written

$$G = \frac{E}{2(1 + \nu)}$$

thus E, G and ν are not independent properties of a linear elastic material

3.7 Transmission of Power by Circular Shafts

the most important use of circular shafts is to transmit mechanical power, such as drive shaft of an automobile, propeller shaft of a ship, axle of bicycle, torsional bar, etc.

a common design problem is the determination of the required size of a shaft so that it will transmit a specified amount of power at a specified speed of revolution without exceeding the allowable stress

consider a motor drive shaft, rotating at angular speed ω, it is transmitting a torque T, the work done is

$$W = T\phi \qquad \text{[T is constant for steady state]}$$

where ϕ is angular rotation in radians, ant the power is dW / dt

$$P = \frac{dW}{dt} = T\frac{d\phi}{dt} = T\omega \qquad \omega \ : \ \text{rad / s}$$

$\because \ \omega = 2\pi f \qquad f$ is frequency of revolution $\qquad f : \text{Hz} = \text{s}^{-1}$

$\therefore \ P = 2\pi f T$

denote n the number of revolution per minute (rpm), then $n = 60 f$

thus $$P = \frac{2 n \pi T}{60} \qquad (n = \text{rpm}, T = \text{N-m}, P = \text{W})$$

in U.S. engineering practice, power is often expressed in horsepower (hp), 1 hp = 550 ft-lb / s, thus the horsepower H being transmitted by a rotating shaft is

$$H = \frac{2 n \pi T}{60 \text{ x } 550} = \frac{2 n \pi T}{33{,}000} \qquad (n = \text{rpm}, T = \text{lb-ft}, H = \text{hp})$$

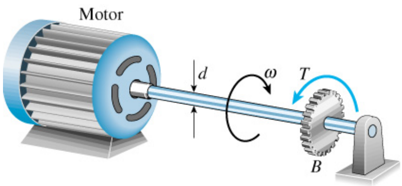
1 hp = 550 lb-ft/s = 550 x 4.448 N x 0.305 m/s = 746 N-m / s
= 746 W (W : watt)

Example 3-7

$P = 30$ kW, $\tau_{all} = 42$ MPa

(a) $n = 500$ rpm, determine d

(b) $n = 4000$ rpm, determine d



(a) $$T = \frac{60 P}{2\pi n} = \frac{60 \text{ x } 30 \text{ kW}}{2\pi \text{ x } 500} = 573 \text{ N-m}$$

$$\tau_{max} = \frac{16 T}{\pi d^3} \qquad d^3 = \frac{16 T}{\pi \tau_{all}} = \frac{16 \text{ x } 573 \text{ N-m}}{\pi \text{ x } 42 \text{ MPa}} = 69.5 \text{ x } 10^{-6} \text{ m}^3$$

$$d = 41.1 \text{ mm}$$

(b)
$$T = \frac{60\,P}{2\,\pi\,n} = \frac{60 \text{ x } 30 \text{ kW}}{2\,\pi \text{ x } 4000} = 71.6 \text{ N-m}$$

$$d^3 = \frac{16\,T}{\pi\,\tau_{all}} = \frac{16 \text{ x } 71.6 \text{ N-m}}{\pi \text{ x } 42 \text{ MPa}} = 8.68 \text{ x } 10^{-6} \text{ m}^3$$

$$d = 20.55 \text{ mm}$$

the higher the speed of rotation, the smaller the required size of the shaft

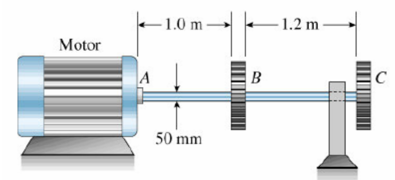
Example 3-8

a solid steel shaft ABC, $d = 50$ mm

motor A transmit 50 kW at 10 Hz

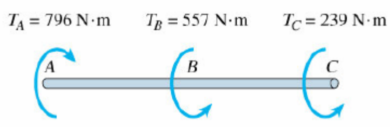
$P_B = 35$ kW, $P_C = 15$ kW

determine τ_{max} and ϕ_{AC}, $G = 80$ GPa



(a)

$$T_A = \frac{P_A}{2\,\pi\,f} = \frac{50 \text{ x } 10^3}{2\,\pi\,10} = 796 \text{ N-m}$$



(b)

similarly $P_B = 35$ kN $T_B = 557$ N-m

$$P_C = 15 \text{ kN} \qquad T_C = 239 \text{ N-m}$$

then $T_{AB} = 796$ N-m $T_{BC} = 239$ N-m

shear stress and angle of twist in segment AB

$$\tau_{AB} = \frac{16\,T_{AB}}{\pi\,d^3} = \frac{16 \text{ x } 796}{\pi\,50^3} = 32.4 \text{ MPa}$$

$$\phi_{AB} = \frac{T_{AB}\,L_{AB}}{G\,I_p} = \frac{796 \text{ x } 1.0}{80 \text{ x } 10^9 \dfrac{\pi}{32} 0.05^4} = 0.0162 \text{ rad}$$

shear stress and angle of twist in segment BC

$$\tau_{BC} = \frac{16\, T_{BC}}{\pi\, d^3} = \frac{16 \text{ x } 239}{\pi\, 50^3} = 9.7 \text{ MPa}$$

$$\phi_{AB} = \frac{T_{BC}\, L_{BC}}{G\, I_p} = \frac{239 \text{ x } 1.2}{80 \text{ x } 10^9 \dfrac{\pi}{32} 0.05^4} = 0.0058 \text{ rad}$$

$$\therefore \quad \tau_{max} = \tau_{AB} = 32.4 \text{ MPa}$$

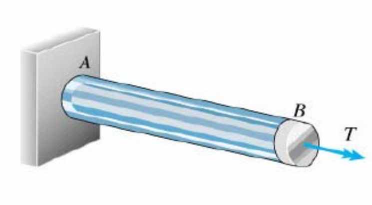
$$\phi_{AC} = \phi_{AB} + \phi_{BC} = 0.0162 + 0.0058 = 0.022 \text{ rad} = 1.26^{\circ}$$

3.8 Statically Indeterminate Torsional Members

torsional member may be statically indeterminate if they are constrained by more supports than are required to hold them in static equilibrium, or the torsional member is made by two or more kinds of materials

flexibility and stiffness methods may be used

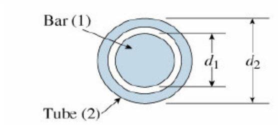
only flexibility method is used in the later discussion



(a)

consider a composite bar AB fixed at A

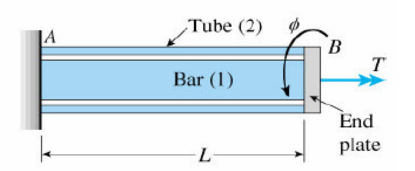
the end plate rotates through an angle ϕ



(b)

T_1 and T_2 are developed in the solid bar and tube, respectively

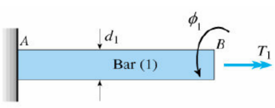
equation of equilibrium



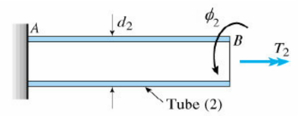
(c)

$$T_1 + T_2 = T$$

equation of compatibility



(d)



(e)

$$\phi_1 = \phi_2$$

torque-displacement relations

$$\phi_1 = \frac{T_1 L}{G_1 I_{p1}} \qquad \phi_2 = \frac{T_2 L}{G_2 I_{p2}}$$

then the equation of compatibility becomes

$$\frac{T_1 L}{G_1 I_{p1}} = \frac{T_2 L}{G_2 I_{p2}}$$

now we can solve for T_1 and T_2

$$T_1 = T\left(\frac{G_1 I_{p1}}{G_1 I_{p1} + G_2 I_{p2}}\right) \quad T_2 = T\left(\frac{G_2 I_{p2}}{G_1 I_{p1} + G_2 I_{p2}}\right)$$

and

$$\phi = \frac{T L}{G_1 I_{p1} + G_2 I_{p2}}$$

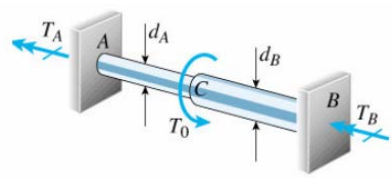
Example 3-9

a bar ACB is fixed at both ends

T_0 is applied at point C

AC : d_A, L_A, I_{pA}

CB : d_B, L_B, I_{pB}

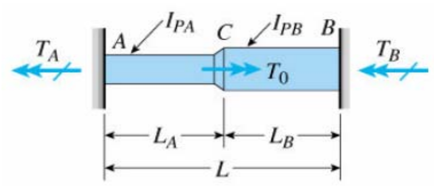


(a)

determine (a) T_A, T_B (b) τ_{AC}, τ_{CB} (c) ϕ_C

equation of equilibrium

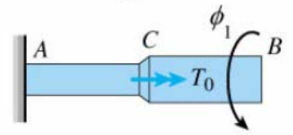
$$T_A + T_B = T_0$$



(b)

equation of compatibility

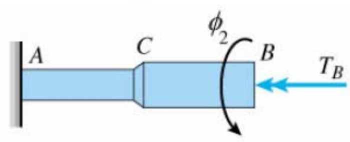
$$\phi_1 + \phi_2 = 0$$



(c)

torque-displacement equations

$$\phi_1 = T_0 L_A / G I_{pA}$$



(d)

$$\phi_2 = -\frac{T_B L_A}{G I_{pA}} - \frac{T_B L_B}{G I_{pB}}$$

then the equation of compatibility becomes

$$\frac{T_0 L_A}{G I_{pA}} - \frac{T_B L_A}{G I_{pA}} - \frac{T_B L_B}{G I_{pB}} = 0$$

T_A and T_B can be solved

$$T_A = T_0 \left(\frac{L_B I_{pA}}{L_B I_{pA} + L_A I_{pB}} \right) \qquad T_B = T_0 \left(\frac{L_A I_{pB}}{L_B I_{pA} + L_A I_{pB}} \right)$$

if the bar is prismatic, $I_{pA} = I_{pB} = I_p$

then $$T_A = \frac{T_0 L_B}{L} \qquad T_B = \frac{T_0 L_A}{L}$$

maximum shear stress in AC and BC are

$$\tau_{AC} = \frac{T_A d_A}{2 I_{pA}} = \frac{T_0 L_B d_A}{2 (L_B I_{pA} + L_A I_{pB})}$$

$$\tau_{CB} = \frac{T_B d_B}{2 I_{pB}} = \frac{T_0 L_A d_B}{2 (L_B I_{pA} + L_A I_{pB})}$$

angle of rotation at section C is

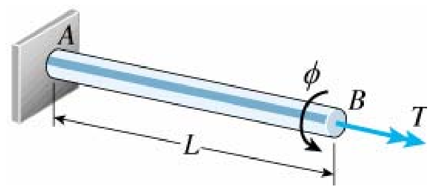
$$\phi_C = \frac{T_A L_A}{G I_{pA}} = \frac{T_B L_B}{G I_{pA}} = \frac{T_0 L_A L_B}{G (L_B I_{pA} + L_A I_{pB})}$$

if the bar is prismatic, $I_{pA} = I_{pB} = I_p$

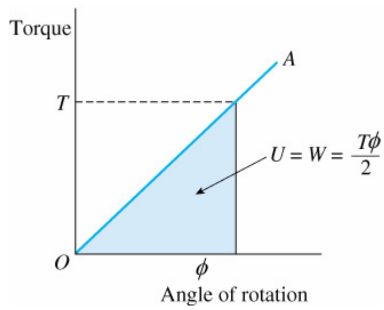
then $$\phi_C = \frac{T_0 L_A L_B}{G L I_p}$$

3.9 Strain Energy in Torsion and Pure Shear

consider a prismatic bar AB subjected to a torque T, the bar twists an angle ϕ



if the bar material is linear elastic, then the strain energy U of the bar is



$$U \;=\; W \;=\; T\,\phi\,/\,2$$

$$\because \qquad \phi \;=\; T\,L\,/\,G\,I_p$$

then
$$U \;=\; \frac{T^2\,L}{2\,G\,I_p} \;=\; \frac{G\,I_p\,\phi^2}{2\,L}$$

if the bar is subjected to nonuniform torsion, then

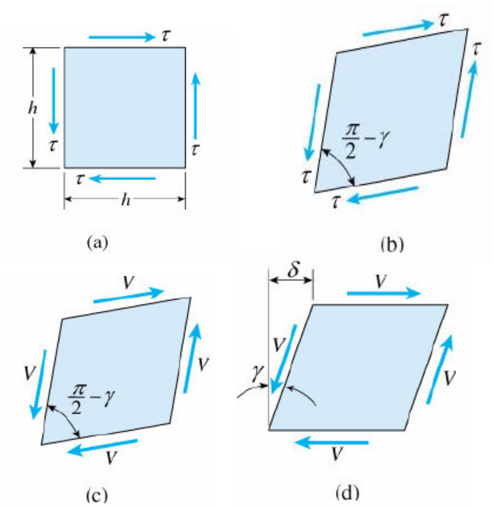
$$U \;=\; \sum_{i=1}^{n} U_i \;=\; \sum_{i=1}^{n} \frac{T_i^2\,L_i}{2\,G_i\,I_{pi}}$$

if either the cross section or the torque varies along the axis, then

$$dU \;=\; \frac{[T(x)]^2\,dx}{2\,G\,I_p(x)} \qquad U \;=\; \int dU \;=\; \int_0^L \frac{[T(x)]^2\,dx}{2\,G\,I_p(x)}$$

strain energy density in pure shear

consider a stressed element with each side having length h and thickness t, under shear stress τ with shear strain γ



the shear force V is

$$V \;=\; \tau\,h\,t$$

and the displacement δ is

$$\delta = h\,\gamma$$

for linear elastic material, strain energy stored in this element is

$$U = W = \frac{V\,\delta}{2} = \frac{\tau\,\gamma\,h^2\,t}{2}$$

and the strain energy density $u = U$ / per unit volume, then

$$u = \tau\,\gamma / 2 = \tau^2 / 2\,G = G\,\gamma^2 / 2$$

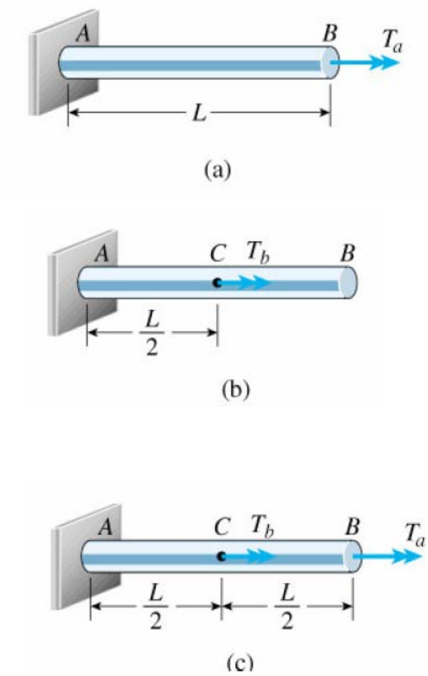
Example 3-10

a solid circular bar AB of length L

(a) torque T_a acting at the free end

(b) torque T_b acting at the midpoint

(c) both T_a and T_b acting simultaneously



T_a = 100 N-m $\quad$ T_b = 150 N-m

L = 1.6 m $\quad$ G = 80 GPa

$I_p = 79.52 \times 10^3$ mm^4

determine the strain energy in each case

(a)

$$U_a = \frac{T_a^2\,L}{2\,G\,I_p} = \frac{100^2 \times 10^6 \times 1.6 \times 10^3}{2 \times 80 \times 10^3 \times 79.52 \times 10^3} = 1.26 \text{ J} \quad \text{(N-m)}$$

(b)

$$U_b = \frac{T_b^2\,(L/2)}{2\,G\,I_p} = \frac{T_b^2\,L}{4\,G\,I_p} = 2.83 \text{ J}$$

(c)

$$U_c = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2 G_i I_{pi}} = \frac{T_a^{\,2} (L/2)}{2 G I_p} + \frac{(T_a + T_b)^2 (L/2)}{2 G I_p}$$

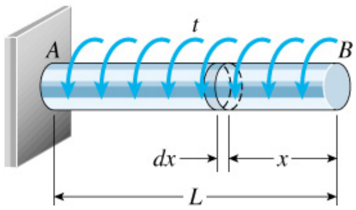
$$= \frac{T_a^{\,2} L}{2 G I_p} + \frac{T_a T_b L}{2 G I_p} + \frac{T_b^{\,2} L}{4 G I_p}$$

$$= 1.26 \text{ J} + 1.89 \text{ J} + 2.83 \text{ J} = 5.98 \text{ J}$$

Note that (c) is not equal to (a) + (b), because $U \sim T^{\,2}$

Example 3-11

a prismatic bar AB is loaded by a distributed torque of constant intensity t per unit distance



$t = 480$ lb-in/in $\quad L = 12$ ft

$G = 11.5 \times 10^6$ psi $\quad I_p = 18.17$ in^4

determine the strain energy

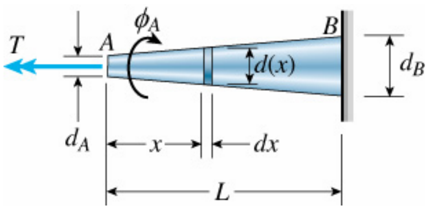
$T(x) = t\,x$

$$U = \int_0^L \frac{[(tx)]^2\,dx}{2 G I_p} = \frac{1}{2 G I_p} \int_0^L (tx)^2\,dx = \frac{t^2 L^3}{6 G I_p}$$

$$= \frac{480^2 \times (12 \times 12)^3}{6 \times 11.5 \times 10^6 \times 17.18} = 580 \text{ in-lb}$$

Example 3-12

a tapered bar AB of solid circular cross section is supported a torque T



$d = d_A \sim d_B$ from left to right

determine ϕ_A by energy method

$$W = \frac{T\,\phi_A}{2}$$

$$I_p(x) = \frac{\pi}{32}[d(x)]^4 = \frac{\pi}{32}\left(d_A + \frac{d_B - d_A}{L}x\right)^4$$

$$U = \int_0^L \frac{[T(x)]^2\,dx}{2\,G\,I_p(x)} = \frac{16\,T^2}{\pi\,G}\int_0^L \frac{dx}{\left(d_A + \frac{d_B - d_A}{L}x\right)^4}$$

$$= \frac{16\,T^2\,L}{3\,\pi\,G\,(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

with $U = W$, then ϕ_A can be obtained

$$\phi_A = \frac{32\,T\,L}{3\,\pi\,G\,(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

same result as in example 3-5

3-10 Thin-Walled Tubes

3-11 Stress Concentrations in Torsion

3-12 Nonlinear Torsion of Circular Bars

UNIT 3 SPRINGS

Introduction

A spring is defined as an elastic body, whose function is to distort when loaded and to recover its original shape when the load is removed. The various important applications of springs are as follows:

1. To cushion, absorb or control energy due to either shock or vibration as in car springs, railway buffers, air-craft landing gears, shock absorbers and vibration dampers.
2. To apply forces, as in brakes, clutches and spring-loaded valves.
3. To control motion by maintaining contact between two elements as in cams and followers.
4. To measure forces, as in spring balances and engine indicators.
5. To store energy, as in watches, toys, etc.

Types of Springs

Though there are many types of the springs, yet the following, according to their shape, are important from the subject point of view.

1. *Helical springs*. The helical springs are made up of a wire coiled in the form of a helix and are primarily intended for compressive or tensile loads. The cross-section of the wire from which the spring is made may be circular, square or rectangular. The two forms of helical springs are ***compression helical spring*** as shown in Fig (*a*) and ***tension helical spring*** as shown in Fig (*b*).

(*a*) Compression helical spring. (*b*) Tension helical spring.

Helical springs.

The helical springs are said to be ***closely coiled*** when the spring wire is coiled so close that the plane containing each turn is nearly at right angles to the axis of the helix and the wire is subjected to torsion. In other words, in a closely coiled helical spring, the helix angle is very small, it is usually less than 10°. The major stresses produced in helical springs are shear stresses due to twisting. The load applied is parallel to or along the axis of the spring.

In ***open coiled helical springs***, the spring wire is coiled in such a way that there is a gap between the two consecutive turns, as a result of which the helix angle is large. Since the application of open coiled helical springs are limited, therefore our discussion shall confine to closely coiled helical springs only.

The helical springs have the following advantages:

(***a***) These are easy to manufacture.

(***b***) These are available in wide range.

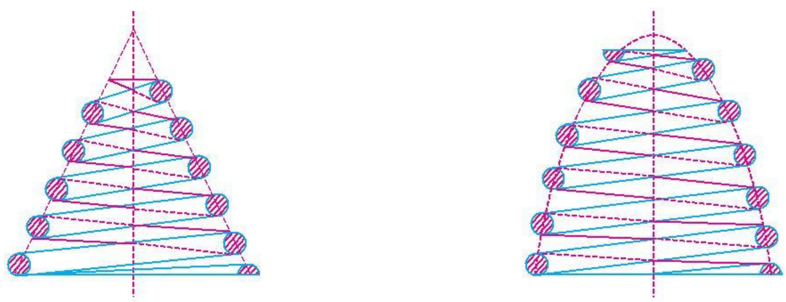
(***c***) These are reliable.

(***d***) These have constant spring rate.

(***e***) Their performance can be predicted more accurately.

(***f***) Their characteristics can be varied by changing dimensions.

