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SCHOOL OF PHARMACY

DEPARTMENT OF PHARMACY

COURSE NAME: REMEDIAL MATHEMATICS

COURSE CODE: BP106RMT

UNIT – I – PARTIAL FRACTIONS – BP106RMT

UNIT-1

PARTIAL FRACTION

An algebraic fraction can be broken down into simpler parts known as "partial fractions". Consider an algebraic fraction, $(3x+5)/(2x^2-5x-3)$. This expression can be split into simple form like $(2)/(x-3) – (1)/(2x+1)$.

The simpler parts $[(2)/(x-3)]-[(1)/(2x+1)]$ are known as partial fractions.

This means that the algebraic expression can be written in the form of:

$(3x+5)/(2x^2-5x-3) = ((2)/(x-3))-((1)/(2x+1))$

Note: The partial fraction decomposition only works for the proper rational expression (the degree of the numerator is less than the degree of the denominator). In case, if the rational expression is in improper rational expression (the degree of the numerator is greater than the degree of the denominator), first do the division operation to convert into proper rational expression. This can be achieved with the help of polynomial long division method.

PARTIAL FRACTION FORMULA

The procedure or the formula for finding the partial fraction is:

1. While decomposing the rational expression into the partial fraction, begin with the proper rational expression.
2. Now, factor the denominator of the rational expression into the linear factor or in the form of irreducible quadratic factors (Note: Don't factor the denominators into the complex numbers).
3. Write down the partial fraction for each factor obtained, with the variables in the numerators, say A and B.
4. To find the variable values of A and B, multiply the whole equation by the denominator.
5. Solve for the variables by substituting zero in the factor variable.
6. Finally, substitute the values of A and B in the partial fractions.

PARTIAL FRACTIONS FROM RATIONAL FUNCTIONS

Any number which can be easily represented in the form of p/q, such that p and q are integers and $q\neq0$ is known as a rational number. Similarly, we can define a rational function as the ratio of two polynomial functions P(x) and Q(x), where P and Q are polynomials in x and $Q(x)\neq0$. A rational function is known as proper if the degree of P(x) is less than the degree of Q(x); otherwise, it is known as an improper rational function. With the help of the long division process, we can reduce improper rational functions to proper rational functions. Therefore, if P(x)/Q(x) is improper, then it can be expressed as:

$P(x)Q(x)=A(x)+R(x)Q(x)$

Here, A(x) is a polynomial in x and R(x)/Q(x) is a proper rational function.

We know that the integration of a function f(x) is given by F(x) and it is represented by:

$\int f(x)dx = F(x) + C$

Here R.H.S. of the equation means integral of f(x) with respect to x.

PARTIAL FRACTIONS DECOMPOSITION

In order to integrate a rational function, it is reduced to a proper rational function. The method in which the integrand is expressed as the sum of simpler rational functions is known as decomposition into partial fractions. After splitting the integrand into partial fractions, it is integrated accordingly with the help of traditional integrating techniques. Here the list of Partial fractions formulas is given.

PARTIAL FRACTION OF IMPROPER FRACTION

An algebraic fraction is improper if the degree of the numerator is greater than or equal to that of the denominator. The degree is the highest power of the polynomial. Suppose, m is the degree of the denominator and n is the degree of the numerator. Then, in addition to the partial fractions arising from factors in the denominator, we must include an additional term: this additional term is a polynomial of degree n − m.

Note:

- A polynomial with zero degree is K, where K is a constant
- A polynomial of degree 1 is Px + Q
- A polynomial of degree 2 is Px^2+Qx+K

Case	Fraction $\frac{N(x)}{D(x)}$	Form of denominator, D(x)	Partial Fraction Form (where A, B and C are unknown constants)
1	$\frac{N(x)}{(ax+b)(cx+d)}$	Linear Factors	$\frac{A}{ax+b}+\frac{B}{cx+d}$
2	$\frac{N(x)}{(ax+b)^2}$	Repeated Linear Factors	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
	$\frac{N(x)}{(ax+b)(cx+d)^2}$	Linear and Repeated Linear Factors	$\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}$
3	$\frac{N(x)}{(ax+b)(x^2+c^2)}$	Linear and Quadratic (which cannot be factorised) Factors	$\frac{A}{ax+b}+\frac{Bx+C}{x^2+c^2}$

Type: 1

When the factors of the denominator are all linear and distinct i.e., non repeating.

Example 1:

Resolve $\frac{7x - 25}{(x - 3)(x - 4)}$ into partial fractions.

Solution:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{A}{x-3} + \frac{B}{x-4} \text{------------------}(1)$$

Multiplying both sides by L.C.M. i.e., $(x - 3)(x - 4)$, we get

$$7x - 25 = A(x - 4) + B(x - 3) \text{--------------} \quad (2)$$

$$7x - 25 = Ax - 4A + Bx - 3B$$

$$7x - 25 = Ax + Bx - 4A - 3B$$

$$7x - 25 = (A + B)x - 4A - 3B$$

Comparing the co-efficients of like powers of x on both sides, we have

$$7 = A + B \text{ and}$$

$$-25 = -4A - 3B$$

Solving these equation we get

$$A = 4 \quad \text{and} \quad B = 3$$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$$7(4) - 25 = A(4 - 4) + B(4 - 3)$$

$$28 - 25 = 0 + B(1)$$

$$B = 3$$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$$7(3) - 25 = A(3 - 4) + B(3 - 3)$$

$$21 - 25 = A(-1) + 0$$

$$-4 = -A$$

$$A = 4$$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

2. Solve 3x+1/(x-2)(x+1) into partial fractions

$$\frac{3x+1}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

$$\Rightarrow \frac{3x+1}{(x-2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$\Rightarrow \quad 3x+1 = A(x+1) + B(x-2) \quad ...(1)$$

Putting $x = -1$ in (1) we get,

$$-3+1 = B(-3)$$

$$\Rightarrow \quad -2 = -3B \quad \Rightarrow \boxed{B = \frac{2}{3}}$$

Putting $x = 2$ in (1) we get,

$$6+1 = A(2+1)$$

$$\Rightarrow \quad 7 = 3A \Rightarrow \boxed{\frac{7}{3} = A}$$

$$\therefore \frac{3x+1}{(x-2)(x+1)} = \frac{\frac{7}{3}}{x-2} + \frac{\frac{2}{3}}{x+1}$$

$$= \frac{7}{3(x-2)} + \frac{2}{3(x+1)}$$

3. Resolve $1/x^2$-1 into partial fractions

Solutios: $\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$

$$1 = A(x+1) + B(x-1) \qquad (1)$$

Put $x - 1 = 0$, $\Rightarrow$ $x = 1$ in equation (1)

$1 = A(1+1) + B(1-1)$ $\Rightarrow$ $A = \frac{1}{2}$

Put $x + 1 = 0$, $\Rightarrow$ $x = -1$ in equation (1)

$1 = A(-1+1) + B(-1-1)$

$1 = -2B$, $\Rightarrow$ $B = \frac{1}{2}$

$$\frac{1}{x^2 - 1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Type: 2

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\dfrac{x^2 - 3x + 1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x-1)^2(x-2)$, we get

$$x^2 - 3x + 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2 \quad \text{(I)}$$

Putting $x - 1 = 0 \quad \Rightarrow \quad x = 1$ in (I), then

$$(1)^2 - 3(1) + 1 = B(1-2)$$
$$1 - 3 + 1 = -B$$
$$-1 = -B$$
$$\Rightarrow \quad B = 1$$

Putting $x - 2 = 0 \quad \Rightarrow \quad x = 2$ in (I), then

$$(2)^2 - 3(2) + 1 = C(2-1)^2$$
$$4 - 6 + 1 = C(1)^2$$
$$\Rightarrow \quad -1 = C$$

Now $x^2 - 3x + 1 = A(x^2 - 3x + 2) + B(x - 2) + C(x^2 - 2x + 1)$

Comparing the co-efficient of like powers of x on both sides, we get

$$A + C = 1$$
$$A = 1 - C$$
$$= 1 - (-1)$$
$$= 1 + 1 = 2$$
$$\Rightarrow \quad A = 2$$

Hence the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x-2}$$

Type 3:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x+3)(x^2+1)$, we get

$$9x-7 = A(x^2+1) + (Bx+C)(x+3) \qquad \text{(I)}$$

Put $x+3=0 \Rightarrow x=-3$ in Eq. (I), we have

$$9(-3)-7 = A((-3)^2+1) + (B(-3)+C)(-3+3)$$

$$-27-7 = 10A+0$$

$$A = -\frac{34}{10} \Rightarrow \boxed{A = -\frac{17}{5}}$$

$$9x-7 = A(x^2+1) + B(x^2+3x) + C(x+3)$$

Comparing the co-efficient of like powers of x on both sides

$$A+B=0$$
$$3B+C=9$$

Putting value of A in Eq. (i)

$$-\frac{17}{5}+B=0 \Rightarrow \boxed{B = \frac{17}{5}}$$

From Eq. (iii)

$$C = 9-3B = 9-3\left(\frac{17}{4}\right)$$

$$= 9-\frac{51}{5} \Rightarrow \boxed{C = -\frac{6}{5}}$$

Hence the required partial fraction are

$$\frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

LOGARITHMS

The logarithmic function is an inverse of the exponential function. It is defined as:

$y=\log_a x$, if and only if $x=a^y$; for $x>0$, $a>0$, and $a\neq1$.

Natural logarithmic function: The log function with base e is called natural logarithmic function and is denoted by $\log_e$.

$f(x) = \log_e x$

The questions of logarithm could be solved based on the properties, given below

Product rule: $\log_b MN = \log_b M + \log_b N$

Quotient rule: $\log_b M/N = \log_b M - \log_b N$

Power rule: $\log_b M^p = P \log_b M$

Zero Exponent Rule: $\log_a 1 = 0$

Change of Base Rule: $\log_b (x) = \ln x / \ln b$ or $\log_b (x) = \log_{10} x / \log_{10} b$

Logarithm Properties

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_a b = \frac{1}{\log_b a}$$

The following can be derived from the above properties

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$\log_a a^r = r$$

$$\log_a \frac{1}{b} = -\log_a b$$

$$\log_{\frac{1}{a}} b = -\log_a b$$

$$\log_a b \log_b c = \log_a c$$

Solved Examples:

1. Express $5^3 = 125$ in logarithm form.

Solution:

$5^3 = 125$

As we know,

$a^b = c \Rightarrow \log_a c=b$

Therefore;

$\mathrm{Log}_5 125 = 3$

2. Express $\log_{10} 1 = 0$ in exponential form.

Solution:

Given, $\log_{10} 1 = 0$

By the rule, we know;

$\log_a c=b \Rightarrow a^b = c$

Hence,

$10^0 = 1$

3. Find the log of 32 to the base 4.

Solution: $\log_4 32 = x$

$4^x = 32$

$(2^2)^x = 2x2x2x2x2$

$2^{2x} = 2^5$

$2x=5$

$x=5/2$

Therefore,

$\log_4 32 =5/2$

4. Find x if $\log_5(x-7)=1$.

Solution: Given,

$\log_5(x-7)=1$

Using logarithm rules, we can write;

$5^1 = x-7$

$5 = x-7$

$x=5+7$

$x=12$

5. If $\log_a m=n$, express a^{n-1} in terms of a and m.

Solution:

$\log_a m=n$

$a^n=m$

$a^n/a=m/a$

$a^{n-1}=m/a$

6. Solve for x if $\log(x-1)+\log(x+1)=\log_2 1$

Solution: $\log(x-1)+\log(x+1)=\log_2 1$

$\log(x-1)+\log(x+1)=0$

$\log[(x-1)(x+1)]=0$

Since, $\log 1 = 0$

$(x-1)(x+1) = 1$

$x^2-1=1$

$x^2=2$

$x=\pm \sqrt{2}$

Since, log of negative number is not defined.

Therefore, $x=\sqrt{2}$

7. Express $\log(75/16)-2\log(5/9)+\log(32/243)$ in terms of log 2 and log 3.

Solution: $\log(75/16)-2\log(5/9)+\log(32/243)$

Since, $n\log_a m=\log_a m^n$

$\Rightarrow\log(75/16)-\log(5/9)^2+\log(32/243)$

$\Rightarrow\log(75/16)-\log(25/81)+\log(32/243)$

Since, $\log_a m-\log_a n=\log_a(m/n)$

$\Rightarrow\log[(75/16)\div(25/81)]+\log(32/243)$

$\Rightarrow\log[(75/16)\times(81/25)]+\log(32/243)$

$\Rightarrow\log(243/16)+\log(32/243)$

Since, $\log_a m+\log_a n=\log_a mn$

$\Rightarrow \log(32/16)$

$\Rightarrow \log 2$

8. Express 2logx+3logy=log a in logarithm free form.

Solution: $2\log x+3\log y=\log a$

$\log x^2+\log y^3=\log a$

$\log x^2y^3=\log a$

$x^2y^3=\log a$

9. Prove that: 2log(15/18)-log(25/162)+log(4/9)=log2

Solution: $2\log(15/18)-\log(25/162)+\log(4/9)=\log 2$

Taking L.H.S.:

$\Rightarrow 2\log(15/18)-\log(25/162)+\log(4/9)$

$\Rightarrow \log(15/18)^2-\log(25/162)+\log(4/9)$

$\Rightarrow \log(225/324)-\log(25/162)+\log(4/9)$

$\Rightarrow \log[(225/324)(4/9)]-\log(25/162)$

$\Rightarrow \log[(225/324)(4/9)]/(25/162)$

$\Rightarrow \log(72/36)$

$\Rightarrow \log 2$ (R.H.S)

10. Express $\log_{10}(2+1)$ in the form of $\log_{10}x$.

Solution: $\log_{10}(2+1)$

$=\log_{10}2+\log_{10}1$

$=\log_{10}(2\times10)$

$=\log_{10}20$

11. Find the value of x, if $\log_{10}(x-10)=1$.

Solution: Given, $\log_{10}(x-10)=1$.