2. *Conical and volute springs*. The conical and volute springs, as shown in Fig. 23.2, are used in special applications where a telescoping spring or a spring with a spring rate that increases with the load is desired. The conical spring, as shown in Fig (*a*), is wound with a uniform pitch whereas the volute springs, as shown in Fig. (*b*), are wound in the form of paraboloid with constant pitch



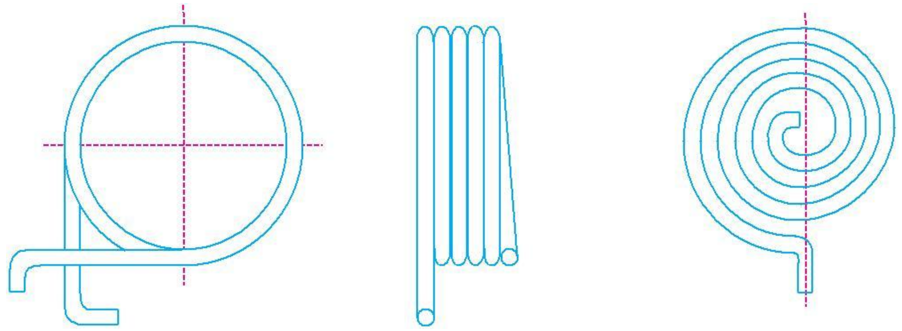
(*a*) Conical spring. (*b*) Volute spring.

and lead angles. The springs may be made either partially or completely telescoping. In either case, the number of active coils gradually decreases. The decreasing number of coils results in an increasing spring rate. This characteristic is sometimes utilised in vibration problems where springs are used to support a body that has a varying mass.

The major stresses produced in conical and volute springs are also shear stresses due to twisting.

3. ***Torsion springs*****.** These springs may be of ***helical*** or ***spiral*** type as shown in Fig. The **helical type** may be used only in applications where the load tends to wind up the spring and are used in various electrical mechanisms. The **spiral type** is also used where the load tends to increase the number of coils and when made of flat strip are used in watches and clocks.

The major stresses produced in torsion springs are tensile and compressive due to bending.



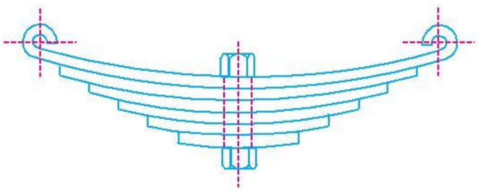
(*b*) Spiral torsion spring.

(*a*) Helical torsion spring.

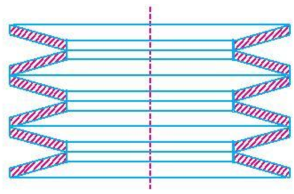
Torsion springs.

4. ***Laminated or leaf springs*****.** The laminated or leaf spring (also known as ***flat spring*** or ***carriage spring***) consists of a number of flat plates (known as leaves) of varying lengths held together by means of clamps and bolts, as shown in Fig. These are mostly used in automobiles.

The major stresses produced in leaf springs are tensile and compressive stresses.



Laminated or leaf springs.



Disc or bellevile springs.

5. ***Disc or bellevile springs*****.** These springs consist of a number of conical discs held together against slipping by a central bolt or tube as shown in Fig. These springs are used in applications where high spring rates and compact spring units are required.

The major stresses produced in disc or bellevile springs are tensile and compressive stresses.

6. Special purpose springs. These springs are air or liquid springs, rubber springs, ring springs etc. The fluids (air or liquid) can behave as a compression spring. These springs are used for special types of application only.

Uses of springs:

(a) To apply forces and to control motions as in brakes and clutches.

(b) To measure forces as in spring balance.

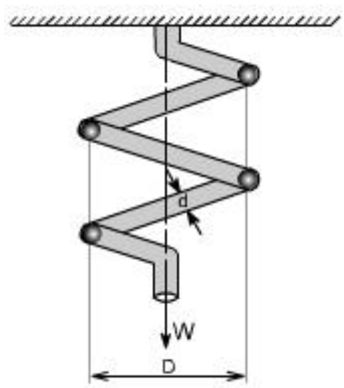
(c) To store energy as in clock springs.

(d) To reduce the effect of shock or impact loading as in carriage springs.

(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

Derivation of the Formula :

In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load W.



Let

W = axial load

D = mean coil diameter

d = diameter of spring wire

n = number of active coils

C = spring index = D / d for circular wires

l = length of spring wire

G = modulus of rigidity

x = deflection of spring

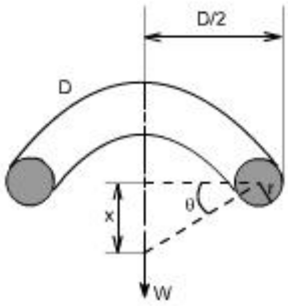
q = Angle of twist

When the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that

x = D / 2. q

Again l = p D n [consider, one half turn of a close coiled helical spring]



Assumptions: (1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a spring will be assumed to lie in a plane which is nearly ^r to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical. Hence to maintain equilibrium of a segment of the spring, only a shearing force V = F and Torque T = F. r are required at any X – section. In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is uniformly distributed and is negligible

So applying the torsion formula.

Using the torsion formula i.e

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

$$\text{and substitituting } J = \frac{\pi d^4}{32}; T = w.\frac{d}{2}$$

$$\theta = \frac{2.x}{D}; l = \pi.D.x$$

SPRING DEFLECTION

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{G.2x/D}{\pi.D.n}$$

Thus,

$$x = \frac{8w.D^3.n}{G.d^4}$$

Spring stiffness: The stiffness is defined as the load per unit deflection therefore

$$k = \frac{w}{x} = \frac{w}{\frac{8w.D^3.n}{G.d^4}}$$

Therefore

$$k = \frac{G.d^4}{8.D^3.n}$$

Shear stress

$$\frac{\frac{w.d/2}{\frac{\pi d^4}{32}}}{} = \frac{\tau_{max^m}}{d/2}$$

$$\text{or } \tau_{max^m} = \frac{8wD}{\pi d^3}$$

WAHL'S FACTOR:

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

K = Wahl's factor and is defined as $K = \frac{4c-1}{4c-4} + \frac{0.615}{c}$

Where C = spring index = D/d

If we take into account the Wahl's factor than the formula for the shear stress becomes $\tau_{max^m} = \frac{16.T.k}{\pi d^3}$

Strain Energy: The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion

$$U = \frac{T^2L}{2EI}$$

$$L = \pi Dn$$

$$I = \frac{\pi d^4}{64}$$

so after substitution we get

$$U = \frac{32T^2Dn}{E.d^4}$$

Example: A close coiled helical spring is to carry a load of 5000N with a deflection of 50 mm and a maximum shearing stress of 400 N/mm^2 .if the number of active turns or active coils is 8.Estimate the following:

(i) Wire diameter

(ii) Mean coil diameter

(iii) Weight of the spring.

Assume G = 83,000 N/mm^2 ; r = 7700 kg/m^3

Solution:

(i) For wire diameter if W is the axial load, then

$$\frac{\frac{w.d/2}{\frac{\pi d^4}{32}}}{} = \frac{\tau_{max^m}}{d/2}$$

$$D = \frac{400}{d/2} \cdot \frac{\pi d^4}{32} \cdot \frac{2}{W}$$

$$D = \frac{400.\pi d^3.2}{5000.16}$$

$$D = 0.0314\, d^3$$

Further, deflection is given as

$$x = \frac{8wD^3.n}{G.d^4}$$

on substituting the relevant parameters we get

$$50 = \frac{8.5000.(0.0314\, d^3)^3.8}{83{,}000.d^4}$$

$$d = 13.32\,mm$$

Therefore,

$D = .0314 \times (13.317)^3 mm$

$=74.15mm$

$D = 74.15$ mm

Weight

mass or weight = volume. density

= area.length of the spring.density of spring material

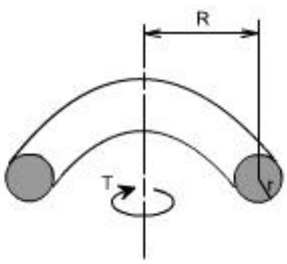
$$= \frac{\pi d^2}{4} . \pi D n.\rho$$

On substituting the relevant parameters we get

Weight = 1.996 kg

$\simeq 2.0$ kg

Close – coiled helical spring subjected to axial torque T or axial couple.



In this case the material of the spring is subjected to pure bending which tends to reduce Radius R of the coils. In this case the bending moment is constant through out the spring and is equal to the applied axial Torque T. The stresses i.e. maximum bending stress may thus be determined from the bending theory.

$$\sigma_{max} = \frac{M.y}{I} = \frac{T.d/2}{\frac{\pi d^4}{64}}$$

$$\sigma_{max} = \frac{32T}{\pi d^3}$$

Deflection or wind – up angle:

Under the action of an axial torque the deflection of the spring becomes the “wind – up” angle of the spring which is the angle through which one end turns relative to the other. This will be equal to the total change of slope along the wire, according to area – moment theorem

$$\theta = \int_0^L \frac{MdL}{EI} \text{ but } M = T$$

$$= \int_0^L \frac{T.dL}{EI} = \frac{T}{EI}\int_0^L dL$$

Thus, as 'T' remains constant

$$\theta = \frac{T.L}{EI}$$

Futher

$$L = \pi D.n$$

$$I = \frac{\pi d^4}{64}$$

Therefore, on substitution, the value of θ obtained is

$$\boxed{\theta = \frac{64T.D.n}{E.d^4}}$$

Springs in Series: If two springs of different stiffness are joined endon and carry a common load W, they are said to be connected in series and the combined stiffness and deflection are given by the following equation.

$$\frac{W}{k} = x_1 + x_2 = \frac{W}{k_1} + \frac{W}{k_2}$$

or

$$\boxed{\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}}$$

Springs in parallel: If the two springs are joined in such a way that they have a common deflection ‘x’; then they are said to be connected in parallel. In this case the load carried is shared between the two springs and total load $W = W_1 + W_2$

$$x = \frac{W}{k} = \frac{W_1}{k_1} = \frac{W_2}{k_2}$$

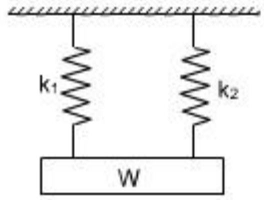
Thus $W_1 = \dfrac{Wk_1}{k}$

$W_2 = \dfrac{Wk_2}{k}$

Futher

$W = W_1 + W_2$

thus $\boxed{k = k_1 + k_2}$



Terms used in Compression Springs

The following terms used in connection with compression springs are important from the subject point of view.

1. *Solid length.* When the compression spring is compressed until the coils come in contact with each other, then the spring is said to be ***solid.*** The solid length of a spring is the product of total number of coils and the diameter of the wire. Mathematically,

Solid length of the spring,

$L_S = n'.d$

where n' = Total number of coils, and

d = of the wire.

2. *Free length.* The free length of a compression spring, as shown in Fig. is the length of the spring in the free or unloaded condition. It is equal to the solid length plus the maximum deflection or compression of the spring and the clearance between the adjacent coils (when fully compressed). Mathematically,

Free length of the spring,

L_F = Solid length + Maximum compression + *Clearance between adjacent coils (or clash allowance)

$= n'.d + \delta_{max} + 0.15\, \delta_{max}$

The following relation may also be used to find the free length of the spring, *i.e.*

$L_F = n'.d + \delta_{max} + (n' - 1) \times 1$ mm

In this expression, the clearance between the two adjacent coils is taken as 1 mm.

3. *Spring index.* The spring index is defined as the ratio of the mean diameter of the coil to the diameter of the wire. Mathematically,

Spring index, $C = D / d$

where D = Mean diameter of the coil, and

d = Diameter of the wire.

4. *Spring rate.* The spring rate (or stiffness or spring constant) is defined as the load required per unit deflection of the spring. Mathematically,

Spring rate, $k = W / \delta$

where W = Load, and

δ = Deflection of the spring.

5. *Pitch.* The pitch of the coil is defined as the axial distance between adjacent coils in uncompressed state. Mathematically,

Pitch of the coil, $p = \dfrac{\text{Free length}}{n' - 1}$

The pitch of the coil may also be obtained by using the following relation, *i.e.*

Pitch of the coil, $$p = \frac{L_F - L_S}{n'} + d$$

where L_F = Free length of the spring,

L_S = Solid length of the spring,

n' = Total number of coils, and

d = Diameter of the wire.

In choosing the pitch of the coils, the following points should be noted :

(*a*) The pitch of the coils should be such that if the spring is accidently or carelessly compressed, the stress does not increase the yield point stress in torsion.

(*b*) The spring should not close up before the maximum service load is reached.

Note : In designing a tension spring (See Example 23.8), the minimum gap between two coils when the spring is in the free state is taken as 1 mm. Thus the free length of the spring,

$$L_F = n.d + (n - 1)$$

and pitch of the coil, $$p = \frac{L_F}{n - 1}$$

Example 1. *Design a helical compression spring for a maximum load of 1000 N for a deflection of 25 mm using the value of spring index as 5.*

The maximum permissible shear stress for spring wire is 420 MPa and modulus of rigidity is 84 kN/mm^2.

Take Wahl's factor, $K = \frac{4C - 1}{4C - 4} + \frac{0.615}{C}$*, where C = Spring index.*

Solution. Given: W = 1000 N; δ = 25 mm; $C = D/d = 5$; τ = 420 MPa = 420 N/mm^2 ; G = 84 kN/mm^2 = 84 × 10^3 N/mm^2

1. ***Mean diameter of the spring coil***

Let D = Mean diameter of the spring coil, and

d = Diameter of the spring wire.

We know that Wahl's stress factor,

$$K = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} = \frac{4 \cdot 5 - 1}{4 \cdot 5 - 4} + \frac{0.615}{5} = 1.31$$

and maximum shear stress (τ),

$$420 = K \cdot \frac{8\,W.C}{\pi\,d^2} = 1.31 \cdot \frac{8 \cdot 1000 \cdot 5}{\pi\,d^2} = \frac{16\,677}{d^2}$$

$\therefore$ d^2 = 16 677 / 420 = 39.7 or d = 6.3 mm

we shall take a standard wire of size *SWG* 3 having diameter (d) = 6.401 mm.

$\therefore$ Mean diameter of the spring coil,

$D = C.d = 5\,d = 5 \times 6.401$ = 32.005 mm **Ans.** ... ($\because$ $C = D/d = 5$)

and outer diameter of the spring coil,

$D_o = D + d$ = 32.005 + 6.401 = 38.406 mm **Ans.**

2. ***Number of turns of the coils***

Let n = Number of active turns of the coils.

we know that compression of the spring (δ),

$$25 = \frac{8W.C^3.n}{G.d} = \frac{8 \cdot 1000\,(5)^3\,n}{84 \cdot 10^3 \cdot 6.401} = 1.86\,n$$

$\therefore$ n = 25 / 1.86 = 13.44 say 14 **Ans.**

For squared and ground ends, the total number of turns, n'

= n + 2 = 14 + 2 = 16 **Ans.**

3. ***Free length of the spring***

We know that free length of the spring

$= n'.d + \delta + 0.15\,\delta = 16 \times 6.401 + 25 + 0.15 \times 25$

$= 131.2$ mm **Ans.**

4. ***Pitch of the coil***

We know that pitch of the coil

$$= \frac{\text{Free length}}{n' - 1} = \frac{131.2}{16 - 1} = 8.75 \text{ mm } \textbf{Ans.}$$

Example 2. *Design a close coiled helical compression spring for a service load ranging from 2250 N to 2750 N. The axial deflection of the spring for the load range is 6 mm. Assume a spring index of 5. The permissible shear stress intensity is 420 MPa and modulus of rigidity, G = 84 kN/mm^2.*

Neglect the effect of stress concentration. Draw a fully dimensioned sketch of the spring, show-ing details of the finish of the end coils.

Solution. Given : $W_1 = 2250$ N ; $W_2 = 2750$ N ; $\delta = 6$ mm ; $C = D/d = 5$; $\tau = 420$ MPa = 420 N/mm^2 ; $G = 84$ kN/mm^2 = 84×10^3 N/mm^2

1. ***Mean diameter of the spring coil***

Let D = Mean diameter of the spring coil for a maximum load of $W_2 = 2750$ N, and

d = Diameter of the spring wire.

We know that twisting moment on the spring,

$$T = W_2 \cdot \frac{D}{2} = 2750 \cdot \frac{5d}{2} = 6875\, d \qquad \ldots \because C = \frac{D}{d} = 5$$

We also know that twisting moment (T),

$$6875\, d = \frac{\pi}{16} \cdot \tau \cdot d^3 = \frac{\pi}{16} \cdot 420 \cdot d^3 = 82.48\, d^3$$

$\therefore$ $d^2 = 6875 / 82.48 = 83.35$ or $d = 9.13$ mm

From Table 23.2, we shall take a standard wire of size *SWG* 3/0 having diameter (d) = 9.49 mm. $\therefore$ Mean diameter of the spring coil,

$$D = 5d = 5 \times 9.49 = 47.45 \text{ mm } \textbf{Ans.}$$

We know that outer diameter of the spring coil,

$$D_o = D + d = 47.45 + 9.49 = 56.94 \text{ mm } \textbf{Ans.}$$

and inner diameter of the spring coil,

$$D_i = D - d = 47.45 - 9.49 = 37.96 \text{ mm } \textbf{Ans.}$$

2. ***Number of turns of the spring coil***

Let n = Number of active turns.

It is given that the axial deflection (δ) for the load range from 2250 N to 2750 N (*i.e.* for W = 500 N) is 6 mm.

We know that the deflection of the spring (δ),

$$6 = \frac{8\, W . C^3 . n}{G . d} = \frac{8 \cdot 500\, (5)^3\, n}{84 \cdot 10^3 \cdot 9.49} = 0.63\, n$$

$\therefore$ $n = 6 / 0.63 = 9.5$ say 10 **Ans.**

For squared and ground ends, the total number of turns, $n' = 10 + 2 = 12$ **Ans.**

3. ***Free length of the spring***

Since the compression produced under 500 N is 6 mm, therefore maximum compression produced under the maximum load of 2750 N is

$$\delta_{max} = 6/500*2750 = 33 \text{ mm}$$

We know that free length of the spring,

$$L_F = n'.d + \delta_{max} + 0.15\,\delta_{max}$$
$$= 12 \times 9.49 + 33 + 0.15 \times 33$$
$$= 151.83 \text{ say } 152 \text{ mm}$$ **Ans.**

4. ***Pitch of the coil***

We know that pitch of the coil

$$= \frac{\text{Free length}}{n' - 1} = \frac{152}{12 - 1} = 13.73 \text{ say } 13.8 \text{ mm}$$ **Ans.**

Problem 16.39. *A closely coiled helical spring of mean diameter 20 cm is made of 3 cm diameter rod and has 16 turns. A weight of 3 kN is dropped on this spring. Find the height by which the weight should be dropped before striking the spring so that the spring may be compressed by 18 cm. Take $C = 8 \times 10^4$ N/mm².*

Sol. Given :

Mean dia. of coil, $D = 20$ cm $= 200$ mm

$\therefore$ Mean radius of coil, $R = \frac{200}{2} = 100$ mm

Dia. of spring rod, $d = 3$ cm $= 30$ mm

Number of turns, $n = 16$

Weight dropped, $W = 3$ kN $= 3000$ N

Compression of the spring, $\delta = 18$ cm $= 180$ mm

Modulus of rigidity, $C = 8 \times 10^4$ N/mm²

Let h = Height through which the weight W is dropped

W = Gradually applied load which produces the compression of spring equal to 180 mm.

Now using equation (16.26),

$$\delta = \frac{64W.R^3.n}{Cd^4}$$

or $$180 = \frac{64 \times W \times 100^3 \times 16}{8 \times 10^4 \times 30^4}$$

or $$W = \frac{180 \times 8 \times 10^4 \times 30^4}{64 \times 100^3 \times 16} = 11390 \text{ N}$$

Work done by the falling weight on spring

$$= \text{Weight falling } (h + \delta) = 3000\,(h + 180) \text{ N-mm}$$

Energy stored in the spring $= \frac{1}{2} W \times \delta$

$$= \frac{1}{2} \times 11390 \times 180 = 1025100 \text{ N-mm.}$$

Equating the work done by the falling weight on the spring to the energy stored in the spring, we get

$$3000(h + 180) = 1025100$$

or $$h + 180 = \frac{1025100}{3000} = 341.7 \text{ mm}$$

$\therefore$ $$h = 341.7 - 180 = \mathbf{161.7\ mm.} \quad \textbf{Ans.}$$

Problem 16.43. *A closely coiled helical spring made of 10 mm diameter steel wire has 15 coils of 100 mm mean diameter. The spring is subjected to an axial load of 100 N. Calculate :*

(*i*) *The maximum shear stress induced,*

(*ii*) *The deflection, and*

(*iii*) *Stiffness of the spring.*

Take modulus of rigidity, $C = 8.16 \times 10^4\ N/mm^2$.

(AMIE, Winter 1990 ; Converted to S.I. units)

Sol. Given :

Dia. of wire, $d = 10$ mm

Number of coils, $n = 15$

Mean dia. of coil, $D = 100$ mm

$\therefore$ Mean radius of coil, $R = \dfrac{100}{2} = 50$ mm

Axial load, $W = 100$ N

Modulus of rigidity, $C = 8.16 \times 10^4$ N/mm^2.

(*i*) *Maximum shear stress induced*

Using equation (16.24), $\tau = \dfrac{16WR}{\pi d^3} = \dfrac{16 \times 100 \times 50}{\pi \times 10^3} = \mathbf{24.46\ N/mm^2}$. **Ans.**

(*ii*) *The deflection* (δ)

Using equation (16.26),

$$\delta = \frac{64W \times R^3 \times n}{C \times d^4} = \frac{64 \times 100 \times 50^3 \times 15}{8.16 \times 10^4 \times 10^4}$$

$$= \mathbf{14.7\ mm.} \quad \textbf{Ans.}$$

(*iii*) *Stiffness of the spring*

$$\text{Stiffness} = \frac{\text{Load on spring}}{\text{Deflection of spring}}$$

$$= \frac{\text{Load on spring}}{\text{Deflection of spring}} = \frac{100}{14.7} = \mathbf{6.802\ N/mm.} \quad \textbf{Ans.}$$

UNIT 4 - TORSION OF SHAFTS

Torsion occurs when any shaft is subjected to a torque. This is true whether the shaft is rotating (such as drive shafts on engines, motors and turbines) or stationary (such as with a bolt or screw). The torque makes the shaft twist and one end rotates relative to the other inducing shear stress on any cross section. Failure might occur due to shear alone or because the shear is accompanied by stretching or bending.

1.1. TORSION EQUATION

The diagram shows a shaft fixed at one end and twisted at the other end due to the action of a torque T.

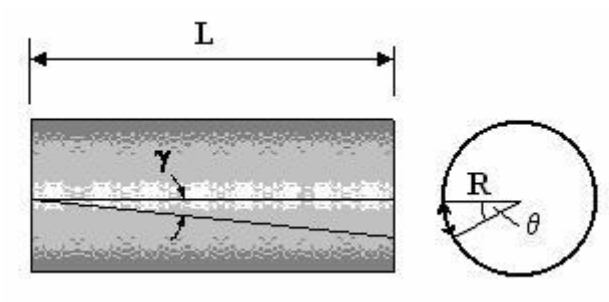


Figure 1

The radius of the shaft is R and the length is L.

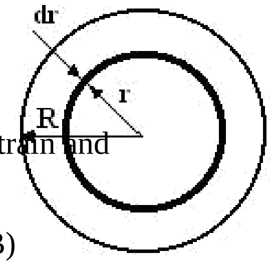
Imagine a horizontal radial line drawn on the end face. When the end is twisted, the line rotates through an angle θ. The length of the arc produced is $R\theta$.

Now consider a line drawn along the length of the shaft. When twisted, the line moves through an angle γ. The length of the arc produced is $L\gamma$.

If we assume that the two arcs are the same it follows that $R\theta = L\gamma$

Hence by equating $L\gamma = R\theta$ we get $\gamma = \dfrac{R\theta}{L}$ ....(1A)

If you refer to basic stress and strain theory, you will appreciate that γ is the shear



strain on the outer surface of the shaft. The relationship between shear strain and shear stress is

$$G = \frac{\tau}{\gamma} \quad \text{....................(1B)}$$

τ is the shear stress and G the modulus of rigidity.

G is one of the elastic constants of a material. The equation is only true so long as the material remains
elastic.

$$G\theta/L = T/R \quad \text{....................(1C)}$$

Since the derivation could be applied to any radius, it follows that shear stress is directly proportional to radius 'r' and is a maximum on the surface. Equation (1C) could be written as

$$\frac{G\theta}{L} = \frac{\tau}{r} \quad \text{....................(1D)}$$

Now let's consider how the applied torque 'T' is balanced by the internal stresses of the material.

Consider an elementary ring of material with a shear stress τ acting on it at radius r.

The area of the ring is $dA = 2\pi\, r\, dr$

The shear force acting on it tangential is $dF = \tau\, dA = \tau\, 2\pi r\, dr$

This force acts at radius r so the torque produced is $dT = \tau\, 2\pi r2\, dr$

Since $\tau = \frac{G\theta\, r}{L}$ from equation (1D) then $dT = \frac{G\theta}{L} 2\pi\, r^3 dr$

Figure 2

The torque on the whole cross section resulting from the shear stress is $T = \frac{G\theta}{L} 2\pi \int_0^R r^3 dr$

The expression $2\pi \int^R r^3 dr$ is called the polar second moment of area and denoted as 'J'. The Torque equation

reduces to $T = \frac{G\theta}{L} J$ and this is usually written as $\frac{T}{J} = \frac{G\theta}{L}$ (1E)

Combining (1D) and (1E) we get the torsion equation $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r}$ ....(1F)

1.2 POLAR SECOND MOMENTS OF AREA

This tutorial only covers circular sections. The formula for J is found by carrying out the integration or may be found in standard tables.

For a shaft of diameter D the formula is $J = \frac{\pi D^4}{32}$

This is not to be confused with the second moment of area about a diameter, used in bending of beams (I) but it should be noted that J = 2 I.

1.3 HOLLOW SHAFTS

Since the shear stress is small near the middle, then if there is no other stress considerations other than torsion, a hollow shaft may be used to reduce the weight.

The formula for the polar second moment of area is $J = \frac{\pi}{32}\left(D_O{}^4 - D_I{}^4\right)$

D_O is the outside diameter and D_I the inside diameter.

1.4 MECHANICAL POWER TRANSMISSION BY A SHAFT

In this section you will derive the formula for the power transmitted by a shaft and combine it with torsion theory.

Mechanical power is defined as work done per second. Work done is defined as force times distance moved. Hence

P = Fx/t where P is the Power
F is the force
x is distance moved.
t is the time taken.

Since distance moved/time taken is the velocity of the force we may write

P = F v ....(2A) where v is the velocity.

When a force rotates at radius R it travels distance moved in one revolution is one circumference in the time of one revolution. Hence the x = 2πR

If the speed is N rev/second then the time of one revolution is 1/N seconds. The mechanical power is hence P = F 2πR/(1/N) = 2πNFR

Since FR is the torque produced by the force this reduces to

P = 2πNT ....(2B)

Since 2πN is the angular velocity ω radians/s it further reduces to

P = ωT.................(2C)

Note that equations (2C) is the angular equivalent of equation (2A) and all three equations should be remembered.

WORKED EXAMPLE No.1

A shaft 50 mm diameter and 0.7 m long is subjected to a torque of 1200 Nm. Calculate the shear stress and the angle of twist. Take G = 90 GPa.

SOLUTION

Important values to use are D = 0.05 m, L = 0.7 m, T = 1200 Nm, G = 90 x10^9 Pa

$$J = \frac{\pi D^4}{32} = \frac{\pi \text{ x } 0.05^4}{32} = 613.59 \text{ x } 10^{-9} \text{ m}^4$$

$$\tau_{max} = \frac{TR}{J} = \frac{1200 \text{ x } 0.025}{613.59 \text{ x } 10^{-9}} = 48.89 \text{ x } 10^6 \text{ Pa or } 48.89 \text{ MPa}$$

$$\theta = \frac{TL}{J} = \frac{1200 \text{ x } 0.7}{90 \text{ x } 10^9 \text{ x } 613.59 \text{ x } 10^{-9}} = 0.0152 \text{ radian}$$

$$\text{Alternately } \theta = \frac{\tau L}{GR} = \frac{48.89 \text{ x } 10^6 \text{ x } 0.7}{90 \text{ x } 10^9 \text{ x } 0.025} = 0.0152 \text{ radian}$$

$$\text{Converting to degrees } \theta = 0.0152 \text{ x } \frac{180}{\pi} = 0.871^\circ$$

WORKED EXAMPLE No.2

Repeat the previous problem but this time the shaft is hollow with an internal diameter of 30 mm.

$$J = \frac{\pi\left(D^4 - d^4\right)}{32} = \frac{\pi \text{ x } \left(0.05^4 - 0.03^4\right)}{32} = 534.07 \text{ x } 10^{-9} \text{ m}^4$$

$$\tau_{max} = \frac{TR}{J} = \frac{1200 \text{ x } 0.025}{534.07 \text{ x } 10^{-9}} = 56.17 \text{ x } 10^6 \text{ Pa or } 56.17 \text{ MPa}$$

$$\theta = \frac{TL}{J} = \frac{1200 \text{ x } 0.7}{90 \text{ x } 10^9 \text{ x } 534.07 \text{ x } 10^{-9}} = 0.0175 \text{ radian}$$

$$\text{Alternately } \theta = \frac{\tau L}{GR} = \frac{56.17 \text{ x } 10^6 \text{ x } 0.7}{90 \text{ x } 10^9 \text{ x } 0.025} = 0.0175 \text{ radian}$$

$$\text{Converting to degrees } \theta = 0.0152 \text{ x } \frac{180}{\pi} = 1^\circ$$

Note that the answers are nearly the same even though there is much less material in the shaft.

WORKED EXAMPLE No.3

A shaft 40 mm diameter is made from steel and the maximum allowable shear stress for the material is 50 MPa. Calculate the maximum torque that can be safely transmitted. Take G = 90 GPa.

SOLUTION

Important values to use are:

$D = 0.04$ m, $R = 0.02$ m, $\tau = 50 \times 10^6$ Pa and $G = 90 \times 10^9$ Pa

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r}$$

$$J = \frac{\pi D^4}{32} = \frac{\pi \times 0.04^4}{32} = 251.32 \times 10^{-9}\ \text{m}^4$$

The complete torsion equation is $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$ Rearrange and ignore the middle term.

$$T = \frac{\tau_{max} J}{R} = \frac{50 \times 10^6 \times 251.32 \times 10^{-9}}{0.02} = 628.3\ \text{Nm}$$

WORKED EXAMPLE No.4

A shaft is made from tube. The ratio of the inside diameter to the outside diameter is 0.6. The material must not experience a shear stress greater than 500 kPa. The shaft must transmit 1.5 MW of mechanical power at 1500 rev/min. Calculate the shaft diameters.

SOLUTION

The important quantities are $P = 1.5 \times 10^6$ Watts, $\tau = 500 \times 10^3$ Pa, N = 1500 rev/min and d = 0.6D.

$N = 1500\ \text{rev/min} = 1500/60 = 25\ \text{rev/s} \qquad P = 2\pi N T$

$$\text{hence } T = \frac{P}{2\pi N} = \frac{1.5 \times 10^6}{2\pi \times 25} = 9549.3\ \text{Nm}$$

$$J = \frac{\pi(D^4 - d^4)}{32} = \frac{\pi\{D^4 - (0.6D)^4\}}{32} = \frac{\pi\{D^4 - 0.36D^4\}}{32} = 0.08545D^4$$

$$\frac{T}{J} = \frac{\tau}{R} = \frac{2\tau}{D} \quad \text{hence} \quad \frac{9549.3}{0.08545D^4} = \frac{2 \times 500 \times 10^3}{D} = \frac{9549.3}{0.08545 \times 2 \times 500 \times 10^3} = \frac{D^4}{D} = D^3$$

$$D^3 = 0.11175 \quad D = \sqrt[3]{0.11175} = 0.4816\ \text{m} = 481.6\ \text{mm} \qquad d = 0.6D = 289\ \text{mm}$$

Columns and Struts

Introduction

A machine part subjected to an axial compressive force is called a ***strut.*** A strut may be horizontal, inclined or even vertical. But a vertical strut is known as a ***column, pillar*** or ***stanchion***. The machine members that must be investigated for column action are piston rods, valve push rods, connecting rods, screw jack, side links of toggle jacketc. In this chapter, we shall discuss the design of piston rods, valve push rods and connecting rods.

Failure of a Column or Strut

It has been observed that when a column or a strut is subjected to a compressive load and the load is gradually increased, a stage will reach when the column will be subjected to ultimate load. Beyond this, the column will fail by crushing and the load will be known as ***crushing load.***

Types of End Conditions of Columns

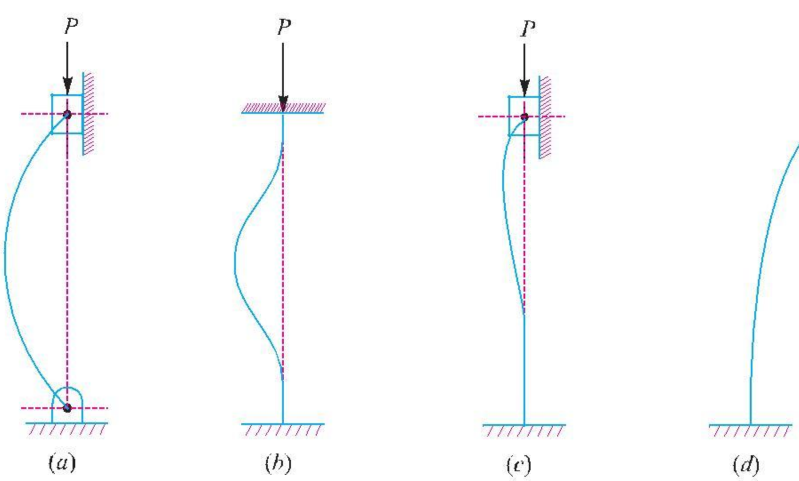
In actual practice, there are a number of end conditions for columns. But we shall study the Euler's column theory on the following four types of end conditions which are important from the subject point of view:

1.3 Both the ends hinged or pin jointed as shown in Fig (*a*),

1.4 Both the ends fixed as shown in Fig.(*b*),

1.5 One end is fixed and the other hinged as shown in Fig.(*c*), and

1.6 One end is fixed and the other free as shown in Fig. (*d*).



Euler's Column Theory

The first rational attempt, to study the stability of long columns, was made by Mr. Euler. He derived an equation, for the buckling load of long columns based on the bending stress. While deriving this equation, the effect of direct stress is neglected. This may be justified with the statement, that the direct stress induced in a long column is negligible as compared to the bending stress. It may be noted that Euler's formula cannot be used in the case of short columns, because the direct stress is considerable, and hence cannot be neglected.

16.5 Assumptions in Euler's Column Theory

The following simplifying assumptions are made in Euler's column theory :

1. Initially the column is perfectly straight, and the load applied is truly axial.
2. The cross-section of the column is uniform throughout its length.

3. The column material is perfectly elastic, homogeneous and isotropic, and thus obeys Hooke's law.
4. The length of column is very large as compared to its cross-sectional dimensions.
5. The shortening of column, due to direct compression (being very small) is neglected.
6. The failure of column occurs due to buckling alone.
7. The weight of the column itself is neglected.

16.6 Euler's Formula

According to Euler's theory, the crippling or buckling load (W_{cr}) under various end conditions is represented by a general equation,

$$W_{cr} = \frac{C\pi^2 EI}{l^2} = \frac{C\pi^2 EAk^2}{l^2} \qquad \ldots (\because I = A.k^2)$$

$$= \frac{C\pi^2 EA}{(l/k)^2}$$

where E = Modulus of elasticity or Young's modulus for the material of the column,

A = Area of cross-section,

k = Least radius of gyration of the cross-section,

l = Length of the column, and

C = Constant, representing the end conditions of the column or end fixity coefficient.

The following table shows the values of end fixity coefficient (C) for various end conditions.

Table 1. Values of end fixity coefficient (C).

S. No.	*End conditions*	*End fixity coefficient (C)*
1.	Both ends hinged	1
2.	Both ends fixed	4
3.	One end fixed and other hinged	2
4.	One end fixed and other end free	0.25

Slenderness Ratio

In Euler's formula, the ratio l / k is known as ***slenderness ratio.*** It may be defined as the ratio of the effective length of the column to the least radius of gyration of the section.

Limitations of Euler's Formula

A little consideration will show that the crippling stress will be high, when the slenderness ratio is small. We know that the crippling stress for a column cannot be more than the crushing stress of the column material. It is thus obvious that the Euler's fromula will give the value of crippling stress of the column (equal to the crushing stress of the column material) corresponding to the slenderness ratio. Now consider a mild steel column. We know that the crushing stress for mild steel is 330 N/mm^2 and Young's modulus for mild steel is 0.21 × 10^6 N/mm^2.

Now equating the crippling stress to the crushing stress, we have

$$\frac{C\pi^2 E}{(l/k)^2} = 330$$

$$\frac{1 \cdot 9.87 \cdot 0.21 \cdot 10^6}{(l/k)^2} = 330 \qquad \ldots (\text{Taking } C = 1)$$

r $$(l / k)^2 = 6281$$

$$\therefore \quad l / k = 79.25 \text{ say } 80$$

Hence if the slenderness ratio is less than 80, Euler's formula for a mild steel column is not valid.

Sometimes, the columns whose slenderness ratio is more than 80, are known as ***long columns***, and those whose slenderness ratio is less than 80 are known as ***short columns.*** It is thus obvious that the Euler's formula holds good only for long columns.

16.9 Equivalent Length of a Column

Sometimes, the crippling load according to Euler's formula may be written as

$$W_{cr} = \frac{\pi^2 E I}{L^2}$$

where L is the equivalent length or effective length of the column. The equivalent length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends to that of the given column. The relation between the equivalent length and actual length for the given end conditions is shown in the following table.

Table 2. Relation between equivalent length (L) and actual length (l).

S.No.	*End Conditions*	*Relation between equivalent length (L) and actual length (l)*
1.	Both ends hinged	$L = l$
2.	Both ends fixed	$L = \frac{l}{2}$
3.	One end fixed and other end hinged	$L = \frac{l}{\sqrt{2}}$
4.	One end fixed and other end free	$L = 2l$

Example 16.1. *A T-section 150 mm × 120 mm × 20 mm is used as a strut of 4 m long hinged at both ends. Calculate the crippling load, if Young's modulus for the material of the section is 200 kN/mm².*

Solution. Given : $l = 4$ m $= 4000$ mm ; $E = 200$ kN/mm^2 $= 200 \times 10^3$ N/mm^2

First of all, let us find the centre of gravity (G) of the T-section as shown in Fig. 16.2.

Let $\bar{y}$ be the distance between the centre of gravity (G) and top of the flange,

We know that the area of flange,

$$a_1 = 150 \times 20 = 3000 \text{ mm}^2$$

Its distance of centre of gravity from top of the flange,

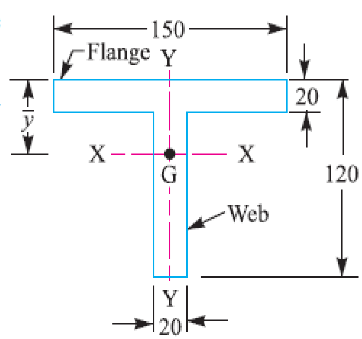
$$y_1 = 20 / 2 = 10 \text{ mm}$$

Area of web, $a_2 = (120 - 20)\, 20 = 2000 \text{ mm}^2$

Its distance of centre of gravity from top of the flange,

$$y_2 = 20 + 100 / 2 = 70 \text{ mm}$$

$$\therefore \quad \bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{3000 \times 10 + 2000 \times 70}{3000 + 2000} = 34 \text{ mm}$$



All dimensions in mm.

Fig. 16.2

We know that the moment of inertia of the section about X-X,

$$I_{XX} = \left[\frac{150\,(20)^3}{12} + 3000\,(34-10)^2 + \frac{20\,(100)^3}{12} + 2000\,(70-34)^2\right]$$

$$= 6.1 \times 10^6 \text{ mm}^4$$

and
$$I_{YY} = \frac{20\,(150)^3}{12} + \frac{100\,(20)^3}{12} = 5.7 \times 10^6 \text{ mm}^4$$

Since I_{YY} is less than I_{XX}, therefore the column will tend to buckle in Y-Y direction. Thus we shall take the value of I as $I_{YY} = 5.7 \times 10^6$ mm^4.

Moreover as the column is hinged at its both ends, therefore equivalent length,

$$L = l = 4000 \text{ mm}$$

We know that the crippling load,

$$W_{cr} = \frac{\pi^2 E\, I}{L^2} = \frac{9.87 \times 200 \times 10^3 \times 5.7 \times 10^6}{(4000)^2} = 703 \times 10^3 \text{ N} = 703 \text{ kN}$$ **Ans.**

RANKINE'S FORMULA:

We have already discussed that Euler's formula gives correct results only for very long columns. Though this formula is applicable for columns, ranging from very long to short ones, yet it does not give reliable results. Prof. Rankine, after a number of experiments, gave the following empirical formula for columns.

$$\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} \qquad \text{...}(i)$$

where W_{cr} = Crippling load by Rankine's formula,

W_C = Ultimate crushing load for the column = $\sigma_c \times A$,

W_E = Crippling load, obtained by Euler's formula = $\dfrac{\pi^2 E I}{L^2}$

A little consideration will show, that the value of W_C will remain constant irrespective of the fact whether the column is a long one or short one. Moreover, in the case of short columns, the value of W_E will be very high, therefore the value of $1 / W_E$ will be quite negligible as compared to $1 / W_C$. It is thus obvious, that the Rankine's formula will give the value of its crippling load (*i.e.* W_{cr}) approximately equal to the ultimate crushing load (*i.e.* W_C). In case of long columns, the value of W_E will be very small, therefore the value of $1 / W_E$ will be quite considerable as compared to $1 / W_C$. It is thus obvious, that the Rankine's formula will give the value of its crippling load (*i.e.* W_{cr}) approximately equal to the crippling load by Euler's formula (*i.e.* W_E). Thus, we see that Rankine's formula gives a fairly correct result for all cases of columns, ranging from short to long columns.

From equation (*i*), we know that

$$\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} = \frac{W_E + W_C}{W_C \times W_E}$$

$$\therefore \qquad W_{cr} = \frac{W_C \times W_E}{W_C + W_E} = \frac{W_C}{1 + \dfrac{W_C}{W_E}}$$

Now substituting the value of W_C and W_E in the above equation, we have

$$W_{cr} = \frac{\sigma_c \times A}{1 + \dfrac{\sigma_c \times A \times L^2}{\pi^2 E I}} = \frac{\sigma_c \times A}{1 + \dfrac{\sigma_c}{\pi^2 E} \times \dfrac{A.L^2}{A.k^2}} \qquad \text{... } (\because I = A.k^2)$$

$$= \frac{\sigma_c \times A}{1 + a\left(\dfrac{L}{k}\right)^2} = \frac{\text{Crushing load}}{1 + a\left(\dfrac{L}{k}\right)^2}$$

where σ_c = Crushing stress or yield stress in compression,

A = Cross-sectional area of the column,

a = Rankine's constant = $\dfrac{\sigma_c}{\pi^2 E}$,

L = Equivalent length of the column, and

k = Least radius of gyration.

The following table gives the values of crushing stress and Rankine's constant for various materials.

Table 16.3. Values of crushing stress (σ_c) and Rankine's constant (a) for various materials.

S.No.	Material	σ_c in MPa	$a = \frac{\sigma_c}{\pi^2 E}$
1.	Wrought iron	250	$\frac{1}{9000}$
2.	Cast iron	550	$\frac{1}{1600}$
3.	Mild steel	320	$\frac{1}{7500}$
4.	Timber	50	$\frac{1}{750}$

Problem 19.4 (a). *A simply supported beam of length 4 metre is subjected to a uniformly distributed load of 30 kN/m over the whole span and deflects 15 mm at the centre. Determine the crippling loads when this beam is used as a column with the following conditions :*

(*i*) *one end fixed and other end hinged*

(*ii*) *both the ends pin jointed.* (Annamalai University, 1990)

Sol. Given :

Length, $L = 4$ m $= 4000$ mm

Uniformly distributed load, $w = 30$ kN/m $= 30{,}000$ N/m

$$= \frac{30{,}000}{1000} \text{ N/mm} = 30 \text{ N/mm}$$

Deflection at the centre, $\delta = 15$ mm.