$\log_{10}(x-10) = \log_{10}10$

$x-10 = 10$

$x=10+10$

$x=20$

12. Find the value of x, if log(x+5)+log(x-5)=4log2+2log3

Solution: Given,

$\log(x+5)+\log(x-5)=4\log 2+2\log 3$

$\log(x+5)(x-5) = 4\log 2+2\log 3$ [log mn=log m+log n]

$\log(x^2-25) = \log 2^4+\log 3^2$

$\log(x^2-25) = \log 16+\log 9$

$\log(x^2-25)=\log(16\times9)$

$\log(x^2-25)=\log 144$

$x^2-25=144$

$x^2=169$

$x=\pm\sqrt{169}$

$x=\pm13$

13. Solve for x, if log(225/log15) = log x

Solution: $\log x = \log(225/\log 15)$

$\log x=\log[(15\times15)]/\log 15$

$\log x = \log 15^2/\log 15$

$\log x = 2\log 15/\log 15$

$\log x = 2$

Or

$\log_{10}x=2$

$10^2=x$

$x=10\times10$

$x=100$

Solved Examples

Evaluate the expression below using Log Rules.

$$\log_3 162 - \log_3 2$$

(i)Sol:

$$\begin{aligned}\log_3 162 - \log_3 2 &= \log_3\left(\frac{162}{2}\right)\\ &= \log_3(81)\\ &= \log_3 3^4\\ &= 4\log_3 3\\ &= 4(1)\\ \log_3 162 - \log_3 2 &= 4\end{aligned}$$

$\log_2 8 + \log_2 4$

(ii)sol:

$$\begin{aligned}\log_2 8 + \log_2 4 &= \log_2 2^3 + \log_2 2^2\\ &= 3\log_2 2 + 2\log_2 2\\ &= 3(1) + 2(1)\\ &= 3 + 2\\ \log_2 8 + \log_2 4 &= 5\end{aligned}$$

Some Important Expansions

1. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots.$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots.$

3. $a^x = 1 + x\log a + \frac{x^2}{2!}(log\ a)^2 + \ldots\ldots.$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots\ldots$

5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\ldots$

6. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$

Example: $\lim_{x \to 1} (3x^2 + 4x + 5)$ = $3(1)^2 + 4(1) + 5 = 12$

Example: $\lim_{x \to 2} \frac{x^2-4}{x+3} = \frac{4-4}{2+3} = \frac{0}{5} = 0$

Example: $\lim_{x \to 2} \frac{x^2-5x+6}{x^2-4}$

Solution: $\lim_{x \to 2} \frac{(x-2)(x-3)}{(x+2)(x-2)} \Rightarrow \lim_{x \to 2} \frac{x-3}{x+2} = \frac{-1}{4}$

Continuity of a Function

A function *f* is said to be continuous at the point *x* = *a* if the following conditions are true:

- f(a) is defined
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$.
- Both side limits must also be equal i.e. $\lim_{x \to a-} f(x) = f(a) = \lim_{x \to a+} f(x)$.

IMPORTANT RESULTS:

RESULT:1

Show that

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1},$$

when, n is any integer.

Case I. Firstly suppose that n is a *positive integer*. By actual division,

$$\frac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}a+\ldots\ldots+a^{n-1}$$

the equality being valid for every value of x other than a. As the limit does not depend upon the value for $x=a$, we can write

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\left(x^{n-1}+x^{n-2}a+\ldots\ldots+a^{n-1}\right).$$

Being a polynomial, the function

$$x^{n-1}+x^{n-2}a+\ldots\ldots+a^{n-1}$$

is continuous for every value of x and, as such, its limit when $x\to a$ must be equal to its value for $x=a$. Thus

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=a^{n-1}+a^{n-2}a+\ldots\ldots+a^{n-1}=na^{n-1}$$

Case II. Now suppose that n is a *negative integer*, say $-m$, where m is a positive integer. We have

$$\frac{x^n-a^n}{x-a}=\frac{x^{-m}-a^{-m}}{x-a}=\frac{a^m-x^m}{x-a}\cdot\frac{1}{a^mx^m}$$

$$=-\frac{x^m-a^m}{x-a}\,\frac{1}{a^mx^m}.$$

$$\therefore\ \lim_{x\to a}\frac{x^n-a^n}{x-a}=-\lim_{x\to a}\frac{x^m-a^m}{x-a}\ \lim_{x\to a}\frac{1}{a^mx^m}$$

$$=-ma^{m-1}\cdot\frac{1}{a^ma^m}=-ma^{-m-1}=na^{n-1},$$

RESULT:1

Example: $\lim\limits_{x\to 2}\frac{x^{10}-2^{10}}{x^5-2^5}$

$$\lim_{x\to 2}\frac{\dfrac{x^{10}-{}^{10}}{x-2}}{\dfrac{x^5-2^5}{x-2}}=\frac{\overset{2}{\cancel{10}}.2^9}{\cancel{5}.2^4}=2^6$$

RESULT: 2 Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof:

$\sin x < x < \tan x.$

$\therefore \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$

$\therefore 1 < \frac{x}{\sin x} < \frac{\sin x}{\cos x \ \sin x} = \frac{1}{\cos x}$

$\lim_{x \to 0} \frac{1}{\cos x} = 1$

$\therefore \lim_{x \to 0} \frac{x}{\sin x} = 1$

$\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1$

FUNCTIONS

DEFINITION:

Function (mathematics) is defined as if each element of set A is connected with the elements of set B, it is not compulsory that all elements of set B are connected; we call this relation as function.

f: A → B (f is a function from A to B)

Types of function:

One-one Function or Injective Function :

If each elements of set A is connected with different elements of set B, then we call this function as One-one function.

Many-one Function :

If any two or more elements of set A are connected with a single element of set B, then we call this function as Many one function.

Onto function or Surjective function :

Function f from set A to set B is onto function if each element of set B is connected with set of A elements.

Into Function :

Function f from set A to set B is Into function if at least set B has a element which is not connected with any of the element of set A.

Constant Function:

A function f: A→B is a constant function if the range of f contains only one element.
f(x)=4

Identity Function:

Let A be a non - empty set then f: A→A defined by f(x)=x, called the identity function on A

EXAMPLE

1. What is the range of the following function?

f(x) = {(4,6) ,(5,7),(6,8),(7,9)}

Solution:-

Given a set of ordered pairs, the range is found by identifying the y-coordinates from the set.

So the range is {6, 7, 8, 9}.

2. What is the domain of the following function?

f(x) = {(4,6) ,(5,7),(6,8),(7,9)}

Solution:-

The domain contains the x-coordinates of a set of ordered pairs.

{4,5,6,7}

3. Identify one to one and onto function

Which of the following functions from A to B are one-one and onto.

(i) $f_1 = \{(1,3)(2,5),(3,7)\}$; $A = \{1,2,3\}$, $B = \{3,5,7\}$

(ii) $f_2 = \{(2,a)(3,b),(4,c)\}$; $A = \{2,3,4\}$, $B = \{a,b,c\}$

(iii) $f_3 = \{(a,x)(b,x),(c,z),(d,z)\}$; $A = \{a,b,c,d\}$, $B = \{x,y,z\}$

Solution:

i) $f_1 = \{(1,3), (2,5), (3,7)\}$

$A = \{1, 2, 3\}$, $B = \{3, 5, 7\}$

We can earily observe that in f_1 every element of A has different image from B.

$\therefore$ f_1 in one-one

also, each element of B is the image of some element of A.

$\therefore$ f_1 in onto.

ii)

$f_2 = \{(2,a), (3,b), (4,c)\}$

$A = \{2, 3, 4\}$ $B = \{a, b, c\}$

It in clear that different elements of A have different images in B

$\therefore$ f_2 in one-one

Again, each element of B is the image of some element of A.

$\therefore$ f_2 in onto

iii) $f_3 = \{(a,x), (b,x), (c,z)(d,z)\}$

$A = \{a, b, c, d\}$ $B = \{x, y, z\}$

Since, $f_3(a) = x = f_3(b)$ and $f_3(c) = z = f_3(d)$

$\therefore$ f_3 in not one-one

Again, $y \in B$ in not the image of any of the element of A

$\therefore$ f_3 in not onto

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COURSE NAME: REMEDIAL MATHEMATICS

COURSE CODE: BP106RMT

UNIT – II – MATRICES & DETERMINANTS – BP106RMT

UNIT-2

MATRICES

Sir ARTHUR CAYLEY (1821-1895) of England was the first Mathematician to introduce the term MATRIX in the year 1858. But in the present day applied Mathematics in overwhelmingly large majority of cases it is used, as a notation to represent a large number of simultaneous equations in a compact and convenient manner.

Matrix Theory has its applications in Operations Research, Economics and Psychology. Apart from the above, matrices are now indispensible in all branches of Engineering, Physical and Social Sciences, Business Management, Statistics and Modern Control systems.

Definition of a Matrix

A rectangular array of numbers or functions represented by the symbol

$$\begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{m1} & a_{m2} & \dots a_{mn} \end{pmatrix}$$

is a MATRIX.

The numbers or functions a_{ij} of this array are called elements, may be real or complex numbers, where as m and n are positive integers, which denotes the number of Rows and number of Column.

For example

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} x^2 & \sin x \\ \sqrt{x} & \frac{1}{x} \end{pmatrix} \text{ are the matrices}$$

Order of a Matrix

A matrix A with m rows and n columns is said to be of the order m by n (m x n).

Symbolically

$A = (a_{ij})_{m\times n}$ is a matrix of order m x n. The first subscript i in (a_{ij}) ranging from 1 to m identifies the rows and the second subscript j in (a_{ij}) ranging from 1 to n identifies the columns.

Types of Matrices

SQUARE MATRIX

When the number of rows is equal to the number of columns, the matrix is called a Square Matrix.

For example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ is a Matrix of order } 2 \times 3 \text{ and}$$

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \text{ is a Matrix of order } 2 \times 2$$

$$C = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \text{ is a Matrix of order } 2 \times 2$$

$$D = \begin{pmatrix} 0 & 22 & 30 \\ -4 & 5 & -67 \\ 78 & -8 & 93 \end{pmatrix} \text{ is a Matrix of order } 3 \times 3$$

ROW MATRIX

A matrix having only one row is called Row Matrix

For example

A = (2 0 1) is a row matrix of order 1 x 3

B = (1 0) is a row matrix or order 1 x 2

COLUMN MATRIX

A matrix having only one column is called Column Matrix.

For example

$$A = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \text{ is a column matrix of order } 3 \times 1 \text{ and}$$

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is a column matrix of order } 2 \times 1$$

ZERO OR NULL MATRIX

A matrix in which all elements are equal to zero is called Zero or Null Matrix and is denoted by O.

For example

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ is a Null Matrix of order } 2 \times 2 \text{ and}$$

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is a Null Matrix of order } 2 \times 3$$

DIAGONAL MATRIX

A square Matrix in which all the elements other than main diagonal elements are zero is called a diagonal matrix.

Consider the square matrix

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 5 & -2 & -4 \\ 3 & 6 & 5 \end{pmatrix}$$

Here 1, -2, 5 are called main diagonal elements and 3, -2, 7 are called secondary diagonal elements.

For example

$A = \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}$ is a Diagonal Matrix of order 2 and

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a Diagonal Matrix of order 3

UNIT MATRIX OR IDENTITY MATRIX

A scalar Matrix having each diagonal element equal to 1 (unity) is called a Unit Matrix and is denoted by I.

For example

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a Unit Matrix of order 2

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a Unit Matrix of order 3

Equality of matrices

Two matrices are said to equal when

i) they have the same order and
ii) the corresponding elements are equal.

Addition of matrices

Addition of matrices is possible only when they are of same order (i.e., conformal for addition). When two matrices A and B are of same order, then their sum (A+B) is obtained by adding the corresponding elements in both the matrices.

Subtraction of matrices

Subtraction of matrices is also possible only when they are of same order. Let A and B be the two matrices of the same order. The matrix A - B is obtained by subtracting the elements of B from the corresponding elements of A.

Multiplication of matrices

Multiplication of two matrices is possible only when the number of columns of the first matrix is equal to the number of rows of the second matrix (i.e. conformable for multiplication) For example

Let A = (a_{ij}) be an m x p matrix, and let B = (b_{ij}) be an p x n matrix

For example

$$\text{if } A = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 6 & 7 \end{pmatrix}_{3\times 2} \quad B = \begin{pmatrix} 5 & -7 \\ -2 & 4 \end{pmatrix}_{2\times 2}$$

$$\text{then } AB = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 5 & -7 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 5 + 5\text{x}(-2) & 3\text{x}(-7) + 5\text{x}(5) \\ 2 \times 5 + (-1) \times (-2) & 2 \times (-7) + (-1) \times (4) \\ 6 \times 5 + 7\text{x}\ (-2) & 6\text{x}\ (-7) + 7\text{x}\ (4) \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 12 & -18 \\ 16 & -14 \end{pmatrix}$$

Transpose of a matrix

Let A = (a_{ij}) be a matrix of order mxn. The transpose of A, denoted by A^T of order nxm is obtained by interchanging rows into columns of A.