For a simply supported beam, carrying U.D.L. over the whole span, the deflection at the centre is given by,

$$\delta = \frac{5}{384} \times \frac{w \times L^4}{EI}$$

or
$$15 = \frac{5}{384} \times \frac{30 \times 4000^4}{EI}$$

$\therefore$
$$EI = \frac{5}{384} \times \frac{30 \times 4000^4}{15}$$

$$= \frac{5}{384} \times \frac{3 \times 256}{15} \times 10^{13} = \frac{2}{3} \times 10^{13} \text{ N mm}^2.$$

(*i*) *Crippling load when the beam is used as a column with one end fixed and other end hinged*.

The crippling load P for this case in terms of actual length is given by equation (19.4) as

$$P = \frac{2\pi^2 \times EI}{L_e^2}, \text{ where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{2 \times \pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \textbf{8224.5 kN.} \quad \textbf{Ans.}$$

(*ii*) *Crippling load when both the ends are pin-jointed*

This is given by equation (19.1) in terms of actual length as

$$P = \frac{2\pi^2 \times EI}{l^2} \quad \text{where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{\pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \textbf{4112.25 kN.} \quad \textbf{Ans.}$$

Problem 19.5. *A solid round bar 4 m long and 5 cm in diameter was found to extend 4.6 mm under a tensile load of 50 kN. This bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.0.*

Sol. Given :

Actual length of bar, $L = 4$ m $= 4000$ mm

Dia. of bar, $d = 5$ cm

$\therefore$ Area of bar, $A = \frac{\pi}{4} \times 5^2 = 6.25\pi \text{ cm}^2 = 6.25\pi \times 10^2 \text{ mm}^2 = 625\pi \text{ mm}^2$

Extension of bar, $\delta L = 4.6$ mm

Tensile load, $W = 50$ kN $= 50000$ N.

In this problem, the values of Young's modulus (E) is not given. But it can be calculated from the given data.

$$\therefore \text{ Young's modulus, } E = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{\left(\dfrac{\text{Tensile load}}{\text{Area}}\right)}{\left(\dfrac{\text{Extension of bar}}{\text{Length of bar}}\right)}$$

$$\left(\because \text{ Stress} = \frac{\text{Load}}{\text{Area}} \text{ and strain} = \frac{\delta L}{L}\right)$$

$$= \frac{\left(\dfrac{W}{A}\right)}{\dfrac{\delta L}{L}} = \frac{W}{A} \times \frac{L}{\delta L} = \frac{50000}{625\pi} \times \frac{4000}{4.6} = 2.214 \times 10^4 \text{ N/mm}^2.$$

Since the strut is hinged at its both ends,

$\therefore$ Effective length, L_e = Actual length = 4000 mm

Let P = Crippling or buckling load.

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2}$$

$$= \frac{\pi^2 \times 2.214 \times 10^4 \times \frac{\pi}{64} \times 5^4 \times 10^4}{4000 \times 4000} \qquad \left(\because I = \frac{\pi}{64} \times 5^4 \times 10^4 \text{ mm}^4\right)$$

= **4189.99 say 4190 N. Ans.**

And safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{4190}{4} =$ **1047.5 N. Ans.**

Problem 19.13. *The external and internal diameter of a hollow cast iron column are 5 cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the values of $\sigma_c = 550$ N/mm^2 and $a = \frac{1}{1600}$ in Rankine's formula.*

Sol. Given :

External dia., $D = 5$ cm

Internal dia., $d = 4$ cm

$\therefore$ Area, $A = \frac{\pi}{4}(5^2 - 4^2) = 2.25\pi \text{ cm}^2 = 2.25\pi \times 10^2 \text{ mm}^2 = 225\pi \text{ mm}^2$

Moment of Inertia, $I = \frac{\pi}{64}[5^4 - 4^4] = 5.7656\,\pi \text{ cm}^4$

$= 5.7656\pi \times 10^4 \text{ mm}^4 = 57656\pi \text{ mm}^4$

$\therefore$ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\pi}{225\pi}} = 25.625 \text{ mm}$$

Length of column, l = 3 m = 3000 mm

As both the ends are fixed,

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{3000}{2} = 1500$ mm

Crushing stress, σ_c = 550 N/mm^2

Rankine's constant, $a = \frac{1}{1600}$

Let P = Crippling load by Rankine's formula

Using equation (19.9), we have

$$P = \frac{\sigma_c \cdot A}{1+\left(\frac{L_e}{k}\right)^2} = \frac{550 \times 225\pi}{1 + \frac{1}{1600} \times \left(\frac{1500}{25.625}\right)^2}$$

$$= \frac{550 \times 225\pi}{3.1415} = \mathbf{123750 \text{ N.}} \quad \textbf{Ans.}$$

Problem 19.9. *Determine the crippling load for a T-section of dimensions 10 cm × 10 cm × 2 cm and of length 5 m when it is used as strut with both of its ends hinged. Take Young's modulus, $E = 2.0 \times 10^5 \ N/mm^2$.*

Sol. Given :

Dimensions of T-section= 10 cm × 10 cm × 2 cm

Length actual, l = 5 m = 5000 mm

Young's modulus, E = 2.0 × 10^5 N/mm^2.

First of all, calculate the C.G. of the section. The given section is symmetrical about the axis Y-Y, hence the C.G. of the section will lie on Y-Y axis.

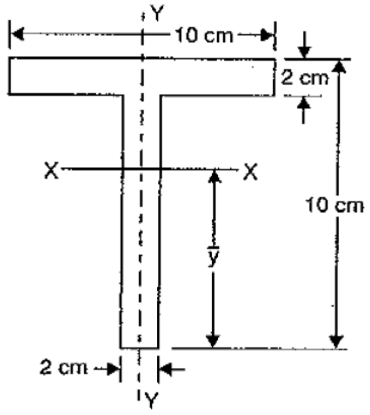


Fig. 19.10

Let $\bar{y}$ = Distance of C.G. of the section from bottom end.

For the flange, we have $a_1 = 10 \times 2 = 20$ cm^2

y_1 = Distance of C.G. of area a_1 from the bottom end
$= 8 + 1 = 9$ cm

For the web, we have $a_2 = 8 \times 2 = 16$ cm^2

y_2 = Distance of C.G. of area a_2 from bottom end $= \frac{8}{2} = 4$ cm

Using the relation, $\quad \bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$

$$= \frac{20 \times 9 + 16 \times 4}{20 + 16} = \frac{180 + 64}{36} = 6.777 \text{ cm}$$

Moment of inertia of the section about the axis X-X,

$$I_{XX} = \left(\frac{10 \times 8^3}{12} + 20 \times 2.223^2\right) + \left(\frac{2 \times 8^3}{12} + 16 \times 2.777^2\right)$$

$$= (6.667 + 98.834) + (85.333 + 123.387) = 314.221 \text{ cm}^4.$$

Moment of inertia of the section about the axis Y-Y,

$$I_{YY} = \frac{2 \times 10^3}{12} + \frac{8 \times 2^3}{12} = 166.67 + 5.33 = 172 \text{ cm}^4.$$

Least value of moment of inertia is about Y-Y axis

$$\therefore \quad I = 172 \text{ cm}^4 = 172 \times 10^4 \text{ mm}^4$$

Since the strut is hinged at both of its end

$\therefore$ Effect length, $L_e = l = 5000$ mm

Let P = Crippling load

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 2.0 \times 10^5 \times 172 \times 10^4}{5000^3} = \mathbf{135805.7 \text{ N.}} \quad \textbf{Ans.}$$

Problem 19.17. *Find the Euler crushing load for a hollow cylindrical cast iron column 20 cm external diameter and 25 mm thick if it is 6 m long and is hinged at both ends. Take $E = 1.2 \times 10^6$ N/mm².*

Compare the load with the crushing load as given by the Rankine's formula, taking $\sigma_c = 550$ N/mm² and $a = \frac{1}{1600}$; for what length of the column would these two formulae give the same crushing load ? (AMIE, Winter 1984)

Sol. Given :

External dia., $D = 20$ cm

Thickness, $t = 25$ mm $= 2.5$ cm

$\therefore$ Internal dia., $d = (D - 2 \times t) = 20 - 2 \times 2.5 = 15$ cm.

Area, $$A = \frac{\pi}{4}(20^2 - 15^2) = \frac{175\,\pi}{4} = 137.44 \text{ cm}^2 = 13744 \text{ mm}^2$$

Moment of inertia, $$I = \frac{\pi}{64}[20^4 - 15^4] = \frac{\pi}{64}(160000 - 50625)$$
$$= 5368.93 \text{ cm}^4 = 53689300 \text{ mm}^4$$

$\therefore$ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{53689300}{15744}} = 62.5 \text{ mm}$$

Length of column, $l = 6$ m $= 6000$ mm

End conditions = Both ends are hinged

$\therefore$ Effective length, $L_e = l = 6000$ mm

Value of $E = 1.2 \times 10^5$ N/mm².

Euler's crushing load is given by equation (19.5),

$$P = \frac{\pi^2 EI}{L_e^2}$$
$$= \frac{\pi^2 \times 1.2 \times 10^5 \times 53689300}{6000^2} = \mathbf{1766307\ N.}\quad \textbf{Ans.}$$

Crushing load by Rankine's formula

The value of $\sigma_c = 550$ N/mm²

Value of $a = \frac{1}{1600}$

Let P = Crushing load by Rankine's formula

Using equation (19.9),

$$P = \frac{\sigma_c \,.\, A}{1 + a\,.\left(\frac{L_e}{k}\right)^2} = \frac{550 \times 13744}{1 + \frac{1}{1600} \times \left(\frac{6000}{62.5}\right)^2} = \mathbf{1118224.8\ N.\ Ans.}$$

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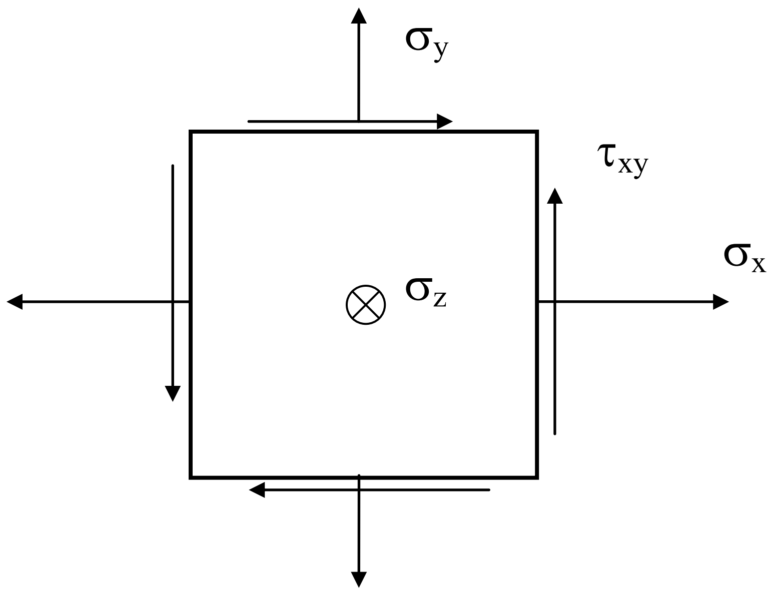
SCHOOL OF MECHANICAL ENGINEERING
DEPARTMENT OF AERONAUTICAL ENGINEERING

UNIT 5-INTRODUCTION TO AIRCRAFT STRUCTURES-SAEA1305

Strength of Materials and Failure Theories

State of Stress



This is a 2D state of stress – only the independent stress components are named. A single stress component σ_z can exist on the z-axis and the state of stress is still called 2D and the following equations apply. To relate failure to this state of stress, three important stress indicators are derived: Principal stress, maximum shear stress, and VonMises stress.

Principal stresses:

$$\boxed{\begin{aligned}\sigma_1,\sigma_2 &= \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \sigma_3 &= Given\ or\ known\end{aligned}}$$

If σ_y=0 (common case) then

$$\boxed{\begin{aligned}\sigma_1,\sigma_2 &= \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} \\ \sigma_3 &= Given\ or\ known\end{aligned}}$$

If $\sigma_x = \sigma_y = 0$ then $\sigma_1 = \sigma_2 = +/- \tau_{xy}$. If $\sigma_y = \tau_{xy} = 0$, then $\sigma_1 = \sigma_x$ and $\sigma_2 = 0$.

Maximum shear stress – Only the absolute values are important.

$$Max(\tau_{\max,12}, \tau_{\max,13}, \tau_{\max,23})$$

$$\tau_{\max,12} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{\max,1,3} = \frac{\sigma_1 - \sigma_3}{2} \qquad \tau_{\max,23} = \frac{\sigma_2 - \sigma_3}{2}$$

If σ_3=0, the

$$\tau_{\max,12} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{\max,1,3} = \frac{\sigma_1}{2} \qquad \tau_{\max,23} = \frac{\sigma_2}{2}$$

The Vom Mises stress:

$$\sigma_v = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}}$$

When σ_3=0, the von Mises stress is:

$$\sigma_v = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}$$

When only σ_x, and τ_{xy} are present (as in combined torsion and bending/axial stress or pure torsion), there is no need to calculate the principal stresses, the Von Mises stress is:

$$\sigma_v = \sqrt{\sigma_x^2 + 3\tau_{xy}^2}$$

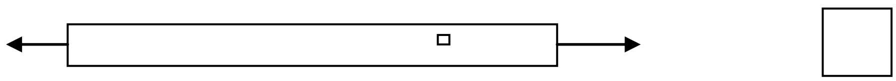
Note that in pure shear or pure torsion σ_x =0. If σ_x =0, then

$$\sigma_v = \sqrt{3\tau_{xy}^2} = \sqrt{3}\tau_{xy}$$

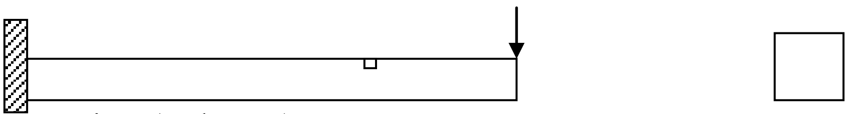
According to distortion energy theory, yielding occurs when σ_v reached the yield strength S_y. Therefore in pure shear, yielding occurs when τ_{xy} reaches 58% of S_y.

Common loading applications and stresses (when oriented properly)

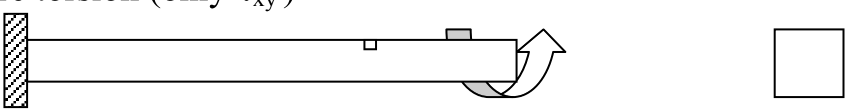
Direct Tension/Compression (only σ_x)



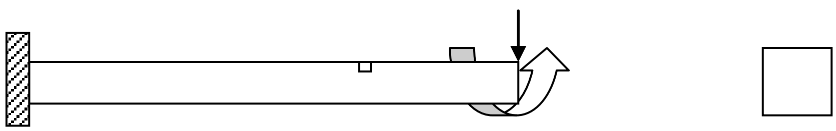
Beam bending (only σ_x on top/bottom)



Pure torsion (only τ_{xy})



Rotating shafts (bending + torsion) – (σ_x and τ_{xy})



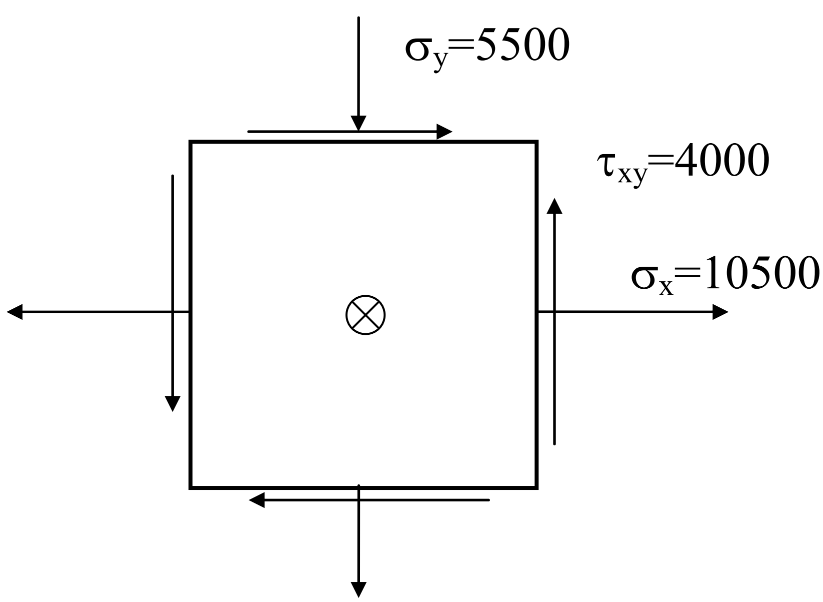
Problem #S1

A member under load has a point with the following state of stress:

$$\sigma_x = 10500 \quad psi, Tensile \quad \sigma_y = 5500 \quad psi, Compressive$$

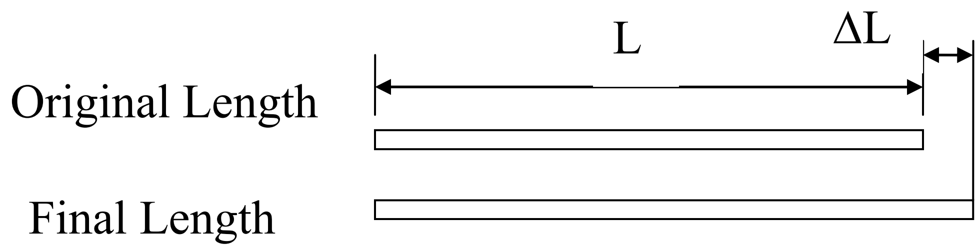
$$\tau_{xy} = 4000 \quad psi \quad \sigma_3 = 0$$

Determine $\sigma 1$, $\sigma 2$, τ_{max} (Ans: 11444 tensile, 6444 Compressive, 8944 psi)



Strain (one dimensional)

A bar changes length under the influence of axial forces and temperature changes.



Total strain definition:

$$\varepsilon_{total} = \varepsilon_t = \frac{\Delta L}{L}$$

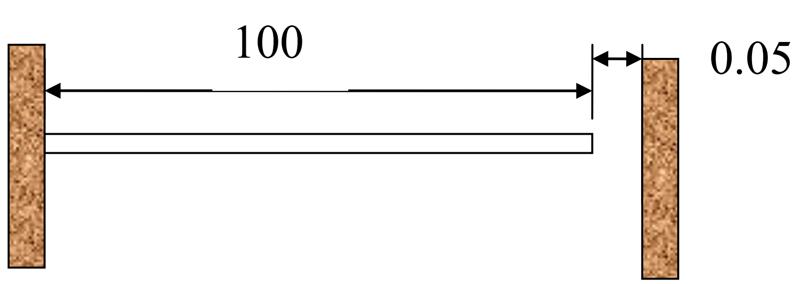
Total strain is a combination of mechanical and thermal strains:

$$\varepsilon_t = \varepsilon_M + \varepsilon_T = \frac{F}{EA} + \alpha \Delta T$$

Both the mechanical and the thermal strains are algebraic values. ΔT is positive for an increase in temperature. F is positive when it is a tensile force.

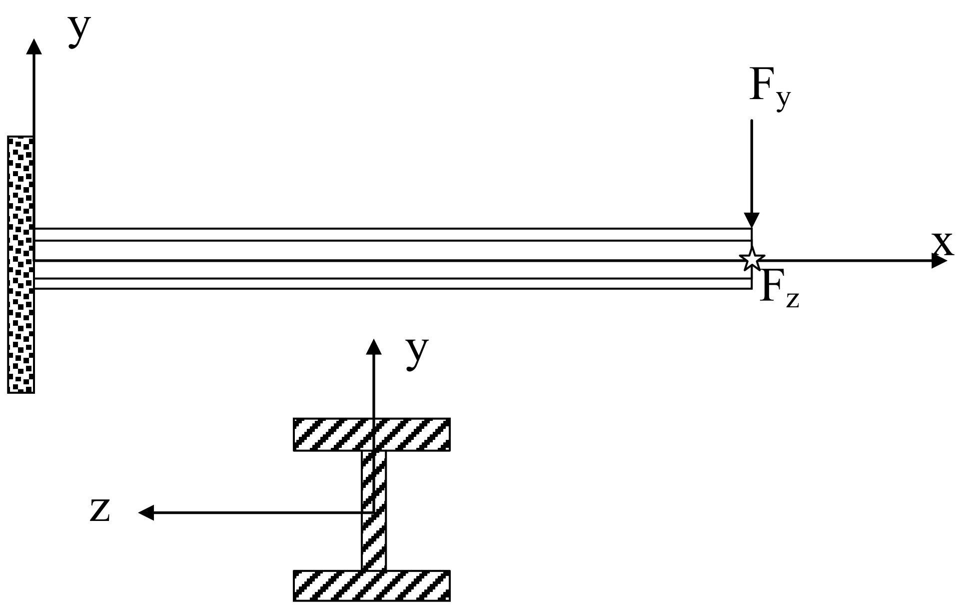
Problem #S2

The end of the steel bar has a gap of 0.05" with a rigid wall. The length of the bar is 100" and its cross-sectional area is 1 in^2. The temperature is raised by 100 degrees F. Find the stress in the bar. ANS: 4500 Psi Comp.



Bending of "straight" beams

Bending formulas in this section apply when the beam depth (in the plane of bending) is small (by at least a factor or 20) compared to the beam radius of curvature.



Bending stress for bending about the Z-axis:

$$\sigma_x = \frac{M_z y}{I_z} \qquad M_z = F_y L$$

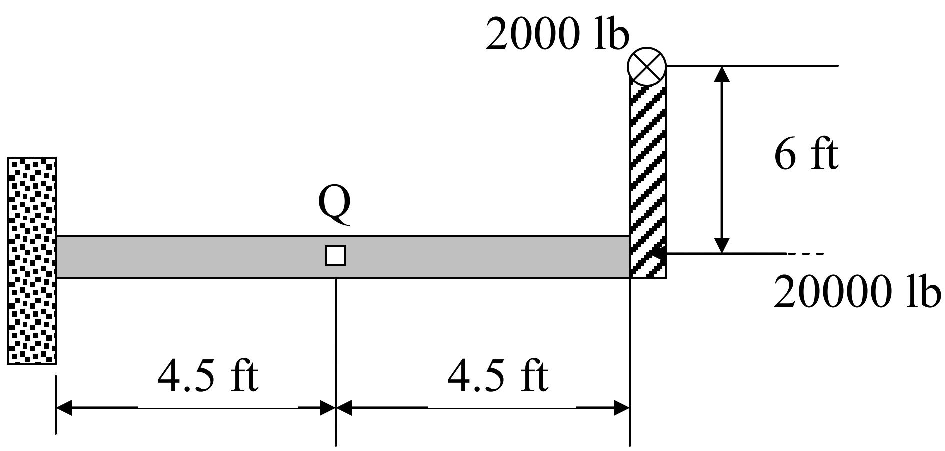
I_z is area moments of inertias about the z and represents resistance to rotation about z axis. Bending stress for bending about the Y-axis:

$$\sigma_x = \frac{M_y z}{I_y} \qquad M_y = F_z L$$

I_y is area moments of inertias about the y and represents resistance to rotation about y axis. Use tables to look up moments of inertia for various cross-sections. The parallel axis theorem can be used to find moment of inertia w/r a parallel axis.

Problem #S3

The solid circular steel bar with R=2" (diameter 4") is under two loads as shown. Determine the normal stress σ_x at point Q. Point Q is on the surface closest to the observer and the 2000 lb goes into the paper.



[The most common stress analysis problems in exams involve simple bending, simple torsion, or a combination of the two. This is an example of the combination – the torsion analysis would be treated later.]

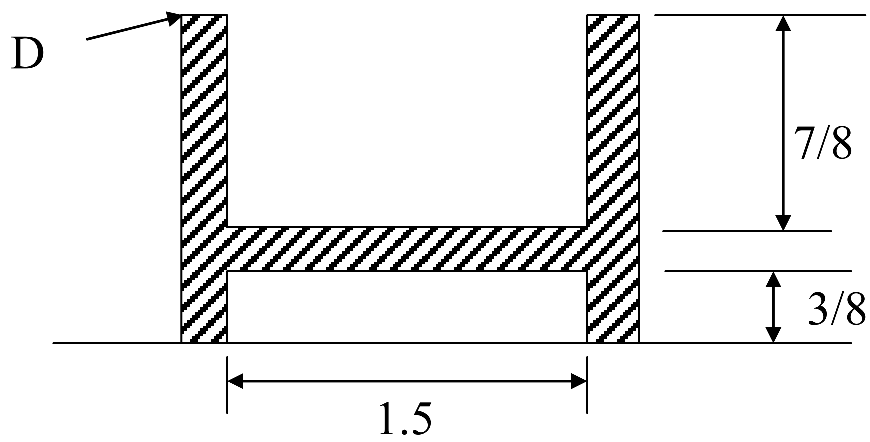
Answer: 15600 psi

Problem #S4

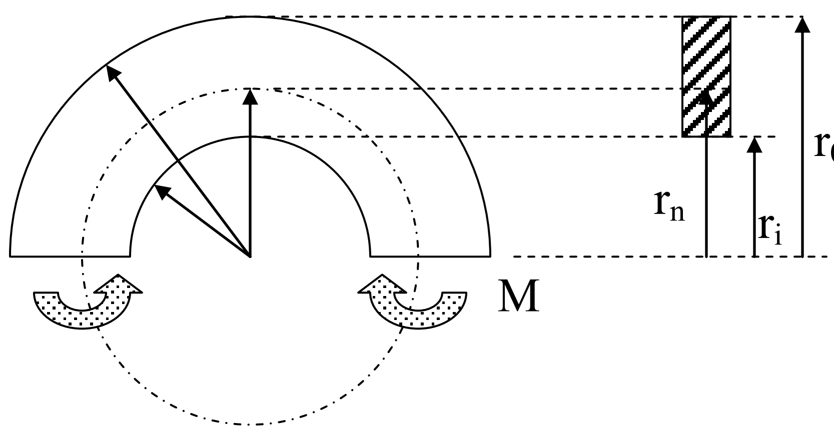
A beam with the cross-section shown is under a bending moment of $FL=M_z=10000$ lb-in acting on this cross-section. The thicknesses of all webs are 0.25 inches.

Determine:
a) The location of the neutral axis (0.667 from bottom)
b) The moment of inertia about the z-axis (0.158 in^4)
c) Bending stress at D (52700 psi)
d) Solve part b) if the cross-section was H-shaped

[Finding area moments of inertias are popular exam questions. This problem is a little longer than typical ones but it is a good preparation exercise]



Bending Stresses in Curved Beams



Maximum bending stresses occur at r_i and r_o - The magnitude is largest at r_i

$$\sigma_i = \frac{M(r_n - r_i)}{eAr_i}$$

The stress at the outer surface is similar but with r_o replacing r_i. In this expression, M is the bending moment at the section, A is the section area and e is the distance between the centroidal axis and neutral axis. These two axes were the same in straight beams.

$$e = \bar{r} - r_n$$

The radius of the neutral axis for a rectangular section can be obtained as:

$$r_n = \frac{r_o - r_i}{\ln(r_o / r_i)}$$

Refer to Shigley or other design handbooks for other cross-sections:

- Circular
- Trapezoidal
- T-shaped
- Hollow Square
- I-Shaped

Note: When finding bending moment of forces, the exact moment arm is r_n but the centroidal radius is also close enough to be a good approximation.

For a circular shape with a radius of R, r_n is:

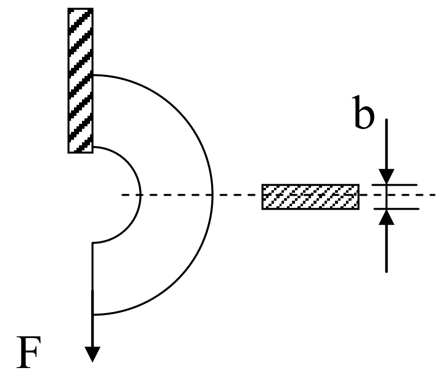
$$r_n = \frac{R^2}{2(r_c - \sqrt{r_c^2 - R^2})}$$

Where $r_c = R + r_i$

Check Shigley for other cross-section forms such as T-shaped beams.

Problem #S5

Given:	$r_i = 2$ in	$r_o = 4$ in
	$b = 1$ in	
	$F = 10000$ lb	
Find:	maximum bending stress	
	Maximum total stress	
Answer:	57900 psi (bending only)	
	62900 psi (total)	



Torque, Power, and Torsion of Circular Bars

Relation between torque, power and speed of a rotating shaft:

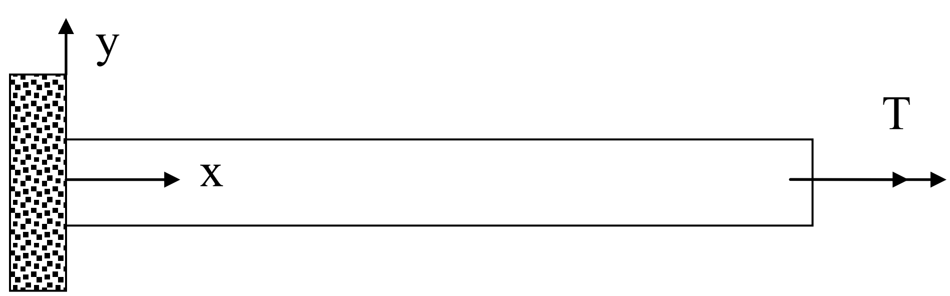
$$H = \frac{Tn}{63000}$$

H is power in Hp, *T* is torque in lb-in, and *n* is shaft speed in rpm. In SI units:

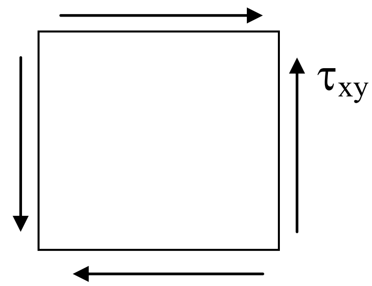
$$H = T\omega$$

H is power in Watts, *T* is torque in N-m, and ω is shaft speed in rad/s.

The shear stress in a solid or tubular round shaft under a torque:



The shear stress is a maximum on the surface of the bar. The state of stress can be represented as a case of pure shear:



The shear stress is:

$$\tau = \frac{Tr}{J}$$

J is the area polar moment of inertia and for a solid (d_i=0) or hollow section,

$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

The Von Mises stress in pure shear is:

$$\sigma_V = \sqrt{3\tau_{xy}^2} = \sqrt{3}\tau_{xy}$$

When the behavior is ductile, yielding occurs when σ_v reaches the yield strength of the material. This is based on the distortion energy theory which is the best predictor of yielding. According to this, yielding occurs when:

$$\sigma_V = S_y \Rightarrow \quad \sqrt{3}\tau_{xy} = S_y$$

$$\Rightarrow \quad \tau_{xy} = \frac{1}{\sqrt{3}} S_y \quad Or \quad \tau_{xy} = 0.58 S_y$$

This predicts that yielding in pure shear occurs when the shear stress reaches 58% of the yield strength of the material.

The angle of rotation of a circular shaft under torque

$$\theta = \frac{TL}{GJ}$$

The angle of rotation is in radians, L is the length of the bar, and G is a constant called the shear modulus. The shear modulus can be obtained from the modulus of elasticity E, and the poisson's ration ν:

$$G = \frac{E}{2(1+\nu)}$$

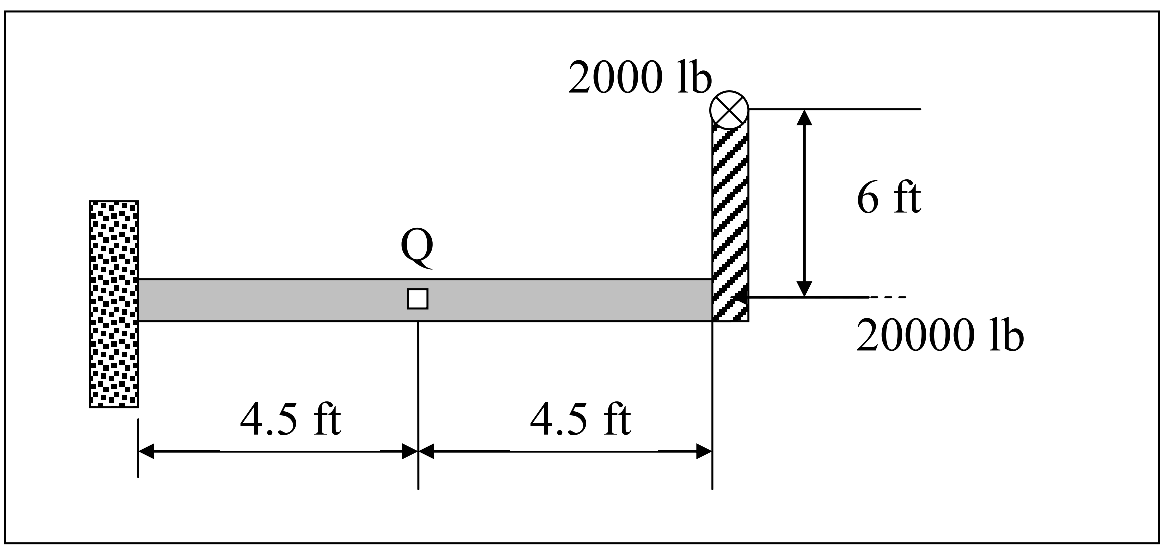
For steels, this value is $11.5*10^6$ psi.

Problem #S6

Consider the loading situation shown in Problem #S3. Determine:

a) the torsional shear stress for an element on the shaft surface.
b) The maximum shear stress at point Q. Use the given (as answer in Problem #S3) maximum normal stress at point Q to estimate the maximum shear stress.

Answers: a) 11460, b)13860



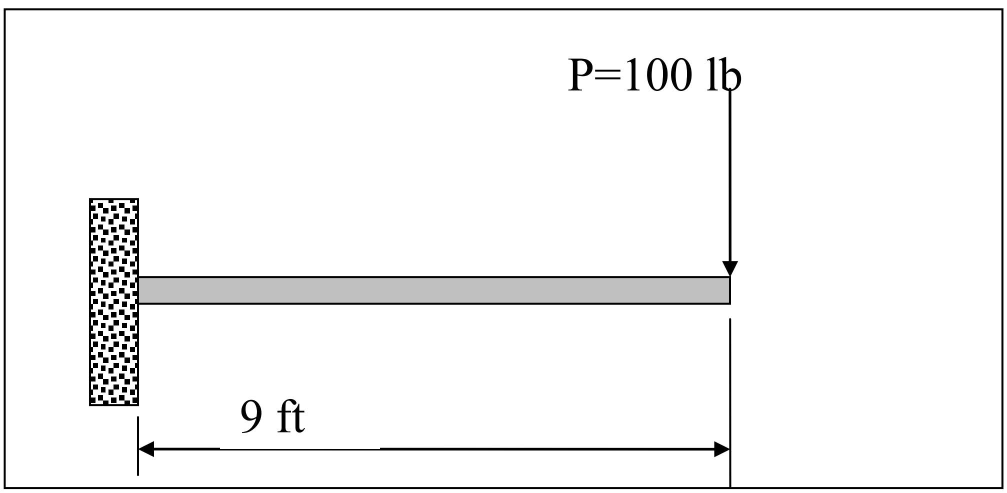
Beam and Frame Deflection - Castigliano's Theorem

"When a body is elastically deflected by any combination of loads, the deflection at any point and in any direction is equal to the rate of change of strain energy with respect to the load located at that point and acting in that direction" – even a fictitious load.

When torsion or bending is present, they dominate the strain energy. The deflection due to torsional and bending loads is:

$$\delta = \int_0^L \frac{T \frac{\partial T}{\partial F}}{GJ} dx + \int_0^L \frac{M \frac{\partial M}{\partial F}}{EI} dx$$

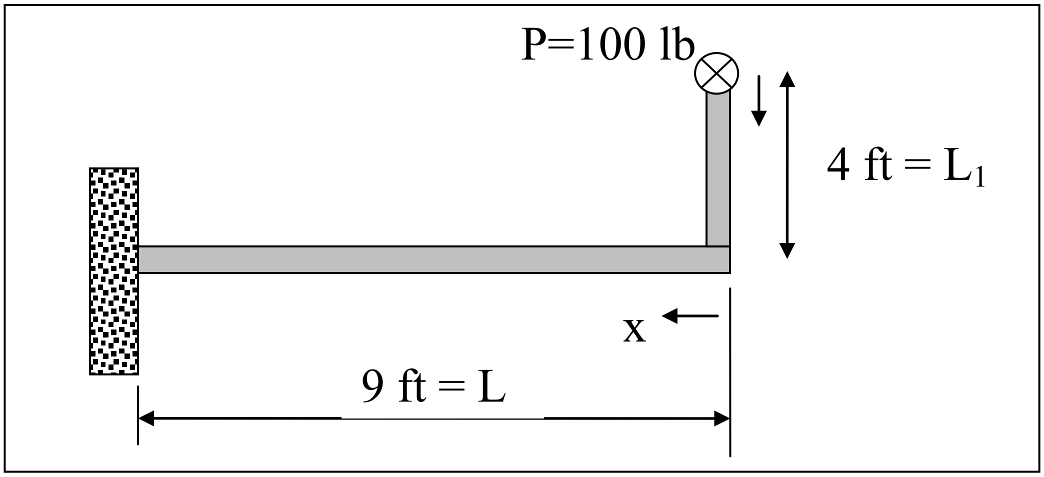
Example: Solid steel tube with ID=1.75 and OD= 2.75 inches. Determine the deflection of the end of the tube.



$$\delta = \int_0^L \frac{M \frac{\partial M}{\partial F}}{EI} dx \quad where \quad M = Px$$

$$\delta = \int_0^L \frac{Px(x)}{EI} dx = \frac{PL^3}{3EI} = \frac{100(9*12)^3}{3(30*10^6)(2.347)} = 0.6\ in$$

Example: Solid steel tube with ID=1.75 and OD= 2.75 inches. Determine the deflection of the end of the tube.



Deflection from bending in the 9-ft span

$$\delta = \int_0^L \frac{M\frac{\partial M}{\partial F}}{EI}dx \quad where \quad M = Px$$

$$\delta = \int_0^L \frac{Px(x)}{EI}dx = \frac{PL^3}{3EI} = \frac{100(9*12)^3}{3(30*10^6)(2.347)} = 0.596$$

Deflection from bending in the 4-ft span

$$\delta = \int_0^{L_1} \frac{M\frac{\partial M}{\partial F}}{EI}dx_1 \quad where \quad M = Px_1$$

$$\delta = \int_0^{L_1} \frac{Px_1(x_1)}{EI}dx_1 = \frac{PL_1^{\ 3}}{3EI} = \frac{100(4*12)^3}{3(30*10^6)(2.347)} = 0.157$$

Deflection from torsion in the 9-ft span

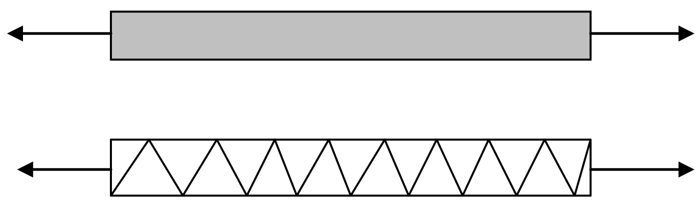
$$\delta = \int_0^L \frac{T\frac{\partial T}{\partial F}}{EI}dx \quad where \quad T = PL_1$$

$$\delta = \int_0^L \frac{PL_1(L_1)}{EI}dx = \frac{PL_1^{\ 2}}{EI}L = \frac{100(4*12)^2(9*12)}{(30*10^6)(2.347)} = 0.353$$

Total Deflection = 0.596 + 0.157 + 0.353 = 1.1 in

Deflections, Spring Constants, Load Sharing

Axial deflection of a bar due to axial loading

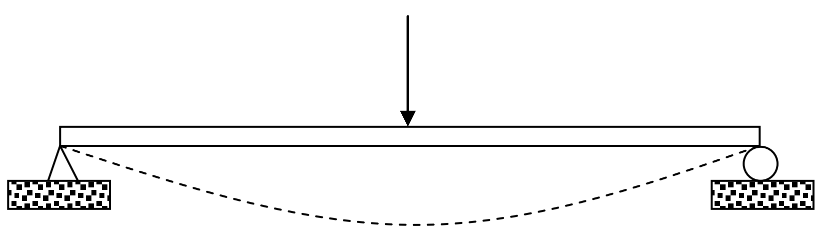


The spring constant is:

$$K = \frac{EA}{L}$$

Lateral deflection of a beam under bending load

A common cases is shown. The rest can be looked up in deflection tables.



$$K = \frac{48EI}{L^3}$$

For cantilevered beams of length L:

$$K = \frac{3EI}{L^3}$$

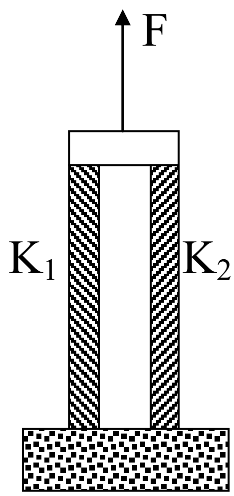
Torsional stiffness of a solid or tubular bar is:

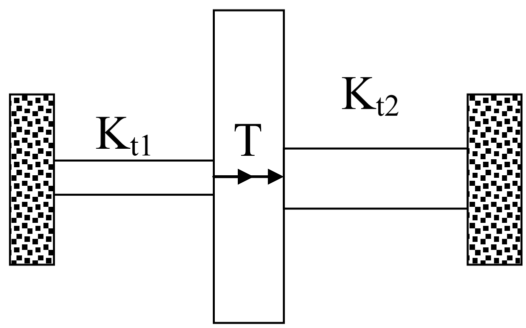
$$K_t = \frac{GJ}{L}$$

The units are in-lbs per radian.

Load Distribution between parallel members

If a load (a force or force couple) is applied to two members in parallel, each member takes a load that is proportional to its stiffness.





The force F is divided between the two members as:

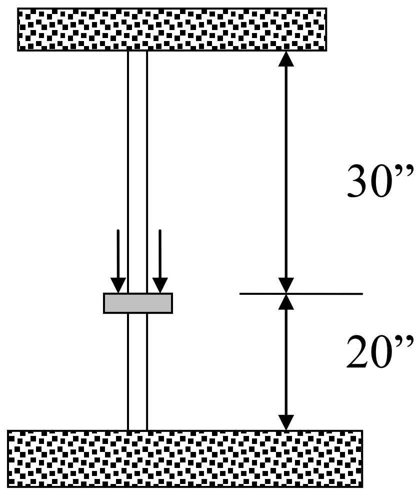
$$F_1 = \frac{K_1}{K_1 + K_2} F \qquad F_2 = \frac{K_2}{K_1 + K_2} F$$

The torque T is divided between the two bars as:

$$T_1 = \frac{K_{t1}}{K_{t1} + K_{t2}} T \qquad T_2 = \frac{K_{t2}}{K_{t1} + K_{t2}} T$$

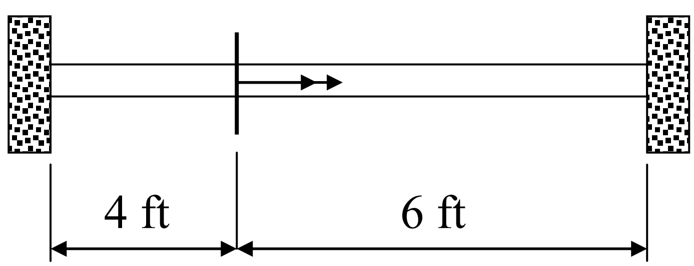
Problem #S7

A one-piece rectangular aluminum bar with 1 by ½ inch cross-section is supporting a total load of 800 lbs. Determine the maximum normal stress in the bar.