For example

$$\text{If} \quad A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}_{2\times 3}, \text{ then}$$

$$A^T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$$

Problems based on Addition , Subtraction and Multiplication of Matrices

Example 1

$$\text{If } A = \begin{pmatrix} 5 & 9 & 6 \\ 6 & 2 & 10 \end{pmatrix} \text{ and } B = \begin{pmatrix} 6 & 0 & 7 \\ 4 & -8 & -3 \end{pmatrix}$$

find A + B and A-B

Solution :

$$A+B = \begin{pmatrix} 5+6 & 9+0 & 6+7 \\ 6+4 & 2+(-8) & 10+(-3) \end{pmatrix} = \begin{pmatrix} 11 & 9 & 13 \\ 10 & -6 & 7 \end{pmatrix}$$

$$A-B = \begin{pmatrix} 5-6 & 9-0 & 6-7 \\ 6-4 & 2-(-8) & 10-(-3) \end{pmatrix} = \begin{pmatrix} -1 & 9 & -1 \\ 2 & 10 & 13 \end{pmatrix}$$

Example 2

If $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 7 & 9 \\ 1 & 6 & 4 \end{pmatrix}$ **and** $B = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 2 & 5 \\ 6 & -2 & 7 \end{pmatrix}$

show that $5(A+B) = 5A + 5B$

Solution :

$$A+B = \begin{pmatrix} 5 & 4 & 7 \\ 8 & 9 & 14 \\ 7 & 4 & 11 \end{pmatrix} \quad \therefore 5(A+B) = \begin{pmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{pmatrix}$$

$$5A = \begin{pmatrix} 10 & 15 & 25 \\ 20 & 35 & 45 \\ 5 & 30 & 20 \end{pmatrix} \text{ and } 5B = \begin{pmatrix} 15 & 5 & 10 \\ 20 & 10 & 25 \\ 30 & -10 & 35 \end{pmatrix}$$

$$\therefore 5A+5B = \begin{pmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{pmatrix} \quad \therefore 5(A+B) = 5A + 5B$$

Example 3

If $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$ **, then compute** $A^2 - 5A + 3I$

Solution:

$$A^2 = A.A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} -5 & 6 \\ -9 & 10 \end{pmatrix}$$

$$5A = 5\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -10 \\ 15 & -20 \end{pmatrix}$$

$$3I = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\therefore A^2 - 5A + 3I = \begin{pmatrix} -5 & 6 \\ -9 & 10 \end{pmatrix} - \begin{pmatrix} 5 & -10 \\ 15 & -20 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 16 \\ -24 & 30 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 16 \\ -24 & 33 \end{pmatrix}$$

DETERMINANTS

An important attribute in the study of Matrix Algebra is the concept of **Determinant,** ascribed to a square matrix. A knowledge of **Determinant** theory is indispensable in the study of Matrix Algebra.

The determinant associated with each square matrix A = (a_{ij}) is a **scalar** and denoted by the symbol det.A The scalar may be real or complex number, positive, Negative or Zero. A matrix is an array and has **no numerical** value, but a determinant has **numerical value.**

For example

when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then determinant of A is

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and the determinant value is = ad - bc

Example 1

Evaluate $\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}$$

$$= 1 \times (-2) - 3 \times (-1) = -2 + 3 = 1$$

Example 2

Evaluate $\begin{vmatrix} 2 & 0 & 4 \\ 5 & -1 & 1 \\ 9 & 7 & 8 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 0 & 4 \\ 5 & -1 & 1 \\ 9 & 7 & 8 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 7 & 8 \end{vmatrix} -0\begin{vmatrix} 5 & 1 \\ 9 & 8 \end{vmatrix} +4\begin{vmatrix} 5 & -1 \\ 9 & 7 \end{vmatrix}$$

$$= 2\ (-1 \times 8 - 1 \times 7) - 0\ (5 \times 8 - 9 \times 1) + 4\ (5\text{x}7 - (-1) \times 9)$$
$$= 2\ (-8 - 7) - 0\ (40 - 9) + 4\ (35 + 9)$$
$$= -30 - 0 + 176 = 146$$

Properties of Determinants

(i) If all the elements in a row or in a (column) of a determinant are multiplied by a constant k, then the value of the determinant is multiplied by k.

(ii) The value of the determinant is unaltered when a constant multiple of the elements of any row (column), is added to the corresponding elements of a different row (column) in a determinant.

(iii) If each element of a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant is expressed as the sum of two or more determinants of the same order

(iv) If any two rows or columns of a determinant are proportional, then the value of the

(vi) If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes only in sign.

(vii) If the determinant has two identical rows (columns), then the value of the determinant is zero.

CRAMERS RULE

1. Solve the following equations by using Cramer's rule

(i) $2x + 3y = 7$; $3x + 5y = 9$

(ii) $5x + 3y = 17$; $3x + 7y = 31$

(iii) $2x + y - z = 3$, $x + y + z = 1$, $x - 2y - 3z = 4$

(iv) $x + y + z = 6$, $2x + 3y - z = 5$, $6x - 2y - 3z = -7$

Solution:

(i) $2x + 3y = 7$, $3x + 5y = 9$

Solution:

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1 \neq 0.$$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 7 & 3 \\ 9 & 5 \end{vmatrix} = 7(5) - 9(3)$$

$$= 35 - 27 = 8$$

$$\Delta y = \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} = 2(9) - 3(7)$$

$$= 18 - 21 = -3$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{8}{1} = 8$$

$$y = \frac{\Delta y}{\Delta} = \frac{-3}{1} = -3$$

(ii) $5x + 3y = 17$; $3x + 7y = 31$

Solution:

$$\Delta = \begin{vmatrix} 5 & 3 \\ 3 & 7 \end{vmatrix} = 5(7) - 3(3)$$

$$= 35 - 9 = 26$$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

z

$$\Delta x = \begin{vmatrix} 17 & 3 \\ 31 & 7 \end{vmatrix} = 17(7) - 31\,(3)$$

$$= 119 - 93 = 26$$

$$\Delta y = \begin{vmatrix} 5 & 17 \\ 3 & 31 \end{vmatrix} = 5\,(31) - 17\,(3)$$

$$= 155 - 51 = 104.$$

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{26}}{\cancel{26}} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{4}{\cancel{104}}}{\cancel{26}} = 4$$

(iii) $2x + y - z = 3, x + y + z = 1, x - 2y - 3z = 4$

Solution:

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = 2$$

$$\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 2\,(-3+2) - 1\,(-3\,-1)\,-1\,(-2-1)$$

$$= 2(-1) - 1\,(-4) - 1\,(-3)$$

$$= -2 + 4 + 3 = 5.$$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$x = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}$$

$$= 3(-3 + 2) - 1(-3\,-4) - 1(-2\,-4)$$

$$= 3\,(-1) - 1\,(-7) - 1\,(-6)$$

$$= -3 + 7 + 6 = 10.$$

$$\Delta y = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= 2(-3\,-4) - 3\,(-3\,-1) - 1\,(4-1)$$

$$= 2\,(-7) - 3\,(-4) - 1(3)$$

$$= -14 + 12 - 3 = -5$$

$$\Delta z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$= 2(4+2) - 1(4-1) + 3(-2-1)$

$= 2(6) - 1(3) + 3(-3)$

$= 12 - 3 - 9$

$= 0$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{\overset{2}{\cancel{10}}}{\cancel{5}} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{\cancel{-5}}{\cancel{5}} = -1$$

$$z = \frac{\Delta z}{\Delta} = \frac{0}{5} = 0$$

(iv) $x + y + z = 6,\ 2x + 3y - z = 5,\ 6x - 2y - 3z = -7.$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 6 & -2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & -1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 6 & -2 \end{vmatrix}$$

$= 1(-9-2) - 1(-6+6) + 1(-4-18)$

$= 1(-11) - 1(0) + 1(-22)$

$= -11 - 22 = -33 \neq 0$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 6 & 1 & 1 \\ 5 & 3 & -1 \\ -7 & -2 & -3 \end{vmatrix}$$

$$= 6\begin{vmatrix} 3 & -1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 5 & -1 \\ -7 & -3 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ -7 & -2 \end{vmatrix}$$

$= 6(-9-2) - 1(-15-7) + 1(-10+21)$

$= 6(-11) - 1(-22) + 1(11)$

$= -66 + 22 + 11 = -33$

$$\Delta y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 5 & -1 \\ 6 & -7 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & -1 \\ -7 & -3 \end{vmatrix} - 6\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 1\begin{vmatrix} 2 & 5 \\ 6 & -7 \end{vmatrix}$$

$= 1(-15 -7) -6(-6 +6) + 1(-14 -30)$

$= 1(-22) -6(0) + 1 (-44)$

$= -22 - 44 = - 66$

$$\Delta z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & 5 \\ 6 & -2 & -7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 6 & -7 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 6 & -2 \end{vmatrix}$$

$= 1(-21 +10) -1(-14 -30) +6 (-4 -18)$
$=1(-11) -1(-44) +6(-22)$
$= -11 + 44 - 132 = - 99$

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{-33}}{\cancel{-33}} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{2}{\cancel{-66}}}{\cancel{-33}} = 2$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{3}{\cancel{-99}}}{\cancel{-33}} = 3$$

CHARACTERISTIC EQUATION

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get nroots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution Vector

2. Find the characteristic equation of $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

<u>Solution</u>: Let A $= \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equationl of A is $\lambda^2 - S_1\lambda + S_2 . S_1 = sum of the main diagonal elements = 3 + 2 = 5$ and $S_2 = Determinant of A = |A| = 3(2) - 1(-1) = 7$

Working rule to find characteristic

equation: For a 3 x 3 matrix:

Method 1:

The characteristic equation is $|A-\lambda I|=0$

Method 2:

Its characteristic equation can be written as $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$ where
S1 = sum of the main diagonal elements,
S2 = sum of the minors of the main diagonal elements,
S3 = Determinant of A =|A|

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A-\lambda I|=0$

Method 2:

Its characteristic equation can be written as $\lambda^2-S_1\lambda+S_2=0$ where S1 = sum of the main diagonal elements, S2 = Determinant of A

Examples:

1. Find the characteristic equation of $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Solution: Its characteristic equation is $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$,

where S1 = sum of the main diagonal elements = 8 + 7 + 3 = 18,

S2 = sum of the minors of the main diagonal elements=45

S3 = Determinant of A =| | A =0

Therefore, the characteristic equation is $\lambda^3-18\lambda^2+45\lambda=0$.

CAYLEY-HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A.

(2) To calculate the inverse of a square matrix A.

Problems:

1. Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation

Solution:Let A $=\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$$S_1 = Sum\, of\, the\, main\, diagonal\, elements = 1 + 1 = 2$$

$$S_2 = |A| = 1 - (-4) = 5$$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$$

$$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Therefore, the given matrix satisfies its own characteristic equation.

2. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and hence find its inverse.**

Solution: The characteristic polynomial of A is p(λ) = $\lambda^2 - \lambda - 1$.

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^2 - A - I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^2 - A - I = 0,$$

Multiplying by A^{-1} we get $A - I - A^{-1} = 0$,

$$A^{-1} = A - I$$

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

3. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ and hence find is inverse.**

Solution: The characteristic polynomial of A is p(λ) = $\lambda^3 - 2\lambda^2 - 5\lambda + 6$.

$$A^2 = \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}, A^3 = \begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix}$$

To verify $A^3 - 2A^2 - 5A + 6I = 0$ -------------- (1)

$$A^3 - 2A^2 - 5A + 6I =$$

$$\begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix} - 2\begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix} - 5\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply equation (1) by A^{-1}

We get $A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = 5I + 2A - A^2$

$$6A^{-1} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2) = 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

SCHOOL OF PHARMACY

DEPARTMENT OF PHARMACY

COURSE NAME: REMEDIAL MATHEMATICS

COURSE CODE: BP106RMT

UNIT – III – CALCULUS– BP106RMT

Unit - III

Differential Calculus

Introduction

Derivatives are a fundamental tool of calculus. For example, the derivative of the position of a moving object with respect to time is the object's velocity. This measures how quickly the position of the object changes when time is advanced.

Differentiation

The rate at which a function changes with respect to independent variable is called the derivative of the function. The derivative of a function $y = f(x)$is defined in terms of the limit involving increments of the independent and dependent variables and this limit is denoted by$\frac{dy}{dx}$. The process of finding the derivative of a function is called the derivative of a function.

First derivative formulas

S. No.	$y = f(x)$	$\frac{dy}{dx}$
1.	x^n	nx^{n-1}
2.	e^x	e^x
3.	$log\, x$	$\frac{1}{x}$
4.	$sin\, x$	$cos\, x$
5.	$cos\, x$	$-sin\, x$
6.	$\tan x$	$sec^2\, x$
7.	$\cot x$	$-cosec^2\, x$
8.	$sec\, x$	$sec\, x\, tanx$
9.	$cosec\, x$	$-co\, sec\, x\, cotx$
10.	k	0
11.	ku	$k\frac{du}{dx}$
12.	$u \pm v$	$\frac{du}{dx} \pm \frac{dv}{dx}$
13.	uv	$u\frac{dv}{dx} \pm v\frac{du}{dx}$
14.	$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
15.	a^x	$a^x logx$

Example 1:

Differentiate the following with respect to x

(i) $x+\frac{1}{x}$ (ii) $7x^3 + 4x^2 - 3x + 2$ (iii) $x^2(2x-1)$ (iv)$5x^2 e^x logx$

(v) $\frac{2logx}{x}$

Solution

(i) Let $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - 1.x^{-2}$

$= 1 - \frac{1}{x^2}$.

(ii) Let $y = 7x^3 + 4x^2 - 3x + 2$

$\frac{dy}{dx} = 7.3x^2 + 4.2x - 3 + 0$

$= 21x^2 + 8x - 3.$

(iii)Let $y = x^2(2x-1)$ $\frac{dy}{dx} = x^2 \frac{d(2x-1)}{dx} + (2x-1)\frac{d(x^2)}{dx}$

$=x^2 .2 + (2x-1)2x$

$= 2x^2 + 4x^2 - 2x$

$= 6x^2 - 2x.$

(iv)Let$y = 5x^2 e^x logx$

$$\frac{dy}{dx} = 5x^2 e^x \frac{d(logx)}{dx} + 5x^2 logx \frac{d(e^x)}{dx} + e^x logx \frac{d(5x^2)}{dx}$$

$= 5x^2 e^x \frac{1}{x} + 5x^2 logx e^x + e^x logx. 10x$

$$= 5xe^x(\frac{1}{x} + xlogx + 2logx).$$

(v)Let $y = \frac{2logx}{x}$ $\frac{dy}{dx} = \frac{x\frac{d(2logx)}{dx} - 2logx\frac{d(x)}{dx}}{x^2}$

$= \frac{x.\frac{2}{x} - 2logx.1}{x^2} = \frac{2(1-logx)}{x^2}$.

Example 2

Differentiate $6x^4 - 7x^3 + 3x^2 - x + 8$ with respect to x.

Solution:

Let $y = 6x^4 - 7x^3 + 3x^2 - x + 8$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^4) - \frac{d}{dx}(7x^3) + \frac{d}{dx}(3x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(8)$$

$$= 6\frac{d}{dx}(x^4) - 7\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(8)$$

$$= 6(4x^3) - 7(3x^2) + 3(2x) - (1) + 0$$

$$\frac{dy}{dx} = 24x^3 - 21x^2 + 6x - 1$$

Example 14

Find the derivative of $3x^{2/3} - 2\log_e x + e^x$

Solution:

Let $y = 3x^{2/3} - 2\log_e x + e^x$

$$\frac{dy}{dx} = 3\frac{d}{dx}(x^{2/3}) - 2\frac{d}{dx}(\log_e x) + \frac{d}{dx}(e^x)$$

$$= 3(2/3)x^{-1/3} - 2(1/x) + e^x$$

$$= 2x^{-1/3} - 2/x + e^x$$

Example 4

Differentiate : cosx . logx with respect to x

Solution:

Let $y = \cos x \,.\, \log x$

$$\frac{dy}{dx} = \cos x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\cos x)$$

$$= \cos x\,\frac{1}{x} + (\log x)(-\sin x)$$

$$= \frac{\cos x}{x} - \sin x \log x$$

Example 5

Differentiate $x^2 e^x \log x$ with respect to x

Solution:

Let $y = x^2 e^x \log x$

$$\frac{dy}{dx} = x^2 e^x \frac{d}{dx}(\log x) + x^2 \log x \frac{d}{dx}(e^x) + e^x \log x \frac{d}{dx}(x^2)$$

$$= (x^2 e^x)(1/x) + x^2 \log x\,(e^x) + e^x \log x\,(2x)$$

$$= x e^x + x^2 e^x \log x + 2x e^x \log x$$

$$= x e^x (1 + x \log x + 2 \log x)$$

7.3.11 Successive Differentiation

Let y be a function of x, and its derivative $\frac{dy}{dx}$ is in general another function of x. Therefore $\frac{dy}{dx}$ can also be differentiated. The derivative of $\frac{dy}{dx}$ namely $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the derivative of the second order. It is written as $\frac{d^2y}{dx^2}$ (or) y_2. Similarly the derivative of $\frac{d^2y}{dx^2}$ namely $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is called the third order derivative and it is written as $\frac{d^3y}{dx^3}$ and so on.

Derivatives of second and higher orders are called higher derivatives and the process of finding them is called Successive differentiation.

Example 29

If $y = e^x \log x$ find y_2

Solution:

$$y = e^x \log x$$

$$y_1 = e^x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(e^x)$$

$$= \frac{e^x}{x} + \log x\,(e^x)$$

$$y_1 = e^x\left(\frac{1}{x} + \log x\right)$$

$$y_2 = e^x \frac{d}{dx}\left(\frac{1}{x} + \log x\right) + \left(\frac{1}{x} + \log x\right)\frac{d}{dx}(e^x)$$

$$y_2 = e^x\left\{-\frac{1}{x^2} + \frac{1}{x}\right\} + \left(\frac{1}{x} + \log x\right)e^x$$

$$= e^x\left\{-\frac{1}{x^2} + \frac{1}{x} + \frac{1}{x} + \log x\right\}$$

$$= e^x\left\{\left(\frac{2x-1}{x^2}\right) + \log x\right\}$$

MAXIMA AND MINIMA

EXAMPLE:

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

(i) $f(x) = 4x^2 - 4x + 4$ on R

Solution:

Given $f(x) = 4x^2 - 4x + 4$ on R

$= 4x^2 - 4x + 1 + 3$

By grouping the above equation we get,

$= (2x - 1)^2 + 3$

Since, $(2x - 1)^2 \geq 0$

$= (2x – 1)^2 + 3 \geq 3$

$= f(x) \geq f(½)$

Thus, the minimum value of f(x) is 3 at x = ½

Since, f(x) can be made large. Therefore maximum value does not exist.

(ii) $f(x) = x^3 – 6x^2 + 9x + 15$

Solution:

Given, $f(x) = x^3 – 6x^2 + 9x + 15$

Differentiate with respect to x, we get, $f'(x) = 3x^2 – 12x + 9 = 3(x^2 – 4x + 3)$

$= 3 (x – 3) (x – 1)$

For all maxima and minima,

$f'(x) = 0$

$= 3(x – 3) (x – 1) = 0$

$= x = 3, 1$

At x = 1 f '(x) changes from negative to positive

Since, x = – 1 is a point of Maxima

At x =3 f'(x) changes from negative to positive

Since, x =3 is point of Minima.

Hence, local maxima value $f (1) = (1)^3 – 6(1)^2 + 9(1) + 15 = 19$

Local minima value $f (3) = (3)^3 – 6(3)^2 + 9(3) + 15 = 15$

(iii) $f(x) = x^3 – 3x$

Solution:

Given, $f (x) = x^3 – 3x$

Differentiate with respect to x then we get,

$f' (x) = 3x^2 – 3$

Now, $f'(x) =0$

$= 3x^2 = 3 \Rightarrow x = \pm 1$

Again differentiate $f'(x) = 3x^2 – 3$

$f''(x)= 6x$

$f''(1)= 6 > 0$

f’’ (– 1)= – 6 > 0

By second derivative test, x = 1 is a point of local minima and local minimum value of g at

x = 1 is f (1) = 1^3 – 3 = 1 – 3 = – 2

However, x = – 1 is a point of local maxima and local maxima value of g at

x = – 1 is f (– 1) = $(– 1)^3$ – 3(– 1)

= – 1 + 3

= 2

Hence, the value of minima is – 2 and maxima is 2.

Applications of derivative

Example 6:

The total cost function for the production of x units of an item is given by $T = 10- 4x^3 + 3x^4$. Find (a) average cost (b) marginal cost (c) marginal average cost.

Solution

Given $T = 10- 4x^3 + 3x^4$

(a) Average cost = $\frac{T}{x} = \frac{10- 4x^3+ 3x^4}{x}$ (b) marginal cost = $\frac{dT}{dx} = - 12x^3 + 12x^3$ (c) marginal average cost = $d\left(\frac{T}{x}\right) = \frac{x(- 12x^3+ 12x^3)-(10- 4x^3+ 3x^4).1}{x^2} = \frac{9x^4- 8x^3-10}{x^2}$.

Example 7:

Let the cost function of a firm be given by the following equation: $C(x) = 300x - 10x^2 + \frac{1}{3}x^3$ where $C(x)$ stands for the cost function and x for output. Calculate (i) output at which marginal cost is minimum (ii) output at which average cost is minimum (iii) output at which average cost is equal marginal cost.

Solution

Given $C(x) = 300x - 10x^2 + \frac{1}{3}x^3$ (i) Marginal cost (MC) = $\frac{dC}{dx} = 300 - 20x + x^2 \frac{d(MC)}{dx} = -20 + 2x \frac{d^2(MC)}{dx^2} = -20 \frac{d(MC)}{dx} = 0$ gives $-20 + 2x = 0 \therefore x = 10 \frac{d^2(MC)}{dx^2} < 0$ ∴MC is maximum when the output is 10. (ii) Average cost (AC) = $\frac{C(x)}{x}$ = $300 - 10x + \frac{x^2}{3} \frac{d(AC)}{dx} = -10 + \frac{2}{3}x \frac{d^2(AC)}{dx^2} = \frac{2}{3} \frac{d(AC)}{dx} = 0$ gives $-10 + \frac{2}{3}x = 0 \therefore x = 15$ and $\frac{d^2(C)}{dx^2} > 0$ ∴ AC is minimum when the output is 15. (iii) When average cost = marginal cost$300 - 10x + \frac{x^2}{3}$ = $300 - 20x + x^2$i.e.,$2x^2 - 30x = 02x(x - 15) = 0 \therefore x = 0$

or $\therefore x = 15 x = 0$ is inadmissible.∴Output at which average cost is equal to marginal cost is 15 units.

Example 8:

The cost function for producing x units of a product is $C(x) = x^3 - 12x^2 + 48x + 11$ (in rupees) and the revenue function is R$(x) = 83x - 4x^2 - 21$. Find the output for which profit is maximum.

Solution

Given $C(x) = x^3 - 12x^2 + 48x + 11$ and R$(x) = 83x - 4x^2 - 21$

Profit function P = R – C = $(83x - 4x^2 - 21) - x^3 - 12x^2 + 48x + 11 = -x^3 + 8x^2 + 35x - 32 \frac{dP}{dx} = -3x^2 + 16x + 35 \frac{d^2P}{dx^2} = -6x + 16$

When $\frac{dP}{dx} = 0, \quad -3x^2 + 16x + 35 = 03x^2 - 16x - 35 = 03x^2 - 21x + 5x - 35 = 03x(x-7) + 5(x-7) = 0(3x+5)(x-7) = 0x = 7$ or $\frac{-5}{3} x = \frac{-5}{3}$ is inadmissible. Also $\frac{d^2P}{dx^2} < 0$ ∴The profit is maximum when the output is 7 units.

The maximum profit $= -7^3 + 8.7^2 + 35\,.7 - 32 = -343 + 392 + 245 - 32 = Rs.262/-$

SCHOOL OF PHARMACY

DEPARTMENT OF PHARMACY

COURSE NAME: REMEDIAL MATHEMATICS

COURSE CODE: BP106RMT

UNIT – IV – ANALYTICAL GEOMETRY– BP106RMT

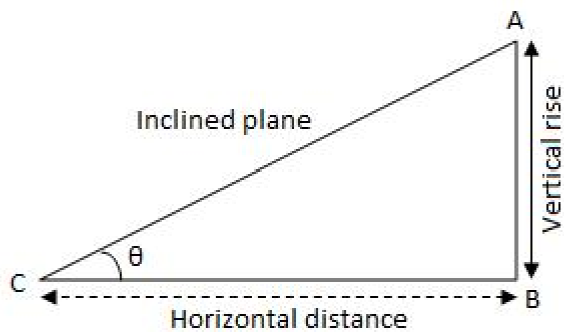
UNIT-4

ANALYTICAL GEOMETRY

We will discuss here about the slope of a line or gradient of a line.

Concept of slope (or gradient):

If θ (≠ 90°) is the inclination of a straight line, then tan θ is called its slope or gradient. The slope of any inclined plane is the ratio between the vertical rise of the plane and its horizontal distance.



slope = verticalrise/horizontaldistance =AB/BC = tan θ

Where θ is the angle which the plane makes with the horizontal

Slope of a straight line:

The slope of a straight line is the tangent of its inclination and is denoted by letter 'm' i.e. if the inclination of a line is θ, its slope m = tan θ.

Condition for perpendicularity

We will discuss here about the condition of perpendicularity of two straight lines.

Let the lines AB and CD be perpendicular to each other. If the inclination of AB with the positive direction of the x-axis is θ then the inclination of CD with the positive direction of the x-axis will be 90° + θ.

Therefore, the slope of AB = tan θ, and the slope of CD = tan (90° + θ).

From trigonometry, we have, tan (90° + θ) = - cot θ

Therefore, if the slope of AB is m_1 and
the slope CD = m_2 then
$m_1 = \tan\theta$ and $m_2 = -\cot\theta$.
So, $m_1 \cdot m_2 = \tan\theta \cdot (-\cot\theta) = -1$
Two lines with slopes m_1 and m_2 are perpendicular to each other if and only if $m_1 \cdot m_2 = -1$

1. Find the equation of the line passing through the point (-2, 0) and perpendicular to the line $4x - 3y = 2$.

Solution:

First we need to express the given equation in the form $y = mx + c$.

Given equation is $4x - 3y = 2$.

$-3y = -4x + 2$

$y = 4/3x - 2/3$
Therefore, the slope (m) of the given line = 4/3
Let the slope of the required line be m_1.

According to the problem the required line is perpendicular to the given line.

Therefore, from the condition of perpendicularity we get,

$m_1 \cdot 4/3 = -1$
$\Rightarrow m_1 = -3/4$
Thus, the required line has the slope -3/4 and it passes through the point (-2, 0).

Therefore, using the point-slope form we get

$y - 0 = -3/4\{x - (-2)\}$
$\Rightarrow y = -3/4(x + 2)$

$\Rightarrow 4y = -3(x + 2)$

$\Rightarrow 4y = -3x + 6$

$\Rightarrow 3x + 4y + 6 = 0$, which is the required equation.

Condition of parallelism

If two lines are parallel then they are inclined at the same angle θ with the positive direction of the x-axis. So, their slopes are equal.

Two lines with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$

Note: If the slope of a line is m then any line parallel to it will also have the slope m.

1. Prove that the lines $3x - 2y - 1 = 0$ and $9x - 6y + 5 = 0$ are parallel.

Solution:

The slope of the lines can be found by comparing the equations with $y = mx + c$.

Equation of the first straight line $3x - 2y - 1 = 0$

Now we need to express the given equation in the form $y = mx + c$.

$3x - 2y - 1 = 0$

$\Rightarrow -2y = -3x + 1$

$\Rightarrow y = -3/-2x + 1/-2$
$\Rightarrow y = 3/2x - 1/2$
Therefore, the slope (m_1) of the given line = 3/2

Equation of the second line $9x - 6y + 5 = 0$

Now we need to express the given equation in the form $y = mx + c$.