Answer: 960 psi

Problem #S8
A solid steel bar with 1” diameter is subjected to 1000 in-lb load as shown. Determine the reaction torques at the two end supports.



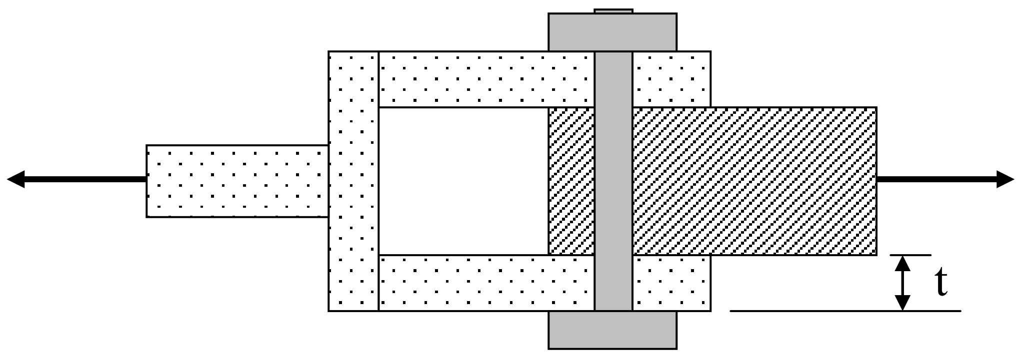
Answer: 600 on the left, 400 on the right.

Direct shear stress in pins

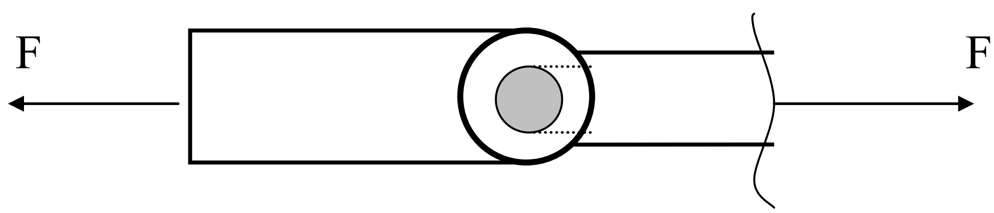
Pins in double shear (as in tongue and clevis) is one of the most common method of axial connection of parts.

The shear stress in the pin and bearing stresses are approximately uniformly distributed and are obtained from:

$$\tau = \frac{F}{2A_{pin}} \qquad \sigma_b = \frac{F}{2td}$$



The clevis is also under tear-out shear stress as shown in the following figure (top view):



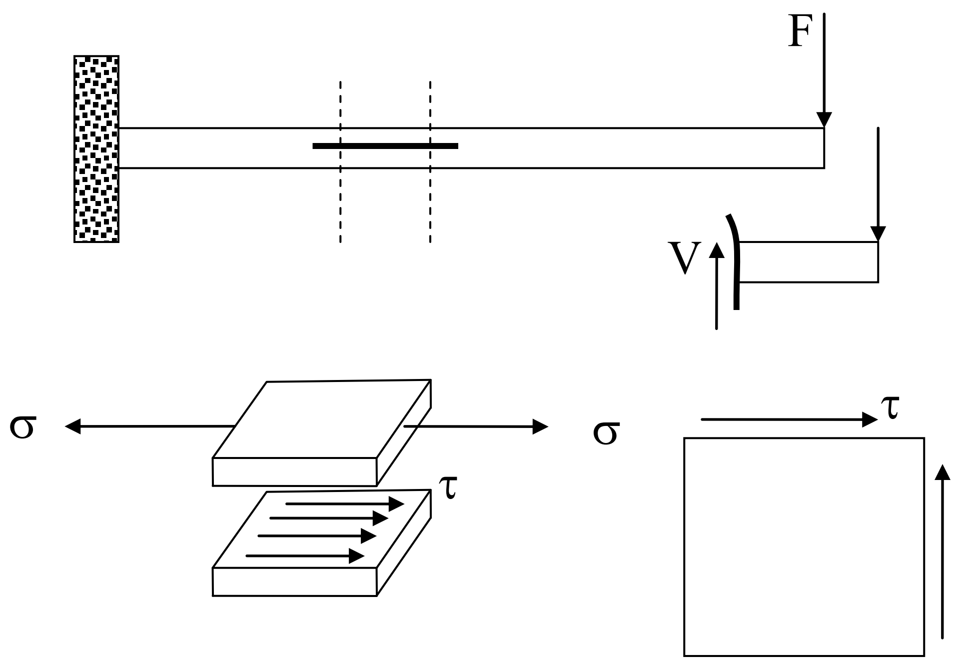
Tear-out shear stress is:

$$\tau = \frac{F}{4A_{clevis}}$$

In this formula $A_{clevis}=t(R_o-R_i)$ is approximately and conservatively the area of the dotted cross-section. R_o and R_i are the outer and inner radii of the clevis hole. Note that there are 4 such areas.

Shear stresses in beams under bending forces

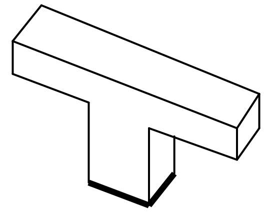
When a beam is under a bending force, its "layers" like to slide on one-another as a deck of cards would do if bent. Since the beam "layers" can not slide relative to each other, a shear stress develops within the beam just as shear stresses develop between card faces if they were glued together. This is shown below. The shear stress in beams is relatively small and can be ignored for one-piece beams. But for composite beams that are glued, welded, riveted, bolted, or somehow attached together, this shear stress can be significant enough to tear off the welding or bolts.

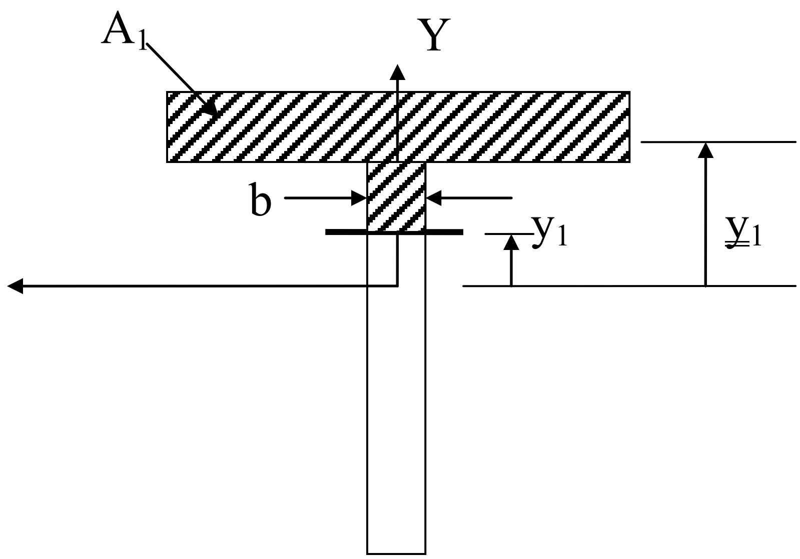


The value of the shear stress depends on the following:

- The shear force *V* acting on the cross-section of interest. In the above figure, the shear force is F in all cross-sections. The larger the force, the larger the stress.
- The width of the beam *b* at the cross-section. The wider the beam, the lower the stress.
- The area moment of inertia of the entire cross-section w/r to neutral axis. The more moment of inertia, the less the stress.
- The last parameter is Q which is the "bending stress balance factor". The more Q, the more bending stress has to be balanced by shear.

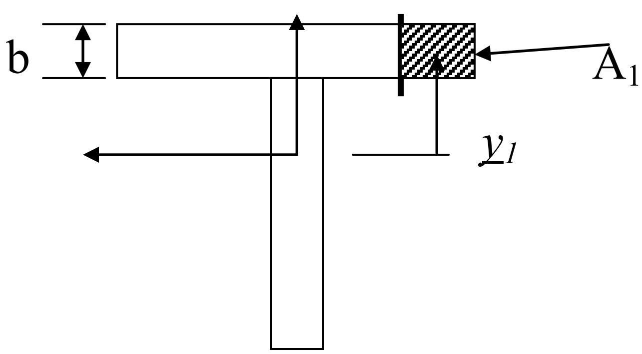
$$\tau = \frac{VQ}{I_z b} \qquad Q = A_1\bar{y}_1$$



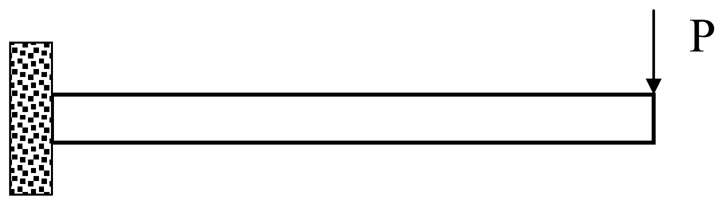


A_1 is the area of the cross-section left hanging and $\bar{y}_1$ is the distance between the centroid of A_1 and the neutral axis (which is the same as the centroidal axis of the entire cross-section).

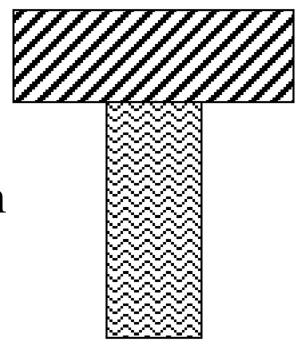
The following is another example.



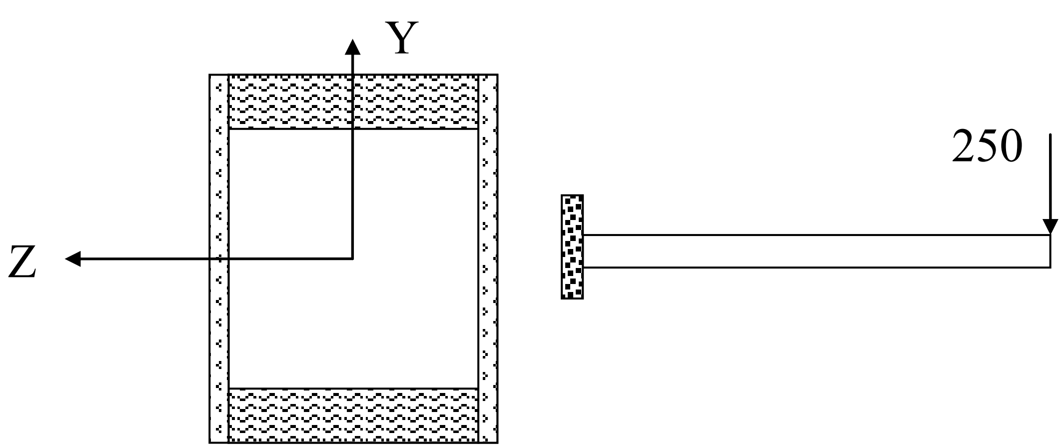
Problem # S9 : 2 by 4 Pine wood boards have been glued together to create a composite beam as shown. Assume the dimensions are 2" by 4" (in reality they are less than the nominal value). If the shear strength of the glue is 11 psi, determine the largest load P that the beam can carry w/o glue failure. Assume beam is long enough for the classical beam theory to apply. Do not consider failure due to bending stresses. Answer:90.4 lbs



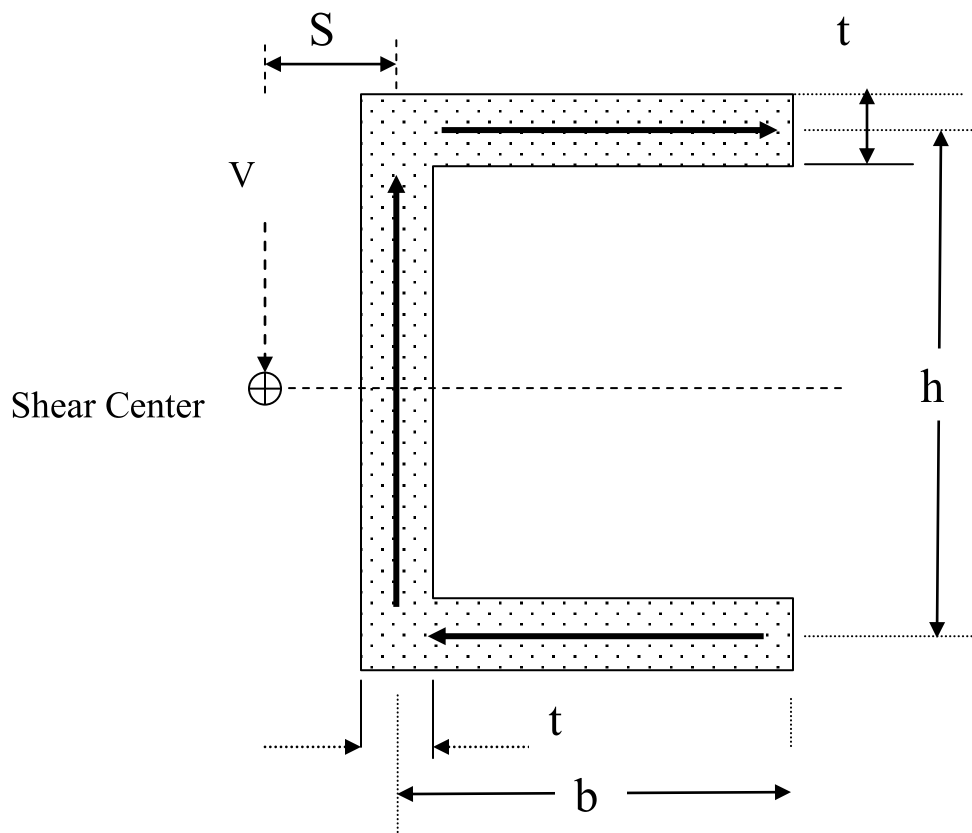
Cross-section



Problem #S10: A composite beam is glued as shown. Horizontal members are 1 by 6 inch and the vertical members are ¼ by 10 inch. Transverse load at this cross-section is F=250 lbs. Determine the required minimum glue strength in shear. Answer: 11.8 psi



Shear Center of a C-Channel



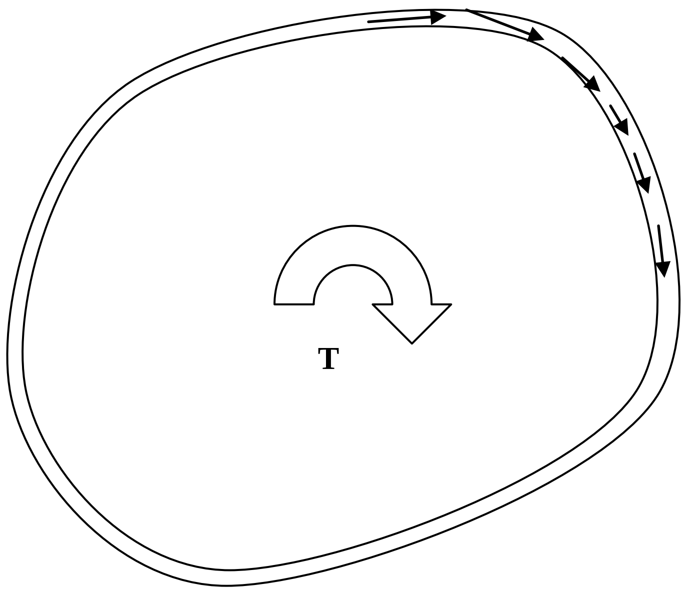
Transverse loads on non-symmetric sections can create twisting torques and warp beam flanges. If such transverse loads are applied at an offset location, the shear forces balance and do not twist the beam. This location is called the Shear Center. For the C-channel shown

$$S = \frac{h^2 b^2 t}{4I}$$

For a semi-circular cross-section, the shear center is at:

$$s = r\left(\frac{4}{\pi} - 1\right)$$

Torsion of Thin-walled Tubes



Shear stress in thin-walled tubes (left for closed tubes – right for open tubes)

$$\tau = \frac{T}{2At} \qquad\qquad \tau = \frac{3T}{St^2}$$

Where T is the torque, t is the wall thickness, S is the perimeter of the midline, and A is the cross-sectional area defined by the midline of the tube wall. Using area or perimeter of the inner or outer boundary is also acceptable since the wall thickness is small.

For a member of constant cross-section, the angle of twist in radians is

$$\phi = \frac{TSL}{4A^2Gt}$$

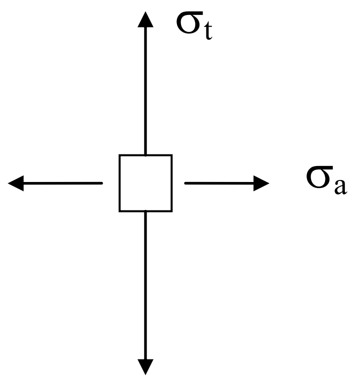
Where S is the perimeter of the midline, L is the length of the beam, and G is shear modulus. There is a similar formula for open tubes. [Shigley]

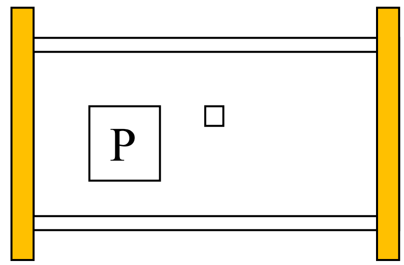
Problem #S11: A square tube of length 50 cm is fixed at one end and subjected to a torque of 200 Nm. The tube is 40 mm square (outside dimension) and 2 mm thick. Determine the shear stress in the tube and the angle of its rotation.

Answer: Stress 34.6 Mpa
Rotation (twist of the beam end): 0.011 radians or 0.66 degrees

Stress in Thin-Walled Cylinders

If the thickness t is less than 1/20th of the mid radius of the pressure vessel, the stresses can be closely approximated using the following simple formulas. The critical stress point in pressure vessels is always on the inner surface.





The tangential or hoop stress is:

$$\sigma_t = \frac{Pd_i}{2t}$$

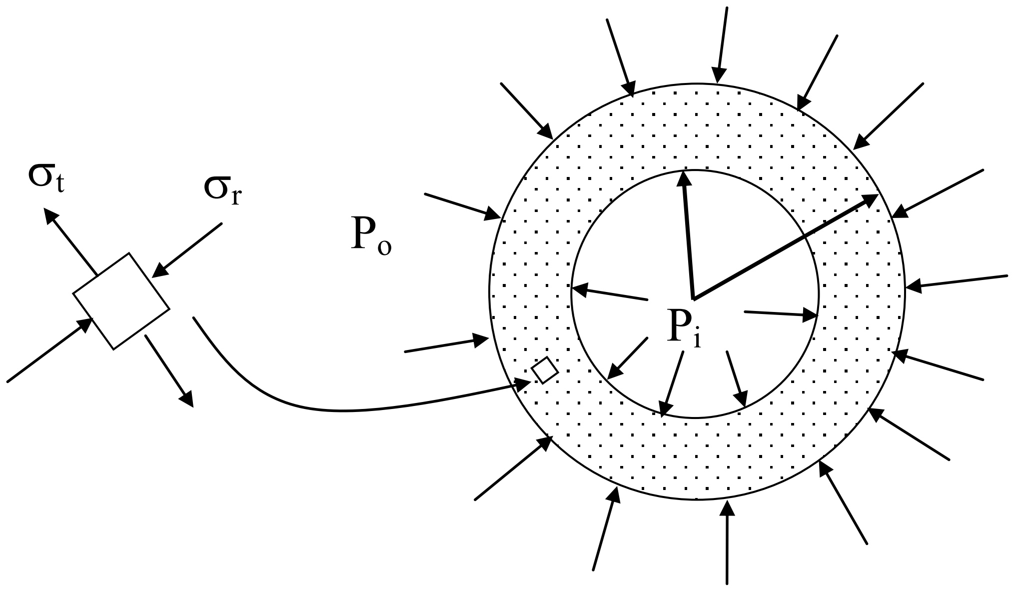
P is the internal pressure, t is the wall thickness, and d_i is the inner diameter. The axial stress is:

$$\sigma_a = \frac{Pd_i}{4t}$$

The radial stress on the inner surface is P which is ignored as it is much smaller than the hoop stress.

Stresses in Thick-walled Cylinders

In thick-walled cylinders the tangential and radial stresses vary exponentially with respect to the radial location within the cylinder and if the cylinder is closed the axial stress would be a constant. All the three stresses are principal stresses when stress element is cut as a pie piece – they occur on surfaces on which shear stresses are zero. The critical stress point is on the inner surface.



The tangential stress:

$$\sigma_t = \frac{P_i r_i^2 - P_o r_o^2 - r_i^2 r_o^2 \left(\dfrac{P_o - P_i}{r^2} \right)}{r_o^2 - r_i^2}$$

The radial stress is:

$$\sigma_r = \frac{P_i r_i^2 - P_o r_o^2 + r_i^2 r_o^2 \left(\dfrac{P_o - P_i}{r^2} \right)}{r_o^2 - r_i^2}$$

When the external pressure is zero, the stresses on the inner surface are:

$$\sigma_t = \frac{P_i (r_i^2 + r_o^2)}{r_o^2 - r_i^2}$$

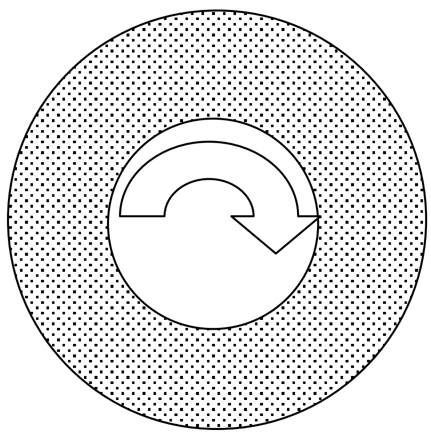
$$\sigma_r = \frac{P_i (r_i^2 - r_o^2)}{r_o^2 - r_i^2} = -P_i$$

When the ends are closed, the external pressure is often zero and the axial stress is

$$\sigma_a = \frac{P_i r_i^2}{r_o^2 - r_i^2}$$

Problem #S12: A steel cylinder with a yield strength of 57 ksi is under external pressure only. The dimensions are: ID=1.25" and OD=1.75". If the external pressure is 11200 psi, what is the factor of safety guarding against yielding. Use the distortion energy theory. Answer: 1.25.

Stresses in rotating disks



A rotating disk develops substantial inertia-caused stresses at high speeds. The tangential and radial stresses in a disk rotating at ω rad/sec is as follows:

$$\sigma_t = \rho\omega^2 (\frac{3+\nu}{8})(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2)$$

and

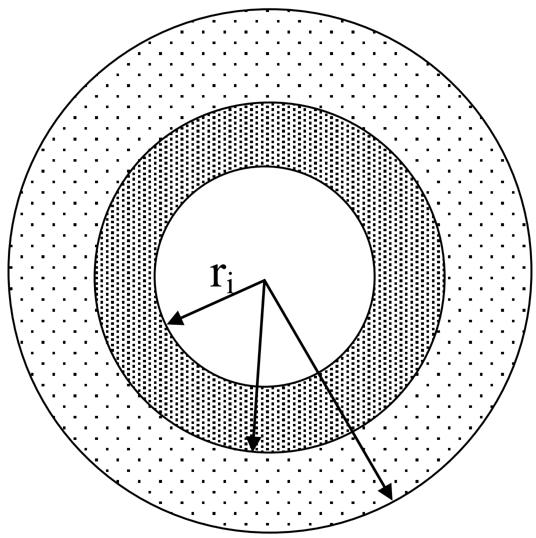
$$\sigma_r = \rho\omega^2 (\frac{3+\nu}{8})(r_i^2 + r_o^2 - \frac{r_i^2 r_o^2}{r^2} - r^2)$$

where ρ is the mass density and ν is the Poisson's ratio. The disk thickness is to be less than 1/10 of the outer radius.

Problem #S13: A disk is rotating at 2069 rpm. The disk's OD=150 mm and its ID is 25 mm. The Poisson's ratio is 0.24 and the disk's mass density is 3320 kg/m^3. Determine the maximum tensile stress in the disk as a result of rotation. Answer: 0.715 Mpa.

Interface pressure as a result of shrink or press fits

When the internal pressure is high, shrink-fit cylinders lower the induced stresses. When two cylinders with a radial interference of δ_r are press or shrink fitted, an interface pressure develops as follows:



The interface pressure for same material cylinders with interface nominal radius of R and inner and outer radii of r_i and r_o:

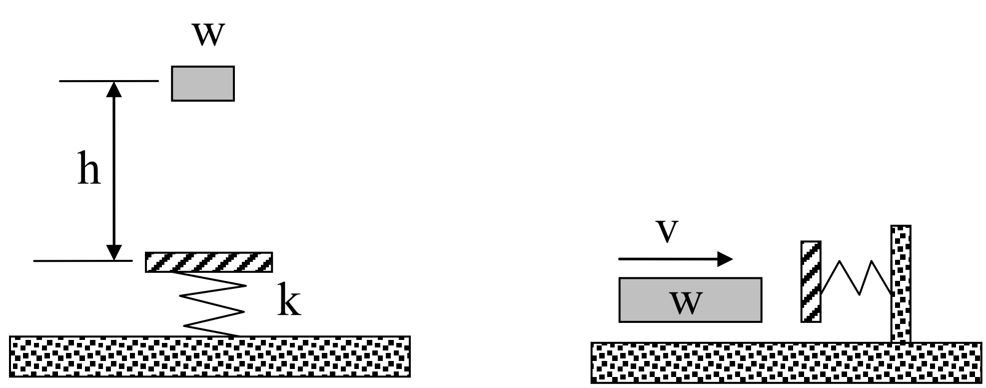
$$P = \frac{E\delta_r}{R}\left(\frac{(r_o^2 - R^2)(R^2 - r_i^2)}{2R^2(r_o^2 - r_i^2)}\right)$$

Problem #S14: A collar is press-fitted on a solid shaft. Both parts are made of steel. The shaft diameter is 40.026 mm and the collar diameter is 40 mm. The outer diameter of the collar is 80 mm. Find the interface pressure. Answer: 50 Mpa.

When both shrink fit and internal pressure is combined, the method of superposition must be used.

Impact Forces

The equivalent static load created by an object falling and impacting another object can be very large. Equations of energy in dynamics can be used to determine such loads. Two common cases involve an object falling from a height and a speeding object impacting a structure. In both cases the damping is assumed to be small.



For a falling weight (ignoring the energy loss during impact):

$$F_e = \left(1 + \sqrt{1 + \frac{2hk}{W}}\right)W$$

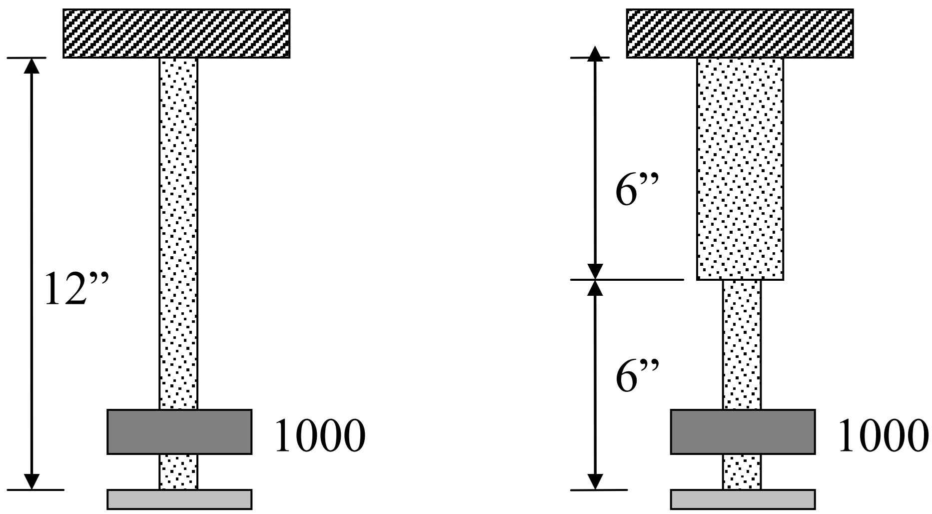
$$F_e = \left(1 + \sqrt{1 + \frac{2h}{\delta_{st}}}\right)W$$

If $h=0$, the equivalent load is 2W. For a moving body with a velocity of V before impact, the equivalent force (ignoring energy losses) is:

$$F_e = V\sqrt{mk}$$

These are conservative values as ignoring the energy loss leads to larger equivalent forces.

Problem #S15: A 1000 lb weight drops a distance of 1-in on a platform supported by a 1 in^2 steel bar of length 12 inches. What is the theoretical tensile stress that would develop in the bar. Answer: 70.7 ksi.



\# S15   \# S16

Problem #S16: This is the same problem as #S15 but the bar is made up of two segments. The upper segment has an area of 2 in^2. Determine the maximum theoretical stress developing in the bar as the result of the weight dropping on the platform. Answer: 81.6 ksi.

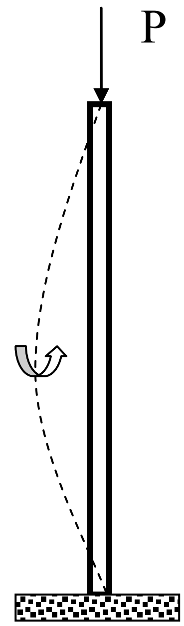
Exercise Question: You have made grocery shopping and the cashier placed all your items in a paper bag. The bag's dead weight is now 15 lbs. What force would the bag handles experience if you:

a) Lift the bag gently and lower it?
b) Slide the bag off the countertop and suddenly resist the weight of the bag at a rate of 30 lbs/in of drop?
c) Let the bag slide off and drop 5” before you suddenly resist it at a rate of 30 lbs per/in of drop.
d) Same as c) but rate of resistance is 60 lbs/in.

Failure of columns under compressive load (Buckling)

A beam under axial compressive load can become unstable and collapse. This occurs when the beam is long and its internal resistance to bending moment is insufficient to keep it stable. The internal resistance is a function of area moment of inertia, I, and the stiffness of the material.

Note that the longer the beam, the more bending moment is created at the center and for the beam to remain stable, it needs to be stiffer or have more bending resistance area.



For every long beams there is a critical load beyond which even a tiny nudge would result is a collapse. This critical load can be found using Euler formula.

In shorter columns the critical load may cause stresses well above the yield strength of the material before the Euler load is reached. For such cases, Johnson formula is used which relates the failure to yielding rather than instability.

The critical Euler load for a beam that is long enough is:

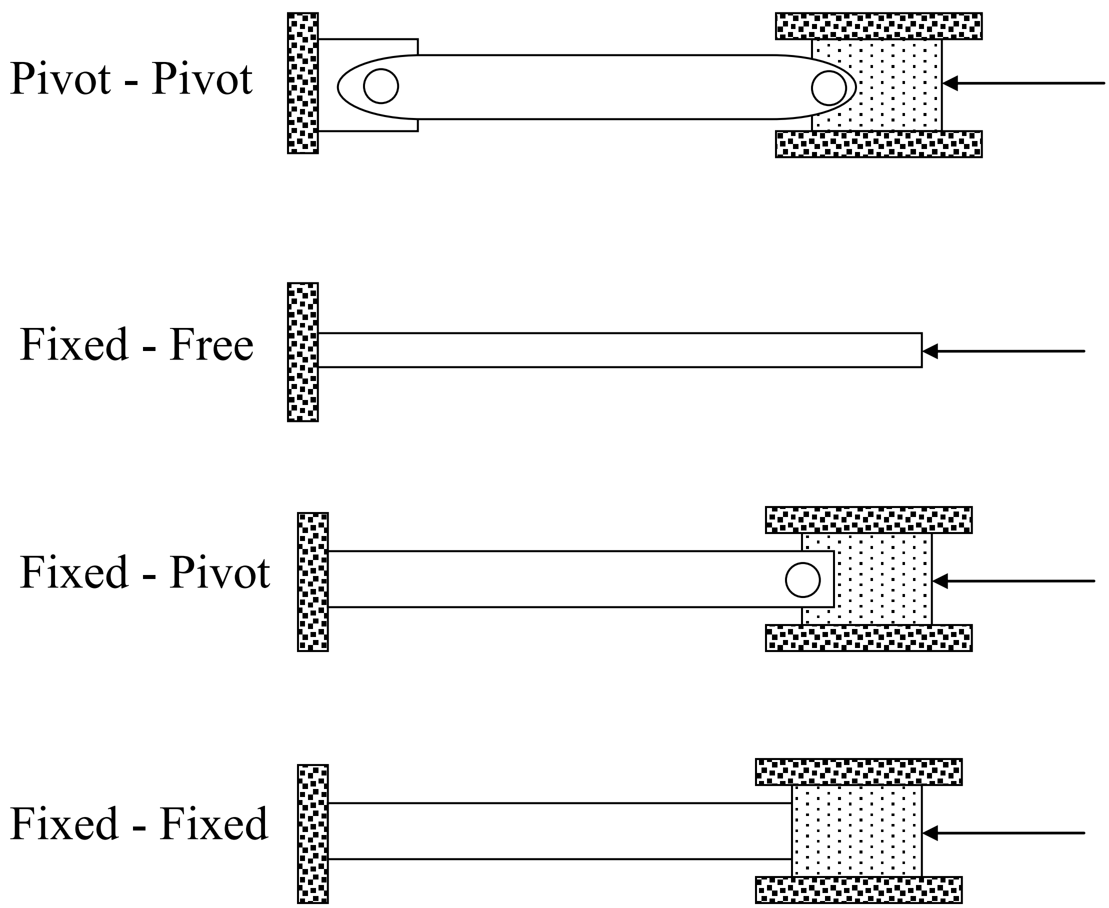
$$P_{cr} = C\frac{\pi^2 EI}{L^2}$$

C is the *end-condition number*. The following end-condition numbers should be used for given cases:

- When both ends are free to pivot use C=1. Free to pivot means the end can rotate but not move in lateral direction. Note that even if the ends are free to rotate a little, such as in any bearing, this condition is applicable.
- When one end is fixed (prevented from rotation) and the other is free, the beam buckles easier. Use C= 1/4 .

- When one end is fixed and the other end can pivot, use C=2 when the fixed end is truly fixed in concrete. If the fixed end is attached to structures that might flex under load, use C=1.2 (recommended).
- When both ends are fixed (prevented from rotation and lateral movement), use C=4. Again, a value of C=1.2 is recommended when there is any chance for pivoting.

These conditions are depicted below:



An alternate but common form of the Euler formula uses the "slenderness ratio" which is defined as follows:

$$\textit{Slenderness Ratio} = \left(\frac{L}{k}\right) \quad \textit{where} \quad k = \sqrt{\frac{I}{A}}$$

k is the area radius of gyration of the cross-section.

Range of validity of the Euler formula

Experimentation has shown that the Euler formula is a good predictor of column failure when:

$$\frac{L}{k} >= \sqrt{\frac{2\pi^2 EC}{S_y}}$$

If the slenderness ratio is less than the value in the formula, then the better predictor of failure is the Johnson formula:

$$P_{cr} = A\left[S_y - \left(\frac{S_y L}{2\pi k}\right)^2 \frac{1}{CE}\right]$$

Alternatively, we can calculate the critical load from both the Euler and the Johnson formulas and pick the one that is lower.

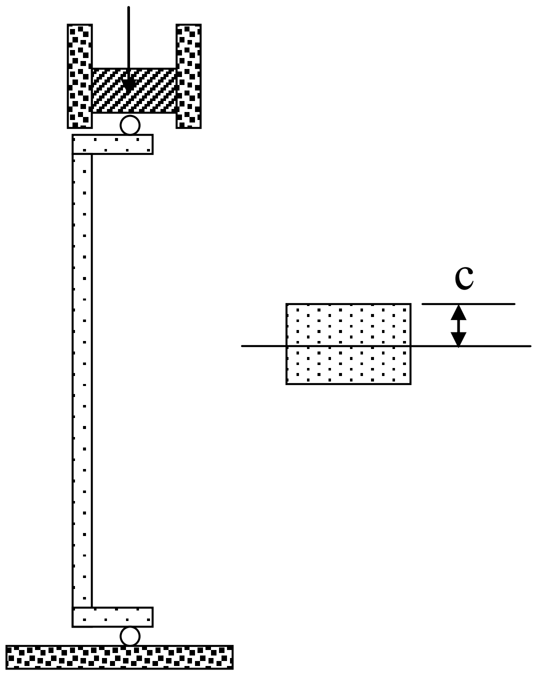
Problem #S17: The axial load on a round solid steel bar in compression is 5655 lbs. The material is AISI 1030 HR. Assume the end conditions are pin-pin or pivot-pivot. Determine the factor of safety against failure for the following two conditions:

a) L=60" and D=diameter=1.5"
b) L=18" and D= 7/8 "

Answers: a) 3.6 and b) 4.4

Note: When a beam is under compression, it would buckle about the axis with smaller area moment of inertia.

Eccentrically loaded columns



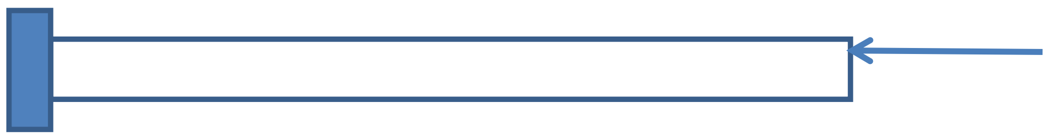
The more general case of column loading is when the load is applied eccentrically. This eccentric load exacerbates the situation as it induces more bending moment due to its eccentricity. The prediction formula is known as the *Secant Formula* which is essentially a classical bending stress formula although it may not look like it. The secant formula is:

$$P_{cr} = \frac{AS_y}{1 + \frac{ec}{k^2}\sec\left(\frac{L}{Ck}\sqrt{\frac{P_{cr}}{4EA}}\right)}$$

where e is the eccentricity, c is the distance from the outer layer to the neutral axis, and the rest of the symbols have already been defined.

A slight technical difficulty with this formula is that P_{cr} appears on both sides of the equation resulting in the need to use trial-and-error or use a non-linear equation solver. However, usually the load is given and you would calculate the stress (in place of S_y in the formula).

Example: A column has a fixed end and the other end is free and unsupported. The column length is 8 feet long. The beam cross-section is a square tube with outer dimensions of 4 by 4 inches. The area of the cross-section is calculated to be 3.54 in^2 and its smallest area moment of inertia is 8 in^4. Determine the maximum compressive stress when the beam is supporting 31.1 kips at an eccentricity of 0.75 inches off the beam axis.



Solution

We find the stress σ from the secant formula. The area radius of gyration is:

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{8}{3.54}} = 1.5\,in$$

The formula is

$$P_{cr} = \frac{AS_y}{1 + \frac{ec}{k^2}\sec\left(\frac{L}{Ck}\sqrt{\frac{P_{cr}}{4EA}}\right)}$$

For this problem, P=31100 lbs is known and S_y becomes the unknown σ_{max}. Substituting the numbers:

$$31100 = \frac{3.54(\sigma_{max})}{1 + \frac{0.75(2)}{(1.5)^2}\sec\left(\frac{8(12)}{(0.25)(1.5)}\sqrt{\frac{31100}{4(29)(10^6)(3.54)}}\right)}$$

Calculating for σ_{max} we get:

$$\sigma_{max} = 22000 \text{ psi}$$

Notes:

1. The end condition is C=0.25 (some books do not apply C but instead they use an equivalent length L_{eq} which is L divided by square root of C.
2. The argument of the secant function is in radians. Convert to degrees first before taking cosines.
3. The angle in degrees in secant function must be between 0 and 90 degrees (0 and $\pi/4$ in radians). Add or subtract multiples of 90 degrees until the angle is between 0 and 90 degrees. In this problem the angle is 126 degrees.

Failure Theories

Failure under load can occur due to excessive elastic deflections or due to excessive stresses. Failure prediction theories due to excessive stresses fall into two classes: Failure when the loading is static or the number of load cycles is one or quite small, and failure due to cyclic loading when the number of cycles is large often in thousands of cycles.

Failure under static load

Parts under static loading may fail due to:

a) **Ductile behavior:** Failure is due to bulk yielding causing permanent deformations that are objectionable. These failures may cause noise, loss of accuracy, excessive vibrations, and eventual fracture. In machinery, bulk yielding is the criteria for failure. Tiny areas of yielding are OK in ductile behavior in static loading.
b) **Brittle behavior:** Failure is due to fracture. This occurs when the materials (or conditions) do not allow much yielding such as ceramics, grey cast iron, or heavily cold-worked parts.

Theories of ductile failure (yielding)

Yielding is a shear stress phenomenon. That means materials yield because the shear stresses on some planes causes the lattice crystals to slide like a deck of cards. In pure tension or compression, maximum shear stresses occur on 45-degree planes – these stresses are responsible for yielding and not the larger normal stresses.

The best predictor of yielding is the maximum distortion energy theory (DET). This theory states that yielding occurs when the Von Mises stress reaches the yield strength. The more conservative predictor is the maximum shear stress theory (MST), which predicts yielding to occur when the shear stresses reach $S_y/2$. For example in a pure torsion situation, the DET predicts the yielding to start when τ reaches 58% of S_y. But the MST predicts yielding to start when τ reaches 50% of S_y. Use of DET is more common in design work.
Note that in static loading and ductile behavior, stress concentrations are harmless as they only create small localized yielding which do not lead to

any objectionable dimensional changes. The material “yielding” per se is not harmful to materials as long as it is not repeated too many times.

Problem # S18: A 2” diameter steel bar with Sy=50 ksi is under pure torsion of a 20,000 in-lb. Find the factor of safety guarding against yielding based on: a) Distortion energy theory, and b) Max shear stress theory. Rounded answers: 2.3 and 2.

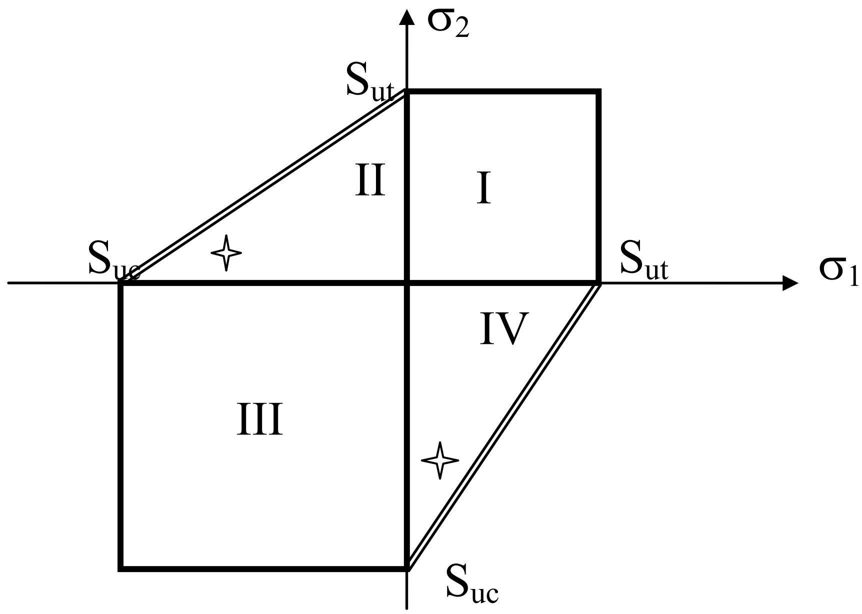
Theories of brittle failure

There are two types of theories for brittle failure. The classical theories assume that the material structure is uniform. If the material structure is non-uniform, such as in many thick-section castings, and that the probability of large flaws exist, then the theory of fracture mechanics predicts the failure much more accurately. Many old ship hulls have split into two while the existing classical theories predicted that they should not. We will only look at the classical brittle failure theories.

An important point to remember is that brittle materials often show much higher ultimate strength in compression than in tension. One reason is that, unlike yielding, fracture of brittle materials when loaded in tension is a normal stress phenomenon. The material fails because eventually normal tensile stresses fracture or separate the part in the direction normal to the plane of maximum normal stress (or principal stress – see Page 1).