$9x - 6y + 5 = 0$

$\Rightarrow -6y = -9x - 5$

$\Rightarrow y = -9/-6x - 5/-6$
$\Rightarrow y = 3/2x + 5/6$
Therefore, the slope (m_2) of the given line = 3/2

Now we can clearly see that the slope of the first line m_1 = the slope of the second line m_2

Therefore, the given two lines are parallel.

Slope of a line joining two points

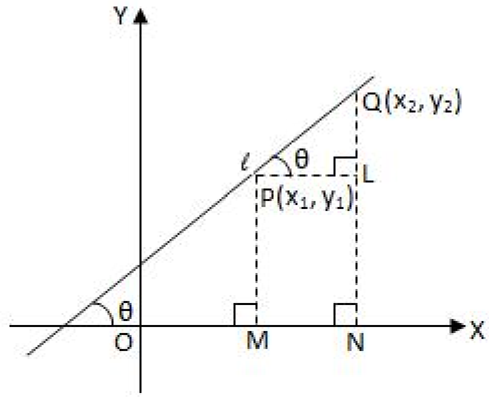
We will discuss here about the slope of the line joining two points.

To find the slope of a non-vertical straight line passing through two given fixed points:

Let P (x_1,y_1) and Q (x_2,y_2) be the two given points. According to the problem, the straight line PQ is non-vertical $x_2 \neq x_1$.

Required to find, the slope of the line through P and Q.

From P, Q draw perpendiculars PM, QN on x-axis and PL ⊥ NQ. Let θ be the inclination of the line PQ, then ∠LPQ = θ.



From the above diagram, we have

PL = MN = ON - OM = x_2-x_1
LQ = = NQ - NL = NQ - MP = y_2-y_1

Therefore, the slope of the line PQ = tan θ = LQ/PL= y_2-y_1/x_2-x_1

Hence, the slope (m) of a non-vertical line passing through the points P (x_1,y_1) and Q (x_2,y_2) is given by

slope = m = y_2-y_1/x_2-x_1

1. Find the slope of the line passing through the points M (-2, 3) and N (2, 7).

Solution:

Let M (-2, 3) = (x_1, y_1) and N (2, 7) = (x_2, y_2)

We know that the slope of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is

m = y_2-y_1/x_2-x_1

Therefore, slope of MN = y_2-y_1/x_2-x_1=7-3/2+2=4/4=1

2. Find the slope of the line passing through the pairs of points (-4, 0) and origin.

Solution:

We know that the coordinate of the origin is (0, 0)

Let P (-4, 0) = (x_1, y_1) and O (0, 0) = (x_2, y_2)

We know that the slope of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is

m = y_2-y_1/x_2-x_1

Therefore, slope of PO = y_2-y_1/x_2-x_1= 0/4=0

We will learn how to find the slope and y-intercept of a line.

slope and y-intercept of a given line:

Step I: Convert the given equation of the line in the slope-intercept form y = mx + c.

Step II: Then, the co-efficient of x is slope (m) and the constant term term with its proper sign is y-intercept (c).

1. Find the slope and y-intercept of the line 2x - 3y - 4 = 0.

Solution:

Given equation is 2x - 3y - 4 = 0

⟹ -3y = -2x + 4

⟹ y = 2/3x – 4/3
Therefore, the slope (m) of the given line = 2/3 and its y-intercept (c) = - 4/3

2. Find the slope and y-intercept of the line y = 4

Solution:

First we need to express the given equation in the form y = mx + c.

Given equation is y = 4

⟹ y = 0x + 4

Therefore, the slope (m) of the given line = 0 and its y-intercept (c) = 4

UNIT-4

INTEGRAL CALCULUS

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions. If a function f is differentiable in an interval I, i.e., its derivative f' exists at each point of I, then a natural question arises that given f' at each point of I, can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the indefinite integral of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types: (a) the problem of finding a function whenever its derivative is given, (b) the problem of finding the area bounded by the graph of a function under certain conditions. These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the Integral Calculus.

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down immediately the corresponding formulae (referred to as standard formulae) for the integrals of these functions, as listed below which will be used to find integrals of other functions.

Derivatives	**Integrals (Anti derivatives)**
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$;	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C$, $n\neq -1$
Particularly, we note that	
$\frac{d}{dx}(x)=1$;	$\int dx=x+C$
(ii) $\frac{d}{dx}(\sin x)=\cos x$;	$\int \cos x\,dx=\sin x+C$
(iii) $\frac{d}{dx}(-\cos x)=\sin x$;	$\int \sin x\,dx=-\cos x+C$
(iv) $\frac{d}{dx}(\tan x)=\sec^2 x$;	$\int \sec^2 x\,dx=\tan x+C$
(v) $\frac{d}{dx}(-\cot x)=\operatorname{cosec}^2 x$;	$\int \operatorname{cosec}^2 x\,dx=-\cot x+C$
(vi) $\frac{d}{dx}(\sec x)=\sec x\tan x$;	$\int \sec x\tan x\,dx=\sec x+C$
(vii) $\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x\cot x$;	$\int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+C$
(viii) $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1} x+C$
(ix) $\frac{d}{dx}(-\cos^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1} x+C$
(x) $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=\tan^{-1} x+C$
(xi) $\frac{d}{dx}(-\cot^{-1} x)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=-\cot^{-1} x+C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right)=\frac{1}{x\sqrt{x^2-1}}$; $\int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1} x+\text{C}$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right)=\frac{1}{x\sqrt{x^2-1}}$; $\int\frac{dx}{x\sqrt{x^2-1}}=-\operatorname{cosec}^{-1}x+\text{C}$

(xiv) $\frac{d}{dx}(e^x)=e^x$; $\int e^x dx=e^x+\text{C}$

(xv) $\frac{d}{dx}(\log|x|)=\frac{1}{x}$; $\int\frac{1}{x}\,dx=\log|x|+\text{C}$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{log\ a}\right)=a^x$; $\int a^x dx=\frac{a^x}{log\ a}+\text{C}$

I. Find the following integrals.

1. $\int(5x^2-8x+5)dx=\boxed{\frac{5x^3}{3}-4x^2+5x+C}$

2. $\int(-6x^3+9x^2+4x-3)dx=\boxed{\frac{-3x^4}{2}+3x^3+2x^2-3x+C}$

3. $\int(x^{\frac{3}{2}}+2x+3)dx=\boxed{\frac{2x^{\frac{5}{2}}}{5}+x^2+3x+C}$

4. $\int\left(\frac{8}{x}-\frac{5}{x^2}+\frac{6}{x^3}\right)dx=\int\left(\frac{8}{x}-5x^{-2}+6x^{-3}\right)dx$

$=8Ln(x)-\frac{5x^{-1}}{-1}+\frac{6x^{-2}}{-2}=\boxed{8Ln(x)+\frac{5}{x}-\frac{3}{x^2}+C}$

5. $\int(\sqrt{x}+\frac{1}{3\sqrt{x}})dx=\int\left(x^{\frac{1}{2}}+\frac{1}{3}x^{-\frac{1}{2}}\right)dx$

$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{1}{3}\frac{x^{\frac{1}{2}}}{\frac{1}{2}}=\boxed{\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{3}x^{\frac{1}{2}}+C}$

6. $\int (12x^{\frac{3}{4}} - 9x^{\frac{5}{3}})dx = \boxed{\frac{48x^{\frac{7}{4}}}{7} - \frac{27x^{\frac{8}{3}}}{8} + c}$

7. $\int \frac{x^2+4}{x^2} dx = \int 1 + 4x^{-2} \ \ dx = \boxed{x - \frac{4}{x} + C}$

8. $\int \frac{1}{x\sqrt{x}} dx = \int x^{-\frac{3}{2}} dx = \boxed{-\frac{2}{\sqrt{x}} + C}$

9. $\int (1+3t)t^2\, dt = \int t^2 + 3t^3 \ \ dt = \boxed{\frac{t^3}{3} + \frac{3t^4}{4} + C}$

Methods of Integration In previous section, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function F whose derivative is f which led us to the integral of f. However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on: 1. Integration by Substitution 2. Integration using Partial Fractions 3. Integration by Parts 7.3.1 Integration by substitution

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\int \tan x \, dx = \log|\sec x| + C$

We have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Put $\cos x = t$ so that $\sin x \, dx = -dt$

Then $\int \tan x \, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$

or $\int \tan x \, dx = \log|\sec x| + C$

(ii) $\int \cot x \, dx = \log|\sin x| + C$

We have $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

Put $\sin x = t$ so that $\cos x \, dx = dt$

Then $\int \cot x \, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$

(iii) $\int \sec x \, dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x \, dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

Put $\sec x + \tan x = t$ so that $\sec x\,(\tan x + \sec x)\, dx = dt$

Therefore, $\int \sec x \, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

We have

$$\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} \, dx$$

Put $\operatorname{cosec} x + \cot x = t$ so that $-\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\, dx = dt$

So $\int \operatorname{cosec} x \, dx = -\int \frac{dt}{t} = -\log|t| = -\log|\operatorname{cosec} x + \cot x| + C$

$$= -\log\left|\frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x - \cot x}\right| + C$$

$$= \log\left|\operatorname{cosec} x - \cot x\right| + C$$

INTEGRALS OF SOME PARTICULAR FUNCTION

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1) $\displaystyle \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(2) $\displaystyle \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(3) $\displaystyle \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(4) $\displaystyle \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(5) $\displaystyle \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

(6) $\displaystyle \int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

We now prove the above results:

(1) We have $\dfrac{1}{x^2-a^2}=\dfrac{1}{(x-a)(x+a)}$

$$=\frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right]=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

Therefore, $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\left[\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right]$

$$=\frac{1}{2a}\left[\log|(x-a)|-\log|(x+a)|\right]+C$$

$$=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$$

Therefore, $\displaystyle\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\left[\int\frac{dx}{a-x}+\int\frac{dx}{a+x}\right]$

$$=\frac{1}{2a}[-\log|a-x|+\log|a+x|]+C$$

$$=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2-x^2}=\frac{1}{2a}\left[\frac{(a+x)+(a-x)}{(a+x)(a-x)}\right]=\frac{1}{2a}\left[\frac{1}{a-x}+\frac{1}{a+x}\right]$$

Therefore, $\displaystyle\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\left[\int\frac{dx}{a-x}+\int\frac{dx}{a+x}\right]$

$$=\frac{1}{2a}[-\log|a-x|+\log|a+x|]+C$$

$$=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$$

8.2.7 Integration by parts

If u and v are functions of x such that u is differentiable and v is integrable, then

$$\int u \; dv = uv - \int v \; du$$

Observation:

(i) When the integrand is a product, we try to simplify and use addition and subtraction rule. When this is not possible we use integration by parts.

(ii) While doing integration by parts we use 'ILATE' for the relative preference of u . Here,

I $\rightarrow$ Inverse trigonometic function

L $\rightarrow$ Logarithmic function

A $\rightarrow$ Algebraic function

T $\rightarrow$ Trigonometric function

E $\rightarrow$ Exponential function

Example 18

Evaluate $\int x.e^x \; dx$

Solution :

Let $u = x \quad , \quad dv = e^x \; dx$

$du = dx \quad , \quad v = e^x$

$$\int x.e^x \; dx = x\,e^x - \int e^x \; dx$$

$$= x\,e^x - e^x + C$$

$$= e^x\,(x-1) + C$$

Evaluate $\int x.\sin 2x \; dx$

Solution:

$$\text{Let } u = x \quad , \quad \sin 2x \, dx = dv$$

$$du = dx \quad , \quad \frac{-\cos 2x}{2} = v$$

$$\int x.\sin 2x \quad dx = \frac{-x\cos 2x}{2} + \int \frac{\cos 2x}{2} \, dx$$

$$= \frac{-x\cos 2x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2}$$

$$= \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} + C$$

AN APPLICATION OF INTEGRATION

Marginal cost, Total cost and Average cost:

Let the cost C of producing and marketing x units of a commodity be $C = f(x)$ The Average cost per unit is $AC = \frac{c}{x} = \frac{f(x)}{x}$.

The Marginal cost is MC= $\frac{dc}{dx} = f'(x)$.

The Marginal cost is derivative with respect to x of the total cost function $C = f(x)$.

Therefore the total cost function C is the integral with respect to x of the marginal cost function.

i.e $\int f'(x)dx + c$.

The constant of integration can be obtained by knowing an initial condition. Usually this is determined by knowing the fixed cost, that is the cost when $x = 0$

Marginal Revenue, Total Revenue and Average Revenue :

Let $y = f(x)$ be the price unit where x is the number of units sold.. Then the total revenue is given by $R = xy = xf(x)$.

The Average Revenue or the revenue per unit is given by $AR = \frac{R}{x} = f(x)$

The Marginal Revenue is given by $MR = \frac{dR}{dx} = R(x)$.

The marginal function is the derivative of revenue function. Thus the revenue function is the integral of the marginal revenue.

i.e $R(x) = \int R'(x)dx + c$.

The constant of integration is determined by an initial condition. The initial condition is that the revenue is zero if the demand is zero.

NOTE: Total profit function is given by $P(x) = R(x) - c(x)$.

EXAMPLE 1:

The Marginal cost function for production is $\frac{dc}{dx} = 10 + 24x - 3x^2$. If the total cost of producing one unit is Rs. 25.Find the total cost function and average cost function.

Sol:

$$\frac{dc}{dx} = 10 + 24x - 3x^2.$$

Integrating, C=$\int(10 + 24x - 3x^2)dx + c_1$

$$= 10x + 12x^2 - x^3 + c_1$$

To find $c_1$we take the conditions that c = 25 when x=1.

Therefore 25= 10+12-1+c_1

$\therefore c_1 = 4$

$\therefore$ The total cost function is C= $10x + 12x^2 - x^3 + 4$.