In compression the story is quite different. When a brittle material is loaded in compression, the normal stress cannot separate the part along the direction normal to the plane of maximum normal stress. In the absence of separating normal stresses, shear stresses would have to do the job and separate or fracture the material along the direction where the shear stresses are maximum. In pure compression, this direction is at 45 degrees to the plane of loading. Brittle materials, however, are very strong in shear. The bottom line is that it takes a lot more compressive normal stress to create a fracture.

We only discuss these theories for a 2D state of stress – 3D is similar but is more formula-based. Theories of failure in brittle fracture divide the σ_1-σ_2 region into 4 quadrants. In the first quadrant, both principal stresses are positive.



When both σ_1 and σ_2 are positive (tensile), the fracture is predicted to occur when one of the two principal stresses reaches S_{ut}. When both σ_1 and σ_2 are negative (compressive), the fracture occurs when the magnitude of one of the two principal stresses reaches S_{uc}. The magnitude of S_{uc} is often more than S_{ut} as the prior discussion indicated.

In the other two quadrants, where one principal stress is positive and the other is negative, the Columb-Mohr theory is a conservative theory for failure prediction. It is also easy to use. The Columb-Mohr theory failure line simply connects the failure points as shown in the figure as double lines. <u>Using only the magnitudes</u> of the stresses, in Quadrant II or IV:

$$\frac{\sigma_1}{S_{ut}} + \frac{\sigma_2}{S_{uc}} = \frac{1}{n}$$

In this formula (σ_1,σ_2) is the load point (two principal stresses), and *n* is the factor of safety associated with that load point. The positive principal stress is associated with S_{ut} and the negative principal stress is associated with S_{uc}.

Problem #S19: A flywheel made of Grade 30 cast iron has the following dimensions: ID=6”, OD=10” and thickness=0.25”. What is the speed that would lead to the flywheel’s fracture? Answer: 13600 rpm

Summary of Failure Theories

Ductile Failure Definition

- Macroscopic and measurable bulk deformation
- Slight change in geometry

Conditions for ductile failure

- Metals (Except cast irons and P/M parts)
- At least 2% strain before fracture

Cause of failure (deformation)

- Excessive SHEAR stresses

Prediction Theories

- Maximum DET
  - Yielding occurs when $\boxed{\sigma_V = S_y}$
- Maximum Shear Stress Theory
  - Yielding occurs when $\boxed{\tau_{\max} = \frac{S_y}{2}}$

What to do with stress concentration?

- IGNORE them – They cause small areas of yielding and do not cause macroscopic and measurable bulk deformation.

Brittle Failure Definition

- Fracture

Conditions for Brittle failure

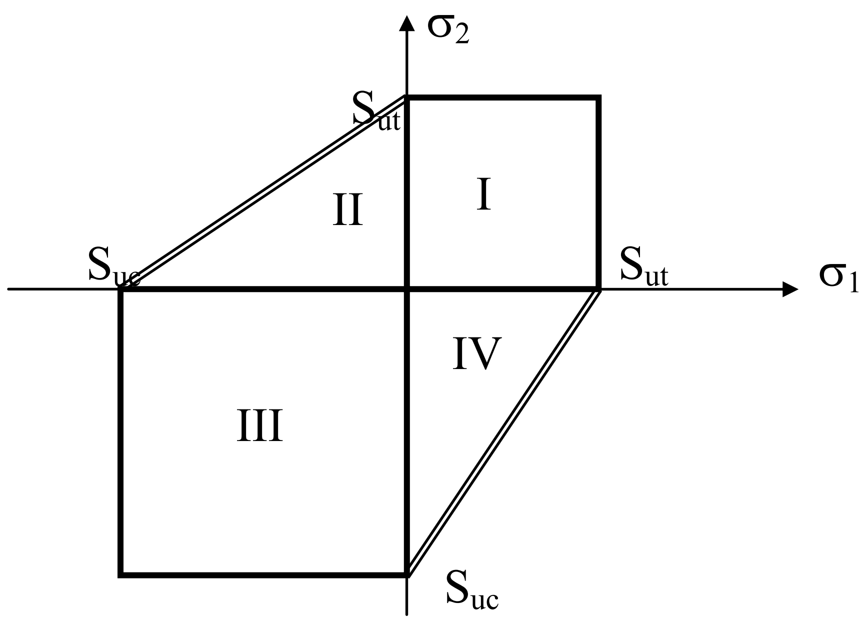
- Gray cast irons and P/M parts [I], ceramics [II]
- Other metals in special conditions:
  - Extreme cold or extreme impact
  - Extreme cold-working or extreme heat treatment

Cause of failure (fracture)

- Excessive normal stresses in tension, shear in compression

Prediction Theories

- Columb-Mohr theory



$$\frac{\sigma_1}{S_{ut}} + \frac{\sigma_2}{S_{uc}} = \frac{1}{n}$$

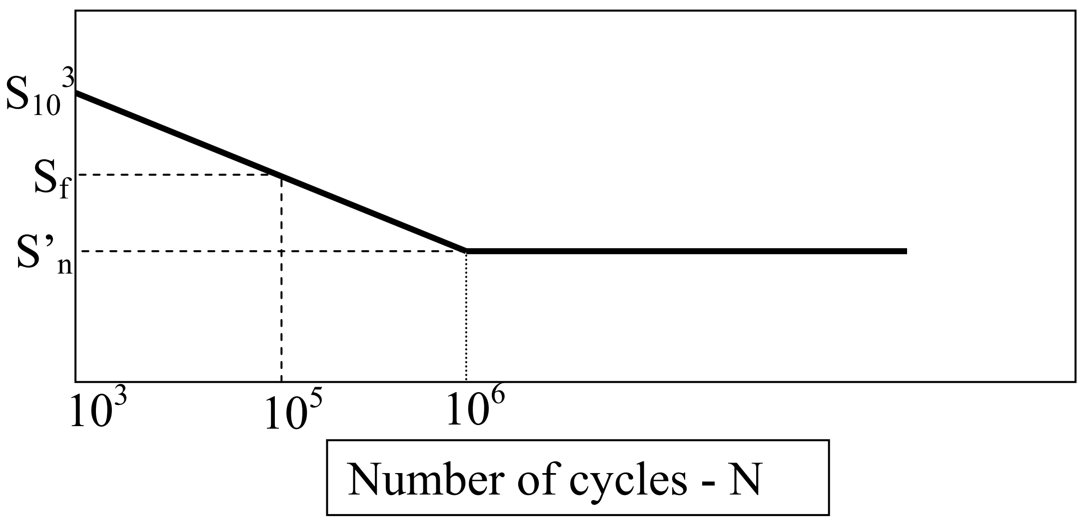
What to do with stress concentration?

- Ignore for [I] –their strength is already reduced, Apply for [II]

Fatigue Failure

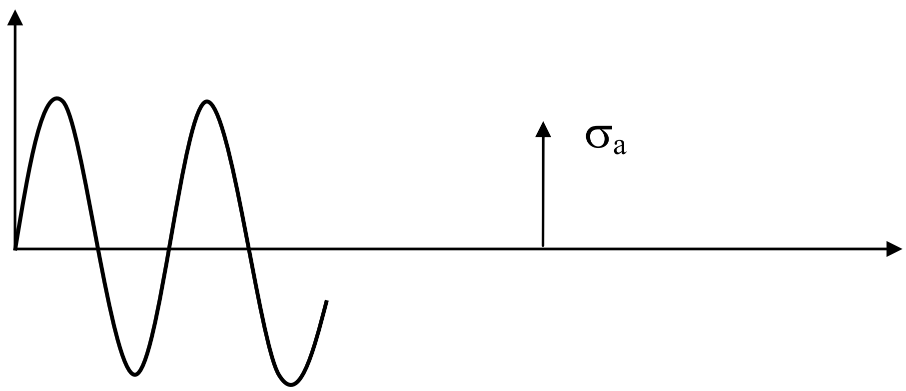
Repeated loading can lead to fatigue failure at loads much less than those leading to static failure. Fatigue failure is sensitive to the magnitude of the stress regardless of how localized and small the stress area is. Therefore, stress concentrations play an important role in fatigue failure. Note: If the material bulk itself is full of unseen stress raisers (such as in grey cast iron), the geometric stress raisers must be ignored.

Design for infinite life starts with test results of the material in rotating bending test (known as Moore test). The Moore test stress limit is called the rotating bending endurance limit, S'_n. This is the stress for which no failure occurs regardless of the number of cycles. In the absence of direct experimental data, Moore test endurance limit is 50% of the ultimate stress for steels.



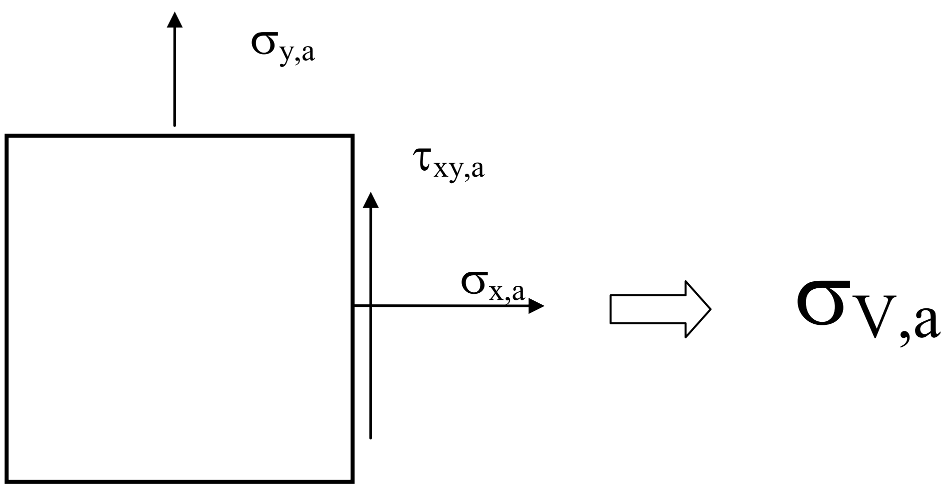
The rotating bending or Moore test endurance limit has to be corrected for the actual part loading and conditions. This includes corrections for surface roughness, gradient effect, and size of the part (in Moore test the specimens are polished, under rotating bending, and are 0.3" in diameter). The result of these corrections is the endurance limit S_n. Another notation for endurance limit is S_e

Purely Alternating Load

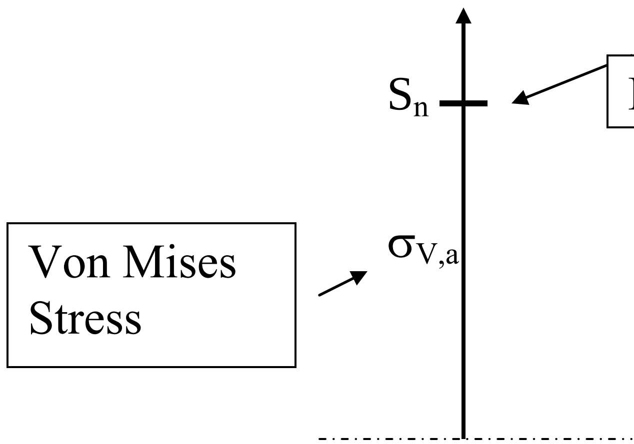


Combined Alternating Loading

When the state of stress is known, the Von Mises stresses can be analyzed. In the case of this figure all stresses are purely alternating.



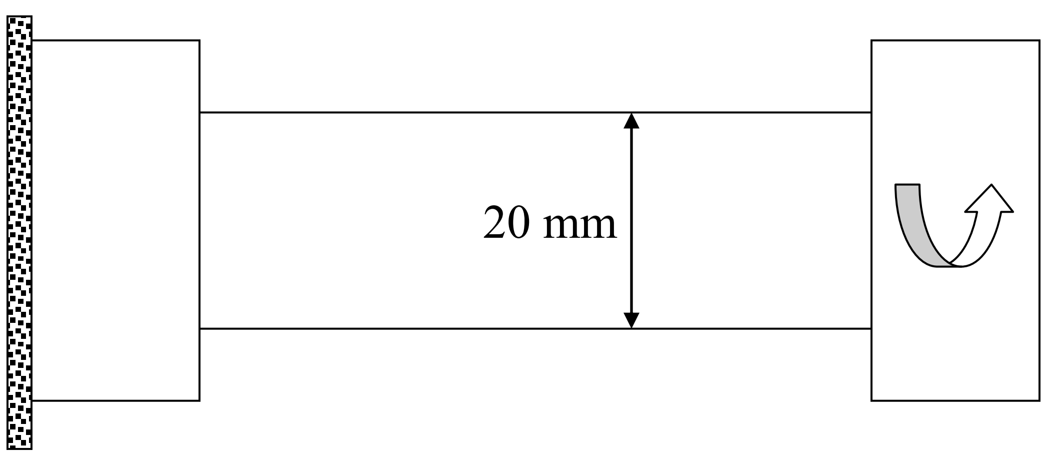
Most common loadings in shafts involves σ_x, τ_{xy}, or both.



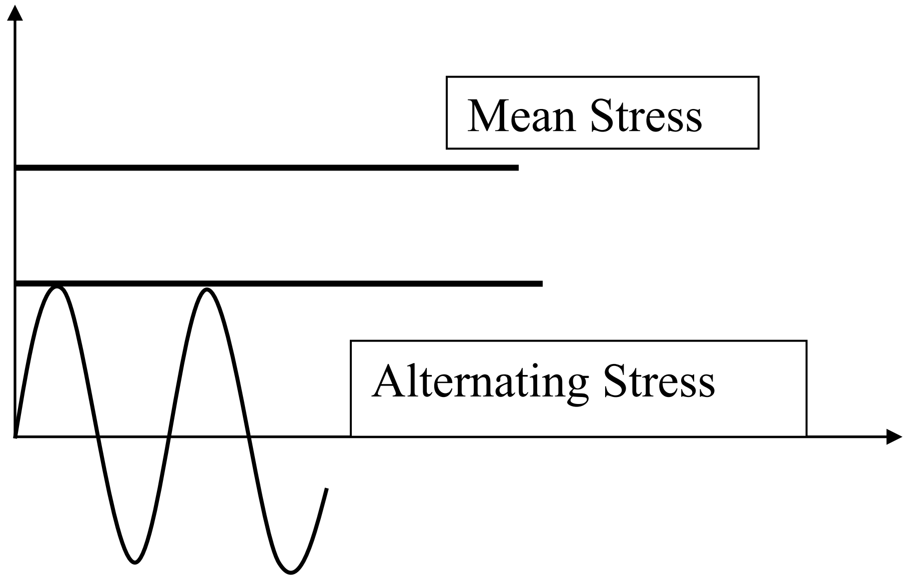
The index *a* in the above formula emphasizes that the loading is *purely alternating*.

Problem #S21

The steel shaft shown below is under purely alternating torque of 56 N-m. The torque fluctuates between 56 Nm CW and 56 Nm CCW. Assume S_{ut}=518 MPa, and the correction factors of 0.9 and 0.78 apply for gradient and surface finish. Also assume a fatigue stress concentration factor of 1.48 for the shoulder fillets. Answer: About 2



Fluctuating and Steady Loads (optional)



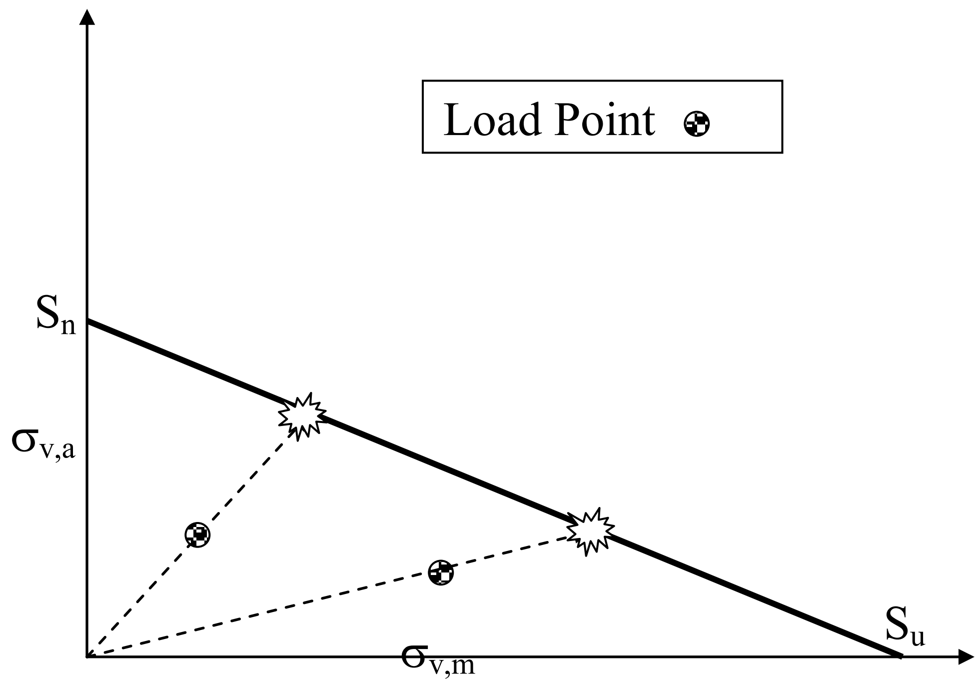
When both mean and fluctuating loads are present, the Goodman criterion is used to determine how much the mean loading affects (reduces) the endurance limit. To begin the analysis, determine the mean and alternating Von Mises stresses. These are actual maximum stresses and they do include the fatigue stress concentration factors. As a result we should be able to calculate the following:

$$\sigma_{V,m}$$

$$\sigma_{V,a}$$

The mean Von Mises is only due to mean loads and the alternating Von Mises is only due to alternating loads. In power transmission shafts the loading includes a steady shear (power torque) and an alternating bending stress (due to shaft flexure and rotating just like Moore test set up).

The load points plot in the Goodman diagram as shown below:



$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_u} = \frac{1}{n}$$

To determine the factor of safety guarding against fatigue failure, we must consider the overload mechanism. If both the steady and alternating components of stress are subject to increase as shown, the margin of safety is determined by the Goodman line.

Fatigue Failure Definition

- Fracture

Conditions for Fatigue failure

- Repeated loading
- All metals

Cause of failure (fracture)

- Excessive LOCALIZED SHEAR stresses causing repeated yielding → Local brittle fracture → Crack growth

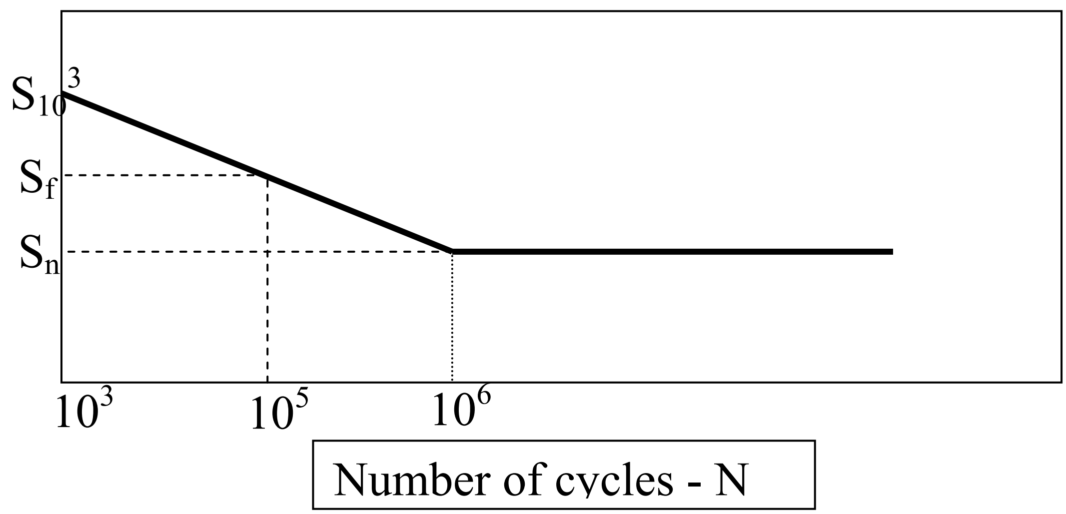
Prediction Theories

- Failure occurs when the local VonMises stress reaches the Endurance Limit.

What to do with stress concentration?

- Apply to all (mean and alternating stresses) except gray cast iron or other materials with type-I internal structure

Endurance Limit



Cumulative Fatigue Damage (Miner's or Palmgren Rule)

If a part is stressed to a load for which the fatigue life is 10^3 cycles, then each cycle takes 0.001 of the life of the part. If stressed to a load for which the fatigue life is 10^4 cycles, then each cycle takes 0.0001 of the life of the part and so on. This inference leads to the following cumulative fatigue damage formula:

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \ldots + \frac{n_k}{N_k} = 1$$

In this relation, n_1 is the number of cycles in a loading that would have a fatigue life of N_1 cycles, etc.

Example: A critical point of a landing gear is analyzed for fatigue failure. Experiments show that in each landing a "compound load cycle" is applied to the member consisting of 5 cycles of 80 ksi stress, 2 cycles of 90 ksi, and 1 cycle at 100 ksi stress. All stress cycles are fully reversed (no mean component). An experimental S-N curve is also available for this part (this curve can also be constructed using Moore test but for critical parts it is always best to spend the money and create a true S-N curve). The S-N curve shows the fatigue lives of the component at the loading stresses to be as follows:

Stress Level	Number of cycles	Fatigue life
80 Ksi	5	10^5 cycles
90 Ksi	2	38000 cyc
100 Ksi	1	16000 cyc

Determine the life of this part in the number of compound cycles.

Solution: Each compound cycle takes the following fraction of life out of the part:

$$\frac{5}{10^5} + \frac{2}{38000} + \frac{1}{16000} = 0.0001651$$

The number of cycles is reciprocal of this value which is 6059 cycles.

Unit Conversions

Problem #S11: Length: 1.640 feet
Torque: 147.4 ft-lbOD: 1.575 in
Thickness: 0.07874 in Answer (Stress): 5 Ksi

Problem #S14: Shaft Diameter: 1.5758" Collar diameter: 1.5748"
OD of collar: 3.1496" Answer (Pressure): 7.25 Ksi