The average cost function AC is

$$AC = \frac{C}{x} = \frac{10x + 12x^2 - x^3 + 4}{x} = \frac{4}{x} + 10 + 12x - x^2$$

DEFINITE INTEGRAL

The definite integral of the continuous function f(x) between the limits x = a and x = b is defined as $\int_a^b f(x) \ dx = [\ F(x)\ ]_a^b$ = F (b) - F (a) where 'a' is the lower limit and 'b' is the upper limit and F(x) is the integral of f(x) .

To evaluate the definite integral, integrate the given function as usual . Then obtain the difference between the values by substituting the upper limit first and then the lower limit for x.

Example 25

Evaluate $\int_1^2 (4x^3 + 2x + 1) \ dx$

Solution:

$$\int_1^2 (4x^3 + 2x + 1) \ dx = \left[4\frac{x^4}{4} + 2\frac{x^2}{2} + x\right]_1^2$$

$$= (2^4 + 2^2 + 2) - (1 + 1 + 1)$$

$$= (16 + 4 + 2) - 3$$

$$= 19$$

Evaluate $\displaystyle\int_2^3 \frac{\mathbf{2x}}{\mathbf{1+x^2}}\ \mathbf{dx}$

Solution:

$$\int_2^3 \frac{2x}{1+x^2}\ dx = \int_5^{10} \frac{dt}{t}$$

Put $1 + x^2 = t$

$2x\,dx = dt$

When $x = 2$; $t = 5$

$x = 3$; $t = 10$

$$= [\log t]_5^{10} = \log 10 - \log 5$$

$$= \log_e \frac{10}{5}$$

$$= \log_e 2$$

SCHOOL OF PHARMACY

DEPARTMENT OF PHARMACY

COURSE NAME: REMEDIAL MATHEMATICS

COURSE CODE: BP106RMT

UNIT – V – DIFFERENTIAL EQUATIONS– BP106RMT

UNIT-5
DIFFERENTIAL EQUATIONS

Exact differential equation.

A first order differential equation of type $M(x,y)dx + N(x,y)dy = 0$ is called an *exact differential equation* if there exists a function of two variables $u(x,y)$ with continuous partial derivatives such that $du(x,y) = M(x,y)dx + N(x,y)dy$

The general solution of an exact equation is given by $u(x,y) + \int f(y)dy = c$,where c is an arbitrary constant

Test for Exactness

Let functions $M(x,y)$ and $N(x,y)$ have continuous partial derivatives in a certain domain *D*.

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Algorithm for Solving an Exact Differential Equation

1. First it's necessary to make sure that the differential equation is *exact* using the *test for exactness*:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

2. Integrate M with respect to x keeping y constant ie $\int Mdx$
3. Integrate those terms in N not containing x with respect to y.ie $\int \left[N - \frac{\partial}{\partial y} \int Mdx \right] dy$
4. The general solution of the exact differential equation is given by $\int Mdx + \int \left[N - \frac{\partial}{\partial y} \int Mdx \right] dy = c$

Example1. Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$$M = 5x^4 + 3x^2y^2 - 2xy^3 \qquad N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\Rightarrow \frac{\partial M}{\partial y} = 6x^2y - 6xy^2 \qquad and \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2 \qquad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \ \therefore the\ given\, equation\ is\, exact.$$

The required solution is given by $\int M dx + \int [terms\ of\ N\ not\ containing\ x]dy = c$

$\int (5x^4 + 3x^2y^2 - 2xy^3)dx \quad + \quad \int (-5y^4)dy = c$

$x^5 + x^3y^2 - x^2y^3 - y^5 = c$

Equations Reducible to Exact equations.

Rule1. $If\ \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is function of x alone, say f(x) $then\ I.F = e^{\int f(x)dx}$

Rule2. $If\ \frac{-1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is function of y alone, say f(y) $then\ I.F = e^{\int f(y)dy}$

Rule3. $If\ M\ is\ of\ the\ form\ M = yf_1(xy)\ N\ is\ of\ the\ form\ N = xf_2(xy), then\ I.F = \frac{1}{Mx - Ny}$

Rule4. $If Mdx + Ndy = 0\ is\ a$ hom$ogeneous\ equation\ in\ x\ and\ y\ then\ I.F = \frac{1}{Mx + Ny}$

Example2. Solve $(2x\log x - xy)dy + 2ydx = 0.$

Solution . Given $(2x\log x - xy)dy + 2ydx = 0.$ (1)

Here M = 2y, N = 2xlogx-xy.

$$\Rightarrow \frac{\partial M}{\partial y} = 2 \quad and \quad \frac{\partial N}{\partial x} = 2(1+\log x) - y \Rightarrow$$

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{-2\log x + y}{2x\log x - xy} = -\frac{1}{x} = f(x).$$

$$I.F = e^{\int f(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$$

$$(1) I.F \Rightarrow \frac{2y}{x}dx + (2\log x - y)dy = 0 \Rightarrow mdx + ndy = 0\ which\ is\ exact.$$

The required solution is given by $\int mdx + \int [terms\ of\ n\ not\ containing\ x]dy = c$

$\Rightarrow$ The required solution is given by $\int \frac{2y}{x}dx + \int (-y)dy = 0$.

$\Rightarrow$ The required solution is given by $2y\log x - \frac{y^2}{2} = 0$.

Example3. Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$

Solution . Given $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$ (1)

Here $M = y^4 + 2y \quad N = xy^3 + 2y^4 - 4x$

$$\Rightarrow \frac{\partial M}{\partial y} = 4y^3 + 2 \quad and \quad \frac{\partial N}{\partial x} = y^3 - 4 \Rightarrow \quad \frac{-1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = -\frac{(4y^3+2)-(y^3-4)}{y^4+2y} = -\frac{3}{y} = f(y).$$

$$I.F = e^{\int f(y)dy} = e^{\int \frac{-3}{y}dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$$

$$(1) I.F \Rightarrow (y + \frac{2}{y^2})dx + (x + 2y - \frac{4x}{y^3})dy = 0 \Rightarrow mdx + ndy = 0\ which\ is\ exact.$$

The required solution is given by $\int mdx + \int [terms\ of\ n\ not\ containing\ x]dy = c$

$\Rightarrow$ The required solution is given by $\int (y+\frac{2}{y^2})dx + \int (2y)dy = c$.

The required solution is given by $x(y+\frac{2}{y^2})+y^2 = c$

Example4. Solve $y(xy+2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$

Solution . Given $y(xy+2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$ (1)

$\Rightarrow y(1+2xy)dx + x(1-xy)dy = 0$

$M = y(1+2xy) = yf_1(xy)$ and $N = x(1-xy) = xf_2(xy)$,

Then I.F= $\frac{1}{Mx - Ny} == \frac{1}{y(1+2xy)x - x(1-xy)y} = \frac{1}{3x^2y^2}$

$(1) I.F \Rightarrow (\frac{1}{3x^2y}+\frac{2}{3x})dx + (\frac{1}{3xy^2}-\frac{1}{3y})dy = 0 \Rightarrow mdx + ndy = 0$ *which is exact.*

The required solution is given by $\int mdx + \int [terms\ of\ n\ not\ containing\ x]dy = c$

$\Rightarrow$ The required solution is given by $\int (\frac{1}{3x^2y}+\frac{2}{3x})dx + \int (-\frac{1}{3y})dy = c$.

$\Rightarrow$ The required solution is given by $-\frac{1}{3xy}+\frac{2\log x}{3}-\frac{\log y}{3} = c.$

Example5. Solve $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$.

Solution . Given $(x^3+y^3)dx + (xy^2)dy = 0.$ (1)

$Here\ M = (x^3+y^3)\ and\ N = -(xy^2)\ which\ are$ hom$ogeneous\ in\ x\ and\ y.$

$then\ I.F = \frac{1}{Mx+Ny} = \frac{1}{(x^3+y^3)x + (-xy^2)y} = \frac{1}{x^4}$

$(1) I.F \Rightarrow (\frac{1}{x}+\frac{y^3}{x^4})dx - (\frac{y^2}{x^3})dy = 0 \Rightarrow mdx + ndy = 0$ *which is exact.*

The required solution is given by $\int mdx + \int [terms\ of\ n\ not\ containing\ x]dy = c$

$\Rightarrow$ The required solution is given by $\int (\frac{1}{x}+\frac{y^3}{x^4})dx + \int (0)dy = c$.

$\Rightarrow$ The required solution is given by $\log x - \frac{y^3}{3x^3} = c.$

Example6. *Solve* $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$

UNIT 4

LAPLACE TRANSFORMS

1. Introduction

A transformation is mathematical operations, which transforms a mathematical expressions into another equivalent simple form. For example, the transformation logarithms converts multiplication division, powers into simple addition, subtraction and multiplication respectively.

The Laplace transform is one which enables us to solve differential equation by use of algebraic methods. Laplace transform is a mathematical tool which can be used to solve many problems in Science and Engineeing. This transform was first introduced by Laplace, a French mathematician, in the year 1790, in his work on probability theory. This technique became very popular when heaveside funcitons was applied to the solution of ordinary differential equation in electrical Engeneering problems.

Many kinds of transformation exist, but Laplace transform and fourier transform are the most well known. The Laplace transform is related to fourier transform, but whereas the fourier transform expresses a function or signal as a series of mode of vibrations, the Laplace transform resolves a function into its moments.

Like the fourier transfrom, the Laplace transform is used for solving differential and integral equations. In Physics and Engineering it is used for analysis of linear time invariant systems such as electrical circuits, harmonic oscillators, optical devices and mechanical systems. In such analysis, the Laplace transform is often interpreted as a transformation form the time domain in which inputs and outputs are functions of time, to the frequency domain, where the same inputs and outputs are functions of complex angular frequency in radius per unit time. Given a simple mathematical or functional discription of an input or output to a system, the Laplace transform provides an alternative functional discription that often simplifies the process of analyzing the behaviour of the system or in synthesizing a new system based on a set of specification. The Laplace transform belongs to the family of integral transforms. The solutions of mechanical or electrical problems involving discontinuous force function are obtained easily by Laplace transforms.

1.1 Definition of Laplace Transforms

Let $f(t)$ be a functions of the variable t which is defined for all positive values of t. Lets be the real constant. If the integral $\int_0^\infty e^{-st} f(t)dt$ exist and is equal to F(s), then F(s) is called the Laplace transform of $f(t)$ and is denoted by the symbol L[f(t)].

i.e. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F[s]$

The Laplace Transform of $f(t)$ is said to exist if the integral converges for some values of s, otherwise it does not exist.

Here the operator L is called the Laplace transform operator which transforms the functions $f(t)$ into F(s).

Remark: $\lim_{S\to\infty} F(s) = 0$.

1.2 Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous in any interval $[a,b]$ if it is defined on that interval, and the interval can be divided into a finite number of sub intervals in each of which $f(t)$ is continuous.

In otherwords piecewise continuous means *f(t)* can have only finite number of finite discontinuities.

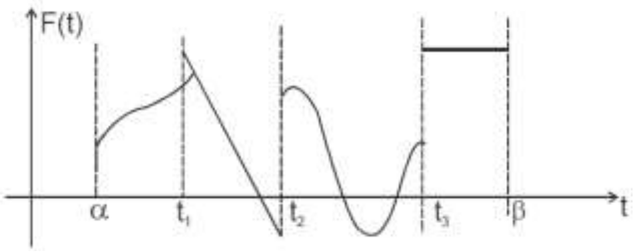


Figure 1.1

An example of a function which is periodically or sectional continuous is shown graphically in Fig 1.1 above. This function has discontinuities at t_1, t_2 and t_3.

1.3 Definition of Exponential order

A function $f(t)$ is said to be of exponential order if $\lim_{t\to\infty} e^{-st} f(t) = 0$.

1.4 Sufficient conditions for the existence of the Laplace Transforms

Let $f(t)$ be defined and continuous for all positive values of t. The Laplace Transform of $f(t)$ exists if the following conditions are satisfied.

(i) $f(t)$ is piecewise continuous (or) sectionally continuous.

(ii) $f(t)$ should be of exponential order.

1.5 Seven Indeterminates

1.	$\frac{0}{0}$	4.	$\infty \times \infty$	7.	0^0.
2.	$\frac{\infty}{\infty}$	5.	1^{∞}		
3.	$0 \times \infty$	6.	∞^0.		

2. Laplace Transform of Standard functions

1. Prove that $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$ or $s > -a$

Proof:

By definition $L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt$

$$L[e^{-at}] = \int_0^{\infty} e^{-st}.e^{-at}\, dt = \int_0^{\infty} e^{-t(s+a)}\, dt$$

Hence $L[e^{-at}] = \frac{1}{s+a}$

2. Prove that $L[e^{at}] = \frac{1}{s-a}$ where $s > a$

Proof:

By the defn of $L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt$

$$L[e^{+at}] = \int_0^{\infty} e^{-st}.e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt = \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^{\infty}$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^{0}\right] = \frac{1}{s+a}$$

Hence $L[e^{at}]$ $= \frac{1}{s-a}$

3. $L(\cos at) = \int_0^\infty e^{-st} \cos at\, dt$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty = 0 - \frac{1}{s^2+a^2}(-S)$$

$$= \frac{s}{s^2+a^2} \qquad \because \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx - b\sin bx]$$

Hence $L(\cos at) = \frac{s}{s^2+a^2}$

4. $L(\sin at) = \int_0^\infty e^{-st} \sin at\, dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at + a\cos at)\right]_0^\infty$

$$= 0 - \frac{1}{s^2+a^2}(0-a)$$

$L(\sin at) = \frac{a}{s^2+a^2}$

5. $L(\cos hat) = \frac{1}{2}L\left(e^{at} + e^{-at}\right)$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) + \frac{1}{2}\left(\frac{s+a+s-a}{(s+a)(s-a)}\right)$$

$$= \frac{s}{s^2-a^2}$$

$L(\cos hat) = \frac{s}{s^2-a^2}$

6. $L(\sin hat) = \frac{1}{2}L\left(e^{at} - e^{-at}\right) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$

$$= \frac{1}{2}\left(\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right) = \frac{a}{s^2-a^2}$$

L(sin hat) $= \dfrac{a}{s^2-a^2}$

7. *L(1)* $= \int_0^\infty e^{-st}.1.\,dt = \left[\dfrac{e^{-st}}{-s}\right]_0^\infty$

$$= \left(0-\frac{1}{-s}\right) = \frac{1}{s}$$

L(1) $= \dfrac{1}{s}$

8. $L(t^n)$ $= \int_0^\infty e^{-st}t^n\,dt$

$$= \left[(t^n)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty nt^{n-1}\left(\frac{e^{-st}}{-s}\right)dt = (0-0)+\frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

8. $L(t^t)$ $= \int_0^\infty e^{-st}t^n dt$

$$= \left[(t^n)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty nt^{n-1}\left(\frac{e^{-st}}{-s}\right)dt = (0-0)+\frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

$$= \frac{n}{s}L\left(t^{n-1}\right)$$

$L(t^n)$ $= \dfrac{n}{s}L\left(t^{n-1}\right)$

$L(t^{n-1})$ $= \dfrac{n-1}{s}L\left(t^{n-2}\right)$

$L(t^3)$ $= \dfrac{3}{s}L\left(t^2\right)$

$L(t^2)$ $= \dfrac{2}{s}L(t)$

$L(t^n)$ $= \dfrac{n}{s}.\dfrac{n-1}{s}.\dfrac{n-2}{s}.....\dfrac{3}{s}.\dfrac{2}{s}.\dfrac{1}{s}.L(1).$

$$= \frac{n!}{s^n} L[1] = \frac{n!}{s^n} \cdot \frac{1}{s}$$

$$L(t^n) = \frac{n!}{s^{n+1}} \, or \, \frac{\sqrt{(n+1)}}{s^{n+1}}$$

In particular n = 1,2,3….

we get $L(t) = \frac{1}{s^2}$

$$L(t^2) = \frac{2!}{s^3}$$

$$L(t^3) = \frac{3!}{s^4}$$

2.1 Linear propertyof Laplace Transform

1. $L(f(t) \pm g(t)) = L(f(t) \pm L(g(t))$

2. $L(Kf(t) \pm g(t)) = KL(f(t)$

Proof (1): By the defn of L.T

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L(f(t) \pm g(t)] = \int_0^\infty e^{-st}\left[f(t) \pm g(t)\right]dt$$

$$= \int_0^\infty e^{-st} f(t)dt \pm \int_0^\infty e^{-st} g(t)dt$$

$$= L\left[f(t)\right] \pm L\left[g(t)\right]$$

Hence $L[f(t) \pm g(f)] = L[f(t)] \pm L[g(t)]$

(2) $L[Kf(t)] = KL[f(t)]$

By the defn of L.T

$$L[Kf(t)] = \int_0^\infty e^{-st} Kf(t)dt$$

$$= \quad K\int_0^\infty e^{-st} f(t)dt$$

$$= \quad KL[f(t)]$$

Hence $L[K(t)] \quad = \quad KL[f(t)]$

2.2 Recall

1. $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

2. $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

3. $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

4. $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

5. $\sin^2 A = \dfrac{1-\cos 2A}{2}$

6. $\cos^2 A = \dfrac{1+\cos 2A}{2}$

7. $\sin 3A = 3\sin A - 4 \sin^3 A$

8. $\cos 3A = 4\cos^3 A - 3 \cos A$

9. $\sin (A+B) = \sin A \cos B + \cos A \sin B$

10. $\sin (A-B) = \sin A \cos B - \cos A \sin B$

11. $\cos (A-B) = \cos A \cos B + \sin A \sin B$

12. $\cos (A+B) = \cos A \cos B - \sin A \sin B$

3.1 Problems

1. Find Laplace Transform of $\sin^2 t$

Solution

$$L(\sin^2 t) \quad = \quad L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \quad \frac{1}{2}L(1-\cos 2t) = \quad \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

2. Find $L(cos^3t)$

Solution:

We know that $\cos^3 A = 4\cos^3 A - 3\cos A$

hence $\cos^2 A \quad = \quad \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$

$$L(\cos^2 t) = \frac{1}{4}L(3\cos t + \cos 3t) = \frac{1}{4}\left(\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right)$$

3. Find $L(\sin 3t \cos t)$

Solution:

We know that $\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

hence $\sin 3t \cos t \quad = \quad \frac{1}{2}(\sin 4t + \sin 2t)$

$$L(\sin 3t \cos t) = \frac{1}{2}L(\sin 4t + \sin 2t) = \frac{1}{2}\left(\frac{4}{s^2+16} + \frac{2}{s^2+4}\right)$$

$$= \frac{2}{s^2+16} + \frac{1}{s^2+4}$$

4. Find $L(1 + e^{-3t} - 5e^{4t})$

Solution

$$L[1+e^{-3t} - 5e^{4t}] = L[1]\ L[e^{-3t}] + 5L\ (e^{4t}]$$

$$= \frac{1}{s} + \frac{1}{s+3} - \frac{5}{s-4}$$

6. Find $L(3 + e^{6t} + \sin 2t - 5\cos 3t)$

Solution:

$$L(3 + e^{6t} + \sin 2t - 5\cos 3t) = 3L(1) + L(e^{6t}) + L(\sin 2t) - 5L(\cos 3t)$$

$$= 3.\frac{1}{s} + \frac{1}{s-6} + \frac{2}{s^2+4} - \frac{5s}{s^2+9}$$

7. Find $L(\sin(2t + 3))$

Solution:

$$L(\sin(2t+3)) = L(\sin 2t \cos 3 + \sin 3 \cos 2t)$$

$$= \cos 3 L(\sin 2t) + \sin 3 L(\cos 2t)$$

$$= \cos 3 \frac{2}{s^2+4} + \sin 3 \frac{s}{s^2+4}$$

8. Find $L(\sin 4t + 3 \sin h2t - 4 \cos h5t + e^{-5t})$

Solution:

$$L(\sin 4t + 3 \sin h2t - 4 \cos h5t + e^{-5t})$$

$$= L(\sin 4t) + 3L(\sin h2t) - 4L(\cos h5t) + L(e^{-5t})$$

$$= \frac{4}{s^2+16} + 3.\frac{2}{s^2-4} - 4.\frac{s}{s^2-25} + \frac{1}{s+5}$$

$$= \frac{4}{s^2+16} + \frac{6}{s^2-4} - \frac{4s}{s^2-25} + \frac{1}{s+5}$$

9. Find $L((1 + t)^2)$

Solution:

$$L((1+t)^2) = L(1 + 2t + t^2)$$

$$= L(1) + 2L(t) + L(t^2)$$

$$= \frac{1}{s} + 2.\frac{1}{s^2} + \frac{2!}{s^3}$$

3.2 Note

1. $\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$ (By definition)

$\Gamma(n+1) = n!$, $n = 1,2,3,\ldots$

$\Gamma(n+1) = n\Gamma(n)$, $n > 0$

12. Find $L\left(\frac{1}{\sqrt{t}} + t^{3/2}\right)$

Solution:

$$L\left(\frac{1}{\sqrt{t}}+t^{3/2}\right) = L(t^{-1/2}) + L(t^{3/2})$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}+\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}+\frac{3}{2}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}+\frac{3}{4}\frac{\sqrt{\pi}}{s^{\frac{5}{2}}}$$

4. First Shifting Theorem (First translation)

1. If $L(f(t)) = F(s)$. then $L(e^{-at} f(t)) = F(s+a)$

4.1 Corollary: $L(e^{at} f(t)) = F(s-a)$

4.2 Note:

1. $L(e^{-at} f(t)) = L[f(t)]_{s\to s+a}$

$= [F(s)]_{s\to s+a}$

$= F(s+a)$

2. $L(e^{at} f(t)) = L[f(t)]_{s\to s-a}$

$= [F(s)]_{s\to s-a}$

$= F(s-a)$

4.3 Problems

1. Find $L(te^{2t})$

Solution:

$$L(te^{2t}) = [L(t)]_{s\to s-2} = \left(\frac{1}{s^2}\right)_{s\to s-2} = \frac{1}{(s-2)^2}$$

2. Find $L(t^5 e^{-t})$

Solution:

$L(t^5e^{-t}) \quad = \quad [L(t^5)]_{s\to s+1} = \left(\frac{5!}{s^6}\right)_{s\to s+1} \quad = \quad \frac{5!}{(s+1)^6}$

3. Find $L(e^{-2t} \sin 3t)$

Solution:

$L(e^{-2t} \sin 3t) \quad = \quad L(\sin 3t)]_{s\to s+2}$

$= \left(\frac{3}{s^2+9}\right)_{s\to s+2} \quad = \quad \frac{3}{(s+2)^2+9}$

$= \frac{s+1}{(s+1)^2-16}$

5. Theorem

If $L(f(t)) = F(s)$, then $L(tf(t)) = \frac{-d}{ds}(F(s))$

similarly $L(t^2 f(t)) \quad = (-1)^2 \frac{d^2}{ds^2}F(s)$

$L(t3 f(t)) \quad = \quad (-1)^3 \frac{d^3}{ds^3}F(s)$

In general, $L(t^n f(t)) \quad = \quad (-1)^n \frac{d^n}{ds^n}F(s)$

5.1 Problems

1. Fine $L(te^{3t})$

Solution:

We know that $L(tff(t)) = \frac{-d}{ds}L(f(t))$

Here $f(t) = e^{3t}$

L $(te^{3t}) \quad = \quad \frac{-d}{ds}L(e^{3t}) = \quad \frac{-d}{ds}\left(\frac{1}{s-3}\right)$

$= \left(-\frac{(s-3)(0)-(1)}{(s-3)^1}\right) \quad = \quad \frac{1}{(s-3)^2}$

2. Find $L(t\sin 3t)$

Solution:

$$L(tf(t)) = \frac{-d}{ds}L(f(t))$$

$$L(tf(t)) = \frac{-d}{ds}L(\sin 3t) = \frac{-d}{ds}\left(\frac{3}{s^2+9}\right) = \left(\frac{-(s^2+9)(0)+3(2s)}{(s^2+9)^2}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

3. Find $L(te^{-2t}\sin 3t)$

Solution:

$$L(e^{-2t}(t\sin 3t)) = L(t\sin 3t)_{s\to s+2}$$

$$= \left\{\frac{-d}{ds}L(\sin 3t)\right\}_{s\to s+2} = \left\{\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}\right\}_{s\to s+2} = \frac{6(s+2)}{((s+2)^2+9)^2}$$

6. Theorem

If $L(f(t)) = F(s)$ and if $\underset{t\to 0}{Lt}\frac{f(t)}{t}$ exist then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty e^{-st}f(t)ds$

Recall

1. $\log(AB) = \log A + \log B$
2. $\log(A/B) = \log A - \log B$
3. $\log A^B = B\log A$
4. $\log 1 = 0$
5. $\log 0 = -\infty$
6. $\log\infty = \infty$

7. $\int \frac{1}{x} dx = \log x$

8. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

9. $\tan^{-1}(\infty) = \frac{\pi}{2}$

10. $\cot^{-1}\left(s/a\right) = \frac{\pi}{2} - \tan^{-1}\left(s/a\right)$

Problems

1. Find $L\left(\frac{1-e^{2t}}{t}\right)$

Solutions:

$$\lim_{t \to 0} \frac{1-e^{2t}}{t} = \frac{0}{0} \text{ (Interminate form)}$$

Apply L – Hospital Rule

$$\lim_{t \to 0} \frac{-2e^{2t}}{1} = -2$$

$\therefore$ the given function exists in the limit t $\to$ 0

$$L\left(\frac{1-e^{2t}}{t}\right) = \int_s^\infty L\left(1-e^{2t}\right) ds$$

$$= \int_s^\infty \left(L(1) - L\left(e^{2t}\right)\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-2}\right) ds$$

$$= \left(\log s - \log(s-2)\right)_s^\infty$$

$$= \left[\log\left(\frac{s}{s-2}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s\left(1-\frac{2}{s}\right)}\right)\right]_s^\infty = \log\left(\frac{1}{1-\frac{2}{s}}\right)_s^\infty$$

$$= 0 - \log\frac{s}{s-2}$$

$$= \log\left(\frac{s}{s-2}\right)^{-1}$$

$$= \log\left(\frac{s-2}{s}\right)$$

2. Find $L\left(\frac{1-\cos at}{t}\right)$

Solution:

$\lim_{t\to 0}\frac{1-\cos at}{t} = \frac{0}{0}$ (Indeterminte form)

Apply L – Hospital Rule.

$\lim_{t\to 0}\frac{a\ \sin at}{1} = 0$ (finite)

$\therefore$ the given function exists in the limit t $\rightarrow$ 0

$$L\left(\frac{1-\cos at}{t}\right) = \int_s^\infty L(1-\cos at)\,ds = \int_s^\infty \left(L(1) - L(\cos at)\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right)ds = \left(\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right)_s^\infty$$

$$= \left(\log s - \log\left(s^2+a^2\right)^{\frac{1}{2}}\right)_s^\infty = \left(\log\frac{s}{\sqrt{s^2+a^2}}\right)_s^\infty$$

$$= \left(\log\frac{s}{s\sqrt{1+\frac{a^2}{s^2}}}\right)_s^\infty = \left(\log\frac{1}{\sqrt{1+\frac{a^2}{s^2}}}\right)_s^\infty$$

$$= \left(\log 1 - \log\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{s}{\sqrt{s^2+a^2}}\right) = \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)^{-1} = \log\left(\frac{\sqrt{a^2+s^2}}{s}\right)$$

11. Laplace Transform of Derivations

Here, we explore how the Laplace transform interacts with the basic operators of calculus differentation and integration. The greatest interest will be in the first identity that we will derive. This relates the transform of a derivative of a function to the transform of the original function, and will allow to convert many initial - value problems to easily solved algebraic Equations. But there are useful relations involving the Laplace transform and either differentiation (or) integration. So we'll look at them too.

11.1. Theorem

If $L(f(t)) = F(s)$ Then

(i) $L(f'(t)) = sL(f(t)) - f$

(ii) $L(f''(t)) = s^2L(f(t)) - sf(0) - f'(0)$

and in genereal

$$L(f^n(t)) = s^nL(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots\ldots f^{n-1}(0)$$

11.2 Note

We have, $L(f'(t)) = sL(f(t)) - f(0)$ …. (1) and

$L(f''(t)) = s^2L(f(t)) - sf(0) - f'(0)$ ….(2)

When $f(0) = 0$ and $f'(0) = 0$

(1) & (2) becomes

$Lf'(t) = sLf(t)$ and $Lf''(t) = s^2Lf(t)$

This shows that under certain conditions, the process of Laplace transform replaces differentiation by multiplication by the factor s and s 2 respectively.

13 Definition

If the Laplace transform of a function $f(t)$ is $F(S)$ (ie) $L(f(t)) = F(S)$ then $f(t)$ is called an inverse laplace transform of $F(s)$ and is denoted by

$$f(t) = L^{-1}(F(s))$$

Here L^{-1}1 is called the inverse Laplace transform operator.

14. Standard results in inverse Laplace transforms

Laplace Transform	**Inverse Laplace Transform**
$L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
$L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
$L(t) = \frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right) = t$
$L(t^2) = \frac{2!}{s^3}$	$L^{-1}\left(\frac{2!}{s^3}\right) = t^2$
$L(t^3) = \frac{3!}{s^4}$	$L^{-1}\left(\frac{3!}{s^4}\right) = t^3$
$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

where n is a +ve integer

$L(\sin at) = \frac{a}{s^2 + a^2}$	$L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at$
$L(\cos at) = \frac{s}{s^2 + a^2}$	$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$
$L(\sin hat) = \frac{a}{s^2 - a^2}$	$L^{-1}\left(\frac{a}{s^2 - a^2}\right) = \sin hat$
$L(\cos hat) = \frac{s}{s^2 - a^2}$	$L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cos hat$

15 Partial Fraction

The rational fraction P(x)/Q(x) is said to be resolved into partial fraction if it can be expressed as the sum of difference of simple proper fractions.

Rules for resolving a Proper Fraction P(x) / Q(x) into partial fractions

Rule 1

Corresponding to every non repeated, linear factor (ax+b) of the denomiator Q(x), there exists a partial fraction of the form $\frac{A}{ax+b}$ where A is a constant, to be determined.

For Example

(i) $$\frac{2x-7}{(x-2)(3x-5)} = \frac{A}{x-2} + \frac{B}{3x-5}$$

(ii) $$\frac{5x^2+18x+22}{(x-1)(x+2)(2x+3)} = \frac{A}{x-1} + \frac{A}{x+2} + \frac{C}{2x+3}$$

Rule 2

Corresponding to every repeated linear factor $(ax\ b)^k$ of the denominator Q(x), there exist k partial fractions of the forms,

$$\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \frac{A_3}{(ax+b)^3}, \cdots \frac{A_k}{(ax+b)^k}$$

where $A_1, A_2, \ldots\ldots A_k$ are constants to be detemined.

For example

(i) $$\frac{4x-3}{(x+2)(2x-3)^2} = \frac{A}{x+2} + \frac{B}{2x-3} + \frac{C}{(2x-3)^2}$$

(ii) $$\frac{x+2}{(x-1)(2x-1)^3} = \frac{A}{x-1} + \frac{B}{(2x+1)} + \frac{C}{(2x-1)^2} + \frac{D}{(2x+1)^3}$$

Rule 3

Corresponding to every non-repeated irreducible quadratic factor $ax^{2\ +}\ bx + c$ of the denominator Q(x) there exists a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.

$(ax^2 + bx + c)$ is said to be an irreducible quadratic factor, if it cannot be factorized into two linear fractors with real coefficients.

Example

(i) $$\frac{x^2+1}{(x^2+4)(x^2+9)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+9}$$

(ii) $$\frac{8x^3-5x^2+2x+4}{(2x-1)^2(3x^2+4)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{Cx+D}{3x^2+4}$$

In the case of an improper fraction, by division, it can be expressed as the sum of integral function and a proper fraction and then proper fraction is resolved into partial fractions.

Inverse Laplace Transform using Partial Fractions

1. Find $L^{-1}\left(\frac{1}{(s+1)(s+3)}\right)$

Solution:

Let $F(s) = \left(\frac{1}{(s+1)(s+3)}\right)$

Let us split $F(S)$ into partial fractions,

$$\frac{1}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$1 = A(S+3) + B(S+1)$$

Putting $S = -1$

$A = \frac{1}{2}$

Putting $S = -3$

$B = -\frac{1}{2}$

$$\therefore \frac{1}{(s+1)(s+3)} = \frac{\frac{1}{2}}{(s+1)} + \frac{-\frac{1}{2}}{(s+3)}$$

$$\therefore \left(\frac{1}{(s+1)(s+3)}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

$$= \frac{1}{2}\left(e^{-t} - e^{-3t}\right)$$

2. Find $L^{-1}\left(\dfrac{s^2+s-2}{s(s+3)(s-2)}\right)$

Solution:

Consider, $\dfrac{s^2+s-2}{s(s+3)(s-2)}=\dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{C}{s-2}$

$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A(s+3)(s-2)+Bs(s-2)+Cs(s+3)}{s(s+3)(s-2)}$$

$$s^2+s-2=A(s+3)(s-2)+Bs(s-2)+Cs(s+3)$$

put s = -3

9 – 3 – 2 -= B(-3)(5)

4=15B

$B = \dfrac{4}{15}$

put s = 2

4 + 2 – 2 -= C(2)(5)

4 = 10C

$\therefore C = \dfrac{4}{10}$

$C = \dfrac{2}{5}$

put s = 0

-2 = A(3)(-2)

$A = \dfrac{1}{3}$

$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{1}{3}.\frac{1}{s}+\frac{4}{15}.\frac{1}{s+3}+\frac{2}{5}.\frac{1}{s-2}$$

$$\therefore L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right)=\frac{1}{3}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right)+\frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right)$$

$$=\frac{1}{3}(1)+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$

3. Find $L^{-1}\left(\dfrac{s}{s^2+5s+6}\right)$

Solution:

Consider, $\dfrac{s}{s^2+5s+6}=\dfrac{s}{(s+2)(s+3)}=\dfrac{A}{(s+2)}+\dfrac{B}{(s+3)}$

$S = A\ (s + 3) + B(s + 2)$

Put $s = -3$ Put $s = -2$

$-3 = A(0) + b(-1)$ $-2 = A(1) + B(0)$

$-3 = -B$ $A = -2$

$B = 3$

$$\frac{s}{(s+2)(s+3)} = \frac{-2}{(s+2)} + \frac{3}{(s+3)}$$

$$\therefore L^{-1}\left(\frac{1}{(s+2)(s+3)}\right) = 2L^{-1}\left(\frac{1}{(s+2)}\right) + 3L^{-1}\left(\frac{B}{(s+3)}\right)$$

$$= -2e^{-2t} + 3e^{-3t}$$

4. Find $L^{-1}\left(\frac{s}{(s+1)^2}\right)$

Solution:

Consider, $\frac{s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$

$$\frac{s}{(s+1)^2} = \frac{A(s+1)+B}{(s+1)^2}$$

$s = A(s + 1) + B$

Put $s = -1$ Put $s = 0$

$B = -1$ $0 = A + B$

$0 = A - 1$

$A = 1$

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$L^{-1}\left(\frac{s}{(s+1)^2}\right) = L^{-1}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$= L^{-1}\left(\frac{1}{(s+1)}\right) - L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= \quad e^{-t} - e^{-t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= \quad e^{-t} - e^{-t}(t) = e^{-t}(1-t)$$

16. Convolution of two functions

If $f\,(t)$ and $g(t)$ are given functions, then the convolution of $f\,(t)$ and $g(t)$ is defined as $\int_0^t f(u)g(t-u)du.$ It is denoted by $f\,(t) * g\,(t)$.

16.1 Convolution Theorem

If $f\,(t)$ and $g(t)$ are functions defined for $t \geq 0$, then $L(f(t) * g(t)) = L(\,f\,(t))L(g(t))$

(ie) $L(f(t) * g(t)) = F(s).\ G(s)$

where $F(s) = L(f(t)),\ G(s) = L(g(t))$

Proof :

By definition of Laplace Transform,

We have $L(f(t)) * g(t) \quad = \quad \int_0^\infty \left\{e^{-st} f(t) * g(t)\right\} dt$

$$= \quad \int_0^\infty e^{-st} \left\{ \int_0^t f(u)(t-u)du \right\} dt$$

$$= \quad \int_0^\infty \int_0^t e^{-st} f(u) g(t-u) du dt$$

on changing the order of integration,

$$= \quad \int_0^\infty f(u) \left\{ \int_u^\infty e^{-st} g(t-u) du \right\} dt$$

Put $\quad t - u = v \qquad\qquad$ When $\quad t = u, v = 0$

$\quad dt = dv \qquad\qquad$ When $\quad t = \infty,\ v = \infty$

$$L(f(t)) * g(t) \quad = \quad \int_0^\infty f(u) \left\{ \int_0^\infty e^{-s(u+v)} g(v) dv \right\} du$$

$$= \quad \int_0^\infty f(u) e^{-su} \left\{ \int_0^\infty e^{sv} g(v) dv \right\} du$$

$$= \int_0^\infty e^{su} f(u)du \int_0^\infty e^{-sv} g(v)dv$$

$$= \int_0^\infty e^{-st} f(t)dt \int_0^\infty e^{-st} g(t)dv$$

$$= L(f)(t))L(g(t))$$

$$\therefore L(f(t)) * g(t) = F(s).G(s)$$

Corollary

Using the above theorem

We get,

$$L^{-1}(F(s).G(s)) = f(t) * g(t)$$

$$= L^{-1}(F(s) * L^{-1}(G(s))$$

Note

$$f(t) * g(t) = g(t) * f(t)$$

1. Use convolution theorem to find $L^{-1}\left(\frac{1}{(s+a)(s+b)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s+a)(s+b)}\right) = L^{-1}\left(\frac{1}{(s+a)}\right) * L^{-1}\left(\frac{1}{(s+b)}\right) = e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)} du = \int_0^t e^{-au} e^{-bt+bu} du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t = \frac{e^{-bt}}{-(a-b)}(e^{-(a-b)t} - 1)$$

$$= \frac{e^{-bt}}{-(a-b)} + \frac{e^{-bt}}{(a-b)} = \frac{1}{(a-b)}(e^{-bt} e^{-at})$$

2. Use convolution theorem to find $L^{-1}\frac{1}{s(s^2+1)}$

Solution:

$$L^{-1}\frac{1}{s(s^2+1)} = L^{-1}\left(\frac{1}{s}\right)*L^{-1}\left(\frac{1}{s^2+1}\right) = 1*\sin t$$

$$= \int_0^t \sin(t-u)du = \left[\frac{-\cos(t-u\_}{-1}\right]_0^t$$

$$= \cos 0 - \cos \mathrm{t} = 1 - \cos t$$

3. Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$ using convolution theorem

Solution:

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right)$$

$$= L^{-1}\left(\frac{s}{(s^2+a^2)}\right)*L^{-1}\left(\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right)$$

$$= \cos at*\frac{1}{a}\sin at$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)du$$

$$= \frac{1}{a}\int_0^t \left(\frac{\sin a(t-u+u)+\sin a(t-u-u)}{2}\right)du$$

$$= \frac{1}{2a}\int_0^t (\sin at + \sin a(t-du))du$$

$$= \frac{1}{2a}\left[u\sin at + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$= \frac{t\sin at}{2a}$$

APPLICATIONS OF INVERSE LAPLACE TRANSFORMS

1. Using L.T solve $y''-3y'+2y=e^{-t}$ given y(0) −1, $y'(0)=0$

Solution:

$$y''-3y'+2y=e^{-t} \text{ and } y(0)=1,\ y'(0)=0$$

Taking L.T on bothsides,

$$L[y''(t)]-3L[y'(t)]+2L[y(t)]=L[e^{-t}]$$

$$s^2L[y(t)]-sy(0)-y'(0)-3[sL[y(t)]-y(0)]+2L[y(t)]=\frac{1}{s+1}$$

$$s^2L[y(t)]-s-0-3sL[y(t)]+3+2L[y(t)]=\frac{1}{s+1}$$

$$(s^2-3s+2)L[y(t)]=\frac{1}{s+1}+s-3$$

$$(s-1)(s-2)L[y(t)]=\frac{s^2-2s-2}{s+1}$$

$$L[y(t)]=\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}=\frac{A}{s+1}+\frac{B}{s-1}+\frac{C}{s-2}$$

$$s^2-2s-2=A(s-1)(s-2)+B(s+1)(s-2)+C(s+1)(s-1)$$

Put $s=1$, *we get*

$$1-2-2=-2B$$
$$-3=-2B$$

$$-3=-2B$$
$$B=\frac{3}{2}$$

Put $s=2$, *we get*

$$4-4-2=3C$$
$$C=\frac{-2}{3}$$

Put $s=-1$, *we get*

$$1+2-2=6A$$
$$A=\frac{1}{6}$$

$$L[y(t)]=\frac{1/6}{s+1}+\frac{3/2}{s-1}+\frac{2/3}{s-2}$$

$$=\frac{1}{6}\frac{1}{s+1}+\frac{3}{2}\frac{1}{s-1}+\frac{2}{3}\frac{1}{s-2}$$

$$y(t)=\frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right]+\frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right]+\frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$=\frac{1}{6}e^{-t}+\frac{3}{2}e^{t}+\frac{2}{3}e^{2t}$$